FEATURE COLUMN Monthly essays on mathematical topics

Congressional Redistricting and Gerrymandering

We'll look at some measures that have been created to constrain gerrymandering...

David Austin
Grand Valley State University
Email David Austin

Introduction

In the 2012 election for the U.S. House of Representatives, the Republican Party won 234 seats to the Democratic Party's 201. With a total of 435 seats, the Republicans' 33 seat majority represents a 7.5% advantage over the Democratics.

At the same time, Democratic candidates received 1,711,566 more votes than their Republican opponents. With nearly 125 million votes cast in the elections for congressional representatives, Democratic candidates received about 1.7% more votes.

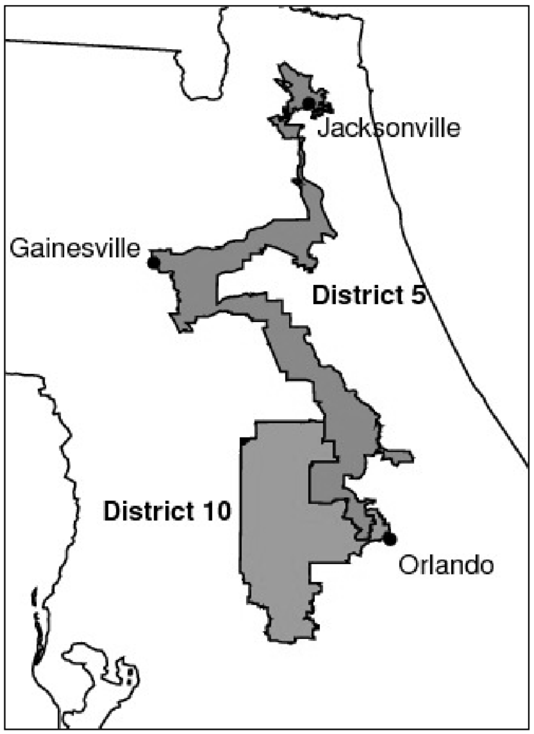
Why is there such a disparity between the votes cast and the seats won? Though this is a complicated issue with several factors cited as influential, the process by which congressional districts are drawn is frequently cited as the most important.

In 34 states, congressional districts are drawn by the state legislature (seven states have only one representative so that the entire state forms a district), which can give the political party controlling the state legislature the power to draw districts that favor candidates of their own party.

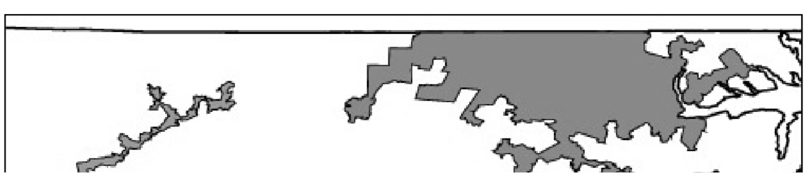
Many states place limits on the legislature's power to draw congressional maps. For instance, most states require that districts be *contiguous,* meaning that one can walk between any two points in a district without leaving that district. Other requirements include respecting local political boundaries and the previous congressional map so that

- *packing*, which concentrates an overwhelming majority of the political opposition in a single district. In essence, you give up one district in exchange for creating several districts in which your party enjoys a smaller, but safe, majority.

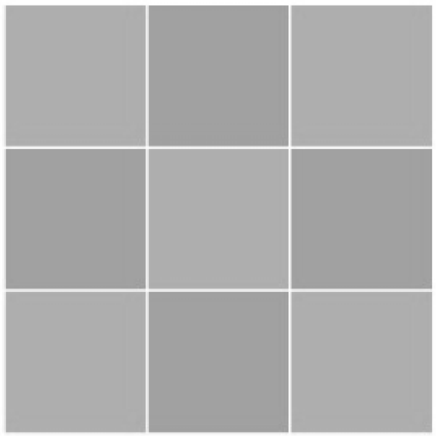
  For instance, a circuit court judge recently ruled that the Florida congressional map is invalid and that Districts 5 and 10, and consequently the entire map, will have to be redrawn. In particular, District 5 appears to have been drawn with the intention of concentrating minority voters into a single district.



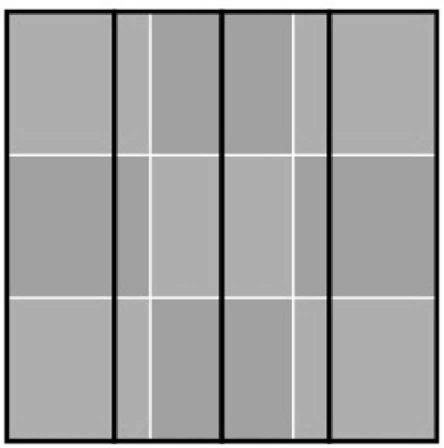
  Similarly, voters in North Carolina have filed suit challenging their state's congressional map.



Suppose that the support for two parties in a rectangular state that is allotted four representatives is geographically dispersed as shown. Notice that a 5 to 4 majority support the red party.



If we simply slice the state into four vertical districts, each party wins two seats.



However, if we slice the state once horizontally and once vertically, the red party wins all four seats.

Besides showing that the results can vary considerably depending on which algorithm is used, this example shows how an algorithm can lead to the very effect we are trying to avoid.

Comments?

Compactness measures

Instead, we'll look at some measures that have been created to constrain gerrymandering. Many states require congressional districts to satisfy a *compactness* condition. While there is no single definition of compactness, the concept attempts to identify oddly shaped regions that may be the result of gerrymandering. We'll look at a couple of different notions that have been used to measure compactness before considering a separate notion called *bizarreness*.

We will consider four widely-cited measures of compactness. These measures are created by comparing a district to an "ideal" simple figure, such as a rectangle or circle, with the assumption that these figures are ideal because they involve a small number of choices making them of less use for gerrymandering. A tabulation of these four measures may be found here and here.

All four measures vary from 0 to 1 with districts scoring lower judged to be less compact and hence more possibly the result of gerrymandering.

- The first and most widely used measure, due to Polsby and Popper, relies on the isoperimetric inequality, which says this: for any simple closed curve in the plane with perimeter P and bounding area A,then

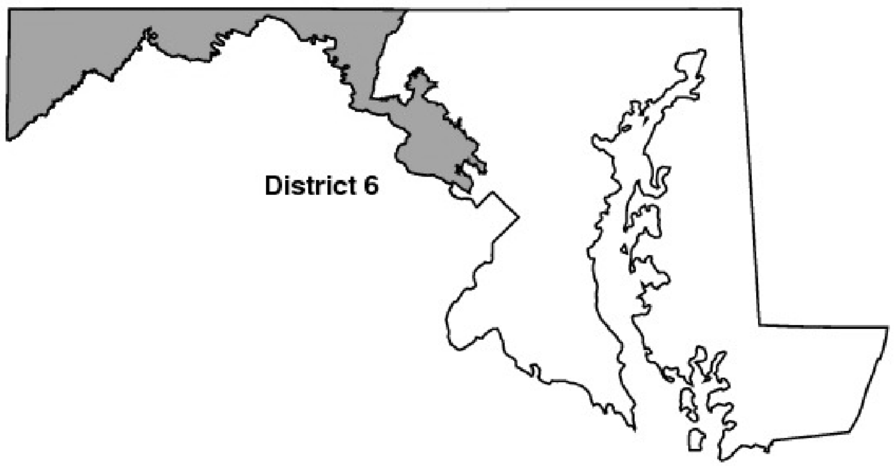
  $$4\pi A \leq P^2$$

  and equality holds exactly when the curve is a circle. This implies that among all simple closed planar curves of the same perimeter, the circle encloses the largest area. The Polsby-Popper measure is computed as

  $$\frac{4\pi A}{P^2} \leq 1$$

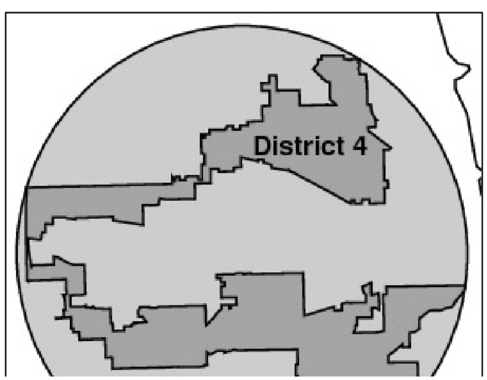
  The effect is to compare the area of a congressional district to the area of a circle having

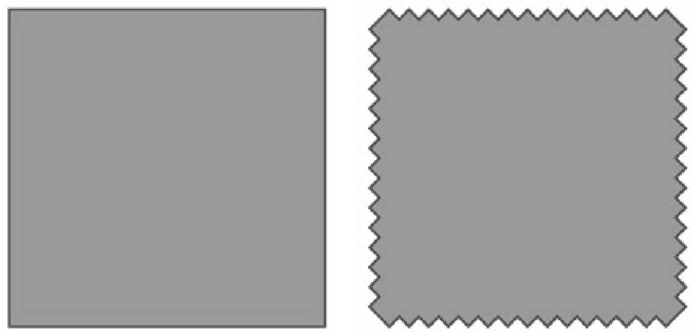
With a Polsby-Popper score of 0.071, Maryland's 6th District stands out. However, this appears to result from the fact that the district's boundary is partially formed by the Potomac River rather than an attempt to gain a political advantage through congressional redistricting.



The Polsby-Popper measure is easily computed using cartographic boundary files made publicly available by the U.S. Census Bureau. In those files, the districts are described as polygonal shapes with given vertices (λ_i, ϕ_i) in longitude-latitude coordinates.

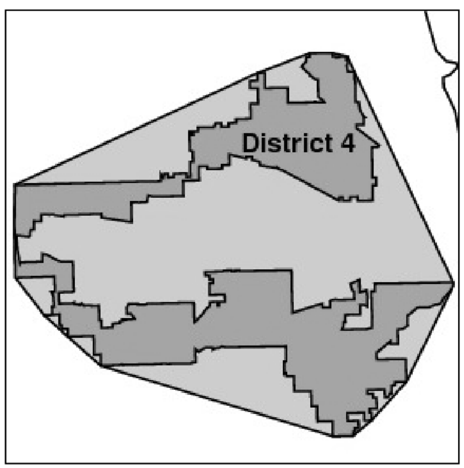
- Our second measure of compactness, due to Reock, is the ratio of the area of the district to the area of the smallest circle circumscribing the district. As in the Polsby-Popper measure, each congressional district is compared to an ideal circular district.





The Schwartzberg measure is quite sensitive to small variations in the boundary of the region. Remember that districts are often required to follow local political boundaries, such as townships, which could naturally cause an increase in the perimeter.

- The Convex Hull measure is the ratio of the area of a district to that of the conex hull of the district. We may, therefore, think of this as a measurement of the convexity of the district. Convex districts, for instance, would be assigned a measure of 1, whereas districts such as the Illionois 4$^{\text{th}}$ District would be significantly lower and hence less compact.



Comments?

The bizareness of a district

Miller define the bizarreness of a district to be the probability that the shortest path within the state between any two people in the district stays within the district. For instance, a convex district would have a bizarreness score of 1. Since we consider only the shortest path within the state, states with irregular boundaries are not penalized.

To be more precise, suppose that x_i is the location of the i^{th} person in the district. We will also define $d_S(x_i, x_j)$ to be the length of the shortest path within the state between the i^{th} and j^{th} person and $d_D(x_i, x_j)$ the length of the shortest path within the district. The ratio

$$\frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} \leq 1,$$

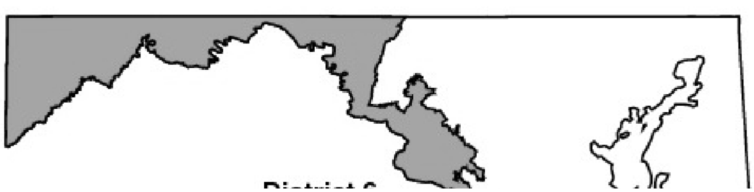
and equality holds exactly when the shortest path within the state lies within the district. If there are a total of N people in the district, we then define the bizarreness of the district to be

$$\frac{1}{N^2} \sum_{i,j} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)}.$$

Notice that the bizarreness lies between 0 and 1 and equals 1 when the district is convex. Lower values of the bizarreness may indicate gerrymandering.

The bizarreness may be practically computed using census tracts. Suppose that a census tract within the district has population p_α and is centered at x_α. Then the bizarreness may be computed by

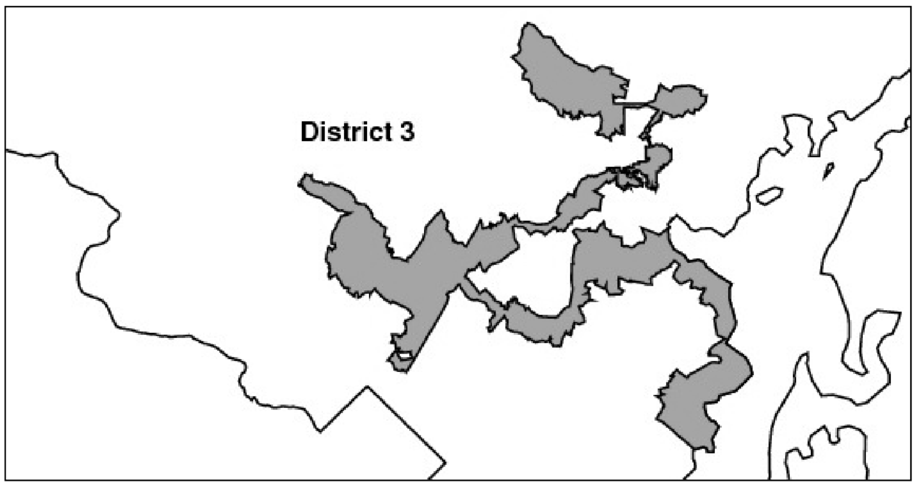
$$\frac{1}{N^2} \sum_{\alpha,\beta} \frac{d_S(x_\alpha, x_\beta)}{d_D(x_\alpha, x_\beta)} p_\alpha p_\beta.$$



Polsby-Popper	0.071
Reock	0.121
Schwartberg	0.119
Convex Hull	0.562
Bizarreness	0.926

While the irregular shape of this district results from the state's boundary, the first four measures indicate that this district may be the result of gerrymandering. The bizarreness measure, however, indicates that it is most likely not.

If we look instead at Maryland's 3rd District, we find that all five measures indicate the possibility of gerrymandering.



Measure	Score
Polsby-Popper	0.032
Reock	0.194
Schwartberg	0.029
Convex Hull	0.402
Bizarreness	0.140

Hodge, Marshall, and Patterson have amended Chambers and Miller's definition of bizarreness to make it somewhat easier to compute. Generally speaking, it takes some work to determine the shortest path between two points within a congressional district. Hodge and his collaborators sidestep this issue by defining a notion of semi-visibility.

about the nature of the problem.

Instead, we've looked at a few of the common measures that are used to indicate the possibility of gerrymandering. These have, in fact, formed part of the basis for legal arguments challenging some congressional maps.

While none of the measures is perfect, taken together they may be effective for evaluating redistricting plans. A quick Google search will reveal any number of alternative measures.

Of course, there is always what one writer has humorously called the Grofman Interocular Test: like pornography, you recognize gerrymandering when you see it.

Comments?

References

- **Christopher Chambers and Alan Miller.** A Measure of Bizarreness. RepRap Home Page. *Quarterly Journal of Political Science,* Vol. 5, no. 1, pp. 27-44, 2010.

- **Jonathan Hodge, Emily Marshall, and Geoff Patterson.** Gerrymandering and Convexity. *College Mathematics Journal,* vol. 41, no. 4, pp. 312-24, 2010.

- **Alan Maceachren.** Compactness of Geographic Shape: Comparison and Evaluation of Measures. *Geografiska Annaler,* Vol. 67, no. 1, pp. 53-67, 1985.

- **William Vickrey.** On the Prevention of Gerrymandering. *Political Science Quarterly,* Vol. 76, no. 1, pp. 105-10, 1961.

- **Edward Tufte.** The Relationship between Seats and Votes in Two-Party Systems. *The American Political Science Review,* Vol. 67, no. 2, pp. 540-54, 1973.

- 2012 Federal Election Results.