Algunas identidades de álgebra y análisis vectorial

En lo que sigue, ϕ y ψ representan escalares; $\mathbf{a}$, $\mathbf{b}$ *y* $\mathbf{c}$ *vectores.*

1. **Triples productos y productos mixtos**

   i) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

   ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

   iii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \quad (\equiv [\mathbf{a}, \mathbf{b}, \mathbf{c}])$

   iv) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$

2. **Identidades cuadráticas en ∇**

   i) $\nabla \cdot (\nabla \times \mathbf{a}) = 0$

   ii) $\nabla \times (\nabla \phi) = \mathbf{0}$

   iii) $\nabla^2 \mathbf{a} \equiv \nabla(\nabla \cdot \mathbf{a}) - \nabla \times (\nabla \times \mathbf{a})$
   (Laplaciano de un campo vectorial).

3. **Reglas de producto**

   i) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

   ii) $\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$

   iii) $\nabla \cdot (\phi \mathbf{a}) = \phi \nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla \phi$

   iv) $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$

   v) $\nabla \times (\phi \mathbf{a}) = \phi \nabla \times \mathbf{a} - \mathbf{a} \times \nabla \phi$

   vi) $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a}$

   vii) $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) - \mathbf{b} \times (\nabla \times \mathbf{a}) - (\mathbf{a} \times \nabla) \times \mathbf{b} + (\mathbf{b} \times \nabla) \times \mathbf{a}$

4. **Teoremas integrales**

   i) $\int_{\mathbf{r}_1}^{\mathbf{r}_2} (\nabla \phi) \cdot d\mathbf{l} = \phi(\mathbf{r}_2) - \phi(\mathbf{r}_1)$

   ii) $\int_V (\nabla \cdot \mathbf{a}) d^3 r = \int_{S(V)} \mathbf{a} \cdot d\mathbf{s}$

   iii) $\int_{S(C)} (\nabla \times \mathbf{a}) \cdot d\mathbf{s} = \oint_C \mathbf{a} \cdot d\mathbf{l}$

   iv) $\int_V (\nabla \phi)\, d^3 r = \int_{S(V)} d\mathbf{s}\, \phi$

   v) $\int_V (\nabla \times \mathbf{a})\, d^3 r = \int_{S(V)} d\mathbf{s} \times \mathbf{a}$

5. **Coordenadas curvilíneas ortogonales**

   Son sistemas de coordenadas $u_i(x, y, z)$ $(i = 1, 2, 3)$ tales que las líneas coordenadas que pasan por cada punto, son ortogonales. Es decir, para un desplazamiento infinitesimal $d\mathbf{l}$, la longitud de arco correspondiente ds verifica: $ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$ (donde los h_i son, en gral., funciones de u_1, u_2 y u_3). El elemento de volumen será entonces: $d^3 r = h_1 h_2 h_3\, du_1 du_2 du_3$

   Operadores vectoriales

$$\nabla\phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}\hat{\mathbf{u}}_3, \qquad \nabla \cdot \mathbf{a} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial\,(a_1 h_2 h_3)}{\partial u_1} + \frac{\partial\,(a_2 h_1 h_3)}{\partial u_2} + \frac{\partial\,(a_3 h_1 h_2)}{\partial u_3}\right)$$

$$\nabla \times \mathbf{a} = \frac{1}{h_2 h_3}\left[\frac{\partial\,(a_3 h_3)}{\partial u_2} - \frac{\partial\,(a_2 h_2)}{\partial u_3}\right]\hat{\mathbf{u}}_1 + \frac{1}{h_1 h_3}\left[\frac{\partial\,(a_1 h_1)}{\partial u_3} - \frac{\partial\,(a_3 h_3)}{\partial u_1}\right]\hat{\mathbf{u}}_2 + \frac{1}{h_1 h_2}\left[\frac{\partial\,(a_2 h_2)}{\partial u_1} - \frac{\partial\,(a_1 h_1)}{\partial u_2}\right]\hat{\mathbf{u}}_3$$

Ejemplos

Coordenadas cartesianas

$d\mathbf{l} = dx\ \hat{\mathbf{x}} + dy\ \hat{\mathbf{y}} + dz\ \hat{\mathbf{z}}$
$d^3r = dx\ dy\ dz$
$\nabla\phi = \partial_x\phi\ \hat{\mathbf{x}} + \partial_y\phi\ \hat{\mathbf{y}} + \partial_z\phi\ \hat{\mathbf{z}}$
$\nabla\cdot\mathbf{a} = \partial_x a_x + \partial_y a_y + \partial_z a_z$
$\nabla\times\mathbf{a} = (\partial_y a_z - \partial_z a_y)\ \hat{\mathbf{x}} + (\partial_z a_x - \partial_x a_z)\ \hat{\mathbf{y}} + (\partial_x a_y - \partial_y a_x)\ \hat{\mathbf{z}}$
$\nabla^2\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi$

Coordenadas cilíndricas

$d\mathbf{l} = dr\ \hat{\mathbf{r}} + r\ d\varphi\ \hat{\varphi} + dz\ \hat{\mathbf{z}}$
$d^3r = r\ dr\ d\varphi\ dz$
$\nabla\phi = \partial_r\phi\ \hat{\mathbf{r}} + r^{-1}\partial_\varphi\phi\ \hat{\varphi} + \partial_z\phi\ \hat{\mathbf{z}}$
$\nabla\cdot\mathbf{a} = r^{-1}\partial_r(ra_r) + r^{-1}\partial_\varphi a_\varphi + \partial_z a_z$
$\nabla\times\mathbf{a} = [r^{-1}\partial_\varphi a_z - \partial_z a_\varphi]\ \hat{\mathbf{r}} + [\partial_z a_r - \partial_r a_z]\ \hat{\varphi} + r^{-1}[\partial_r(ra_\varphi) - \partial_\varphi a_r]\ \hat{\mathbf{z}}$
$\nabla^2\phi = r^{-1}\partial_r(r\partial_r\phi) + r^{-2}\partial_\varphi^2\phi + \partial_z^2\phi$

Coordenadas esféricas

$d\mathbf{l} = dr\ \hat{\mathbf{r}} + r\ d\theta\ \hat{\theta} + r\ \text{sen}\ \theta\ d\varphi\ \hat{\varphi}$
$d^3r = r^2\text{sen}\ \theta\ dr\ d\theta\ d\varphi$
$\nabla\phi = \partial_r\phi\ \hat{\mathbf{r}} + r^{-1}\partial_\theta\phi\ \hat{\theta} + (r\ \text{sen}\ \theta)^{-1}\partial_\varphi\phi\ \hat{\varphi}$
$\nabla\cdot\mathbf{a} = r^{-2}\partial_r(r^2 a_r) + (r\ \text{sen}\ \theta)^{-1}\partial_\theta(\text{sen}\ \theta\ a_\theta) + (r\ \text{sen}\ \theta)^{-1}\partial_\varphi a_\varphi$
$\nabla\times\mathbf{a} = (r\ \text{sen}\ \theta)^{-1}[\partial_\theta(\text{sen}\ \theta\ a_\varphi) - \partial_\varphi a_\theta]\ \hat{\mathbf{r}} + r^{-1}[(\text{sen}\ \theta)^{-1}\partial_\varphi a_r - \partial_r(ra_\varphi)]\ \hat{\theta} + r^{-1}[\partial_r(ra_\theta) - \partial_\theta a_r]\ \hat{\varphi}$
$\nabla^2\phi = r^{-2}\partial_r(r^2\partial_r\phi) + (r^2\text{sen}\ \theta)^{-1}\partial_\theta(\text{sen}\ \theta\partial_\theta\phi) + (r^2\text{sen}^2\theta)^{-1}\partial_\varphi^2\phi$