

8.6.1 Appendix 8.A: The ESPRIT algorithm

An important example for high-resolution algorithms is the *ESPRIT* (Estimation of Signal Parameters by Rotational Invariance Techniques) algorithm. The basic idea is as simple as it is ingenious. Consider an antenna array with N_r elements. Now create a subarray with the elements $1, 2, \dots, N_r - 1$, and a subarray with elements $2, 3, \dots, N_r$. It is clear from Eq. (8.37) that the steering matrices of the subarrays are linked by the following matrix equation:

$$\mathbf{A}_2 = \mathbf{A}_1 \mathbf{\Phi} \quad (8.48)$$

where $\mathbf{\Phi}$ is a diagonal matrix whose main diagonal entries are $\exp(-jk_0 d_a \cos(\phi_i))$. Define now a selection matrix \mathbf{J}_k so that $\mathbf{J}_k \mathbf{A} = \mathbf{A}_k$; this allows us to write Eq. (8.48) as:

$$\mathbf{J}_1 \mathbf{A} \mathbf{\Phi} = \mathbf{J}_2 \mathbf{A} \quad (8.49)$$

In the next step we compute the signal space – i.e., the vector space spanned by the receive vectors \mathbf{r} (for a general discussion of vector spaces, see Strang [1988]). A basis for these vector spaces can be obtained by eigenvalue decomposition EVD of the covariance matrix:

$$\mathbf{R}_{rr} \approx \mathbf{U} \mathbf{\Lambda} \mathbf{U}^\dagger \quad (8.50)$$

where the columns of the matrix \mathbf{U} are the basis of the signal space, and $\mathbf{\Lambda}$ is a diagonal matrix. The equation becomes exact when noise is negligible.

If the signal is noisy, the basis for the signal subspace is defined by the columns of \mathbf{U} that correspond to the signal subspace \mathbf{U}_s . In practice, these correspond to the K dominant eigenvalues of \mathbf{R}_{rr} (finding the number of incident waves N , also called “source order”, is non-trivial; for a discussion see, e.g., Haardt and Nossek [1995]).

Assume now that N uncorrelated planewaves impinge on a ULA of N_r sensor elements and $N \leq N_r$. The EVD of the covariance matrix is given as:

$$\begin{aligned} \mathbf{R}_{rr} &= \mathbf{U} \mathbf{\Lambda} \mathbf{U}^\dagger \\ &= \mathbf{U}_s \mathbf{\Lambda}_s \mathbf{U}_s^\dagger + \mathbf{U}_n \mathbf{\Lambda}_n \mathbf{U}_n^\dagger \\ &= \mathbf{U}_s \mathbf{\Lambda}_s \mathbf{U}_s^\dagger + \sigma_n^2 \mathbf{U}_n \mathbf{U}_n^\dagger \end{aligned}$$

Therefore, the basis for the signal subspace is defined by either the columns of \mathbf{U}_s or, equivalently, only those columns of \mathbf{U} which correspond to the N dominant eigenvalues of \mathbf{R}_{rr} .

It is also obvious that the matrix \mathbf{A} is in the signal space spanned by \mathbf{U}_s . Thus there exists a matrix \mathbf{T} so that \mathbf{A} can be represented as $\mathbf{A} = \mathbf{U}_s \mathbf{T}$. Thus, Eq. (8.49) can be written as:

$$\mathbf{J}_1 \mathbf{U}_s \mathbf{\Psi} = \mathbf{J}_2 \mathbf{U}_s \quad (8.51)$$

where

$$\mathbf{\Psi} = \mathbf{T} \mathbf{\Phi} \mathbf{T}^{-1} \quad (8.52)$$

The matrix $\mathbf{\Psi}$ is created from the matrix $\mathbf{\Phi}$ by a transformation that preserves the eigenvalues. In other words, if we know the eigenvalues of $\mathbf{\Psi}$, we also know the eigenvalues of $\mathbf{\Phi}$. The eigenvalues of $\mathbf{\Phi}$ give the angles of incidence via the relationship $\exp(-jk_0 d_a \cos(\phi_i))$. This completes the search for directions of arrival.

Thus, the ESPRIT algorithm uses the following steps:

- Determine the covariance matrix of the received signal, \mathbf{R}_{rr} .
- Perform an eigenvalue decomposition, and separate signal subspace from noise subspace, in order to determine the eigenbase \mathbf{U}_s .
- Determine the matrix $\mathbf{\Psi}$ by solving the (usually overdetermined) system of equations (8.51).
- Find an eigendecomposition of the matrix $\mathbf{\Psi}$.

- Determine the directions $\phi_i = \arccos[\arg(\text{eigval}(\mathbf{\Psi})) / (-k_0 d_a)]$.

It has to be stressed that ESPRIT relies on shift invariance of the antenna structure, and thus is mainly suitable for measurements with ULAs where all elements have the same pattern.

This description covers just the basic principles. Details of implementation, as well as generalizations to the multi-dimensional case, estimation of the number of incident signals, etc., are advanced research topics.