BE 3317 Final Project

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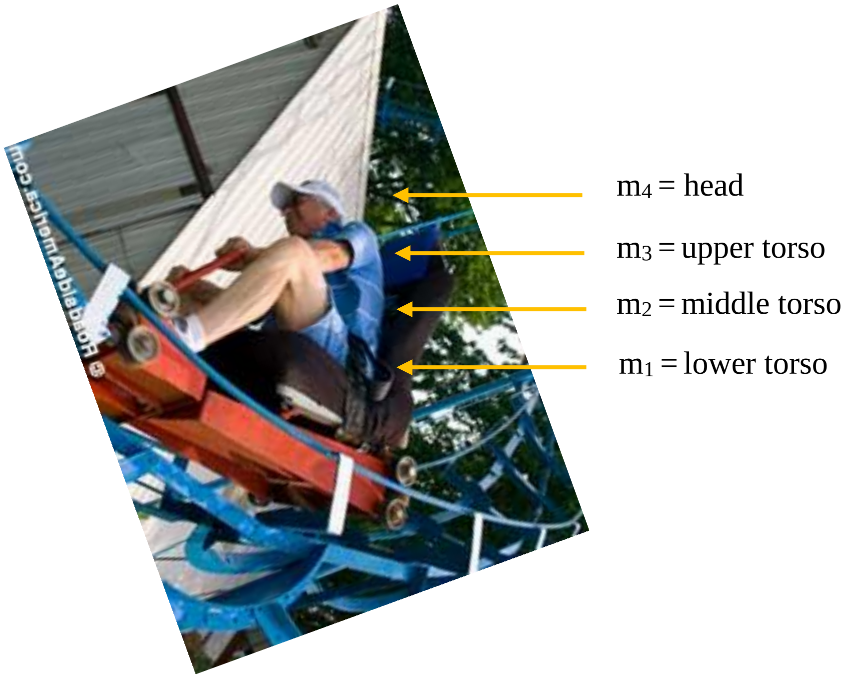
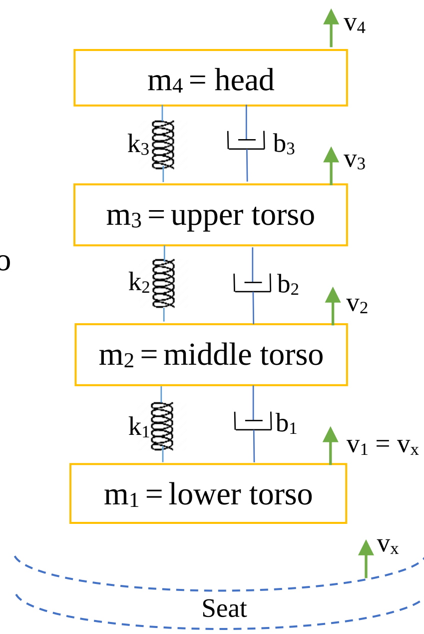
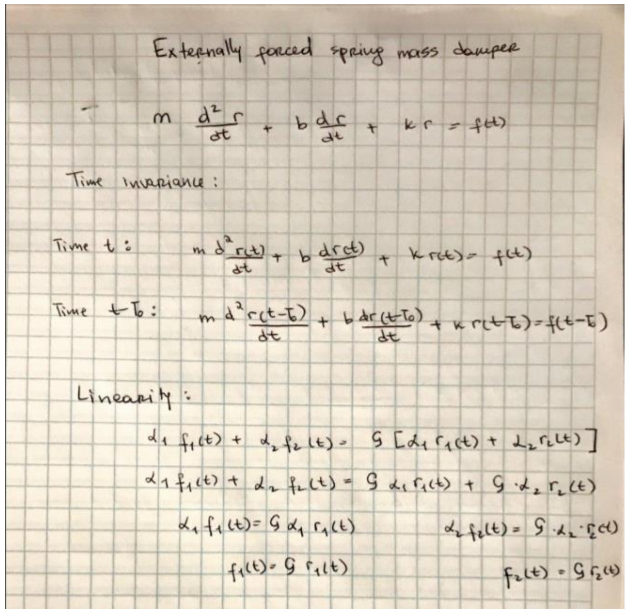
1) Biomechanical model of the roller coaster system (SMD model):



SMD Model:



2) The system is linear and time invariant (Figure 1).
   - Laplace transform is not strictly valid for time varying system. Unilateral Laplace transform used for the physically realizable systems excited by causal signals.
   - LCCDE is used for Laplace transformation.
   - Impulse response h(t) is unique for LTI system. In this case, impulse response is unique and its analog in Laplace transform, transfer function equals to V1(s)/ V4(s).
   - Impulse response is same for multiple nonlinear systems. This is not the case with this biomechanical system.
   - LTI system does not depend on order of sub-systems unlike non-linear systems. In our example we have 3 subsystems (lower, middle, upper torso) before the system of interest (head) is affected.
   - The representation of externally forced spring-mass-damper provided has been tested for time invariance and linearity (Gavin 2019).



Externally forced spring mass damper

$$m \frac{d^2 r}{dt} + b \frac{dr}{dt} + k r = f(t)$$

Time invariance:

Time t: $m \frac{d^2 r(t)}{dt} + b \frac{dr(t)}{dt} + k r(t) = f(t)$

Time $t-T_0$: $m \frac{d^2 r(t-T_0)}{dt} + b \frac{dr(t-T_0)}{dt} + k r(t-T_0) = f(t-T_0)$

Linearity:

$\alpha_1 f_1(t) + \alpha_2 f_2(t) = G[\alpha_1 r_1(t) + \alpha_2 r_2(t)]$

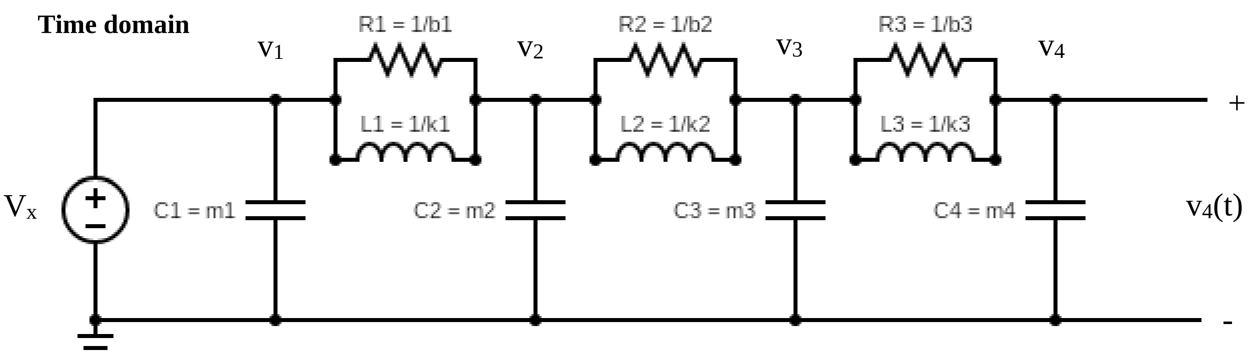
$\alpha_1 f_1(t) + \alpha_2 f_2(t) = G \alpha_1 r_1(t) + G \cdot \alpha_2 r_2(t)$

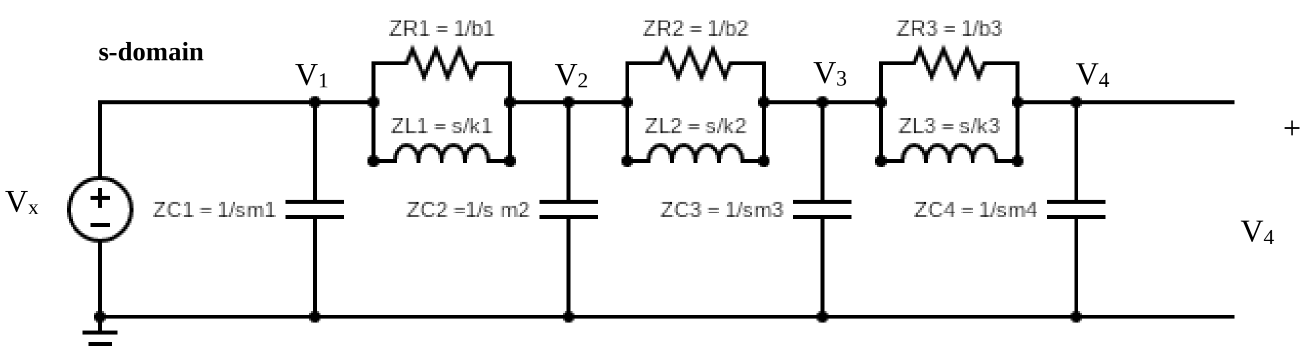
$\alpha_1 f_1(t) = G \alpha_1 r_1(t)$ $\qquad$ $\alpha_2 f_2(t) = G \cdot \alpha_2 \cdot r_2(t)$

$f_1(t) = G r_1(t)$ $\qquad$ $f_2(t) = G r_2(t)$

Figure 1: Checking for linear and time invariance of the system

3) Electrical circuit analogue and s-domain circuit:





4) In this study, we based our variables and constants from a 2012 Poland experiment investigating the biomechanics of a driver when the magnetorheological suspension of a vehicle seat is controlled. Individual models of drivers were measured at various parameters such as the stiffness and damping of the lumbar spine, spinal ligaments and muscles, intervertebral discs, and the masses of the body structures. We decided to use the data pertaining to Driver 2's masses and stiffness and damping of spine discs for the purpose of this project (Table 1). In the human body, the muscle and other soft tissue represent dampers, and bones, such as the spinal cord, represent springs. In our investigation, it is necessary to include dampers when determining the vertical head displacement since soft tissue is responsible for absorbing the shock waves exerted by the kinetic energy induced by the roller coaster cart. Moreover, the kinetic energy will be converted into heat energy, thus reducing the risk of bone fracture and other bodily injury that can be inflicted.

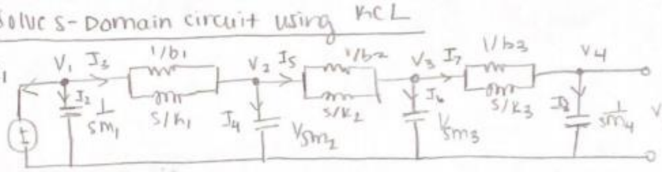
Table 1: Driver 2 data extrapolated according to the SMD model variables in Question 1 (Gągorowski, 2012)

Mass	**Springs/Stiffness of spine disc (N/m)**	**Dampers/ Damping of spine disc (N/m)**
Head (m_4) = 6.51 kg	Upper Torso/Head (k_3) = 85,400	Upper Torso/Head (b_3) = 2.30
Upper Torso (m_3) = 30.61 kg	Middle/Upper Torso (k_2) = 248,000	Middle/Upper Torso (b_2) = 4.66
Middle Torso (m_2) = 0.42 kg	Lower/Middle Torso (k_1) = 218,000	Lower/Middle Torso (b_1) = 3.45
Lower Torso (m_1) = 0.3 kg		

5) Output of the system:

Solving for the output voltage requires an application of Kirchoff's Current law at each of the four nodes (V1,V2,V3,V4). Below is the proof of solving for the output voltage (V4).

Solve s-Domain circuit using KCL



let $V_x = V_1$ for simplicity

① $I_1 = \boxed{V_1 - V_x}$ ② $I_2 = \frac{V_1}{\frac{1}{sm_1}} \to \boxed{V_1(sm_1)}$

③ $I_3 = \frac{V_1 - V_2}{\left(\frac{1}{\frac{1}{b_1}} + \frac{1}{\frac{s}{k_1}}\right)^{-1}} \to \frac{V_1 - V_2}{\left(b_1 + \frac{k_1}{s}\right)^{-1}} \to \boxed{(V_1 - V_2)\left(\frac{b_1 s + k_1}{s}\right)}$

④ $I_4 = \frac{V_2}{\frac{1}{sm_2}} \to \boxed{V_2(sm_2)}$ ⑤ $I_5 = \frac{V_2 - V_3}{\left(\frac{1}{\frac{1}{b_2}} + \frac{1}{\frac{s}{k_2}}\right)^{-1}} \to \frac{(V_2 - V_3)(b_2 s +}{s}$

⑥ $I_6 = \frac{V_3}{\frac{1}{sm_3}} \to \boxed{V_3(sm_3)}$

⑦ $I_7 = \frac{V_3 - V_4}{\left(\frac{1}{\frac{1}{b_3}} + \frac{1}{\frac{s}{k_3}}\right)^{-1}} \to \boxed{\frac{(V_3 - V_4)(b_3 s + k_3)}{s}}$ ⑧ $I_8 = \frac{V_4}{\frac{1}{sm_4}}$

$\boxed{I_8 = V_4(sm_4)}$

$\to I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 = 0$

Equation #1: $V_1\left(\frac{b_1 s + k_1}{s}\right) - V_2\left(\frac{b_1 s + k_1}{s}\right) + V_2 sm_2$

Equation #2 + $V_2\left(\frac{b_2 s + k_2}{s}\right) - V_3\left(\frac{b_2 s + k_2}{s}\right) + V_3 sm_3$

Equation #3 + $V_3\left(\frac{b_3 s + k_3}{s}\right) - V_4\left(\frac{b_3 s + k_3}{s}\right) + V_4 sm_4$)

allow $a_1 = b_1 + \frac{k_1}{s}$ Eq 1: $[V_1(a_1) - V_2(a_1) + V_2 sm_2] - 1$

$a_2 = b_2 + \frac{k_2}{s}$ Eq 2: $[V_2(a_2) - V_3(a_2) + V_3 sm_3] x - 1$

$a_3 = b_3 + \frac{k_3}{s}$ Eq 3: $[V_3(a_3) - V_4(a_3) + V_4 sm_3] x - 1$

The output of the system was solved by solving the s-domain circuit as a system of equations using Kirchhoff's current law for each of the four nodal points. The work is demonstrated as follows:

Compute the matrix

$$\begin{array}{ccc} \text{Equation \#1} & \text{Equation \#2} & \text{Equation \#3} \end{array}$$

$$\begin{bmatrix} a_1 + a_2 + sm_2 & -a_2 & 0 \\ -a_2 & a_2 + a_3 + sm_3 & -a_3 \\ 0 & -a_3 & a_3 + sm_4 \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} a_1 V_x \\ 0 \\ 0 \end{bmatrix}$$

$$x(t) = e^{-2t} \sin^2(4\pi t)$$

$$v_x(t) = \frac{dx(t)}{dt} = 2e^{2t} \sin(4\pi t)\left[\sin(4\pi t) + \cos(4\pi t)\right]$$

$$\Downarrow$$

$$V_x(s) = \frac{32\pi^2 s}{(s-2)\left((s-2)^2 + 64\pi^2\right)}$$

After solving the matrix, we can produce the following voltage functions in s-domain: V2, V3, and V4. The functions provided satisfy s>2 range. If needed other functions can be provided upon request.

V2:

$$\begin{aligned}&(78956.8\, s\, (1.37497\times 10^{8}\, s^{5} + 9.51055\times 10^{10}\, s^{4} + 8.43423\times 10^{12}\, s^{3} +\\ &\quad 2.20547\times 10^{15}\, s^{2} + 5.69505\times 10^{16}\, s + 9.25955\times 10^{18}))/\\ &(((s-2)^2 + 631.655)\,(4.18469\times 10^{7}\, s^{7} + 8.31507\times 10^{10}\, s^{6} +\\ &\quad 5.1336\times 10^{13}\, s^{5} + 3.82317\times 10^{15}\, s^{4} + 9.41049\times 10^{17}\, s^{3} +\\ &\quad 1.23398\times 10^{19}\, s^{2} + 2.28641\times 10^{21}\, s - 4.62977\times 10^{21}))\end{aligned}$$

$$\begin{aligned}&(1.57914\times 10^{7}\, s\, (1.04661\times 10^{7}\, s^{4} + 1.25713\times 10^{10}\, s^{3} +\\ &\quad 4.09763\times 10^{12}\, s^{2} + 2.84753\times 10^{14}\, s + 4.62977\times 10^{16}))/\\ &(((s-2)^2 + 631.655)\,(4.18469\times 10^{7}\, s^{7} + 8.31507\times 10^{10}\, s^{6} +\\ &\quad 5.1336\times 10^{13}\, s^{5} + 3.82317\times 10^{15}\, s^{4} + 9.41049\times 10^{17}\, s^{3} +\\ &\quad 1.23398\times 10^{19}\, s^{2} + 2.28641\times 10^{21}\, s - 4.62977\times 10^{21}))\end{aligned}$$

V3:

Solving for voltage at the fourth node is equivalent to the output voltage of the circuit. Output voltage is a representation of the vertical velocity of the head movement throughout the entire roller coaster ride. The equation for the output voltage in the s-domain is as follows:

V4:

$$
\begin{aligned}
&(1.57914\times10^{10}\, s\,(23\, s+8540)\,(16077\, s^{2}+1.87428\times10^{7}\, s+5.42128\times10^{9}))/\\
&(((s-2)^{2}+631.655)\,(4.18469\times10^{7}\, s^{7}+8.31507\times10^{10}\, s^{6}+\\
&\quad 5.1336\times10^{13}\, s^{5}+3.82317\times10^{15}\, s^{4}+9.41049\times10^{17}\, s^{3}+\\
&\quad 1.23398\times10^{19}\, s^{2}+2.28641\times10^{21}\, s-4.62977\times10^{21}))
\end{aligned}
$$

After typing the Laplace domain equation into the online graphing calculator, we can produce a visual representation of the output voltage system below:

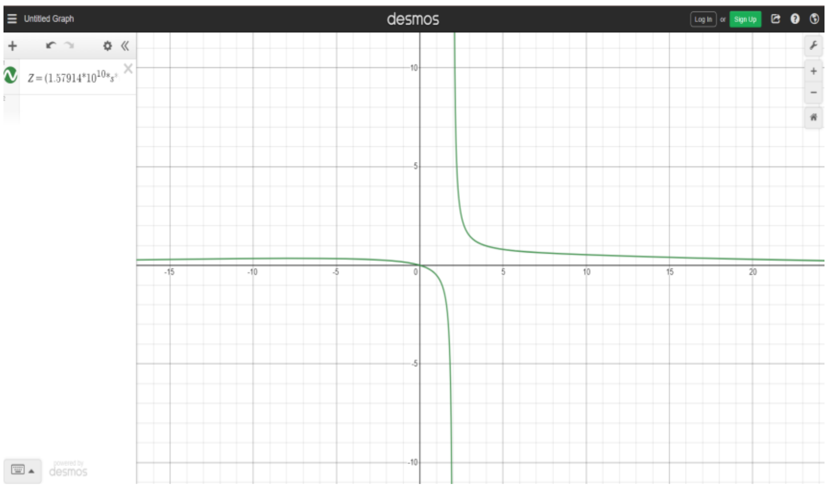


Figure 2: Plot of output voltage for the system with dampers in s-domain

In addition to the circuit diagram demonstrated above containing both dampers and inductors, the group also wanted to determine what the effect would be on the output of the circuit if the dampers were removed from the model. Dampers represent the damping stiffnesses of the spine disc and are responsible for absorbing shock, as stated earlier. Thus, without these dampers there would be an overwhelming amount of kinetic energy exerting shock on various parts of the body.

V4 (without dampers):

$$(1.04443\times 10^{23}\, s)/((s^2 - 4\, s + 635.655)\,(5\,978\,133\, s^7 - 11\,956\,266\, s^6 + 6\,784\,956\,930\,000\, s^5 - 13\,569\,913\,860\,000\, s^4 + 131\,497\,499\,200\,000\,000\, s^3 - 262\,994\,998\,400\,000\,000\, s^2 + 330\,698\,080\,000\,000\,000\,000\, s - 661\,396\,160\,000\,000\,000\,000)) \wedge s \neq 0$$

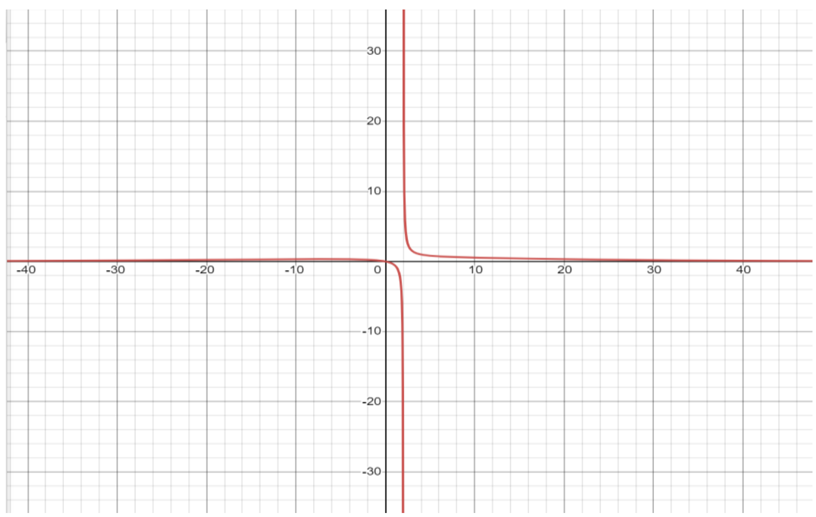


Figure 3: Plot of output voltage without dampers in s-domain

As demonstrated by the graphs, including dampers in the model has a very minimal effect on the voltage output of the system. The reason for this outcome is due to the large values of the masses and spring constants from the researched literature. Hence, the roller coaster system does not need to have dampers due to the very large constant values and due to the small height of the roller coaster off the ground. Because the input velocity reaches a maximum of 0.8 meters off the ground, this could be a logical factor that would allow us to dismiss the consideration of dampers in the model.

References

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