DIRICHLET-NEUMANN BRACKETING FOR FRACTIONAL RANDOM SCHRŌDINGER OPERATORS

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We proof Dirichlet-Neumann bracketing for handling fractional random Schrōdinger Operators. The Equations (3.1) and (3.2) of [1] combined with the Equations (1.1) and (1.2) of [2] lead to the following

Definition. For each $\alpha \in (0,2)$ and bounded domain $\Lambda \in \mathbb{R}^d$ we define the *fractional Neumann Laplacian* by the system

$$-\Delta_N^{\alpha/2}|_\Lambda[\psi](x) := \left(\int_{\mathbb{R}^d} \frac{1-\cos(\zeta_1)}{|\zeta|^{d+\alpha}}\, d\zeta\right)^{-1} P.V. \int_{\mathbb{R}^d} \frac{\psi(x)-\psi(y)}{|x-y|^{d+\alpha}}\, dy = f \text{ in } \Lambda,$$

$$\mathcal{N}_\alpha[\psi](x) := \left(\int_{\mathbb{R}^d} \frac{1-\cos(\zeta_1)}{|\zeta|^{d+\alpha}}\, d\zeta\right)^{-1} \int_{\Lambda} \frac{\psi(x)-\psi(y)}{|x-y|^{d+\alpha}}\, dy = 0 \text{ in } \overline{\Lambda}^{\complement} \tag{1}$$

for every $\psi \in H^{2\cdot\frac{\alpha}{2}}(\Lambda) = H^\alpha(\Lambda)$, the fractional Sobolev space of order α. Here ζ_1 denotes the first entry of the vector ζ and *P.V.* stands for Cauchy's Pricipal Value. The expression $\mathcal{N}_\alpha$ is called the *nonlocal normal derivative*. In the following we express the normalizing constant in both lines just by $C_{d,\alpha}$.

Lemma. *Let Λ_1, Λ_2 be disjoint open subsets of an open bounded Lipschitz domain $\Lambda \subset \mathbb{R}^d$ such that*

$$\overset{\circ}{\overline{\Lambda_1 \cup \Lambda_2}} = \Lambda.$$

Then for each $\alpha \in (0,2)$ one has

$$0 \le \mu_n(-\Delta_N^{\alpha/2}|_{\Lambda_1\cup\Lambda_2}) \le \mu_n(-\Delta_N^{\alpha/2}|_\Lambda)$$

where $\mu_n(-\Delta_N^{\alpha/2})$ denotes the n-th eigenvalue of the Neumann fractional Laplacian counting multiplicity. Thus adding Neumann surfaces lowers the eigenvalues of the fractional Laplacian.

Proof. We follow the proof of Proposition XIII.15.4 (c) in [5]. According to the Definition on page 269 of [5] we show that

$$0 \le -\Delta_N^{\alpha/2}|_{\Lambda_1\cup\Lambda_2} \le -\Delta_N^{\alpha/2}|_\Lambda$$

which holds per definition iff (i) $Q(-\Delta_N^{\alpha/2}|_\Lambda) \subseteq Q(-\Delta_N^{\alpha/2}|_{\Lambda_1\cup\Lambda_2})$ for the form domain Q of the operators and (ii) for any $\psi \in Q(-\Delta_N^{\alpha/2}|_\Lambda)$ one has

$$0 \le (\psi, -\Delta_N^{\alpha/2}|_{\Lambda_1\cup\Lambda_2}[\psi]) \le (\psi, -\Delta_N^{\alpha/2}|_\Lambda[\psi]).$$

Then the claim follows from part (b) of the Lemma just before Proposition XIII.15.4 in [5] on page 270.

Proof of (i): We determine $Q(-\Delta_N^{\alpha/2}|_\Lambda)$ first. The domain of the classical Laplacian $-\Delta$ is the Sobolev space H^2, i.e. $\mathscr{D}(-\Delta) = H^2$. We can express the Laplacian also in terms of the quadratic form $(\psi, -\Delta[\psi]) = (\nabla[\psi], \nabla[\psi])$ where $(f, g) = \int f(x)g(x)dx$ due to integration by parts, so $Q(-\Delta) = H^1$ in a weak sense. In the strong sense we have $\mathscr{D}(-\Delta_N^{\alpha/2}|_\Lambda) = H^{2\cdot\frac{\alpha}{2}}(\Lambda) = H^\alpha(\Lambda)$, see the discussion in section 7 of [2]. In the same way as above we can define the fractional Neumann Laplacian via

$$(\psi, -\Delta_N^{\alpha/2}|_\Lambda[\psi]).$$

We can bring half of the regularity on the left side of this form in the following way: Since Λ is a bounded Lipschitz domain it is an extension domain for H^α where $\alpha \in (0, 1)$, see Theorem 5.4 in [1]. This means that $H^\alpha(\Lambda)$ is continuously embedded in $H^\alpha(\mathbb{R}^d)$ for $\alpha \in (0, 1)$, i.e. for each $\psi \in H^\alpha(\Lambda)$ there exists some $\tilde{\psi} \in H^\alpha(\mathbb{R}^d)$ such that $\tilde{\psi}|_\Lambda = \psi$ and $||\tilde{\psi}||_{H^\alpha(\mathbb{R}^d)} \leq C_{d,\alpha,\Lambda}||\psi||_{H^\alpha(\Lambda)}$. For those $\alpha, \psi, \tilde{\psi}$ we can write

$$\begin{aligned}
&(\psi, -\Delta_N^{\alpha/2}|_\Lambda[\psi]) \\
= &\int_\Lambda \psi(x) \cdot \{-\Delta_N^{\alpha/2}|_\Lambda\}[\psi](x)\ dx \\
= &\int_{\mathbb{R}^d} \tilde{\psi}(x) \cdot \{-\Delta^{\alpha/2}\}[\tilde{\psi}](x) \cdot \mathbb{1}_\Lambda(x)\ dx \cdot \mathbb{1}_{\{\mathcal{N}_\alpha[\tilde{\psi}](x)\equiv 0 \text{ in } \overline{\Lambda}^\complement\}}(\tilde{\psi}) \\
= &\int_{\mathbb{R}^d} \mathscr{F}^{-1}[\mathscr{F}[\tilde{\psi}]](x) \cdot \mathscr{F}^{-1}[|\xi|^\alpha \mathscr{F}[\tilde{\psi}]](x) \cdot \mathbb{1}_\Lambda(x)\ dx \cdot \mathbb{1}_{\{\mathcal{N}_\alpha[\tilde{\psi}](x)\equiv 0 \text{ in } \overline{\Lambda}^\complement\}}(\tilde{\psi}) \\
= &\int_{\mathbb{R}^d} \mathscr{F}^{-1}[|\xi|^{\alpha/2} \mathscr{F}[\tilde{\psi}]](x) \cdot \{-\Delta^{\alpha/4}\}[\tilde{\psi}](x) \cdot \mathbb{1}_\Lambda(x)\ dx \cdot \mathbb{1}_{\{\mathcal{N}_\alpha[\tilde{\psi}](x)\equiv 0 \text{ in } \overline{\Lambda}^\complement\}}(\tilde{\psi}) \\
= &\int_\Lambda \{-\Delta_N^{\alpha/4}|_\Lambda\}[\psi](x) \cdot \{-\Delta_N^{\alpha/4}|_\Lambda\}[\psi](x)\ dx,
\end{aligned}$$

where $\mathcal{N}_\alpha$ denotes the nonlocal normal derivative defined in Equation (2). Notice that the condition $\mathcal{N}_\alpha[\tilde{\psi}](x) \equiv 0$ in $\overline{\Lambda}^\complement$ is included in $H^\alpha(\Lambda)$, cf. [2]. This leads to

$$(\psi, -\Delta_N^{\alpha/2}|_\Lambda[\psi]) = (-\Delta_N^{\alpha/4}|_\Lambda[\psi], -\Delta_N^{\alpha/4}|_\Lambda[\psi]), \tag{2}$$

thus $Q(-\Delta_N^{\alpha/2}|_\Lambda) = H^{2\cdot\frac{\alpha}{4}}(\Lambda) = H^{\alpha/2}(\Lambda)$ for $\alpha \in (0, 1)$. The case $\alpha \in \mathbb{N}$ is classical, for $\alpha \in (1, 2)$ consider

$$-\Delta_N^{\alpha/2}|_\Lambda[\psi] = -\Delta_N^{\alpha/2-1}|_\Lambda[\nabla[\psi]].$$

Therefore $Q(-\Delta_N^{\alpha/2}|_\Lambda) = H^{\alpha/2}(\Lambda)$ for each $\alpha \in (0, 2)$.

Thus in order to proof (i) we must show

$$H^{\alpha/2}(\Lambda) \subseteq H^{\alpha/2}(\Lambda_1 \oplus \Lambda_2).$$

We can define these spaces by the quadratic form

$$(f,g)_{H^{\alpha/2}(\Lambda)} := \int_\Lambda f(x)g(x)dx + \int_\Lambda \int_\Lambda \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}}\, dx\; dy,$$

see Equation (3.2) in [2]. Let $\psi \in H^{\alpha/2}(\Lambda)$. We have to show that

$$||\psi||_{H^{\alpha/2}(\Lambda_1 \oplus \Lambda_2)} \leq ||\psi||_{H^{\alpha/2}(\Lambda)} < \infty.$$

This is indeed the case due to

$$||\psi||_{H^{\alpha/2}(\Lambda_1 \oplus \Lambda_2)} = \sqrt{(\psi,\psi)}_{H^{\alpha/2}(\Lambda_1 \dot\cup \Lambda_2)} = \sqrt{(\psi,\psi)}_{H^{\alpha/2}(\Lambda)} = ||\psi||_{H^{\alpha/2}(\Lambda)} < \infty \tag{3}$$

since $\Lambda \backslash (\Lambda_1 \dot\cup \Lambda_2)$ has Lebesgue measure zero.

Proof of (ii): We must show that for any $\psi \in H^{\alpha/2}(\Lambda)$ one has

$$0 \leq (\psi, -\Delta_N^{\alpha/2}|_{\Lambda_1 \dot\cup \Lambda_2}[\psi]) \leq (\psi, -\Delta_N^{\alpha/2}|_\Lambda[\psi]).$$

The first inequality follows from Equation (3) and the second inequality is due to Equation (4). □

In order to proof the full Dirichlet-Neumann Bracketing we can use the function space

$$X_0^{\alpha/2,2}(\Lambda) := \{\psi \in H^{\alpha/2}(\mathbb{R}^d) \text{ and } \psi \equiv 0 \text{ in } \Lambda^\complement\}$$

from Equation (1.6) in [3]. The authors proved in Theorem 6 of [3] for continuous boundaries the dense embedding

$$C_c^\infty(\Lambda) \overset{d}{\hookrightarrow} X_0^{\alpha/2,2}(\Lambda).$$

Thus we have the same dense embedding as in the classical case where $\alpha = 2$ and this space fits the argumentation of Proposition XIII.15.4 in [5].

In order to prove Proposition XIII.15.4 (a) take the definition of the fractional Laplacian via the quadratic form

$$\mathcal{E}(f,g) = C_{d,\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(y)-f(x))(\bar g(y) - \bar g(x))}{|x-y|^{d+\alpha}} dx\; dy,$$

see Theorem 1.1 (g) of [4]. Restrictions to some $\Lambda' \subset \Lambda \subseteq \mathbb{R}^d$ are worked into the integration areas.

In order to prove Proposition XIII.15.4 (b) notice that

$$C_c^\infty(\Lambda) \subset H^{\alpha/2}(\Lambda).$$

References

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