ON THE INTEGRATED DENSITY OF STATES OF FRACTIONAL RANDOM SCHRÖDINGER OPERATORS

PETER KERN AND LEONARD PLESCHBERGER

Abstract. We proof the existence of the Integrated Density of States (IDS) for fractional random Schrödinger operators with Gaussian potential based on the theory of isotropic α-stable Lévy processes. Further we show that the IDS exhibits Lifshitz tails and determine its asymptotics at the right end of the spectrum. Isotropic α-stable Lévy processes are self-similar but not continuous. Thus self-similarity is a fundamental concept for these quantum operators.

1. Introduction

For fixed $\alpha \in (0, 2]$ we consider the fractional random Schrödinger operator

$$H_\omega[\psi] := (-\Delta)^{\alpha/2}[\psi] + V_\omega \cdot \psi \tag{1.1}$$

acting on suitably regular functions ψ on $\mathbb{R}^d$, where V_ω is either a Gaussian or Poissonian random potential on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$ rigorously defined in the sequel. The fractional Laplacian $(-\Delta)^{\alpha/2}$ will be introduced in Section 2 and reduces to the negative Laplacian $-\Delta$ when $\alpha = 2$ but in contrast to the ordinary Laplacian, the fractional Laplacian is a nonlocal operator. We are interested in the spectrum of the operator H_ω which consists of random eigenvalues λ_ω that fulfill the corresponding eigenvalue problem $H_\omega[\psi] = \lambda_\omega \cdot \psi$ for some $\psi \neq 0$. Since the operator H_ω is selfadjoint, it only has real eigenvalues. We want to calculate the average number of quantum states λ_ω per volume up to a certain barrier $\lambda \in \mathbb{R}$. For that purpose we restrict the operator H_ω to a box Λ containing the origin $0 \in \mathbb{R}^d$ and choose zero Dirichlet exterior conditions. This results in a countable number of quantum states $\lambda_{\omega,\Lambda}$ bounded from below such that the spectrum of the restricted operator $H_{\omega,\Lambda}$ can

Date: June 24, 2025.

2010 *Mathematics Subject Classification.* Primary 82B44; Secondary 47B80, 47D08, 60G52, 60H25, 81Q10.

Key words and phrases. Schrödinger operator, fractional Laplacian, stable Lévy process, Gaussian potential, Poissonian potential, integrated density of states, Lifshitz tails.

be ordered as

$$\sigma(H_{\omega,\Lambda}) = \{\lambda^{(1)}_{\omega,\Lambda} \leq \lambda^{(2)}_{\omega,\Lambda} \leq \dots\}. \tag{1.2}$$

Define the normalized eigenvalue counting function of $H_{\omega,\Lambda}$ by

$$\mathsf{IDS}_{\omega,\Lambda}(\lambda) := \frac{1}{|\Lambda|} \sum_{k=1}^{\infty} \mathbb{1}_{\{\lambda^{(k)}_{\omega,\Lambda} \leq \lambda\}}, \tag{1.3}$$

where $|\Lambda|$ denotes the volume of the box. Enlarging the box to the whole $\mathbb{R}^d$ denoted by $|\Lambda| \to \infty$ and taking expectations this results in the so-called integrated density of states (IDS)

$$\mathsf{IDS}(\lambda) := \lim_{|\Lambda|\to\infty} \mathbb{E}[\mathsf{IDS}_{\omega,\Lambda}(\lambda)]. \tag{1.4}$$

Nakao proved in [16] the existence of the IDS for random Schrödinger operators with random potentials. He works in the setting where $\alpha = 2$, i.e. the free part is the classical Laplacian. Further he proves the asymptotics of the IDS at the left and right end of the spectrum of random Schrödinger operators both for Poissonian and Gaussian potentials. Thus we can exclude the case $\alpha = 2$ in our considerations. Nakao's work is based on Pastur [19] and the important work of Donsker and Varadhan [5] on the Wiener sausage. Ōkura gereneralized Nakao's work [18] to a larger class of nonlocal operators with random potential including the fractional Laplacian. In detail he proves the existence of the IDS for a large class of random potentials and especially for Poissonian potentials. Further he determines the asymptotics at the left end of the spectrum for Poissonian potentials.

In Section 3 we proof the existence of the IDS for the fractional Schrödinger operator with Gaussian potentials based on Ōkura's general result from [18]. Then we analyze its asymptotics at the left end of the spectrum as $\lambda \to -\infty$. In Section 4 we follow the idea of Nakao [16] in proving Lifshitz tails for the fractional random Schrödinger operator with Gaussian potential, i.e. exponential decay of the number of eigenstates at the left end of the spectrum. For that purpose we use a technique developed by Pastur [20] which was generalized by Ōkura [18]. Then we analyze the asymptotics at the right end of the spectrum as $\lambda \to +\infty$ for both Gaussian and Poissonian potentials in Sections 5 and 6. Our stochastic proof mostly differs from Nakao's technique in [16]. It turns out that the asymptotics at the left end of the spectrum does not depend on $\alpha \in (0, 2)$ and thus is completely determined by the random potential, whereas the asymptotics at the right end of the spectrum only depends on $\alpha \in (0,2)$ in case of both Gaussian and Poissonian potentials.

2. The fractional Laplacian and α-stable Lévy processes

In this section we recall known results on the fractional Laplacian in $\mathbb{R}^d$ and in bounded domains. In $\mathbb{R}^d$ there exist plenty (almost) equivalent definitions of the fractional Laplacian, see [14] or [15]. We use a pseudo-differential approach which is most suitable for our needs. Throughout this text we define the Fourier transform by

$$\mathscr{F}[\psi](\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} \psi(x) dx \quad \text{for all } \xi \in \mathbb{R}^d \tag{2.1}$$

acting on $L^1(\mathbb{R}^d)$. For non-negative ψ with $\|\psi\|_1 = 1$ it coincides with the characteristic function of a random variable Y with probability density ψ up to a constant, since

$$\mathbb{E}\left[e^{i\langle \xi, Y\rangle}\right] = \int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} \psi(x) dx = (2\pi)^{d/2} \mathscr{F}[\psi](\xi).$$

The Fourier transform translates differential operators into polynomials and vice versa. Extending this to fractional powers, for fixed $\alpha \in (0,2)$ we define the fractional Laplacian $(-\Delta)^{\alpha/2}$ as

$$(-\Delta)^{\alpha/2}[\psi](x) := \mathscr{F}^{-1}\left[|\xi|^\alpha \cdot \mathscr{F}[\psi]\right](x), \tag{2.2}$$

where $|\xi|$ denotes the Euclidean norm of $\xi \in \mathbb{R}^d$, with domain

$$\mathscr{D}\left((-\Delta)^{\alpha/2}\right) := \left\{\psi \in L^2(\mathbb{R}^d) : (-\Delta)^{\alpha/2}[\psi] \in L^2(\mathbb{R}^d)\right\}$$

and $\mathscr{D}((-\Delta)^{\alpha/2}) = H^{\alpha/2}(\mathbb{R}^d) = W^{\alpha/2,2}(\mathbb{R}^d)$ coincides with the Bessel potential space, respectively the fractional Sobolev space; see section 3 in [23] and the references therein. The fractional Laplacian $(-\Delta)^{\alpha/2}$ generates a strongly continuous semigroup

$$T_t[\psi](x) := e^{-t(-\Delta)^{\alpha/2}}[\psi](x) \tag{2.3}$$

which can also be expressed by its Markov transition kernel $p(t,x,y)$ given by the Lebesgue density function $p(t,y)$ of an isotropic α-stable Lévy process $X = (X_t)_{t\geq 0}$ on a probability space $(\Omega', \mathscr{A}', \mathbb{P}')$. The connection between the analytic, potential-theoretic and stochastic perspective is given by

$$T_t[\psi](x) = \int_{\mathbb{R}^d} p(t,x,y)\psi(y) dy = \mathbb{E}_x[\psi(X_t)], \tag{2.4}$$

where $\mathbb{E}_x$ denotes the expected value with respect to $\mathbb{P}'_x := \mathbb{P}'(\,\cdot\,|X_0 = x)$ and $p(t,x,y) = p(t,x-y)$ due to the stationary increments of the Lévy process X. Hence, from an analytic perspective the fractional Laplacian is the negative generator of the strongly continuous semigroup and the transition kernel is its Green's function solution. According to the Lévy-Chinčin formula, the isotropic α-stable Lévy process X is also characterised by its Lévy exponent $Q(\xi) = |\xi|^\alpha$ which by (2.2) coincides with the symbol of $(-\Delta)^{\alpha/2}$ in Fourier space, such that the characteristic function of X yields the identity

$$\mathbb{E}[\exp(i\langle \xi, X_t\rangle)] = \exp(-tQ(\xi)) = \exp(-t|\xi|^\alpha).$$

The integrability of this function ensures the applicability of the Fourier inversion formula. Therefore, for $t > 0$ the process X possesses the continuous density

$$p(t,x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} e^{-tQ(\xi)} d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} e^{-t|\xi|^\alpha} d\xi \tag{2.5}$$

which for $\alpha \in (0,2)$ cannot be expressed in simple terms but belongs to $C^\infty(\mathbb{R}^d)$ with all derivatives in $L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$; see [24]. Combining the fractional Laplacian with a Gaussian or Poissonian random potential to the fractional random Schrödinger operator H_ω in (1.1), the corresponding evolution semigroup has a probabilistic interpretation by means of the Feynman-Kac formula

$$e^{-tH_\omega}[\psi](x) = \mathbb{E}_x\left[e^{-\int_0^t V_\omega(X_s)\,ds}\,\psi(X_t)\right]$$

as laid out in [4].

Now we want to restrict the fractional Laplacian $(-\Delta)^{\alpha/2}$ to a bounded open box $\Lambda \subset \mathbb{R}^d$ containing the origin. Since the Fourier transform is only defined in the whole space, the pseudo-differential approach fails. Exterior conditions are necessary, since X exits Λ almost surely with a jump into $\Lambda^\complement$ and does not touch $\partial\Lambda$. We choose zero Dirichlet conditions on the exterior $\Lambda^\complement$ corresponding to the first exit time $\tau_\Lambda := \inf\{t \geq 0 : X_t \in \Lambda^\complement\}$. In analogy to (2.3) and (2.4) we define the restricted fractional Laplacian $(-\Delta)^{\alpha/2}_\Lambda$ by its evolution semigroup

$$e^{-t(-\Delta)^{\alpha/2}_\Lambda}[\psi](x) = \int_\Lambda p_\Lambda(t,x,y)\psi(y)dy = \mathbb{E}_x[\psi(X_t)\mathbb{1}_{\{t<\tau_\Lambda\}}] \tag{2.6}$$

for $\psi \in L^2(\Lambda)$ and the kernel is given by

$$p_\Lambda(t,x,y) = p(t,x,y)\,\mathbb{E}^t_{x,y}[\mathbb{1}_{\{t<\tau_\Lambda\}}] \quad \text{for all } x,y \in \Lambda \text{ and } t > 0,$$

where $\mathbb{E}^t_{x,y}$ denotes expectation with respect to $\mathbb{P}'(\,\cdot\,|X_0 = x,\, X_t = y)$. For the construction of this α-stable bridge measure we refer to [3]. This yields again a probabilistic interpretation of the restricted fractional random Schrödinger operator $H_{\omega,\Lambda}$ for our Gaussian or Poissonian random potential by means of the Feynman-Kac formula

$$e^{-tH_{\omega,\Lambda}}[\psi](x) = \mathbb{E}_x\left[e^{-\int_0^t V_\omega(X_s)\,ds}\,\psi(X_t)\,\mathbb{1}_{\{t<\tau_\Lambda\}}\right]$$

with $\psi \in L^2(\Lambda)$; see [4]. Since this operator is Hilbert-Schmidt due to $|\Lambda| < \infty$ and boundedness of the kernel

$$p_{\omega,\Lambda}(t,x,y) = p(t,x,y)\,\mathbb{E}^t_{x,y}\left[e^{-\int_0^t V_\omega(X_s)\,ds}\,\mathbb{1}_{\{t<\tau_\Lambda\}}\right]$$

for all $x, y \in \Lambda$ and $t > 0$, this indeed results in a spectrum of the form (1.2) for the restricted operator $H_{\omega,\Lambda}$.

3. Existence of the IDS for Gaussian Potentials

First of all we make sure that the IDS actually exists in case of our fractional random Schrödinger operators with Gaussian potential. Throughout this text we assume that the Gaussian potential $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ is a stationary centered Gaussian random field on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with $\mathbb{P}$-almost surely continuous paths. Then V_ω is determined by its covariance function $c(x) := \mathbb{E}[V_\omega(0)V_\omega(x)]$ which is continuous at the origin and assumed to be strictly positive. As an example we may consider the stationary centered Ornstein-Uhlenbeck field $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ with covariance function $c(x) = \sigma^2 e^{\langle a,|x|\rangle}$ for $\sigma > 0$ and $a \in (0,\infty)^d$ which can easily be constructed from a standard Wiener field $W_\omega = (W_\omega(x))_{x\in\mathbb{R}^d}$ with covariance function $\mathbb{E}[W_\omega(x)W_\omega(y)] = \prod_{k=1}^d (x_k \wedge y_k)$ by setting $V_\omega(x) := \sigma\, e^{\langle a,x\rangle} W_\omega(e^{2a_1x_1}, \ldots, e^{2a_dx_d})$; e.g. see [1] for details.

We will apply the following general existence theorem of Ōkura [18] which further proves a representation of the Laplace-Stieltjes transform

$$\mathcal{L}[\mathsf{IDS}](t) := \int_{\mathbb{R}} e^{-\lambda t} d\mathsf{IDS}(\lambda). \tag{3.1}$$

Theorem 3.1 ([18], Theorem 5.1)**.** *Let L be the generator of a d-dimensional symmetric Lévy process $X = (X_t)_{t\geq 0}$ on a probability space $(\Omega', \mathscr{A}', \mathbb{P}')$ with Lévy exponent $Q(\xi)$ and let $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ be a stationary random field over a probability space*

$(\Omega, \mathscr{A}, \mathbb{P})$. *Suppose that the following two conditions are satisfied:*

$$e^{-t\sqrt{Q(\xi)}} \in L^1(\mathbb{R}^d) \quad \text{for every } t > 0 \tag{3.2}$$

and there exists a constant $r > 2$ such that

$$\exp\left(\int_0^t V_\omega^-(X_s(\eta))ds\right) \in L^r\left(\mathbb{P}(d\omega) \otimes \mathbb{P}_0'(d\eta)\right) \quad \text{for every } t > 0, \tag{3.3}$$

where $V_\omega^- := \max\{-V_\omega, 0\}$. Then the IDS for the operator $H_\omega := -L + V_\omega$ exists as a right-continuous nondecreasing function $\mathsf{IDS}(\lambda)$ on $\mathbb{R}$ with $\lim_{\lambda\downarrow-\infty} \mathsf{IDS}(\lambda) = 0$ such that for every continuity point λ of IDS we have

$$\lim_{|\Lambda|\to\infty} \mathbb{E}[\mathsf{IDS}_{\omega,\Lambda}(\lambda)] = \mathsf{IDS}(\lambda). \tag{3.4}$$

Moreover, for every $t > 0$ we have

$$\mathcal{L}[\mathsf{IDS}](t) = p(t,0,0)\ \mathbb{E}(d\omega) \times \mathbb{E}_{0,0}^t(d\eta)\left[e^{-\int_0^t V_\omega(X_s(\eta))ds}\right]. \tag{3.5}$$

Note that condition (3.2) implies $e^{-tQ(\xi)} \in L^1(\mathbb{R}^d)$ for every $t > 0$ and thus by (2.5) and stationarity of the increments guarantees the existence of $p(t, 0, 0)$ in (3.5). Since our isotropic α-stable process X and our Gaussian random field V_ω are within the setting of Theorem 3.1, as an application we get:

Theorem 3.2. *For fixed $\alpha \in (0, 2)$ consider the fractional random Schrödinger operator H_ω in (1.1) with Gaussian potential V_ω as above. Then the IDS of H_ω exists as a nondecreasing càdlàg function with $\lim_{\lambda\downarrow-\infty} \mathsf{IDS}(\lambda) = 0$. Further, for every $t > 0$ its Laplace-Stieltjes transform is represented by*

$$\mathcal{L}[\mathsf{IDS}](t) = p(t,0,0)\ \mathbb{E}(d\omega) \times \mathbb{E}_{0,0}^t(d\eta)\left[e^{-\int_0^t V_\omega(X_s(\eta))ds}\right]. \tag{3.6}$$

Proof. We only have to check the conditions (3.2) and (3.3) of Theorem 3.1 in our model. For the isotropic α-stable process X we have $Q(\xi) = |\xi|^\alpha$ and thus

$$\int_{\mathbb{R}^d} \left|e^{-t\sqrt{Q(\xi)}}\right| d\xi = \int_{\mathbb{R}^d} e^{-t|\xi|^{\alpha/2}}\, d\xi < \infty \quad \text{for every } t > 0$$

shows that condition (3.2) is satisfied. In order to check condition (3.3), note that by Jensen's inequality for the normalized Lebesgue measure on $[0, t]$ and monotone convergence we have

$$\mathbb{E}(d\omega) \times \mathbb{E}_0(d\eta)\left[\exp\left(r\int_0^t V_\omega^-(X_s(\eta))\, ds\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}(d\omega) \times \mathbb{E}_0(d\eta) \left[\sum_{k=0}^{\infty} \frac{(rt)^k}{k!} \left(\int_0^t V_\omega^-(X_s(\eta)) \, \frac{ds}{t} \right)^k \right] \\
&\leq \mathbb{E}(d\omega) \times \mathbb{E}_0(d\eta) \left[\sum_{k=0}^{\infty} \frac{(rt)^k}{k!} \int_0^t \left(V_\omega^-(X_s(\eta)) \right)^k \, \frac{ds}{t} \right] \\
&= \frac{1}{t} \, \mathbb{E}(d\omega) \times \mathbb{E}_0(d\eta) \left[\int_0^t \exp\left(rt \, V_\omega^-(X_s(\eta)) \right) ds \right]
\end{aligned}
$$

By the Tonelli-Fubini theorem and stationarity $\mathbb{P}_{V_\omega(x)} = \mathcal{N}_{0,c(0)}$ of the centered Gaussian field we further get for every $t > 0$ and arbitrary $r > 2$

$$
\begin{aligned}
&\mathbb{E}(d\omega) \times \mathbb{E}_0(d\eta) \left[\exp\left(r \int_0^t V_\omega^-(X_s(\eta)) \, ds \right) \right] \\
&\quad = \frac{1}{t} \int_0^t \int_{\mathbb{R}^d} \int_\Omega \exp\left(rt \, V_\omega^-(x) \right) \, d\mathbb{P}(\omega) \, d\mathbb{P}'_{X_s^\alpha | X_0^\alpha = 0}(x) \, ds \\
&\quad = \frac{1}{t} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \exp(rt \max\{-y, 0\}) \, d\mathcal{N}_{0,c(0)}(y) \, d\mathbb{P}'_{X_s^\alpha | X_0^\alpha = 0}(x) \, ds \\
&\quad = \int_{\mathbb{R}} \exp(rt \max\{-y, 0\}) \, d\mathcal{N}_{0,c(0)}(y) \\
&\quad = \int_0^\infty 1 \, d\mathcal{N}_{0,c(0)}(y) + \int_{-\infty}^0 \exp(-rty) \, d\mathcal{N}_{0,c(0)}(y) \\
&\quad = \frac{1}{2} + \int_0^\infty \exp(rty) \, d\mathcal{N}_{0,c(0)}(y) < \infty,
\end{aligned}
$$

since Gaussian random variables have finite exponential moments. Thus condition (3.3) is fulfilled and a direct application of Theorem 3.1 concludes the proof. □

4. Lifshitz Tails of the IDS for Gaussian Potentials

In this section we derive the precise asymptotics of the IDS with Gaussian potential at the left end of the spectrum, i.e. the decay of the IDS as $\lambda \to -\infty$. For this purpose, we use the following result of Ōkura [18] on lower and upper bounds for the Laplace-Stieltjes transform of the IDS. Thereafter we derive the asymptotics of the IDS itself by using a Tauberian theorem stated in [7] which translates the behavior of the Laplace-Stieltjes transform of the IDS as $t \to +\infty$ to the behavior of the IDS itself as $\lambda \to -\infty$. In the following let $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ denote the Dirichlet form of a symmetric

Lévy process with Lévy exponent $Q(\xi)$ given by

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} Q(\xi)\left|\mathscr{F}[f](\xi)\right|^2 d\xi \tag{4.1}$$

with domain

$$\mathscr{D}(\mathcal{E}) = \left\{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} Q(\xi)\left|\mathscr{F}[f](\xi)\right|^2 d\xi < \infty \right\}.$$

Ōkura's theorem states:

Theorem 4.1 ([18], Theorem 7.1)**.** *Let $X = (X_t)_{t\geq 0}$ be a d-dimensional symmetric Lévy process on a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ with Lévy exponent $Q(\xi)$ satisfying $e^{-tQ(\xi)} \in L^1(\mathbb{R}^d)$ for all $t > 0$ and let $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ be the Dirichlet form of X. Further, let $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ be a stationary random field defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that*

$$\mathbb{E}[\exp(-tV_\omega(0))] < \infty \quad \text{for all } t > 0. \tag{4.2}$$

Then for all $f \in \mathscr{D}(\mathcal{E})$ with $||f||_2 = 1$ and $t > 0$ we have

$$\begin{aligned} ||f||_1^{-2} e^{-t\mathcal{E}(f,f)-\Phi_t(f)} &\leq p(t,0,0)\,\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\,ds\right)\right] \\ &\leq p(t,0,0)\,\mathbb{E}\left[e^{-tV_\omega(0)}\right], \end{aligned} \tag{4.3}$$

where

$$\Phi_t(f) = -\ln\mathbb{E}\left[\exp\left(-t\int_{\mathbb{R}^d} V_\omega(x)\,|f(x)|^2\,dx\right)\right].$$

Note that in our case we have $Q(\xi) = |\xi|^\alpha$ for the isotropic α-stable Lévy process such that $e^{-tQ(\xi)} \in L^1(\mathbb{R}^d)$ for all $t > 0$ and condition (4.2) is automatically fulfilled for our Gaussian potential, since exponential moments of Gaussian random variables exist. According to Theorem 3.2 the expression in the middle of the inequality (4.3) is exactly the Laplace-Stieltjes transform of the IDS. This enables us to prove:

Lemma 4.2. *For the Laplace-Stieltjes transform of the IDS of the fractional random Schrödinger operator H_ω with Gaussian potential from Theorem 3.2 we have*

$$\lim_{t\to+\infty} \frac{\ln\mathcal{L}[\mathsf{IDS}](t)}{t^2} = \frac{c(0)}{2}. \tag{4.4}$$

Proof. Upper bound: Since $p(t,0,0) = p(t,0) \le 1$ for t large enough and for a centered Gaussian random variable Y we have $\mathbb{E}[\exp(Y)] = \exp(\frac{1}{2}\,\mathbb{E}[Y^2])$ it follows from (4.3) that for large t we have

$$\begin{aligned} \ln \mathcal{L}[\mathsf{IDS}](t) &\le \ln p(t,0,0) + \ln \mathbb{E}[\exp(-tV_\omega(0))] \\ &\le \ln \mathbb{E}[\exp(-tV_\omega(0))] = \frac{1}{2}\,\mathbb{E}[(-tV_\omega(0))^2] = \frac{c(0)}{2}\,t^2 \end{aligned}$$

which yields

$$\limsup_{t\to\infty} \frac{\ln \mathcal{L}[\mathsf{IDS}](t)}{t^2} \le \frac{c(0)}{2}. \tag{4.5}$$

Lower bound: We choose a testfunction $\psi \in \mathscr{D}(\mathcal{E})$ with $||\psi||_2 = 1$. Following the argumentation in Theorem 9.3 of [21] let us define $R = R(t) := t^{-\frac{1}{2}+\beta}$ for some $\beta \in (0,\frac{1}{2})$ and $\psi_R(x) := R^{-d/2}\psi(R^{-1}x)$. Plugging this function into (4.3) we get

$$\begin{aligned} \ln \mathcal{L}[\mathsf{IDS}](t) &\ge \ln ||\psi_R||_1^{-2} - t\,\mathcal{E}(\psi_R,\psi_R) + \ln \mathbb{E}\left[\exp\left(-t\int_{\mathbb{R}^d} V_\omega(x)\,|\psi_R(x)|^2\,dx\right)\right] \\ &=: (I) - (II) + (III) \end{aligned}$$

and consider these three parts separately.
First part (I): By a change of variables $y = R^{-1}x$ we get

$$\begin{aligned} \ln ||\psi_R||_1^{-2} &= -2\ln \int_{\mathbb{R}^d} |R^{-d/2}\psi(R^{-1}x)|\,dx \\ &= -2\ln\left(R^{-d/2}\int_{\mathbb{R}^d} R^d|\psi(y)|\,dy\right) \\ &= -2\ln\left(R^{d/2}||\psi||_1\right) \\ &= -d\ln t^{-\frac{1}{2}+\beta} - 2\ln||\psi||_1 \\ &= d(\tfrac{1}{2}-\beta)\ln t - 2\ln||\psi||_1. \end{aligned} \tag{4.6}$$

Second part (II): By (4.1) and a change of variables $\eta = R\xi$ we obtain

$$
\begin{aligned}
t\,\mathcal{E}(\psi_R,\psi_R) &= t\int_{\mathbb{R}^d} |\xi|^\alpha |\mathscr{F}[\psi_R(\cdot)](\xi)|^2 d\xi \\
&= t\int_{\mathbb{R}^d} |\xi|^\alpha |\mathscr{F}[R^{-d/2}\psi(R^{-1}\cdot)](\xi)|^2 d\xi \\
&= t\int_{\mathbb{R}^d} |\xi|^\alpha R^{-d} |R^d \mathscr{F}[\psi](R\xi)|^2 d\xi \\
&= t\int_{\mathbb{R}^d} |\xi|^\alpha R^{d} |\mathscr{F}[\psi](R\xi)|^2 d\xi \\
&= t\int_{\mathbb{R}^d} |R^{-1}\eta|^\alpha |\mathscr{F}[\psi](\eta)|^2 d\eta \\
&= t\,R^{-\alpha}\mathcal{E}(\psi,\psi) \\
&= t^{\frac{\alpha}{2}-\alpha\beta+1}\mathcal{E}(\psi,\psi).
\end{aligned} \tag{4.7}
$$

Third part (III): First note that $Y(\omega) = -t\int_{\mathbb{R}^d} V_\omega(x)\,|\psi_R(x)|^2\,dx$ is a centered Gaussian random variable and thus $\mathbb{E}[\exp(Y)] = \exp(\frac{1}{2}\mathbb{E}[Y^2])$ which yields

$$
\begin{aligned}
&\ln\mathbb{E}\left[\exp\left(-t\int_{\mathbb{R}^d} V_\omega(x)\,|\psi_R(x)|^2\,dx\right)\right] \\
&= \frac{t^2}{2}\,\mathbb{E}\left[\left(\int_{\mathbb{R}^d} V_\omega(x)\,|\psi_R(x)|^2\,dx\right)^2\right] \\
&= \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{E}[V_\omega(x)V_\omega(y)]\,|\psi_R(x)|^2|\psi_R(y)|^2\,dx\,dy \\
&= \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(x-y)|R^{-\frac{d}{2}}\psi(R^{-1}x)|^2|R^{-\frac{d}{2}}\psi(R^{-1}y)|^2\,dx\,dy \\
&= \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(x-y)R^{-2d}|\psi(R^{-1}x)|^2|\psi(R^{-1}y)|^2\,dx\,dy \\
&= \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c(R(u-v))\,|\psi(u)|^2|\psi(v)|^2\,du\,dv.
\end{aligned} \tag{4.8}
$$

Alltogether, by (4.6), (4.7) and (4.8) we get the lower bound

$$
\begin{aligned}
\ln\mathcal{L}[\mathsf{IDS}](t) \geq\ & d(\tfrac{1}{2}-\beta)\ln t - 2\ln||\psi||_1 - t^{\frac{\alpha}{2}-\alpha\beta+1}\mathcal{E}(\psi,\psi) \\
&+ \frac{t^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} c\left(t^{\beta-\frac{1}{2}}(u-v)\right)\,|\psi(u)|^2|\psi(v)|^2\,du\,dv.
\end{aligned}
$$

Since $\beta \in (0, \frac{1}{2})$, c is continuous at the origin and $c(x) \leq c(0)$ by the Cauchy-Schwarz inequality, using dominated convergence we finally get

$$\liminf_{t\to\infty} \frac{\ln \mathcal{L}[\mathsf{IDS}](t)}{t^2} \geq \frac{c(0)}{2} \tag{4.9}$$

which together with (4.5) concludes the proof. □

Now we apply the following Tauberian theorem which was first stated by Fukushima, Nagai and Nakao in [7] and later proven by Nagai in [17].

Theorem 4.3 ([17], Corollary 2)**.** *Let $H(\lambda)$ be a non-decreasing function on $\mathbb{R}$ such that $H(-\infty) = 0$ and let $\mathcal{L}[H](t)$ be its Laplace-Stieltjes transform. Then we have*

$$H(\lambda) \sim e^{-A|\lambda|^\beta} \quad \text{as } \lambda \downarrow -\infty \qquad \Longleftrightarrow \qquad \mathcal{L}[H](t) \sim e^{Bt^\gamma} \quad \text{as } t \uparrow +\infty \tag{4.10}$$

for constants $A, B > 0$ and $\beta, \gamma > 1$ fulfilling

$$\gamma = \frac{\beta}{\beta - 1} \qquad \Longleftrightarrow \qquad \beta = \frac{\gamma}{\gamma - 1} \tag{4.11}$$

and

$$B = (\beta - 1)\beta^{\beta/(1-\beta)} A^{1/(1-\beta)} \qquad \Longleftrightarrow \qquad A = (\gamma - 1)\gamma^{\gamma/(1-\gamma)} B^{1/(1-\gamma)}. \tag{4.12}$$

A direct application of Theorem 4.3 to the situation of Lemma 4.2 shows the occurrence of Lifshitz tails:

Theorem 4.4. *For the IDS of the fractional random Schrödinger operator H_ω from Theorem 3.2 we have*

$$\lim_{\lambda\downarrow-\infty} \frac{\ln \mathsf{IDS}(\lambda)}{\lambda^2} = -\frac{1}{2c(0)}. \tag{4.13}$$

Proof. In view of (4.4) the right-hand side of (4.10) is fulfilled with $\gamma = 2$ and $B = c(0)/2$. This yields $\beta = 2$ and $A = (2c(0))^{-1}$ by (4.11) and (4.12) from which the assertion easily follows by (4.10). □

5. Asymptotic Behaviour of the IDS at $+\infty$ for Gaussian Potentials

In this section we derive the precise asymptotics of the IDS at the right end of the spectrum, i.e. as $\lambda \to +\infty$. For this purpose we determine the asymptotics of the positive unilateral Laplace-Stieltjes transform of the IDS as $t \downarrow 0$ and use an appropriate Tauberian theorem.

Lemma 5.1. *In the situation of Theorem 3.2 we have*

$$\lim_{t\downarrow 0}\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\,ds\right)\right]=1. \tag{5.1}$$

Proof. As in the proof of Theorem 3.2 we get

$$\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\,ds\right)\right]\le\frac{1}{2}+\int_0^\infty \exp(ty)\,d\mathbb{P}_{\mathcal{N}_{0,c(0)}}(y).$$

and thus by dominated convergence we have

$$\limsup_{t\downarrow 0}\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\,ds\right)\right]\le 1. \tag{5.2}$$

On the other hand Jensen's inequality with $V_\omega^+=\max\{V_\omega,0\}$ and an application of the Tonelli-Fubini theorem yields

$$\begin{aligned}
&\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\,ds\right)\right]\\
&\quad\ge\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[\exp\left(-\int_0^t V_\omega^+(X_s(\eta))\,ds\right)\right]\\
&\quad\ge\exp\left(\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[-\int_0^t V_\omega^+(X_s(\eta))\,ds\right]\right)\\
&\quad=\exp\left(-\int_0^t\int_{\mathbb{R}}\int_\Omega V_\omega^+(x)\,d\mathbb{P}(\omega)\,d\mathbb{P}'_{X_s|X_0=0=X_t}(x)\,ds\right)\\
&\quad=\exp\left(-\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\max\{y,0\}\,d\mathcal{N}_{0,c(0)}(y)\,d\mathbb{P}'_{X_s|X_0=0=X_t}(x)\,ds\right)\\
&\quad=\exp\left(-t\int_0^\infty y\,d\mathcal{N}_{0,c(0)}(y)\right)=\exp\left(t\sqrt{\tfrac{c(0)}{2\pi}}\right).
\end{aligned}$$

Thus we also have

$$\liminf_{t\downarrow 0}\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\,ds\right)\right]\ge 1$$

which together with (5.2) concludes the proof. □

Now we decompose the Laplace-Stieltjes transform of the IDS into the positive and negative unilateral Laplace-Stieltjes transforms given by

$$\mathcal{L}^+[\mathsf{IDS}](t)=\int_0^\infty e^{-\lambda t}\,d\mathsf{IDS}(\lambda)\quad\text{and}\quad\mathcal{L}^-[\mathsf{IDS}](t)=\int_{-\infty}^0 e^{-\lambda t}\,d\mathsf{IDS}(\lambda)$$

for which the asymptotics as $t\to 0$ will be derived separately.

Lemma 5.2. *In the situation of Theorem 3.2 we have*

$$\lim_{t\downarrow 0}\frac{\mathcal{L}^-[\mathsf{IDS}](t)}{t^{-d/\alpha}}=0. \tag{5.3}$$

Proof. Since by Theorem 4.4 we have $\mathsf{IDS}(\lambda)\sim\exp(-\frac{\lambda^2}{2c(0)})$ as $\lambda\downarrow-\infty$ by partial integration for Riemann-Stieltjes integrals we get for every $t>0$

$$\begin{aligned}0\le\frac{\mathcal{L}^-[\mathsf{IDS}](t)}{t^{-d/\alpha}}&=t^{d/\alpha}\int_{-\infty}^{0}e^{-\lambda t}\,d\mathsf{IDS}(\lambda)\\&=t^{d/\alpha}\left[e^{-\lambda t}\mathsf{IDS}(\lambda)\right]_{\lambda=-\infty}^{0}+t^{d/\alpha+1}\int_{-\infty}^{0}e^{-\lambda t}\mathsf{IDS}(\lambda)\,d\lambda.\\&=t^{d/\alpha}\mathsf{IDS}(0)+t^{d/\alpha+1}\int_{-\infty}^{0}e^{-\lambda t}\mathsf{IDS}(\lambda)\,d\lambda.\end{aligned}$$

Choose $R>0$ such that by Theorem 4.4 we have

$$\mathsf{IDS}(\lambda)\le 2\exp(-\tfrac{\lambda^2}{2c(0)})\quad\text{ for all }\lambda\le -R$$

and choose $t>0$ sufficiently small such that $t/R\le(4c(0))^{-1}$ then we get

$$\begin{aligned}0&\le\frac{\mathcal{L}^-[\mathsf{IDS}](t)}{t^{-d/\alpha}}\\&\le t^{d/\alpha}\mathsf{IDS}(0)+2t^{d/\alpha+1}\int_{-\infty}^{-R}e^{-\lambda t}\exp(-\tfrac{\lambda^2}{2c(0)})\,d\lambda+t^{d/\alpha+1}\int_{-R}^{0}e^{-\lambda t}\mathsf{IDS}(\lambda)\,d\lambda\\&\le t^{d/\alpha}\mathsf{IDS}(0)+2t^{d/\alpha+1}\int_{-\infty}^{-R}\exp(-\lambda^2(\tfrac{1}{2c(0)}-\tfrac{t}{R}))\,d\lambda+t^{d/\alpha+1}\mathsf{IDS}(0)\int_{-R}^{0}e^{-\lambda t}\,d\lambda\\&\le t^{d/\alpha}\mathsf{IDS}(0)+2t^{d/\alpha+1}\int_{-\infty}^{-R}\exp(-\tfrac{\lambda^2}{4c(0)})\,d\lambda+t^{d/\alpha}\mathsf{IDS}(0)(e^{Rt}-1)\to 0\end{aligned}$$

as $t\downarrow 0$ as desired. □

Lemma 5.3. *In the situation of Theorem 3.2 we have*

$$\lim_{t\downarrow 0}\frac{\mathcal{L}^+[\mathsf{IDS}](t)}{t^{-d/\alpha}}=p(1,0). \tag{5.4}$$

Proof. By (3.6), Lemma 5.2 and Lemma 5.1 we get

$$\begin{aligned}\lim_{t\downarrow 0}\frac{\mathcal{L}^+[\mathsf{IDS}](t)}{t^{-d/\alpha}}&=\lim_{t\downarrow 0}\frac{\mathcal{L}[\mathsf{IDS}](t)}{t^{-d/\alpha}}\\&=\lim_{t\downarrow 0}t^{d/\alpha}p(t,0,0)\ \mathbb{E}(d\omega)\times\mathbb{E}_{0,0}^{t}(d\eta)\left[e^{-\int_0^t V_\omega(X_s(\eta))ds}\right]\end{aligned}$$

$$= \lim_{t\downarrow 0} t^{d/\alpha} p(t,0) = p(1,0)$$

due to the scaling property of the isotropic α-stable Lévy process. $\square$

Now we are able to use the following Tauberian theorem of Hardy-Littlewood type.

Theorem 5.4 ([25], Theorem 4.6)**.** *Let f be a function on $\mathbb{R}_+$ such that $\mathcal{L}^+[f](t)$ exists for every $t>0$ and for some constants $K, A, \gamma > 0$ the function $\lambda \mapsto f(\lambda) + K\lambda^\gamma$ is non-decreasing on $\mathbb{R}_+$ and fulfills*

$$\lim_{t\downarrow 0} \frac{\mathcal{L}^+[f](t)}{t^{-\gamma}} = A. \tag{5.5}$$

Then we have

$$\lim_{\lambda\to\infty} \frac{f(\lambda)}{\lambda^\gamma} = \frac{A}{\Gamma(\gamma+1)}. \tag{5.6}$$

Theorem 5.5. *The IDS of the fractional random Schrödinger operator H_ω with Gaussian potential defined in Theorem 3.2 exhibits the asymptotics*

$$\lim_{\lambda\to\infty} \frac{\mathsf{IDS}(\lambda)}{\lambda^{d/\alpha}} = \frac{p(1,0)}{\Gamma(\frac{d}{\alpha}+1)}. \tag{5.7}$$

Proof. By Lemma 5.3 the conditions of Theorem 5.4 are fulfilled for $f = \mathsf{IDS}|_{[0,\infty)}$ and constants $\gamma = d/\alpha$, $A = p(1,0)$ and arbitrary $K>0$. A direct application of Theorem 5.4 yields (5.7). $\square$

Corollary 5.6. *In dimension $d=1$ we have for all $\alpha \in (0,2)$*

$$\lim_{\lambda\to\infty} \frac{\mathsf{IDS}(\lambda)}{\lambda^{1/\alpha}} = \frac{1}{\pi}. \tag{5.8}$$

Proof. In dimension $d=1$ we can describe the constant $p(1,0)$ in (5.7) explicitly. For $\alpha \neq 1$ equations (14.30) and (14.33) in [24] yield $p(1,x) = \pi^{-1}\Gamma(\frac{1}{\alpha}+1) + \mathcal{O}(x)$ as $x \to 0$ and for $\alpha = 1$ the symmetric Cauchy density fulfills $p(1,0) = \pi^{-1} = \pi^{-1}\Gamma(2)$. Plugging this into (5.7) proves the claim. $\square$

6. Asymptotic Behavior of the IDS at $+\infty$ for Poissonian Potentials

In case of a fractional random Schrödinger operator with Poissonian potential we derive the same asymptotics at $+\infty$ as in the case of Gaussian potentials. The Poisson potential is given by convolution of a shape function Poisson random measure.

Theorem 6.1. *For fixed $\alpha \in (0,2)$ consider the fractional random Schrödinger operator H_ω in (1.1) with Poissonian potential $V_\omega = (V_\omega(x))_{x\in\mathbb{R}^d}$ given by*

$$V_\omega(x) = \int_{\mathbb{R}^d} \varphi(x-y)\; d\mathsf{P}_\omega(dy), \tag{6.1}$$

where P_ω denotes a Poisson random measure with Lebesgue intensity on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\varphi \in L^1(\mathbb{R}^d)$ is a nonnegative continuous function.
Then the IDS of the operator H_ω exhibits the asymptotics

$$\lim_{\lambda\to\infty} \frac{\mathsf{IDS}(\lambda)}{\lambda^{d/\alpha}} = \frac{p(1,0)}{\Gamma(\frac{d}{\alpha}+1)}. \tag{6.2}$$

Proof. Ōkura showed the existence of the integrated density of states for fractional Schrödinger operators with Poissonian potential in Theorem 6.1 of [18]. It yields the representation

$$\mathcal{L}[\mathsf{IDS}](t) = p(t,0)\, \mathbb{E}(d\omega) \times \mathbb{E}^t_{0,0}(d\eta) \left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\, ds\right)\right]. \tag{6.3}$$

We will first show that (5.1) remains true in case of our Possonian potential. As in the proof of Theorem 3.2 we get

$$\begin{aligned}
&\mathbb{E}(d\omega) \times \mathbb{E}^t_{0,0}(d\eta) \left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\, ds\right)\right] \\
&\le \frac{1}{t}\int_0^t \mathbb{E}(d\omega) \times \mathbb{E}^t_{0,0}(d\eta) \left[\exp(V_\omega^-(X_s(\eta))\right]\, ds = 1
\end{aligned}$$

since the negative part of the Poissonian potential disappears. Thus we have

$$\limsup_{t\downarrow 0} \mathbb{E}(d\omega) \times \mathbb{E}^t_{0,0}(d\eta) \left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\, ds\right)\right] \le 1. \tag{6.4}$$

Furthermore, by Jensen's inequality, the Tonelli-Fubini theorem and stationarity of the potential we get

$$\begin{aligned}
&\mathbb{E}(d\omega) \times \mathbb{E}^t_{0,0}(d\eta) \left[\exp\left(-\int_0^t V_\omega(X_s(\eta))\, ds\right)\right] \\
&\ge \exp\left(\mathbb{E}(d\omega) \times \mathbb{E}^t_{0,0}(d\eta) \left[-\int_0^t V_\omega(X_s(\eta))\, ds\right]\right) \\
&= \exp\left(-\int_0^t \int_{\mathbb{R}} \int_{\Omega} V_\omega(x)\, d\mathbb{P}(\omega)\, d\mathbb{P}'_{X_s|X_0=0=X_t}(x)\, ds\right) \\
&= \exp(-t\mathbb{E}[V_\omega(0)]) = \exp\left(-t\mathbb{E}\left[\int_{\mathbb{R}^d} \varphi(0-y)\, \mathsf{P}_\omega(dy)\right]\right)
\end{aligned}$$

$$= \exp\left(-t\int_{\mathbb{R}^d}\varphi(-y)\,dy\right) = \exp\left(-t||\varphi||_1\right),$$

where in the first but last step we used the moment formula for Poisson point processes with Lebesgue intensity measure, e.g. see Corollary 24.15(i) in [13]. Thus we also have

$$\liminf_{t\downarrow 0}\mathbb{E}(d\omega)\times\mathbb{E}^t_{0,0}(d\eta)\left[\exp\left(-\int_0^t V_\omega(X_s(\eta))ds\right)\right]\geq 1$$

which together with (6.4) shows (5.1) and (6.3) yields

$$\lim_{t\downarrow 0}\frac{\mathcal{L}[\mathsf{IDS}](t)}{p(t,0)} = 1.$$

By the scaling property of an isotropic α-stable Lévy process and the fact that the spectrum of H_ω is contained in the positive real line we get

$$\lim_{t\downarrow 0}\frac{\mathcal{L}^+[\mathsf{IDS}](t)}{t^{-d/\alpha}} = \lim_{t\downarrow 0}\frac{\mathcal{L}[\mathsf{IDS}](t)}{t^{-d/\alpha}} = p(1,0).$$

This shows that the conditions of Theorem 5.4 are fulfilled for $f=\mathsf{IDS}$ and constants $\gamma=d/\alpha$, $A=p(1,0)$ and arbitrary $K>0$. A direct application of Theorem 5.4 yields (6.2). □

References

[1] Barcy, M. (2018) Examples of random fields that can be represented as space-domain scaled stationary Ornstein-Uhlenbeck fields. *Mathematica Slovaca* **68**(1), 197–210.

[2] Carmona, R.; and Lacroix, J. (1990) *Spectral Theory of Random Schrödinger Operators.* Birkhäuser, Boston.

[3] Chaumont, L.; and Uribe Bravo, G. (2011) Markovian bridges: Weak continuity and pathwise constructions. *Ann. Probab.* **39**(2) 609–647.

[4] Demuth, M.; and van Casteren, J.A. (2000) *Stochastic Spectral Theory for Self-adjoint Feller Operators. A Functional Analysis approach.* Birkhäuser, Basel.

[5] Donsker, M.D.; Varadhan, S.R.S. (1975) Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.* **28** (1975) 525–565.

[6] Fiscella, A; Servadei, R.; Valdinoci, E. (2015) Density properties for fractional Sobolev spaces. *Annales – Academiae Scientiarum Fennicae Mathematica* **40**(1) (2015) 235 – 253.

[7] Fukushima, M.; Nagai, H.; and Nakao S. (1975) On an asymptotic property of spectra of a random difference operator. *Proc. Japan Acad.* **51** (1975) 100–102.

[8] Gebert, M.; and Rojas-Molina, C. (2020) Lifshitz tails for the fractional Anderson model. Preprint, available at: `https://arxiv.org/abs/1910.02077`

[9] Kaleta, K.; and Pietruska-Pałuba, K. (2020) Lifshitz tail for alloy-type models driven by the fractional Laplacian. Preprint, available at: `https://arxiv.org/abs/1906.03419`

[10] Kaleta, K.; and Pietruska-Pałuba, K. (2020) Lifshitz tail for continuous Anderson models driven by Lévy operators. Preprint, available at: `https://arxiv.org/abs/1910.01153`

[11] Kirsch, W.; and Martinelli, F. (1982) On the density of states of Schrödinger operators with random potential. *J. Phys. A* **15** 2139–2156.

[12] Kirsch, W.; and Veselić, I. (2010) Lifshitz tails for a class of Schrödinger operators with random breather-type potential. Lett. Math. Phys. **94** 27–39.

[13] Klenke, A. (2013) *Probability Theory – A Comprehensive Course.* 2nd ed. Springer, London.

[14] Kwaśnicki, M. (2017) Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.* **20**(1) 7–51.

[15] Lischke, A.; Pang, G.; Gulian, M.; Song, F.; Glusa, C.; Zheng, X.; Mao, Z.; Cai, W.; Meerschaert, M.M.; Ainsworth, M.; Karniadakis, G.E. (2020) What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys.* **404** 109009.

[16] Nakao, S. (1977) On the spectral distribution of the Schrödinger operator with random potential. *Japan. J. Math.* **3**(1) 111–139.

[17] Nagai, H. (1977) A remark on the Minlos-Povzner Tauberian theorem. *Osaka J. Math.* **14**(1) 117–123.

[18] Ōkura, H. (1979) On the spectral distributions of certain integro-differential operators with random potential. *Osaka J. Math.* **16** 633–666.

[19] Pastur, L.A. (1973) Spectra of random self-adjoint operators. *Russian Math. Surveys.* **28** 1–67.

[20] Pastur, L.A. (1977) Behavior of some Wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential. *Theoret. Math. Phys.* **32** 615–620.

[21] Pastur, L.A.; and Figotin, A. (1992) *Spectra of Random and Almost-Periodic Operators.* Springer, Berlin.

[22] Reed, M.; and Simon, B. (1978) *Methods of Modern Mathematical Physics. IV: Analysis of Operators.* Academic Press, New York.

[23] Song, R.; and Vondraček, Z. (2003) Potential theory of subordinate killed Brownian motion in a domain. *Probab. Theory Relat. Fields* **125** 578–592.

[24] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.

[25] Widder, D.V. (1966) *The Laplace Transform.* Princeton University Press, Princeton.

Peter Kern, Mathematical Institute, Heinrich-Heine-University Düsseldorf, Universitätsstr. 1, 40225 Düsseldorf, Germany
Email address: `kern@hhu.de`

Leonard Pleschberger, Mathematical Institute, Heinrich-Heine-University Düsseldorf, Universitätsstr. 1, 40225 Düsseldorf, Germany
Email address: `Leonard.Pleschberger@uni-duesseldorf.de`