Parabolic Fractal Geometry of Stable Lévy Processes with Drift

Leonard Pleschberger

Heinrich Heine University Düsseldorf

21.05.2024

Agenda

1. Introduction: α-stable Lévy processes $X = (X_t)_{t \geq 0}$ in $\mathbb{R}^d$
2. Stochastic formulation of PDEs
3. α-parabolic Hausdorff dimension
4. Formula: Hausdorff dimension of $X + f$
5. Estimates for the parabolic Hausdorff dimension

Introduction: α-Stable Lévy Processes $X = (X_t)_{t \geq 0}$

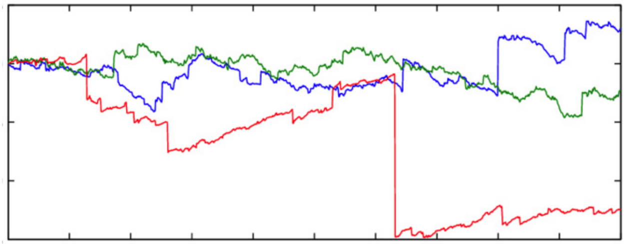
1. Time-continuous stochastic process where $\alpha \in (0, 2]$;
   $\alpha = 2$: Brownian Motion

2. Lévy process: Independent and stationary increments, stochastically continuous, starts a.s. in $0 \in \mathbb{R}^d$

3. Transition density $p_t(x, y) = p_t(x - y)$ with characteristic function

$$\mathbb{E}\left[e^{i\langle \xi, X_t \rangle}\right] = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} \cdot p_t(x)\, dx = e^{-t||\xi||^\alpha}$$

Introduction: α-Stable Lévy Processes $X = (X_t)_{t \geq 0}$



α-Stable Lévy Processes: Main Properties

1. $\alpha = 2$: Brownian motion
2. Non-continuous stochastic process for $\alpha \in (0, 2)$
3. X is self-similar, i.e. $X_t \stackrel{\text{f.d.}}{=} t^{1/\alpha} \cdot X_1$
4. Models non-local phenomena, i.e. jumps are involved
5. Translates fractional heat equation to particle scale

Translation of PDEs to Stochastic Processes

The fractional heat equation

$$\dot{v} + (-\Delta)^{\alpha/2}[v] = 0, \; v|_{t=0} = v_0$$

is solved by

$$e^{-t\,(-\Delta)^{\alpha/2}}[v_0](x) = \int_{\mathbb{R}^d} p(t,x,y) \cdot v_0(y)\,dy = \mathbb{E}[v_0(x + X_t)]$$

Stable Lévy Processes are useful

1. Model non-local particle movement by including jumps
2. Subsumes Brownian motion and jump processes
3. (Fractional) Brownian motion is self-similar **and** continuous
4. All other stable processes are self-similar, **not** continuous
5. **Stable processes are testfunctions for self-similarity!**

Stochastic Formulation of PDEs

PDE with initial value $v|_{t=0} = v_0$

Rule for v_t without time derivatives

SDE with initial value $X_0(x) = x$

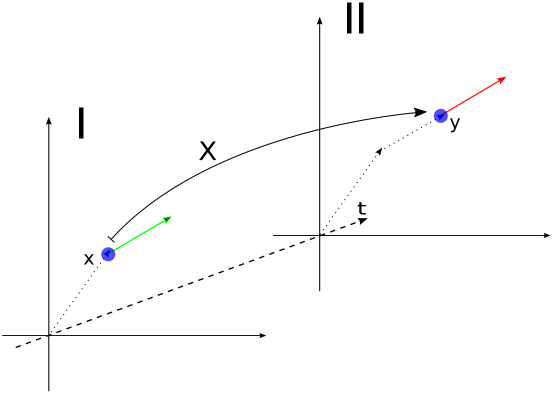
Inviscid Burgers Equation

$$\dot{v} + (v \cdot \nabla)v - \mu\Delta v + \nabla p = 0, \quad \operatorname{div} v = 0$$

$$v_t(y) = \mathbb{E}\mathbf{P}[\nabla Y_t^{-1}(y) \cdot v_0(Y_t^{-1}(y))]$$

$$Y_t(x) = x + \int_0^t v_s(x)\,ds + \sqrt{2\mu}B_t = y$$

Inviscid Burgers Equation



Viscid Burgers Equation

$$\dot{v} + (v \cdot \nabla)v - \mu\Delta v + \nabla p = 0, \quad \operatorname{div} v = 0$$

$$v_t(y) = \mathbb{E}\mathbf{P}[\nabla Y_t^{-1}(y) \cdot v_0(Y_t^{-1}(y))]$$

$$Y_t(x) = x + \int_0^t v_s(x)\, ds + \sqrt{2\mu} B_t = y$$

Inviscid Burgers Equation

$$\dot{v} + (v \cdot \nabla)v - \mu\Delta v + \nabla p = 0, \quad \operatorname{div} v = 0$$

$$v_t(y) = \mathbb{E}\mathbf{P}[\nabla Y_t^{-1}(y) \cdot v_0(Y_t^{-1}(y))]$$

$$Y_t(x) = x + \int_0^t v_s(x)\, ds + \sqrt{2\mu}B_t = y$$

Euler Equations

$$\dot{v} + (v \cdot \nabla) v - \mu \Delta v + \nabla p = 0, \quad \operatorname{div} v = 0$$

$$v_t(y) = \mathbb{E}\mathbf{P}[\nabla Y_t^{-1}(y) \cdot v_0(Y_t^{-1}(y))]$$

$$Y_t(x) = x + \int_0^t v_s(x)\, ds + \sqrt{2\mu} B_t = y$$

Navier-Stokes Equations

$$\dot{v} + (v \cdot \nabla)v - \mu\Delta v + \nabla p = 0, \quad \mathrm{div}\; v = 0$$

$$v_t(y) = \mathbb{E}\mathbf{P}[\nabla Y_t^{-1}(y) \cdot v_0(Y_t^{-1}(y))]$$

$$Y_t(x) = x + \int_0^t v_s(x)\, ds + \sqrt{2\mu} B_t = y$$

Stochastic Process plus Deterministic Driftfunction

$$\dot{v} + (v \cdot \nabla)v - \mu \Delta v + \nabla p = 0, \quad \mathrm{div}\, v = 0$$

$$v_t(y) = \mathbb{E}\mathbf{P}[\nabla Y_t^{-1}(y) \cdot v_0(Y_t^{-1}(y))]$$

$$Y_t(x) = x + \int_0^t v_s(x)\, ds + \sqrt{2\mu} B_t = y$$

Parabolic Hausdorff Measure and Dimension

Restricted Hausdorff measure:

$$\mathcal{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A) := \lim_{\delta \downarrow 0} \inf\left\{ \sum_{n=1}^{\infty} |\mathsf{P}_n|^{\beta} : A \subseteq \bigcup_{n=1}^{\infty} \mathsf{P}_n,\ \mathsf{P}_n \in \mathcal{P}^{\alpha},\ |\mathsf{P}_n| \leq \delta \right\}$$

$$\mathcal{P}^{\alpha}\text{-}\dim A := \sup\left\{\beta : \mathcal{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A) = \infty\right\} = \inf\left\{\beta : \mathcal{P}^{\alpha}\text{-}\mathcal{H}^{\beta}(A) = 0\right\}$$

Parabolic Cylinders

Distinct non-linear scaling between time and space:

[*Taylor*, *Watson*, 1985]

$$\mathcal{P}^2 = \left\{ [t, t + r^2] \times \prod_{i=1}^{d} [x_i, x_i + r],\ c \in (0, 1] \right\}$$

[*Peres*, *Sousi*, 2016]

$$\mathcal{P}^\alpha = \left\{ [t, t + c] \times \prod_{i=1}^{d} [x_i, x_i + c^{1/\alpha}],\ c \in (0, 1] \right\}$$

Formulas: Hausdorff Dimension for $X + f$

Let f be a Borel measurable function, $\varphi_\alpha := \mathcal{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ where $\varphi_1 = \dim \mathcal{G}_T(f)$. Then a.s.

$$\dim \mathcal{G}_T(X+f) = \begin{cases} \varphi_1, & \alpha \in (0,1], \\ \varphi_\alpha \ \wedge \ \frac{1}{\alpha} \cdot \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d, & \alpha \in [1,2]. \end{cases}$$

and

$$\dim \mathcal{R}_T(X+f) = \begin{cases} \alpha \cdot \varphi_\alpha \ \wedge \ d, & \alpha \in (0,1], \\ \varphi_\alpha \ \wedge \ d, & \alpha \in [1,2]. \end{cases}$$

Upper Bounds: Geometric Measure Theory

One a.s. has

$$\mathcal{P}^{\alpha}\text{-}\dim \mathcal{G}_T(X+f) \leq \mathcal{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f).$$

Proof idea:

Let $\beta := \varphi_\alpha$, let $M_k(\omega)$ be the random number of hypercubes with sidelength $c_k^{1/\alpha}$ that the path $t \mapsto X_t(\omega)$ hits.

$$\mathbb{E}\left[\mathcal{P}^{\alpha}\text{-}\mathcal{H}^{\beta+\varepsilon}(\mathcal{G}_T(X+f))\right] \leq \mathbb{E}\left[\sum_{k=1}^{\infty} |\mathrm{P}^{\omega}|^{\beta+\varepsilon}\right]$$

$$\lesssim \sum_{k=1}^{\infty} \mathbb{E}[M_k(\omega)] \cdot c_k^{\beta+\varepsilon} \lesssim \sum_{k=1}^{\infty} c_k^{\beta+\delta} < \infty.$$

Hence $\mathcal{P}^{\alpha}\text{-}\dim \mathcal{G}_T(X+f) \leq \mathcal{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f)$, a.s. $\square$

Lower Bounds: Potential Theory

Let $\varphi_\alpha := \mathcal{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ where $\varphi_1 = \dim \mathcal{G}_T(f)$. Then a.s.

$$\dim \mathcal{G}_T(X+f) \geq \begin{cases} \varphi_1, & \alpha \in (0,1], \\ \varphi_\alpha \ \wedge \ \frac{1}{\alpha} \cdot \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d, & \alpha \in [1,2]. \end{cases}$$

Proof idea:

Let $K^\beta(t,x) = \mathbb{E}\left[||(t, X_t(\omega) + x)||^{-\beta}\right]$. Then one a.s. has

$$\mathcal{E}_{K^\beta}(\mu) = \int\int_{\mathcal{G}_T(f)\times\mathcal{G}_T(f)} K^\beta(t-s, f(t)-f(s))\,\mathrm{d}\mu(s,x)\,\mathrm{d}\mu(t,y) < \infty$$

for some probability measure $\mu \in \mathcal{M}^1(\mathcal{G}_T(f))$. $\square$

Parabolic Version of Frostman's Lemma

Let $A \in \mathcal{B}(\mathbb{R}^{1+d})$. If $\mathcal{P}^\alpha\text{-}\dim A > \beta$, then there exists $\mu \in \mathcal{M}^1(A)$ such that

$$\mu\left(\left[t,\, t+c\right] \times \prod_{i=1}^{d} \left[x_i,\, x_i + c^{1/\alpha}\right]\right) \lesssim \begin{cases} c^{\beta}, & \alpha \in (0,1], \\ c^{\beta/\alpha}, & \alpha \in [1,\infty) \end{cases}$$

for every $c \in (0,1]$ and $t, x_1, \ldots, x_d \in \mathbb{R}$.

Equalities for the Parabolic Hausdorff Dimension

Define $\varphi_f := \mathcal{P}^{\alpha}\text{-}\dim \mathcal{G}_T(f)$. If $f \equiv C \in \mathbb{R}^d$, then a.s.

$$\varphi_f = (\alpha \vee 1) \cdot \dim T.$$

If $f \equiv X \in \mathbb{R}^d$, then a.s.

$$\varphi_X = (\alpha \vee 1) \cdot \dim T.$$

Formulas: Hausdorff Dimension for X

Since $X = X + 0^d$ one has a.s.

$$\dim \mathcal{G}_T(X) = \begin{cases} \dim T + 1 - 1/\alpha, & d = 1,\ \alpha \cdot \dim T > 1, \\ (\alpha \vee 1) \cdot \dim T, & d = 1,\ \alpha \cdot \dim T \leq 1 \text{ or } d \geq 2, \end{cases}$$

which was proved by *Blumenthal* and *Getoor* (1960).

It fits the result from *Taylor* (1953) for Brownian motion $B = (B_t)_{t \geq 0}$:

$$\dim \mathcal{G}_{[0,1]}(B) = \begin{cases} 3/2 & \text{for } d = 1, \\ 2, & \text{for } d \geq 2. \end{cases}$$

Estimates for the Parabolic Hausdorff Dimension

Define $\varphi_\alpha := \mathcal{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$ where $\varphi_1 = \dim \mathcal{G}_T(f)$. Then a.s.

$$\varphi_\alpha \leq \begin{cases} \varphi_1 + \left(\frac{1}{\alpha} - 1\right) \cdot d \ \wedge \ d+1, & \alpha \in (0,1], \\ \varphi_1 + \alpha - 1 \ \wedge \ d+1, & \alpha \in [1,\infty) \end{cases}$$

and

$$\varphi_\alpha \geq \begin{cases} \varphi_1 \ \vee \ \frac{1}{\alpha} \cdot \varphi_1 + 1 - \frac{1}{\alpha}, & \alpha \in (0,1], \\ \varphi_1 \ \vee \ \alpha \cdot \varphi_1 + (1-\alpha) \cdot d, & \alpha \in [1,\infty). \end{cases}$$

Estimates for the Parabolic Hausdorff Dimension

Define $\varphi_\alpha := \mathcal{P}^\alpha\text{-}\dim \mathcal{G}_T(f)$, $\beta \in (0,1]$ and $f \in C^\beta\left(T, \mathbb{R}^d\right)$. Then a.s.

$$\varphi_\alpha \leq \begin{cases} \dim T + d \cdot \left(\frac{1}{\alpha} - \beta\right) \ \wedge \ \frac{\dim T}{\alpha\beta} \ \wedge \ d+1, & \alpha \in (0,1], \\ \alpha \cdot \dim T + d \cdot (1 - \alpha\beta) \ \wedge \ \frac{\dim T}{\beta} \ \wedge \ d+1, & \alpha \in \left[1, \frac{1}{\beta}\right], \\ \alpha \cdot \dim T \ \wedge \ \frac{1}{\beta} \cdot (\dim T - 1) + \alpha \ \wedge \ d+1, & \alpha \in \left[\frac{1}{\beta}, \infty\right) \end{cases}$$

Conclusion

1. Parabolic fractal geometry lets the stable process disappear.
2. Difficult in lower dimension and easier in higher dimensions.
3. Aim: Make use of the results for PDEs via polar sets.

Literature

1. **Peres, Sousi** (2016)
   Dimension of Fractional Brownian motion with variable drift.
   Probab. Theory Related Fields **165**, 771–94.

2. **Pruitt, Taylor** (1969)
   Sample Path Properties of Processes with Stable Components.
   Z. Wahrscheinlichkeitstheorie verw. Gebiete **12**, 267–289.

3. **Taylor, Watson** (1985)
   A Hausdorff measure classification of polar sets for the heat equation.
   Math. Proc. Cambridge Phil. Soc. **97**, 325–44.

leonard.pleschberger@gmail.com