

Teams in Online Scheduling Polls: Game-Theoretic Aspects*

Robert Bredereck¹, Jiehua Chen², Rolf Niedermeier², Svetlana Obraztsova³, and
Nimrod Talmon⁴

¹University of Oxford, United Kingdom, robert.bredereck@cs.ox.ac.uk

²TU Berlin, Germany, {jiehua.chen, rolf.niedermeier}@tu-berlin.de

³I-CORE, Hebrew University of Jerusalem, Israel, svetlana.obraztsova@gmail.com

⁴Weizmann Institute of Science, Israel, nimrodtalmon77@gmail.com

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Abstract

Consider an important meeting to be held in a team-based organization. Taking availability constraints into account, an online scheduling poll is being used in order to decide upon the exact time of the meeting. Decisions are to be taken during the meeting, therefore each team would like to maximize its relative attendance in the meeting (i.e., the proportional number of its participating team members). We introduce a corresponding game, where each team can declare (in the scheduling poll) a lower total availability, in order to improve its relative attendance—the pay-off. We are especially interested in situations where teams can form coalitions.

We provide an efficient algorithm that, given a coalition, finds an optimal way for each team in a coalition to improve its pay-off. In contrast, we show that deciding whether such a coalition exists is NP-hard. We also study the existence of Nash equilibria: Finding Nash equilibria for various small sizes of teams and coalitions can be done in polynomial time while it is coNP-hard if the coalition size is unbounded.

1 Introduction

An organization is going to hold a meeting, where people are to attend. Since people come from different places and have availability constraints, an open online scheduling poll is taken to decide upon the meeting time. Each individual can approve or disapproval of each of the suggested time slots. In order to have the highest possible attendance, the organization will schedule the meeting at a time slot with the maximum sum of declared availabilities. During the meeting, proposals will be discussed and decisions will be made. Usually, people have different interests in the decision making, e.g. they are from different teams that each want their own proposals to be put through. We consider people with the same interest as from the same *team*

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and as a result, each team (instead of each individual) may declare (in the scheduling poll) the number of its members that can attend the meeting at each suggested time slot.

For a simple illustration, suppose that three teams, t_1 , t_2 , and t_3 , are about to hold a meeting, either at 9am or at 10am. Two members from t_1 , one member from t_2 , and three members from t_3 are available at 9am, while exactly two members of each team are available at 10am. The availabilities of the teams can be illustrated as an integer matrix:

$$A := \begin{array}{cc} & \begin{matrix} c_1 & c_2 \end{matrix} \\ \begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 3 & 2 \end{pmatrix} & \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} \end{array}$$

A time slot is a winner if it receives the maximum sum of declared availabilities. Thus, if the three teams declare their true availabilities, then both 9am and 10am co-win (since six people in total are available at 9am and 10am each), and the meeting will be scheduled at either 9am or 10am.

Now, if a team (i.e. people with the same interest) wants to influence any decision made during the meeting, then it will want to send as many of its available team members to the meeting as possible because this will maximize its *relative power*—the proportion of its own attendees. For our simple example, if the meeting is to be held at 9am, then the relative powers of teams t_1 , t_2 , and t_3 are $1/3$, $1/6$, and $1/2$, respectively. This again means that a sophisticated team will change its availabilities declared in the poll from time to time. However, each team must not report a number which is higher than its true availability since it cannot send more members than available. Given this constraint, it is interesting to know whether any team can increase its relative attendance by misreporting about its availability.

Aiming to have more power in the meeting, our teams can declare a different number than their true availabilities, possibly changing the winning time slot to one where their relative powers are maximized. For the case where several time slots co-win, however, it is not clear which co-winning time slot will be used. To be on the safe side, the teams must maximize the relative power of *each* co-winning time slot. In other words, our teams are pessimistic and consider their *pay-off* as the *minimum* over all the relative powers at each co-winning time slot. In our example, this means that the pay-off of team t_2 would be $1/6$, since this is its relative power at 9am, which is smaller than its relative power, $1/3$, at 10am. The pay-offs of teams t_1 and t_3 are both $1/3$. In this case, team t_2 can be strategic by updating its availability and declare zero availability at 9am; as a result, the meeting would be held at 10am, where team t_2 has better pay-off with relative number of $1/3$.

We do not allow arbitrary deviations from the real availabilities of teams; specifically, we do not allow a team to declare as available a higher number than actually available. Further, we do not allow a team to send more team members to the meeting than it declared as available, because this is often mandated by the circumstances. As some examples, we mention that the organizer might need to arrange a meeting room and specify the number of participants in the meeting up-front (similarly, if the meeting is to be carried in a restaurant, the number of chairs at the table shall be decided beforehand); or the organizer might need to obtain buses to transport the participants. Thus, the teams must send exactly the declared number of members to the meeting. For instance, it is not possible for team t_2 to declare 3 at 9am since only one of its

team members is available. A formal description of the corresponding game, called *team power game* (TPG, in short), and a discussion on our example are given in Section 2.

As already remarked, to improve the pay-off, a team may lie about the number of its available members. Sometimes, teams can even form a coalition and update their availabilities strategically. In our example, after team t_2 misreported its availabilities such that each team receives a pay-off of $1/3$, teams t_1 and t_3 may collaborate: if both teams keep their declared availabilities at 9am but declare zero availability at 10am (note that team t_2 does not change its updated availabilities), then 9am will be the unique winner (with total availability of 5); as a result, t_1 and t_3 receive better pay-offs of $2/5$ and $3/5$, respectively. Such a successful deviation from the declared availabilities of the teams in a coalition (while keeping the declared availabilities of the teams not in the coalition unchanged) is called an *improvement step*.

After some teams perform an improvement step, other teams may also want to update their availabilities to improve. This iterative process leads to the question of whether there is a stable situation, where improvement is impossible—a *Nash equilibrium*. Of course, when searching for equilibria, it is natural to ask how hard it is to decide whether an improvement step is possible.

In this paper, we are interested in the computational complexity of the following problems: (1) finding an improvement step (if it exists) for a specific coalition, (2) finding an improvement step (if it exists) for any coalition, and (3) finding a *t-strong Nash equilibrium* (if it exists).

Main Contributions. We show that, depending on the size of the coalition (i.e., the amount of collaboration allowed), the computational complexity of finding an improvement step for a given coalition and deciding whether an improvement step exists for an arbitrary coalition ranges from being polynomial-time solvable to being NP-hard; further, deciding whether an improvement step exists for any coalition of size at most t is W[2]-hard when parameterizing by the coalition size t . We show that a 1-strong Nash equilibrium always exists for some special profiles and we provide a simple polynomial-time algorithm for finding it in these cases. Finally, we show that deciding whether a t -strong Nash equilibrium exists is coNP-hard. Our results are summarized in Table 1.

Related Work. Recently, online scheduling polls such as Doodle / Survey Monkey caught the attention of several researches. Reinecke et al. [14] initiated empirical investigations of scheduling polls and identified influences of national culture on people’s scheduling behavior, by analyzing actual Doodle polls from 211 countries. Zou, Meir, and Parkes [15] also analyzed actual Doodle polls, and devised a model to explain their experimental findings. They observed that people participating in open polls tend to be more “cooperative” and additionally approve time slots that are very popular or unpopular; this is different to the behavior of people participating in closed polls. Obraztsova et al. [13] formally modeled the behavior observed by Zou, Meir, and Parkes [15] as a game, where approving additional time slots may result in pay-off increase. While the game introduced by Obraztsova et al. [13] captures the scenario that each individual player tries to appear to be cooperative, our team power game models the perspective that each individual team (player) as a whole tries to maximize its relative power in the meeting, which means that approving more time slots is not necessarily a good strategy.

Quite different in flavor, Lee [9] considered a computational problem from the point of view of the poll initiator, whose goal is to choose the time slots to poll over, in order to optimize a specific cost function. Finally, since scheduling polls might be modeled as approval elections, we mention the vast amount of research done on approval elections in general, e.g., [1] and on iterative approval voting in particular, e.g., [2, 11, 12].

Table 1: Complexity results for the team power game. “Unary” (resp. “Binary”) means that the input and the strategy profiles are encoded in unary (resp. binary). Variable t stands for the number of teams in a coalition, while a_{\max} stands for the maximum true availability. An entry labeled with “P” means polynomial-time solvability. An entry labeled with “FPT for k ” means solvability in $f(k) \cdot |I|^{O(1)}$ time, where f is a function solely depending on k and $|I|$ denotes the size of the given input. An entry labeled with “W[2]-hard for k ” implies that the corresponding problem is not “FPT for k ” unless $W[2] = \text{FPT}$ (this is considered unlikely in parameterized complexity theory).

(1) Finding an improvement step for a given coalition	
Unary	in P (Thm. 1)
Binary	in FPT for t (Thm. 2) ¹
(2) Deciding the existence of an improvement step for any coalition	
Binary	in P for constant t (Cor. 1)
$a_{\max} = 1$	NP-complete (Thm. 3)
$a_{\max} = 1$	W[2]-hard for t (Thm. 3)
(3) Finding a 1-strong Nash equilibrium	
$a_{\max} \leq 3$	in P, always exists (Thm. 4)
$a_{\max} \geq 4$	open (Rem. 2)
Finding a 2-strong Nash equilibrium	
$a_{\max} = 1$	in P, always exists (Prop. 1)
$a_{\max} \geq 2$	open, does not always exist
Deciding the existence of a t -strong Nash equilibrium	
$a_{\max} = 2$	coNP-hard (Thm. 5)

¹We conjecture it to be even in P. Strong NP-hardness is excluded by Theorem 1.

2 Preliminaries

We begin this section by defining the rules of the game which is of interest here. Then, we formally define the related computational problems we consider in this paper. Throughout, given a number $n \in \mathbb{N}$, by $[n]$ we mean the set $\{1, 2, \dots, n\}$.

Rules of the Game. The game is called the *team power game* (TPG, in short). It consists of n players, the *teams*, t_1, t_2, \dots, t_n , and m possible *time slots*, c_1, c_2, \dots, c_m . Each team t_i is associated with a *true availability vector* $A_i = (a_i^1, a_i^2, \dots, a_i^m)$, where $a_i^j \in \mathbb{N}$ is the (true) availability of team t_i for time slot c_j . Importantly, each team is only aware of its own availability vector. During the game, each team t_i announces a *declared (availability) vector* $B_i = (b_i^1, b_i^2, \dots, b_i^m)$, where $b_i^j \leq a_i^j$ is the declared availability of team t_i for time slot c_j ; using standard game-theoretic terms, we define the *strategy* of team t_i to be its declared availability vector B_i . We use A and B to denote the matrices consisting of a row for each team’s true and declared availability vectors. That is, for $i \in [n]$ and $j \in [m]$, $A := (a_i^j)$, $B := (b_i^j)$. Given a declared availability matrix B , the co-winners of the corresponding scheduling poll, denoted as $\text{winners}(B)$, are the

time slots with the maximum sum of declared availabilities:

$$\text{winners}(B) := \arg \max_{c_j \in \{c_1, c_2, \dots, c_m\}} \left\{ \sum_{i \in [n]} b_i^j \right\}.$$

Before we define the *pay-off* of each team, we introduce the notion of *relative power*. The relative power $\text{team-power}(B, t_i, c_j)$ of team t_i at time slot c_j equals the number of members from t_i who will attend the meeting at time slot c_j , divided by the total number of attendees at this time slot:

$$\text{team-power}(B, t_i, c_j) := \frac{b_i^j}{\sum_{k \in [n]} b_k^j}.$$

In order to define the pay-off of each team, we need to decide how to proceed when several time slots tie as co-winners. In this paper we consider a *maximin* version of the game, where ties are broken adversarially. That is, the pay-off of team t_i is defined to be the minimum, over all co-winners, of its *relative power*:

$$\text{pay-off}(B, t_i) := \min_{c_j \in \text{winners}(B)} \text{team-power}(B, t_i, c_j).$$

When we refer to an *input* for TPG, we mean a true availability matrix $A \in \mathbb{N}^{n \times m}$ where each row A_i represents the true availability of a team t_i for the m time slots. When we refer to a *strategy profile* (in short, *strategy*) for input A we mean a declared availability matrix $B \in \mathbb{N}^{n \times m}$ where each row B_i represents the declared availability vector of team t_i .

Computational Problems Related to the Game. Given a *coalition*, i.e., a subset of teams, a deviation of the teams in the coalition from their current strategy profile is an *improvement step* if, by this deviation, each team in the coalition strictly improves its pay-off. Given a positive integer t , a *t-strong Nash equilibrium* for some input A is a strategy profile B such that no coalition of at most t teams has an improvement step wrt. B . We are interested in the following computational questions:

1. Given an input, a strategy profile, and a coalition of at most t teams, does this coalition admit an improvement step compared to the given strategy profile?
2. Given an input, a strategy profile, and a positive integer t , is there any coalition of at most t teams which has an improvement step compared to the given strategy profile?
3. Given an input and a positive integer t , does a *t-strong Nash equilibrium* for this input exist?

We are particularly interested in understanding the dependency of the computational complexity of the above problems on the number t of teams in a coalition. Specifically, we consider (1) t being a constant (modeling situations where not too many teams are willing to cooperate or where cooperation is costly) and (2) t being unbounded.

Illustrating Example. Consider the input matrix A given in Section 1, which specifies the true availabilities of three teams t_1, t_2, t_3 over two time slots c_1, c_2 . If all teams declared their true availabilities, then both time slots win with total availability 6. The pay-offs of the teams t_1, t_2, t_3 are $1/3, 1/6, 1/3$, respectively. Team t_2 can improve its pay-off by declaring $(0, 2)$ (i.e., declaring 0 for c_1 and 2 for c_2). As a result, c_2 would become the unique winner with total availability 6

and team t_2 would receive a better pay-off: $1/3$. Thus, the profile B for A where all teams declare their true availabilities (i.e., where $B = A$) is not a 1-strong Nash equilibrium. Nevertheless, A does admit the following 1-strong Nash equilibrium:

$$B' := \begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 3 & 0 \end{pmatrix}$$

The declared availability matrix B' , however, is not a 2-strong Nash equilibrium, since if team t_1 and t_2 would form a coalition and declare the same availability vector $(0, 2)$, then c_2 would be the unique winner with total availability 4 and both t_1 and t_2 would have a better pay-off: $1/2$.

3 Improvement Steps

We begin with the following lemma, which basically says that, in search for an improvement step, a fixed coalition of teams needs only to focus on a single time slot.

Lemma 1. *If a coalition has an improvement step wrt. a strategy profile B , then it also has an improvement step $E = (e_i^j)$ wrt. B , where there is one time slot c_k such that each team t_i in the coalition declares zero availability for all other time slots (i.e., $e_i^j = 0$ holds for each team t_i in the coalition and each time slot $c_j \neq c_k$).*

The missing proof for Lemma 1 can be found in Appendix A.1.

By Lemma 1, we know that if a fixed coalition has an improvement step for a strategy profile B , then it admits an improvement step that involves only one time slot. Assume that it is time slot c_k . In order to compute an improvement step for the coalition, we first declare zero availabilities for the teams in the coalition, for all other time slots. Then, we have to declare specific availabilities for the teams in the coalition, for time slot c_k . This is where the collaboration between the teams comes into play: even though each team, in order to improve its pay-off, might wish to declare as high as possible availability for time slot c_k (i.e., its true availability), the teams shall collaboratively decide on the declared availabilities, since a too-high declared availability for one team might make it impossible for another team (even when declaring the maximum possible amount, i.e., the true availability) to improve its pay-off. It turns out that this problem is basically equivalent to the following problem (which, in our eyes, is interesting also on its own).

Relation to Horn Constraint Systems. Using Lemma 1, we know that if a coalition of t teams admits an improvement step for a strategy profile B , then it admits an improvement step that involves only one time slot. Let c_k be such a time slot. Then, finding an improvement step that involves only time slot c_k reduces to the following number problem. given a natural number vector $(a_1, a_2, \dots, a_t) \in \mathbb{N}^t$, a rational number vector $(p_1, p_2, \dots, p_t) \in \mathbb{Q}^t$ with $\sum_{i \in [t]} p_i \leq 1$, and a natural number $p \in \mathbb{N}$, searches for a natural number vector (x_1, x_2, \dots, x_t) where for each $i \in [t]$ the following holds: (1) $1 \leq x_i \leq a_i$ and (2) $x_i / (p + \sum_{i \in [t]} x_i) > p_i$.

Intuitively, the vector (a_1, \dots, a_t) corresponds to the true availabilities of the teams in the coalition in time slot c_k , while the solution vector (x_1, \dots, x_t) corresponds to the declared availabilities of the teams in the coalition in time slot c_k ; accordingly, the first constraint makes

sure that each declared availability is upper-bounded by its true availability. Further, the vector (p_1, \dots, p_t) corresponds to the current pay-offs of the teams in the coalition, while p corresponds to the sum of the declared availabilities of the teams not in the coalition at time slot c_k ; accordingly, the second constraint makes sure that, for each team in the coalition, the new pay-off is strictly higher than its current pay-off. More formally, we argue that the coalition $\{t_1, t_2, \dots, t_t\}$ has an improvement step compared to strategy profile B , involving only time slot c_k , if and only if the instance (A^*, P, p) for t -THRESHOLD COVERING has a solution, where $A^* := (a_1^k, \dots, a_t^k)$, $P := (\text{pay-off}(B, t_1), \dots, \text{pay-off}(B, t_t))$, and $p := \sum_{i \in [n] \setminus [t]} b_i^k$.

Remark 1. Since the values p_i ($i \in [t]$) are rational numbers, we can rearrange the second constraint in the description of t -THRESHOLD COVERING to obtain an integer linear feasibility problem. This means that t -THRESHOLD COVERING is a special variant of the so-called HORN CONSTRAINT SYSTEM problem which, given a matrix $U = (u_{i,j}) \in \mathbb{R}^{n' \times m'}$ with each row having at most one positive element, a vector $b \in \mathbb{R}^{n'}$, and a positive integer d , decides the existence of an integer vector $x \in \{0, 1, \dots, d\}^{m'}$ such that $U \cdot x \geq b$; HORN CONSTRAINT SYSTEM is weakly NP-hard and can be solved in pseudo-polynomial time [8, 8].

Taking a closer look at t -THRESHOLD COVERING, we observe the following: if we would know the sum of the variables (x_1, \dots, x_t) , then we would be able to directly solve our problem by checking every constraint and taking the smallest feasible value (i.e., given $\sum_{i \in [t]} x_i$, we would set each x_i to be the minimum over all values satisfying $x_i / (p + \sum_{i \in [t]} x_i) > p_i$). This yields a simple polynomial-time algorithm for finding an improvement step for the likely case where all availabilities are polynomially upper-bounded in the input size; technically, this means where the input profile A is encoded in unary.

Theorem 1. Consider an input A and a strategy profile B . Let s be the sum of all entries in A . Finding an improvement step (if it exists) for a given coalition is solvable in $O(s^2)$ time.

The proof for Theorem 1 can be found in Appendix A.2.

Indeed, t -THRESHOLD COVERING can be reduced to finding the sum $\sum_{i \in [t]} x_i$. If the input is encoded in binary, however, then this sum might be exponentially large in the number of bits that encode our input, thus we cannot simply enumerate all possible values. If the coalition size t or a certain parameter ℓ that measures the number of “large” true availabilities is a constant, then we still have polynomial-time algorithms for which the degree of the polynomial in the running time does not depend on the specific parameter value. Specifically, by the famous Lenstra’s theorem ([10], later improved by Kannan [7] and by Frank and Tardos [4]), we have the following result.

Theorem 2. Consider an input A and a strategy profile B . Let L be the length of the binary encoding of A . For each of the following times T , there is a T -time algorithm that finds an improvement step, compared to B , for a given coalition of t teams:

1. $T = O(t^{2.5t+o(t)} \cdot L^2)$ and
2. for each constant value c , $T = f(\ell_c) \cdot t^2 \cdot L^{c^2+2}$,

where f is a computable function and $\ell_c := \max_j |\{i \in [t] : a_i^j > L^c\}|$ is the maximum over the numbers of teams t_i in the coalition that have true availabilities a_i^j with $a_i^j > L^c$ for the same time slot c_j .

The proof for Theorem 2 can be found in Appendix A.3.

Using Theorem 2, and checking all $\sum_{i=1}^t \binom{n}{i}$ possible coalitions of size at most t , we obtain the following.

Corollary 1. *Given an input and a strategy profile, we can find, in polynomial time, a coalition of a constant number of teams and, for this coalition, find an improvement step compared to the given profile.*

In general, however, deciding whether an improvement step exists is computationally intractable as the next result shows. We briefly note that, under standard complexity assumptions, a problem being $W[2]$ -hard for parameter k presumably excludes any algorithm with running time $f(k) \cdot |I|^{O(1)}$, where f is a computable function depending only on k and $|I|$ is the size of the input.

Theorem 3. *Given an input and a strategy profile, deciding whether there is a coalition of size t that has an improvement step is $W[2]$ -hard wrt. t even if all teams are of size one. It remains NP-complete if there is no restriction on the coalition size.*

Proof. (Sketch). To show $W[2]$ -hardness, we provide a parameterized reduction from the SET COVER problem, which is $W[2]$ -complete wrt. the set cover size k [3]: Given sets $\mathcal{F} = \{S_1, \dots, S_m\}$ over a universe $U = \{u_1, \dots, u_n\}$ of elements and a positive integer k , SET COVER asks whether there is a size- k set cover $\mathcal{F}' \subseteq \mathcal{F}$, i.e., $|\mathcal{F}'| = k$ and $\bigcup_{S_i \in \mathcal{F}'} S_i = U$. The idea of such a parameterized reduction is, given a SET COVER instance (\mathcal{F}, U, k) , to produce, in $f(k) \cdot (|\mathcal{F}| + |U|)^{O(1)}$ time, an equivalent instance (A, B, t) such that $t \leq g(k)$, where f and g are two computable functions. Let (\mathcal{F}, U, k) denote a SET COVER instance. For technical reasons, we assume that each set cover contains at least three sets.

Time slots. For each element $u_j \in U$, we create one *element slot* e_j . Let $E := \{e_1, \dots, e_n\}$ denote the set containing all element slots. We create two special time slots: α (the original winner) and β (the potential new winner).

Teams and true availabilities $A = (a_i^j)$. For each set $S_i \in \mathcal{F}$, we create a *set team* t_i that has true availability 1 at time slot α , at time slot β , and at each element slot e_j with $u_j \in S_i$. We introduce several dummy teams, as follows. Intuitively, the role of these dummy teams is to allow to set specific sums of availabilities for the time slots; the crucial observation in this respect is that the dummy teams do not have any incentive to change their true availabilities, therefore we can assume that they do not participate in any coalition. For each element u_j , let $\#(u_j)$ denote the number of sets from \mathcal{F} that contain u_j . For each element slot e_j , we create $(2m - 1 - \#(u_j))$ dummy teams such that each of these dummy teams has availability 1 at element slot e_j and availability 0 for all other time slots. Similarly, for time slot α , we create m additional dummy teams, each of which has availability 1 for time slot α and availability 0 for all other time slots. For time slot β , we create $2m - 1 - k$ further dummy teams, each of which has availability 1 for time slot β and availability 0 for all other time slots.

Declared availabilities $B = (b_i^j)$. Each dummy team declares availability for the time slot where it is available. Each set team declares availability for all time slots where it is available except for time slot β where all set teams declare availability 0.

We set the size of the coalition t to be k . This completes the reduction which can be computed in polynomial time. Indeed, it is also a parameterized reduction. A formal correctness proof as well as the extension to the case of unrestricted coalition sizes to show the NP-hardness result are deferred to Appendix A.4. \square

Taking a closer look at the availability matrix constructed in the proof of Theorem 3, we observe the following.

Corollary 2. *Deciding the existence of an improvement step for any coalition is NP-hard, even for very sparse availability matrices, i.e., even if each team has only one team member and is truly available at no more than four time slots.*

4 Nash equilibria

We move on to consider the existence of Nash equilibria. Somewhat surprisingly, it seems that, a 1-strong Nash equilibrium always exists. Unfortunately, we can only prove this when the maximum availability $a_{\max} := \max_{i \in [n], j \in [m]} a_i^j$ is at most three. Extending our proof strategy to $a_{\max} \geq 4$ seems to require a huge case analysis.

Theorem 4. *If the maximum availability a_{\max} is at most three, then TPG always admits a 1-strong Nash equilibrium.*

Proof. (Sketch). Let $A = (a_i^j)$ be the input profile. We begin by characterizing two simple cases for which 1-strong Nash equilibria always exist.

Safe single-team slot. Suppose that a time slot c_j exists where only one team, t_i , is available with some availability a^* (i.e., $a_i^j = a^*$), all other teams are not available in this time slot (i.e., $a_{i'}^j = 0$ for all $i' \neq i$), and no other team, $t_{i'}, i' \neq i$, is available with availability greater than a^* at any time slot. Then, we obtain a 1-strong Nash equilibrium $B = (b_i^l)$ by setting $b_i^j := a_i^j$, and, for each $i' \in [n]$ and each $j' \neq j$, setting $b_{i'}^{j'} := 0$; to see why B is a Nash equilibrium, notice that the only team (namely t_i) i.e. available at time slot c_j already receives the best possible pay-off (namely 1) and no other team can prevent c_j from being a co-winner, which would be necessary to improve their pay-off (which is 0). We call such time slot c_j a *safe single-team slot*.

Safe multiple-team slot. Suppose that a time slot c_j exists where multiple teams have non-zero true availabilities and no single team is powerful enough to prevent c_j from co-winning, by declaring zero availability. That is, for each team t_i and each time slot $c_{j'} \neq c_j$, it holds that $a_i^{j'} \leq \sum_{i' \neq i} a_{i'}^{j'}$. Again, we obtain a 1-strong Nash equilibrium $B = (b_i^j)$ by setting $b_i^j := a_i^j$ for each team t_i and setting $b_i^{j'} := 0$ for each other time slot $c_j \neq c_{j'}$. We call such time slot c_j a *safe multiple-team slot*. For example, the following input profile contains two safe multiple-team slots, namely c_1 and c_4 :

$$A := \begin{array}{cccc} & c_1 & c_2 & c_3 & c_4 \\ \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix} & t_1 & t_2 & t_3 & t_4 \end{array}$$

We are ready to consider instances $a_{\max} \leq 3$.

Instances with $a_{\max} = 2$. Consider the maximum availability sum x of all time slots, i.e., the maximum column sum of the matrix A . Clearly, $x \geq a_{\max}$. We proceed by considering the different possible values of x .

Cases with $x = 2$: If x is two, then there is a time slot where only one team is available with availability $a_{\max} = 2$. Thus, there is a safe single-team slot.

Cases with $x = 3$: If x is three, then, since we have $a_{\max} = 2$, it follows that either (1) there is a safe single-team slot where only one team is available with availability $a_{\max} = 2$ or (2) there is a time slot c_j where a single team t_i has availability $a_{\max} = 2$ and another team $t_{i'}$ has availability 1. In the first case, there is a safe single-team, so let us consider the second case. To this end, let c_j be the time slot such that a single team t_i has availability $a_{\max} = 2$ and another team $t_{i'}$ has availability 1. Next, we show how to construct a 1-strong Nash equilibrium $B = (b_i^j)$. First, for each team t_i , set $b_i^j := a_i^j$ and $b_i^{j'} := 0, j' \neq j$. This makes time slot c_j the unique winner. Team t_i receives pay-off $2/3$ and team $t_{i'}$ receives pay-off $1/3$. Second, for each time slot $c_{j'} \neq c_j$, if $a_{i'}^{j'} > 0$, then set $b_{i'}^{j'} := 1$; otherwise, find any team $t_k \neq t_i$ with non-zero availability $a_k^{j'} = 1$ and set $b_k^{j'} := 1$. In this way, every time slot except c_j has total availability one (if there is at least one team with non-zero availability for this slot). Thus, c_j remains a unique winner and the declared total availabilities of other time slots make it impossible for any team to improve: First, team t_i cannot improve because it would receive the same pay-off $2/3$ for every time slot which it could make a new single winner (recall that no safe single-team slot exists). Second, team $t_{i'}$ also cannot improve because it cannot create a new single winner at all. Last, neither of the remaining teams can improve because they cannot prevent c_j from co-winning. Hence, we have a 1-strong Nash equilibrium.

Cases with $x \geq 4$: Every time slot c_j with availability sum x is a safe multiple-team slot since $\forall j': a_i^{j'} \leq a_{\max} = 2$ and $\forall i: \sum_{i' \neq i} a_{i'}^j \geq x - a_{\max} \geq 2$.

We defer the proof details for instances with $a_{\max} = 3$ to Appendix A.5. \square

Remark 2. *We do not know any instances without 1-strong Nash equilibria. However, we could not generalize our proof even for instances with $a_{\max} = 4$. Nevertheless, some general observations from our proof hold for every a_{\max} . In particular, if there is a column with only one entry with a_{\max} (a special case of a safe single-team slot) or if the maximum column sum is at least $2a_{\max}$ (a special case of a safe multiple-team slot), then a 1-strong Nash equilibrium exists.*

Since our proof is constructive, we obtain the following.

Corollary 3. *If the maximum availability a_{\max} is at most three, then a 1-strong Nash equilibrium for TPG can be found in polynomial time.*

The situation where $t \geq 2$ is quite different already with only two teams. By a proof similar to the case of $t = 1$ and $a_{\max} = 2$, we can show that a 2-strong Nash equilibrium always exists for $t = 2$ and $a_{\max} = 1$:

Proposition 1. *If the maximum availability a_{\max} is one, then a 2-strong Nash equilibrium for TPG always exists and can be found in polynomial time.*

Complementing Theorem 4, we demonstrate that 2-strong Nash equilibria do not always exist, even when $a_{\max} = 2$; to this end, consider the following example.

Example 1. *We provide in the following an example with maximum true availability two, and show that it does not admit a 1-strong Nash equilibrium. Consider the following input for TPG.*

$$A := \begin{matrix} & \begin{matrix} c_1 & c_2 \end{matrix} \\ \begin{pmatrix} 2 & 0 \\ 2 & 2 \\ 0 & 2 \end{pmatrix} & \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} \end{matrix}$$

Informally, The main crux of this example is that t_1 (or, symmetrically, t_3) can cooperate with t_2 ; in such cooperation, t_2 can choose whether to be ‘in favor’ of t_1 or t_3 , by declaring either $b_1^2 = 2$ and $b_2^2 = 0$ (favoring t_1), or $b_1^2 = 0$ and $b_2^2 = 2$ (favoring t_3). Moreover, t_1 or t_3 can ‘reward’ t_2 by not declaring its true availability, which is 2, but only 1. In such a cooperation, both t_2 and t_1 (or t_2 and t_3) strictly improve their pay-offs.

More formally, let us consider all possibilities for t_2 . By symmetry, we can assume without loss of generality, that t_2 declares either $(0, 1)$, $(0, 2)$, $(1, 1)$, $(1, 2)$, or $(2, 2)$ (declaring $(0, 0)$ is not possible in a Nash equilibrium—see the analog $(1, 1)$ case)).

To this end, we denote the declared availability matrix B simply by the symbolic vector $[b_1^1 b_1^2, b_2^1 b_2^2, b_3^1 b_3^2]$ (e.g. the declared availability matrix $B = A$ is denoted by $[20, 22, 02]$). We write $B \rightarrow^X B'$ if the coalition X receives a better pay-off by changing the declared availabilities from B to B' .

Now, we consider each of the possible strategies that team t_2 may take:

t_2 declares $(0, 1)$:

$$[x0, 01, 0y] \rightarrow^{\{t_3\}} [x0, 01, 02] \quad (0 \leq x \leq 2, 0 \leq y \leq 1); \quad [x0, 01, 02] \rightarrow^{\{t_2\}} [x0, 02, 02] \quad (0 \leq x \leq 2).$$

t_2 declares $(0, 2)$:

$$[x0, 02, 0y] \rightarrow^{\{t_3\}} [x0, 02, 02] \quad (0 \leq x \leq 2, 0 \leq y \leq 1); \\ [x0, 02, 02] \rightarrow^{\{t_1, t_2\}} [10, 20, 02] \quad (0 \leq x \leq 2).$$

t_2 declares $(1, 1)$:

$$[x0, 11, 0y] \rightarrow^{\{t_2\}} [x0, 20, 0y] \quad (x > y); \\ [x0, 11, 0y] \rightarrow^{\{t_2\}} [x0, 20, 0y] \quad (x = y); \\ [x0, 11, 0y] \rightarrow^{\{t_2\}} [x0, 02, 0y] \quad (x < y).$$

t_2 declares $(1, 2)$:

$$[x0, 12, 0y] \rightarrow^{\{t_3\}} [x0, 12, 02] \quad (0 \leq x \leq 2, 0 \leq y \leq 1); \\ [x0, 12, 02] \rightarrow^{\{t_2\}} [x0, 20, 02] \quad (0 \leq x \leq 1); \\ [20, 12, 02] \rightarrow^{\{t_1, t_2\}} [10, 20, 02].$$

t_2 declares $(2, 2)$:

$$[x0, 22, 0y] \rightarrow^{\{t_1\}} [20, 22, 0y] \quad (0 \leq x \leq 1, 0 \leq y \leq 1); \\ [20, 22, 02] \rightarrow^{\{t_1, t_2\}} [10, 20, 02]; \\ [20, 22, 01] \rightarrow^{\{t_2\}} [20, 02, 01]; \\ [10, 22, 02] \rightarrow^{\{t_2\}} [10, 20, 02].$$

The above case analysis cover (by symmetry) all possible cases, and thus, shows that there are no 2-strong Nash equilibria for the instance A .

Naturally, there are profiles for which t -strong Nash equilibria do exist: consider, for example, A being the all-one matrix.

Next, we show that if the coalition size is unbounded, then finding a Nash equilibrium becomes coNP-hard.

Theorem 5. *Deciding whether a Nash equilibrium exists for a given input is coNP-hard.*

Proof. (Sketch). We reduce from the complement of the following NP-complete problem [6]: RESTRICTED X3C, which given sets $\mathcal{F} = \{S_1, \dots, S_{3n}\}$, each containing exactly 3 elements from $E = \{e_1, \dots, e_{3n}\}$ such that (1) $n \geq 2$ and (2) each element e_i appears in exactly 3 sets, asks whether there is a size- n exact cover $\mathcal{F}' \subseteq \mathcal{F}$, i.e., $|\mathcal{F}'| = n$ and $\bigcup_{S_i \in \mathcal{F}'} S_i = U$.

Given an instance (\mathcal{F}, E) of the complement of RESTRICTED X3C we construct a game. For each element e_i ($i \in [3n]$) we construct a time slot e_i . We construct one additional time slot, denoted by α . For each set S_j ($j \in [3n]$) we construct a team s_j . For a team s_j , we set its availability for time slot e_i , namely a_i^j , to be n if $e_i \in S_j$, and otherwise 0. We set the availability of all teams to be 1 in time slot α . We consider $2n$ -strong Nash-equilibria; thus, we consider coalitions containing up to $2n$ teams. This finishes the description of the polynomial-time reduction. The correctness proof can be found in Appendix A.6. \square

5 Conclusion

We introduced a game considering power of teams (referred to as TPG) that is naturally motivated by online scheduling polls where teams declare and update their availabilities in a dynamic process to increase their relative power. Our work leads to several directions for future research.

Tie-breaking rules: In this paper the teams are pessimistic, i.e., in case of several co-winners, the pay-off is defined as the *minimum* of the relative number, over the co-winners. This corresponds to situations where ties are broken adversarially. We chose this tie-breaking as a standard and natural one, and as one which models teams which are pessimistic in nature, where having too low power in the team might have very bad consequences. Naturally, one might study other tie-breaking rules such as breaking ties uniformly at random or breaking ties lexicographically; we mention that most of our results seem to transfer to lexicographic tie-breaking.

More refined availability constraints: In the online scheduling polls considered in this paper, the availability constraints expressed by the participants are dichotomous: each participant can only declare either “available” or “not available” at each time slot. Sometimes, the availability constraints of people participating in scheduling polls are more fine-grained; for example, a participant might not be sure whether she is available or not for some of the suggested time slots, but can only provide a “maybe available” answer for these time slots. Correspondingly, it is interesting to study TPG when we allow participants to express more refined availability constraints, maybe even allowing them to fully rank the time slots according to their constraints.

Nash modification problem: Taking the point of view of the poll convener (who desires to reach a Nash equilibrium), we suggest to study the following problem: given an input for TPG, what is the minimum number of time slots that shall be removed so that the modified input will have a Nash equilibrium?

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A Missing Proofs

Below we provide proofs missing from the main text.

A.1 Proof of Lemma 1

Proof. Suppose that, for a certain coalition, $D = (d_i^j)$ is an improvement step for $B = (b_i^j)$. Let c_{j_1} and c_{j_2} be two time slots such that at least one team t_i in the coalition has $d_i^{j_1} \neq 0$ and at least one team $t_{i'}$ (possibly different) in the coalition has $d_{i'}^{j_2} \neq 0$.

We distinguish between two cases. If $c_{j_1} \notin \text{winners}(D)$ (or $c_{j_2} \notin \text{winners}(D)$), then the pay-offs of all teams remain the same even if each team t_i in the coalition with non-zero declared availability $d_i^{j_1}$ (or $d_i^{j_2}$) for c_{j_1} (or c_{j_2}) will declare zero instead. Otherwise, both c_{j_1} and c_{j_2} are co-winners in D . Fix any team t_i in the coalition. Since strategy D is an improvement step compared to B , it follows that the relative powers of t_i in c_{j_1} and in c_{j_2} for strategy D are both strictly larger than the pay-off of t_i for strategy B . Thus, we can change the improvement step D to make all teams in the coalition declare zero availability, say, in c_{j_2} , and we would still obtain an improvement step. By repeating the above reasoning, we fix a desired improvement step where all teams in the coalition have non-zero availability only in the same single time slot. \square

A.2 Proof of Theorem 1

Proof. Following Lemma 1, we begin by guessing the unique time slot c_k for which the teams in the coalition will declare non-zero availability.

Further, we guess the value of $w := \sum_{i=1}^t x_i$. Crucially, this value is upper-bounded by $\sum_{i=1}^t a_i^j \leq sn$; recall that s is the sum of all true availabilities and n is the number of teams. Then, for each team t_i in the coalition, we compute the minimum value needed for t_i to get strictly better pay-off than pay-off(B, t_i), i.e., $\min\{x \leq a_i^j \mid x/(p+w) > \text{pay-off}(B, t_i)\}$, where p is the sum of the declared availabilities of the teams not in the coalition; we pick these values as the x_i 's. If as a result we obtain $\sum_{i=1}^t x_i \leq w$, then we return (x_1, \dots, x_t) . Otherwise, we proceed to the next guess. If all guesses fail, then we reject. The running time is $O(s^2)$. \square

A.3 Proof of Theorem 2

Proof. Following Lemma 1, we begin by guessing the unique time slot c_k for which the teams in the coalition will declare non-zero availability.

After guessing the unique time slot c_k for which the teams in the coalition would declare non-zero availabilities, we can run the integer linear program (ILP, in short) specified for the t -THRESHOLD COVERING problem. By the famous result of Lenstra [10] (later improved by Kannan [7] and by Frank and Tardos [4]), we know that an ILP with ρ variables and z input bits can be solved in $O(\rho^{2.5\rho+o(\rho)} \cdot z)$ time. Since the ILP specified for t -THRESHOLD COVERING has t variables and can be represented in $O(|A'| \cdot |B'|)$ bits, where $|A'|$ and $|B'|$ denote the numbers of binary bits needed to encode the true and the declared availabilities at time slot c_j , respectively, we obtain an algorithm with running time $O(t^{2.5t+o(t)} \cdot L^2)$.

As for the second running time, after guessing the unique time slot c_k , we additionally guess the sum w' of declared availabilities of the teams in the coalition whose true availabilities are upper-bounded by L^c ; we call these teams *small teams*. Then, we modify the ILP specified for t -THRESHOLD COVERING to search for the declared availabilities of the remaining teams. Using the declared availabilities for the remaining teams, we can calculate the declared availabilities for the small teams as described in the first algorithm. Again, by Lenstra's result, we can solve the ILP in $g(\ell_c) \cdot |A'_j| \cdot |B'_j|$ time, where $|A'_j|$ and $|B'_j|$ denote the numbers of binary bits needed to encode the true and declared availabilities at time slot c_k , respectively. Searching for the declared availabilities for the small teams can be done in $O(t^2 \cdot L^{c^2})$ time. In total, we obtain an algorithm with running time $f'(\ell_c) \cdot t^2 \cdot L^{c^2+2}$. \square

A.4 Proof of Theorem 3 (correctness, unrestricted coalition size)

Proof. Let us state the total availabilities of the various time slots. Each element slot has total availability $2m - 1$, time slot β has total availability $2m - 1 - k$, and time slot α has total availability $2m$. Indeed, time slot α is a unique winner. Each set team receives a pay-off of $1/(2m)$ and each dummy team receives pay-off zero.

Next, we prove the correctness of the reduction. For the “if” case, assume that there is a size- k set cover $\mathcal{F}' \subseteq \mathcal{F}$. We show that the coalition that corresponds to the set cover \mathcal{F}' (recall that $t = k$) can improve by making time slot β the unique winner. To this end, each set team t_i with $S_i \in \mathcal{F}'$ changes its declared availability for time slot α and for all element slots to 0 and changes its declared availability for time slot β to 1. As a result, time slot α has total availability $2m - k$, each element slot has total availability at most $2m - 2$ (since the coalition corresponds to a set cover), and time slot β has total availability $2m - 1$. Thus, time slot β is the unique winner and each set team receives a pay-off of $1/(2m - 1)$; this is a strict improvement for all teams in the coalition.

For the “only if” case, assume that there is a coalition of t teams that can improve their pay-offs by changing their declared availabilities. We observe that time slot α cannot be a (co-)winner since if it was, then either no team would improve or the pay-off of at least one coalition member would be zero. Now, we show that the subfamily \mathcal{F}' corresponds to the coalition is a set cover of size $t = k$. For this, we distinguish between two cases, depending on whether time slot β is a unique winner or not.

First, suppose that the coalition makes time slot β become the new unique winner. Then, after changing the coalition's declared availabilities, time slot β has total availability of at most $2m - 1$. It follows that each element time slot has total availability of at most $2m - 2$. This implies that \mathcal{F}' , which corresponds to the set teams of the coalition, forms a set cover, since otherwise there would be one element slot for which the coalition cannot decrease its total availability to be at most $2m - 2$.

Next, suppose that β is not a unique winner which means that there is some subset $E' \subseteq E$ of element slots within the co-winner set (including the case that some element slot e_j is a unique winner). Note that if the coalition contains only one team t^* , then time slot α would still be a (co-)winner which is not possible by the arguments above. Thus, let us assume that the coalition has at least two teams that make all element slots from E' become (co-)winners. All coalition members must still declare availability 1 for all element slots from E' , since otherwise

the pay-off of some coalition members would decrease to zero. Furthermore, for each element slot $e_j \notin E'$ that is not a co-winner, there is at least one coalition member that changes its declared availability from 1 to 0, since otherwise e_j would be a co-winner. Hence, subfamily \mathcal{F}' is a set cover (recall that it corresponds to the teams in the coalition): each element corresponding to an element slot from E' is covered by all sets from \mathcal{F}' and each element corresponding to an element slot from $E \setminus E'$ is covered by at least one set from \mathcal{F}' .

By the analysis of the two cases above, the existence of a coalition that can improve always implies the existence of a set cover. This completes the proof for the case that the size of the coalition is at most t .

Unrestricted coalition size. Our problem is in NP as we can check in polynomial-time whether a specific strategy is an improvement step for a given coalition. If there is no restriction on the coalition size, then NP-hardness does not immediately transfer from the above construction, but it can be obtained by a slight modification as follows.

The above proof would almost work through: a size- k set cover in the original instance still implies a coalition of size t that can improve by making β be the new unique winner. Furthermore, by the same arguments as above, it still follows that every coalition that can improve still consists of set teams which correspond to a set cover. We cannot, however, exclude the existence of a coalition which corresponds to a set cover of size larger than t . To fix this, we assume that each element in our SET COVER instance occurs in at most three sets. (This variant remains NP-hard since VERTEX COVER remains NP-hard even if each vertex has degree at most three [5]; we did not assume this in the W[2]-hardness proof since this variant is not W[2]-hard.) We further assume, without loss of generality, that $k \geq 3$.

The reasoning for the restricted coalition size case still holds, so it only remains to show that no coalition of size larger than t can improve. Assume, towards a contradiction, that there is a coalition of size at least $t + 1$ that can improve. First, notice again that time slot α cannot be a (co-)winner. Second, assume that there is some subset $E' \subseteq E$ of element slots within the set of co-winners (including the case that some element slot e_j is a single winner). All coalition members must still declare availability 1 for all element slots from E' , since otherwise the pay-off of some team member would decrease to zero. However, this is not possible since each element occurs in at most three sets and $t + 1 = k + 1 > 3$. Third, assume that the coalition makes time slot β be the new unique winner. Then, after changing the coalition's declared availabilities, time slot β has total availability of at least $2m$ since otherwise some coalition members would decrease their pay-off. However, this also means that no coalition member improves its pay-off, which is at most the same as the original pay-off $1/(2m)$. This completes the proof for the case of unrestricted coalition size. \square

A.5 Proof of Theorem 4 (instances with $a_{\max} \neq 2$)

Proof. Instances with $a_{\max} = 1$. Considering the above two cases, the proof is relatively simple for inputs with $a_{\max} = 1$: either there is a column j in A where only one team has availability 1, implying that time slot c_j is a safe single-team slot, or every time slot c_j is a safe multiple-team slot since $\forall j': a_i^{j'} \leq a_{\max} = 1$ and $\forall i: \sum_{i' \neq i} a_{i'}^j \geq 1$.

Instances with $a_{\max} = 3$. We consider the case where the maximum true availability a_{\max} is three. Again, let x be the maximum sum of availabilities over all time slots, and notice that

$x \geq a_{\max}$. If x is at least six, then every time slot c_j with availability sum x is a safe multiple-team slot, because $\forall j': a_i^{j'} \leq a_{\max} = 3$ and $\forall i: \sum_{i' \neq i} a_{i'}^j \geq x - a_{\max} \geq 3$. If x is three, then there is a time slot where only one team is available with availability $a_{\max} = 3$, i.e., there is a safe single-team slot.

Now, assume that x is four or five. We distinguish between four cases and implicitly assume that the k th case does not hold in the $(k+1)$ th case. First, there is a safe single-team slot (which implies a 1-strong Nash equilibrium by our observation). Note that this case includes time slots where only one team is available with availability $a_{\max} = 3$ as well as the situation that there is a time slot c_j where only one team t_i is available with availability 2 and t_i is the only team with availability 2 for every time slot.

Second, there is a time slot c_j where one team t_i has availability $a_{\max} = 3$ and another team $t_{i'}$ has availability one while every remaining team has availability zero. Analogously to the case with $a_{\max} = 2$ and $x = 3$, first, for each team t_ℓ , we set $b_\ell^j := a_\ell^j$ and $b_\ell^{j'} := 0$ ($j' \neq j$) to make time slot c_j a single winner. Team t_i receives pay-off $3/4$ and team $t_{i'}$ receives pay-off $1/4$. Now, for each team t_ℓ and for each time slot $c_{j'} \neq c_j$, we modify the declared availabilities $b_\ell^{j'}$ as follows. If there is some $\ell \notin \{i, i'\}$ with $a_\ell^{j'} = 3$, then set $b_\ell^{j'} := a_\ell^{j'}$. Otherwise, if $a_{i'}^{j'} > 0$, then set $b_{i'}^{j'} := a_{i'}^{j'}$ and if $a_{i'}^{j'} = 0$, then set $b_\ell^{j'} := a_\ell^{j'}$ for the first position $\ell \notin \{i, i'\}$ with $a_\ell^{j'} > 0$. This does not prevent c_j from being the unique winner but makes it impossible for the teams to improve their pay-offs: team t_i cannot improve, because it would receive at most the same pay-off $3/4$ for every time slot which it could make a new single winner. Note that, since we are not in the first case, it follows that there is no time slot where only team t_i is available with availability $a_{\max} = 3$. Furthermore, if there is a time slot c_{j^*} where team t_i is available with availability 2 and no other team is available, then there is some time slot $c_{j''}$ with $a_{i''}^{j''} = 3$ ($i'' \neq i$) and, hence, $b_{i''}^{j''} = 3$. (This slot $c_{j''}$ must exist since otherwise we would be in the first case and have a safe single-team slot c_{j^*} .) Thus, team t_i cannot make c_{j^*} become a new single winner. Team $t_{i'}$ also cannot improve, because it cannot create a new single winner at all. Hence, we have a 1-strong Nash equilibrium.

Third, there is a time slot c_j where one team t_i has availability $a_{\max} = 3$ and another team $t_{i'}$ has availability two, implying that the remaining teams are not available at slot c_j . Similarly to the previous case, for each team t_ℓ , we first set $b_\ell^j := a_\ell^j$ and $b_\ell^{j'} := 0$ ($j' \neq j$) to make time slot c_j a single winner. Team t_i receives pay-off $3/5$ and team $t_{i'}$ receives pay-off $2/5$. Now, for each team $t_\ell \neq t_i$ and for each time slot $c_{j'} \neq c_j$ with $a_i^{j'} = a_{\max} = 3$, we modify its declared availability $b_\ell^{j'}$ such that the total declared availability sum of the teams other than t_i is always two at time slot $c_{j'}$.

- If $a_{i'}^{j'} \geq 2$, then set $b_{i'}^{j'} := a_{i'}^{j'}$.
- If $a_{i'}^{j'} = 1$, then set $b_{i'}^{j'} := a_{i'}^{j'}$ and let $t_\ell \notin \{t_i, t_{i'}\}$ be the first team with $0 < a_\ell^{j'} < a_{\max}$ and set $b_\ell^{j'} := a_\ell^{j'}$. Note that such a team t_ℓ must exist since we are not in the second case.
- If $a_{i'}^{j'} = 0$, then set $b_\ell^{j'} := a_\ell^{j'}$ for the first team $t_\ell \notin \{t_i, t_{i'}\}$ with $a_\ell^{j'} > 1$ or for the first two teams $t_\ell \notin \{t_i, t_{i'}\}$ with $a_\ell^{j'} = 1$. Again, such teams t_ℓ exist(s) since we are not in the second case. This does not prevent c_j from being the unique winner but makes it impossible for the teams to improve their pay-offs: team t_i cannot improve, because it would receive at

most the same pay-off $3/5$ for every time slot which it could make a new (co-)winner. Each of the remaining teams (including $t_{i'}$) also cannot improve, because it cannot create a new single winner at all (note that $x = 5$).

Hence, we have a 1-strong Nash equilibrium.

Fourth, there is a time slot c_j where one team t_i has availability $a_{\max} = 3$ and there are another two teams, $t_{i'}$ and $t_{i''}$, both with availability one. Similarly to the previous case, for each team t_ℓ , we first set $b_\ell^j := a_\ell^j$ and $b_\ell^{j'} := 0$ ($j' \neq j$) to make time slot c_j a single winner. Team t_i receives pay-off $3/5$ and teams $t_{i'}$ and $t_{i''}$ both receive pay-off $1/5$. Then, for each time slot $c_{j'} \neq c_j$ such that $a_i^{j'} = a_{\max} = 3$, we modify the declared availabilities $b_\ell^{j'}$ of some teams ℓ :

- If $a_{i'}^{j'} > 0$ and $a_{i''}^{j'} > 0$, then set $b_{i'}^{j'} := 1$ and set $b_{i''}^{j'} := 1$.
- If $a_{i'}^{j'} = 0$ and $a_{i''}^{j'} > 1$, then set $b_{i''}^{j'} := a_{i''}^{j'}$.
- If $a_{i'}^{j'} = 0$ and $a_{i''}^{j'} = 1$, then set $b_{i''}^{j'} := a_{i''}^{j'}$, let t_ℓ be the first team with $0 < a_\ell^{j'} < a_{\max}$, and set $b_\ell^{j'} := a_\ell^{j'}$. Note that such a team t_ℓ must exist since we are not in the second case.
- If $a_{i'}^{j'} = 0$ and $a_{i''}^{j'} = 0$, then let $t_\ell, t_{\ell'}$ be the first two teams with $a_\ell^{j'} = a_{\ell'}^{j'} = 1$ and set $b_\ell^{j'} := a_\ell^{j'}$. Note that such a team t_ℓ must exist since we are not in the first three cases. The cases with $a_{i''}^{j'} = 0$ follow analogously. The new declared availabilities still make c_j a unique winner but make it impossible for the teams to improve their pay-offs: team t_i cannot improve, because it would receive at most the same pay-off $3/5$ for every time slot which it could make a new (co-)winner. Neither of the remaining teams (including $t_{i'}$ and team $t_{i''}$) can improve, because they cannot create a new single winner at all.

Hence, we have a 1-strong Nash equilibrium. \square

A.6 Proof of Theorem 5 (correctness)

We will use the following observation.

Observation 1. *Let B be a Nash-equilibrium for some input A . Then, for each time slot $c_j \in \text{winners}(B)$, all teams declare their true availabilities.*

Proof. Assume, towards a contradiction, that the declared availability of some team t_i in time slot c_j is strictly less than its true availability. Then, this team can improve its payoff by declaring its true availability. \square

Now we are ready to proof the correctness of the construction for Theorem 5.

Proof. (of Theorem 5). Now, we show that a $2n$ -Nash-equilibrium exists if and only if an exact cover does not exist.

We show the “only if” part by showing that existence of an exact cover implies non-existence of a Nash-equilibrium. To this end, we assume that there is an exact cover and prove that a Nash-equilibrium does not exist.

Let \mathcal{F}' be an exact cover. Consider the remaining sets $\bar{\mathcal{F}}' = \{S_1, \dots, S_{3n}\} \setminus \mathcal{F}'$. It holds that $|\mathcal{F}'| = n$ and $|\bar{\mathcal{F}}'| = 2n$. Notice further that, while \mathcal{F}' covers each element exactly once, $\bar{\mathcal{F}}'$ covers each element exactly twice.

We further assume, towards a contradiction, that a Nash-equilibrium does exist; let us denote it by B . In what follows, we consider several cases concerning the structure of profile B , which is a Nash-equilibrium, and show an improvement step for each of these cases, contradicting the assumption that B is indeed a Nash-equilibrium.

Consider the set of winning time slots of B . First we consider the case where $\alpha \in B$.

Case 1. In this case, $\alpha \in \text{winners}(B)$. By Observation 1 we have that the declared availabilities of all teams in α is 1; thus, the sum of declared availabilities for all winning slots is $3n$. Below, we consider three different sub-cases.

Case 1a. In this case, α is the only winning time slot, i.e., $\alpha = \text{winners}(B)$. Consider any time slot, say e_1 , and consider the three teams who are available in it, say s_i, s_j , and s_k . We claim that these teams can improve their pay-off, as follows: if s_i, s_j , and s_k , will declare availability of 0 for all time slots except for e_1 for which they will declare availability of n , then their pay-off would increase from $\frac{1}{3n}$ to $\frac{1}{3}$.

Case 1b. In this case, there are at most three other time-slots, besides α , with sum of declared availabilities being $3n$, and there exists a team s_i such that $a_j^i = n$ for each j such that $E_j \in \text{winners}(B)$. We claim that s_i can improve its pay-off, as follows: if s_i will declare availability of 0 for time slot α , then its pay-off would increase from $\frac{1}{3n}$ to $\frac{1}{3}$.

Case 1c. In this case, α is indeed a winning slot, but neither Case 1a nor Case 1b hold. It follows that, for every team s_j there must exist $E_i \in \text{winners}(B)$ such that $a_i^j = 0$. Thus, $\text{pay-off}(s_j, B) = 0$ for each team s_j . Recall that $\bar{\mathcal{F}}'$ contains those sets which are not part of the exact cover for the instance of RESTRICTED X3C. We claim that the set of teams corresponding to the sets in $\bar{\mathcal{F}}'$ can improve their pay-off as follows: if each such team will declare availability of 0 for all time slots E_i and availability of 1 for α , then their pay-offs would increase to $\frac{1}{3n}$. This follows since $\bar{\mathcal{F}}'$ covers each element twice, thus, by deviating as described above, the sum of declared availabilities of all time slots E_i would decrease to being at most n , while the sum of availabilities of α would be at least $2n$, making α a unique winner.

Next we consider the case where α is not winning.

Case 2. In this case, $\alpha \notin \text{winners}(B)$. By Observation 1 we have that the declared availabilities in the winning time slots of all teams is n , thus, the sum of declared availabilities for all winning time slots is $3n$. Below, we consider two sub-cases.

Case 2a. In this case, there exists a team s_i such that $a_j^i = n$ for each j such that $E_j \in \text{winners}(B)$. Since in the instance of RESTRICTED X3C we have that no two sets are the same, it follows that there are four other teams, besides s_i , denoted by s_j, s_f, s_p , and s_q , and two different time slots, denoted by e_x and by e_y , such that it the following conditions hold:

1. $a_x^i = a_x^j = a_x^f = n$;
2. $a_y^i = a_y^p = a_y^q = n$;
3. $j \notin \{p, q\}$;
4. $E_y \in \text{winners}(B)$.

Notice that $\text{pay-off}(s_j, B) = 0$, $\text{pay-off}(s_i, B) = \frac{1}{3}$, and $\text{pay-off}(s_f, B) \leq \frac{1}{3}$. We claim that s_i, s_j, s_f can improve their pay-offs as follows: if s_i, s_f will declare availabilities of n for time slot E_x and 0 for all other slots while s_j will declare availability of $n - 1$ for time slot E_x and 0 for all other slots, then E_x would become the unique winning time slot; as a result, the pay-offs of s_i, s_f would increase to $\frac{n}{3n-1}$ while the pay-off of s_j would increase to $\frac{n-1}{3n-1}$.

Case 2b. In this case, no team exists i.e. available in all winning time slots. In this case, the improvement step described in Case 1c is an improvement step, and the reasoning is the same, thus omitted.

For the “if” part, we have to that if no exact cover exists, then there is a Nash equilibrium. To this end, we will describe a profile B and would argue that, if indeed an exact cover does not exist, then B is a Nash-equilibrium.

The profile B is as follows. Each team s_j declares availability of n for all time slots e_i and availability of 0 for α , making each time slot a co-winner. Thus, the pay-off of each team is 0. Let us assume, towards a contradiction, that B is not a $2n$ -Nash-equilibrium, i.e., that an improvement step with respect to B , involving at most $2n$ teams, exists, denoted by B' .

First of all, we observe that no time slot e_i is a winning time slot (i.e., $e_i \notin \text{winners}(B')$ for all e_i) because otherwise after the improvement step B' , at least $3n - 3$ teams will have pay-off 0 (this follows since, in the instance of RESTRICTED X3C, each element is included in exactly three sets). However, no coalition of at most 3 teams can decrease the sum of availabilities of all other time slots as to make e_i the unique winner; thus, we conclude that, after the improvement step, α shall be the unique winning slot.

The maximum sum of availabilities which time slot α might get after an improvement step involving at most $2n$ teams is $2n$. Therefore, to have that such a deviation is a profitable deviation, i.e., an improvement step, the sum of declared availabilities of all time slots e_i has to decrease by at least $n + 1$. This could happen only if, for each time slot e_i , at least two teams with non-zero availabilities would decrease their declared availabilities. Thus, an exact cover must exist. \square