Multi-Player Diffusion Games on Graph Classes

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Abstract

We study competitive diffusion games on graphs introduced by Alon et al. [1] to model the spread of influence in social networks. Extending results of Roshanbin [8] for two players, we investigate the existence of pure Nash equilibria for at least three players on different classes of graphs including paths, cycles, grid graphs and hypercubes; as a main contribution, we answer an open question proving that there is no Nash equilibrium for three players on $m \times n$ grids with $\min\{m, n\} \geq 5$. Further, extending results of Etesami and Basar [3] for two players, we prove the existence of pure Nash equilibria for four players on every d-dimensional hypercube.

1 Introduction

Social networks, and the diffusion of information within them, yields an interesting and well-researched field of study. Among other models, competitive diffusion games have been introduced by Alon et al. [1] as a game-theoretic approach towards modelling the process of diffusion (or propagation) of influence (or information in general) in social networks. Such models have applications in "viral marketing" where several companies (or brands) compete in influencing as many customers (of products) or users (of technologies) as possible by initially selecting only a "small" subset of target users that will "infect" a large number of other users. Herein, the network is modelled as an undirected graph where the vertices correspond to the users, with edges modelling influence relations between them. The companies, being the players of the corresponding

diffusion game, choose an initial subset of target vertices which then influence other neighboring vertices via a certain propagation process. More concretely, a vertex adopts a company's product at some specific time during the process if he is influenced by (that is, connected by an edge to) another vertex that already adopted this product. After adopting a product of one company, a vertex will never adopt any other product in the future. However, if a vertex gets influenced by several companies at the same time, then he will not adopt any of them and he is removed from the game. See Section 1.3 for the formal definitions of the game.

In their initial work, Alon et al. [1] studied how the existence of pure Nash equilibria is influenced by the diameter of the underlying graph. Following this line of research, Roshanbin [8] investigated the existence of Nash equilibria for competitive diffusion games with two players on several classes of graphs such as paths, cycles and grid graphs. Notably, she proved that on sufficiently large grids, there always exists a Nash equilibrium for two players, further conjecturing that there is no Nash equilibrium for three players on grids. We extend the results of Roshanbin [8] for two players to three or more players on paths, cycles and grid graphs, proving the conjectured non-existence of a pure Nash equilibrium for three players on grids as a main result. Etesami and Basar [3] also followed this line of research, by inverstigating the existence of Nash equilibria for competitive diffusion games with two players on d-dimensional hypercubes. We extend their results by showing that there always exists a Nash equilibrium for four players on any d-dimensional hypercube.

An overview of our results is given in Section 1.2. After introducing the preliminaries in Section 1.3, we discuss our results for paths and cycles in Section 2, followed by the proof of our main contribution regarding grids in Section 3. We discuss hypercubes in Section 4 and finish with some statements concerning general graphs in Section 5.

1.1 Related Work

The study of influence maximization in social networks was initiated by Kempe et al. [6]. Several game-theoretic models have been suggested, including our model of reference, introduced by Alon et al. [1]. Some interesting generalizations of this model are the model by Tzoumas et al. [12], who considered a more complex underlying diffusion process, and the model studied by Etesami and Basar [3], allowing each player to choose multiple vertices. Dürr and Thang [2] and Mavronicolas et al. [7] studied so-called Voronoi games, which are closely related to our model (but not similar; there, players can share vertices). Recently, Ito et al. [4] considered the competitive diffusion game on weighted graphs, including negative weights. Concerning our model, Alon et al. [1] claimed the existence of pure Nash equilibria for any number of players on graphs of diameter at most two, however, Takehara et al. [11] gave a counterexample consisting of a graph with nine vertices and diameter two with no Nash equilibrium for two players.

Our main point of reference is the work of Roshanbin [8], who studied the

existence (and non-existence) of pure Nash equilibria mainly for two players on special graph classes (paths, cycles, trees, unicycles, and grids); indeed, our work can be seen as an extension of that work to more than two players. Small [9] already showed that there is a Nash equilibrium for any number of players on any star or clique. Small and Mason [10] proved that there is always a pure Nash equilibrium for two players on a tree, but not always for more than two players. Janssen and Vautour [5] considered safe strategies on trees and spider graphs, where a safe strategy is a strategy which maximizes the minimum payoff of a certain player, when the minimum is taken over the possible unknown actions of the other players.

1.2 Our Results

We begin by characterizing the existence of Nash equilibria for paths and cycles, showing that, except for three players on paths of length at least six, a Nash equilibrium exists for any number of players playing on any such graph (Theorem 1 and 2). We then prove Conjecture 1 of Roshanbin [8], showing that there is no Nash equilibrium for three players on $G_{m \times n}$, as long as both m and n are at least 5 (Theorem 3). Then, we prove the existence of Nash equilibria for four players on any d-dimensional hypercube (Theorem 4). Finally, we investigate the minimum number of vertices such that there is an arbitrary graph with no Nash equilibrium for k players. We prove an upper bound showing that there always exists a tree on $\lfloor \frac{3}{2}k \rfloor + 2$ vertices with no Nash equilibrium for k players (Theorem 5).

1.3 Preliminaries

Notation.   For $i, j \in \mathbb{N}$ with $i < j$, we define $[i, j] := \{i, \dots, j\}$ and $[i] := \{1, \dots, i\}$. We consider simple, finite, undirected graphs $G = (V, E)$ with vertex set V and edge set $E \subseteq \{\{u, v\} \mid u, v \in V\}$. For two vertices $u, v \in V$, we define the *distance* $\mathrm{dist}_G(u, v)$ between u and v to be the length of a shortest path from u to v in G.

A *path* $P_n = (V, E)$ on n vertices is the graph with $V = [n]$ and $E = \{\{i, i+1\} \mid i \in [n-1]\}$. A *cycle* $C_n = (V, E)$ on n vertices is the graph with $V = [n]$ and $E = \{\{i, i+1\} \mid i \in [n-1]\} \cup \{\{n, 1\}\}$. For $m, n \in \mathbb{N}$, the $m \times n$ *grid* $G_{m \times n} = (V, E)$ is a graph with vertices $V = [m] \times [n]$ and edges $E = \{\{(x, y), (x', y')\} \mid |x - x'| + |y - y'| = 1\}$. We use the term *position* for a vertex $v \in V$. Note that the distance of two positions $v = (x, y)$, $v' = (x', y') \in V$ is $\mathrm{dist}_{G_{m \times n}}(v, v') = \|v - v'\|_1 := |x - x'| + |y - y'|$. For $d \in \mathbb{N}$, $d \geq 1$, the d-*dimensional hypercube* $H_d = (V, E)$ is defined on the vertex set $V = \{0, 1\}^d$, that is, a vertex $x = x_1 \dots x_d \in V$ is a binary string of length d. The set of edges is defined as $E = \{\{x, y\} \mid \Delta(x, y) = 1\}$, where $\Delta(x, y) := |\{i \in [d] \mid x_i \neq y_i\}|$ is the *Hamming distance* of x and y, that is, the number of positions in which x and y have different bits. Note that $\mathrm{dist}_{H_d}(x, y) = \Delta(x, y)$.
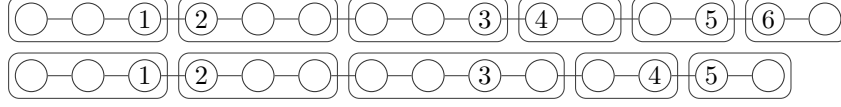
Figure 1: Illustrations for Theorem 1, showing a Nash equilibrium for 6 players on P_{15} (top) and a Nash equilibrium for 5 players on P_{14} (bottom). The boxes show the colored regions of each player.

Diffusion Game on Graphs. A *competitive diffusion game* $\Gamma = (G, k)$ is defined by an undirected graph $G = (V, E)$ and a number k of players (we name the players Player 1, ... , Player k), each having its distinct color in $[k]$. The *strategy space* of each player is V, such that each Player i selects a single vertex $v_i \in V$ at time 0, which is then colored by her color i. If two players choose the same vertex v, then this vertex is removed from the graph. For Player i, we use the terms strategy and position interchangeably, to mean its chosen vertex. A *strategy profile* is a tuple $(v_1, \ldots, v_k) \in V^k$ containing the initially chosen vertex for each player. The *payoff* $U_i(v_1, \ldots, v_k)$ of Player i is the number of vertices with color i after the following propagation process. At time $t + 1$, any so far uncolored vertex that has only uncolored neighbors and neighbors colored in i (and no neighbors with other colors $j \in [k] \setminus \{i\}$) is colored in i. Any uncolored vertex with at least two different colors among its neighbors is removed from the graph. The process terminates when the coloring of the vertices does not change between consecutive steps. A strategy profile $(v_1, \ldots, v_k)$ is a (pure) *Nash equilibrium* if, for any player $i \in [k]$ and any vertex $v' \in V$, it holds that $U_i(v_1, \ldots, v_{i-1}, v', v_{i+1}, \ldots, v_k) \leq U_i(v_1, \ldots, v_k)$.

2 Paths and Cycles

In this section, we fully characterize the existence of Nash equilibria on paths and cycles, for any number k of players.

Theorem 1. *For any $k \in \mathbb{N}$ and any $n \in \mathbb{N}$, there is a Nash equilibrium for k players on P_n, except for $k = 3$ and $n \geq 6$.*

The general idea of the proof is to pair the players and distribute these pairs evenly. In the rest of the section, we prove three Lemmas whose straight-forward combination proves Theorem 1.

Lemma 1. *For any even $k \in \mathbb{N}$ and any $n \in \mathbb{N}$, there is a Nash equilibrium for k players on P_n.*

Proof. If $n \leq k$, then a strategy profile where each vertex of the path is chosen by at least one player is clearly a Nash equilibrium.

Otherwise, if $n > k$, then the idea is to build pairs of players, which are then placed such that two paired players are neighbors and the distance of any two consecutive pairs is roughly equal (specifically, differs by at most two). See

Figure 1 for an example. Intuitively, this yields a Nash equilibrium since each player obtains roughly the same payoff (specifically, differing by at most one), therefore no player can improve. Since we have n vertices, we want each player's payoff to be at least $z := \lfloor \frac{n}{k} \rfloor$. This leaves $r := n \mod k$ other vertices, which we distribute between the first r players such that the payoff of any player is at most $z + 1$. This can be achieved as follows. Let $p_i \in [n]$ denote the position of Player i, that is, the index of the chosen vertex on the path. We define

$$p_i := \begin{cases} z \cdot i + \min\{i, r\} & \text{if } i \text{ is odd,} \\ p_{i-1} + 1 & \text{if } i \text{ is even.} \end{cases}$$

Note that, by construction, it holds that $p_1 \in \{z, z+1\}$ and $p_k = n - z + 1$. Moreover, for each odd indexed player $i \geq 3$, we have that $2z - 1 \leq p_i - p_{i-1} \leq 2z + 1$. We claim that $u_i := U_i(p_1, \ldots, p_k) \in \{z, z+1\}$ holds for each $i \in [k]$. Clearly, $u_1 = p_1 \in \{z, z+1\}$ and $u_k = n - p_k + 1 = z$. For all odd $i \geq 3$, it is not hard to see that $u_i = u_{i-1} = 1 + \lfloor (p_i - p_{i-1} - 1)/2 \rfloor \in \{z, z+1\}$, which proves the claim.

To see that the strategy profile $(p_1, \ldots, p_k)$ is a Nash equilibrium, consider an arbitrary player i and any other strategy $p'_i \in [n]$ that she picks. Clearly, we can assume $p'_i \neq p_j$ for all $j \neq i$ since otherwise Player i's payoff is zero. If $p'_i < p_1$ or $p'_i > p_k$, then Player i gets a payoff of at most z. If $p_j < p'_i < p_{j+1}$ for some even $j \in [2, k-2]$, then her payoff is at most $1 + \lfloor (p_{j+1} - p_j - 2)/2 \rfloor \leq z$. $\square$

We can modify the construction given in the proof of Lemma 1 to also work for odd numbers k greater than three.

Lemma 2. *For any odd $k > 3 \in \mathbb{N}$ and for any $n \in \mathbb{N}$, there is a Nash equilibrium for k players on P_n.*

Proof. We give a strategy profile based on the construction for an even number of players (proof of Lemma 1). The idea is to pair the players, placing the remaining lonely player between two consecutive pairs.

This is best explained using a reduction to the even case. Specifically, given the strategy profile $(p'_1, \ldots, p'_{k+1})$ for an even number $k + 1$ of players on P_{n+1} as constructed in the proof of Lemma 1, we define the strategy profile $(p_1, \ldots, p_k) := (p'_1, \ldots, p'_{k-2}, p'_k - 1, p'_{k+1} - 1)$. To see why this results in a Nash equilibrium, let $z := \lfloor (n+1)/(k+1) \rfloor$ and note that by construction it holds that $p_1 \in \{z, z+1\}$, $p_k = n - z + 1$, and $2z - 1 \leq p_{i+1} - p_i \leq 2z + 1$ for all $i \in [2, k-1]$. Moreover, each player receives a payoff of at least z, therefore all players (except for Player $(k-2)$) cannot improve by the same arguments as in the proof of Lemma 1. Regarding Player $(k-2)$, note that her payoff is

$$1 + \lfloor (p_{k-1} - p_{k-2} - 1)/2 \rfloor + \lfloor (p_{k-2} - p_{k-3} - 1)/2 \rfloor \geq 2z - 1.$$

Hence, she clearly cannot improve by choosing any position outside of $[p_{k-3}, p_{k-1}]$. Also, she cannot improve by choosing any other position in $[p_{k-3}, p_{k-1}]$. To see this, note that her maximum payoff from any position in $[p_{k-3}, p_{k-1}]$ is

$$1 + \lfloor (p_{k-1} - p_{k-3} - 2)/2 \rfloor = 1 + \lfloor (p_{k-1} - p_{k-2} - 1 + p_{k-2} - p_{k-3} - 1)/2 \rfloor,$$

which is equal to the above payoff since $p_{k-1} - p_{k-2}$ and $p_{k-2} - p_{k-3}$ cannot both be even, by construction. □

It remains to discuss the fairly simple (non)-existence of Nash equilibria for three players. Note that Roshanbin [8] already stated without proof that there is no Nash equilibrium for three players on $G_{2 \times n}$ and $G_{3 \times n}$ and Small and Mason [10] showed that there is no Nash equilibrium for three players on P_7. For the sake of completeness, we prove the following lemma.

Lemma 3. *For three players, there is a Nash equilibrium on P_n if and only if $n \leq 5$.*

Proof. If $n \leq 3$, then a strategy profile where each vertex of the path is chosen by at least one player is clearly a Nash equilibrium. For $n \in \{4, 5\}$, the strategy profile $(2, 3, 4)$ is a Nash equilibrium.

To see that there is no Nash equilibrium for $n \geq 6$, consider an arbitrary strategy profile (p_1, p_2, p_3). Without loss of generality, we can assume that $p_1 < p_2 < p_3$ and consider the following two cases. First, we assume that $p_2 = p_1 + 1$ and $p_3 = p_2 + 1$. If $p_1 > 2$, then Player 2 increases her payoff by choosing $p_1 - 1$. Otherwise, it holds that $p_3 < n - 1$ and Player 2 increases her payoff by moving to $p_3 + 1$. Therefore, this case does not yield a Nash equilibrium. For the remaining case, it holds that $p_1 < p_2 - 1$ or $p_3 > p_2 + 1$. If $p_1 < p_2 - 1$, then Player 1 increases her payoff by moving to $p_2 - 1$, while if $p_3 > p_2 + 1$, then Player 3 increases her payoff by moving to $p_2 + 1$. Thus, this case does not yield a Nash equilibrium as well, and we are done. □

We close this section with the following result considering cycles. Interestingly, for cycles there exists a Nash equilibrium also for three players.

Theorem 2. *For any $k, n \in \mathbb{N}$, there is a Nash equilibrium for k players on C_n.*

Proof. It is an easy observation that the constructions given in the proofs of Lemma 1 and 2 also yield Nash equilibria for cycles, that is, when the two endpoints of the path are connected by an edge. Thus, it remains to show a Nash equilibrium for $k = 3$ players for any C_n. We set $p_1 := 1$, $p_2 := n$ and

$$p_3 := \begin{cases} \lfloor n/2 \rfloor & \text{if } n \mod 4 = 1, \\ \lceil n/2 \rceil & \text{else.} \end{cases}$$

It is not hard to check that (p_1, p_2, p_3) is a Nash equilibrium. □

3 Grid Graphs

In this section we consider three players on the $m \times n$ grid $G_{m \times n}$ and we prove the following theorem.

Theorem 3. *If $m \geq 5$ and $n \geq 5$, then there is no Nash equilibrium for three players on $G_{m \times n}$.*

Before proving the theorem, let us first introduce some general definitions and observations. Throughout this section, we denote the strategy of Player i, that is, the initially chosen vertex of Player i, by $p_i := (x_i, y_i) \in [m] \times [n]$. Note that any strategy profile where more than one player chooses the same position is never a Nash equilibrium since in this case each of these players gets a payoff of zero, and can improve its payoff by choosing any free vertex (to obtain a payoff of at least one). Therefore, we will assume without loss of generality that $p_1 \neq p_2 \neq p_3$. Further, note that the game is highly symmetric with respect to the axes. Specifically, reflecting coordinates along a dimension, or rotating the grid by 90 degrees, yields the same outcome for the game. Thus, in what follows, we only consider possible cases up to the above symmetries.

We define $\Delta_x := \max_{i,j \in [k]} |x_i - x_j|$ and $\Delta_y := \max_{i,j \in [k]} |y_i - y_j|$ to be the maximum coordinate-wise differences among the positions of the players. We say that a player *strictly controls* the other two players, if both reside on the same side of the player, in both dimensions.

Definition 1. *Player i* strictly controls *the other players, if for each other Player j with $j \neq i$, either*

$$
\begin{aligned}
(x_i < x_j) \ &\wedge\ (y_i < y_j), \ or \\
(x_i < x_j) \ &\wedge\ (y_i > y_j), \ or \\
(x_i > x_j) \ &\wedge\ (y_i < y_j), \ or \\
(x_i > x_j) \ &\wedge\ (y_i > y_j) \ holds.
\end{aligned}
$$

We now prove Theorem 3.

Proof of Theorem 3. Let $m \geq 5$ and $n \geq 5$. We perform a case distinction based on the relative positions of the three players. As a first case, we consider strategy profiles where the players are playing "far" from each other, that is, there are two players whose positions differ by at least four in some coordinate (formally $\max\{\Delta_x, \Delta_y\} \geq 3$). For these profiles, we distinguish two subcases, namely, whether there existss a player who strictly controls the others (Lemma 4) or not (Lemma 5). We prove that none of these cases yields a Nash equilibrium by showing that there always exists a player who can improve is payoff. Notably, the improving player always moves closer to the other two players. We are left with the case where the players are playing "close" to each other, in the sense that their positions all lie inside a 3×3 subgrid (that is, $\max\{\Delta_x, \Delta_y\} \leq 2$). For these strategy profiles, we show that there always exists a player who can improve her payoff (Lemma 6), however the improving position depends not only on the relative positions between the players, but also on the global positioning of this subgrid on the overall grid. This leads to a somewhat erratic behaviour, which we overcome by considering all possible close positions (up to symmetries) in the proof of Lemma 6. Altogether, Lemmas 4, 5 and 6 cover all possible strategy profiles (ruling them out as Nash equilibria), thus implying the theorem. $\square$

In order to conclude Theorem 3, it remains to prove the lemmas mentioned in the case distinction discussed above. To this end, we start with two easy
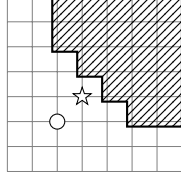
Figure 2: Example of a strategy profile where Player 1 (white circle) has both other players to her top right with distance at least three (the shaded region denotes the possible positions for Player 2 and 3). Player 1 can increase her payoff by moving closer to the others (star).

preliminary results. First, we observe that a vertex for which there is a unique player with the shortest distance to it is colored in that player's color (note that this is true in general for every graph and any number of players).

Observation 1. *Let $G = (V, E)$ be an undirected graph and let $(p_1, \ldots, p_k)$ be a strategy profile. Let $v \in V$ be a vertex for which there exists an $i \in [k]$ such that $dist_G(p_i, v) =: \delta < dist(p_j, v)$ holds for all $j \in [k]$, $j \neq i$, then v will be colored in color i at time δ.*

Proof. The proof is by induction on the distance δ. For $\delta = 0$, it clearly follows from the definition of the propagation process that $v = p_i$ has color i at time 0. For all $\delta > 0$, it follows from the induction hypothesis that v has a neighbor u with $dist(p_i, u) = \delta - 1$ that is colored in color i at time $\delta - 1$. Moreover, for all neighbors w of v it holds $dist(p_j, w) > \delta - 1$ for all $j \neq i$. This implies that no neighbor of v has a different color than i at time $\delta - 1$, and thus, v has color i at time δ. $\square$

Based on Observation 1, we show that whenever a player has distance at least three to the other players and both of them are positioned on the same side of that player (with respect to both dimensions), then she can improve her payoff by moving closer to the others (see Figure 2 for an illustration).

Proposition 1. *If $x_1 \leq x_j$, $y_1 \leq y_j$, and $\|p_1 - p_j\|_1 \geq 3$ holds for $j \in \{2, 3\}$, then Player 1 can increase her payoff by moving to $(x_1 + 1, y_1 + 1)$.*

Proof. Let $p'_1 := (x_1 + 1, y_1 + 1)$ and $x \in [x_1] \times [y_1]$. Note that $\|p'_1 - x\|_1 = \|p_1 - x\|_1 + 2 < \|p_j - x\|_1 = \|p_1 - p_j\|_1 + \|p_1 - x\|_1 \geq \|p_1 - x\|_1 + 3$ holds for $j \in \{2, 3\}$. Hence, from position p'_1, Player 1 still has the unique shortest distance to x. By Observation 1, x gets color 1. Moreover, for any other position $x \notin [x_1] \times [y_1]$, there is a shortest path from p_1 to x going through at least one of the positions $(x_1 + 1, y_1)$, $(x_1, y_1 + 1)$, or p'_1. Clearly, there is also a shortest path from p'_1 to x of at most the same length going through one of these positions. Thus, if x was colored with color 1 before, then x is still colored in color 1.

To see that Player 1 strictly increases her payoff from p'_1, note that $\|p'_1 - x\|_1 = \|p_1 - x\|_1 - 2$ holds for all $x \in [x_1 + 1, m] \times [y_1 + 1, n]$. Hence, Player 1 now has the unique shortest distance to all those positions where the distance
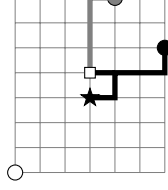
Figure 3: A profile where Player 1 (white) has distance 6 to the vertex v (black star) and Player 2 (gray) and Player 3 (black) have distance 5 to v. The vertex u (white square) on the shortest path from Player 2 to v indicated by thick lines is removed during the propagation due to the shortest path from Player 3 to u. Player 3, however, still reaches v by a path of length 5. Thus, v is not colored by Player 1.

from p_1 was at most one larger than the shortest distance from any other player. We claim that there exists a position $v \in [x_1 + 1, m] \times [y_1 + 1, n]$ with $\|p_1 - v\|_1 - \min_{j \in \{2,3\}} \|p_j - v\|_1 \in \{0, 1\}$ which is not colored in color 1 before. By Observation 1, Player 1 then gets v when moving to p_1'. To verify the claim, assume first without loss of generality that $\|p_2 - p_1\|_1 \leq \|p_3 - p_1\|_1$. Then, on any shortest path from p_2 to p_1, there exists a "middle" vertex $v = (v_x, v_y) \in [x_1, x_2] \times [y_1, y_2]$ with $\|p_1 - v\|_1 - \|p_2 - v\|_1 \in \{0, 1\}$ and $\|p_2 - v\|_1 \leq \|p_3 - v\|_1$. Assume towards a contradiction that v is colored in color 1. Then this implies that on every shortest path from p_2 to v a vertex has been removed during the propagation process since otherwise either v would have been removed or colored in color 2. But this can only happen due to the position of Player 3 since Player 1 has a larger distance than Player 2 to all vertices in the subgrid $[v_x, x_2] \times [v_y, y_2]$. Hence, on every shortest path from p_2 to v there exists a vertex with the same distance to p_3, which implies that p_3 and p_2 have the same distance to v. It follows, that also on every shortest path from p_3 to v a vertex has to be removed during propagation because of Player 2. But this is not possible due to the structure of a grid (Figure 3 depicts a typical situation). Since p_2 and p_3 have the same distance to v, it follows from $p_2 \neq p_3$ that $x_2 \neq x_3$ and $y_2 \neq y_3$. Assume without loss of generality that $x_2 < x_3$ (and thus $y_2 > y_3$) and consider the shortest path $v, (v_x, v_y + 1), \ldots, (v_x, y_2), (v_x + 1, y_2), \ldots, p_2$ from v to p_2. At least one of the inner vertices has to be removed during the propagation. The vertices $(v_x, y_2), \ldots, (x_2 - 1, y_2)$ cannot be removed because their distance to p_3 is strictly larger than the distance to p_2. Hence, a vertex $u = (u_x, u_y) \in \{(v_x, v_y + 1), \ldots, (v_x, y_2 - 1)\}$ has to be removed. Assume that u has the maximum y-coordinate among the removed vertices. It follows that there is a shortest path from p_3 to u going through $(u_x, u_y - 1)$ or through $(u_x + 1, u_y)$ on which all inner vertices are colored in color 3. But this implies that there is also a shortest path from p_3 to v going through $(u_x, u_y - 1)$ or $(u_x + 1, u_y)$ that is colored with color 3. Hence, v cannot have color 1. $\quad\square$

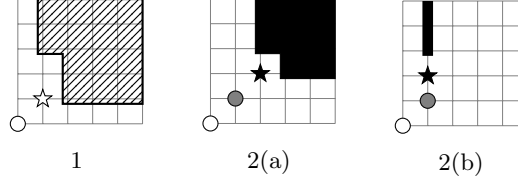We go on to prove the lemmas, starting with the case that the players play

Figure 4: Possible cases (up to symmetry) for Player 1 (white) strictly control-ling Player 2 (gray) and Player 3 (black). Circles denote the player's strategies. The shaded region contains the possible positions of both Player 2 and 3, whereas the black regions denote possible positions for Player 3 only. A star marks the position improving the payoff of the respective player.

far from each other. The following lemma handles the first subcase, that is, where one of the players strictly controls the others.

Lemma 4. *A strategy profile with* $\max\{\Delta_x, \Delta_y\} \geq 3$ *where one of the players strictly controls the others is not a Nash equilibrium.*

Proof. We assume without loss of generality that Player 1 strictly controls Player 2 and Player 3, specifically, we assume that $x_1 < x_2$ and $y_1 < y_2$ and $x_1 < x_3$ and $y_1 < y_3$ holds. Figure 4 depicts the three possible cases for the positions of Player 2 and Player 3. For each case, we show that a player which can improve her payoff exists.

Case 1: We assume that $(x_2, y_2) \neq (x_1+1, y_1+1)$ and $(x_3, y_3) \neq (x_1+1, y_1+1)$. By Proposition 1, Player 1 gets a higher payoff from $(x_1 + 1, y_1 + 1)$.

Case 2: We assume without loss of generality that $(x_2, y_2) = (x_1 + 1, y_1 + 1)$.

(a) We assume $x_2 < x_3$ and $y_2 < y_3$. Then, $x_3 > x_2+1$ or $y_3 > y_2+1$ holds since $\max\{\Delta_x, \Delta_y\} \geq 3$. Note that Player 3 strictly controls Player 1 and Player 2 and that this case is symmetric to Case 1.

(b) We assume $x_2 \geq x_3$ or $y_2 \geq y_3$. Then, it holds that $x_3 = x_2$ or $y_3 = y_2$. We assume $x_3 = x_2$ (the argument for $y_3 = y_2$ being analogous). Since $\max\{\Delta_x, \Delta_y\} \geq 3$, we have $y_3 > y_2 + 1$, thus Player 3 can improve by moving to $(x_2, y_2 + 1)$ because then all positions in $[m] \times [y_2+1, n]$ are colored in color 3, and before only a strict subset of these positions were colored in her color.

$\square$

The other subcase, where no player strictly controls the others, is handled by the following lemma.

Lemma 5. *A strategy profile with* $\max\{\Delta_x, \Delta_y\} \geq 3$ *where no player strictly controls the others is not a Nash equilibrium.*
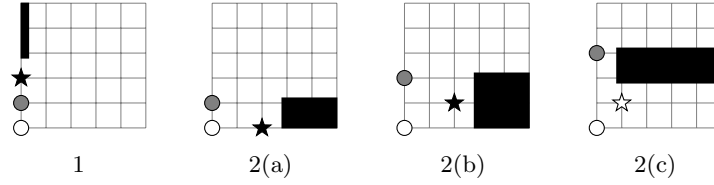
Figure 5: Possible cases (up to symmetry) when no player strictly controls the others. Circles denote the positions of Player 1 (white) and Player 2 (gray). The black regions contain the possible positions for Player 3. A star marks the position improving the payoff of the respective player.

Proof. If no player strictly controls the others, then it follows that at least two players have the same coordinate in at least one dimension. We perform a case distinction on the cases as depicted in Figure 5.

Case 1: All three players have the same coordinate in one dimension. We assume that $x_1 = x_2 = x_3$ (the case $y_1 = y_2 = y_3$ is analogous). Without loss of generality also $y_1 < y_2 < y_3$ holds. Since $\max\{\Delta_x, \Delta_y\} \geq 3$, it follows that $y_{i+1} - y_i \geq 2$ holds for some $i \in \{1, 2\}$, say for $i = 2$. Clearly, Player 3 can improve her payoff by choosing $(x_3, y_2 + 1)$ (analogous to Case 2b in the proof of Lemma 4).

Case 2: There is a dimension where two players have the same coordinate but not all three players have the same coordinate in any dimension. We assume $x_1 = x_2 < x_3$ and $y_1 < y_2$ (all other cases are analogous). We also assume that $y_1 \leq y_3 \leq y_2$, since otherwise Player 3 strictly controls the others, and this case is handled by Lemma 4.

    (a) We assume that $y_2 = y_1 + 1$. Then $x_3 \geq x_1 + 3$ holds since $\max\{\Delta_x, \Delta_y\} \geq 3$. Player 3 increases her payoff by moving to $(x_1 + 2, y_1)$ (analogous to Case 2b in the proof of Lemma 4).

    (b) We assume that $y_2 = y_1 + 2$. Then $x_3 \geq x_1 + 3$ holds since $\max\{\Delta_x, \Delta_y\} \geq 3$. Player 3 increases her payoff by moving to $(x_1 + 2, y_1 + 1)$ (analogous to Case 2b in the proof of Lemma 4).

    (c) We assume that $y_2 > y_1 + 2$ and $|y_2 - y_3| \leq |y_1 - y_3|$. That is, without loss of generality, Player 3 is closer to Player 2. Then, by Proposition 1, Player 1 increases her payoff by moving to $(x_1 + 1, y_1 + 1)$.

$\square$

It remains to consider the cases where the players play close to each other.

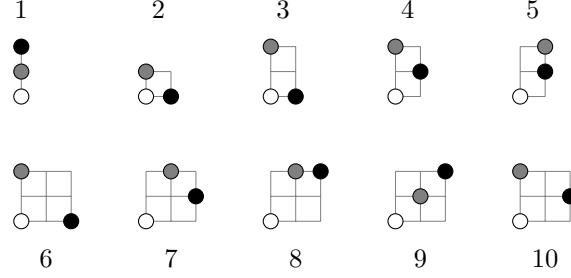**Lemma 6.** *A strategy profile with* $\max\{\Delta_x, \Delta_y\} \leq 2$ *is not a Nash equilibrium.*

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Figure 6: Possible positions (up to symmetry) of three players playing inside a subgrid of size at most 3×3. The position of Player 1 (white) is denoted (x, y).

Proof. First, we assume that $\Delta_x + \Delta_y \geq 2$, as otherwise there would be at least two players on the same position (so each one of them can improve by moving to any free vertex). Without loss of generality, we also assume that $\Delta_x \leq \Delta_y$, leaving the following cases to consider (depicted in Figure 6).

Case 1: Let $(x_1, y_1) = (x, y) \in [m] \times [n-2]$, $(x_2, y_2) = (x, y+1)$, and $(x_3, y_3) = (x, y+2)$ be the positions of the three players. Clearly, with these positions, Player 1 gets a payoff of my, Player 2's payoff is m, and Player 3 gets a payoff of $m(n-y-1)$. Now, if $y \geq 3$, then Player 2 can improve by choosing $(x, y-1)$ thus achieving a payoff of $m(y-1) > m$. If $y = 1$, then Player 2 improves by playing on $(x, y+3)$ with a payoff of $m(n-y-2) > m$ (remember that $n \geq 5$). Also, if $y = 2$ and $n > 5$, then Player 2 gets a higher payoff by choosing $(x, y+3)$. For $y = 2$ and $n = 5$, we observe that either $x \geq \lceil m/2 \rceil$ or $m - x \geq \lceil m/2 \rceil$ and assume, without loss of generality, that $x \geq \lceil m/2 \rceil$ holds. Then, applying Observation 1, it holds that, if Player 2 chooses $(x-1, y)$, then all positions in $[x-1] \times [y+1]$ are colored in color 2, Thus, the payoff of Player 2 is at least $(\lceil m/2 \rceil - 1)3 > m$ (remember that $m \geq 5$).

Case 2: Let $(x_1, y_1) = (x, y) \in [m-1] \times [n-1]$, $(x_2, y_2) = (x, y+1)$, and $(x_3, y_3) = (x+1, y)$ be the positions of the three players. Note that, due to symmetry, this is the only case we have to consider (without loss of generality). Clearly, exactly the positions in $[x] \times [y]$ are colored in color 1, therefore Player 1 has a payoff of xy. If $x < m/2$, then position $(x+2, y)$ yields a payoff of $(m-x-1)n \geq (\lceil m/2 \rceil - 1)n$ for Player 1 since she colors all vertices in $[x+2, m] \times [n]$. Note that $\lceil m/2 \rceil - 1 \geq x$ and $n > y$, thus Player 1's payoff improved. Analogously, for $y < n/2$, Player 1's payoff from position $(x, y+2)$ is at least $m(\lceil n/2 \rceil - 1) > xy$. If $x > \lceil m/2 \rceil$, then Player 3 can improve by choosing $(x-1, y)$. To see that this is true, note first that by choosing (x_3, y_3), Player 3 colors only positions in $[x+1, m] \times [n]$. Now, observe that if position (x', y') is colored in color 3 when Player 3 chooses (x_3, y_3), then it holds also that position $(x - (x' - x), y)$ is

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colored in color 3 when Player 3 chooses $(x - 1, y) = (x - (x_3 - x), y)$, due the symmetries, and since the distances from this position to the players positions are all identical. Hence, Player 3 colors at least the same number of positions in $[x - (m - x), x - 1] \times [y]$. Note that $x - (m - x) \geq 2$ since $x > \lceil m/2 \rceil$. By Observation 1, Player 3 additionally colors position $(1, 1)$ yielding a strictly greater payoff. By analogous arguments, for $y > \lceil n/2 \rceil$, the position $(x, y - 1)$ yields a better payoff for Player 2. Finally, assume $x = \lceil m/2 \rceil$ and $y = \lceil n/2 \rceil$. Now, Player 1 can improve by choosing $(x-1, y+1)$, thus coloring at least all positions in $[x-1] \times [n]$, giving a payoff of at least $(\lceil m/2 \rceil - 1)n > \lceil m/2 \rceil \cdot \lceil n/2 \rceil = xy$ for $m \geq 5$ and $n \geq 5$.

Case 3: Let $(x_1, y_1) = (x, y) \in [m - 1] \times [n - 2]$, $(x_2, y_2) = (x, y + 2)$, and $(x_3, y_3) = (x + 1, y)$ be the positions of the three players. First, note that Player 1 colors all positions in $[x] \times [y]$, gaining a payoff of xy, Player 2 colors $[m] \times [y + 2, n]$, gaining a payoff of $m(n - y - 1)$, and Player 3 gets all positions in $[x + 1, n] \times [y + 1]$, gaining a payoff of exactly $(m - x)(y + 1)$. Now, if $y = 1$, then Player 1 can choose $(x, 4)$ to obtain a payoff of $m(n - 3) > x$, since $m > x$ and $n - 3 \geq 2$. Hence, assume $y > 1$, and observe that both the payoff of Player 1 from $(x + 1, y - 1)$ and the payoff of Player 3 from $(x, y - 1)$ equals $m(y - 1)$. Assuming that we have a Nash equilibrium, we obtain the two inequations $xy \geq m(y - 1)$ and $(m - x)(y + 1) \geq m(y - 1)$, which yield $m(y - 1)/y \leq x \leq 2m/(y + 1)$. Note that we obtain a contradiction for $y \geq 3$. Hence, we can assume that $y = 2$ and $\lceil m/2 \rceil \leq x \leq \lfloor 2m/3 \rfloor$. If $n \geq 6$, then Player 1 can improve by choosing $(x, 5)$ achieving a payoff of $m(n-4) \geq 2m > 2x$. Thus, we also assume $n = 5$. Now, Player 1 can choose position $(x - 1, 4)$ to color all but three positions in $[x - 1] \times [5]$. The only positions which she does not color are $(x - 1, 2)$, $(x - 1, 1)$ and $(x - 2, 1)$. Her payoff is thus $5(x - 1) - 3$, which, for all $x \geq \lceil m/2 \rceil \geq 3$, is more than $2x$.

Case 4: Let $(x_1, y_1) = (x, y) \in [m - 1] \times [n - 2]$, $(x_2, y_2) = (x, y + 2)$, and $(x_3, y_3) = (x+1, y+1)$ be the positions of the three players. It is easy to see that, if $x = m - 1$, then Player 3's payoff is exactly one, and she can gain more by choosing $(x - 1, y + 1)$ instead. For $x < m - 1$, note that, apart from (x_3, y_3), Player 3 colors only positions (x', y') with $x' \geq x_3 + 1 = x + 2$. Note also that Player 3 does not color all of these positions. For example, at least one of the positions $(x+2, y-1)$ or $(x + 2, y + 3)$ exists on the grid (since $n \geq 5$) and is reached by Player 1 or Player 2 at the same time during the propagation process of the game. However, by choosing $(x + 2, y + 1)$, Player 3 still colors the position (x_3, y_3) and clearly all positions (x', y') with $x' \geq x + 2$, thus improving her payoff.

Case 5: Let $(x_1, y_1) = (x, y) \in [m - 1] \times [n - 2]$, $(x_2, y_2) = (x + 1, y + 2)$, and $(x_3, y_3) = (x+1, y+1)$ be the positions of the three players. If $y = n-2$,

then Player 2's payoff is exactly m. Therefore, she increases her payoff by moving to $(y - 1, x)$, because then her payoff is at least $2m$. Thus, we can assume that $y < n - 2$. If $x \geq m - 2$, then Player 3's payoff is either 1 or 3 (it is 1 if $x = m - 1$ and 3 if $x = m - 2$). Therefore, she increases her payoff by moving to $(x - 2, y)$, because then her payoff is at least n (and $n \geq 5$). Thus, we can assume that $x < m - 2$. We are left only with the case where $y < n - 2$ and $x < m - 2$. In this case, Player 3 increases her payoff by moving to $(x + 2, y + 2)$.

Case 6: Let $(x_1, y_1) = (x, y) \in [m - 2] \times [n - 2]$, $(x_2, y_2) = (x, y + 2)$, and $(x_3, y_3) = (x + 2, y)$ be the positions of the three players. It is clear that only a strict subset of the positions in $[x + 2, m] \times [n]$ are colored in color 3 (for example, the position $(x + 2, y + 2)$ is not colored in color 3). By choosing $(x + 2, y + 1)$, however, all positions in $[x + 2, m] \times [n]$ are colored in color 3, resulting in a strictly higher payoff for Player 3.

Case 7: Let $(x_1, y_1) = (x, y) \in [m - 2] \times [n - 2]$, $(x_2, y_2) = (x + 1, y + 2)$, and $(x_3, y_3) = (x + 2, y + 1)$ be the positions of the three players. By Proposition 1, Player 1 increases her payoff by moving to $(x_1 + 1, y_1 + 1)$.

Case 8: Let $(x_1, y_1) = (x, y) \in [m - 2] \times [n - 2]$, $(x_2, y_2) = (x + 1, y + 2)$, and $(x_3, y_3) = (x + 2, y + 2)$ be the positions of the three players. By Proposition 1, Player 1 increases her payoff by moving to $(x_1 + 1, y_1 + 1)$.

Case 9: Let $(x_1, y_1) = (x, y) \in [m - 2] \times [n - 2]$, $(x_2, y_2) = (x + 1, y + 1)$, and $(x_3, y_3) = (x + 2, y + 2)$ be the positions of the three players. Notice that the payoff of Player 2 is only one. It is clear that Player 2 increases her payoff more by moving to $(x, y + 1)$, because then her payoff is at least two, as she also colors the position $(x, y + 2)$.

Case 10: Let $(x_1, y_1) = (x, y) \in [m - 2] \times [n - 2]$, $(x_2, y_2) = (x, y + 2)$, and $(x_3, y_3) = (x + 2, y + 1)$ be the positions of the three players. Note first that Player 1's colors exactly the positions in $[x + 1] \times [y]$, thus, her payoff is $(x + 1)y$. For $y = 1$, Player 1 can move to $(x, 4)$, achieving a payoff of at least $(x + 1)(n - 3) \geq 2(x + 1)$ since she colors all positions in $[x + 1] \times [4, n]$. Otherwise, if $y \geq 2$, then, by choosing $(x + 1, y)$, Player 1 still colors all positions in $[x + 1] \times [y]$, and additionally also the position $(x + 2, y - 1)$.

$\square$

4   Hypercubes

Etesami and Basar [3] studied diffusion games on hypercubes and proved the existence of a Nash equilibrium for two players on every d-dimensional hypercube. In this section, we extend their result to four players.

Recall that the vertices of H_d are all binary strings of length d, where two vertices are adjacent if and only if they differ in exactly one bit. Moreover, the geodesic distance of two vertices u and v is exactly the Hamming distance $\Delta(u, v)$. By $\overline{a}$, we denote the *complement* of $a = a_1 \ldots a_d$, where $\overline{a}_i := 1 - a_i$ for all $i \in [d]$.

We prove the following theorem.

Theorem 4. *Let $x, y \in \{0, 1\}^d$ be two adjacent vertices of H_d for $d \geq 1$. Then, every strategy profile $(p_1, \ldots, p_4)$ with $\{p_1, \ldots, p_4\} = \{x, \overline{x}, y, \overline{y}\}$ is a Nash equilibrium.*

A first step to prove Theorem 4 is to show that for a strategy profile of the form as described in Theorem 4, it holds that whenever a single player chooses another position, then for the resulting strategy profile the payoff of each player equals exactly the number of vertices to which she has a unique closest distance. Note that we already know from Observation 1 that each player always colors all the vertices to which she has the uniquely closest distance. In Lemma 7 we will show the opposite direction, namely that no player obtains vertices to which the distance is not the unique closest distance among all players (note that this does not hold in general). In a second step, we can then compute the payoffs of all players and show that they are maximal for the strategy profile stated in Theorem 4.

We start with the following lemma.

Lemma 7. *Let $x, y \in \{0, 1\}^d$ be two adjacent vertices in H_d, $d \geq 1$, and let $(p_1, \ldots, p_4)$ be a strategy profile with $\{x, y, \overline{y}\} \subseteq \{p_1, \ldots, p_4\}$ and $p_i \neq p_j$ for all $i, j \in [4]$, $i \neq j$. Let $v \in \{0, 1\}^d$ be a vertex and let $\delta := \min_{i=1,\ldots,4} \Delta(p_i, v)$. If there exist $i, j \in [4]$, $i \neq j$, such that $\Delta(p_i, v) = \Delta(p_j, v) = \delta$, then the vertex v will not be colored by any player at the end of the propagation process.*

Proof. Since the order of the players does not matter, we assume that $p_1 = p \in \{0, 1\}^d$, $p_2 = x$, $p_3 = y$, and $p_4 = \overline{y}$. For the case $\delta \leq 1$, the statement clearly holds by definition of the propagation process. Hence, we consider $\delta \geq 2$. Note that for any two vertices $p_i \neq p_j$ with $\Delta(p_i, v) = \Delta(p_j, v) = \delta$, the distance $\Delta(p_i, p_j)$ must be even (and thus, at least two) since two vertices with an odd distance cannot have the same distance to any other vertex. Hence, x and y can never have the same distance δ to v since their distance is one (they are adjacent). It follows that either two or three players have the same shortest distance δ to v among all players. For both cases, we show that there always exist two neighbors of v each having a different player with a unique closest distance of $\delta - 1$. Using Observation 1, we can then conclude that these neighbors are colored in different colors and that v is thus removed.

First, assume that exactly the two players i and j have distance δ to v, while the other two players have a distance larger than δ. Then, p_i and p_j differ in $\Delta(p_i, p_j) = 2c$ bits for some $c \geq 1$. Note that v equals p_i in exactly c of these bits and equals p_j in the other c bits (otherwise they cannot have the same distance to v). Hence, by swapping one of the c bits where v equals p_j,

we reach a neighbor u of v such that $\Delta(p_i, u) = \delta - 1 < \Delta(p_j, u) = \delta + 1$. Analogously, swapping one of the c bits where v equals p_i yields a neighbour w with $\Delta(p_j, w) = \delta - 1 < \Delta(p_i, w) = \delta + 1$. Note that the other two players have distance at least δ to both u and w since their distance to v is at least $\delta + 1$. Thus, p_i has the unique shortest distance to u, and p_j has the unique shortest distance to w. According to Observation 1, u and w are thus colored in different colors at time $\delta - 1$. Consequently, v is removed at time δ.

Now, assume that exactly three players have the minimum distance δ to v. Since x and y are adjacent, we know that the only two possible cases are that p, $\overline{y}$, and either x or y have distance δ to v.

Case 1: $\Delta(p, v) = \Delta(x, v) = \Delta(\overline{y}, v) = \delta < \Delta(y, v)$. Note that $\Delta(y, v) = d - \Delta(\overline{y}, v)$, which yields $\delta < d/2$. Note further that x and $\overline{y}$ differ in an even number of $\Delta(x, \overline{y}) = d - 1$ bits, which means d is odd. Hence, $\delta \leq (d-1)/2$. Moreover, v equals each of x and $\overline{y}$ in exactly half of their $d - 1$ differing bits (otherwise v cannot have the same distance to both). It follows that $\delta = (d-1)/2$ and it also holds that $x_i = \overline{y}_i \neq v_i$ is not possible for any $i \in [d]$ (this holds since $x_i = \overline{y}$ implied $\overline{y} = v_i$, which holds since $\delta = (d-1)/2$). Now consider the vertex p which also differs in δ bits to v. Clearly, p and v can neither differ in the same δ bits as v and x, nor in the same δ bits as v and $\overline{y}$ since then p_1 would be equal to either x or $\overline{y}$. Hence, there exist indices i and j among the $d - 1$ differing bits of x and $\overline{y}$ such that $v_i = \overline{y}_i = p_i \neq x_i$ and $v_j = x_j = p_j \neq \overline{y}_j$. Thus, by swapping the ith bit of v, we reach a neighbor u with $\Delta(x, u) = \delta - 1 < \Delta(p, u) = \Delta(\overline{y}, u) = \delta + 1$. Similarly, swapping the jth bit of v yields a neighbor w with $\Delta(\overline{y}, w) = \delta - 1 < \Delta(p, w) = \Delta(x, w) = \delta + 1$. Clearly, also y has distance at least δ to both u and w. Hence, by Observation 1, u and w are colored in different colors and thus v is removed.

Case 2: $\Delta(p, v) = \Delta(y, v) = \Delta(\overline{y}, v) = \delta < \Delta(x, v)$. Note that $\Delta(y, \overline{y}) = d$ is even and that v equals both y and $\overline{y}$ in exactly $\delta = d/2$ bits. As in Case 1, the vertex p cannot differ from v in the same δ bits as y or $\overline{y}$. Thus, we again find indices i and j with $v_i = \overline{y}_i = p_i \neq y_i$ and $v_j = y_j = p_j \neq \overline{y}_j$ such that for the corresponding neighbors u and w of v we have $\Delta(y, u) = \delta - 1 < \Delta(p, u) = \Delta(\overline{y}, u) = \delta + 1$ and $\Delta(\overline{y}, w) = \delta - 1 < \Delta(p, w) = \Delta(y, w) = \delta + 1$. Since by assumption also x has distance at least δ from u and w, it follows that u and w are colored in different colors and v is removed.

$\square$

Lemma 7 (together with Observation 1) shows that in every strategy profile as described in Theorem 4 the payoff of each player equals the number of vertices to which she has the unique minimum distance among all players.

The following lemma gives upper bounds on the possible payoffs for players in such a profile.

Lemma 8. *Let $x, y \in \{0,1\}^d$, $d \geq 1$, and let $V_x^{y\overline{y}} := \{v \in \{0,1\}^d \mid \Delta(v,x) < \min\{\Delta(v,y), \Delta(v,\overline{y})\}\}$. Then, $|V_x^{y\overline{y}}| \leq 2^{d-2}$. Moreoever, if d is odd, then the bound is even smaller, that is,*

$$|V_x^{y\overline{y}}| \leq 2^{d-2} - \frac{1}{2}\binom{d-1}{(d-1)/2}.$$

Proof. Let $\alpha := \Delta(x,y)$; thus, $\Delta(x,\overline{y}) = d - \alpha$. For any vertex $v \in V_x^{y\overline{y}}$, it holds $\Delta(x,v) < \Delta(y,v)$ and also $\Delta(x,v) < \Delta(\overline{y},v)$. The first inequality implies that v equals x in more than half of the α bits where x and y differ. Analogously, the second inequality implies that v equals x in more than half of the $d - \alpha$ bits where x and $\overline{y}$ differ. Clearly, the set of bits where x and y differ is disjoint from the set of bits where x and $\overline{y}$ differ. Hence, the number of possible vertices in $V_x^{y\overline{y}}$ is

$$|V_x^{y\overline{y}}| = \left(\sum_{\ell=\lceil(\alpha+1)/2\rceil}^{\alpha}\binom{\alpha}{\ell}\right) \cdot \left(\sum_{\ell=\lceil(d-\alpha+1)/2\rceil}^{d-\alpha}\binom{d-\alpha}{\ell}\right) \tag{1}$$

$$\leq 2^{\alpha-1} \cdot 2^{d-\alpha-1} = 2^{d-2},$$

which proves the general bound. Now, for d being odd, note that either α or $d - \alpha$ is even. We assume without loss of generality that $\alpha \leq d - 1$ is even (that is, $d - \alpha$ is odd). Then, Equation (1) can be written as

$$|V_x^{y\overline{y}}| = \left(\sum_{\ell=\alpha/2+1}^{\alpha}\binom{\alpha}{\ell}\right) \cdot \left(\sum_{\ell=\lceil(d-\alpha)/2\rceil}^{d-\alpha}\binom{d-\alpha}{\ell}\right)$$

$$\leq \left(2^{\alpha-1} - \frac{1}{2}\binom{\alpha}{\alpha/2}\right) \cdot 2^{d-\alpha-1}$$

$$= 2^{d-2} - \left(2^{d-\alpha-2} \cdot \binom{\alpha}{\alpha/2}\right).$$

We now use the following identity for the central binomial coefficient

$$\binom{\alpha}{\alpha/2} = \frac{\alpha!}{((\alpha/2)!)^2} = 2^\alpha \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (\alpha-1)}{2 \cdot 4 \cdot \ldots \cdot \alpha}$$

to obtain

$$|V_x^{y\overline{y}}| \leq 2^{d-2} - \left(2^{d-2} \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (\alpha-1)}{2 \cdot 4 \cdot \ldots \cdot \alpha}\right)$$

$$\leq 2^{d-2} - \left(2^{d-2} \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (d-2)}{2 \cdot 4 \cdot \ldots \cdot (d-1)}\right)$$

$$\leq 2^{d-2} - \frac{1}{2}\binom{d-1}{(d-1)/2}.$$

$\square$

We can now proceed with proving Theorem 4.

Proof of Theorem 4. To start with, observe that for $d = 1$ the statement clearly holds since there are only two vertices in H_1, which gives a payoff of zero for each player, and it is not possible to obtain more than zero vertices for any player (by definition of the diffusion game).

In the following we consider $d \geq 2$. Since the ordering of the players does not matter, we fix the strategy profile $(p_1 = x, p_2 = \overline{x}, p_3 = y, p_4 = \overline{y})$. Moreover, due to the symmetry of the hypercube, we only have to consider the case that Player 1 changes her strategy. Let us first determine the payoff of Player 1 for the above profile. According to Observation 1 and Lemma 7, we know that Player 1 obtains exactly those vertices $v \in \{0,1\}^d$ to which she has the unique minimum distance, that is, $\Delta(x,v) < \min\{\Delta(\overline{x},v), \Delta(y,v), \Delta(\overline{y},v)\}$. Recall that x and y are adjacent, that is, they differ in $\Delta(x,y) = 1$ bit. Therefore, $\Delta(x,v) < \Delta(y,v)$ implies that v equals x in that bit. Thus, we have $\Delta(y,v) = \Delta(x,v) + 1$, $\Delta(\overline{x},v) = d - \Delta(x,v)$, and $\Delta(\overline{y},v) = d - 1 - \Delta(x,v)$. Hence, v has to satisfy $\Delta(x,v) < (d-1)/2$ in order to satisfy $\Delta(x,v) < \Delta(\overline{y},v)$ and $\Delta(x,v) < \Delta(\overline{x},v)$. That is, v is allowed to differ from x in at most $\lfloor (d-1)/2 \rfloor$ of the $d-1$ bits where x is equal to y. The payoff of Player 1 is thus

$$\sum_{\ell=0}^{\lfloor(d-1)/2\rfloor} \binom{d-1}{\ell}.$$

Note that this payoff equals 2^{d-2} if d is even, and $2^{d-2} - \frac{1}{2}\binom{d-1}{(d-1)/2}$ if d is odd.

Now, let Player 1 choose an arbitrary vertex $p_1 \in \{0,1\}^d$. Clearly, we can assume that $p_1 \notin \{\overline{x}, y, \overline{y}\}$ since her payoff is zero otherwise. Hence, by Observation 1 and Lemma 7, we know again that the payoff of Player 1 equals the number of vertices in $\{v \in \{0,1\}^d \mid \Delta(x,v) < \min\{\Delta(\overline{x},v), \Delta(y,v), \Delta(\overline{y},v)\}\}$. By Lemma 8, we know that this number is at most 2^{d-2} if d is even, and at most $2^{d-2} - \frac{1}{2}\binom{d-1}{(d-1)/2}$ if d is odd. Therefore, Player 1 cannot increase her payoff by changing her strategy, which finishes the proof. $\square$

The existence of Nash equilibria on hypercubes for three players as well as for more than four players remains open in general.
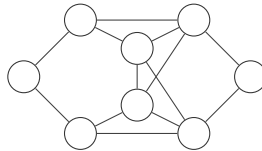
5 General Graphs

Figure 7: A graph on 8 vertices with no Nash equilibrium for two players.
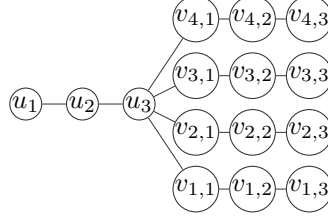
Figure 8: A tree with no Nash equilibrium for 9 players.

In this section, we study the existence of Nash equilibria on arbitrary graphs. Using computer simulations, we found that for two players, a Nash equilibrium exists on any graph with at most $n = 7$ vertices. For $n = 8$, we obtained the graph depicted in Figure 7, for which there is no Nash equilibrium for two players. As it is clear that adding isolated vertices to the graph in Figure 7 does not allow for a Nash equilibrium, we conclude the following.

Corollary 1. *For two players, there is a Nash equilibrium on each n-vertex graph if and only if $n \leq 7$.*

For more than two players, we can show the following.

Theorem 5. *For any $k > 2$ and any $n \geq \lfloor \frac{3}{2}k \rfloor + 2$, there exists a tree with n vertices such that there is no Nash equilibrium for k players.*

Proof. We describe a construction only for $n = \lfloor \frac{3}{2}k \rfloor + 2$, as we can add arbitrarily many isolated vertices without introducing a Nash equilibrium.

We first describe the construction for k being odd. We create one P_3, whose vertices we denote by u_1, u_2, and u_3, such that u_2 is the middle vertex of this P_3. For each $i \in [2, \lceil \frac{k}{2} \rceil]$, we create a copy of P_3, denoted by P_i, whose vertices we denote by $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$, such that $v_{i,2}$ is the middle vertex of P_i. For each $i \in [2, \lceil \frac{k}{2} \rceil]$, we connect $v_{i,1}$ to u_3. An example for $k = 9$ is depicted in Figure 8.

To see that there is no Nash equilibrium for the constructed graph, consider first strategy profiles for which u_3 is free (that is, no player chooses u_3). If also both u_1 and u_2 are free, then there exists some P_i with at least 2 occupied vertices (by the pigeon-hole principle). It is clear that at least one of the players occupying these vertices can increase her payoff by moving to u_3. If u_1 and u_2 are both occupied, then there exists some P_i with at most 1 occupied vertex (again, by the pigeon-hole principle). It is clear that the player occupying this vertex can increase her payoff by moving to u_3. If only one vertex out of u_1 and u_2 is occupied, then the player occupying this vertex can increase her payoff by moving to u_3. Therefore, we can assume that u_3 is occupied. In this case, it holds that if both u_1 and u_2 are free, then at least one player has a payoff of 1, and she gains more by moving to u_2. If both u_1 and u_2 are occupied, then at least one P_i has at most one occupied vertex, and this occupied vertex can only be $v_{i,1}$, therefore the player occupying u_2 gains more by moving to $v_{i,2}$. Lastly, if exactly one out of u_1 and u_2 is free, then at least one P_{i_1} has at most one

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occupied vertex, and this occupied vertex can only be $v_{i_1,1}$. Moreover, at least one P_{i_2} has at least two occupied vertices, therefore a player occupying one of these vertices gains more by moving to $v_{i_1,2}$. Therefore, this graph has no Nash equilibrium for k players.

For k being even, we create one P_2, whose vertices we denote by u_1, u_2. For each $i \in [2, \frac{k}{2} + 1]$, we create a copy of P_3, denoted by P_i, whose vertices we denote by $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$, such that $v_{i,2}$ is the middle vertex of P_i. For each $i \in [2, \frac{k}{2} + 1]$, we connect $v_{i,1}$ to u_2. This graph has no Nash equilibrium for k players, as can be verified by a similar analysis as above. $\qquad\square$

6   Conclusion

We studied a competitive diffusion game for three or more players on several classes of graphs, answering—as a main contribution—an open question concerning the existence of a Nash equilibrium for three players on grids [8] negatively. Further, extending previous results on hypercubes [3], we proved that Nash equilibria always exist for four player on d-dimensional hypercubes. With this work, we provide a first systematic study of this game for more than two players. However, there are several questions left open, of which we mention some here.

An immediate question (generalizing Theorem 3) is whether a Nash equilibrium exists for more than three players on a grid. Computer simulations lead us to the conjecture that there is no Nash equilibrium for four players on a grid of size larger than 6×6. A further immediate question (generalizing Theorem 4) is whether a Nash equilibrium exists for three players or more than four players on a d-dimensional hypercube. Also, giving a lower bound for the number of vertices n such that there is a graph with n vertices with no Nash equilibrium for k players is an interesting question as it is not clear that the upper bounds given in Theorem 5 are optimal. In other words, is it true that $n \leq \frac{3}{2}k + 1$ implies the existence of a Nash equilibrium for k players?

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