

Combinatorial Voter Control in Elections*

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Abstract

Voter control problems model situations such as an external agent trying to affect the result of an election by adding voters, for example by convincing some voters to vote who would otherwise not attend the election. Traditionally, voters are added one at a time, with the goal of making a distinguished alternative win by adding a minimum number of voters. In this paper, we initiate the study of combinatorial variants of control by adding voters: In our setting, when we choose to add a voter v , we also have to add a whole bundle $\kappa(v)$ of voters associated with v . We study the computational complexity of this problem for two of the most basic voting rules, namely the Plurality rule and the Condorcet rule.

1 Introduction

We study the computational complexity of control by adding voters [2, 24], investigating the case where the sets of voters that we can add have some combinatorial structure. The problem of election control by adding voters models situations where some agent (e.g., a campaign manager for one of the alternatives) tries to ensure a given alternative’s victory by convincing some undecided voters to vote. Traditionally, in this problem we are given a description of an election (that is, a set C of alternatives and a set V of voters who decided to vote), and also a set W of undecided voters (for each voter in $V \cup W$ we assume to know how this voter intends to vote which is given by a linear order of the set C ; we might have good approximation of this knowledge from preelection polls). Our goal is to ensure that our preferred alternative p becomes a winner, by convincing as few voters from W to vote as possible (provided that it is at all possible to ensure p ’s victory in this way).

Control by adding voters corresponds, for example, to situations where supporters of a given alternative make direct appeals to other supporters of the alternative to vote (for example, they may stress the importance of voting, or help with the voting process by offering rides to the voting locations, etc.). Unfortunately, in its traditional phrasing, control by adding voters

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does not model larger-scale attempts at convincing people to vote. For example, a campaign manager might be interested in airing a TV advertisement that would motivate supporters of a given alternative to vote (though, of course, it might also motivate some of this alternative’s enemies), or maybe launch viral campaigns, where friends convince their own friends to vote. It is clear that the sets of voters that we can add should have some sort of a combinatorial structure. For instance, a TV advertisement appeals to a particular group of voters and we can add all of them at the unit cost of airing the advertisement. A public speech in a given neighborhood will convince a particular group of people to vote at a unit cost of organizing the meeting or convincing a person to vote will “for free” also convince her friends to vote.

The goal of our work is to formally define an appropriate computational problem modeling a combinatorial variant of control by adding voters and to study its computational complexity. Specifically, we focus on the Plurality rule and the Condorcet rule, mainly because the Plurality rule is the most widely used rule in practice, and it is one of the few rules for which the standard variant of control by adding voters is solvable in polynomial time [2], whereas for the Condorcet rule the problem is polynomial-time solvable for the case of single-peaked elections [18]. For the case of single-peaked elections, in essence, all our hardness results for the Condorcet rule directly translate to all Condorcet-consistent voting rules, a large and important family of voting rules. We defer the formal details, definitions, and concrete results to the following sections. Instead, we state the high-level, main messages of our work:

- Many typical variants of combinatorial control by adding voters are intractable, but there is also a rich landscape of tractable cases.
- Assuming that voters have single-peaked preferences does not lower the complexity of the problem (even though it does so in many election problems [7, 10, 18]). On the contrary, assuming single-crossing preferences does lower the complexity of the problem.

We believe that our setting of combinatorial control, and—more generally—combinatorial voting, offers a very fertile ground for future research and we intend the current paper as an initial step.

Related Work. Bartholdi et al. [2] first studied the concept of election control by adding/deleting voters or alternatives in a given election. They studied the constructive variant of the problem, where the goal is to ensure a given alternative’s victory (and we focus on this variant of the problem as well). The destructive variant, where the goal is to prevent someone from winning, was introduced by Hemaspaandra et al. [24]. These papers focused on the Plurality rule and the Condorcet rule (and the Approval rule, for the destructive case of Hemaspaandra et al. [24]). Since then, many other researchers extended this study to a number of other rules and models [3, 16, 17, 19, 28, 27, 31, 33].

In all previous work on election control, the authors always assumed that one could affect each entity of the election at unit cost only. For example, one could add a voter at a unit cost and adding two voters always was twice as expensive as adding a single voter. Only the paper of Faliszewski et al. [19], where the authors study control in weighted elections, could be seen as an exception: One could think of adding a voter of weight w as adding a group of w voters of unit weight. On the one hand, the weighted election model does not allow one to express rich combinatorial structures as those that we study here, and on the other hand, in our study we consider unweighted elections only (though adding weights to our model would be seamless).

The specific combinatorial flavor of our model has been inspired by the seminal work of

Rothkopf et al. [35]¹ on *combinatorial auctions* (see, e.g., Sandholm [36] for additional information). There, bidders can place bids on combinations of items such that the bid on the combination of a set of items might be less than, equal to, or greater than the sum of the individual bids on each element from the same set of items. While in combinatorial auctions one “bundles” items to bid on, in our scenario one bundles voters.

In the computational social choice literature, combinatorial voting is typically associated with scenarios where voters express opinions over a set of items that themselves have a specific combinatorial structure (typically, one uses CP-nets to model preferences over such alternative sets [6]). For example, Conitzer et al. [11] studied a form of control in this setting and Mattei et al. [30] studied bribery problems. In contrast, we use the standard model of elections where all alternatives and preference orders are given explicitly, but we have a combinatorial structure of the sets of voters that can be added.

2 Preliminaries

We assume familiarity with standard notions regarding algorithms and complexity theory. For each nonnegative integer z , we write $[z]$ to mean $\{1, \dots, z\}$.

Elections. An election $E := (C, V)$ consists of a set C of m alternatives and a set V of $|V|$ voters $v_1, v_2, \dots, v_{|V|}$. Each voter v has a linear order \succ_v over the set C , which we call a *preference order*. For example, let $C = \{c_1, c_2, c_3\}$ be a set of alternatives. The preference order $c_1 \succ_v c_2 \succ_v c_3$ of voter v indicates that v likes c_1 the best (1st position), then c_2 , and c_3 the least (3rd position). We call a voter $v \in V$ a *c-voter* if c is at the first position of her preference order. Given a subset $C' \subseteq C$ of alternatives, if not stated explicitly, we write $\langle C' \rangle$ to denote an arbitrary but fixed preference order over C' .

Voting Rules. A voting rule \mathcal{R} is a function that given an election E outputs a (possibly empty) set $\mathcal{R}(E) \subseteq C$ of the (tied) election winners. We study the Plurality rule and the Condorcet rule. Given an election, the *Plurality score* of an alternative c is the number of voters that have c at the first position in their preference orders; an alternative is a Plurality winner if it has the maximum Plurality score. An alternative c is a *Condorcet winner* [12] if it beats all other alternatives in head-to-head contests. That is, c is a *Condorcet winner* in election $E = (C, V)$ if for each alternative $c' \in C \setminus \{c\}$ it holds that $|\{v \in V \mid c \succ_v c'\}| > |\{v \in V \mid c' \succ_v c\}|$. Condorcet’s rule elects the (unique) Condorcet winner if it exists, and returns an empty set otherwise. A voting rule is *Condorcet-consistent* if it elects a Condorcet winner when there is one (however, if there is no Condorcet winner, then a Condorcet-consistent rule is free to provide any set of winners).

Domain Restrictions. Intuitively, an election is *single-peaked* [5] if it is possible to order the alternatives on a line in such a way that for each voter v the following holds: If c is v ’s most preferred alternative, then for each two alternatives c_i and c_j that both are on the same side of c (with respect to the ordering of the alternatives on the line), among c_i and c_j , v prefers the one closer to c . For example, single-peaked elections arise when we view the alternatives

¹According to google scholar, accessed April 2014, cited more than 1000 times.

on the standard political left-right spectrum and voters form their preferences based solely on alternatives' positions on this spectrum. Formally, we have the following definition.

Definition 1 (Single-peaked elections). *Let C be a set of alternative and let L be a linear order over C (referred to as the societal axis). We say that a preference order \succ (over C) is single-peaked with respect to L if for each three alternative $x, y, z \in C$ it holds that:*

$$((x L y L z) \vee (z L y L x)) \implies ((x \succ y) \implies (y \succ z)).$$

An election (C, V) is single-peaked with respect to L if the preference order of each voter in V is single-peaked with respect to L . An election is single-peaked if there is a societal axis with respect to which it is single-peaked.

There are polynomial-time algorithms that given an election decide if it is single-peaked and, if so, provide a societal axis for it [1, 15]. *Single-crossing* elections, introduced by Roberts [34], capture a similar idea as single-peaked ones, but from a different perspective. This time we assume that it is possible to order the voters so that for each two alternatives a and b either all voters rank a and b identically, or there is a single point along this order where voters switch from preferring one of the alternatives to preferring the other one. Formally, we have the following definition.

Definition 2 (Single-crossing elections). *An election $E = (C, V)$ is single-crossing if there is an order L over V such that for each two alternatives x and y and each three voters v_1, v_2, v_3 such that $v_1 L v_2 L v_3$ it holds that:*

$$(x \succ_{v_1} y \wedge x \succ_{v_3} y) \implies x \succ_{v_2} y.$$

As for the case of single-peakedness, there are polynomial-time algorithms that decide if an election is single-crossing and, if so, produce the voter order witnessing this fact [14, 8].

Combinatorial Bundling Functions. Given a voter set X , a combinatorial bundling function $\kappa : X \rightarrow 2^X$ (abbreviated as *bundling function*) is a function assigning to each voter a subset of voters. For convenience, for each subset $X' \subseteq X$, we let $\kappa(X') = \bigcup_{x \in X'} \kappa(x)$. For $x \in X$, $\kappa(x)$ is called x 's *bundle* (and for this bundle, x is called its *leader*). We assume that $x \in \kappa(x)$ and so $\kappa(x)$ is never empty. We typically write b to denote the maximum bundle size under a given κ (which will always be clear from context). Intuitively, we use combinatorial bundling functions to describe the sets of voters that we can add to an election at a unit cost. For example, one can think of $\kappa(x)$ as the group of voters that join the election under x 's influence. We represent bundling functions explicitly: For each voter x we list the voters in $\kappa(x)$.

We are interested in various special cases of bundling functions. We say that κ is *leader-anonymous* if for each two voters x and y with the same preference order $\kappa(x) = \kappa(y)$ holds. Furthermore, κ is *follower-anonymous* if for each two voters x and y with the same preference orders, and each voter z , it holds that $x \in \kappa(z)$ if and only if $y \in \kappa(z)$. We call κ *anonymous* if it is both leader-anonymous and follower-anonymous. One possible way of thinking about an anonymous bundling function is that it is a function assigning to each preference order appearing in the input a subset of the preference orders appearing in the input. For example, anonymous bundling functions naturally model scenarios such as airing TV advertisements that appeal to particular groups of voters.

The swap distance between two voters v_i and v_j is the minimum number of swaps of consecutive alternatives that transform v_i 's preference order into that of v_j . Given a number $d \in \mathbb{N}$, we call κ a *full- d bundling function* if for each $x \in X$, $\kappa(x)$ is exactly the set of all $y \in X$ such that the swap distance between the preference orders of x and y is at most d .

We introduce the concept of a *bundling graph* of an election, which, roughly speaking, models how the bundles of two voters interact with each other.

Definition 3 (Bundling graphs). *Given an input instance to C-CC-AV, the bundling graph is a simple and directed graph $G = (V(G), E(G))$. For each voter x there is a vertex $u_x \in V(G)$, and for each two distinct voters y and z such that $y \in \kappa(z)$ there is an arc $(u_z \rightarrow u_y) \in E(G)$.*

For arbitrary bundling functions, the bundling graph is a directed graph. However, if κ is a full- d bundling function, that is, for each voter v , $\kappa(v)$ contains all the voters at swap distance d , then the bundling graph can be thought of as being undirected, due to the following.

Lemma 1. *If κ is a full- d bundling function, then for any unregistered voter x and any $y \in \kappa(x)$, it holds that $x \in \kappa(y)$.*

Proof. To see why the statement holds, notice that for any two voters x and y , if $y \in \kappa(x)$, then the swap distance between x and y is at most d , therefore, because κ is a full- d bundling function, x must be in $\kappa(y)$. This implies that for any arc $(u_x \rightarrow u_y)$ in the bundling graph, the corresponding arc $(u_y \rightarrow u_x)$ is also present in the bundling graph, therefore, we can treat the bundling graph as an undirected graph. \square

Notice that this is not always the case for an arbitrary bundling function. For instance, $\kappa(x) = \{x, y\}$, $\kappa(y) = \{y\}$ is a valid possibility for a bundling function.

Central Problem. We consider the following problem for a given voting rule \mathcal{R} :

\mathcal{R} COMBINATORIAL CONSTRUCTIVE CONTROL BY ADDING VOTERS

(\mathcal{R} -C-CC-AV)

Input: An election $E = (C, V)$, a set W of (unregistered) voters with $V \cap W = \emptyset$, a bundling function $\kappa : W \rightarrow 2^W$, a preferred alternative $p \in C$, and a bound $k \in \mathbb{N}$.

Question: Is there a subset of voters $W' \subseteq W$ of size at most k such that $p \in \mathcal{R}(C, V \cup \kappa(W'))$, where $\mathcal{R}(C, X)$ is the set of winners of the election (C, X) under the rule \mathcal{R} ?

We note that we use here a so-called nonunique-winner model. For a control action to be successful, it suffices for p to be one of the tied winners. Throughout this work, we refer to the set W' of voters such that p wins election $(C, V \cup \kappa(W'))$ as the solution and denote k as the solution size.

\mathcal{R} -C-CC-AV is a generalization of the well-studied problem \mathcal{R} CONSTRUCTIVE CONTROL BY ADDING VOTERS (\mathcal{R} -CC-AV) (in which κ is fixed so that for each $w \in W$ we have $\kappa(w) = \{w\}$). The non-combinatorial problem CC-AV is polynomial-time solvable for the Plurality rule [2], but is NP-complete for the Condorcet rule [28], therefore:

Observation 1. CONDORCET-C-CC-AV is NP-hard even if the maximum bundle size b is one.

	m	n	k	b	d
# alternatives (m)	Non-anonymous: $W[2]$ -h wrt. k even if $m = 2$ [Thm. 2] Anonymous: ILP-FPT wrt. m [Thm. 3]				
# unreg. voters (n)	FPT wrt. n				
solution size (k)			XP [Obs. 2] Anonymous & $b = 3$: $W[1]$ -h wrt. k [Thm. 1]		Single-peaked & full-1 κ : $W[1]$ -h wrt. k [Thm. 8]
max. bundle size (b)			$b = 2$: NP-h [Thm. 4] and P for full- d κ [Thm. 5] $b = 3$: NP-h even for full- d κ [Thm. 6] $b \geq 4$: NP-h even for full-1 κ [Thm. 7]		
max. swap dist. (d)			$d = 1$: $W[1]$ -h wrt. k [Thm. 8] Single-crossing & full- d κ : P [Thm. 9]		

Table 1: **Computational complexity classification of Plurality-C-CC-AV** (since the non-combinatorial problem CC-AV is already NP-hard for Condorcet’s rule, we concentrate here on the Plurality rule). Each row and column in the table corresponds to a parameter such that each cell contains results for the two corresponding parameters combined. Due to symmetry, there is no need to consider the cells under the main diagonal, therefore they are painted in gray. ILP-FPT means FPT based on a formulation as an integer linear program.

Parameterized Complexity. An instance (I, k) of a parameterized problem consists of the actual instance I and an integer k being the *parameter* [13, 20, 32]. A parameterized problem is called *fixed-parameter tractable* (is in FPT) if there is an algorithm solving it in $f(k) \cdot |I|^{O(1)}$ time, for an arbitrary computable function f only depending on parameter k , whereas an algorithm with running-time $|I|^{f(k)}$ only shows membership in the class XP (clearly, $FPT \subseteq XP$). If a parameterized problem is fixed-parameter tractable due to a formulation as integer linear program (ILP), then we say that this problem is in ILP-FPT. One can show that a parameterized problem L is (presumably) not fixed-parameter tractable by devising a *parameterized reduction* from a $W[1]$ -hard or a $W[2]$ -hard problem (such as CLIQUE or SET COVER parameterized by the “solution size”) to L . A parameterized reduction from a parameterized problem L to another parameterized problem L' is a function that, given an instance (I, k) , computes in $f(k) \cdot |I|^{O(1)}$ time an instance (I', k') , such that $k' \leq g(k)$ and $(I, k) \in L \Leftrightarrow (I', k') \in L'$. Betzler et al. [4] survey parameterized complexity investigations in voting.

Our Contributions. We introduce a new model for combinatorial control in voting. As \mathcal{R} -C-CC-AV is generally NP-hard even for \mathcal{R} being the Plurality rule, we show several fixed-parameter tractability results for some of the natural parameterizations of \mathcal{R} -C-CC-AV; we almost completely resolve the complexity of C-CC-AV, for the Plurality rule and the Condorcet rule, as a function of the maximum bundle size b and the maximum distance d from a voter v to the farthest element of her bundle. Further, we show that the problem remains hard even when restricting the elections to be single-peaked, but that it is polynomial-time solvable when we focus on single-crossing elections. Our results for Plurality elections are summarized in Table 1.

3 Complexity for Unrestricted Elections

In this section we provide our results for the case of unrestricted elections, where voters may have arbitrary preference orders. In the next section we will consider single-peaked and single-crossing elections that only allow “reasonable” preference orders.

3.1 Number of Voters, Number of Alternatives, and Solution Size

We start our discussion by considering parameters “the number m of alternatives”, “the number n of unregistered voters”, and “the solution size k ”. A simple brute-force algorithm, checking all possible combinations of k bundles, proves that both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV are in XP for parameter k , and in FPT for parameter n (the latter holds because $k \leq n$). Indeed, the same result holds for all voting rules that are XP/FPT-time computable for the respective parameters.

Observation 2. *Both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV are solvable in $O(n^k \cdot n \cdot m \cdot \text{winner})$ time, where winner is the complexity of determining Plurality/Condorcet winners.*

The XP result for PLURALITY-C-CC-AV with respect to the parameter k probably cannot be improved to fixed-parameter tractability. Indeed, for parameter k we show that the problem is W[1]-hard, even for anonymous bundling functions and for maximum bundle size three.

Theorem 1. *PLURALITY-C-CC-AV is NP-hard and W[1]-hard when parameterized by the solution size k , even when the maximum bundle size b is three and the bundling function is anonymous.*

Proof. We provide a parameterized reduction from the W[1]-hard problem CLIQUE parameterized by the parameter h [13], which asks for the existence of a size- h clique in an input graph G .

CLIQUE

Input: An undirected graph $G = (V(G), E(G))$ and $h \in \mathbb{N}$.

Question: Does G admit a size- h clique, that is, a size- h vertex subset $U \subseteq V(G)$ such that $G[U]$ is complete?

Let (G, h) be a CLIQUE instance. Without loss of generality, we assume that G is connected, that $h \geq 3$, and that each vertex in G has degree at least $h - 1$. We construct an election $E = (C, V)$ with $C := \{p, w, g\} \cup \{c_e \mid e \in E(G)\}$. The registered voter set V consists of $\binom{h}{2} + h$ voters each with preference order $w \succ \langle C \setminus \{w\} \rangle$, another $\binom{h}{2}$ voters each with preference order $g \succ \langle C \setminus \{g\} \rangle$, and another h voters each with preference order $p \succ \langle C \setminus \{p\} \rangle$. For each vertex $u \in V(G)$, we define $C(u) := \{c_e \mid e \in E(G) \wedge u \in e\}$, and construct the set W of unregistered voters as follows:

- (i) For each vertex $u \in V(G)$, we add an unregistered g -voter w_u with preference order $g \succ \langle C(u) \rangle \succ \langle C \setminus (\{g\} \cup C(u)) \rangle$, and we set $\kappa(w_u) = \{w_u\}$. We call these unregistered voters *vertex voters*.
- (ii) For each edge $e = \{u, u'\} \in E(G)$, we add an unregistered p -voter w_e with preference order $p \succ c_e \succ \langle C \setminus \{p, c_e\} \rangle$, and we set $\kappa(w_e) = \{w_u, w_{u'}, w_e\}$. We call these unregistered voters *edge voters*.

Since all the unregistered voters have different preference orders (this is so because G is connected, $h \geq 3$, and each vertex has degree at least $h - 1$), every bundling function for our instance, κ included, is anonymous. To finalize our construction, we set $k := \binom{h}{2}$.

We show that G has a size- h clique if and only if $(E = (C, V), W, \kappa, p, k)$ is a yes instance for PLURALITY-C-CC-AV. For the “if” part, suppose that there is a subset W' of at most k voters such that p wins the election $(C, V \cup \kappa(W'))$. We show that the vertex set $U' := \{u \in V(G) \mid w_e \in W' \wedge u \in e\}$ is a size- h clique for G . First, we observe that p needs at least $\binom{h}{2}$ points to become a winner because of the difference in scores between the initial winner w and p . By our construction, only bundles that include the edge voters give points to p and each of such bundles gives p exactly one point. Since we can add at most $k = \binom{h}{2}$ bundles, we must add exactly k bundles of the edge voters. This means that $E(G[U'])$ contains at least k edges. However, in order to ensure p 's victory, $\kappa(W')$ may only give at most h additional points to g . This means that U' contains at most h vertices. With $|E(G[U'])| \geq k$, we conclude that U' is of size h and, hence, is a size- h clique for G .

For the “only if” part, suppose that $U' \subseteq V(G)$ is a size- h clique for G . We construct the subset W' by adding to it any edge voter w_e with $e \in E(G[U'])$. Obviously, $|W'| = k$. Now it is easy to check that p co-wins with both w and g the election $(C, V \cup \kappa(W'))$ with score $\binom{h}{2} + h + 1$. \square

If we drop the anonymity requirement for the bundling function, then we obtain a stronger intractability result. For parameter k , the problem becomes W[2]-hard, even for two alternatives. This is quite remarkable because typically election problems with a small number of alternatives are easy (they can be solved either through brute-force attacks or through integer linear programming attacks employing the famous FPT algorithm of Lenstra [26]; see the survey of Betzler et al. [4] for examples, but note that there are also known examples of problems where a small number of alternatives does not seem to help [9]). Further, since our proof uses only two alternatives, it applies to almost all natural voting rules: For two alternatives almost all of them (including the Condorcet rule) are equivalent to the Plurality rule. The reduction is from the W[2]-complete problem SET COVER parameterized by the solution size [13].

Theorem 2. *Both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV parameterized by the solution size k are W[2]-hard, even for two alternatives.*

Proof. We provide a parameterized reduction from the W[2]-complete problem SET COVER parameterized by the parameter h [13].

SET COVER

Input: A collection $\mathcal{S} = \{S_1, \dots, S_m\}$ of subsets of the universe $X = \{x_1, \dots, x_n\}$ and $h \in \mathbb{N}$.

Question: Is there a size- h subset $\mathcal{S}' \subseteq \mathcal{S}$ that covers the universe, that is, $\bigcup \mathcal{S}' = X$?

Let (\mathcal{S}, X, h) be a SET COVER instance. We construct an election $E = (C, V)$ with $C = \{p, g\}$. The registered voter set V consists of only $(n - k)$ g -voters. We construct the unregistered voter set W as follows:

- (i) For each element $x_i \in X$, we construct one p -voter, denoted by w_i^x (called *element-voter*), and two g -voters, denoted by w_{i1}^d and w_{i2}^d (called *dummy-voters*), and we set $\kappa(w_i^x) = \{w_{i1}^d, w_{i2}^d\}$ and set $\kappa(w_{i1}^d) = \kappa(w_{i2}^d) = \{w_i^x, w_{i2}^d\}$.

- (ii) For each set S_j , we construct one g -voter, denoted by w_j^S (called *set-voter*), and we set $\kappa(w_j^S) = \{w_j^S\} \cup \{w_i^x \mid x_i \in S_j\}$. That is, the bundle for the voter corresponding to a set contains all of the voters corresponding to the elements of the set.

Finally, we set $k := h$ and let d be arbitrary.

The construction is obviously a parameterized reduction, and we show now that there is a size- h subset $\mathcal{S}' \subseteq \mathcal{S}$ that covers the universe if and only if there is a size- k subset W' of unregistered voters, such that if added (with their respective bundles) to the election, p becomes a Plurality winner of the election.

For the “if” part, suppose that there is such a size- k subset W' . Also, if there are some *element-voters* in the solution, then we can simply remove them, as they do not help p win, due to the *dummy-voters*. The only way to achieve the score increase of $n - k$ for p is to have all of the *element-voters* added to the election, and this can be done only by covering all of the universe, with at most k *set-voters*; therefore, the solution corresponds to a set covering of the universe.

For the “only if” part, given a size- h subset $\mathcal{S}' \subseteq \mathcal{S}$ that covers the universe, we choose, for every $S_j \in \mathcal{S}'$, its respective voter w_j^S , and add it to the election. This gives a size- k subset W' of unregistered voters, which easily can be verified to result in p winning the election.

As for the Condorcet rule, we use the same unregistered voters as defined above and we construct the original election with $(2n - k - 1)$ g -voters and $(n - k)$ p -voters. \square

The above proof uses the non-anonymity of the bundling function in a crucial way. If we require the bundling function to be anonymous, then C-CC-AV can be formulated as an integer linear program where the number of variables and the number of constraints are bounded by some function in the number m of alternatives. The idea behind this is that with anonymity we can formulate our problem through an integer linear program where the number of variables and the number of constraints are bounded by some function in m . Such integer linear programs are in FPT with respect to the number of variables [26].

Theorem 3. *For anonymous bundling functions, both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV parameterized by the number m of alternatives are fixed-parameter tractable.*

Proof. We describe an integer linear program (ILP) with at most $m!$ variables and at most $m! + m$ constraints that solves both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV. Fixed-parameter tractability then follows, since any ILP with ρ variables and L input bits is solvable in $O(\rho^{2.5\rho + o(\rho)} L)$ time ([26] and [25]).

With m alternatives, there are at most $m!$ voters with pairwise different preference orders in a given election. For each alternative $a \in C$, let $s(a)$ be its initial score. Since the voters are anonymous, there are at most $m!$ different bundles. Furthermore, we can assume that all voters in W' have pairwise different preference orders (this is because, due to anonymity, there is no additional gain of adding two voters with the same preference order).

Let $\succ_1, \succ_2, \dots, \succ_{m!}$ be an ordering of all of the possible preference orders over m alternatives. For $i \in [m!]$, let N_i be the number of voters with preference order \succ_i in W . For $i \in [m!]$ and $j \in [m!]$, let M_i^j have value 1 if there is a voter with preference order \succ_j that is in the bundle of a voter whose preference order is \succ_i , and otherwise 0. For each alternative a and each $i \in [m!]$, let $B_i^a = 1$ if alternative a is at the first position in the preference order \succ_i (that is, i is a a -voter), and otherwise $B_i^a = 0$.

For each preference order \succ_i , $i \in [m!]$, we introduce one boolean variable x_i , with the intent that the value of x_i will be 1 if and only if W' contains a voter with preference order i . Indeed, an integer linear program usually tries to minimize or maximize a certain function, while here, we write the integer linear program as simply a feasibility problem. It can be easily rewritten with a minimization function instead. Now we are ready to state the integer linear program.

$$\sum_{i \in [m!]} x_i \leq k, \quad (1)$$

$$x_i \leq N_i \quad \forall i \in [m!], \quad (2)$$

$$\sum_{i \in [m!]} \sum_{j \in [m!]} (B_j^a - B_j^p) \cdot N_j \cdot M_i^j \cdot x_i < s(p) - s(a) \quad \forall a \in C \setminus \{p\}. \quad (3)$$

Constraint (1) ensures that at most k voters are added to W' . Constraint (2) ensures that the voters added to W' are available in W . Constraint (3) ensures that no other alternative has a higher Plurality score than alternative p . It can be easily verified that there is a solution for this integer linear program if and only if there is a solution to the input instance. \square

3.2 Combinatorial Parameters

We focus now on the complexity of PLURALITY-C-CC-AV as a function of two combinatorial parameters: (a) the maximum swap distance d between the leader and his followers in one bundle, and (b) the maximum size b of each voter's bundle.

Specifically, we show that if κ is a d -bounded bundling function (that is, it is not required to contain all voters at a given distance), then Plurality-C-CC-AV is polynomial-time solvable if the maximum bundle size is one, but if the maximum bundle size is two, then Plurality-C-CC-AV is NP-hard. However, if κ is a full- d bundling function (that is, it is required to contain all voters at a given distance), then Plurality-C-CC-AV is polynomial-time solvable if the maximum bundle size is two, but if the maximum bundle size is three, then Plurality-C-CC-AV is NP-hard.

First, if $b = 1$, then C-CC-AV reduces to CC-AV and, thus, can be solved by a greedy algorithm in polynomial time [2].

Observation 3. *If the maximum bundle size b is one, then PLURALITY-C-CC-AV is polynomial-time solvable.*

However, for arbitrary bundling functions, PLURALITY-C-CC-AV becomes intractable as soon as $b = 2$.

Theorem 4. *PLURALITY-C-CC-AV is NP-hard even if the maximum bundle size b is two.*

Proof. We provide a reduction from a restricted variant of the NP-complete problem 3SAT, where each clause has either two or three literals, each variable occurs exactly four times, twice as a positive literal, and twice as a negative literal.

(2,2)-3SAT

Input: A collection \mathcal{C} of clauses over the set $\mathcal{X} = \{x_1, \dots, x_n\}$ of variables such that each clause has either two or three literals, and each variable appears exactly four times, twice as a positive literal and twice as a negative literal.

Question: Is there a truth assignment that satisfies all the clauses in \mathcal{C} ?

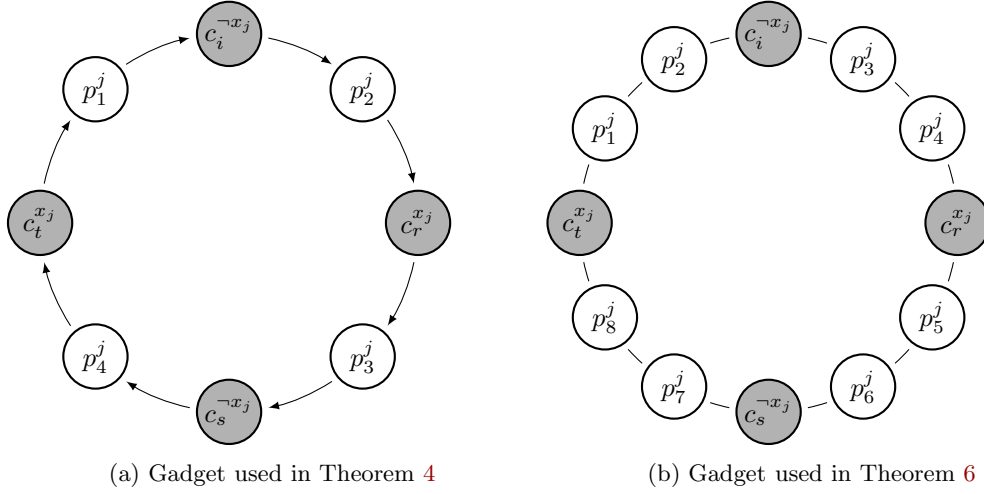


Figure 1: Part of the construction used in Theorem 4 and Theorem 6. Specifically, we show the cycle corresponding to variable x_j which occurs as a negative literal in clauses C_i and C_s and as a positive literal in clauses C_r and C_t .

This variant is still NP-hard since from Tovey [37, Theorem 2.1], one obtained NP-hardness for 3SAT where each clause has either two or three literals, each variable occurs either two or three times, and at most one time as a negative literal.

We can reduce from this problem to (2,2)-3SAT as follows. First, we assume that no variable appears only positively, because if this is the case, we can just set it to true and remove it. For each variable x_i that appears three times (two times positively and one time negatively), we add one new variable y_i , and two new clauses $\{\neg x_i, \neg y_i, \neg y_i\}$ and $\{\neg y_i, \neg y_i\}$. For each variable x_i that appears two times (one time positively and one time negatively), we add one new clause $\{\neg x_i, x_i\}$. It can be verified that the original instance is a yes-instance if and only if the newly constructed instance is a yes-instance for (2,2)-3SAT.

Now, given a (2,2)-3SAT instance $(\mathcal{C}, \mathcal{X})$, where \mathcal{C} is the set of clauses over the set of variables \mathcal{X} , we construct an election (C, V) . We set $k := 4|\mathcal{X}|$, and construct the set C of alternatives to be $C := \{p, w\} \cup \{c_i \mid C_i \in \mathcal{C}\}$, where the c_i are called the *clause* alternatives. We construct the set V of registered voters such that the initial score of w is $4|\mathcal{X}|$, the initial score of the clause alternative c_i is $4|\mathcal{X}| - |C_i| + 1$ (where $|C_i|$ is the number of literals that clause C_i contains), and the initial score of p is zero. We construct the set W of unregistered voters as follows (throughout the rest of the proof, we will often write ℓ_j to refer to a literal that contains variable x_j ; depending on the context, ℓ_j will mean either x_j or $\neg x_j$ and the exact meaning will always be clear):

1. for each variable $x_j \in \mathcal{X}$, we construct four p -voters, denoted by $p_1^j, p_2^j, p_3^j, p_4^j$; we call such voters *variable* voters.
2. for each clause $C_i \in \mathcal{C}$ and each literal ℓ contained in C_i , we construct a c_i -voters, denoted by c_i^ℓ ; we call such voter a *clause* voter. Note that clause C_i has exactly $|C_i|$ corresponding clause voters.

We define the assignment function κ as follows: For each variable $x_j \in \mathcal{X}$ that occurs as a negative literal ($\neg x_j$) in clauses C_i and C_s , and as a positive literal (x_j) in clauses C_r and C_t , we set

$$\begin{aligned}\kappa(p_1^j) &:= \{p_1^j, c_i^{\neg x_j}\}, & \kappa(c_i^{\neg x_j}) &:= \{c_i^{\neg x_j}, p_2^j\}, \\ \kappa(p_2^j) &:= \{p_2^j, c_r^{x_j}\}, & \kappa(c_r^{x_j}) &:= \{c_r^{x_j}, p_3^j\}, \\ \kappa(p_3^j) &:= \{p_3^j, c_s^{\neg x_j}\}, & \kappa(c_s^{\neg x_j}) &:= \{c_s^{\neg x_j}, p_4^j\}, \\ \kappa(p_4^j) &:= \{p_4^j, c_t^{x_j}\}, & \kappa(c_t^{x_j}) &:= \{c_t^{x_j}, p_1^j\}.\end{aligned}$$

Notice that the bundling graph (Definition 3) contains a cycle corresponding to each variable, as depicted in Figure 1a.

The general idea is that in order to let p win, all p -voters must be in $\kappa(W')$ and no clause alternative c_i should gain more than $(|C_i| - 1)$ points. More formally, we show now that $(\mathcal{C}, \mathcal{X})$ has a satisfying truth assignment if and only if there is a size- k subset $W' \subseteq W$ such that p wins election $(C, V \cup \kappa(W'))$ (recall that $k = 4|\mathcal{X}|$).

For the “if” direction, let $\beta : \mathcal{X} \rightarrow \{T, F\}$ be a satisfying truth assignment function for $(\mathcal{C}, \mathcal{X})$. Intuitively, β will guide us through constructing the set W' in the following way: First, for each variable x_j , we put into W' those voters $c_i^{\ell_j}$ for whom β sets ℓ_j to false. This way in $\kappa(W')$ we include $2|\mathcal{X}|$ p -voters and, for each clause c_i , at most $(|C_i| - 1)$ c_i -voters. The former is true because exactly $|\mathcal{X}|$ literals are set to false by β , each literal is included in exactly two clauses, and adding each $c_i^{\ell_j}$ into W' also includes a unique p -voter into $\kappa(W')$; the latter is true because if β is a satisfying truth assignment then each clause C_i contains at most $(|C_i| - 1)$ literals set to false. Then, for each clause voter $c_i^{\ell_j}$ already in W' , we also add the voter p_a^j , $1 \leq a \leq 4$, that contains $c_i^{\ell_j}$ in his or her bundle. This way we include in $\kappa(W')$ additional $2|\mathcal{X}|$ p -voters without increasing the number of clause voters included. Formally, we define W' as follows:

$$\begin{aligned}W' &:= \{c_i^{\neg x_j}, p_a^j \mid \neg x_j \in C_i \wedge \beta(x_j) = T \wedge c_i^{\neg x_j} \in \kappa(p_a^j)\} \cup \\ &\quad \{c_i^{x_j}, p_a^j \mid x_j \in C_i \wedge \beta(x_j) = F \wedge c_i^{x_j} \in \kappa(p_a^j)\}.\end{aligned}$$

As per our intuitive argument, one can verify that all p -voters are contained in $\kappa(W')$ and each clause alternative c_i gains at most $(|C_i| - 1)$ points.

For the “only if” part, let W' be a subset of voters such that p wins election $(C, V \cup \kappa(W'))$.

First, we make the following observation. Let x_j be some variable and consider clauses C_i and C_s where literal $\neg x_j$ appears, and clauses C_r and C_t where literal x_j appears. We claim that we can assume that $\kappa(W')$ contains at most two voters among $c_i^{\neg x_j}$, $c_s^{\neg x_j}$, $c_r^{x_j}$, and $c_t^{x_j}$. First, let us assume that $\kappa(W')$ contains all of these voters. Since p is a winner of election $(C, V \cup \kappa(W'))$, it must be that $\kappa(W')$ also contains all four p -voters of the form p_a^j , $1 \leq a \leq 4$. This means that W' includes at least four voters from the set:

$$Q_j = \{c_i^{\neg x_j}, c_s^{\neg x_j}, c_r^{x_j}, c_t^{x_j}, p_1^j, p_2^j, p_3^j, p_4^j\}.$$

In effect, we can replace W' with W'' defined as

$$W'' := (W' \setminus Q_j) \cup \{c_r^{x_j}\} \cup \{p_a^j \mid c_r^{x_j} \in \kappa(p_a^j)\} \cup \{c_t^{x_j}\} \cup \{p_a^j \mid c_t^{x_j} \in \kappa(p_a^j)\}.$$

Compared to W' , W'' contains at most as many voters as W' does, $\kappa(W'')$ contains the same number of p -voters as $\kappa(W')$ does, and for each clause alternative c , $\kappa(W'')$ contains no more

c -voters than $\kappa(W')$ does. Thus, p is still a winner of election $(C, V \cup \kappa(W''))$ and W'' is a valid solution.

Furthermore, let us assume that exactly three voters among $c_i^{\neg x_j}$, $c_s^{\neg x_j}$, $c_r^{x_j}$, and $c_t^{x_j}$ are included in $\kappa(W')$. For the sake of concreteness, let $c_i^{\neg x_j}$ be the voter not in $\kappa(W')$. We use a similar argument as before. Specifically, since p is a winner of $(C, V \cup \kappa(W'))$, W' must include at least four voters among those in Q_j . Replacing W' with W'' (defined in the previous paragraph) works again. Notice that, replacing W' with W'' would also work if $c_s^{\neg x_j}$ was the voter not included in W' ; if either $c_r^{x_j}$ or $c_t^{x_j}$ were the not-included voter, we would replace W' with

$$W''' := (W' \setminus Q_j) \cup \{c_i^{\neg x_j}\} \cup \{p_a^j \mid c_i^{\neg x_j} \in \kappa(p_a^j)\} \cup \{c_s^{\neg x_j}\} \cup \{p_a^j \mid c_s^{\neg x_j} \in \kappa(p_a^j)\}.$$

We will now argue that for each variable x_j , $\kappa(W')$ contains either the two voters of the form c^{x_j} or the two voters of the form $c^{\neg x_j}$. We start by observing that for each two clauses that contain the same variable but not the same literal, at least one corresponding clause voter must be added to the election (otherwise $\kappa(W')$ would not contain all the unregistered p -voters). Thus, if one clause voter is not contained in $\kappa(W')$, then both of its “neighboring” (in the sense of being adjacent in the bundling graph, depicted in Figure 1a) clause voters must be included in $\kappa(W')$. Together with the arguments from previous paragraphs, this means that for each variable x_j , $\kappa(W')$ either contains the two voters of the form c^{x_j} or the two voters of the form $c^{\neg x_j}$.

This is critical for the sanity of the truth assignment function β we will construct now. In order to let p win, all p -voters must be added to the election. This means that for each two clauses that contain the same variable but not the same literal, at least one corresponding clause voter must be added to the election.

We set $\beta : \mathcal{X} \rightarrow \{T, F\}$ such that $\beta(x_j) := T$ if there is a clause voter $c_i^{x_j} \notin \kappa(W')$, and $\beta(x_j) := F$ if there is a clause voter $c_i^{\neg x_j} \notin \kappa(W')$. Following the previous arguments, function β is well-defined. It is a satisfying truth assignment function for (C, \mathcal{X}) because for each clause C_i , by the fact that p is a winner in election $(C, V \cup \kappa(W'))$, we have that $\kappa(W')$ contains at most $(|C_i| - 1)$ c_i -voters for each clause alternative c_i . This is possible only if each clause contains at least one literal ℓ such that β sets ℓ to truth. \square

The situation is different for full- d bundling functions, because we can extend the greedy algorithm by Bartholdi et al. [2] to bundles of size two.

Theorem 5. *If κ is a full- d bundling function and the maximum bundle size b is two, then PLURALITY-C-CC-AV is polynomial-time solvable.*

Proof. Since $b = 2$ and κ is a full- d bundling function, the bundling graph has maximum degree one. Therefore, it contains only isolated vertices and disjoint edges. We first add the disjoint edges with both end-points corresponding to p -voters. If we have some more budget, then we add isolated vertices corresponding to p -voters. We are left only with isolated vertices corresponding to non- p -voters, which we throw away, disjoint edges with both end-points corresponding to non- p -voters, which we also throw away, and disjoint edges with one end-point corresponding to a p -voter and another end-point corresponding to a non- p -voter, which we treat now. Specifically, we add these disjoint edges with one end-point corresponding to a p -voter and another end-point corresponding to a non- p -voter, sorted ascendingly by the current score of the non- p -voter. \square

However, as soon as $b = 3$, we obtain NP-hardness, by modifying the reduction used in Theorem 4.

Theorem 6. *If κ is a full- d bundling function, then PLURALITY-C-CC-AV is NP-hard even if the maximum bundle size b is three.*

Proof. We use a similar reduction as in the proof of Theorem 4, with the only difference that we introduce eight p -voters for each variable instead of four b -voters. We set the full- d bundling function κ such that each variable voter’s bundle consists of two variable voters and one clause voter, and such that each clause voter’s bundle also consists of two variable voters and one clause voter. Now the cycle corresponding to each variable consists of twelve vertices, as depicted in Figure 1b. Moreover, κ is full- d for some d . The correctness proof is analogous to the one shown for Theorem 4. \square

Taking also the swap distance d into account, we find out that both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV are NP-hard, even if $d = 1$. This stands in contrast to the case where $d = 0$, where \mathcal{R} -C-CC-AV reduces to the CC-AV problem (perhaps for the weighted voters [19]), which, for Plurality voting, is polynomial-time solvable by a simple greedy algorithm.

Theorem 7. *PLURALITY-C-CC-AV is NP-hard even for full-1 bundling functions and even if the maximum bundle size b is four.*

Proof. The theorem follows from the proof of Theorem 8, applied to a reduction from VERTEX COVER, for graphs with maximum vertex degree equal to three. VERTEX COVER remains NP-complete in this case [21]. \square

4 Single-Peaked and Single-Crossing Elections

In this section, we focus on instances with full- d bundling functions, and we do so because without this restriction the hardness results from previous sections easily translate to our restricted domains (at least for the case of the Plurality rule). We find that the results for the combinatorial variant of control by adding voters for single-peaked and single-crossing elections are quite different than those for the non-combinatorial case. Indeed, both for Plurality and for Condorcet, the voter control problems for single-peaked elections and for single-crossing elections are solvable in polynomial time for the non-combinatorial case [7, 18, 29]. For the combinatorial case, we show hardness for both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV for single-peaked elections, but give polynomial-time algorithms for single-crossing elections. We mention that the intractability results can also be seen as regarding anonymous bundling functions because all full- d bundling functions are leader-anonymous and follower-anonymous.

We begin with single-peaked elections.

Theorem 8. *Both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV parameterized by the solution size k are W[1]-hard for single-peaked elections, even for full-1 bundling functions.*

Proof. We provide a parameterized reduction from the W[1]-hard problem PARTIAL VERTEX COVER (PVC) with respect to the “solution size” parameter h [23], which asks for a set of at most h vertices in a graph G , which intersects with at least ℓ edges. More formally:

PARTIAL VERTEX COVER (PVC)

Input: An undirected graph $G = (V(G), E(G))$ and $h, \ell \in \mathbb{N}$.

Question: Does G admits a size- h vertex subset $U \subseteq V(G)$ which intersects at least ℓ edges in G ?

Given a PVC instance (G, h, ℓ) , we set $k := h$, and construct an election $E = (C, V)$ with $C := \{p, w\} \cup \{a_i, \bar{a}_i, b_i, \bar{b}_i \mid u_i \in V(G)\}$ such that the initial score of w is $h + \ell$ and the initial scores of all the other alternatives are zero. We do so by creating $h + \ell$ registered voters who all have the same preference order \succ such that it differs from the following *canonical preference order*:

$$p \succ w \succ a_1 \succ \bar{a}_1 \succ \dots \succ a_{|V(G)|} \succ \bar{a}_{|V(G)|} \succ b_1 \succ \bar{b}_1 \succ \dots \succ b_{|V(G)|} \succ \bar{b}_{|V(G)|}$$

by only the first pair $\{p, w\}$ of alternatives.

For each set P of disjoint pairs of alternatives, neighboring with respect to the canonical preference order, we define the preference order $\text{diff-order}(P)$ to be identical to the canonical preference order, except that all the pairs of alternatives in P are swapped. The unregistered voter set W is constructed as follows:

- (i) for each edge $e = \{u_i, u_j\} \in E(G)$, we create an *edge voter* w_e with preference order $\text{diff-order}(\{\{a_i, \bar{a}_i\}, \{a_j, \bar{a}_j\}\})$ (we say that w_e corresponds to edge e),
- (ii) for each edge $e = \{u_i, u_j\} \in E(G)$, we create a *dummy voter* d_e with preference order $\text{diff-order}(\{\{p, w\}, \{a_i, \bar{a}_i\}, \{a_j, \bar{a}_j\}\})$ (we say that d_e corresponds to edge e), and
- (iii) for each vertex $u_i \in V(G)$, we create a *vertex voter* w_i^u with preference order $\text{diff-order}(\{\{a_i, \bar{a}_i\}\})$ (we say that w_i^u corresponds to u_i).

The preference orders of the voters in $V \cup W$ are single-peaked with respect to the axis

$$\langle \bar{B} \rangle \succ \langle \bar{A} \rangle \succ p \succ w \succ \langle A \rangle \succ \langle B \rangle,$$

where

$$\begin{aligned} \langle \bar{B} \rangle &:= \bar{b}_{|V(G)|} \succ \bar{b}_{|V(G)|-1} \succ \dots \succ \bar{b}_1, & \langle \bar{A} \rangle &:= \bar{a}_{|V(G)|} \succ \bar{a}_{|V(G)|-1} \succ \dots \succ \bar{a}_1, \\ \langle B \rangle &:= b_1 \succ b_2 \succ \dots \succ b_{|V(G)|}, & \langle A \rangle &:= a_1 \succ a_2 \succ \dots \succ a_{|V(G)|}. \end{aligned}$$

Finally, we define the function κ such that it is a full-1 bundling function. To understand how κ works, we carefully calculate the swap distance between the preference orders of all possible pairs of voters in W . We see that:

- (a) any two edge voters have swap distance at least two,
- (b) any edge voter and any dummy voter have swap distance exactly one if they correspond to the same edge, and at least three otherwise,
- (c) any edge voter w_e and any vertex voter w_i^u have swap distance one if $u_i \in e$, and three otherwise,
- (d) any two dummy voters have swap distance at least two,
- (e) any dummy voter and any vertex voter have swap distance at least two, and
- (f) any two vertex voters have swap distance two.

Thus, for each edge $e = \{u_i, u_j\} \in E(G)$ we have $\kappa(w_e) := \{w_e, w_i^u, w_j^u, d_e\}$ and $\kappa(d_e) := \{w_e, d_e\}$, and for each vertex $u_i \in V(G)$ we have $\kappa(w_i^u) := \{w_i^u\} \cup \{w_e \mid u_i \in e \in E(G)\}$.

We show that (G, h, ℓ) is a yes-instance for PVC if and only if there is a size- k subset $W' \subseteq W$ such that p is a Plurality winner of the election $(C, V \cup \kappa(W'))$. Note that all unregistered voters except the dummy voters prefer p over all other alternatives and that p needs at least $h + \ell$ points in order to win.

For the “only if” part, suppose that $X \subseteq V(G)$ is a size- h vertex set and $Y \subseteq E(G)$ is a size- ℓ edge set such that for every edge $e \in Y$ it holds that $e \cap X \neq \emptyset$. We set $W' := \{w_i^u \mid u_i \in X\}$, and it is easy to verify that $\kappa(W')$ consists of h vertex voters and at least ℓ edge voters. Each of them gives p one point if added to the election. This results in p being a winner of the election with score at least $h + \ell$.

For the “if” part, suppose that there is a size- k subset $W' \subseteq W$ such that p is a Plurality winner of the election $(C, V \cup \kappa(W'))$. Observe that if W' contains some dummy voter d_e , then we can replace it with w_e (if w_e is already in W' then we can simply remove d_e from W'). Thus we can assume that W' does not contain any dummy voters. Now, assume that W' contains some edge voter w_e , where $e = \{u_i, u_j\}$. Since, by the previous argument, W' does not contain d_e , we have that d_e is not a member of $\kappa(W' \setminus \{w_e\})$. This means that if both w_{u_i} and w_{u_j} belong to $\kappa(W' \setminus \{w_e\})$ then we can safely remove w_e from W' ; p will still be a winner of the election $(C, V \cup \kappa(W' \setminus \{w_e\}))$. On the other hand, assume that exactly one of w_{u_i}, w_{u_j} does not belong to $\kappa(W' \setminus \{w_e\})$ and let w_u be this voter. It is easy to see that p is a winner of election $(C, V \cup \kappa((W' \setminus \{w_e\}) \cup \{w_u\}))$ (the net effect of including the bundle of w_e is that p 's score increases by at most one, whereas the net effect of including the bundle of w_u is that p 's score increases by at least one). Similarly, if neither u_i nor u_j belong to $\kappa(W' \setminus \{w_e\})$, then it is easy to verify that p is a winner of the election $(C, V \cup \kappa((W' \setminus \{w_e\}) \cup \{w_{u_i}\}))$. All in all, we can assume that W' contains vertex voters only. Since all vertex voters are p -voters, without loss of generality we can assume that W' contains exactly $k = h$ of them.

We define $X := \{u_i \mid w_i^u \in W'\}$ such that $|X| = k$, and $Y := \{e \in E(G) \mid e \cap X \neq \emptyset\}$. By the construction of the edge voters' preference orders, $\kappa(W')$ consists of k vertex voters and $|Y|$ edge voters. This must add up to at least $h + \ell$ voters. Therefore, $|Y| \geq \ell$, implying that at least ℓ edges are covered by X .

As for the Condorcet rule, we use the same unregistered voters as defined above and construct the original election with $h + \ell - 1$ registered voters whose preference orders are $\text{diff-order}(\{w, p\})$. Using the same reasoning as used for the Plurality rule, one can verify that (G, h, ℓ) is a yes-instance for PVC if and only if there is a size- k subset $W' \subseteq W$ such that p is a Condorcet winner of the election $(C, V \cup \kappa(W'))$. \square

We now present some tractability results for single-crossing elections. Consider an \mathcal{R} -C-CC-AV instance $((C, V), W, d, \kappa, p \in C, k)$, containing an election (C, V) and an unregistered voter set W such that $(C, V \cup W)$ is single-crossing, and thus, both (C, V) and (C, W) are single-crossing. This has a crucial consequence for full- d bundling functions: For each unregistered voter $w \in W$, the voters in bundle $\kappa(w)$ appear consecutively along the single-crossing order restricted to only the voters in W .² Using the following lemmas, we can show that PLURALITY-C-CC-AV and CONDORCET-C-CC-AV are polynomial-time solvable in some cases.

²Note that for each single-crossing election, the order of the voters possessing the single-crossing property is, in essence, unique. (modulo voters with the same preference orders and modulo the fact that if an order witnesses the single-crossing property of an election, then its reverse does so as well).

Lemma 2. *Let $I = ((C, V), W, d, \kappa, p \in C, k)$ be a PLURALITY-C-CC-AV instance such that $(C, V \cup W)$ is single-crossing and κ is a full- d bundling function. Then, the following statements hold:*

- (i) *The p -voters are ordered consecutively along the single-crossing order.*
- (ii) *If I is a yes instance, then there is a subset $W' \subseteq W$ of size at most k such that all bundles of voters $w \in W'$ contain only p -voters, except at most two bundles which may contain some non- p -voters.*

Proof. Let $n := |W|$ and let $\alpha := \langle w_1, w_2, \dots, w_n \rangle$ be a single-crossing order of the voters in W . Item (i) follows directly from the definition of the single-crossing property.

As for Item (ii), let $W' \subseteq W$ be a size- k subset of unregistered voters such that p is a Plurality winner in election $(C, V \cup \kappa(W'))$. For each subset $S \subseteq W$ of voters, we use $\text{1st}(S)$ (resp. $\text{2nd}(S)$) to denote the index j (resp. j') of the first voter $w_j \in S$ (resp. the last voter $w_{j'} \in S$) along the single-crossing order. Suppose that there are two bundles, $\kappa(w_i)$ and $\kappa(w_j)$, with $\text{1st}(\kappa(w_i)) \leq \text{1st}(\kappa(w_j))$ such that both contain non- p -voters and the first p -voter along α . If $\text{2nd}(\kappa(w_i)) \leq \text{2nd}(\kappa(w_j))$, then $\kappa(w_i)$ does not contain more p -voters than $\kappa(w_j)$ does, while containing at least as many non- p -voters as $\kappa(w_j)$. Thus, we can remove w_i from W' . Otherwise, $\text{2nd}(\kappa(w_i)) > \text{2nd}(\kappa(w_j))$, which means that $\kappa(w_j) \subset \kappa(w_i)$. Thus, we can remove w_j from W' . In any case, we conclude that W' contains at most one voter w whose bundle $\kappa(w)$ contains a non- p -voter and the first p -voter (along the single-crossing order).

Analogously, we can show that W' contains at most one voter w whose bundle $\kappa(w)$ contains a non- p -voter and the last p -voter (along the single-crossing order). Since for each bundle $\kappa(w)$ with $w \in W'$, if $\kappa(w)$ contains a non- p -voter, then it contains at least one of the first and last voters along α , every bundle $\kappa(w)$ with $w \in W'$ contains at least one p -voter (because if it does not, then we can remove its respective leader voter, as the bundle does not help p), and Item (ii) follows. \square

For Condorcet voting, we use the well-known median-voter theorem (we provide the proof for the sake of completeness).

Lemma 3. *Let $(C, V \cup \kappa(W'))$ be a single-crossing election with single-crossing voter order $\langle x_1, x_2, \dots, x_z \rangle$ and set $X_{\text{median}} := \{x_{\lceil z/2 \rceil}\} \cup \{x_{z/2+1} \text{ if } z \text{ is even}\}$, where $z = |V| + |\kappa(W')|$. Alternative p is a (unique) Condorcet winner in $(C, V \cup \kappa(W'))$ if and only if every voter in X_{median} is a p -voter.*

Proof. Let X_1 be the set of voters $x_1, x_2, \dots, x_{\lceil z/2 \rceil - 1}$ and let X_2 be the set of voters $V \cup \kappa(W') \setminus (X_1 \cup X_{\text{median}})$.

For the “if” part, let c be an arbitrary alternative from $C \setminus \{p\}$. Then, if there is some voter in X_1 which prefers c over p , then all voters in $X_{\text{median}} \cup X_2$ prefer p over c . If there is some voter in X_2 which prefers c over p , then all voters in $X_1 \cup X_{\text{median}}$ prefer p over c . In any case, a strict majority of voters prefer p over c . Thus, p is the (unique) Condorcet winner.

For the “only if” part, suppose for the sake of contradiction that there is a voter in X_{median} which is not a p -voter but a c -voter with $c \in C \setminus \{p\}$. Then, analogously to the reasoning above, at least half of the voters will prefer c over p —a contradiction. \square

With these two lemmas available, we give polynomial-time algorithms for both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV, for the case of single-crossing elections and full- d bundling functions.

Theorem 9. *Both PLURALITY-C-CC-AV and CONDORCET-C-CC-AV are polynomial-time solvable for the single-crossing case with full- d bundling functions.*

Proof. First, we find a (unique) single-crossing voter order for $(C, V \cup W)$ in quadratic time [14, 8]. Due to Lemma 2 and Lemma 3, we only need to store the most preferred alternative of each voter to find the solution set W' . Thus, the running-time from now on only depends on the number of voters. We start with the Plurality rule and let $\alpha := \langle w_1, w_2, \dots, w_{|W|} \rangle$ be a single-crossing voter order.

Due to Lemma 2 (ii), the two bundles in $\kappa(W')$ which may contain non- p -voters appear at the beginning and at the end of the p -voter block, along the single-crossing order. We first guess these two bundles, and after this initial guess, all remaining bundles in the solution contain only p -voters (Lemma 2 (i)). Thus, the remaining task is to find the maximum score that p can gain by selecting k' bundles containing only p -voters. This problem is equivalent to the MAXIMUM INTERVAL COVER problem, which is solvable in $O(|W|^2)$ time (Golab et al. [22, Section 3.2]).

For the Condorcet rule, we propose a slightly different algorithm. The goal is to find a minimum-size subset $W' \subseteq W$ such that p is the (unique) Condorcet winner in $(C, V \cup \kappa(W'))$. Let $\beta := \langle x_1, x_2, \dots, x_z \rangle$ be a single-crossing voter order for $(C, V \cup W)$. Considering Lemma 3, we begin by guessing at most two voters in $V \cup W$ whose bundles may contain the median p -voter (or, possibly, several p -voters) along the single-crossing order of voters restricted to the final election (for simplicity, we define the bundle of each registered voter to be its singleton). The voters in the union of these two bundles must be consecutively ordered. Let those voters be $x_i, x_{i+1}, \dots, x_{i+j}$ (where $i \geq 1$ and $j \geq 0$), let $W_1 := \{x_s \in W \mid s < i\}$, and let $W_2 := \{x_s \in W \mid s > i+j\}$. We guess two integers $z_1 \leq |W_1|$ and $z_2 \leq |W_2|$ with the property that there are two subsets $B_1 \subseteq W_1$ and $B_2 \subseteq W_2$ with $|B_1| = z_1$ and $|B_2| = z_2$ such that the median voter(s) in $V \cup B_1 \cup \{x_i, x_{i+1}, \dots, x_{i+j}\} \cup B_2$ are indeed p -voters (for now, only the sizes z_1 and z_2 matter, not the actual sets). These four guesses cost $O(|V \cup W|^2 \cdot |W|^2)$ time. The remaining task is to find two minimum-size subsets W'_1 and W'_2 such that $\kappa(W'_1) \subseteq W_1$, $\kappa(W'_2) \subseteq W_2$, $|\kappa(W'_1)| = z_1$, and $|\kappa(W'_2)| = z_2$. As already discussed, this can be done in $O(|W|^2)$ time [22]. We conclude that one can find a minimum-size subset $W' \subseteq W$ such that p is the (unique) Condorcet winner in $(C, V \cup \kappa(W'))$ in $O(|V \cup W|^2 \cdot |W|^4)$ time. \square

5 Conclusion

We provide opportunities for future research. First, we did not discuss destructive control and the related problem of combinatorial deletion of voters. For Plurality, we conjecture that combinatorial addition of voters for destructive control, and combinatorial deletion of voters for either constructive or destructive control behave similarly to combinatorial addition of voters for constructive control.

Another, even wider field of future research is to study other combinatorial voting models—this may include controlling the swap distance, “probabilistic bundling”, “reverse bundling”, or using other distance measures than the swap distance. Naturally, it would also be interesting

to consider other problems than election control (with bribery being perhaps the most natural candidate).

Finally, instead of studying a “leader-follower model” as we did, one might also be interested in an “enemy model” referring to control by adding alternatives: The alternatives of an election “hate” each other such that if one alternative is added to the election, then all of its enemies are also added to the election. This scenario of combinatorial candidate control deserves future investigation.

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