Topology Notes

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Basic Definitions

Given a statement of the form $P \to Q$, its **contrapositive** is defined to be the statement $\neg Q \to \neg P$. For example, the contrapositive of the statement:

$$x > 0 \longrightarrow \neg x^3 \neq 0$$

is the statement

$$x^3 = 0 \longrightarrow \neg x > 0$$

.

It's trivial to demonstrate the $P \to Q \leftrightarrow \neg P \to Q$. Another useful definitionn is the **converse** of some statement. Given $P \to Q$, the converse of this statement is $Q \to P$.

Definition 1.1 (1)**.** *A **rule of assignment** is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate of at most one ordered pair belonging to r. Thus, a subset r of $C \times D$ is a rule assignment if:*

$$[(c, d) \in r \text{ and } (c, d') \in r] \to [d = d'] \tag{1}$$

Definition 1.2. *A **function** f is a rule of assignment r, together with a set B that contains the image of r. The domain of A of the rule r is also called the **domain** of the function f; the image set of r is also called the **image set** of f; and the set B is called the **range** of f. If f is a function having domain A and range B, we express the fact by writing*

$$f : A \longrightarrow B$$

Formally, if r is the rule of the function f, then $f(a)$ denotes the unique element of B such that $(a, f(a)) \in r$.

Definition 1.3. *If $f : A \to B$ and if A_0 is a subset of A, we define the **restriction** of f to A_0 to be the function mapping A_0 into B whose rules is*

$$\{(a, f(a)|a \in A_0)\}. \tag{2}$$

It is denoted by $f|A_0$, which is read "f restricted to A_0"

The book stats with a really useful definitions (but I'm lazy, so I'll do it quickly). A function $f : A \to B$ is said to be **injective** (or **one-to-one**) if:

$$[f(a) = f(a')] \to [a = a'] \tag{3}$$

It is said to be **surjective** (or f is a map **onto** B) if:

$$[b \in B] \to [\exists a \in A; b = f(a)] \tag{4}$$

If f is both injective and surjective, it is said to be **bijective** (or is called a **one-to-one correspondence**)

Injectivty of f depends only on the rule of f; surjective depends on the range of f as well. It's valid that composition of functions with same "type" has the same "type" (injective, surjective) If f is bijective, there exists a function from B to A called the **inverse** of f. It is denoted by f^{-1} and is defined by letting $f^{-1}(b)$ be the unique element $a \in A$ for which $f(a) = b$. It's easy to see if f is bijective, then f^{-1} is also bijective.

Lemma 1.1. *Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a$ for every $a \in A$ and $f(h(b)) = b$ for every $b \in B$, the f is bijective and $g = h = f^{-1}$*

Proof. If we have a function g such that $g(f(a)) = a, \forall a \in A$, then f must to be injective. Suppose $m, n in A$ which $g(f(m)) = m \wedge g(f(n)) = n$, if $f(m) = f(n)$, then $g(f(n)) = g(f(m))$ which means $m = n$. Conversely, if $m = n$, then $g(f(n)) = g(f(m))$, which means $f(m) = f(n)$. So $f(m) = f(n) \leftrightarrow m = n$, that proves that f is injective. If we have a function h such that $f(h(b)) = b, \forall b \in B$, then f must to be surjective. □