homework1_1

November 29, 2022

1 Direct Methods for the solution of Linear Systems

Esercizio 1 Given a matrix A R n×n and the vector xtrue = (1, 1, . . . , 1)T R n, write a script that:

- Computes the right-hand side of the linear system b = Axtrue.

- Computes the condition number in 2-norm of the matrix A. It is ill-conditioned? What if we use the ∞-norm instead of the 2-norm?

- Solves the linear system Ax = b with the function np.linalg.solve().

- Computes the relative error between the solution computed before and the true solution xtrue. Remember that the relative error between xtrue and x in R n can be computed as

E(xtrue, x) = ||x − xtrue||2 ||xtrue||2

- Plot a graph (using matplotlib.pyplot) with the relative errors as a function of n and (in a new window) the condition number in 2-norm K2(A) and in ∞-norm, as a function of n.

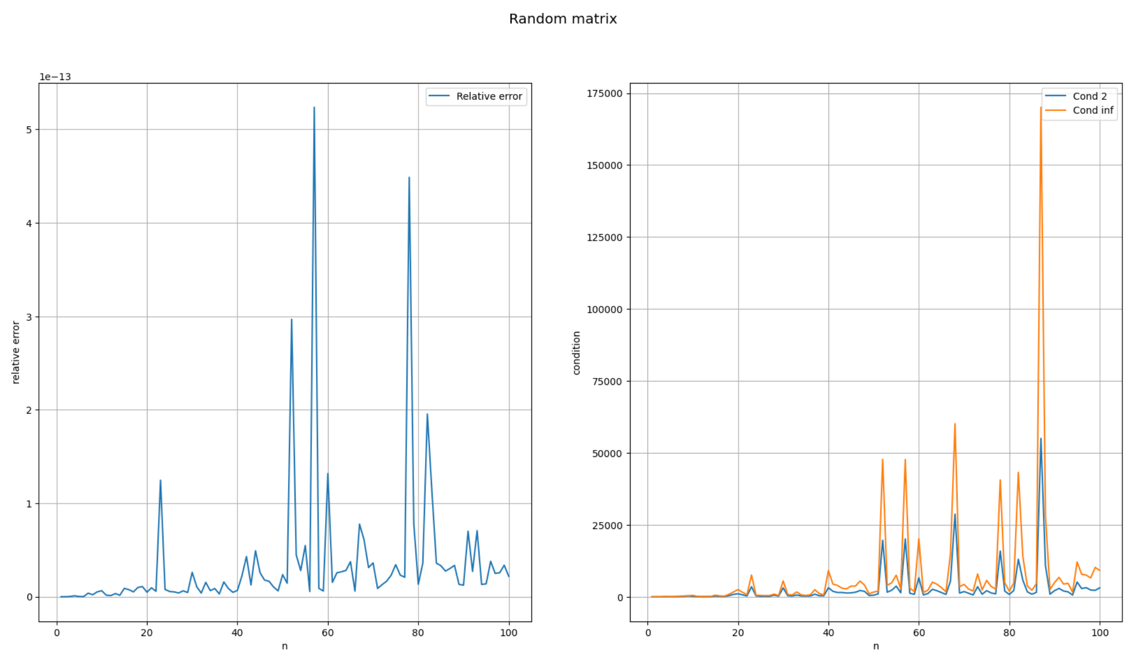
[ ]:
```
import numpy as np
import matplotlib.pyplot as plt

N = 4
A = np.random.randn(N, N)
x_true = np.ones(N)

# Compute b
b = A@x_true

# Compute condition number
cond = np.linalg.cond(A, 2)
cond_inf = np.linalg.cond(A, np.Inf)

print(f"Condition with norm 2: {cond}")
print(f"Condition with norm inf: {cond_inf}")

if cond > 10e6:
    print("A ill-conditioned")
else:
    print("A not ill-conditioned")
```

```python
# Solve system Ax=b
x = np.linalg.solve(A, b)
print(f"Computed x: {x}")

# Relative error
rel_err = np.linalg.norm(x - x_true, 2)/np.linalg.norm(x_true)
print(f"Relative error: {rel_err}")
```

```
Condition with norm 2: 8.666760700434734
Condition with norm inf: 22.369427848512444
A not ill-conditioned
Computed x: [1. 1. 1. 1.]
Relative error: 1.6653345369377348e-16
```

[ ]:
```python
def compute(N):
    A = np.random.rand(N, N)
    x_true = np.ones(N)

    # Compute b
    b = A@x_true

    # Compute condition number
    cond = np.linalg.cond(A, 2)
    cond_inf = np.linalg.cond(A, np.Inf)

    # Solve system Ax=b
    x = np.linalg.solve(A, b)
    # Relative error
    rel_err = np.linalg.norm(x - x_true, 2)/np.linalg.norm(x_true)

    return rel_err, cond, cond_inf


def plot_data(data, n_vector, title=""):
    fig = plt.figure(figsize=(20, 10))
    fig.suptitle(title, fontsize="x-large")
    plt.subplot(1, 2, 1)
    plt.plot(n_vector, data["err_rel"], )
    plt.xlabel("n")
    plt.ylabel("relative error")
    plt.legend(["Relative error"])
    plt.grid()
    plt.subplot(1, 2, 2)
    plt.xlabel("n")
    plt.ylabel("condition")
    plt.plot(n_vector, data["cond"])
```

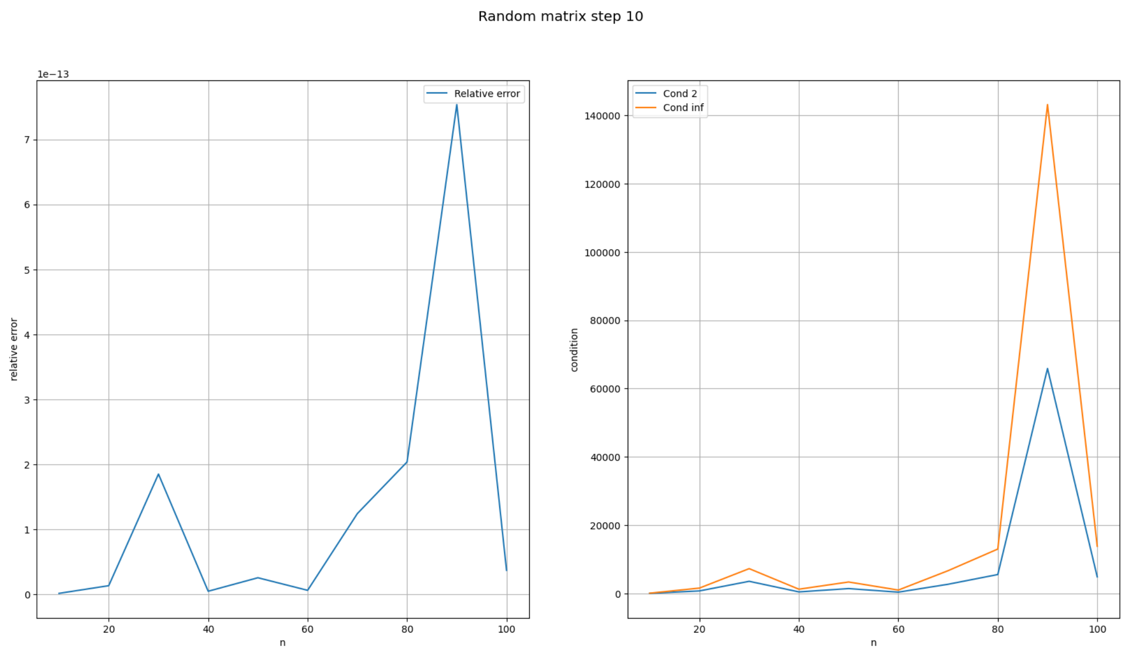
```
    plt.plot(n_vector, data["cond_inf"])
    plt.legend(["Cond 2", "Cond inf"])
    plt.grid()

n = 100

n_vector = np.arange(1, n+1, 1)
data = {"err_rel": [], "cond": [], "cond_inf": []}
for i in range(1, n+1, 1):
    computed = compute(i)
    data["err_rel"].append(computed[0])
    data["cond"].append(computed[1])
    data["cond_inf"].append(computed[2])

plot_data(data, n_vector, title="Random matrix")
```



Esercizio 2 Test the program above with the following choices of A R n×n:

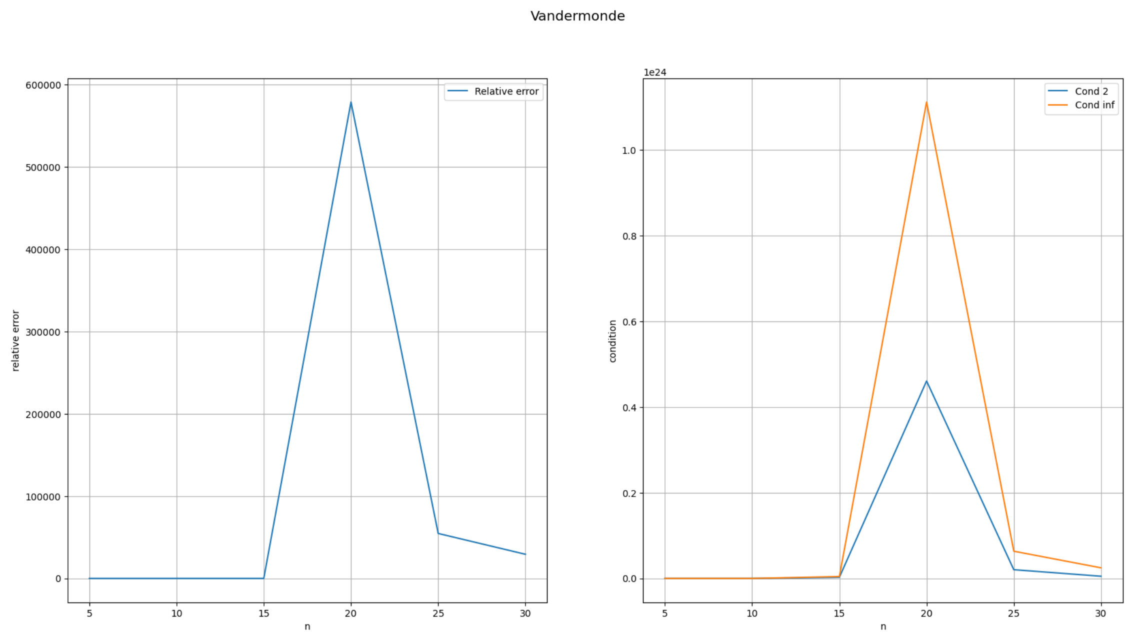
- A random matrix (created with the function np.random.rand()) with size varying with n = {10, 20, 30, . . . , 100}.
- The Vandermonde matrix (np.vander) of dimension n = {5, 10, 15, 20, 25, 30} with respect to the vector x = {1, 2, 3, . . . , n}.
- The Hilbert matrix (scipy.linalg.hilbert) of dimension n = {4, 5, 6, . . . , 12}

[ ]:
```python
n = 100
n_vector = np.arange(10, n+1, 10)
data = {"err_rel": [], "cond": [], "cond_inf": []}
for i in range(10, n+1, 10):
    computed = compute(i)
    data["err_rel"].append(computed[0])
    data["cond"].append(computed[1])
    data["cond_inf"].append(computed[2])

plot_data(data, n_vector, "Random matrix step 10")
```



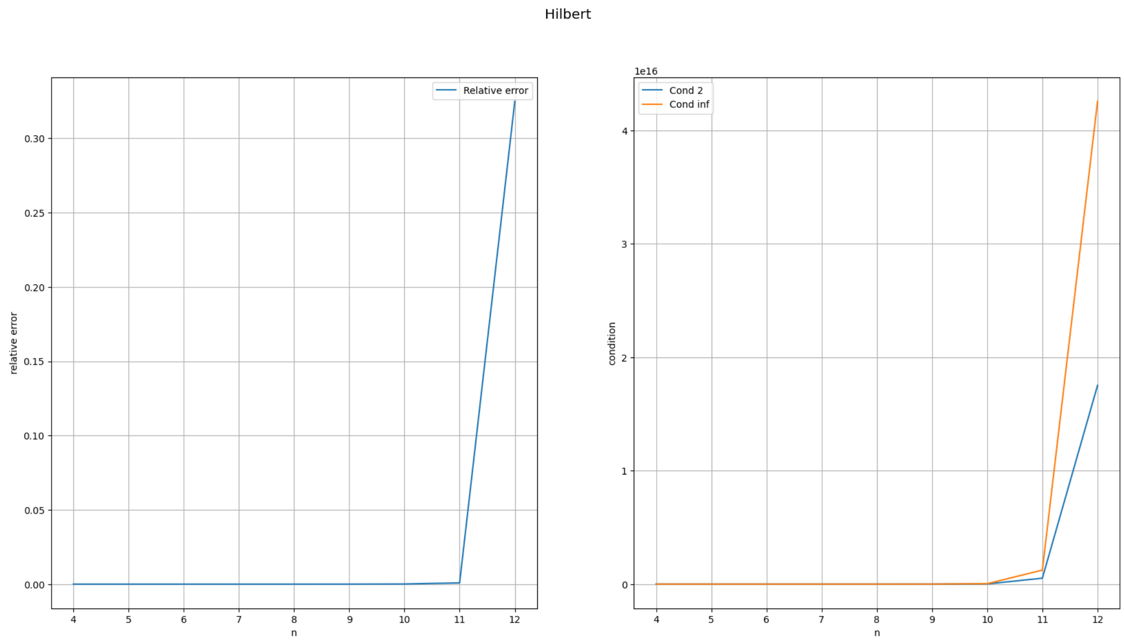
[ ]:
```python
def compute2(A, x_true):
    # Compute b
    b = A@x_true

    # Compute condition number
    cond = np.linalg.cond(A, 2)
    cond_inf = np.linalg.cond(A, np.Inf)

    # Solve system Ax=b
    x = np.linalg.solve(A, b)
    # Relative error
    rel_err = np.linalg.norm(x - x_true, 2)/np.linalg.norm(x_true)

    return rel_err, cond, cond_inf
```

```
dim = np.arange(5, 31, 5)

data_vander = {"err_rel": [], "cond": [], "cond_inf": []}

for i in dim:
    x_true = np.ones(i)
    x_vander = np.arange(1, i+1)
    V = np.vander(x_vander)
    computed = compute2(V, x_true)
    data_vander["err_rel"].append(computed[0])
    data_vander["cond"].append(computed[1])
    data_vander["cond_inf"].append(computed[2])

plot_data(data_vander, dim, title="Vandermonde")
```



[ ]:
```
import scipy as sp

dim = np.arange(4, 13, 1)
data_hilbert = {"err_rel": [], "cond": [], "cond_inf": []}

for i in dim:
    x_true = np.ones(i)
    x_hilbert = np.arange(1, i+1)
    H = sp.linalg.hilbert(i)
    computed = compute2(H, x_true)
    data_hilbert["err_rel"].append(computed[0])
```

```
    data_hilbert["cond"].append(computed[1])
    data_hilbert["cond_inf"].append(computed[2])

plot_data(data_hilbert, dim, title="Hilbert")
```



2 Floating Point Arithmetic

Esercizio 1 The Machine epsilon eps is the distance between 1 and the next floating point number. Compute eps, which is defined as the smallest floating point number such that it holds:

fl(1 + eps) > 1

Tips: use a while structure

[ ]:
```
x = 1
eps = 1
old = 1
while x+eps > 1:
    old = eps
    eps /= 2

print(old)
```

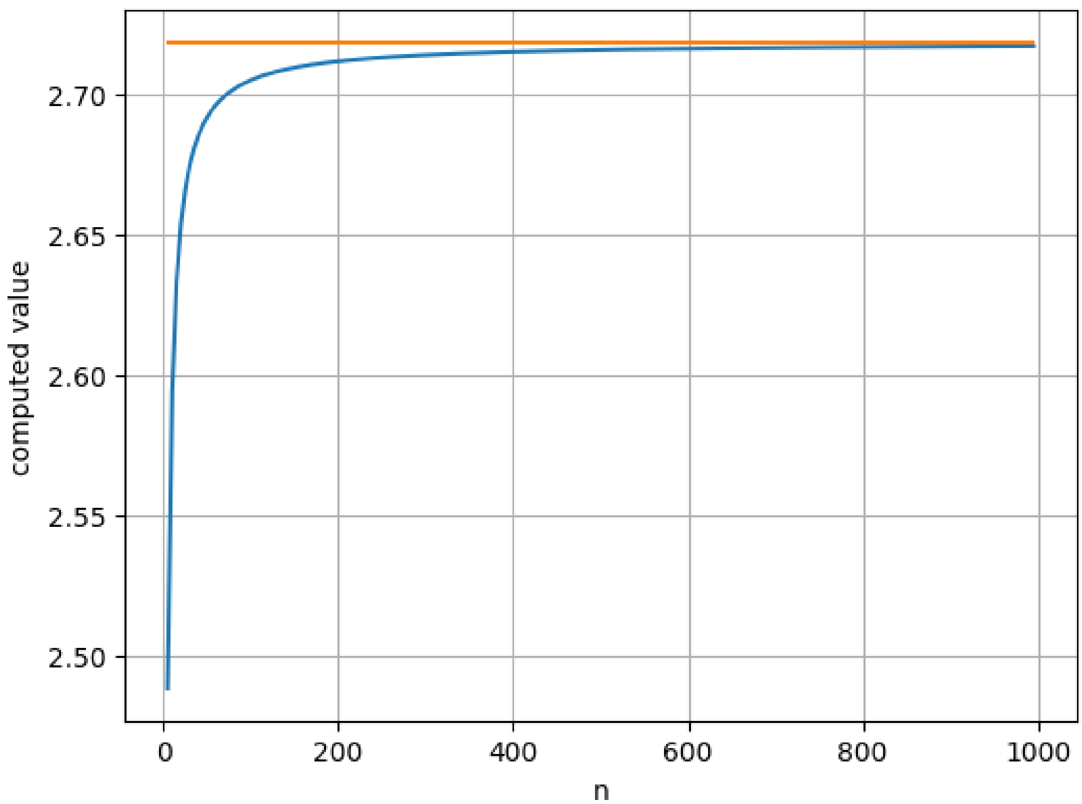
```
2.220446049250313e-16
```

Esercizio 2 Let's consider the sequence an = (1 + 1 n) n. It is well known that:

limn→∞ an = e

where e is the Euler costant. Choose different values for n, compute an and compare it to the real value of the Euler costant. What happens if you choose a large value of n? Guess the reason.

[ ]:
```
def euler(n):
    return (1 + 1/n)**n


n_euler = np.arange(5, 1000, 5)
data_euler = []
for i in n_euler:
    data_euler.append(euler(i))
plt.plot(n_euler, data_euler)
plt.plot(n_euler, [np.e]*n_euler.shape[0])
plt.xlabel("n")
plt.ylabel("computed value")
plt.grid()
plt.show()
```



Esercizio 3 Let's consider the matrices:

A = [[4,2], [1,3]]

B =[[4,2],[2,1]]

Compute the rank of A and B and their eigenvalues. Are A and B full-rank matrices? Can you infer some relationship between the values of the eigenvalues and the full-rank condition? Please, corroborate your deduction with other examples. Tips: Please, have a look at np.linalg.

[ ]:
```
A = [[4, 2], [1, 3]]
B = [[4, 2], [2, 1]]

rank_A = np.linalg.matrix_rank(A)
rank_B = np.linalg.matrix_rank(B)

eig_A = np.linalg.eigvals(A)
eig_B = np.linalg.eigvals(B)

print(f"Rank of A: {rank_A}")
print(f"Rank of B: {rank_B}")

print(f"Eig of A: {eig_A}")
print(f"Eig of B: {eig_B}")
```

```
Rank of A: 2
Rank of B: 1
Eig of A: [5. 2.]
Eig of B: [5. 0.]
```

The rank of a squared matrix is equal to the number of eigenvalues differents from 0

Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

We have two eigenvalues $\lambda_1 = \lambda_2 = 1$ The matrix rank is equal to 2

On the other hand if we take

B = $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

We have $\lambda_1 = 0, \lambda_2 = 3$

The rank of the matrix is equal to the numbers of non-zero eigenvalues, so it is 1