CS 70 Spring 2018 — Discrete Mathematics and Probability Theory — Satish Rao and Babak Ayazifar — DIS 07B

1 Let's Talk Probability

(a) When is $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ true? What is the general rule that always holds?

(b) When is $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ true? What is the general rule that always holds?

(c) If A and B are disjoint, are they independent?

Solution:

(a) In general, we know $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. This is the Inclusion-Exclusion Principle. Therefore if A and B are disjoint, such that $\mathbb{P}(A \cap B) = 0$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ holds.

(b) $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ holds if and only if A and B are independent (by definition).

(c) No, if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let A be the event that we roll a 1, and let B be the event that we roll a 2. Certainly A and B are disjoint, as $\mathbb{P}(A \cap B) = 0$. But these events are not independent: $\mathbb{P}(B \mid A) = 0$, but $\mathbb{P}(B) = 1/6$.

Since disjoint events have $\mathbb{P}(A \cap B) = 0$, we can see that the only time when A and B are independent is when either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

2 Rain and Wind

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS 70, you are curious to play around with these numbers. Find the probability that:

(a) A given day is both windy and rainy.

(b) A given day is rainy.

(c) For a given pair of days, exactly one of the two days is rainy.

Solution:

(a) Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $\mathbb{P}(R \mid W) = 0.3$, $\mathbb{P}(R \mid W^c) = 0.8$ and $\mathbb{P}(W) = 0.2$. Then probability that a given day is both rainy and windy is $\mathbb{P}(R \cap W) = \mathbb{P}(R \mid W)\mathbb{P}(W) = 0.3 \times 0.2 = 0.06$.

(b) Probability that it rains on a given day is $\mathbb{P}(R) = \mathbb{P}(R \mid W)\mathbb{P}(W) + \mathbb{P}(R \mid W^c)\mathbb{P}(W^c) = 0.3 \times 0.2 + 0.8 \times 0.8 = 0.7$.

(c) Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the days are independent, $\mathbb{P}(R_1) = \mathbb{P}(R_2) = \mathbb{P}(R)$. The required probability is $\mathbb{P}(R_1)\mathbb{P}(R_2^c) + \mathbb{P}(R_1^c)\mathbb{P}(R_2) = 2 \cdot 0.7 \cdot 0.3 = 0.42$.

3 Bag of Coins

Your friend Forest has a bag of n coins. You know that k are biased with probability p (i.e. these coins have probability p of being heads). Let F be the event that Forest picks a fair coin, and let B be the event that Forest picks a biased coin. Forest draws three coins from the bag, but he does not know which are biased and which are fair.

(a) What is the probability of three coins being pulled in the order FFB?

(b) What is the probability that the third coin he draws is biased?

(c) What is the probability of picking at least two fair coins?

Solution:

(a) The probability of picking F for the first coin is $(n-k)/n$. The probability of picking F for the second coin, after picking one fair coin already is $(n-k-1)/(n-1)$. The probability of picking B for the third coin is $k/(n-2)$. Thus, the probability of picking the exact sequence FFB is

$$\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

(b) One approach is to condition on the possible outcomes for the first and second coins

$$\{FF, FB, BF, BB\}$$

such that

$$\mathbb{P}(T) = \mathbb{P}(T \cap FF) + \mathbb{P}(T \cap FB) + \mathbb{P}(T \cap BF) + \mathbb{P}(T \cap BB)$$

where T is the event that the third coin is biased.

A simpler approach is to use the notion of symmetry. Since we don't know any information about the first and second coins, the probability that the third coin is biased is the same as the probability that the first coin is biased, which is k/n.

(c) Note that the probability of picking any sequence of two fair coins and a biased coin is the same. It is in fact the probability from part (a). We need to multiply by the number of arrangements of biased and fair coins, however. So, the probability of picking any sequence with two fair coins is

$$\binom{3}{1}\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

We additionally need to consider the probability of getting 3 fair coins.

$$\frac{(n-k)!(n-3)!}{n!(n-k-3)!}$$

We simply sum the two to get our answer:

$$\binom{3}{1}\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}+\frac{(n-k)!(n-3)!}{n!(n-k-3)!}$$

4 Lie Detector

A lie detector is known to be $4/5$ reliable when the person is guilty and $9/10$ reliable when the person is innocent. If a suspect is chosen from a group of suspects of which only $1/100$ have ever committed a crime, and the test indicates that the person is guilty, what is the probability that he is innocent?

Solution:

Let A denote the event that the test indicates that the person is guilty, and B the event that the person is innocent. Note that

$$\mathbb{P}[B]=\frac{99}{100}, \quad \mathbb{P}[\overline{B}]=\frac{1}{100}, \quad \mathbb{P}[A \mid B]=\frac{1}{10}, \quad \mathbb{P}[A \mid \overline{B}]=\frac{4}{5}.$$

Using the Bayesian Inference Rule, we can compute the desired probability as follows:

$$\mathbb{P}[B \mid A]=\frac{\mathbb{P}[B]\mathbb{P}[A \mid B]}{\mathbb{P}[B]\mathbb{P}[A \mid B]+\mathbb{P}[\overline{B}]\mathbb{P}[A \mid \overline{B}]}=\frac{(99/100)(1/10)}{(99/100)(1/10)+(1/100)(4/5)}=\frac{99}{107}$$