CS 70 Spring 2018	Discrete Mathematics and Probability Theory Babak Ayazifar and Satish Rao	DIS 11A

1 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \leq x \leq 1, 0 \leq y \leq 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

Solution:

(a) Since $f(x,y)$ is a probability density function, it must integrate to 1. Then:

$$1 = \int_0^1 \int_0^2 Cxy\,dy\,dx = \int_0^1 2Cx dx = C$$

Therefore, $C = 1$.

(b) To get the marginal distribution of X, we integrate the joint distribution with respect to Y. So:

$$f_X(x) = \int_0^2 f(x,y)dy = \int_0^2 xydy = 2x$$

This is the marginal distribution for $0 \leq x \leq 1$

(c) The conditional distribution of Y given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xy}{2x} = \frac{y}{2}$$

(d) The conditional distribution of Y given $X = x$ does not depend on x, so they are independent.

Alternatively, you could find the marginal distribution of Y and see it is the same as the conditional distribution of Y:

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 xydx = \frac{y}{2}$$

Notice that since X and Y are independent, $f_X(x)f_Y(y) = xy = f_{X,Y}(x,y)$, i.e. the product of the marginal distributions is the same as the joint distribution.

2 Lunch Meeting

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch?

Solution:

Let the random variable A be the time that Alice arrives and the random variable B be the time when Bob arrives. Consider Figure **??**, plotting the space of all outcomes (a, b): The shaded region is

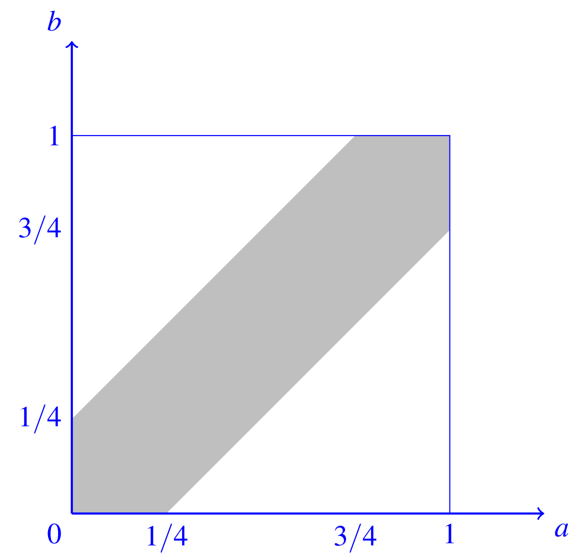

Figure 1: Visualization of joint probability density.

the set of values (a, b) for which Alice and Bob will actually meet for lunch. Since all points in this square are equally likely, the probably they meet is the ratio of the shaded area to the area of the square. If the area of the square is 1, then the area of the shaded region is

$$1 - 2 \times \left[\frac{1}{2} \times \left(\frac{3}{4}\right)^2\right] = \frac{7}{16},$$

since the area of the white triangle on the upper-left is $(1/2) \cdot (3/4)^2$, and the white triangle on the lower-right has the same area. Therefore, the probability that Alice and Bob actually meet is $7/16$.

3 First Exponential to Die

Let X and Y be Exponential(λ_1) and Exponential(λ_2) respectively, independent. What is

$$\mathbb{P}\big(\min(X, Y) = X\big),$$

the probability that the first of the two to die is X?

Solution:

Recall that the CDF of an exponential is $\mathbb{P}[X \le x] = 1 - \exp(-\lambda x)$ for $x \ge 0$.

$$\mathbb{P}\big(\min(X,Y) = X\big) = \mathbb{P}(Y > X) = \int_0^\infty \mathbb{P}(Y > X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-\lambda_2 x} \cdot \lambda_1 \,\mathrm{e}^{-\lambda_1 x}\,\mathrm{d}x$$
$$= -\frac{\lambda_1}{\lambda_1 + \lambda_2}\,\mathrm{e}^{-(\lambda_1+\lambda_2)x}\Big|_{x=0}^{\infty} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$