# 1 Probabilistically Buying Probability Books

Chuck will go shopping for probability books for $K$ hours. Here, $K$ is a random variable and is equally likely to be 1, 2, or 3. The number of books $N$ that he buys is random and depends on how long he shops. We are told that

$$\mathbb{P}[N = n | K = k] = \frac{c}{k}, \qquad \text{for } n = 1, \ldots, k$$

for some constant $c$.

(a) Compute $c$.

(b) Find the joint distribution of $K$ and $N$.

(c) Find the marginal distribution of $N$.

(d) Find the conditional distribution of $K$ given that $N = 1$.

(e) We are now told that he bought at least 1 but no more than 2 books. Find the conditional mean and variance of $K$, given this piece of information.

(f) The cost of each book is a random variable with mean 3. What is the expectation of his total expenditure? *Hint:* Condition on events $N = 1, \ldots, N = 3$ and use the total expectation theorem.

**Solution:**

(a) For any $k$, we know that probabilities conditioned on $K = k$ must sum to 1, i.e

$$\sum_n \mathbb{P}[N = n | K = k] = 1 \ ,$$

so it must be that

$$1 = \sum_{n=1}^{k} \mathbb{P}[N = n | K = k] = k \times \frac{c}{k} = c \ .$$

Thus, $c = 1$.

(b) The joint distribution specifies $\mathbb{P}[N = n \cap K = k]$ for all $n$ and $k$. Note that

$$\mathbb{P}[N = n \cap K = k] = \mathbb{P}[N = n | K = k]\mathbb{P}[K = k]$$

and we know $\mathbb{P}[N = n | K = k]$ and $\mathbb{P}[K = k]$ (it says all $k \in \{1, 2, 3\}$ are equally likely). We use this formula to calculate $\mathbb{P}[N = n \cap K = k]$ for each $n, k$ and list the result in a table:

| $n \setminus k$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{9}$ |
| 2 | 0 | $\frac{1}{6}$ | $\frac{1}{9}$ |
| 3 | 0 | 0 | $\frac{1}{9}$ |

Alternatively, we can define the joint distribution as a formula with specified domain. $\mathbb{P}[N = n, K = k] = \mathbb{P}[N = n \mid K = k]\mathbb{P}[K = k] = \frac{1}{k}\frac{1}{3}$ whenever it is nonzero. So,

$$\mathbb{P}[N = n, K = k] = \begin{cases} \frac{1}{3k} & k \in \{1,2,3\}, n \in \{1,\dots,k\} \\ 0 & \text{otherwise} \end{cases}$$

(c) The marginal distribution of $N$ is given by the value of $\mathbb{P}[N = n]$, for each possible value of $n$. By the total probability rule,

$$\mathbb{P}[N = n] = \mathbb{P}[N = n \cap K = 1] + \mathbb{P}[N = n \cap K = 2] + \mathbb{P}[N = n \cap K = 3] \ .$$

Thus, we get

$$\mathbb{P}[N = n] = \begin{cases} \frac{1}{3} + \frac{1}{6} + \frac{1}{9} & \text{if } n = 1 \\ \frac{1}{6} + \frac{1}{9} & \text{if } n = 2 \\ \frac{1}{9} & \text{if } n = 3 \end{cases} = \begin{cases} \frac{11}{18} & \text{if } n = 1 \\ \frac{5}{18} & \text{if } n = 2 \\ \frac{2}{18} & \text{if } n = 3 \end{cases}$$

(d) By definition, $\mathbb{P}[K = k | N = 1] = \frac{\mathbb{P}[K=k \cap N=1]}{\mathbb{P}[N=1]}$. The numerator comes from the joint distribution of $N$ and $K$ (part (b)), and the denominator comes from the marginal distribution of $N$ (part (c)). Plugging in, we get

$$\mathbb{P}[K = k | N = 1] = \begin{cases} \frac{\frac{1}{3}}{\frac{11}{18}} & \text{if } k = 1 \\ \frac{\frac{1}{6}}{\frac{11}{18}} & \text{if } k = 2 \\ \frac{\frac{1}{9}}{\frac{11}{18}} & \text{if } k = 3 \end{cases} = \begin{cases} \frac{6}{11} & \text{if } k = 1 \\ \frac{3}{11} & \text{if } k = 2 \\ \frac{2}{11} & \text{if } k = 3 \end{cases}$$

(e) We first compute the distribution $\mathbb{P}[K = k | N = 1 \cup N = 2]$ as we did in part (d):

$$\mathbb{P}[K = k | N = 1 \cup N = 2] = \begin{cases} \frac{\frac{1}{3}}{\frac{16}{18}} & \text{if } k = 1 \\ \frac{\frac{1}{6} + \frac{1}{6}}{\frac{16}{18}} & \text{if } k = 2 \\ \frac{\frac{1}{9} + \frac{1}{9}}{\frac{16}{18}} & \text{if } k = 3 \end{cases} = \begin{cases} \frac{3}{8} & \text{if } k = 1 \\ \frac{3}{8} & \text{if } k = 2 \\ \frac{2}{8} & \text{if } k = 3 \end{cases}$$

Now, the mean will be

$$\mathbb{E}[K | N = 1 \cup N = 2] = 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{2}{8} = \frac{15}{8}$$

and the variance will be

$$\begin{aligned}
\operatorname{var}(K|N=1\cup N=2) &= \mathbb{E}\left[K^2|N=1\cup N=2\right] - \mathbb{E}[K|N=1\cup N=2]^2 \\
&= 1^2\times\frac{3}{8}+2^2\times\frac{3}{8}+3^2\times\frac{2}{8}-\left(\frac{15}{8}\right)^2 \\
&= \frac{39}{64} \\
&\approx 0.61
\end{aligned}$$

(f) Let $X$ be his total expenditure. Using the total expectation theorem, we have

$$\mathbb{E}[X] = \mathbb{E}[X|N=1]\mathbb{P}[N=1]+\mathbb{E}[X|N=2]\mathbb{P}[N=2]+\mathbb{E}[X|N=3]\mathbb{P}[N=3]$$

Since each book has an expected price of 3, $\mathbb{E}[X|N=n] = 3\times n$, giving

$$\begin{aligned}
\mathbb{E}[X] = \mathbb{E}[X|N=1]\mathbb{P}[N=1]+\mathbb{E}[X|N=2]\mathbb{P}[N=2]+\mathbb{E}[X|N=3]\mathbb{P}[N=3] \\
= 3\times\frac{11}{18}+6\times\frac{5}{18}+9\times\frac{2}{18} \\
= \frac{9}{2}\ .
\end{aligned}$$

## 2 Continuous Intro

(a) Is

$$f(x) = \begin{cases} 2x, & 0\le x\le 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

(b) Calculate $\mathbb{E}[X]$ and $\operatorname{var}(X)$ for $X$ with the density function

$$f(x) = \begin{cases} 1/\ell, & 0\le x\le \ell, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Suppose $X$ and $Y$ are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0\le x\le 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0\le y\le 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is their joint distribution? (Hint: for this part and the next, we can use independence in much the same way that we did in discrete probability)

(d) Calculate $\mathbb{E}[XY]$ for the above $X$ and $Y$.

**Solution:**

(a) Yes; it is non-negative and integrates to 1. No; a CDF should go to 1 as $x$ goes to infinity and be non-decreasing.

(b) $\mathbb{E}[X] = \int_{x=0}^{\ell} x \cdot (1/\ell)\,\mathrm{d}x = \ell/2$. $\mathbb{E}[X^2] = \int_{x=0}^{\ell} x^2 \cdot (1/\ell)\,\mathrm{d}x = \ell^2/3$.
$\operatorname{var}(X) = \ell^2/3 - \ell^2/4 = \ell^2/12$.

This is known as the continuous uniform distribution over the interval $[0,\ell]$, sometimes denoted $\text{Uniform}[0,\ell]$.

(c) Note that due to independence,

$$\begin{aligned} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \mathbb{P}(X \in [x,x+\mathrm{d}x], Y \in [y,y+\mathrm{d}y]) = \mathbb{P}(X \in [x,x+\mathrm{d}x])\mathbb{P}(Y \in [y,y+\mathrm{d}y]) \\ &\approx f_X(x) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

so their joint distribution is $f(x,y) = 2x$ on the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(d) $\mathbb{E}[XY] = \int_{x=0}^{1} \int_{y=0}^{1} xy \cdot 2x\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{1} x^2\,\mathrm{d}x = 1/3$.
Alternatively, since $X$ and $Y$ are independent, we can compute $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that

$$\mathbb{E}[X] = \int_0^1 x \cdot 2x\,\mathrm{d}x = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3},$$

and $\mathbb{E}[Y] = 1/2$ since the density of $Y$ is symmetric around $1/2$. Hence,

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{3}.$$

## 3 Continuous Computations

Let $X$ be a continuous random variable whose pdf is $cx^3$ (for some constant $c$) in the range $0 \le x \le 1$, and is 0 outside this range.

(a) Find $c$.

(b) Find $\mathbb{P}[1/3 \le X \le 2/3 \mid X \le 1/2]$.

(c) Find $\mathbb{E}(X)$.

(d) Find $\operatorname{var}(X)$.

**Solution:**

(a) Since our total probability must be equal to 1,

$$\int_0^1 cx^3 \, dx = 1 = \frac{1}{4}cx^4\Big|_{x=0}^1 = \frac{c}{4},$$

so $c = 4$.

(b)

$$\begin{aligned}
\mathbb{P}\left[\frac{1}{3} \le X \le \frac{2}{3} \;\Big|\; X \le \frac{1}{2}\right] &= \frac{\mathbb{P}[1/3 \le X \le 2/3 \cap X \le 1/2]}{\mathbb{P}[X \le 1/2]} = \frac{\mathbb{P}[1/3 \le X \le 1/2]}{\mathbb{P}[X \le 1/2]} \\
&= \frac{\int_{1/3}^{1/2} 4x^3 \, dx}{\int_0^{1/2} 4x^3 \, dx} = \frac{[x^4]_{x=1/3}^{1/2}}{[x^4]_{x=0}^{1/2}} = \frac{(1/2)^4 - (1/3)^4}{(1/2)^4} = \frac{65}{81}.
\end{aligned}$$

(c)

$$\mathbb{E}(X) = \int_0^1 x \cdot 4x^3 \, dx = \int_0^1 4x^4 \, dx = \left[\frac{4}{5}x^5\right]_{x=0}^1 = \frac{4}{5}.$$

(d)

$$\operatorname{var}(X) = \int_0^1 x^2 \cdot 4x^3 \, dx - \mathbb{E}(X)^2 = \int_0^1 4x^5 \, dx - \left(\frac{4}{5}\right)^2 = \left[\frac{2}{3}x^6\right]_{x=0}^1 - \frac{16}{25} = \frac{2}{75}.$$