CS 70 Spring 2018 — Discrete Mathematics and Probability Theory — Babak Ayazifar and Satish Rao — DIS 12A

1 Working with the Law of Large Numbers

(a) A fair coin is tossed and you win a prize if there are more than 60% heads. Which is better: 10 tosses or 100 tosses? Explain.

(b) A fair coin is tossed and you win a prize if there are more than 40% heads. Which is better: 10 tosses or 100 tosses? Explain.

(c) A coin is tossed and you win a prize if there are between 40% and 60% heads. Which is better: 10 tosses or 100 tosses? Explain.

(d) A coin is tossed and you win a prize if there are exactly 50% heads. Which is better: 10 tosses or 100 tosses? Explain.

Solution:

(a) 10 tosses. By LLN, the sample mean should have higher probability to be close to the population mean as n increases. Therefore the average proportion of coins that are heads should be closer to 0.50, and has a lower chance of being greater than 0.60 if there are 100 tosses compared with 10 tosses.

(b) 100 tosses. Based on the first part, consider the inverse of the event "more than 60% heads" and the symmetry of heads and tails.

(c) 100 tosses. Based on the first part, consider the union of the events "more than 60% heads" and "more than 60% tails" ("less than 40% heads").

(d) 10 tosses. Compare the probability of getting equal number of heads and tails between $2n$ and $2n+2$ tosses.

$$\begin{aligned}
\mathbb{P}[n \text{ heads in } 2n \text{ tosses}] &= \binom{2n}{n}\frac{1}{2^{2n}} \\
\mathbb{P}[n+1 \text{ heads in } 2n+2 \text{ tosses}] &= \binom{2n+2}{n+1}\frac{1}{2^{2n+2}} = \frac{(2n+2)!}{(n+1)!(n+1)!}\cdot\frac{1}{2^{2n+2}} \\
&= \frac{(2n+2)(2n+1)2n!}{(n+1)(n+1)n!n!}\cdot\frac{1}{2^{2n+2}} \\
&= \frac{2n+2}{n+1}\cdot\frac{2n+1}{n+1}\binom{2n}{n}\cdot\frac{1}{2^{2n+2}} < \left(\frac{2n+2}{n+1}\right)^2\binom{2n}{n}\cdot\frac{1}{2^{2n+2}} \\
&= 4\binom{2n}{n}\cdot\frac{1}{2^{2n+2}} = \binom{2n}{n}\frac{1}{2^{2n}} = \mathbb{P}[n \text{ heads in } 2n \text{ tosses}]
\end{aligned}$$

The larger n is, the less probability we'll get 50% heads. $\square$

Note: By Stirling's approximation, $\binom{2n}{n}2^{-2n}$ is roughly $(\pi n)^{-1/2}$ for large n.

2 Markov's Inequality and Chebyshev's Inequality

A random variable X has variance $\text{var}(X) = 9$ and expectation $\mathbb{E}[X] = 2$. Furthermore, the value of X is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

(a) $\mathbb{E}\left[X^2\right] = 13$.

(b) $\mathbb{P}[X \leq 1] \leq 8/9$.

(c) $\mathbb{P}[X \geq 6] \leq 9/16$.

(d) $\mathbb{P}[X \geq 6] \leq 9/32$.

Solution:

(a) TRUE. Since $9 = \text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 2^2$, we have $\mathbb{E}[X^2] = 9 + 4 = 13$.

(b) TRUE. Let $Y = 10 - X$. Since X is never exceeds 10, Y is a non-negative random variable. By Markov's inequality,

$$\mathbb{P}[10 - X \geq a] = \mathbb{P}[Y \geq a] \leq \frac{\mathbb{E}[Y]}{a} = \frac{\mathbb{E}[10 - X]}{a} = \frac{8}{a}.$$

Setting $a = 9$, we get $\mathbb{P}[X \leq 1] = \mathbb{P}[10 - X \geq 9] \leq 8/9$.

(c) TRUE. Chebyshev's inequality says $\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \text{var}(X)/a^2$. If we set $a = 4$, we have

$$\mathbb{P}[|X - 2| \geq 4] \leq \frac{9}{16}.$$

Now we observe that $\mathbb{P}[X \geq 6] \leq \mathbb{P}[|X - 2| \geq 4]$, because the event $X \geq 6$ is a subset of the event $|X - 2| \geq 4$.

(d) FALSE. Construct a random variable X that satisfies the conditions in the question but does not have an equal chance of being less than -2 or greater than 6. A simple example would be a random variable that takes on 2 values, where $\mathbb{P}[X = a] = p, \mathbb{P}[X = b] = 1 - p$. The expectation must be 2, so we have $pa + (1 - p)b = 2$. The variance is 9, so $\mathbb{E}\left[X^2\right] = 13$ and $pa^2 + (1 - p)b^2 = 13$. Solving for a and b. One example is $\mathbb{P}[X = 0] = 9/13, \mathbb{P}[X = 13/2] = 4/13$.

3 Vegas

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip his or her coin. Assume that each person is independently likely to carry a fair or a trick coin.

1. Given the results of your experiment, how should you estimate p?
2. How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

Solution:

1. We want to construct an estimate $\hat{p}$ such that $\mathbb{E}[\hat{p}] = p$. Then, if we have a large enough sample, we'd expect to get a good estimate of p. Let X_i be the indicator that the ith person's coin flips to a heads. What we observe is the fraction of people whose coin is heads. In other words, we measure $q = \frac{1}{n}\sum_{i=1}^{n} X_i$. How can we use this observation to construct $\hat{p}$?

   First,
   $$\mathbb{E}[q] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \mathbb{E}[X_i] = p \cdot 1 + (1-p)\cdot\frac{1}{2},$$
   where the last equality follows from total probability. Solving for p, we find that
   $$p = 2\,\mathbb{E}[q] - 1 = \mathbb{E}[2q-1].$$
   Thus, our estimator $\hat{p}$ should be $2q-1$.

2. We want to find n such that $P[|\hat{p}-p| \le 0.05] > 0.95$. Another way to state this is that we want
   $$P[|\hat{p}-p| > 0.05] \le 0.05.$$
   Notice that $\mathbb{E}[\hat{p}] = p$ by construction, so we can immediately apply Chebyshev's inequality on $\hat{p}$. What we get is:
   $$P[|\hat{p}-p| > 0.05] \le \frac{\text{var}[\hat{p}]}{0.05^2} \le 0.05$$
   So, we want n such that $\text{var}[\hat{p}] \le 0.05^3$.
   $$\text{var}[\hat{p}] = \text{var}[2q-1] = 4\,\text{var}[q] = \frac{4}{n^2}\,\text{var}\left[\sum_{i=1}^{n} X_i\right] = \frac{4}{n}\,\text{var}[X_1].$$
   But X_i is an indicator (Bernoulli variable), so its variance is bounded by $\frac{1}{4}$. Therefore we have
   $$\text{var}[\hat{p}] \le \frac{4}{n}\frac{1}{4} = \frac{1}{n}.$$
   So, we choose n such that $\frac{1}{n} \le 0.05^3$, so $n \ge \frac{1}{0.05^3} = 8000$.