## Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

*I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.*

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## 1 Faulty Machines

You are trying to use a machine that only works on some days. If on a given day the machine is working, it will break down the next day with probability $0 < b < 1$, and works on the next day $1-b$. If it is not working on a given day, it will work on the next day with probability $0 < r < 1$, and not work on the next day with probability $1-r$. Formulate this process as a Markov chain. As $n \to \infty$, what does the probability that on a given day the machine is working converge to? What properties of the Markov chain allow us to conclude that the probability will actually converge?

**Solution:** We define the following states $\chi = \{W, B\}$ where $W$ is the state that represents the machine working on a given day, and $B$ is the state that represents the machine broken on a given day. The following are the transition probabilities.

$$\mathbb{P}(W,B) = b; \mathbb{P}(W,W) = 1-b; \mathbb{P}(B,W) = r; \mathbb{P}(B,B) = 1-r$$

We know that the Markov chain is finite and irreducible. Hence, it has a unique invariant distribution $\pi$ Furthermore, since the Markov chain has a self-loop of nonzero probability, it is aperiodic.

Hence, for any probability distribution of states at time n, $\pi_n : \lim_{n\to\infty} \pi_n = \pi$

We use the balance equations to find the invariant distribution.

$$\pi = \pi P$$
$$\pi(W) = (1-b)\pi(W) + r\pi(B)$$
$$\pi(B) = b\pi(W) + (1-r)\pi(B)$$
$$\pi(B) + \pi(W) = 1$$
$$\implies \pi(W) = \frac{r}{b+r}, \pi(B) = \frac{b}{b+r}$$

$\implies$ As $n \to \infty$, the probability that on a given day the machine is working is $\pi(W) = \dfrac{r}{b+r}$

# 2 Markov's Coupon Collecting

Courtney is home for Thanksgiving and needs to make some trips to the Traitor Goes grocery store to prepare for the big turkey feast. Each time she goes to the store before the holiday, she receives one of the $n$ different coupons that are being given away. You may recall that we studied how to calculate the expected number of trips to the store needed to collect at least one of each coupon. Using geometric distributions and indicator variables, we determined that expected number of trips to be $n(\ln n + \gamma)$.

Let's re-derive that, this time with a Markov chain model and first-step equations.

(a) Define the states and transition probabilities for each state (explain what states can be transitioned to, and what probabilities those transitions occur with).

(b) Now set up first-step equations and solve for the expected number of grocery store trips Courtney needs to make before Thanksgiving so that she can have at least one of each of the $n$ distinct coupons.

**Solution:**

(a) We model the coupon collector's problem as a Markov chain with states $X_1, X_2, \ldots, X_n, X_{n+1}$ where $X_i$ represents the state we are at if we have collected $i-1$ of the unique coupons and are seeking the $i^{\text{th}}$ coupon. State $X_{n+1}$ represents the terminal state, after we successfully collected all $n$ coupons and don't need to make any more grocery store trips.

If we are at state $X_i$, we either transition back to $X_i$ with probability $(i-1)/n$, or we collect a new coupon and transition to state $X_{i+1}$ with probability $(n-i+1)/n$. Transitioning to any other state is not possible.

(b) For each state $X_i$:

$$\begin{aligned}
\beta(X_i) &= 1 + \frac{i-1}{n} \cdot \beta(X_i) + \frac{n-i+1}{n} \cdot \beta(X_{i+1}) \\
\frac{n-i+1}{n} \cdot \beta(X_i) &= 1 + \frac{n-i+1}{n} \cdot \beta(X_{i+1}) \\
\beta(X_i) &= \frac{n}{n-i+1} + \beta(X_{i+1})
\end{aligned}$$

We know that for the terminal state, $\beta(X_{n+1}) = 0$. Then:

$$\begin{aligned}
\beta(X_n) &= n \\
\beta(X_{n-1}) &= n + \frac{n}{2} \\
&\vdots \\
\beta X_1 &= n + \frac{n}{2} + \cdots + \frac{n}{n} \\
&= n \sum_{i=1}^{n} \frac{1}{i} \\
&= n(\ln n + \gamma).
\end{aligned}$$

# 3 Three Tails

You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting $TTT$?

**Solution:**

We can model this problem as a Markov chain with the following states:

- $S$: Start state, which we are only in before flipping any coins.
- $H$: We see a head, which means no streak of tails currently exists.
- $T$: We've seen exactly one tail in a row so far.
- $TT$: We've seen exactly two tails in a row so far.
- $TTT$: We've accomplished our goal of seeing three tails in a row and stop flipping.

We can write the first step equations and solve for $\gamma(S)$, only counting heads that we see since we are not looking for the total number of flips. The equations are as follows:

$$\begin{aligned}
\gamma(S) &= .5\gamma(T) + .5\gamma(H) \\
\gamma(H) &= 1 + .5\gamma(H) + .5\gamma(T) \\
\gamma(T) &= .5\gamma(TT) + .5\gamma(H) \\
\gamma(TT) &= .5\gamma(H) + .5\gamma(TTT) \\
\gamma(TTT) &= 0
\end{aligned}$$

From the second equation, we see that

$$.5\gamma(H) = 1 + .5\gamma(T)$$

and can substitute that into equation 3 to get

$$.5\gamma(T) = .5\gamma(TT) + 1.$$

Substituting this into equation 4, we can deduce that $\gamma(TT) = 4$. This allows us to conclude that $\gamma(T) = 6$, $\gamma(H) = 8$, and $\gamma(S) = 7$. On average, we expect to see 7 heads before flipping three tails in a row.

## 4 Playing Blackjack

You are playing a game of Blackjack where you start with \$100. You are a particularly risk-loving player who does not believe in leaving the table until you either make \$400, or lose all your money. At each turn you either win \$100 with probability $p$, or you lose \$100 with probability $1 - p$.

(a) Formulate this problem as a Markov chain i.e. define your state space, transition probabilities, and determine your starting state.

(b) Classify your states as being recurrent or transient. If a given state is recurrent, also determine whether it is an absorbing state.

(c) Find the probability that you end the game with \$400.

**Solution:**

(a) Since it is only possible for us to either win or lose \$100, we define the following state space $\chi = \{0, 100, 200, 300, 400\}$. The following are the transition probabilities:

$$\begin{aligned}
\mathbb{P}(0,0) = \mathbb{P}(400,400) &= 1 \\
\mathbb{P}(i, i+100) &= p \text{ for } i \in \{100, 200, 300\} \\
\mathbb{P}(i, i-100) &= 1 - p \text{ for } i \in \{100, 200, 300\}
\end{aligned}$$

(b) The recurrent states are $0, 400$. This is because once we reach one of these two states, we will not be able to leave it. From both these states, we transition to the same state with probability 1. Hence, 0 and 400 are both absorbing states. The transient states are $100, 200, 300$.

(c) We want to find the probability that we are "absorbed" by state 400 before we are absorbed by state 0. We can calculate this probability as follows.

$$\text{Define } a_i = \text{ Probability of reaching state 400 starting at state } i.$$
$$\implies a_0 = 0, a_{400} = 1$$
$$\implies a_i = (1-p)a_{i-100} + pa_{i+100} \text{ for } i \in \{100, 200, 300\}$$
$$a_{100} = pa_{200}$$
$$a_{200} = (1-p)a_{100} + pa_{300} \implies a_{200}[1 - p(1-p)] = pa_{300}$$
$$\implies a_{200} = \frac{pa_{300}}{1 - p(1-p)}$$
$$a_{300} = (1-p)a_{200} + p \implies a_{300} = \frac{(1-p)pa_{300}}{1 - p(1-p)} + p$$
$$\implies a_{300} = \frac{p(1 - p(1-p))}{1 - 2p(1-p)}$$
$$\implies a_{200} = \frac{p^2}{1 - 2p(1-p)}$$
$$\implies a_{100} = \frac{p^3}{1 - 2p(1-p)}$$