CS 70 Spring 2018 | Discrete Mathematics and Probability Theory | Babak Ayazifar and Satish Rao | DIS 13b

1 Allen's Umbrellas

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

We will model this as a Markov chain. Let $\mathscr{X} = \{0,1,2\}$ be the set of states, where the state i represents the number of umbrellas in his current location. Write down the transition matrix, determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

Solution:

Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

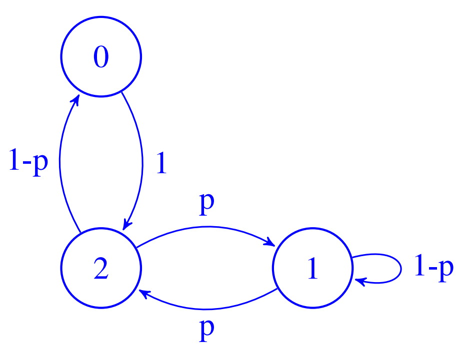
$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 0] = 1.$$

Suppose Allen is in state 1. With probability p, it rains and Allen brings the umbrella, arriving at state 2. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 1] = p, \qquad \mathbb{P}[X_{n+1} = 1 \mid X_n = 1] = 1-p$$

Suppose Allen is in state 2. With probability p, it rains and Allen brings the umbrella, arriving at state 1. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\mathbb{P}[X_{n+1} = 1 \mid X_n = 2] = p, \qquad \mathbb{P}[X_{n+1} = 0 \mid X_n = 2] = 1-p$$



We summarize this with the transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

Observe that the transition matrix has non-zero element in its diagonal, which means the minimum number of steps to transit to state 1 from itself is one. Thus this transition matrix is irreducible and aperiodic, so it converges to its invariant distribution. To solve for the distribution, we set $\pi P = \pi$, or $\pi(P-I) = 0$. This yields the balance equations

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & p \\ 1-p & p & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0) + \pi(1) + \pi(2) = 1$.

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -p & 1 \\ 1-p & p & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Now solve for the distribution:

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} = \frac{1}{3-p} \begin{bmatrix} 1-p & 1 & 1 \end{bmatrix}$$

The invariant distribution also tells us the long-term fraction of time that Allen spends in each state. We can see that Allen spends a fraction $(1-p)/(3-p)$ of his time with no umbrella in his location, so the long-term fraction of time in which he walks through rain is $p(1-p)/(3-p)$.

2 Predicament

Three men are on a boat with cigarettes, but they have no lighter. What do they do?

Solution:

One man throws his cigarette off the boat. The boat is now a cigarette lighter!

3 Guess the Polynomial

Remember the mantra "$d+1$ points uniquely determine a degree $\leq d$ polynomial ($d \in \mathbb{N}$)"?

Write down a polynomial (of any degree) with non-negative, integer coefficients. Your TA will then guess the exact polynomial you have written down, using only information about the polynomial at two points!

Solution:

First, ask for the value of the polynomial at 1. Let x be the value of the polynomial evaluated at 1. Next, ask the student for the value of the polynomial at $x+1$. Expand the student's answer in base $x+1$, and you will have all of the coefficients of the polynomial.

Explanation: If the polynomial is $P(x) = a_n x^n + \cdots + a_0$, where $n = \deg P$, then the base $x+1$ expansion of $P(x+1)$ is a number $d_n \cdots d_0$, such that

$$P(x+1) = \sum_{i=0}^{n} d_i (x+1)^i.$$

We can read off the coefficients of the polynomials from the base $x+1$ expansion: the digit d_i in the expansion equals a_i.

The reason why this works is because we know that $x+1$ is larger than any of the coefficients of the polynomial. Since $x = P(1) = a_n + \cdots + a_0$ and all of the coefficients are non-negative, then $x+1$ is strictly greater than any of the coefficients. Essentially, the purpose of the first step is to gauge how large the coefficients are, and the second step will work as long as you ask for any point which is greater than any individual coefficient (for example, you could just as well ask for the value at $x+2$, but larger numbers means more difficult arithmetic).

This does not contradict the mantra "$d+1$ points uniquely determine a degree $\leq d$ polynomial". In this case we have restricted the coefficients of our polynomials to a very special set (the non-negative integers), so we can do better than asking for $d+1$ points.

4 Which Envelope?

You have two envelopes in front of you containing cash. You know that one envelope contains twice as much money as the other envelope (the amount of money in an envelope is an integer). You are allowed to pick one envelope and see how much cash is inside, and then based on this information, you can decide to switch envelopes or stick with the envelope you already have.

Can you come up with a strategy which will allow you to pick the envelope with more money, with probability strictly greater than $1/2$?

Solution:

Surprisingly, the answer is yes.

First, we are going to choose a random positive integer z. To accomplish this, we flip coins! Let n be the number of flips you need before you see heads. Then, set $z = 2^{n-1} + 0.5$ (okay we lied; this is not an integer, but the 0.5 is just there to break ties later on).

Here is the strategy: look at how much cash is in the envelope you pick. If the amount of cash in the envelope exceeds z, then keep the envelope; otherwise, switch to the other envelope.

Suppose z is smaller than the amount of money in either envelope. Then, you will always keep your original envelope, but since the envelope you chose in the first place is equally likely to be the more lucrative envelope, you still have a $1/2$ probability of choosing the right envelope.

A similar analysis holds if z is greater than the amount of money in either envlope; here you will always switch envelopes, and you have a $1/2$ probability of choosing the right envelope.

However, what happens if z happens to land in between the values of the envelopes? If you initially chose the envelope with less money, then you will switch to the better envelope; if you initially chose the envlope with more money, you will stick with the better envelope. So, in this case, you are guaranteed to end up with the better envelope!

Since z has a positive probability of landing between the values of the envelopes, this strategy gives you a probability of choosing the better envelope which is strictly better than $1/2$.

5 Coloring a Sphere

Consider a sphere in which exactly $1/10$ of the surface of the sphere is colored red (the rest of the sphere is blue). Prove that no matter how the blue is distributed upon the sphere, it is always possible to inscribe a cube inside the sphere so that all of its corners are blue.

Solution:

Suppose we inscribe the cube inside the sphere *randomly*. Let A_i, $i = 1, \dots, 8$, be the event that the ith corner of the cube is red. What is the probability that at least one corner of the cube is colored red? In terms of the events we have defined, this is $\mathbb{P}(A_1 \cup \cdots \cup A_8)$.

The events A_i are not independent, so the probability is tricky to compute, but we can find an upper bound with the union bound.

$$\mathbb{P}(A_1 \cup \cdots \cup A_8) \le \sum_{i=1}^{8} \mathbb{P}(A_i) = \frac{8}{10} < 1.$$

Observe that the probability is strictly less than 1, in other words it is not certain that at least one corner of the cube will be red! The only way this is possible is if there exists some configuration of the cube in which all corners of the cube are blue.

This the idea of a *probabilistic proof*: in order to show that some object exists satisfying some properties, you can consider a random instance of the problem and show that the probability that the object satisfies the properties is strictly greater than 0. This is in fact a widely used technique that was popularized by the mathematician Paul Erdös.