1 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) What is the distribution of X?

(b) What is $\mathbb{P}(X = 7)$?

(c) What is $\mathbb{P}(X \geq 1)$?

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

Solution:

(a) Since we have 20 independent trials, with each trial having a probability $2/5$ of success, $X \sim \text{Binomial}(20, 2/5)$.

(b)

$$\mathbb{P}(X = 7) = \binom{20}{7}\left(\frac{2}{5}\right)^{7}\left(\frac{3}{5}\right)^{13}.$$

(c)

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$\begin{aligned}\mathbb{P}(12 \leq X \leq 14) &= \mathbb{P}(X = 12) + \mathbb{P}(X = 13) + \mathbb{P}(X = 14)\\ &= \binom{20}{12}\left(\frac{2}{5}\right)^{12}\left(\frac{3}{5}\right)^{8} + \binom{20}{13}\left(\frac{2}{5}\right)^{13}\left(\frac{3}{5}\right)^{7} + \binom{20}{14}\left(\frac{2}{5}\right)^{14}\left(\frac{3}{5}\right)^{6}.\end{aligned}$$

2 To Pay or Not to Pay?

Alice goes to Berkeley and she drives to school everyday. Tired of always paying for parking, Alice decides one day not to pay her parking fees. Assume that there is a probability of 0.05 that she gets caught by the meter maid. The parking fee is \$0.25 and if she is caught, her parking ticket is \$10.

(a) How does the expected cost of parking 10 times without paying the meter compare with the cost of paying the meter each time? (*Hint*: Think of Alice getting caught or not as a single biased coin flip with probability 0.05.)

(b) If she parks at the meter 10 times, what is the probability that she will have to pay more than the total amount she could end up saving by not putting the money?

Solution:

(a) If she pays each time, it costs \$2.50. We can see that if Alice doesn't pay, this is a binomial distribution with $n = 10$ and $p = 0.05$, which means the expected number of times she'll get caught in 10 days is $np = 10 \times 0.05 = 0.5$. We multiply that by the cost to see how much he can expect to pay in 10 days if he never pays the meter, which is $0.5 \times \$10 = \$5$. Therefore, It looks like she's better off just paying each time.

(b) This is the probability of Alice getting caught at least one of the 10 days (since the parking ticket is \$10). This is just $1 - (0.95)^{10} \approx 0.40$.

3 Maybe Lossy Maybe Not

Let us say that Alice would like to send a message to Bob, over some channel. Alice has a message of length 4 and sends 5 packets.

(a) Packets are dropped with probability p. What is probability that Bob can successfully reconstruct Alice's message?

(b) Again, packets can be dropped with probability p. The channel may additionally corrupt 1 packet. Alice realizes this and sends 3 additional packets. What is the probability that Bob receives enough packets to successfully reconstruct Alice's message?

(c) Again, packets can be dropped with probability p. This time, packets may be corrupted with probability q. Consider the original scenario where Alice sends 5 packets for a message of length 4. What is probability that Bob can successfully reconstruct Alice's message?

Solution:

(a) Alice's message requires a polynomial of degree 3, which can be uniquely identified by 4 points. Thus, at least 4 points need to make it across the channel. The probability that Bob can recover the message is thus the probability that at most one packet is lost. Since the packets are lost with probability with probability p, we have the probability of losing 1 packet is

$$\binom{5}{1}(1-p)^4 p.$$

The probability of losing 0 packets is $(1-p)^5$. Thus, the probability of losing 0 or 1 packets is

$$\binom{5}{1}(1-p)^4 p + (1-p)^5.$$

This is the probability that Bob receives 4 packets, meaning he can successfully reconstruct the 3-degree polynomial.

(b) Bob needs $n+2k = 6$ packets to guarantee successful reconstruction of Alice's message. There are a total of 8 packets sent, so this guarantee occurs only if 0 packets, 1 packet or 2 packets are lost. The probability of 0 packets lost is

$$(1-p)^8.$$

The probability of one packet lost is

$$\binom{8}{1} p(1-p)^7.$$

The probability of two packets lost is

$$\binom{8}{2} p^2(1-p)^6.$$

Thus, the probability of success is

$$(1-p)^8 + \binom{8}{1} p(1-p)^7 + \binom{8}{2} p^2(1-p)^6.$$

(c) Again, Bob can reconstruct the message if none of the packets are corrupted. We use the same idea as in Part (a). The probability that none of the packets are corrupted is $(1-q)^5$. We know that *on top of* being uncorrupted, we can only at lose at most 1 packet. Thus, we can either lose one packet, which has probability

$$\binom{5}{1} p(1-p)^4.$$

Or, we can lose no packets, which has probability $(1-p)^5$. Yet another possibility is if exactly one packet is corrupted, but that packet is also dropped; in this case, we can recover the message, so long as no other packets are corrupted or dropped. This occurs with probability

$$\binom{5}{1} pq(1-p)^4(1-q)^4.$$

As a result, we have the following.

$$(1-q)^5\left(5p(1-p)^4 + (1-p)^5\right) + 5pq(1-p)^4(1-q)^4.$$