1 Uniform Probability Space

Let $\Omega = \{1,2,3,4,5,6\}$ be a uniform probability space. Let also $X(\omega)$ and $Y(\omega)$, for $\omega \in \Omega$, be the random variables defined in the table:

Table 1: All the rows in the table correspond to random variables.

ω	1	2	3	4	5	6	$\mathbb{E}[\cdot]$
$X(\omega)$	0	0	1	1	2	2	
$Y(\omega)$	0	2	3	5	2	0	
$X^2(\omega)$							
$Y^2(\omega)$							
$XY(\omega)$							
$L[Y \mid X](\omega)$							

(a) Fill in the blank entries of the table. In the column to the far right, fill in the expected value of the random variable.

(b) Are the variables correlated or uncorrelated? Are the variables independent or dependent?

(c) Calculate $\mathbb{E}[(Y - L[Y \mid X])^2]$.

Solution:

(a) See the following table:

ω	1	2	3	4	5	6	$\mathbb{E}[\cdot]$
$X(\omega)$	0	0	1	1	2	2	1
$Y(\omega)$	0	2	3	5	2	0	2
$X^2(\omega)$	0	0	1	1	4	4	$5/3$
$Y^2(\omega)$	0	4	9	25	4	0	7
$XY(\omega)$	0	0	3	5	4	0	2
$L[Y \mid X](\omega)$	2	2	2	2	2	2	2

The third, fourth, and fifth rows can be calculated directly from the corresponding X and Y values. Recall that

$$L[Y \mid X] = \frac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)}\big(X - \mathbb{E}(X)\big) + \mathbb{E}(Y).$$

But $\operatorname{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) = 2 - (1)(2) = 0$, so $L[Y \mid X] = \mathbb{E}(Y) = 2$ for all ω.

(b) Since $\operatorname{cov}(X,Y) = 0$, the variables are uncorrelated. But, we see that $\mathbb{P}(Y = 0) = 1/3$ and $\mathbb{P}(Y = 0 \mid X = 3) = 0$, so the two variables are not independent. Recall that independence implies uncorrelation, but the converse is not true.

(c)
$$\mathbb{E}[(Y - L[Y \mid X])^2] = \mathbb{E}[(Y-2)^2] = \frac{4+0+1+9+0+4}{6} = 3$$

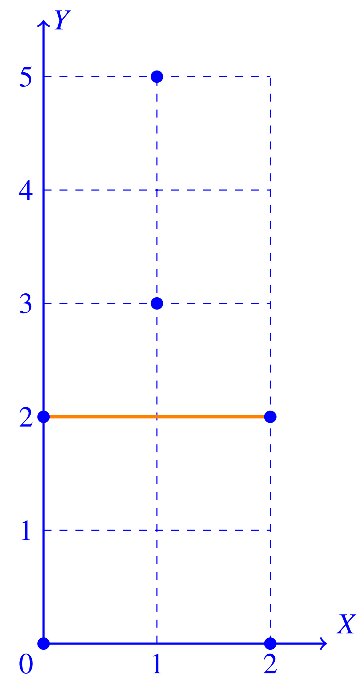

Figure 1: Visualization of regression. The circles are the (X,Y) points. The orange line is the LLSE.

2 LLSE

We have two bags of balls. The fractions of red balls and blue balls in bag A are $2/3$ and $1/3$ respectively. The fractions of red balls and blue balls in bag B are $1/2$ and $1/2$ respectively. Someone gives you one of the bags (unmarked) uniformly at random. You then draw 6 balls from that same bag with replacement. Let X_i be the indicator random variable that ball i is red. Now, let us define $X = \sum_{1 \le i \le 3} X_i$ and $Y = \sum_{4 \le i \le 6} X_i$. Find $L(Y \mid X)$. *Hint*: Recall that

$$L(Y \mid X) = \mathbb{E}(Y) + \frac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)}\big(X - \mathbb{E}(X)\big).$$

Also remember that covariance is bilinear.

Solution:

Note that although the indicator random variables are not independent, we can still apply linearity of expectation. By symmetry, we also know that each indicator follows the same distribution.

Therefore:

$$\begin{aligned}
\mathbb{E}[X] = \mathbb{E}[Y] &= 3 \cdot \mathbb{E}(X_1) = 3 \cdot \mathbb{P}(X_1 = 1) = 3 \cdot \left(\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{7}{4}. \\
\operatorname{cov}(X,Y) &= \operatorname{cov}\Big(\sum_{1 \le i \le 3} X_i, \sum_{4 \le j \le 6} X_j\Big) = 9 \cdot \operatorname{cov}(X_1, X_4) \\
&= 9 \cdot \big(\mathbb{E}(X_1 X_4) - \mathbb{E}(X_1) \cdot \mathbb{E}(X_4)\big). \\
\mathbb{E}(X_1 X_4) - \mathbb{E}(X_1)\,\mathbb{E}(X_4) &= \mathbb{P}(X_1 = 1, X_4 = 1) - \mathbb{P}(X_1 = 1)^2 \\
&= \left[\frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2\right] - \left[\frac{1}{2} \cdot \left(\frac{2}{3}\right) + \frac{1}{2} \cdot \left(\frac{1}{2}\right)\right]^2 = \frac{1}{144}. \\
\operatorname{var}(X) &= \operatorname{cov}\Big(\sum_{1 \le i \le 3} X_i, \sum_{1 \le j \le 3} X_j\Big) \\
&= 3 \cdot \operatorname{var}(X_1) + 6 \cdot \operatorname{cov}(X_1, X_2) = 3\big(\mathbb{E}(X_1^2) - \mathbb{E}(X_1)^2\big) + 6 \cdot \frac{1}{144} \\
&= 3\left[\frac{7}{12} - \left(\frac{7}{12}\right)^2\right] + 6 \cdot \frac{1}{144} = \frac{111}{144}.
\end{aligned}$$

So,

$$L(Y \mid X) = \frac{7}{4} + \frac{9}{111}\left(X - \frac{7}{4}\right) = \frac{3}{37}X + \frac{119}{74}.$$

3 Number of Ones

In this problem, we will revisit dice-rolling, except with conditional expectation.

(a) If we roll a die until we see a 6, how many ones should we expect to see?

(b) If we roll a die until we see a number greater than 3, how many ones should we expect to see?

Solution:

(a) Let Y be the number of ones we see. Let X be the number of rolls we take until we get a 6.

Let us first compute $\mathbb{E}[Y \mid X]$. We know that in each of our $k-1$ rolls before the kth, we necessarily roll a number in $\{1, 2, 3, 4, 5\}$. Thus, we have a $1/5$ chance of getting a one, meaning

$$\mathbb{E}[Y \mid X = k] = \frac{1}{5}(k-1)$$

so

$$\mathbb{E}[Y \mid X] = \frac{1}{5}(X-1).$$

If this is confusing, write Y as a sum of indicator variables.

$$Y = Y_1 + Y_2 + \cdots + Y_k$$

where Y_i is 1 if we see a one on the ith roll. This means

$$\mathbb{E}[Y \mid X = k] = \mathbb{E}[Y_1 \mid X = k] + \mathbb{E}[Y_2 \mid X = k] + \cdots + \mathbb{E}[Y_k \mid X = k].$$

We know for a fact that on the kth roll, we roll a 6, thus $\mathbb{E}[Y_k] = 0$. Thus, we actually consider

$$\begin{aligned}\mathbb{E}[Y_1 \mid X = k] + \mathbb{E}[Y_2 \mid X = k] + \cdots + \mathbb{E}[Y_{k-1} \mid X = k] &= (k-1)\,\mathbb{E}[Y_1 \mid X = k] \\ &= (k-1)\mathbb{P}[Y_1 = 1 \mid X = k] \\ &= (k-1)\frac{1}{5}.\end{aligned}$$

Using the Law of Total Expectation, we know that

$$\begin{aligned}\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] &= \mathbb{E}\left[\frac{1}{5}(X-1)\right] \\ &= \frac{1}{5}\,\mathbb{E}[X-1] \\ &= \frac{1}{5}(\mathbb{E}[X]-1).\end{aligned}$$

Since, $X \sim \text{Geometric}(1/6)$, the expected number of rolls until we roll a 6 is $\mathbb{E}[X] = 6$.

$$\frac{1}{5}(\mathbb{E}[X]-1) = \frac{1}{5}(6-1) = 1.$$

(b) We use the same logic as the first part, except now each of the first $k-1$ rolls can only be 1, 2, or 3, so

$$\mathbb{E}[Y \mid X = k] = \frac{1}{3}(k-1).$$

Then

$$\begin{aligned}\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] &= \mathbb{E}\left[\frac{1}{3}(X-1)\right] \\ &= \frac{1}{3}(\mathbb{E}[X]-1).\end{aligned}$$

Since $X \sim \text{Geometric}(1/2)$, we know that the expected number of rolls until we roll a number greater than 3 is $\mathbb{E}[X] = 2$. This makes $\mathbb{E}[Y] = 1/3$.