CS 70 Discrete Mathematics and Probability Theory
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DIS 0B

1 Contraposition

Prove the statement "if $a+b < c+d$, then $a < c$ or $b < d$".

Solution:

The implication we're trying to prove is $(a+b < c+d) \implies ((a < c) \vee (b < d))$, so the contrapositive is $((a \geq c) \wedge (b \geq d)) \implies (a+b \geq c+d)$. The proof of this is quite straightforward: since we have both that $a \geq c$ and that $b \geq d$, we can just add these two inequalities together, giving us $a+b \geq c+d$, which is exactly what we wanted.

2 Perfect Square

A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Solution:

We will proceed with a direct proof. Let $n = m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m = 2l+1$ for some integer l. Then, $m^2 = 4l^2 + 4l + 1 = 4l(l+1) + 1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.

3 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.
Here, we used the pigeonhole principle because assuming for contradiction that everyone has a

different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0,1,\ldots,n-1$. The objects assigned to these containers are the people at the party. However, container 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.

4 Fermat's Contradiction

Prove that $2^{1/n}$ is not rational for any integer $n \geq 3$. (*Hint*: Use Fermat's Last Theorem.)

Solution:

If not, then there exists an integer $n \geq 3$ such that $2^{1/n} = \frac{p}{q}$ where p,q are positive integers. Thus, $2q^n = p^n$, and this implies

$$q^n + q^n = p^n,$$

which is a contradiction to the Fermat's Last Theorem.

5 Prime Form

Prove that every prime number $m > 3$ is either of the form $6k+1$ or $6k-1$ for some integer k.

Solution: Any integer can be written in the form $6k+i$, where $i \in \{0,1,2,3,4,5\}$. Therefore we have 6 cases:

1. $m = 6k+0$. This can't be true since then m can't be prime (has factors other than itself and 1).
2. $m = 6k+1$. This number could be prime.
3. $m = 6k+2$. This can't be true since $m = 6k+2 = 2(3k+1)$.
4. $m = 6k+3$. This can't be true since $m = 6k+3 = 3(2k+1)$.
5. $m = 6k+4$. This can't be true since $m = 6k+4 = 2(3k+2)$.
6. $m = 6k+5 = 6(k+1)-1$. This number could be prime.

   Therefore if m is a prime number, it must either have the form $6k+1$ or $6k-1$.