CS 70 Spring 2018 — Discrete Mathematics and Probability Theory — Satish Rao and Babak Ayazifar — HW 8

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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1 Faulty Lightbulbs

Box 1 contains 1000 lightbulbs of which 10% are defective. Box 2 contains 2000 lightbulbs of which 5% are defective.

(a) Suppose a box is given to you at random and you randomly select a lightbulb from the box. If that lightbulb is defective, what is the probability you chose Box 1?

(b) Suppose now that a box is given to you at random and you randomly select two light- bulbs from the box. If both lightbulbs are defective, what is the probability that you chose from Box 1?

Solution:

(a) Let:

- D denote the event that the lightbulb we selected is defective.
- B_i denote the event that the lightbulb we selected is from Box i.

We wish to compute $\mathbb{P}[B_1 \mid D]$. Using Bayes' Rule we get:

$$
\begin{aligned}
\mathbb{P}[B_1 \mid D] &= \frac{\mathbb{P}[D \mid B_1] \cdot \mathbb{P}[B_1]}{\mathbb{P}[B_1] \cdot \mathbb{P}[D \mid B_1] + \mathbb{P}[B_2] \cdot \mathbb{P}[D \mid B_2]} \\
&= \frac{0.1 \cdot 0.5}{0.5 \cdot 0.1 + 0.5 \cdot 0.05} \\
&= \frac{2}{3}
\end{aligned}
$$

(b) Let:

- D' denote the event that both the lightbulbs we selected are defective.
- B_i denote the event that the lightbulb we selected is from Box i.

We wish to compute $\mathbb{P}[B_1 \mid D']$. Using Bayes' Rule we get:

$$
\begin{aligned}
\mathbb{P}[B_1 \mid D'] &= \frac{\mathbb{P}[D' \mid B_1] \cdot \mathbb{P}[B_1]}{\mathbb{P}[B_1] \cdot \mathbb{P}[D' \mid B_1] + \mathbb{P}[B_2] \cdot \mathbb{P}[D' \mid B_2]} \\
&= \frac{\frac{100}{1000} \cdot \frac{99}{999} \cdot 0.5}{0.5 \cdot \frac{100}{1000} \cdot \frac{99}{999} + 0.5 \cdot \frac{100}{2000} \cdot \frac{99}{1999}} \\
&= 0.8
\end{aligned}
$$

2 Solve the Rainbow

Your roommate was having Skittles for lunch and they offer you some. There are five different colors in a bag of Skittles: red, orange, yellow, green, and purple, and there are 20 of each color. You know your roommate is a huge fan of the green Skittles. With probability $1/2$ they ate all of the green ones, with probability $1/4$ they ate half of them, and with probability $1/4$ they only ate 5 green ones.

(a) If you take a Skittle from the bag, what is the probability that it is green?

(b) If you take two Skittles from the bag, what is the probability that at least one is green?

(c) If you take three Skittles from the bag, what is the probability that they are all green?

(d) If all three Skittles you took from the bag are green, what are the probabilities that your roommate had all of the green ones, half of the green ones, or only 5 green ones?

(e) If you take three Skittles from the bag, what is the probability that they are all the same color?

Solution:

(a) We will use the law of total probability. Let G be the event that you take a green Skittles from the bag, A be the event that your roommate ate all of the green Skittles, H be the event that your roommate ate half the green Skittles, and F be the event that your roommate ate five green Skittles. Then, we get the total probability as following:

$$\begin{aligned}
\mathbb{P}(G) &= \mathbb{P}(G \cap A) + \mathbb{P}(G \cap H) + \mathbb{P}(G \cap F) && (1)\\
&= \mathbb{P}(G \mid A)\mathbb{P}(A) + \mathbb{P}(G \mid H)\mathbb{P}(H) + \mathbb{P}(G \mid F)\mathbb{P}(F) && (2)\\
&= 0 \cdot \frac{1}{2} + \frac{10}{90} \cdot \frac{1}{4} + \frac{15}{95} \cdot \frac{1}{4} \approx 0.0673 && (3)
\end{aligned}$$

(b) We will consider the complement event, that neither of them are green. Let's call the event that at least one of them is green B, this makes the complement $\overline{B}$ the event that neither Skittles are green. Using the same approach as the previous part, we will get the following:

$$\begin{aligned}
\mathbb{P}(\overline{B}) &= \mathbb{P}(\overline{B} \cap A) + \mathbb{P}(\overline{B} \cap H) + \mathbb{P}(\overline{B} \cap F) && (4)\\
&= \mathbb{P}(\overline{B} \mid A)\mathbb{P}(A) + \mathbb{P}(\overline{B} \mid H)\mathbb{P}(H) + \mathbb{P}(\overline{B} \mid F)\mathbb{P}(F) && (5)\\
&= 1 \cdot \frac{1}{2} + \frac{80}{90} \cdot \frac{79}{89} \cdot \frac{1}{4} + \frac{80}{95} \cdot \frac{79}{94} \cdot \frac{1}{4} \approx 0.874 && (6)
\end{aligned}$$

This makes our final answer the following:

$$\mathbb{P}(B) = 1 - \mathbb{P}(\overline{B}) \approx 0.126$$

(c) Let's call the event of having 3 green Skittles G_3. This event is impossible if our roommate ate all the green Skittles.

If they ate half, we have the probability of G_3 as

$$\frac{10 \times 9 \times 8}{90 \times 89 \times 88}.$$

We can see this by noticing that given our roommate ate half the green Skittles, there will be 10 green Skittles left out of the 90 that are still in the bag. After the first one is removed, there will be 9 out of 89 that are green, and so on.

Similarly, if they ate only five green Skittles, we have the probability of G_3 as

$$\frac{15 \times 14 \times 13}{95 \times 94 \times 93},$$

giving us the final result as:

$$\begin{aligned}
\mathbb{P}(G_3) &= \mathbb{P}(G_3 \mid A)\mathbb{P}(A) + \mathbb{P}(G_3 \mid H)\mathbb{P}(H) + \mathbb{P}(G_3 \mid F)\mathbb{P}(F) && (7)\\
&= 0 \cdot \frac{1}{2} + \frac{10 \times 9 \times 8}{90 \times 89 \times 88} \cdot \frac{1}{4} + \frac{15 \times 14 \times 13}{95 \times 94 \times 93} \cdot \frac{1}{4} && (8)\\
&\approx 0.00108 && (9)
\end{aligned}$$

(d) We can use the Bayes Rule to solve this.

$$\mathbb{P}(A \mid G_3) = \frac{\mathbb{P}(G_3 \cap A)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid A)\mathbb{P}(A)}{\mathbb{P}(G_3)} = \frac{0 \times 1/2}{0.00108} = 0$$

This makes intuitive sense, since if you took three green Skittles out of the bag, it is impossible that your roommate ate all of them. Using it for the two other conditions, we get:

$$\mathbb{P}(H \mid G_3) = \frac{\mathbb{P}(G_3 \cap H)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid H)\mathbb{P}(H)}{\mathbb{P}(G_3)} = \frac{10 \times 9 \times 8}{90 \times 89 \times 88} \cdot \frac{1}{4} \cdot \frac{1}{0.00108} \approx 0.237$$

$$\mathbb{P}(F \mid G_3) = \frac{\mathbb{P}(G_3 \cap F)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid F)\mathbb{P}(F)}{\mathbb{P}(G_3)} = \frac{15 \times 14 \times 13}{95 \times 94 \times 93} \cdot \frac{1}{4} \cdot \frac{1}{0.00108} \approx 0.763$$

Note that the sum of these probabilities add up to 1.

(e) We can divide this into two cases. If the color of all the Skittles is green, we have already calculated the probability in the previous part.

For all other colors, we can notice that the probabilities will have the same structure, and since these are disjoint events, we can add them to get our final result. Let's find the probability for the case of getting three red Skittles, let's call this event R_3. We find this probability as follows:

$$\begin{aligned}
\mathbb{P}(R_3) &= \mathbb{P}(R_3 \mid A)\mathbb{P}(A) + \mathbb{P}(R_3 \mid H)\mathbb{P}(H) + \mathbb{P}(R_3 \mid F)\mathbb{P}(F) && (10)\\
&= \frac{20 \times 19 \times 18}{80 \times 79 \times 78} \cdot \frac{1}{2} + \frac{20 \times 19 \times 18}{90 \times 89 \times 88} \cdot \frac{1}{4} + \frac{20 \times 19 \times 18}{95 \times 94 \times 93} \cdot \frac{1}{4} && (11)\\
&\approx 0.0114 && (12)
\end{aligned}$$

If we call the probability of getting three Skittles of the same color X_3, we can find it by adding the probability for the events for different colors such as G_3, and R_3. The probability for getting 3 of the same color for yellow, orange, and purple will be the same as it was for red. Using the same name convention for red and green for the other colors, this can be summed up as:

$$\begin{aligned}
\mathbb{P}(X_3) &= \mathbb{P}(G_3) + \mathbb{P}(R_3) + \mathbb{P}(Y_3) + \mathbb{P}(O_3) + \mathbb{P}(P_3) && (13)\\
&= \mathbb{P}(G_3) + 4\mathbb{P}(R_3) && (14)
\end{aligned}$$

The above holds since these are all disjoint events, we can't get all three Skittles to be the same color for different colors at the same time. Overall, getting this means we are adding these probabilities, giving us:

$$\mathbb{P}(X_3) = \mathbb{P}(G_3) + 4 \cdot \mathbb{P}(R_3) \approx 0.0468$$

3 Easter Eggs

You made the trek to Soda for a Spring Break-themed homework party, and every attendee gets to leave with a party favor. You're given a bag with 20 chocolate eggs and 40 (empty) plastic eggs. You pick 5 eggs without replacement.

(a) What is the probability that the first egg you drew was a chocolate egg?

(b) What is the probability that the second egg you drew was a chocolate egg?

(c) Given that the first egg you drew was an empty plastic one, what is the probability that the fifth egg you drew was also an empty plastic egg?

Solution:

(a) $\mathbb{P}(\text{chocolate egg}) = \frac{20}{60} = \frac{1}{3}$.

(b) Long calculation using Total Probability Rule: let C_i denote the event that the ith egg is chocolate, and P_i denote the event that the ith egg is plastic. We have

$$\begin{aligned} \mathbb{P}(C_2) &= \mathbb{P}(C_1 \cap C_2) + \mathbb{P}(P_1 \cap C_2) \\ &= \mathbb{P}(C_1)\mathbb{P}(C_2 \mid C_1) + \mathbb{P}(P_1)\mathbb{P}(C_2 \mid P_1) \\ &= \frac{1}{3} \cdot \frac{19}{59} + \frac{2}{3} \cdot \frac{20}{59} \\ &= \frac{1}{3}. \end{aligned} \tag{15}$$

Short calculation: By symmetry, this is the same probability as part (a), $1/3$. This is because we don't know what type of egg was picked on the first draw, so the distribution for the second egg is the same as that of the first. To see this rigorously observe that $\mathbb{P}[C_2 \cap P_1] = \mathbb{P}[P_2 \cap C_1]$ and, thus:

$$\begin{aligned} \mathbb{P}[C_2] &= \mathbb{P}[C_2 \cap C_1] + \mathbb{P}[C_2 \cap P_1] \\ &= \mathbb{P}[C_2 \cap C_1] + \mathbb{P}[P_2 \cap C_1] \\ &= \mathbb{P}[C_1] \ . \end{aligned}$$

(c) By symmetry, since we don't know any information about the 2nd, 3rd, or 4th eggs, $\mathbb{P}(\text{5th egg = plastic} \mid \text{1st egg = plastic}) = \mathbb{P}(\text{2nd egg = plastic} \mid \text{1st egg = plastic}) = 39/59$. Rigorously, notice that $\mathbb{P}[C_5 \cap P_2 \mid P_1] = \mathbb{P}[P_5 \cap C_2 \mid P_1]$ and therefore:

$$\begin{aligned} \mathbb{P}[P_5 \mid P_1] &= \mathbb{P}[P_5 \cap C_2 \mid P_1] + \mathbb{P}[P_5 \cap P_2 \mid P_1] \\ &= \mathbb{P}[C_5 \cap P_2 \mid P_1] + \mathbb{P}[P_5 \cap P_2 \mid P_1] \\ &= \mathbb{P}[P_2 \mid P_1] \ . \end{aligned}$$

Of coures, the Bayes Rule calculation would also work but it is really tedious.

4 Cliques in Random Graphs

Consider a graph $G(V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. So for example if $n=2$, then with probability $1/2$, $G(V,E)$ is the graph consisting of two vertices connected by an edge, and with probability $1/2$ it is the graph consisting of two isolated vertices.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. What is the probability that a particular set of k vertices forms a k-clique?

(c) Prove that the probability that the graph contains a k-clique for $k = 4\lceil \log n \rceil + 1$ is at most $1/n$. You may use the fact that $\binom{n}{k} \le n^k$ without proof.

Solution:

(a) There are two choices for each of the $\binom{n}{2}$ pairs of vertices, so the size of the sample space is $2^{\binom{n}{2}}$.

(b) For a fixed set of k vertices to be a k-clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.

(c) Let A_S denote the event that S is a k-clique, where $S \subseteq V$ is of size k. Then, the event that the graph contains a k-clique can be described as the union of A_S's over all $S \subseteq V$ of size k. Using the union bound,

$$\mathbb{P}\left[\bigcup_{S\subseteq V, |S|=k} A_S\right] \le \sum_{S\subseteq V, |S|=k} \mathbb{P}[A_S] = \sum_{S\subseteq V, |S|=k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size k, the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{k(k-1)/2}} \le \frac{n^k}{\left(2^{(k-1)/2}\right)^k} \le \frac{n^k}{\left(2^{(4\log n+1-1)/2}\right)^k} = \frac{n^k}{\left(2^{2\log n}\right)^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \le \frac{1}{n}.$$

5 Identity Theft

A group of n friends go to the gym together, and while they are playing basketball, they leave their bags against the nearby wall. An evildoer comes, takes the student ID cards from the bags, randomly rearranges them, and places them back in the bags, one ID card per bag. What is the probability that no one receives his or her own ID card back? [*Hint*: Use the generalized inclusion-exclusion principle.]

Then, find an approximation for the probability as $n \to \infty$. You may, without proof, refer to the power series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Solution:

We are looking for the probability of the event that no one receives his or her own ID card back. It is easier to consider the complement of the above event, which is the event that at least one person receives his or her ID card back. Let A_i, $i = 1, \ldots, n$, be the event that the ith friend receives his or her own ID card back, so the event we are considering now is $A_1 \cup \cdots \cup A_n$. We will compute this probability using the generalized inclusion-exclusion formula. Recall that for events a set of n events $B_1, B_2, \ldots, B_n$ this is

$$\mathbb{P}[\bigcup_{i=1}^{n} B_i] = \sum_{i=1}^{n} \mathbb{P}[B_i] - \sum_{i,j} \mathbb{P}[B_i \cap B_j] + \sum_{i,j,k} \mathbb{P}[B_i \cap B_j \cap B_k] - \ldots \pm \mathbb{P}[\bigcap_{i=1}^{n} B_i] \ .$$

- First, we add $\mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n)$. Here, $\mathbb{P}(A_i)$ is the probability that the ith friend receives his or her own ID card back, which is $1/n$. So, we add $n \cdot (1/n) = 1$.
- Next, we subtract $\sum_{(i,j)} \mathbb{P}(A_i \cap A_j)$, where the sum runs over all $(i, j) \in \{1, \ldots, n\}^2$ with $i < j$. Note that $\mathbb{P}(A_i \cap A_j)$ is the probability that both friend i and friend j receive their own ID cards back, which has probability $(n-2)!/n!$. (To see this, observe that once we have decided that friends i and j will receive their own ID cards back, there are $(n-2)!$ ways to permute the ID cards of the $n-2$ other friends, and there are $n!$ total permutations of the n ID cards.) So, we subtract $\sum_{(i,j)} (n-2)!/n!$, but the summation has $\binom{n}{2}$ terms, so we subtract a total of

$$\binom{n}{2} \frac{(n-2)!}{n!} = \frac{n!}{2!(n-2)!} \cdot \frac{(n-2)!}{n!} = \frac{1}{2!}.$$

- At the kth step of the inclusion-exclusion process, we add $(-1)^{k+1} \sum_{(i_1,\ldots,i_k)} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$, where the k-tuples in the summation range over all $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ with $i_1 < \cdots < i_k$. To compute $\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$, note that we have decided that k friends will receive their own ID cards back, the remaining $n-k$ ID cards can be permuted in $(n-k)!$ ways, and there are $n!$ total permutations, so the probability is $(n-k)!/n!$. The summation has a total of $\binom{n}{k}$ terms, so we add

$$(-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = (-1)^{k+1} \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{n!} = (-1)^{k+1} \frac{1}{k!}.$$

Now, adding up all of these probabilities together, we have

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!}.$$

Recall that $A_1 \cup \cdots \cup A_n$ is the *complement* of the event we were originally interested in. So,

$$\mathbb{P}(\text{no friends receive their own ID cards back}) = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Recall the power series for e^x:

$$\mathrm{e}^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Therefore, we have the approximation (which gets better as $n \to \infty$):

$$\mathbb{P}(\text{no friends receive their own ID cards back}) \approx \frac{1}{\mathrm{e}} \approx 0.368.$$

6 Playing Strategically

Bob, Eve and Carol bought new slingshots. Bob is not very accurate hitting his target with probability $1/3$. Eve is better, hitting her target with probability $2/3$. Carol never misses. They decide to play the following game: They take turns shooting each other. For the game to be fair, Bob starts first, then Eve and finally Carol. Any player who gets shot has to leave the game. What is Bob's best course of action regarding his first shot?

(a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first.

(b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.

(c) Compute the probability of the same events for a duel of Bob against Carol.

(d) Assuming that both Eve and Carol play rationally, conclude that Bob's best course of action is to shoot into the air (i.e., intentionally miss)! (Hint: What happens if Bob misses? What if he doesn't?)

Solution:

(a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first.

Observe that:

$$\begin{aligned}\mathbb{P}[E_1] &= \mathbb{P}[\text{Bob hits Eve }] + \mathbb{P}[\text{Bob misses Eve}]\mathbb{P}[\text{Eve misses Bob}]\mathbb{P}[E_1] \\ &= \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3}\mathbb{P}[E_1]\end{aligned}$$

Thus, $\mathbb{P}[E_1] = \frac{3}{7}$.

(b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.

Observe that:

$$\begin{aligned}\mathbb{P}[E_2] &= \mathbb{P}[\text{Eve misses Bob}]\,(\mathbb{P}[\text{Bob hits Eve}] + \mathbb{P}[\text{Bob misses Eve}]\mathbb{P}[E_2]) \\ &= \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3}\mathbb{P}[E_2]\right)\end{aligned}$$

Thus, $\mathbb{P}[E_2] = \frac{1}{7}$.

(c) Compute the probability of the same events for a duel of Bob against Carol alone.

The probability of the event E_3 that Bob, with shot, survives against Carol is:

$$\begin{aligned}\mathbb{P}[E_3] &= \mathbb{P}[\text{Bob hits Carol}] + \mathbb{P}[\text{Bob misses Carol}]\mathbb{P}[\text{Carol misses Bob}]\mathbb{P}[E_3] \\ &= \frac{1}{3} + \frac{2}{3}\cdot 0 \\ &= \frac{1}{3}\ .\end{aligned}$$

The probability of the event E_4 that Bob, without shot, survives against Carol is:

$$\mathbb{P}[E_4] \le \mathbb{P}[\text{Carol misses}] = 0\ .$$

(d) To maximize their chances each player prefers to be left with a weaker opponent. This means that Eve would not shoot at Bob in preference to Carol, and Carol will not shoot at Bob in preference to Eve. Therefore if Bob misses, he will not be shot at until either Eve or Carol lose and he will either be left standing with Eve or Carol, with or without the shot.

So Bob is best off not shooting anyone since the advantage he gains by having the first shot exceeds any possible benefit of facing Eve rather than Carol. He should shoot into the air.