CS 70 Spring 2018 — Discrete Mathematics and Probability Theory — Babak Ayazifar and Satish Rao — HW 11

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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1 More Family Planning

(a) Suppose we have a random variable $N \sim Geom(1/3)$ representing the number of children of a randomly chosen family. Assume that within the family, children are equally likely to be boys and girls. Let B be the number of boys and G the number of girls in the family. What is the joint probability distribution of B, G?

(b) Given that we know there are 0 girls in the family, what is the most likely number of boys in the family?

(c) Now let X and Y be independent random variables representing the number of children in two independently, randomly chosen families. Suppose $X \sim Geom(p)$ and $Y \sim Geom(q)$. Using their joint distribution, find the probability that the number of children in the first family (X) is less than the number of children in the second family (Y). (You may use the convergence formula for a Geometric Series: $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ for $|r| < 1$)

(d) Show how you could obtain your answer from the previous part using an interpretation of the geometric distribution.

Solution:

(a) Conditional on $N = n$, we see that $B \sim Bin(n, \frac{1}{2})$. Then, because we have that $N = X + Y$,

$$\mathbb{P}[B = b, G = g] = \mathbb{P}[B = b, N = b + g]$$

$$= \mathbb{P}[N = b+g]\mathbb{P}[B = b | N = b+g]$$
$$= (1/3)(2/3)^{b+g-1}\binom{b+g}{b}(1/2)^{b+g}$$

(b) We have that

$$\mathbb{P}[B = b | G = 0] = \frac{\mathbb{P}[B = b, G = 0]}{\mathbb{P}[G = 0]} = \frac{(1/3)(2/3)^{b-1}\binom{b}{b}(1/2)^b}{\mathbb{P}[G = 0]} = \frac{2(1/3)^b}{\mathbb{P}[G = 0]}$$

Since this decreases with b, this is maximized when $B = 1$.

(c) We have that for a geometric random variable Y with parameter q, $\mathbb{P}[Y > n] = (1-q)^n$. Now, since X and Y are independent, we have that $\mathbb{P}[X = n, Y > n] = \mathbb{P}[X = n]\mathbb{P}[Y > n] = p(1-p)^{n-1}(1-q)^n$. Then:

$$\mathbb{P}[X < Y] = \sum_{n=1}^{\infty} \mathbb{P}[X = n, Y > n]$$
$$= \sum_{n=1}^{\infty} p(1-p)^{n-1}(1-q)^n$$
$$= p(1-q)\sum_{n=1}^{\infty}(1-p)^{n-1}(1-q)^{n-1}$$
$$= p(1-q)\sum_{n=0}^{\infty}(1-p)^{n}(1-q)^{n}$$
$$= p(1-q)\sum_{n=0}^{\infty}[(1-p)(1-q)]^{n}$$
$$= p(1-q)\cdot\frac{1}{1-(1-p)(1-q)}$$
$$= \frac{p(1-q)}{p+q-pq}$$

(d) If we treat the scenario of two geometric distributions as carrying out two different experiments at the same time (with probabilities p and q of success), we see that finding $\mathbb{P}[Y > X]$ is the same as seeing that Y took more trials for a success than X. Then, we can ignore all trials where both experiments failed (i.e. only consider the first success in either experiment). From this, conditioned on the probability there was indeed one success among the two experiments (which has probability $p+q-pq$ by inclusion-exclusion), the probability that $Y > X$ is just the probability that X succeeded and Y did not, which is $p(1-q)$. Then, $\mathbb{P}[Y > X] = \frac{p(1-q)}{p+q-pq}$

We could alternatively use a recursive definition of the probability that X is less than Y to arrive at the same solution. We let $Z = \mathbb{P}[X < Y]$. Then, on the first trial, for X to be less than Y, either X is successful and Y is not (with probability $p(1-q)$), or both are not successful and we repeat the experiment (with probability $p(1-q)Z$). Then, we get the equation $Z = p(1-q) + (1-p)(1-q)Z$, and we solve for Z to get $Z = \frac{p(1-q)}{p+q-pq}$, the desired answer.

2 Uniform Means

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed uniform random variables on the interval $[0,1]$ (where n is a positive integer).

(a) Let $Y = \min\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Y)$. [*Hint*: Use the tail sum formula, which says the expected value of a nonnegative random variable is $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\,\mathrm{d}x$. Note that we can use the tail sum formula since $Y \geq 0$.]

(b) Let $Z = \max\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Z)$. [*Hint*: Find the CDF.]

Solution:

(a) To calculate $\mathbb{P}(Y > y)$, where $y \in [0,1]$, this means that each X_i is greater than y, for $i = 1, \ldots, n$, so $\mathbb{P}(Y > y) = (1-y)^n$. We then use the tail sum formula:

$$\mathbb{E}(Y) = \int_0^1 \mathbb{P}(Y > y)\,\mathrm{d}y = \int_0^1 (1-y)^n\,\mathrm{d}y = -\frac{1}{n+1}(1-y)^{n+1}\Big|_0^1 = \frac{1}{n+1}.$$

Alternative Solution 1:
As explained above, $\mathbb{P}[Y \leq y] = 1-(1-y)^n$. This gives us the CDF, and if we take its derivative we'll get the probability density function $f(y) = n(1-y)^{n-1}$.

Then

$$\mathbb{E}(Y) = \int_0^1 y \cdot n(1-y)^{n-1}\,\mathrm{d}y.$$

Perform a u substitution, where $u = 1-y$ and $\mathrm{d}u = -\mathrm{d}y$. We see:

$$\begin{aligned}\mathbb{E}(Y) &= n \cdot \int_0^1 -(1-u)\cdot u^{n-1}\,\mathrm{d}u = n \cdot \int_0^1 (u^n - u^{n-1})\,\mathrm{d}u = n\left[\frac{u^{n+1}}{n+1} - \frac{u^n}{n}\right]_{u=0}^1 \\ &= n\left[\frac{(1-y)^{n+1}}{n+1} - \frac{(1-y)^n}{n}\right]_{y=0}^1 = n\left[0 - \left(\frac{1}{n+1} - \frac{1}{n}\right)\right] = n\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{1}{n+1}.\end{aligned}$$

Alternative Solution 2:
Consider adding another independent uniform variable X_{n+1}. $\mathbb{P}(X_{n+1} < Y)$ is just the probability that X_{n+1} is the minimum, which is $1/(n+1)$ by symmetry since all the X_i's are identical. It so happens that because X_{n+1} is a uniform variable on [0,1], this probability is equal to $\mathbb{E}(Y)$. Let f_Y denote the PDF of Y.

$$\begin{aligned}\mathbb{P}(X_{n+1} < Y) &= \int_0^1 \mathbb{P}(X_{n+1} < y \mid Y = y) f_Y(y)\,\mathrm{d}y \\ &= \int_0^1 \mathbb{P}(X_{n+1} < y) f_Y(y)\,\mathrm{d}y && \text{(by independence)} \\ &= \int_0^1 y f_Y(y)\,\mathrm{d}y && \text{(CDF of the uniform distribution)} \\ &= \mathbb{E}(Y).\end{aligned}$$

Alternative Solution 3:
Since $X_1, \dots, X_n$ are i.i.d., their values split the interval $[0,1]$ into $n+1$ sections, and we expect these sections to be of equal length because they are uniformly distributed. Therefore, $\mathbb{E}(Y) = 1/(n+1)$, the position of the smallest indicator.

(b) We could use the tail sum formula, but it turns out that the CDF is in a form that makes it easy to take an integral. If $Z \le z$, where $z \in [0,1]$, each X_i must be less than z, which happens with probability z, so $\mathbb{P}[Z \le z] = z^n$. This gives us the CDF, and if we take its derivative we'll get the probability density function $f(z) = nz^{n-1}$. Then

$$\mathbb{E}(Z) = \int_0^1 z \cdot nz^{n-1}\,\mathrm{d}z = \int_0^1 nz^n\,\mathrm{d}z = \left[n \cdot \frac{z^{n+1}}{n+1}\right]_{z=0}^{1} = \frac{n}{n+1}.$$

Alternative Solution:
As in the previous part, add another independent uniform random variable X_{n+1}. The probability $\mathbb{P}(X_{n+1} > Z)$ is just the probability that X_{n+1} is the maximum, which is $1/(n+1)$ by symmetry.

$$\begin{aligned}
\mathbb{P}(X_{n+1} > Z) &= \int_0^1 \mathbb{P}(X_{n+1} > z \mid Z = z) f_Z(z)\,\mathrm{d}z = \int_0^1 \mathbb{P}(X_{n+1} > z) f_Z(z)\,\mathrm{d}z \\
&= \int_0^1 (1-z) f_Z(z)\,\mathrm{d}z = \int_0^1 f_Z(z)\,\mathrm{d}z - \int_0^1 z f_Z(z)\,\mathrm{d}z \\
\frac{1}{n+1} &= 1 - \mathbb{E}(Z) \\
\mathbb{E}(Z) &= \frac{n}{n+1}
\end{aligned}$$

Alternative Solution 2:
Since $X_1, \dots, X_n$ are i.i.d., their values split the interval $[0,1]$ into $n+1$ sections, and we expect these sections to be of equal length because they are uniformly distributed. The expectation of the smallest X_i is $1/(n+1)$, the expectation of the second smallest is $2/(n+1)$, etc. Therefore, $\mathbb{E}(Z) = n/(n+1)$, the position of the largest indicator.

3 Variance of the Minimum of Uniform Random Variables

Let n be a positive integer and let $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}[0,1]$. Find $\operatorname{var} Y$, where

$$Y := \min\{X_1, \dots, X_n\}.$$

(Hint: If you get stuck with the integral for $\mathbb{E}[Y^2]$, try reviewing how to perform integration by parts.)

Solution:

We know that the density of Y is $f(y) = n(1-y)^{n-1}$, for $y \in [0,1]$, and $\mathbb{E}[Y] = (n+1)^{-1}$. It remains to compute (via integration by parts)

$$\begin{aligned}\mathbb{E}[Y^2] &= \int_0^1 y^2 \cdot n(1-y)^{n-1}\,\mathrm{d}y = n\int_0^1 y^2(1-y)^{n-1}\,\mathrm{d}y = -y^2(1-y)^n\Big|_0^1 + 2\int_0^1 y(1-y)^n\,\mathrm{d}y \\ &= \frac{2}{n+1}\int_0^1 y(n+1)(1-y)^n\,\mathrm{d}y.\end{aligned}$$

Since $g(y) := (n+1)(1-y)^n$ is the density of the minimum of $n+1$ i.i.d. Uniform$[0,1]$ random variables, we recognize the last integral as the expectation of this minimum, which is $1/(n+2)$. Thus,

$$\mathbb{E}[Y^2] = \frac{2}{(n+1)(n+2)}$$

and so

$$\operatorname{var} Y = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{2}{(n+1)(n+2)} - \frac{1}{(n+1)^2} = \frac{2(n+1)-(n+2)}{(n+1)^2(n+2)} = \frac{n}{(n+1)^2(n+2)}.$$

Fun Fact: For a non-negative random variable X with density f_X, one can extend the tail sum formula to give

$$\mathbb{E}[X^2] = \int_0^\infty x^2 f_X(x)\,\mathrm{d}x = \int_0^\infty \left(\int_0^x 2s\,\mathrm{d}s\right) f_X(x)\,\mathrm{d}x = \int_0^\infty 2s\int_s^\infty f_X(x)\,\mathrm{d}x\,\mathrm{d}s = \int_0^\infty 2s\mathbb{P}(X \ge s)\,\mathrm{d}s$$

and this gives another way to compute $\mathbb{E}[Y^2]$ in this problem. You can derive a similar formula to compute any moment $\mathbb{E}[X^k]$ for $k \in \mathbb{N}$.

4 Arrows

You and your friend are competing in an archery competition. You are a more skilled archer than he is, and the distances of your arrows to the center of the bullseye are i.i.d. Uniform$[0,1]$ whereas his are i.i.d. Uniform$[0,2]$. To even out the playing field, you both agree that you will shoot one arrow and he will shoot two. The arrow closest to the center of the bullseye wins the competition. What is the probability that you will win? *Note*: The distances *from the center of the bullseye* are uniform.

Solution:

Let X be the distance of your arrow to the center and Y that of the closest of your friend's arrows. Then, for $x \in [0,1]$ and $y \in [0,2]$,

$$\mathbb{P}[X > x] = 1 - x \quad \text{and} \quad \mathbb{P}[Y > y] = \left(1 - \frac{y}{2}\right)^2.$$

Hence,

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}(1 - \mathbb{P}[X > x]) = -\frac{\mathrm{d}}{\mathrm{d}x}(1-x) = 1, \qquad x \in [0,1].$$

Also,

$$\mathbb{P}[Y > X \mid X = x] = \left(1 - \frac{x}{2}\right)^2.$$

Thus,

$$\begin{aligned}
\mathbb{P}[Y > X] &= \int_0^1 \mathbb{P}(Y > X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^1 \left(1 - \frac{x}{2}\right)^2 f_X(x)\,\mathrm{d}x = \mathbb{E}\left[\left(1 - \frac{X}{2}\right)^2\right] \\
&= \mathbb{E}\left[1 - X + \frac{X^2}{4}\right] = 1 - \frac{1}{2} + \frac{1}{12} = \frac{7}{12},
\end{aligned}$$

since $\mathbb{E}[X] = 1/2$ and $\mathbb{E}[X^2] = \int_0^1 x^2\,\mathrm{d}x = 1/3$.

5 Darts (Again!)

Alvin is playing darts. His aim follows an exponential distribution; that is, the probability density that the dart is x distance from the center is $f_X(x) = \exp(-x)$. The board's radius is 4 units.

(a) What is the probability the dart will stay within the board?

(b) Say you know Alvin made it on the board. What is the probability he is within 1 unit from the center?

(c) If Alvin is within 1 unit from the center, he scores 4 points, if he is within 2 units, he scores 3, etc. In other words, Alvin scores $\lfloor 5 - x \rfloor$, where x is the distance from the center. What is Alvin's expected score after one throw?

Solution:

(a) The CDF of an exponential is $\mathbb{P}[X \le x] = 1 - \exp(-x)$. Therefore,

$$\mathbb{P}[X \le 4] = 1 - \exp(-4).$$

(b) We are given that the dart must be within the board, which means that the dart is at least 4 units away from the center. We can use the definition of conditional probability:

$$\mathbb{P}[X \le 1 \mid X \le 4] = \frac{\mathbb{P}[X \le 1 \cap X \le 4]}{\mathbb{P}[X \le 4]} = \frac{\mathbb{P}[X \le 1]}{\mathbb{P}[X \le 4]} = \frac{1 - \exp(-1)}{1 - \exp(-4)}.$$

(c)

$$\begin{aligned}
\mathbb{E}[\text{score}] &= \int_0^1 4\exp(-x)\,\mathrm{d}x + \int_1^2 3\exp(-x)\,\mathrm{d}x + \int_2^3 2\exp(-x)\,\mathrm{d}x + \int_3^4 \exp(-x)\,\mathrm{d}x \\
&= 4\big(-\exp(-1) + 1\big) + 3\big(-\exp(-2) + \exp(-1)\big) + 2\big(-\exp(-3) + \exp(-2)\big) \\
&\quad + \big(-\exp(-4) + \exp(-3)\big) \\
&= 4 - \exp(-1) - \exp(-2) - \exp(-3) - \exp(-4).
\end{aligned}$$

6 Exponential Distributions: Lightbulbs

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probablity that the new bulb will last at least 30 days?

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

Solution:

(a) Let $X \sim \text{Exponential}(1/50)$ be the time until the bulb is broken. For an exponential random variable with parameter λ, the density function is $f_X(x) = \lambda\,\mathrm{e}^{-\lambda x}$ for $x > 0$. So in this case $\lambda = 1/50$. Thus we can integrate the density to find the probability that the lightbulb broke in the first 30 days:

$$\mathbb{P}[X < 30] = \int_0^{30} \left(\frac{1}{50} \cdot \mathrm{e}^{-x/50}\right) \mathrm{d}x = 1 - \mathrm{e}^{-30/50} = 1 - \mathrm{e}^{-3/5} \approx 0.451.$$

(b) The new bulb's waiting time Y is i.i.d. with the old bulb's. So the answer is

$$\mathbb{P}[Y > 30] = 1 - \mathbb{P}[Y < 30] = 1 - (1 - \mathrm{e}^{-3/5}) = \mathrm{e}^{-3/5} \approx 0.549.$$

(c) The bulb is memoryless, so the probability it will last 60 days given that it has lasted 30 days, is just the probability it will last 30 days:

$$\mathbb{P}[X > 60 \mid X > 30] = \mathbb{P}[X - 30 > 30 \mid X > 30] = \mathbb{P}[X > 30] = \mathrm{e}^{-3/5} \approx 0.549.$$