CS 70 Spring 2018	Discrete Mathematics and Probability Theory Satish Rao and Babak Ayazifar	DIS 07A

1 Flippin' Coins

Suppose we have a biased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 3/4$ and probability of tails $\mathbb{P}[T] = 1/4$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No
- Yes
- Yes
- Yes
- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

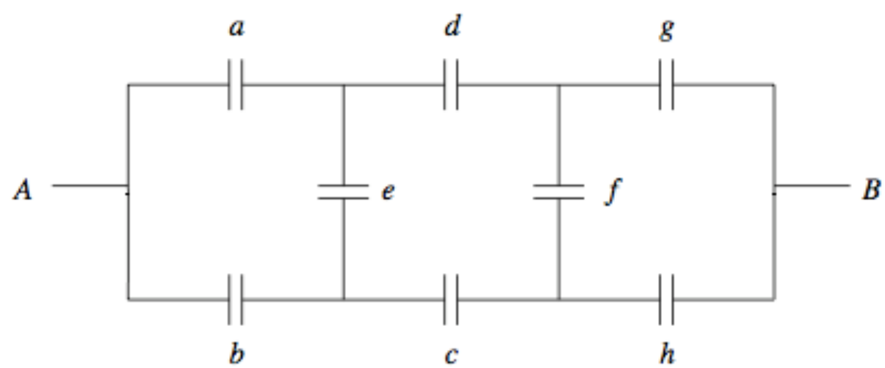
(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) $\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}=\frac{9}{64}$

(f) $\omega\in\{(H,H,T),(H,T,H),(T,H,H)\}$. The probability = $3\cdot\frac{9}{64}=\frac{27}{64}$.

2 Communication Network

In the communication network shown below, link failures are independent, and each link has a probability of failure of p. Consider the physical situation before you write anything. A can communicate with B as long as they are connected by at least one path which contains only in-service links.



(a) Given that exactly five links have failed, determine the probability that A can still communicate with B.

(b) Given that exactly five links have failed, determine the probability that either g or h (*but not both*) is still operating properly.

(c) Given that *a, d* and h have failed (but no information about the information of the other links), determine the probability that A can communicate with B.

Solution:

(a) There are only two paths of 3 links from A to B. There are $\binom{8}{5}$ ways of the links messing up.

So, the probability is $\frac{2}{56}=\frac{1}{28}$.

This is because every single case of exactly 5 links being down has the same probability. So it's a uniform distribution over all possibilities.

(b) Fix g as down and h as working. There are $\binom{6}{4}$ ways to have 4 out of the remaining go down. Symmetric argument for h down and g up.

So, the probability is $\frac{30}{56} = \frac{15}{28}$.

(c) We would just want the 4 on the only remaining path from A to B not to be down.

The probability of this happening is $(1-p)^4$.

3 Venn Diagram

Out of 1000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.

(b) What is the probability that the student belongs to a club?

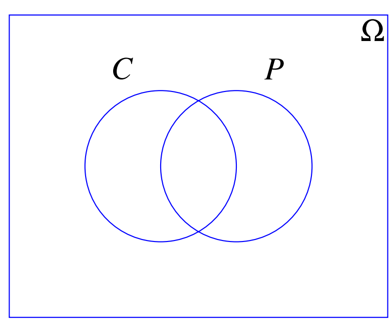
(c) What is the probability that the student works part time?

(d) What is the probability that the student belongs to a club AND works part time?

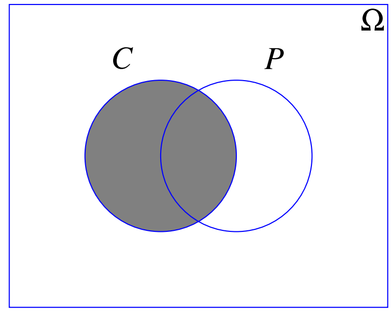
(e) What is the probability that the student belongs to a club OR works part time?

Solution:

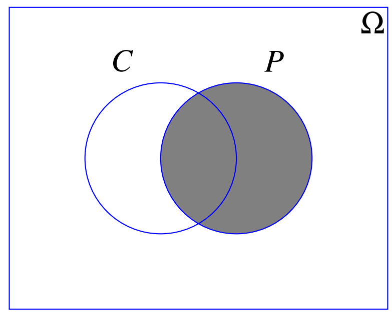
(a) The sample space will be illustrated by a Venn diagram.



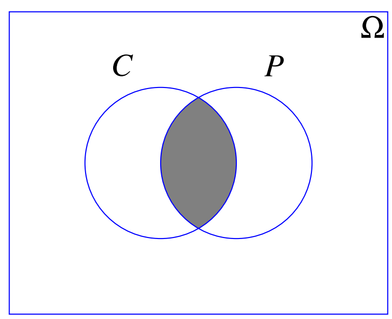
(b) $\mathbb{P}[C] = \frac{|C|}{|\Omega|} = \frac{400}{1000} = \frac{2}{5}$.



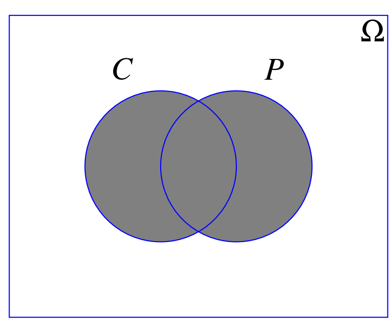
(c) $\mathbb{P}[P] = \frac{|P|}{|\Omega|} = \frac{500}{1000} = \frac{1}{2}$.



(d) $\mathbb{P}[P \cap C] = \frac{|P \cap C|}{|\Omega|} = \frac{50}{1000} = \frac{1}{20}$.



(e) $\mathbb{P}[P \cup C] = \mathbb{P}[P] + \mathbb{P}[C] - \mathbb{P}[P \cap C] = \frac{1}{2} + \frac{2}{5} - \frac{1}{20} = \frac{17}{20}$.



4 Shooting Range

You and your friend are at a shooting range. You ran out of bullets. Your friend still has two bullets left but magically lost his gun. Somehow you both agree to put the two bullets into your six-chambered revolver in successive order, spin the revolver, and then take turns shooting. Your first shot was a blank. You want your friend to shoot a blank too; should you spin the revolver again before you hand it to your friend?

Solution:

No, you shouldn't.

The first chamber fired was one of the four empty chambers. Since the bullets were placed in consecutive order, one of the empty chambers is followed by a bullet, and the other three empty chambers are followed by another empty chamber. So the probability that a bullet will be fired is 1/4.

If you spins the chamber again, the probability that a real bullet is shot would be 2/6, or 1/3, since there are two possible bullets that would be in firing position out of the six possible chambers that would be in position.