LU Decomposition - Crout's Method

$$
\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} L_{00} & & 0 & \\ L_{10} & L_{11} & & \\ L_{20} & L_{21} & L_{22} & \\ L_{30} & L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{01} & U_{02} & U_{03} \\ & 1 & U_{12} & U_{13} \\ & & 1 & U_{23} \\ & & & 1 \end{bmatrix}
$$

So,

$$a_{00} = L_{00}$$

$$a_{01} = L_{00} U_{01} \rightarrow U_{01} = \frac{a_{01}}{L_{00}}$$

$$a_{02} = L_{00} U_{02} \rightarrow U_{02} = \frac{a_{02}}{L_{00}}$$

$$a_{03} = L_{00} U_{03} \rightarrow U_{03} = \frac{a_{03}}{L_{00}}$$

$$a_{10} = L_{10}$$

$$a_{11} = L_{10} U_{01} + L_{11} \rightarrow L_{11} = a_{11} - L_{10} U_{01}$$

$$a_{12} = L_{10} U_{02} + L_{11} U_{12} \rightarrow U_{12} = (a_{12} - L_{10} U_{02}) / L_{11}$$

$$a_{13} = L_{10} U_{03} + L_{11} U_{13} \rightarrow U_{13} = (a_{13} - L_{10} U_{03}) / L_{11}$$

$$a_{20} = L_{20}$$

$$a_{21} = L_{20} U_{01} + L_{21} \rightarrow L_{21} = a_{21} - L_{20} U_{01}$$

$$a_{22} = L_{20} U_{02} + L_{21} U_{12} + L_{22} \rightarrow L_{22} = a_{22} - L_{20} U_{02} - L_{21} U_{12}$$

$$a_{23} = L_{20} U_{03} + L_{21} U_{13} + L_{22} U_{23}$$

$$\rightarrow U_{23} = (a_{23} - L_{20} U_{03} - L_{21} U_{13}) / L_{22}$$

$$a_{30} = L_{30}$$

$$a_{31} = L_{30} U_{01} + L_{31} \rightarrow L_{31} = a_{31} - L_{30} U_{01}$$

$$a_{32} = L_{30} U_{02} + L_{31} U_{12} + L_{32} \rightarrow L_{32} = a_{32} - L_{30} U_{02} - L_{31} U_{12}$$

$$a_{33} = L_{30} U_{03} + L_{31} U_{13} + L_{32} U_{23} + L_{33}$$

$$\rightarrow L_{33} = a_{33} - L_{30} U_{03} - L_{31} U_{13} - L_{32} U_{23}$$

Now, how to code this?

$i=0$ ~ row index for finding either L or U

$j=0$ ~ col " " " " "

$L_{00} = a_{00}$ (on diagonal) $i=j$

$i=0$
$j=1$ $U_{01} = \dfrac{a_{01}}{L_{00}}$ (right of diagonal $j>i$)

$i=0$
$j=2$ $U_{02} = \dfrac{a_{02}}{L_{00}}$ $j>i$

$i=0$
$j=3$ $U_{03} = \dfrac{a_{03}}{L_{00}}$ $j>i$

$$\boxed{\begin{aligned} L_{ij} &= a_{ij} \\ U_{ij} &= \frac{a_{ij}}{L_{ii}} \end{aligned}}$$

for $j=0$

for $j>i$
$j>0$

$i=1$
$j=0$ $L_{10} = a_{10}$ $i>j$

$i=1$
$j=1$ $L_{11} = a_{11} - L_{10}U_{01}$ $i=j$

$i=1$
$j=2$ $U_{12} = (a_{12} - L_{10}U_{02})/L_{11}$ $j>i$

$i=1$
$j=3$ $U_{13} = (a_{13} - L_{10}U_{03})/L_{11}$ $j>i$

$L_{ij} = a_{ij}$ [for $j=0$]

$L_{ij} = a_{ij} - L_{i0}U_{0j}$ [for $i=j$]

$U_{ij} = (a_{ij} - L_{i0}U_{0j})/L_{ii}$
[for $j>i$]

$\left.\begin{matrix} i=2 \\ j=0 \end{matrix}\right\}$ $L_{20} = a_{20}$ $\quad i > j$

$\left.\begin{matrix} i=2 \\ j=1 \end{matrix}\right\}$ $L_{21} = a_{21} - L_{20} U_{01}$ $\quad i > j$

$\left.\begin{matrix} i=2 \\ j=2 \end{matrix}\right\}$ $L_{22} = a_{22} - \cancel{L_{20}} L_{20} U_{02} - L_{21} U_{12}$ $\quad i = j$

$\left.\begin{matrix} i=2 \\ j=3 \end{matrix}\right\}$ $U_{23} = (a_{23} - L_{20} U_{03} - L_{21} U_{13}) / L_{22}$ $\quad j > i$

$L_{ij} = a_{ij} \quad [j=0]$

$L_{ij} = a_{ij} - L_{i0} U_{0j} \quad [j=1]$

$L_{ij} = a_{ij} - L_{i0} U_{0i} - L_{i1} U_{1j} \quad [j=2]$

$U_{ij} = (a_{ij} - L_{i0} U_{0j} - L_{i1} U_{1j}) / L_{ii}$

$(j=3)$

$\left.\begin{matrix} i=3 \\ j=0 \end{matrix}\right\}$ $L_{30} = a_{30}$ $\quad = a_{ij} - \overbrace{0}^{\sum_{k=0}^{j-1} L_{ik} U_{kj}}$ won't run $\quad \underline{\sum_{k=0}^{j-1} L_{ik} U_{kj}}$

$\left.\begin{matrix} i=3 \\ j=1 \end{matrix}\right\}$ $L_{31} = a_{31} - L_{30} U_{01}$ $\quad = a_{ij} - \overline{L_{i0} U_{0j}}$ $\quad \sum_{k=0}^{j-1} L_{ik} U_{kj}$

$\left.\begin{matrix} i=3 \\ j=2 \end{matrix}\right\}$ $L_{32} = a_{32} - L_{30} U_{02} - L_{31} U_{12}$ $\quad = a_{ij}^{*} - \overline{(L_{i0} U_{0j} + L_{i1} U_{1j})}$

$\left.\begin{matrix} i=3 \\ j=3 \end{matrix}\right\}$ $L_{33} = a_{33} - L_{30} U_{03} - L_{31} U_{13} - L_{32} U_{23}$

$= a_{ij} - \underbrace{(L_{i0} U_{0j} + L_{i1} U_{1j} + L_{i2} U_{2j})}_{\sum_{k=0}^{j-1} L_{ik} U_{kj}}$

So, let's see what we have:

$$\begin{bmatrix} L_{00} & & & 0 \\ L_{10} & L_{11} & & \\ L_{20} & L_{21} & L_{22} & \\ L_{30} & L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{01} & U_{02} & U_{03} \\ 0 & 1 & U_{12} & U_{13} \\ & & 1 & U_{23} \\ & & & 1 \end{bmatrix}$$

$$U_{0j} = \frac{a_{0j}}{L_{00}}$$

$$L_{i0} = a_{i0}$$

$$U_{ii} = 1$$

$$a_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}$$

for $j <= i$

$$\frac{a_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{L_{ii}}$$

for $j > i$

$\cancel{i=1,\ j=2,\ U_{12} = a_{12} - (L_{10} U_{01} + }$

ORDER IS:

1. ~~Set~~ Set column Ø: $L_{00}, L_{10}, L_{20}, L_{30}$
2. Set diagonal to 1's: $U_{00}, U_{11}, U_{22}, U_{33}$
3. Go row by row i ..

   Go column by column j

   If $i <= j$

   Set L_{ij}

   If $j > i$

   Set U_{ij}