Taylor Series

Taylor Series Expansions (assume equally spaced points)

$$f(x_{i+1}) = f(x_i) + f^{'}(x_i)h + \frac{f^{''}(x_i)}{2!}h^2 + \frac{f^{'''}(x_i)}{3!}h^3 + \frac{f^{(4)}(x_i)}{4!}h^4 + \ldots$$

$$h = x_{i+1} - x_i$$

There is a value between x_i and x_{i+1}, ξ such that (this is from mean value theorem in calculus):

$$f(x_{i+1}) = f(x_i) + f^{'}(x_i)h + \frac{f^{''}(\xi)}{2!}h^2$$

Sometimes that last part is called the remainder, R_n

$$f(x_{i+1}) = f(x_i) + f^{'}(x_i)h + R_n$$

Note that *n* is the number of terms. In the above example n = 2.

Example: Find $f(x) = \cos(x)$ at x = π/12 using a base point of x = 0 use n = 2 and n = 4

$$\begin{aligned} f'(x) &= -\sin(x) = -\sin(0) = 0 \\ f''(x) &= -\cos(x) = -\cos(0) = -1 \\ f'''(x) &= \sin(x) = \sin(0) = 0 \end{aligned}$$

So for n = 2:

$$\begin{aligned} f(\pi/12) &\approx f(0) + f'(0)h \\ f(\pi/12) &\approx 1 + 0x(\pi/12 - 0) = 1 \end{aligned}$$

Real answer is 0.8439

For n = 4:

$$\begin{aligned} f(\pi/12) &\approx f(0) + f'(0)h + \frac{f''(0)h^2}{2} + \frac{f'''(0)h^3}{6} \\ f(\pi/12) &\approx 1 + 0x(\pi/12 - 0) + \frac{(-1)x(\pi/12)^2}{2} + \frac{(0)x(\pi/12)^3}{6} = 0.966 \end{aligned}$$