MATH 239 Exercises 2

2.1 a) $[x^8]\sum_{n=0}^{\infty}\binom{n+7-1}{7-1}x^n = 3003$

b) $[x^{10}]x^6\sum_{n=0}^{\infty}\binom{n+5-1}{5-1}(2x)^n = 1120$

c) $[x^8](x^3+5x^4)\sum_{k=0}^{6}\binom{6}{k}(3x)^k = \binom{6}{5}(3^5)+\binom{6}{4}(3^4)\cdot 5 = 7533$

d) $[x^9]\cancel{(1-4x)^5} + \sum_{n=0}^{\infty}\binom{n+2-1}{2-1}(3x)^n = 196830$

e) $[x^n]\sum_{j=0}^{\infty}\binom{j+k-1}{k-1}(2tx)^j = \binom{n+k-1}{k-1}(2t)^n$

f) $[x^{n+1}]x^k\sum_{j=0}^{\infty}\binom{j+2k-1}{2k-1}(4x)^j = \binom{n+k}{2k-1}4^{n+1-k}$

g) $[x^n]\,x^k\sum_{j=0}^{\infty}\binom{j+m-1}{m-1}(x^2)^j = \binom{\frac{n-k}{2}+m-1}{m-1}$

h) $[x^n]\sum_{j=0}^{\infty}\binom{j+k-1}{k-1}(x^5)^j + \sum_{j=0}^{\infty}\binom{j+k-1}{k-1}(7x^5)^j = \binom{\frac{n}{5}+k-1}{k-1} + \binom{\frac{n}{5}+k-1}{k-1}\left(7^{\frac{n}{5}}\right)$

2.2 a) $\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty}\binom{n+1}{1}(-1)^n x^n = \sum_{n=0}^{\infty}(n+1)(-1)^n x^n$

b) $\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty}\binom{n+2}{2}x^n$

c) $\frac{1}{(1+x^3)} = \sum_{n=0}^{\infty}\binom{n}{0}(-1)^n x^{3n} = \sum_{n=0}^{\infty}(-1)^n x^{3n}$

d) $\frac{1}{(1-2x)} = \sum_{n=0}^{\infty}\binom{n}{0}(2x)^n = \sum_{n=0}^{\infty}(2x)^n$

e) $\frac{1}{(1+2x^2)} = \sum_{n=0}^{\infty}\binom{n+1}{1}(-2x^2) = \sum_{n=0}^{\infty}(n+1)(-2x^2)$

f) $\frac{1}{(1-2x)^3} = \sum_{n=0}^{\infty}\binom{n+2}{2}(2x)^n$

2.3 a) $\sum_{k=0}^{n}\binom{n}{k}k = \sum_{k=0}^{n}\frac{n!\cdot k}{k!(n-k)!} = \sum_{k=0}^{n}\frac{n(n-1)!}{(k-1)!(n-1-(k-1))!}$

$$= \sum_{k=0}^{n} n\binom{n-1}{k-1} = n\binom{n-1}{0-1} + \sum_{k=1}^{n} n\binom{n-1}{k-1}$$

$$= 0 + n\sum_{k=0}^{n-1}\binom{n-1}{k} = n2^{n-1}$$

b) $$\sum_{n=0}^{n}\binom{n}{k}k(k-1) = \sum_{k=0}^{n}\frac{n!\,k(k-1)}{k!\,(n-k)!} = \sum_{k=0}^{n}\frac{n(n-1)(n-2)!}{(k-2)!(n-2-(k-2))!}$$

$$= \sum_{k=0}^{n} n(n-1)\binom{n-2}{k-2} = n(n-1)\left[\cancel{\binom{n-2}{0-2}} + \cancel{\binom{n-1}{0-1}} + \sum_{k=2}^{n}\binom{n-2}{k-2}\right]$$

$$= n(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k} = n(n-1)2^{n-2}$$

2.4 Let $\Phi_A(x) = (1+x)^{-2} = \sum_{k=0}^{\infty}\binom{k+1}{1}(-x)^k = \sum_{k=0}^{\infty}(k+1)(-x)^k$

$$\Phi_B(x) = (1-2x)^{-2} = \sum_{k=0}^{\infty}\binom{k+1}{1}(2x)^k = \sum_{k=0}^{\infty}(k+1)(2x)^k$$

By the product lemma, the weight function of the product of the two generating series is the sum of their individual ones. To find the coefficient of x^n, we find the pairs in the cross product of the terms of A and B where the exponents of x sum to n. Let w be the weight function of the product of the two generating series. Hence we find the pairs (i, j) where

$$\alpha_i = (i+1)(-1)^i \qquad \omega(a_i, \beta_j) = n = i+j$$

$$\beta_j = (i+1)2^j$$

$$[x^n](1+x)^{-2}(1-2x)^{-2} = \sum_{i=0}^{n}(i+1)(-1)^i(n-i+1)2^{n-i}$$

$$= \frac{1}{27}\left(2^{n+2}(5+3n) + (-1)^n(7+3n)\right)$$

2.5 a) Methodology is the same as 2.4 above.

$$[x^n]\frac{(1+x)^a}{(1-x^2)^a} = [x^n]\left[\sum_{k=0}^{a}\binom{a}{k}x^k\right]\left[\sum_{n=0}^{\infty}\binom{n+a-1}{a-1}x^{2k}\right] \rightarrow n = i+2j$$

$$= \sum^{\lfloor n/2 \rfloor} \binom{a}{\ }\binom{k+a-1}{\ }$$

$$= [x^n] \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{n-2k}\binom{k+a-1}{a-1}$$

$$[x^n]\frac{1}{(1-x)^a} = [x^n]\sum_{k=0}^{\infty}\binom{k+a-1}{a-1}$$

$$= \binom{n+a-1}{a-1}$$

b) ????

2.6 $$\Phi_B(x) = \sum_{\sigma\in B} x^{w(\sigma)} = \sum_{\sigma\in A_1} x^{w(\sigma)} + \sum_{\sigma\in A_2} x^{w(\sigma)} + \cdots$$

$$= \sum_{j=0}^{\infty}\Phi_{A_j}(x)$$

2.7 We can use induction. Base case: 2 sets (proved in Product Lemma 2.12).
Suppose for k sets, we have

$$\Phi_{A_1\times A_2\times\cdots A_k}(x) = \prod_{j=1}^{k}\Phi_{A_j}(x)$$

We prove that for k+1 sets, we have

$$\Phi_{A_1\times\cdots\times A_{k+1}}(x) = \prod_{j=1}^{k+1}\Phi_{A_j}(x)$$

We have already shown that for two sets, the generating series of A x B is the product of the two generating series. Hence we have that

$$\Phi_{(A_1\cdots A_k)\times A_{k+1}}(x) = \left[\prod_{j=1}^{k}\Phi_{A_j}(x)\right]\cdot\Phi_{A_{k+1}}(x) = \prod_{j=1}^{k+1}\Phi_j(x)$$

Hence we have proven the statement.

2.8 $$\sum_{k=0}^{m-n}\binom{m-n}{k}x^k = \frac{\sum_{j=0}^{m}\binom{m}{j}x^j}{\sum_{i=0}^{n}\binom{n}{i}x^i} = \left[\sum_{j=0}^{m}\binom{m}{j}x^j\right]\frac{1}{(1+x)^n} = \left[\sum_{j=0}^{m}\binom{n}{j}x^j\right]\sum_{i=0}^{\infty}\binom{i+n-1}{n-1}(-x)^i$$

$$= \sum_{j=0}^{m}\sum_{i=0}^{\infty}\binom{m}{j}\binom{i+n-1}{n-1}(-1)^i x^{i+j}$$

???

2.9 a) { aaaa, aaab, aaba, abaa, baaa, aabb, abab, abba, baba, baab, bbaa, abbb, babb, bbab, bbba, bbbb }

Weight 0: 5
Weight 1: 10
Weight 2: 1

$$\Phi(x) = 5 + 10x + x^2$$

b) ???

2.10 a)

2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	5	4	3	2	1

Generating series = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^10 + 2x^11 + x^12

$$= \sum_{i=1}^{6} i x^{i+1} + \sum_{i=1}^{5}(6-i)x^{i+7}$$

b) $$\Phi(x) = \sum_{i=1}^{8}\frac{i(i+1)}{2}x^{i+2} + \sum_{i=1}^{8}\frac{(9-i)(10-i)}{2}x^{i+10}$$

2.11

0	1	2	3	4	5
6	10	8	6	4	2

$$\Phi(x) = 6 + \sum_{i=1}^{5}(10-2i)x^i$$

2.12 a) This is a weight function. For every natural number n, there are only a finite number of elements in S that are weight n, since there are only a finite number of pairs (a,b) where a=n, since 0 <= |b| <= a.

$$\Phi_s(x) = \sum_{(a,b)\in S} x^{w(a,b)} = \sum_{(a,b)\in S} x^a = \sum_{a=0}^{\infty} x^a\left[\sum_{b=-a}^{a} 1\right] = \sum_{a=0}^{\infty}(2a+1)x^a$$

$$= \sum_{a=0}^{\infty} x^a + \sum_{a=0}^{\infty} 2ax^a = \sum_{a=0}^{\infty}\binom{a}{0}x^a + 2\sum_{a=0}^{\infty}\binom{a}{1}x^a$$

$$= (1-x)^{-1} + 2(1-x)^{-2} = \frac{3-x}{(1-x)^2}$$

b) This is not a weight function. For every natural number n, there are an infinite number of pairs (a,b) where a+b=n.

c) This is a weight function. For every natural number n, there are only a finite number of elements in S that are weight n, since a <= 2a+b <= 3a.

2.13 a) The generating function for the outcome of one dice is (x + x^2 + x^3 + x^4 + x^5 + x^6). Hence the generating function for the sum of 4 dice is (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.

$$\sum_{n=1}^{6} x^n = \sum_{n=1}^{\infty} x^n - \sum_{n=7}^{\infty} x^n = \frac{x}{1-x} - \frac{x^7}{1-x} = \frac{x-x^7}{1-x}$$

$$\Phi_s(x) = \left(\frac{x-x^7}{1-x}\right)^4$$

b) $$[x^{19}]\Phi_s(x) = [x^{19}]x^4(1-x^6)^4(1-x)^{-4} = [x^{15}]\left[\sum_{k=0}^{4}\binom{4}{k}(-x^6)^k\right]\left[\sum_{n=0}^{\infty}\binom{n+3}{3}x^n\right]$$

$$\alpha_i = \binom{4}{i}(-1)^i$$

$$\beta_j = \binom{j+3}{3} \qquad \omega(\alpha_i, \beta_j) = 15 = 6i+j \;,\; \underbrace{0 \le i \le 4}_{0 \le i \le 2},\; j \ge 0$$

$$[x^{19}]\Phi_s(x) = \sum_{i=0}^{2}\binom{4}{i}(-1)^i\binom{15-6i+3}{3}$$

$$= 56$$

c) $[x^k]\left(\frac{x-x^{d+1}}{1-x}\right)^m = [x^{k-m}]\left[\sum_{i=0}^{m}\binom{m}{i}(-x)^{di}\right]\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}x^n$

$\alpha_i = \binom{m}{i}(-1)^i$ $\quad \omega(\alpha_i, \beta_j) = k-m = di+j$ $\quad k-m-di \geq 0 \rightarrow i \leq \frac{k-m}{d}$

$\beta_j = \binom{j+m-1}{m-1}$ $\quad j = k-m-di,\ j\geq 0,\ 0 \leq i \leq m$

$$[x^k]\Phi(x) = \sum_{i=0}^{\lfloor \frac{k-m}{d} \rfloor}\binom{m}{i}(-1)^i\binom{k-di-1}{m-1}$$

2.14 $\sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j x^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j x^j \cdot x^{k-j} = \Phi_A(x)\sum_{i=0}^{\infty}x^i = \frac{1}{1-x}\Phi_A(x)$

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