MATH 239 W2006 Final

1. a) $[x^n](1-3x^2)^{-2} = [x^n]\sum_{k=0}^{\infty}(k+1)3^k x^{2k}$

$$= \begin{cases} 0 & , n \text{ is odd} \\ 3^{n/2}\left(\frac{n}{2}+1\right), & n \text{ is even} \end{cases}$$

b) $[x^n](1+2x)(1-3x)^{-m} = [x^n](1+2x)\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}3^k x^k$

$$= \binom{n+m-1}{m-1}3^n + \binom{n+m-2}{m-1}3^{n-1}x^{n-1}$$

2. a) Each part must be odd; the allowed parts are P = {1, 3, 3…}. Hence the generating series for a single part is

$$\Phi_P(x) = \frac{x}{1-x^2}$$

By the product lemma, the generating series for 2m parts is

$$\Phi(x) = \Phi_P(x)^{2m} = \frac{x^{2m}}{(1-x^2)^{2m}}$$

The number of compositions of 2n into 2m parts that are odd is hence

$$[x^{2n}]\Phi(x) = [x^{2n-2m}]\sum_{k=0}^{\infty}\binom{k+2m-1}{2m-1}x^{2k}$$

$$= \begin{cases} 0 & , n \text{ is odd} \\ \binom{2n-1}{2m-1} & , n \text{ is even} \end{cases}$$

b) Each part must be at least 2; the allowed parts are P = {2, 3, 4…}. Hence the generating series for a part is

$$\Phi_P(x) = \frac{x^2}{1-x}$$

By the String lemma, the generating series for compositions of n is

$$\Phi(x) = \frac{1}{1-\frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2}$$

3. a) In AB, we can create the string 01.1 = 0.11 two ways, so elements are not uniquely created.

In BA, the elements are {10, 101, 1011, 110, 1101, 11011}. Every element is uniquely created.

b)

$$\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)} = \frac{1}{1-(x+x^2+x^3)}$$

$$\Phi_{B^*}(x) = \frac{1}{1-\Phi_B(x)} = \frac{1}{1-(x+x^2)}$$

$\hookrightarrow$ If not counting repetitions: $\frac{1}{1-x}$

c) The characteristic polynomial is

$$1-7x+10x^2 = (1-5x)(1-2x)$$

The solution is hence of the form

$$b_n = A\cdot 5^n + B\cdot 2^n$$

Setting n=0 and n=1 respectively, we obtain

$$-1 = A + B$$
$$1 = 5A + 2B$$

A=1, B=-2. So we have

$$b_n = 5^n - 2\cdot 2^n$$

4. a) i. The expression (00)* (1*1(00)*00)* 1* is the block decomposition for this set of strings.

ii. The expression (ε ⌣ 1*(00)*0) (1*1(00)*00)* 1* is the block decomposition for this set of strings.

b) $\Phi_T(x) = \frac{1}{1-x}\left(\frac{1}{1-\left(\frac{x^3}{1-x^2}+\frac{x^3}{1-x^2}\right)}\right) = \frac{1+x}{1-x^2-2x^3}$

$$a_n - a_{n-2} - 2a_{n-3} = \begin{cases} 1, & n=0 \\ 1, & n=1 \\ 0, & \text{otherwise} \end{cases}$$

$a_0, a_1, a_2 = 1$

$a_n = a_{n-2} + 2a_{n-3}$

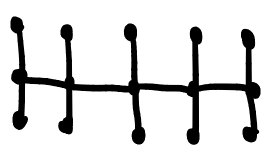
5\. a) H and J are isomorphic. An isomorphism is

1d, 2g, 3b, 4f, 5e, 6a, 7c

b) J and K are not isomorphic. J has 5 triangles, while K only has 3.

6\. a) We can partition the graph into vertices with even numbers and vertices with odd numbers. Two vertices with both even numbers cannot have an edge between them, since there are no even numbers greater than 2 that are prime. The same logic applies for vertices between odd numbers. Hence every edge that exists must be between a vertex with an odd number and one with an even number, and hence P_n is bipartite.

b) T is a tree, and hence has no cycles. Let x1, x2, x3 be the 3 vertices of degree 4, and y1, y2 be the vertices of degree 3. We can connect x1 to x2, and x2 to x3. We cannot connect x1 to x2, otherwise there would be a cycle. We can connect y1 to x1, and y2 to x3. Now each of x1,x2,x3,y1,y2 each need 2 more neighbours. We cannot connect any of them to each other or there would be a cycle. Hence we add 10 more vertices, and we have 15 vertices.



c) Let G be a graph with a bridge e = {u,v}. Suppose G has no vertices of odd degree. We can remove edge e to obtain graph G\e. Vertices u and v must be in different components, and must both have odd degree in G\e. Observe the component containing u, let's call it X. It is a graph, so must have the sum of all the degrees of its vertices be even. But u is the only vertex with odd degree in X, so X must have the sum of all of its degrees be odd. But we have

$$\sum \deg(v) = 2|E(X)|$$

which is a contradiction. Hence G must have a vertex of odd degree.