MATH 239 Exercises 8

8.2.1 We prove this by induction on the number of vertices. Let T be a tree on n vertices.

Base cases: A tree on 1 vertex has no perfect matchings, and a tree on two vertices has one perfect matching.

Inductive hypothesis: Assume for all 2 < k < n, a tree on k vertices has at most one perfect matching. Consider a tree on n vertices. Let v be a leaf of T and u be its parent. The edge uv must be in any perfect matching, since the matching must include v and uv is the only edge incident to v. Hence the number of perfect matchings in T equals the number of perfect matchings in T \ {u, v}. The graph T \ {u, v} is a forest, where each component has less than n vertices. Hence each component is a tree with at most one perfect matching. The number of perfect matchings in a graph is the product of the number of perfect matchings in each component, and hence T \ {u, v} has at most one perfect matching, so T has at most one perfect matching.

8.2.2 For Kn, if n is odd, there are 0 perfect matchings. If n is even, there are (n!/(2^(n/2) (n/2)!) ways to partition the n vertices into pairs. Hence there are

$\dfrac{n!}{2^{n/2}(n/2)!}$ perfect matchings.

For Km,n, if m != n, there are 0 perfect matchings. If m = n, there are n! permutations of n, so there are n! perfect matchings.

8.2.3 We find a recurrence for a_n, where a_n denotes the number of perfect matchings of L_n. For n > 2, we have that we can add a pair of vertices to any perfect matching in a_(n-1). The only perfect matching we can create is to connect the two vertices with a vertical line, since every other vertex is already part of an edge in the perfect matching. We can also take any perfect matching in a_(n-2), and add two pairs of vertices. There are 2 perfect matchings in two pairs of vertices where they are not connected by vertical edges, so we have that for each perfect matching in a_(n-2), there are 2 ways to add a two pairs of vertices. Hence we have

$$a_n = a_{n-1} + 2a_{n-2},\ n \geq 3$$
$$a_1 = 1$$
$$a_2 = 3$$

The characteristic polynomial is

$$1 - x - 2x^2 = (1+x)(1-2x)$$

Hence the solution is of the form

$$A(-1)^n + B \cdot 2^n$$

Substituting n=1 and n=2, we obtain

$$1 = -A + 2B$$

$$3 = A + 4B$$

Solving, we obtain A=1/3, B=2/3. Hence we have

$$a_n = \frac{1}{3}(-1)^n + \frac{2}{3} \cdot 2^n$$

8.2.4 Each vertex in an n-cube is a bitstring (b1,..bn). We can construct a perfect matching by selecting the edge between every vertex (0, b2,..bn) and (1, b2,…bn). This covers every vertex of the n-cube, and each edge only has one end in any vertex.

8.2.5 Each row has 8 squares. We can cover each row with 4 dominos, lined up end to end. Repeat this for each row, and hence we have covered the board in 32 dominos.

8.2.6 Each domino on the board must cover exactly one white and one black square. Removing two opposite corner squares either removes two white or two black squares. Hence there is an uneven number of white and black squares, and we cannot cover them with 31 dominos.

8.2.7 Let G be a graph with an even number of vertices and a Hamilton cycle. We can create a perfect matching by selecting alternating edges in the cycle. Since there is an even number of vertices, every vertex will be paired with exactly one other vertex that is adjacent to it.

8.2.8 If u and v are adjacent to each other in the Hamilton cycle, we can create a perfect matching in G by selecting the alternating edges in the cycle where the edge uv is selected (so no other edges are incident with u or v). Then, H has the same perfect matching except without edge uv.

If u and v are not adjacent to each other in the Hamilton cycle, they must have an even number of vertices in both paths from u to v in the cycle, since adjacent vertices are in different partitions. Hence we can create a perfect matching in G by selecting alternating edges in each path, where neither endpoint u or v is incident with an edge. Since there are an even number of edges in each path, every vertex (except u and v) are in the matching.

8.2.9 The chessboard (without squares removed) can be modeled as a bipartite graph with an even number of vertices. Each square is a vertex and adjacent squares are connected by an edge. Hence we have a bipartition between black and white squares. The graph has a Hamilton cycle. Covering the board with dominos is equivalent to finding a perfect matching. Removing two squares with opposite colours from the chessboard is equivalent to removing two vertices in different partitions (and their edges) from the board. Hence the resulting graph has a perfect matching, and can be covered by 31 dominos.

8.2.10 We prove the statement by induction on n. Our base case, n=2, is true, since it consists of the vertices 1 and 2, which sum to 3 and hence the two are adjacent and that is our perfect matching.

Assume that the statement is true for any 2 < k < n. Observe the prime graph B_n. If n is odd, we are done. If n is even, there must exist a prime number p between n and 2n. We must have that p = n + k for some odd number 1 <= k <= n. Then the set of integers {k, k+1,...n-1, n} can be partitioned into pairs {k, 2n}, {k+1, 2n-1},.. and so on up to {n+floor(k/2), n+ceil(k/2)}. Each of the pairs sums to p=n+k. By the induction hypothesis, the remaining set of integers {1,2,..k-1} has a perfect matching. Thus, the B_n must have a perfect matching.

8.2.11 If C is a cover, every edge of G has at least one end in C. Hence G \ C must have no edges, since removing all edges in C removes all edges incident to a vertex in C, which is every edge. Hence V(G) \ C is a set of pairwise non-adjacent vertices.

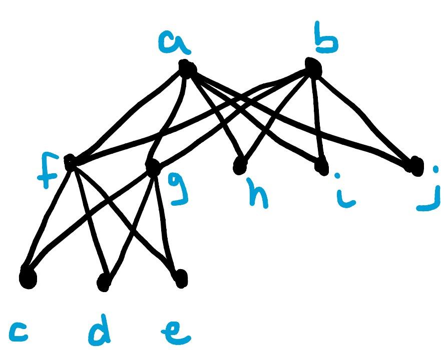
If V(G) \ C is a set of pairwise nonadjacent vertices, it must be that G \ C has no edges. Hence every edge in G must be incident with a vertex in C, and C must be a cover.

8.2.12 We provide a counterexample. Observe K4. It has matchings of size |M|=1 or |M|=2. However, every cover must be of size at least |C|=3. Hence there does not exist an M and C of the same size.

8.2.13 We can formulate this as a graph problem as follows. Let G be the graph representing this problem, with bipartition X, Y. Each non-zero entry in N is a vertex in G in partition X. Each row and column in N is a vertex in G in partition Y. A vertex in X is connected to a vertex in Y if and only if the entry is in the row (or column) of vertex Y. A matching is hence a set of pairs of entries and rows/columns such that each row/column is only matched with one entry. Hence to find the largest-set of non-zero entries with this property, we would like to find the maximum matching in G.

8.2.14 A cover of G is a set of vertices where every edge has at least one end in C, which represents the set of rows and columns that the non-zero entries in N cover.

8.2.15 This graph satisfies all the statements. It has no perfect matching because in a perfect matching, vertices i and j must be adjacent to either a and b. Hence edges f, g, and h must be adjacent to vertices c, d, and e in the perfect matching. But vertex h is only adjacent to vertices a and b, so there cannot be a perfect matching.



8.2.16

Let G be a graph with p=2n vertices and deg(v) >= n for every vertex v. Consider any matching M. If M is perfect, then we are done.

If M is not perfect, then there must exist distinct vertices u and v that are not saturated by M. If u and v are adjacent, then there is an augmenting path of length 1. We can connect these two vertices to create a larger matching. We can continue doing this until M is a matching with no pairs of adjacent vertices not saturated by M. If M is perfect, then we are done.

Otherwise, there must exist distinct vertices u and v are not adjacent and not saturated by M. They must each be adjacent to at least n other vertices. Since every vertex has deg(v) >= n, vertex u must be adjacent to some vertex x, and v adjacent to some vertex y, such that xy is in M. Hence there is an augmented path of length 3, and by Theorem 8.1.1, there is a M' that is larger than M. We can repeat this for all non-adjacent pairs to create a perfect matching.

8.2.17

Let x be the number of vertices incident to edges in M', and y be the number of vertices incident to edges in M. Suppose that |M'| > 2|M|. Then we must have x > 2y. Since at most 2y edges have at least one vertex in M and x > 2y, there exists at least one edge e in M' that have no vertices in M. But this means that e can be added to M to make a larger matching, contradicting the statement that M is not contained in any larger matchings. Hence we must have that |M'| <= 2|M|.

8.3.1

We prove that the minimum size of a cover is at least q/Δ. Suppose C is a cover of G. There can be at most Δ edges incident to any element of C, so there can be at most Δ|C| edges incident with one or more elements of C. But C is a cover, so every edge must be incident with one or more elements of C. Therefore, Δ|C| >= q, or |C| >= q/Δ. Since every cover contains at least q/Δ vertices, the minimum size of a cover is at least q/Δ. By Konig's Theorem, G has a matching of size at least q/Δ.

If G is not bipartite, then we cannot use Konig's Theorem to make this conclusion. K3 is not bipartite, but has q=3 edges and maximum vertex degree of 2. It's maximum matching size is 1, which is less than 3/2.

8.3.2

We have that K3,3 has bipartition 3,3, with q=9 edges. The maximum size of a matching is 3. We have that 3 < 10/3 = (q+1)/n.

8.3.3

Let M be some maximum matching of G. Let X be the vertices in A that are saturated by M, and Y be the vertices in B that are saturated by M. Let kX be the number of edges whose vertices in X have edges to B\Y, and kY be the number of edges whose vertices in A\X have edges to Y. For any edge in M, it cannot be that both its vertex in X has an edge to B\Y and its vertex in Y has an edge to A\X, otherwise we could replace the edge with these two edges and obtain a larger matching. Hence we must have kX + kY <= |M|.

We count the maximal number of edges in G. The n-|M| edges from A\X can have only edges to kY vertices in Y, and the n-|M| edges from B\X can only have edges to kX vertices in X. The |M| vertices in X and |M| vertices in Y can have up to |M|^2 edges between them, except for kXkY edges between the partners of the vertices connected to A\X and B \Y. Hence we have

$$q \leq (n-|M|)k_X + (n-|M|)k_Y + |M|^2 - k_X k_Y$$
$$= (n-|M|)(k_X + k_Y) + |M|^2 - k_X k_Y$$
$$\leq (n-|M|)|M| + |M|^2 - k_X k_Y$$
$$= n|M| - k_X k_Y$$

All vertices in A\X and B\Y must have at least δ edges. Hence either A\X and B\Y are empty, or kX >= δ and kY >= δ. If A\X and B\Y are empty, then we have a perfect matching, and G has a matching of size at least n. Otherwise, we have

$$q \leq n|M| - \delta(|M| - \delta)$$
$$|M| \geq \frac{q - \delta^2}{n - \delta}$$

Hence G has a matching of size at least the minimum of n and (q - δ^2)/(n - δ).

8.3.4

We have that K8,8 has q=64 edges. The maximum size of a matching is 8. We have that 8 < 61/6 = (q-3)/6.

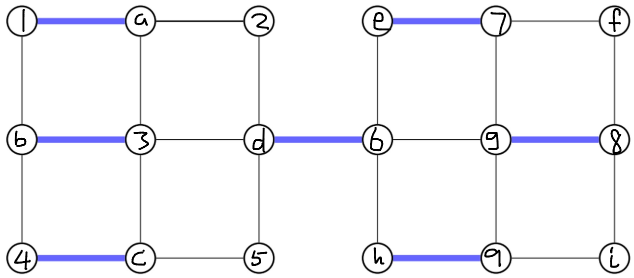
8.3.5

We start with
X = {1, 2}, Y = {}.
1 has an edge to b, so we add it to Y. 2 has an edge to e, so we add it to Y.
X = {1, 2}, Y = {b, e}.
b has an edge to 3, so we add it to X. e has an edge to 5, so we add it to X.
X = {1, 2, 3, 5}, Y = {b, e}.
3 has an edge to c, so we add it to Y. c is an unsaturated vertex in Y, so we have the augmenting path 1b3c.

We replace the matching with a larger matching {1b, 3c, 4d, 5e}. We start with
X = {2}, Y = {}.
2 has edges to b and e, so we add them to Y.
X = {2}, Y = {b, e}.
b has an edge to 1, so we add it to X. e has an edge to 5, so we add it to X.
X = {1, 2, 5}, Y = {b, e}.
We cannot add anything else to Y, so we terminate.

Our maximum matching is {1b, 3c, 4d, 5e} and our minimum cover is {3, 4, b, e}.

8.3.6



We start with
X = {2, 5}, Y = {}
2 has edges to a and d, so we add them to Y. 5 has an edge to c, so we add it to Y.
X = {2, 5}, Y = {a, c, d}
a has an edge to 1, so we add it to X. c has an edge to 4, so we add it to Y. d has an edge to 6, so we add it to Y.
X = {2, 5, 1, 4, 6}, Y = {a, c, d}
1 has an edge to b, so we add it to Y. 6 has edges to e and h, so we add them to Y.
X = {2, 5, 1, 4, 6}, Y = {a, c, d, b, e, h}
b has an edge to 3, so we add it to X. e has an edge to 7, so we add it to X. h has an edge to 9, so we add it to X.
X = {2, 5, 1, 4, 6, 3, 7, 9}, Y = {a, c, d, b, e, h}
7 has an edge to f, so we add it to Y. f is an unsaturated vertex in Y, so we have the augmenting path 2d6e7f.

We replace the matching with a larger matching {1a, 2d, 3b, 4c, 6e, 7f, 8g, 9h}. We start with
X = {5}, Y = {}
5 has an edges to d and c, so we add them to Y.
X = {5}, Y = {c, d}
c has an edge to 4, so we add it to X. d has an edge to 2, so we add it to X.
X = {5, 2, 4}, Y = {c, d}
2 has an edge to a, so we add it to Y. 4 has an edge to b, so we add it to Y.
X = {5, 2, 4}, Y = {c, d, a, b}
a has an edge to 1, so we add it to X. b has an edge to 3, so we add it to X.
X = {5, 2, 4, 1, 3}, Y = {c, d, a, b}
We cannot add anything else to Y, so we terminate.

Our maximum matching is {1a, 2d, 3b, 4c, 6e, 7f, 8g, 9h} and our minimum cover is {6, 7, 8, 9, c, d, a, b}.

8.3.7

Lol nope.

8.3.8

We prove that for any edge in G, if it has no end in (B intersect (C U C')), then it has an end in (A intersect C intersect C'). Every edge has an end in A, an end in B, and at least one end in each of C and C'. For any edge in G, if it has no end in (B intersect (C U C')), then its end in B is not in C or C'. This means that its end in A must be in both C and C', since C and C' are covers. Hence this end is in (A intersect C intersect C').

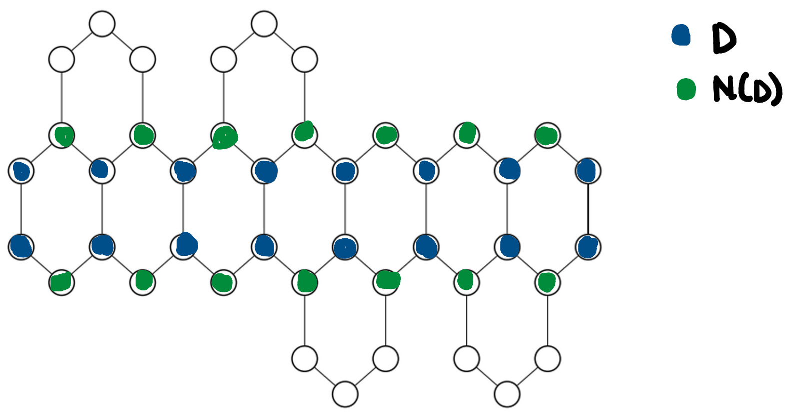
8.3.9

We prove the contrapositive: if Ĉ is not a minimum cover, then one of C and C' is not a minimum cover. Since Ĉ is not a minimum cover, then there exists some vertex v in Ĉ such that Ĉ\v is a cover. v must be in either (A intersect C intersect C') or (B intersect (C U C')). If v is in (A intersect C intersect C'), then v is in both C and C'. Since Ĉ\v is a cover, every edge incident with v has its other end in (B intersect (C U C')). Hence every edge has its other end in either C or C', and C\v or C'\v is a cover, so C or C' is not a minimal cover. If v is in (B intersect (C U C')), then every edge incident with v has its other end in (A intersect C intersect C'). Hence every edge incident with v has an end in both C and C', and C\v and C'\v are covers, and C and C' are not minimal covers. Hence we have proved the statement.

8.6.1

D = {1, 2, 3, 4, 5}, N(D) = {a, b, c, d} (see labeled nodes in q8.3.6)

8.6.2



8.6.3

We have that |M| <= |A|. If |N(D)| >= |D|, then we are done. Otherwise, there exists some vertices in D that do not each have distinct neighbours in B. There can only be as many matchings as there are distinct neighbours of any subset D of A, so we have |M| <= |A| - |D| + |N(D)|.

8.6.4

Suppose the maximum size of a matching is not the minimum of |A|-|D|+|N(D)|. Then there exists some D' with |N(D')| - |D'| > |N(D)| - |D|. From Exercises 8.6.3 above, we have that |M| <= |A| - |D'| + |N(D)'|. But |N(D')| - |D'| > |N(D)| - |D|, so |A| - |D'| + |N(D)'| > |A|-|D|+|N(D)|, a contradiction. Hence we must have that |A|-|D|+|N(D)| is minimum over subsets D of A.

8.6.5

Let e=xy be some edge in G where x is in A, and y is in B. Consider the bipartite graph G' = G\{x,y}. For any subset D of A\x, its neighbour set is the same in G' as in G, except it may be missing the vertex y. Hence we have |N(D)| in G' >= |N(D)|-1 in G. Since for every D of A, |N(D)| > |D| in G, we have that |N(D)| >= |D| in G'. Hence by Hall's theorem, we have a matching that saturates D for any subset D of A. Hence we have a matching that saturates A \x, and by Theorem 8.6.1, G' has a perfect matching M. Hence G has the perfect matching consisting of M plus the edge e.

8.6.6

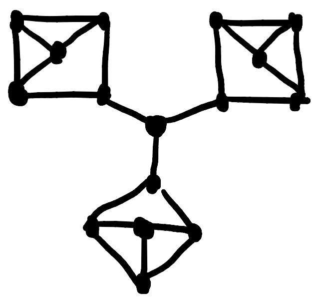
Let A,B be a bipartition of G. Since every edge has one end in A and the other in B, we have that the sum of degrees of vertices in A is equal to the sum of degrees of vertices in B. This is obviously true for multigraphs as well. Hence k|A| = k|B|, and since k > 0, |A|=|B|. Let D be a subset of A. Then since every edge incident with a vertex in D has its other end in N(D), we have that the sum of degrees of vertices in D is greater than or equal to the sum of degrees of vertices in N(D). Hence k|D| <= k|N(D)|, and since k > 0, |D| <= |N(D)|. Hall's Theorem holds for multigraphs, so the statement holds for multigraphs.

8.6.7

We can model this with a bipartite multigraph G with bipartition A, B. Each column is a vertex in A. Each value of card is a vertex in B. Hence we have |A|=|B|=13. There is an edge between vertices x in A and y in B in G for every instance where there is a card of value y in column x.

To find 13 cards where no 2 are in the same column and no 2 are of the same value, we want to find a perfect matching in G. Every set of k rows contains 4k cards, and therefore must contain at least k distinct values. Hence we have that in G, for any subset D of A, |N(D)| >= |D|. Hence by Hall's Theorem and Corollary 8.6.1, there exists a perfect matching.

8.6.8



8.6.9 The largest set of mutually nonadjacent vertices in a graph is the complement of the minimum cover. We can use algorithm for maximum matchings for bipartite graphs to find the minimum cover, and take its complement.

8.6.10 ???

8.6.11

Let G be a k-regular bipartite graph with bipartition A, B. Since G is k-regular, we must have that |A|=|B|. Every vertex cover in G has at least |A| vertices, since each vertex can cover at most k edges, and there are k|A| edges. Hence by Konig's Theorem, the maximum matching has at least |A| edges, so G must have a perfect matching.

8.6.12 ???