# Exercises 5

1. a) $W^{-nk} + W^{-(N-n)k} = W^{-nk} + W^{-Nk}W^{nk}$

$$= W^{-nk} + W^{nk} = e^{\frac{-2\pi i nk}{N}} + e^{\frac{2\pi i nk}{N}}$$

$$= \cos\left(\frac{-2\pi nk}{N}\right) + i\sin\left(-\frac{2\pi nk}{N}\right) + \cos\left(\frac{2\pi nk}{N}\right) + i\sin\left(\frac{2\pi nk}{N}\right)$$

$$= \cos\left(\frac{2\pi nk}{N}\right) \cancel{- i\sin\left(\frac{2\pi nk}{N}\right)} + \cos\left(\frac{2\pi nk}{N}\right) \cancel{+ i\sin\left(\frac{2\pi nk}{N}\right)}$$

$$= 2\cos\left(\frac{2\pi nk}{N}\right)$$

b) [Assignment 4 question]

$$F_k = \sum_{n=0}^{N-1} f_n W^{-nk} = \sum_{n=0}^{N/2-1} f_n W^{-nk} + \sum_{n=N/2}^{N-1} f_n W^{-nk}$$

$$= \sum_{n=0}^{N/2-1} f_n W^{-nk} + \sum_{n=0}^{N/2-1} f_{N-n} W^{-(N-n)k}$$

$$= \sum_{n=0}^{N/2-1} f_n W^{-nk} + f_n W^{-(N-n)k}$$

$$= \sum_{n=0}^{N/2-1} f_n \left(W^{-nk} + W^{-(N-n)k}\right)$$

$$= \sum_{n=0}^{N/2-1} 2 f_n \cos\left(\frac{2\pi nk}{N}\right)$$

We can clearly see that $F_k$ is real.

2. $$F_{2k} = \sum_{n=0}^{N-1} f_n W^{-2nk} = \sum_{n=\frac{N}{4}}^{\frac{3N}{4}-1} W^{-2nk}$$

$$= \frac{W^{k(2-\frac{3N}{2})}(W^{Nk}-1)}{W^{2k}-1}$$

$$= 0$$

3. We have that $W^0 = 1$, so $F_0$ is just equal to the sum of the $f_i$ s.

4. Let $F_k$ be the DFT of the original signal $f_0 \ldots f_{N-1}$. Let $G_k$ be the DFT of the new signal.

$$G_k = \sum_{n=0}^{N-1} (f_n + c) W^{-nk} = F_k + c \sum_{n=0}^{N-1} W^{-nk}$$

$$= F_k + c\left(\frac{1-W^{-Nk}}{1-W^{-k}}\right)$$

$$= F_k$$

5.
i. $F[k] = (6, -1-i, 0, -1+i)$
ii. $F[k] = (8, -2, 0, -2)$
I have no idea why it's real

6. $$\frac{1}{N} \sum_{k=0}^{N-1} F_k \bar{F}_k = \sum_{n=0}^{N-1} f_n \bar{f}_n \longleftarrow$$ Correct version that our slides cover

$$\frac{1}{N} \sum_{k=0}^{N-1} F_k \bar{F}_k = \frac{1}{N} \sum_{k=0}^{N-1} F_k \overline{\sum_{n=0}^{N-1} f\, W^{-nk}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} F_k \sum_{n=0}^{N-1} \bar{f}_n W^{nk}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \bar{f}_n \sum_{k=0}^{N-1} F_k W^{nk}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \overline{f_n} N f_n$$

$$= \sum_{n=0}^{N-1} f_n \overline{f_n}$$

7. Properties of Fourier transform:

- Signal that is purely real has $F_k = \overline{F_{N-k}}$
- Signal that is purely imaginary has $F_k = -\overline{F_{N-k}}$

Our algorithm works as follows. We multiply the second signal by $j$ and add it to the first. Then we take the FFT of this sum.

The FFT of the first signal is $F'_k = \frac{F_k + \overline{F_{N-k}}}{2}$. The contributions of the first (real) signal will add, and the contributions of the second (imaginary) signal will cancel out.

The FFT if the second signal is $F''_k = j\frac{F_k - \overline{F_{N-k}}}{2}$. The contributions of the first (real) signal will cancel out, and the contributions of the second (imaginary) signal will add. We also need to multiply by $j$, since our FFT has real and imaginary parts swapped (since we multiplied our second signal by $j$ originally).