Exercises 4

1. False. An unconditionally stable solution can still have a large local error for large step sizes.

2. $Y_1(x) = y(x)$

$Y_1'(x) = Y_2(x) = y'(x)$

$Y_2'(x) = y''(x)$

$y''(x) = -x^2 y(x) - x y'(x)$

$y''(x) + x y'(x) + x^2 y(x)$

3. Let $Y_1(t) = y(t)$

$Y_1' = Y_2$

$Y_2' = Y_3$

$Y_3' = -\sin(t) Y_3 - g(t) Y_2 + g(t) Y_1$

4. Let $U_1(t) = u(t)$, $V_1(t) = v(t)$

$U_1' = U_2$ $\quad U_2 = u'(t)$ $\quad U_2' = u''(t)$

$V_1' = V_2$

$U_2' + 3V_2 + 4U_1 + V_1 = t$

$V_2' - V_2 + U_1 + V_1 = \cos(t)$

5. Forward Euler:

$$y_{n+1} = y_n - 5 y_n h$$
$$= y_n(1 - 5h)$$

For a stable computation, we must have

$$\lim_{n\to\infty} y_n = 0 \implies |1-5h| < 1$$

$$-1 < 1-5h < 1$$

$$-2 < -5h < 0$$

$$-2 < -5h \qquad \text{, since } h>0$$

$$h < 2/5$$

Modified Euler:

$$y_{n+1} = y_n + \frac{h}{2}\left(-5y_n - 5(y_n - 5y_n h)\right)$$

$$= y_n - 5y_n h + \frac{25}{2} y_n h^2$$

$$= y_n\left(1 - 5h + \frac{25}{2}h^2\right)$$

For a stable computation, we must have

$$\lim_{n\to\infty} y_n = 0 \implies \left|1 - 5h + \frac{25}{2}h^2\right| < 1$$

$$-1 < 1 - 5h + \frac{25}{2}h^2 < 1$$

$$-2 < -5h + \frac{25}{2}h^2 < 0$$

$$-2 < -5h + \frac{25}{2}h^2 \qquad \text{, since } h>0$$

$$h\left(1 - \frac{5}{2}h\right) < 2/5$$

$$\frac{5}{2}\left(h - \frac{1}{5}\right)^2 + \frac{1}{10} < 2/5$$

This inequality is true for all h.

6. The local truncation error for step h, for some constant A, is

$$\ell^h = |y(t_{i+1}) - y^h_{t_{i+1}}|$$

$$= O(h^3)$$

$$= Ah^3 + (\text{smaller terms})$$

$$\approx Ah^3$$

The local truncation error for two steps of size h/2 is

$$\ell^{h/2} = A\left(\frac{h}{2}\right)^3 + A\left(\frac{h}{2}\right)^3$$

$$= \frac{A}{4}h^3$$

We have that the difference between these two is the difference between the computed y-values.

$$\left|\ell^{h} - \ell^{h/2}\right| = \left| y(t_{i+1}) - y^{h}_{t_{i+1}} + (y(t_{i+1}) - y^{h/2}_{t_{i+1}})\right|$$

$$= \left|y^{h}_{t_{i+1}} - y^{h/2}_{t_{i+1}}\right|$$

$$\left|Ah^3 - \frac{A}{4}h^3\right| = 10^{-4}$$

$$A = 40/3$$

Plugging A back into our equation, we get that the local truncation error is

$$\ell^{h} = 1/7500$$