

# MATH 239 W2012 Midterm

$$1. a) [x^n](1+3x^2)^{-5} = [x^n] \sum_{n=0}^{\infty} \binom{n+4}{4} (-3)^n x^{2n}$$

$$= \begin{cases} 0 & n \text{ is odd} \\ \binom{\frac{n}{2}+4}{4} (-3)^{n/2}, & n \text{ is even} \end{cases}$$

$$b) [x^5](1+3x^3)^3(1-x)^{-2} = [x^5] \left[ \sum_{k=0}^3 \binom{3}{k} 3^k x^{3k} \right] \sum_{j=0}^{\infty} (j+1) x^j$$

$$= \binom{3}{0} 3^0 (5+1) + \binom{3}{1} 3^1 (2+1) = 33$$

$$c) [x^n](1-5x)^{-3}(1+x)^m = [x^n] \left[ \sum_{j=0}^{\infty} \binom{n+2}{2} \cdot 5^j x^j \right] \sum_{k=0}^m \binom{m}{k} x^k$$

$$= \sum_{j=0}^n \binom{j+2}{2} \cdot 5^j \binom{m}{k-j}$$

$$d) [x^{73}](1+x^2)^{10}(1-x^4)^2 = [x^{73}] \left[ \sum_{k=0}^{10} \binom{10}{k} x^{2k} \right] (1-2x^4+x^8)$$

$$= 0$$

2. a) The generating function for a part of at least 3 is

$$\overline{\Phi}(x) = \frac{x^3}{1-x}$$

By the product lemma, the generating function for compositions of  $n$  of these parts is

$$(\overline{\Phi}(x))^{100} = \left( \frac{x^3}{1-x} \right)^{100} = \frac{x^{300}}{(1-x)^{100}}$$

b) By the string lemma, the generating function is

$$\frac{1}{1-\Phi(x)} = \frac{1}{1-\frac{x^3}{1-x}} = \frac{1-x}{1-x-x^3}$$

$$\begin{aligned} c) \sum_{k=0}^{100} \Phi(x)^k &= \sum_{k=0}^{\infty} \Phi(x)^k - \sum_{k=101}^{\infty} \Phi(x)^k = \frac{1}{1-\Phi(x)} - \frac{\Phi(x)^{101}}{1-\Phi(x)} \\ &= \frac{1-\Phi(x)^{101}}{1-\Phi(x)} = \frac{1-\frac{x^{303}}{(1-x)^{101}}}{1-\frac{x^3}{1-x}} \\ &= \frac{(1-x)^{101} - x^{303}}{(1-x)^{100}(1-x-x^3)} \end{aligned}$$

3. a)  $(\varepsilon - (00)^*00 - 0 - 000)((11)^*11 - 1 - 111)((00)^*00 - 0 - 000)^*(\varepsilon - (11)^*11 - 1 - 111)$

$$\begin{aligned} \Phi(x) &= \left(1 + \frac{x^2}{1-x^2} + x + x^3\right)^2 \frac{1}{1-\left(\frac{x^2}{1-x^2} + x + x^3\right)^2} \\ &= \frac{1+x-x^5}{1-x-2x^2+x^5} \end{aligned}$$

b)  $1^*(000^*1 - 01^*1)^*0^*$

$$\Phi(x) = \frac{1}{1-x} \left( \frac{1}{1-\left(\frac{x^3}{1-x} + \frac{x}{1-x}\right)} \right) \frac{1}{1-x} = \frac{1}{1-2x+x^4}$$

$$4. a) (1+x^2) \frac{1}{1-\left(x\left(1+\frac{x}{1-x^2}\right)\right)} = \frac{1-x^4}{1-x+x^3}$$

b) It does not create 00111 - where an even block of zeros is followed by an odd block of 1s.

5. a) The characteristic polynomial is

$$1+3x-4x^2 = (1-x)(1+4x)$$

The solution is hence of the form

$$b_n = A + B(-4)^n$$

- b) To find a particular solution of this recurrence, we try a solution of the form  $b_n = \alpha n$  where  $\alpha$  is some constant. Substituting into the recurrence, we get

$$\alpha n + 3\alpha(n-1) - 4\alpha(n-2) = 5$$

$$5\alpha = 5$$

$$\alpha = 1$$

Hence a particular solution is  $b_n = n$ .

The characteristic polynomial of the corresponding homogenous linear recurrence relation is

$$1 + 3x - 4x^2 = (1-x)(1+4x)$$

The general solution is hence of the form

$$b_n = A + B(-4)^n + n$$

Setting  $n=0$  and  $n=1$  respectively, we get

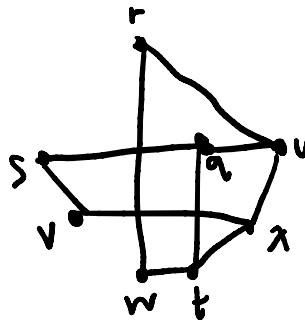
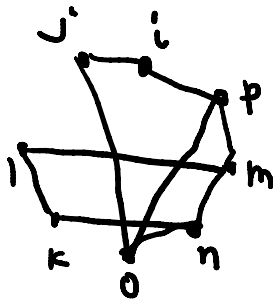
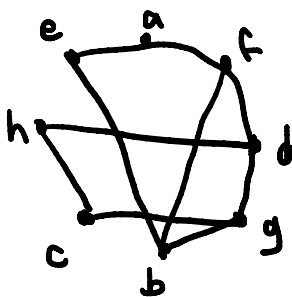
$$1 = A + B(-4)^0 + 0$$

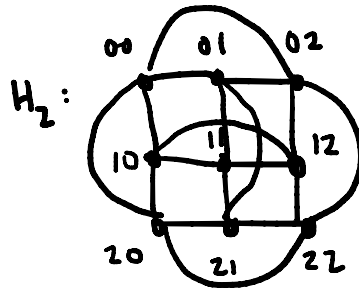
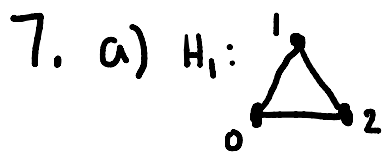
$$2 = A - 4B + 1$$

We get  $A=1$  and  $B=0$ . Hence

$$b_n = 1 + n$$

6. G and H are isomorphic. J is not isomorphic to G, since G has a cycle of 6, while the longest cycle in J is 5. Hence J is also not isomorphic to H.





b) For each character in the string, there are 3 possibilities for which character it could be. There are  $n$  characters. Hence there are  $3^n$  vertices in  $H_n$ .

c) For each string of length  $n$ , there are two possibilities each character could change to, and there are  $n$  characters that we can flip one at a time. Hence each vertex has degree  $2n$ .

$$|E(H_n)| = \frac{1}{2} \sum_{v \in V} 2n = \frac{1}{2} \cdot 3^n \cdot 2n = n3^n$$