MATH 239 F2008 Final

1. Each part must not be equal to 1; the allowed parts are P = {2, 3, 4…}. Hence the generating series for a part is

$$\Phi_p(x) = \frac{x^2}{1-x}$$

The generating function for compositions of n with an even number of parts is

$$\sum_{i=0}^{\infty} \Phi_p(x)^{2i} = \sum_{i=0}^{\infty}\left(\frac{x^4}{(1-x)^2}\right)^i = \frac{1-2x+x^2}{1-2x+x^2-x^4}$$

2. a) i. The expression (ε ⌣ 0 ⌣ 00) ((1 ⌣ 11) (0 ⌣ 00))* (ε ⌣ 1 ⌣ 11) is the block decomposition for this set of strings.

ii. The expression (ε ⌣ 0*000) ((1*111) (0*000))* (ε ⌣ 1*111) is the block decomposition for this set of strings.

b) 010000

c) The elements are not uniquely created. The string 00.00.00 = 0.0.0.0.0.0 can be created these two ways. An expression that creates these strings uniquely is (0 ⌣ 001)*

d) $\Phi_S(x) = \dfrac{1}{1-(x+x^3)}$

3. By theorem 4.8, a_n satisfies the linear recurrence relation with initial conditions given by

$$a_n - 2a_{n-1} + a_{n-2} - a_{n-3} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 1, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

$$a_0, a_1, a_2 = 1$$

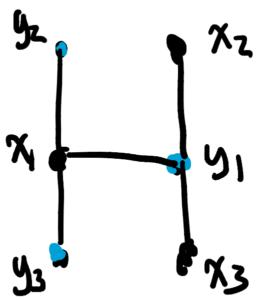
$$a_3 = 3$$

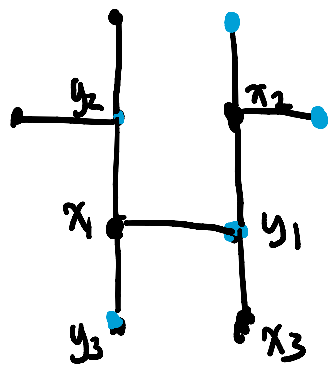
$$a_n = 2a_{n-1} - a_{n-2} + a_{n-3}, \quad n \geq 4$$

4. Let G be a connected graph. We prove that if an edge e is a bridge, it belongs to every spanning tree of G. Let e be an edge in G that is a bridge. Suppose that e does not belong to every spanning tree of G. Let T be a spanning tree not containing e. Then the tree T is a spanning subgraph of G\e. Let u and v be any two vertices of G\e. There must be a unique path from u to v in T, which is also a unique path from u to v in G\e. Then G\e must be connected, which is a contradiction, since e is a bridge. Hence e must belong to every spanning tree of G.

We prove that if an edge e belongs to every spanning tree of G, it is a bridge. Let e be an edge belonging to every spanning tree of G. Suppose e is not a bridge. Then G\e is connected, and G\e must have a spanning tree T. Since V(G) = V(G\e), T is also a spanning tree of G. This is a contradiction, since T does not contain e and e belongs to every spanning tree of G. Hence if e belongs to every spanning tree of G, it must be a bridge.

5. a) Since G is bipartite, G must not contain any odd cycles. Let X, Y be a bipartition of G. Let x1 be a vertex in G in X. It must have degree at least 3, so it must have 3 neighbours in Y. Let y1, y2, y3 be these neighbours. y1 must have 3 neighbours in X, one of which is x1. Let the other two neighbours be x2 and x3. If y2 is adjacent to x2 or x3, we have a path of length 5. Otherwise, x2 and y2 must each have 2 distinct neighbours, and we have a path of length 5.





b) K3,3 is a bipartite graph with degree at least 3 with no path of length 6.

c) From part a, G must have a path of length at least 5. Suppose G does not contain a path of length 6. Let P={v1,v2,v3,v4,v5} be a path of length 5. P must alternate between vertices in bipartitions X and Y. Without loss of generality, let v1 be in X. v1 is adjacent to v2, and since G has degree at least 3, it must be adjacent to 2 other vertices. These 2 other vertices must be in P, otherwise we could extend P to length 6 by adding the vertex to it. But this is a contradiction, since P is of length 5 and only has 2 vertices in Y, one of which is v2. Hence G must contain a path of length at least 6.

6. a) Euler's formula states that p-q+f=2 for a connected planar graph. Suppose P has k components. Then we have

$$\sum_{i=1}^{k}(|V_i| - |E_i| + f_i) = 2k$$

$$|V| - |E| + f = 2k \geq 2 \quad (\text{since } k \geq 1)$$

$$102 - 300 + f \geq 2$$

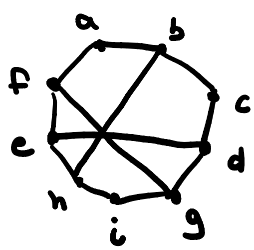
$$f \geq 200$$

b) Suppose P does not contain any cycles. Then we have E = V-k, where k is the number of components in P. Hence we have E > V. This is a contradiction, so P must contain at least one cycle.

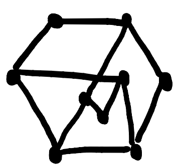
c) Since P contains cycles, the boundary of each face of P must contain a cycle. Hence there must be at least 3 edges in every face, so it must have degree at least 3.

d) Since all faces have degree 3, the outer face must also have degree 3. Suppose G is not connected. Then the outer face is the sum of the boundary walks of the components. Each component must have more than 3 edges. Hence there is only one component. This is a contradiction, so G must be connected.

7. a) H is not planar. This subgraph of H contains an edge subdivision of K3,3, so by Kuratowski's theorem, is nonplanar.



b)



8. a) 1, 2, 4, 3, 5, 11, 7, 6, 8, 10, 12, 14, 9, 17, 13, 15, 16, 18

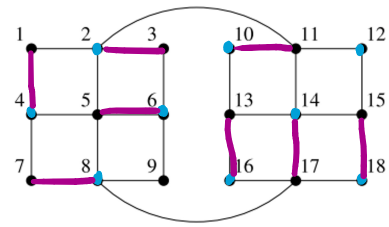
b) X0: 7, 9
X: 7, 9
Y:

X: 7, 9
Y: 4, 6, 8
X: 7, 9, 1, 5, 17
Y: 4, 6, 8, 2, 14, 16, 18
X: 7, 9, 1, 5, 17, 3, 11, 13, 15
Y: 4, 6, 8, 2, 14, 16, 18, 10

10 is unsaturated, we create new matching with augmenting path 7 8 17 14 11 1

X0: 9
X: 9
Y:

X: 9
Y: 6, 8
X: 9, 5, 7
Y: 6, 8, 2, 4
X: 9, 5, 7, 3, 1
Added no vertices to Y, we terminate.



Our maximum matching is
{(1,4),(2,3),(5,6),(7,8),(10,11),(13,16),(14,17),(15,18)}.
Our minimum cover is {2,4,6,8,11,13,15,17}

9. D = { 1, 3, 4, 7 }
N(D) = { a, c, f }
|N(D)| = 3 < 4 = |D|