MATH 239 S2006 Midterm

1\. a) $[x^{10}]\dfrac{x^3}{(1-x)^5} = [x^7]\displaystyle\sum_{n=0}^{\infty}\binom{n+4}{4}x^n = \binom{7+4}{4} = 330$

b) $[x^{13}]\dfrac{(1-2x^2)^6}{(1-x)^4} = [x^{13}]\left[\displaystyle\sum_{k=0}^{6}\binom{6}{k}(-2)^k x^{2k}\right]\sum_{n=0}^{\infty}\binom{n+3}{3}x^n$

$= \displaystyle\sum_{k=0}^{6}\binom{6}{k}(-2)^k\binom{16-2k}{3}$

c) $[x^t]\displaystyle\sum_{r+s=t}(-1)^r\binom{n+r-1}{r}\binom{m}{s}x^t = [x^t]\left[\sum_{r=0}^{\infty}(-1)^r\binom{n+r-1}{r}x^r\right]\sum_{s=0}^{m}\binom{m}{s}x^s$

$= [x^t]\dfrac{1}{(1+x)^n}(1+x)^m = [x^t]\,(1+x)^{m-n} = [x^t]\displaystyle\sum_{j=0}^{m-n}\binom{m-n}{j}x^j$

$= \binom{m-n}{t}$

2\. a) (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)

b) The generating function for an even part is

$\Phi_E(x) = x^2 + x^4 \dots \quad = \dfrac{x^2}{1-x^2}$

By the string lemma, the generating function for the number of compositions of n where each part is even is

$$\frac{1}{1-\Phi_E(x)} = \frac{1}{1-\frac{x^2}{1-x^2}} = \frac{1-x^2}{1-2x^2}$$

3. a) 0* (1*111 0*0)* (ε ∪ 1*111)

b) $A(x) = x^2 + x^4 \cdots = \dfrac{x^2}{1-x^2}$

Alternatively,

$A = 01 \cup 0A1$

$A(x) = x^2 + x^2 A(x)$

$A(x) = \dfrac{x^2}{1-x^2}$

$$S(x) = \frac{1}{1-x}\left(\frac{1}{1-\left(\frac{A(x)}{1-x}\right)}\right)\frac{1}{1-x} = \frac{1}{(1-x)^2}\left(\frac{1}{1-\frac{x^2}{(1-x)(1-x^2)}}\right)$$

$$= \frac{1+x}{1-x-2x^2+x^3}$$

4. a) $a_n - 2a_{n-1} + a_{n-4} - a_{n-5} = \begin{cases} 1, & n=0 \\ 1, & n=4 \\ 0, & \text{otherwise} \end{cases}$

$a_0 = 1$ $\quad a_3 = 8$

$a_1 = 2$ $\quad a_4 = 16$

$a_2 = 4$

$a_n = 2a_{n-1} - a_{n-4} + a_{n-5}, \; n \geq 5$

b) The characteristic polynomial for this recurrence relation is

$$1 - 6x + 12x^2 - 8x^3 = (1-2x)^3$$

The solution is hence of the form

$$b_n = (A + Bn + Cn^2)\, 2^n$$

Setting n=0, n=1, and n=2 respectively, we obtain

$$3 = (A + 0 + 0) \cdot 2^0$$

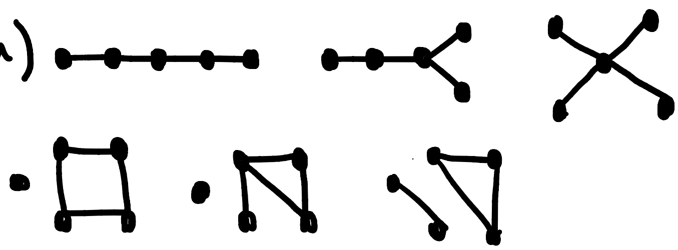
$$4 = (A + B + C) \cdot 2^1$$

$$12 = (A + 2B + 4C) \cdot 2^2$$

Solving the system of equations yields A=3, B=-2, C=1.
Hence we have

$$b_n = (3 - 2n + n^2) \cdot 2^n$$

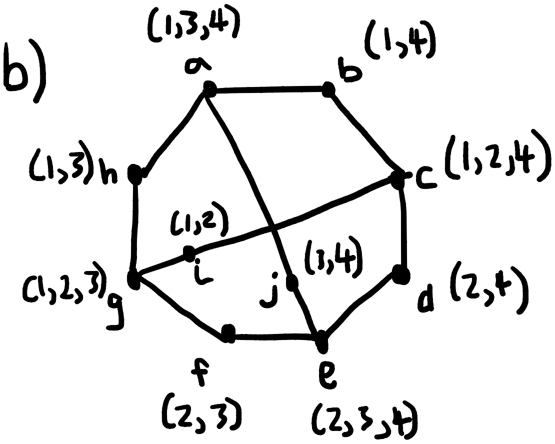
5. a)



b) Let G be a graph where every vertex has at least degree k. Suppose that P = (v0,…vn) has length n < k. Observe vn. It must not be connected to any vertices outside P, otherwise P would not be the longest path in G. Hence it can only be connected to vertices v0,…vn, and so it has a degree no larger than n. But it must have degree k, and n < k, which contradicts our original statement. Hence if every vertex of a graph has at least degree k, then it has a path of length at least k.

6. a) There are (2n choose n+1) + (2n choose n) vertices. Each n+1-element set has n+1 n-element subsets, and so has degree n+1. n-element sets cannot contain n+1-element sets. The graph is bipartite, since no n+1-element vertex can contain another n+1-element set, and no n-element set can contain another n-element set. Hence we can create a bipartition of n+1-element sets and n-element sets. Hence every edge from a n+1-element set has its other end in an n-element set.
Hence the number of edges is (n+1)(2n choose n+1)

b)



a: (1,3,4)
b: (1,4)
c: (1,2,4)
d: (2,4)
e: (2,3,4)
f: (2,3)
g: (1,2,3)
h: (1,3)
i: (1,2)
j: (3,4)

c) The graph is bipartite, since no n+1-element vertex can contain another n+1-element set, and no n-element set can contain another n-element set. Hence we can create a bipartition of n+1-element sets and n-element sets. Hence A_n is bipartite.