MATH 239 Exercises 7

7.3.1 Let G be a graph with a planar embedding, with p vertices, q edges, and f faces. If G has a vertex with degree less than or equal to 3, we have satisfied the statement.

Otherwise, every vertex has degree of 4 or more. Then we have

$$2q = \sum_{v \in V(G)} \deg(v) \geq 4p$$
$$q \geq 2p$$

Suppose G does not have a face of degree 3; ie every face is of degree 4 or more. Then we have

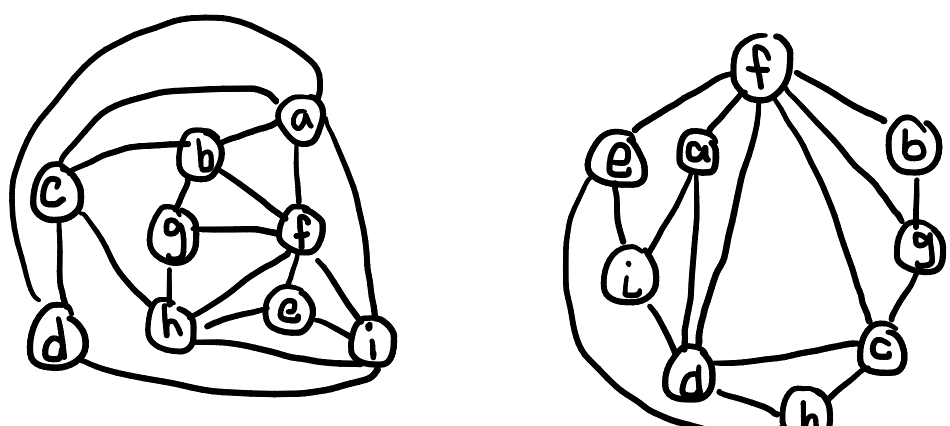
$$2q = \sum_{i=0}^{f} \deg(f_i) \geq 4f$$
$$q \geq 2f$$

Combining the two equations, we obtain

$$2q \geq 2p + 2f$$
$$q \geq p + f$$

But by Euler's theorem, we have p-q+f=2, so p+f=2+q, and we cannot have p+f <= q. Hence the planar embedding must have a face of degree 3.

7.3.2



7.3.3 The convex n-gon and its diagonals form a planar graph, with new vertices being the intersection points of the diagonals. The number of regions is hence the number of faces of this planar embedding.

Let v1=n be the number of vertices of the polygon, each of which has degree n-1. Let v2 be the number of vertices formed by the intersections of diagonals. Any 4 n-gon vertices uniquely determines one intersection point. Hence we have v2=(n choose 4), each of which has degree 4. Let q be the number of edges in this planar graph. We have that

$$q = \frac{1}{2}\sum_{v} \deg(v) = \frac{1}{2}\left(v_1(n-1) + 4v_2\right)$$

By Euler's theorem, we have

$$v_1 + v_2 - q + f_n = 2$$

$$n + \binom{n}{4} - \frac{1}{2}\left(n(n-1) + 4\binom{n}{4}\right) + f_n = 2$$

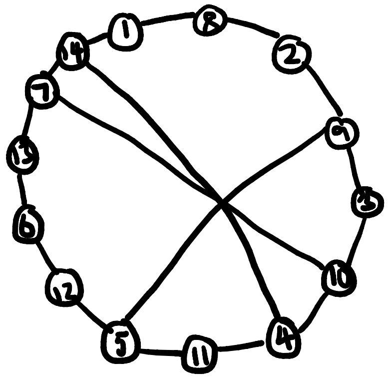
$$f_n = \binom{n}{4} + \frac{1}{2}n^2 - \frac{3}{2}n + 2$$

7.4.1 We have q=12, p=6, f=8. Since each face has degree 3, no vertex can be repeated in a boundary walk, so each face boundary is a 3-cycle. Let C=(v0, v1, v2, v0) be the boundary walk of one of the faces. Each edge of C must border exactly one other face. Each of these 3 neighbouring faces has one vertex not in C - let these be v3, v4, v5. Hence we have the 6 vertices, and also 9 of the edges. More edges can only be drawn among v3, v4, and v5, since the vertices in C already have degree 4. Hence we add 3 edges, connecting each v3, v4, and v5 with each other, resulting in 8 faces. Hence this is the only way to construct this graph.

7.4.2 We have q=30, p=20, f=12. Since each face has degree 5, no vertex can be repeated in a boundary walk, so each face boundary is a 5-cycle. Let C=(v0, v1, v2, v3, v4) be the boundary walk of one of the faces. Each vertex in C cannot be connected to other vertices other than those adjacent to it in the walk, otherwise we no longer have 5-cycle faces. Since all vertices have degree 3, each vertex in C must be connected to one new vertex not in C. Since C must border exactly 5 other faces, we now have 6 total faces and 15 vertices. We have 5 vertices that are only degree 2 - since every other vertex already have degree 3, we must have 5 new vertices, and these are our 20 vertices. Hence these 5 newest vertices must be connected to each other, forming the last 5 faces of our 12. Hence there is only one way to construct this graph.

7.4.3 (Do not recommend reading this, it is extremely cancer and the logic of the previous two is basically the same). We have q=30, p=12, f=20. Since each face has degree 3, no vertex can be repeated in a boundary walk, so each face boundary is a 3-cycle. Let C=(v0, v1, v2, v0) be the boundary walk of one of the faces. Each edge of C must border exactly one other face. Each of these 3 neighbouring faces has one vertex not in C - let these be v3, v4, v5. Hence we have the 6 vertices, and also 9 of the edges. Since each vertex has degree 5, each vertex in C is connected to one more new vertex - let these be v6, v7, v8. Each face must have degree 3, so we must connect v3 with v6 and v7 to form two new faces, and same for symmetric connections amongst v3, v4, v5 and v6, v7, v8. v3, v4, v5 now have degree 5, while v6, v7, and v8 have degree 3, while all vertices in C have degree 5, so we must add 3 new vertices. Hence we have all 12 vertices. To fulfill degrees, we must connect each of these 3 new vertices to one of v3, v4, v5, and two of v6, v7, v8, to form two new faces each. There are no more vertices left, so we connect each of these 3 newest vertices to each other so they each have degree 5, and we have 20 faces. Hence there is only one way to construct this graph.

7.6.1 **a)** We have the following subgraph, which is an edge subdivision of K3,3. Hence by Kuratowski's theorem, the graph is not planar.



b) Do the rest with similar methods :P

7.6.2 Since G has girth k, every face of the graph must have at least k edges. Since each edge is contained in exactly 2 faces, we have 2q >= kf. By Euler's formula, we have

$$2q \geq k(2+q-p)$$
$$q \geq \frac{k(p-2)}{k-2}$$

If q = k(p-2)/(k-2), then we have

$$p - \frac{k(p-2)}{k-2} + f = 2$$
$$f = \frac{2(p-2)}{k-2}$$

Since G has girth k, by the Faceshaking lemma, we have

$$2q = \frac{2k(p-2)}{k-2} = \sum_{i=0}^{f} \deg(F_i) = kf + \sum_{i=0}^{f} \deg(F_i) - k$$

$$\frac{2k(p-2)}{k-2} = \frac{2k(p-2)}{k-2} + \sum_{i=0}^{f} \deg(F_i) - k$$

$$\sum_{i=0}^{f} \deg(F_i) - k = 0$$

Hence all faces of G have degree k.

7.6.3 We have that the Petersen graph has p=10 vertices, q=15 edges, and girth k=5. Suppose the Petersen graph is planar. Then by Euler's Theorem, we have 10-15+f=2 -> f=7, so it would have 7 faces in its planar embedding. By the Faceshaking lemma, we must have

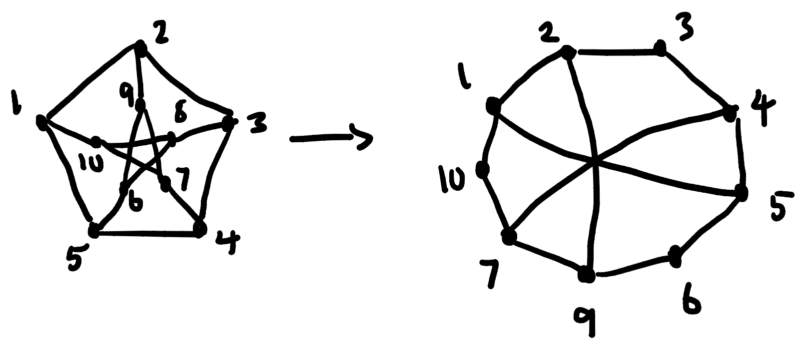
$$2q = \sum_{i=0}^{f} \deg(F_i) \geq kf$$

$$2 \cdot 15 \geq 5 \cdot 7$$

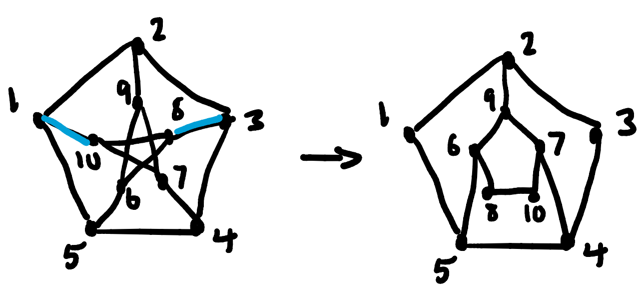
$$30 \geq 35$$

This is clearly a contradiction, so the Petersen graph must be non-planar.

7.6.4 **a)** This subgraph is an edge subdivision of K3,3:



b) We delete two edges to obtain the planar graph as follows:



7.6.5 The n-cube has vertices all with degree n. Since a planar graph has vertices with degree at most 5, the n-cube is not planar for n=6.

For n=4, we have that p=16 and q=32. The 4-cube is bipartite. We have

$$q = 32 > 28 = 2p - 4$$

Hence the n-cube is non-planar for n=4.

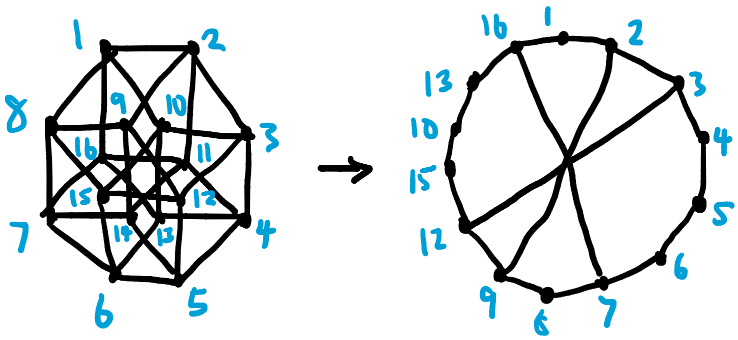
For n=5, we have that p=32 and q=80. The 5-cube is bipartite. We have

$$q = 80 > 60 = 2p - 4$$

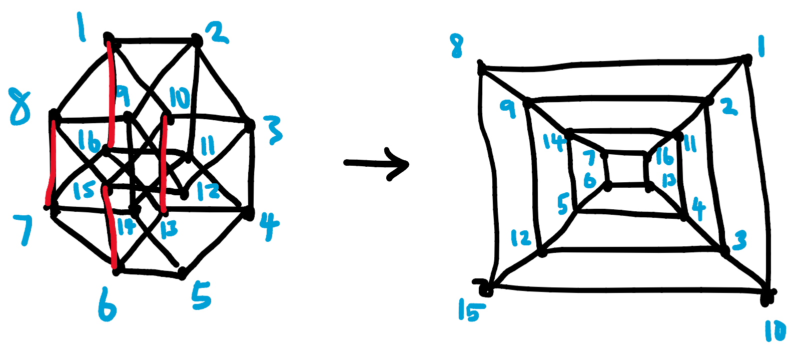
$$q=80>60=2p-4$$

Hence the n-cube is non-planar for n=5.

7.6.6 a) This subgraph is an edge subdivision of K3,3:



b) We delete 4 edges to obtain the planar graph as follows:

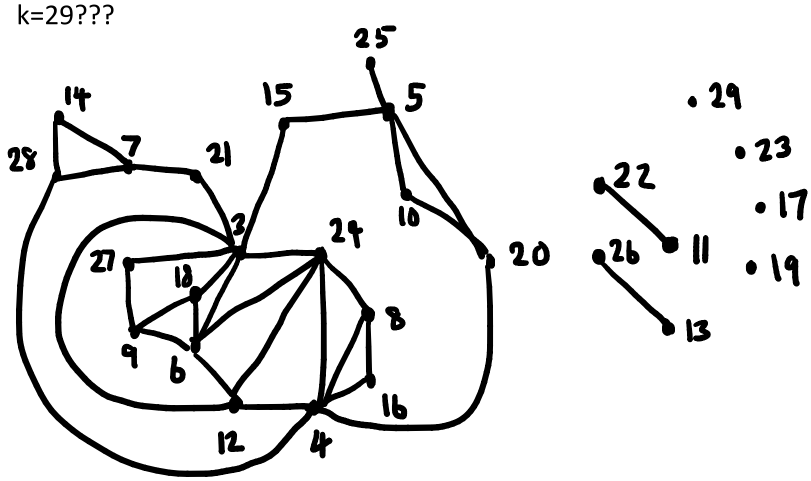


c) The 4-cube is a bipartite graph with 16 vertices and 32 edges. Deleting any 3 edges from the 4-cube gives us a bipartite graph with p=16 vertices and q=29 edges. We have

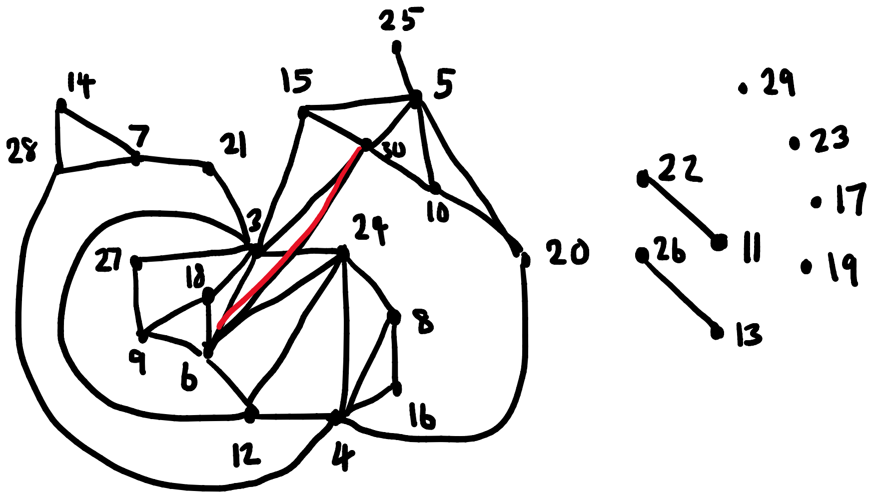
$$q=29>28=2p-4$$

Hence deleting any 3 edges from the 4-cube results in a non-planar graph.

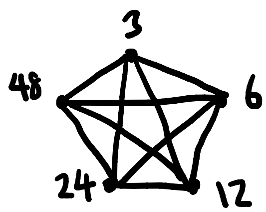
7.6.7 a) k=29???



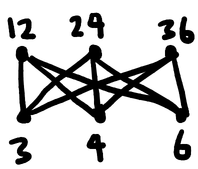
b) But I can't prove this one and this problem is too cancer :P



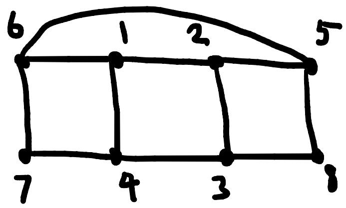
c) For n=48, we have the following subgraph, which is K5



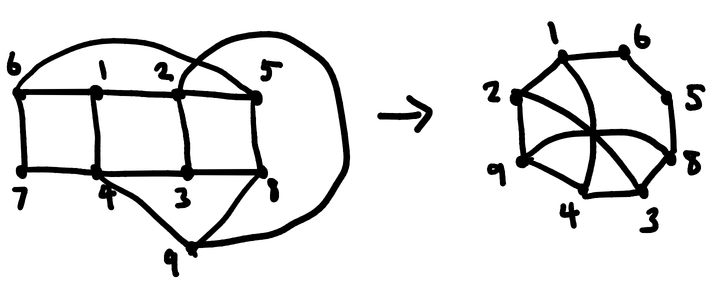
d) For n=36, we have the following subgraph, which is K3,3



7.6.8 a) This is a planar embedding of B8.



b) We have this subgraph, which is an edge division of K3,3.



7.6.9 This is true if |V(G)| <= 2. If p = |V(G)| >= 3 and q = |E(G)| then by Theorem 7.5.6,

$$q \leq 2p-4$$

$$\frac{2q}{p} \leq 4-\frac{8}{p}$$

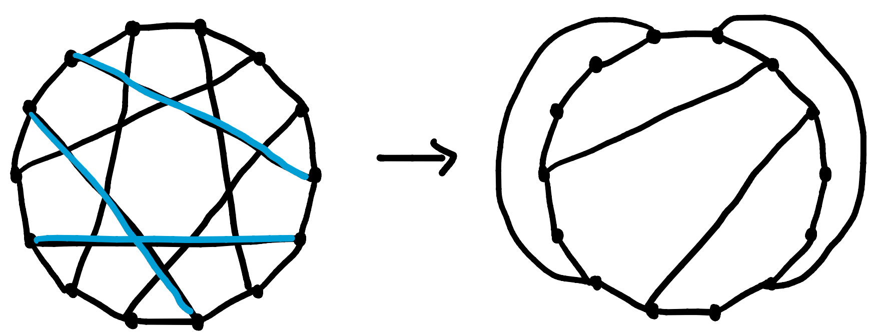
This shows that the average degree of a vertex in G is less than 4, and G contains a vertex of degree at most 3.

7.6.10 a) G has 14 vertices and 28 edges. G is bipartite - we can create a 2-colouring by colouring every vertex an alternating colour as we go around the circle. Hence any H obtained from G by deleting two edges is also bipartite, and has p=14 vertices and q=26 edges. Suppose H is planar. Then we have

$$q = 26 \leq 24 = 2p-4$$

This is clearly false, so H cannot be planar.

b) We can delete 3 edges to obtain the planar graph as follows:



7.6.11 Since G has girth at least 6, every face of the graph must contain at least 6 edges - hence each face has degree at least 6. By lemma 7.5.2, we have

$$(6-2)q \leq 6(p-2)$$

$$\frac{2q}{p} \leq 3-\frac{6}{p}$$

This shows that the average degree of a vertex in G is less than 3, and G contains a vertex of degree at most 2.

The dodecahedron is a counterexample for when the girth is 5 - it is planar with a girth of 5, but every vertex has degree 3.

7.6.12 Since G is planar, we have that

$$q \leq 3p-6$$

By the Handshaking lemma, we have

$$2q = \sum_v \deg(v) \geq 5p$$

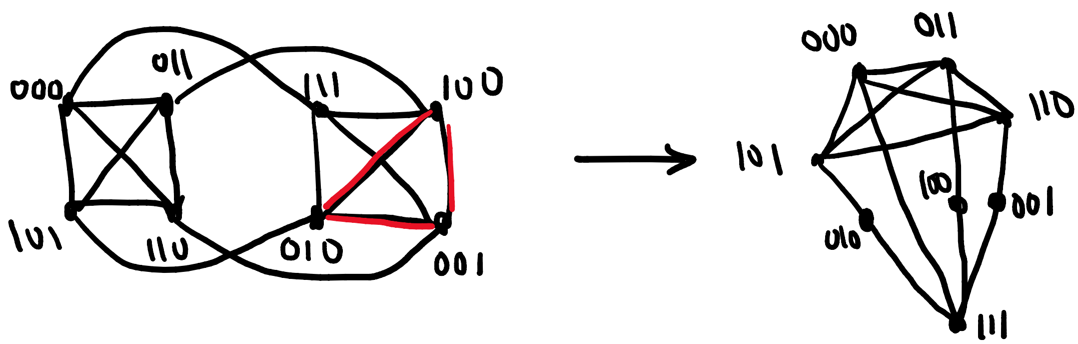
Combining the two equations, we obtain

$$5p \leq 6p-12$$

$$p \geq 12$$

Hence p=|V(G)| >= 12. One such graph with |V(G)|=12 is the icosahedron.

7.6.13 a) The complement of the 3-cube has this subgraph that is an edge subdivision of K5. Hence by Kuratowski's Theorem, it is non-planar.



b) If G is planar and has p >= 11 vertices and q edges, we must have

$$q \leq 3p-6$$

Its complement !G must have (p choose 2)-q edges. Suppose !G is also planar. Then we have

$$\frac{p(p-1)}{2} - q \leq 3p-6$$

Combining the 2 equations, we obtain

$$\frac{p(p-1)}{2} \leq 6p-12$$

$$p \leq 10.772$$

That contradicts our statement that p >= 11. Hence at least one of G and !G must be nonplanar.

7.8.1 a) Let G be a planar graph without a triangle with p vertices and q edges. If G has no cycles, G must be a forest, and hence must have a leaf.

If G has cycles, it has girth 4, so every face of the graph must contain at least 4 edges - hence each face has degree at least 4. By lemma 7.5.2, we have

$$(4-2)q \leq 4(p-2)$$

$$\frac{2q}{p} \leq 4 - \frac{8}{p}$$

This shows that the average degree of a vertex in G is less than 4, and G contains a vertex of degree 3 or less.

b) We prove the statement by induction on the number of vertices.

Base case: All graphs on one vertex are 4-colourable.

Inductive hypothesis: Assume the result is true for all planar graphs without a triangle on p <= k vertices, where k >= 1.

Inductive step: Consider a planar graph G on p=k+1 vertices. From part a, we have that G must have a vertex of degree 3 or less, v. Suppose we remove vertex v, and all edges incident to v, from G, and call the resulting graph G'. Then G' has k vertices, and is a planar graph with no triangles, so must be 4-colourable. There are at most 3 vertices in G' adjacent to v in G, so we can assign these vertices 3 different colours in the 4-colouring of G'. Thus there is at least one of the 4 colours remaining. Assign one of these remaining colours to v, so that v has a different colour from all of its adjacent vertices in G. Thus we have a 4-colouring of G, and the result is true for p=k+1. Hence we have proved the result by induction.

7.8.2 Since G is a planar graph with girth at least 6, G contains a vertex of degree at most 2. We prove the statement by induction on the number of vertices.

Base case: There is one graph on 6 vertices with girth at least 6 - the 6-cycle. It is 3 colourable.

Inductive hypothesis: Assume the result is true for all planar graphs with girth at least 6 on p <= k vertices, where k >= 1.

Inductive step: Consider a planar graph G on p=k+1 vertices. We have that G must have a vertex of degree 2 or less, v. Suppose we remove vertex v, and all edges incident to v, from G, and call the resulting graph G'. Then G' has k vertices, and is a planar graph with girth at least 6, so must be 3-

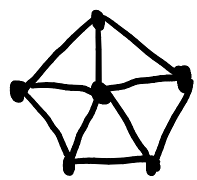
colourable. There are at most 2 vertices in G' adjacent to v in G, so we can assign these vertices 2 different colours in the 3-colouring of G'. Thus there is at least one of the 3 colours remaining. Assign one of these remaining colours to v, so that v has a different colour from all of its adjacent vertices in G. Thus we have a 3-colouring of G, and the result is true for p=k+1. Hence we have proved the result by induction.

7.8.3 a) Let G have p vertices, q edges, and f faces. Since its dual graph G* is isomorphic to G, it must have the same number of vertices and faces, ie p=f. Then by Euler's theorem, we have

$$p-q+f=2=p-q+p$$

$$q=2p-2$$

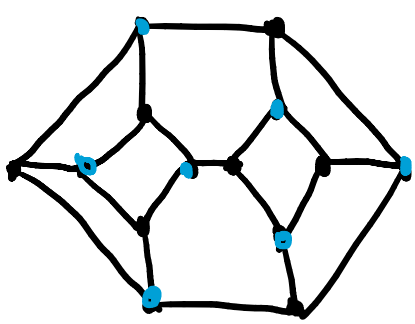
b)



7.8.4 If G has no cycles, then G has no odd cycles, and must be bipartite.

If G contains cycles, we prove that none of them are odd. Suppose G contains an odd cycle. Let C be the shortest odd cycle. It cannot be a face boundary, since every face has even degree. Then C must contain at least two faces, and hence must have a diagonal. But then this diagonal splits C into two cycles, one of which must have odd length, since the sum of the lengths of the two must be odd. But this contradict our original statement that C was the shortest odd cycle. Hence G must not contain any odd cycles, and therefore must be bipartite.

7.8.5



7.8.6 Let G be a planar graph with p > 2 vertices and q=2p-3 edges. Suppose G is 2-colourable. Then by Theorem 7.7.2, G is bipartite. Since G is planar, we must have q <= 2p-4. But this is a contradiction, since 2p-3 > 2p-4. Hence G cannot be 2-colourable.

7.8.7 Let G be a graph with 2m vertices and m^2+1 edges. Suppose G is 2-colourable. Then by Theorem 7.7.2, G is bipartite. From Exercise 4.4.7d, we must have that G has at most floor((2m)^2/4) = m^2 edges. But this contradicts our statement that G has m^2+1 edges. Hence G cannot be 2-colourable.

7.8.8 We can contract the 5 edges highlighted in blue to obtain K5. We have that G/e is planar whenever G is. The contrapositive of this statement is that if G/e is non-planar, G is non-planar. Since K5 is non-planar, the Petersen graph must also be non-planar.