

## MATH 239 Exercises 2

2.1 a)  $[x^8] \sum_{n=0}^{\infty} \binom{n+7-1}{7-1} x^n = 3003$

b)  $[x^{10}] x^6 \sum_{n=0}^{\infty} \binom{n+5-1}{5-1} (2x)^n = 1120$

c)  $[x^5] (x^3 + 5x^4) \sum_{k=0}^{\infty} \binom{6}{k} (3x)^k = \binom{6}{5}(3^5) + \binom{6}{4}(3^4) = 2673$

d)  $[x^9] (-4x)^5 + \sum_{n=0}^{\infty} \binom{n+2-1}{2-1} (3x)^n = 196830$

e)  $[x^n] \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} (2+x)^j = \binom{n+k-1}{k-1} (2+x)^n$

f)  $[x^{m+1}] x^k \sum_{j=0}^{\infty} \binom{j+2k-1}{2k-1} (4x)^j = \binom{n+k}{2k-1} 4^{n+1-k}$

g)  $[x^n] x^k \sum_{j=0}^{\infty} \binom{j+m-1}{m-1} (x^2)^j = \binom{\frac{n+k}{2} + m - 1}{m-1}$

h)  $[x^n] \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} (x^2)^j + \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} (7x^3)^j = \binom{\frac{n}{2} + k - 1}{k-1} + \binom{\frac{n}{3} + k - 1}{k-1} (7^{\frac{n}{3}})$

2.2 a)  $\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} (-1)^n x^n = \sum_{n=0}^{\infty} (n+1)(-1)^n x^n$

b)  $\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$

c)  $\frac{1}{(1+x^3)} = \sum_{n=0}^{\infty} \binom{n}{0} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n x^{3n}$

d)  $\frac{1}{(1-2x)} = \sum_{n=0}^{\infty} \binom{n}{0} (2x)^n = \sum_{n=0}^{\infty} (2x)^n$

e)  $\frac{1}{(1+2x^2)} = \sum_{n=0}^{\infty} \binom{n+1}{1} (-2x^2)^n = \sum_{n=0}^{\infty} (n+1)(-2x^2)^n$

f)  $\frac{1}{(1-2x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} (2x)^n$

2.3 a)  $\sum_{k=0}^n \binom{n}{k} k = \sum_{k=0}^n \frac{n! \cdot k}{k!(n-k)!} = \sum_{k=0}^n \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!}$

$$= \sum_{k=0}^n n \binom{n-1}{k-1} = n \binom{n-1}{0-1} + \sum_{k=1}^n n \binom{n-1}{k-1}$$

$$= 0 + n \sum_{k=0}^{n-1} \binom{n-1}{k} = n 2^{n-1}$$

b)  $\sum_{n=0}^n \binom{n}{k} k(k-1) = \sum_{k=0}^n \frac{n! k(k-1)}{k!(n-k)!} = \sum_{k=0}^n \frac{n(n-1)(n-2)!}{(k-2)!(n-2-(k-2))!}$

$$= \sum_{k=0}^n n(n-1) \binom{n-2}{k-2} = n(n-1) \left[ \binom{n-2}{0-2} + \binom{n-1}{0-1} + \sum_{k=2}^n \binom{n-2}{k-2} \right]$$

$$= n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} = n(n-1) 2^{n-2}$$

2.4 Let  $\Phi_A(x) = (1+x)^{-2} = \sum_{k=0}^{\infty} \binom{k+1}{1} (-x)^k = \sum_{k=0}^{\infty} (k+1)(-x)^k$

$$\Phi_B(x) = (1-2x)^{-2} = \sum_{k=0}^{\infty} \binom{k+1}{1} (2x)^k = \sum_{k=0}^{\infty} (k+1)(2x)^k$$

By the product lemma, the weight function of the product of the two generating series is the sum of their individual ones. To find the coefficient of  $x^n$ , we find the pairs in the cross product of the terms of A and B where the exponents of  $x$  sum to  $n$ . Let  $w$  be the weight function of the product of the two generating series. Hence we find the pairs  $(i, j)$  where

$$a_i = (i+1)(-1)^i \quad w(a_i, \Phi_j) = n = i+j$$

$$\Phi_j = (j+1) 2^j$$

$$[x^n] (1+x)^{-2} (1-2x)^{-2} = \sum_{i=0}^n (i+1)(-1)^i (n-i+1) 2^{n-i}$$

$$= \frac{1}{27} (2^{n+2} (5+3n) + (-1)^n (7+3n))$$

2.5 a) Methodology is the same as 2.4 above.

$$[x^n] \frac{(1+x)^a}{(1-x^2)^a} = [x^n] \left[ \sum_{k=0}^a \binom{a}{k} x^k \right] \left[ \sum_{n=0}^{\infty} \binom{n+a-1}{a-1} x^{2n} \right] \rightarrow n=i+2j$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i} \binom{n+a-1}{a-1}$$

$$[x^n] \sum_{k=0}^{\lfloor \frac{n}{a} \rfloor} \binom{a}{n-2k} \binom{k+a-1}{a-1}$$

$$[x^n] \frac{1}{(1-x)^a} = [x^n] \sum_{k=0}^{\infty} \binom{k+a-1}{a-1}$$

$$= \binom{n+a-1}{a-1}$$

b) ?????

$$2.6 \quad \underline{\Phi}_B(x) = \sum_{r \in B} x^{w(r)} = \sum_{o \in A_1} x^{w(o)} + \sum_{o \in A_2} x^{w(o)} + \dots$$

$$= \sum_{j=0}^{\infty} \underline{\Phi}_{A_j}(x)$$

2.7 We can use induction. Base case: 2 sets (proved in Product Lemma 2.12). Suppose for k sets, we have

$$\underline{\Phi}_{A_1 \times A_2 \times \dots \times A_k}(x) = \prod_{j=1}^k \underline{\Phi}_{A_j}(x)$$

We prove that for k+1 sets, we have

$$\underline{\Phi}_{A_1 \times \dots \times A_{k+1}}(x) = \prod_{j=1}^{k+1} \underline{\Phi}_{A_j}(x)$$

We have already shown that for two sets, the generating series of  $A \times B$  is the product of the two generating series. Hence we have that

$$\underline{\Phi}_{(A_1 \times A_2 \times \dots \times A_k) \times A_{k+1}}(x) = \left[ \prod_{j=1}^k \underline{\Phi}_{A_j}(x) \right] \cdot \underline{\Phi}_{A_{k+1}}(x) = \prod_{j=1}^{k+1} \underline{\Phi}_j(x)$$

Hence we have proven the statement.

$$\begin{aligned}
 2.8 \sum_{k=0}^{m-n} \binom{m-n}{k} x^k &= \frac{\sum_{j=0}^m \binom{m}{j} x^j}{\sum_{i=0}^n \binom{n}{i} x^i} = \left[ \sum_{j=0}^m \binom{m}{j} x^j \right] \frac{1}{(1+x)^n} = \left[ \sum_{j=0}^m \binom{m}{j} x^j \right] \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} (-x)^i \\
 &= \sum_{j=0}^m \sum_{i=0}^{\infty} \binom{m}{j} \binom{i+n-1}{n-1} (-1)^i x^{i+j}
 \end{aligned}$$

???

- 2.9 a) { aaaa, aaab, aaba, abaa, baaa, aabb, abab, abba, baba, baab, bbaa, abbb, babb, bbab, bbba, bbbb }

Weight 0: 5

Weight 1: 10

Weight 2: 1

$$\underline{\Phi}(x) = 5 + 10x + x^2$$

b) ???

- 2.10 a)

2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	5	4	3	2	1

Generating series =  $x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$

$$= \sum_{i=1}^6 i x^{i+1} + \sum_{i=1}^5 (6-i) x^{i+7}$$

$$b) \underline{\Phi}(x) = \sum_{i=1}^8 \frac{i(i+1)}{2} x^{i+2} + \sum_{i=1}^8 \frac{(9-i)(10-i)}{2} x^{i+10}$$

- 2.11

0	1	2	3	4	5
6	10	8	6	4	2

$$\underline{\Phi}(x) = 6 + \sum_{i=1}^5 (10-2i) x^i$$

2.12 a) This is a weight function. For every natural number  $n$ , there are only a finite number of elements in  $S$  that are weight  $n$ , since there are only a finite number of pairs  $(a,b)$  where  $a=n$ , since  $0 \leq |b| \leq a$ .

$$\begin{aligned}\Psi_s(x) &= \sum_{(a,b) \in S} x^{w(a,b)} = \sum_{(a,b) \in S} x^a = \sum_{a=0}^{\infty} x^a \left[ \sum_{b=-a}^a 1 \right] = \sum_{a=0}^{\infty} (2a+1)x^a \\ &= \sum_{a=0}^{\infty} x^a + \sum_{a=0}^{\infty} 2ax^a = \sum_{a=0}^{\infty} \binom{a}{0} x^a + 2 \sum_{a=0}^{\infty} \binom{a}{1} x^a \\ &= (1-x)^{-1} + 2(1-x)^{-2} = \frac{3-x}{(1-x)^2}\end{aligned}$$

b) This is not a weight function. For every natural number  $n$ , there are an infinite number of pairs  $(a,b)$  where  $a+b=n$ .

c) This is a weight function. For every natural number  $n$ , there are only a finite number of elements in  $S$  that are weight  $n$ , since  $a \leq 2a+b \leq 3a$ .

2.13 a) The generating function for the outcome of one dice is  $(x + x^2 + x^3 + x^4 + x^5 + x^6)$ . Hence the generating function for the sum of 4 dice is  $(x + x^2 + x^3 + x^4 + x^5 + x^6)^4$ .

$$\sum_{n=1}^6 x^n = \sum_{n=1}^6 x^n - \sum_{n=7}^{\infty} x^n = \frac{x}{1-x} - \frac{x^7}{1-x} = \frac{x-x^7}{1-x}$$

$$\Psi_s(x) = \left( \frac{x-x^7}{1-x} \right)^4$$

$$b) [x^{19}] \Psi_s(x) = [x^9] x^4 (1-x^6)^4 (1-x)^{-4} = [x^5] \left[ \sum_{k=0}^4 \binom{4}{k} (-x^6)^k \right] \sum_{n=0}^{\infty} \binom{n+3}{3} x^n$$

$$\alpha_i = \binom{4}{i} (-1)^i$$

$$\beta_j = \binom{j+3}{3} \quad \omega(\alpha_i \beta_j) = 15 = 6i + j \Rightarrow \underbrace{0 \leq i \leq 4}_{0 \leq j \leq 15}$$

$$[x^{19}] \Psi_s(x) = \sum_{i=0}^2 \binom{4}{i} (-1)^i \binom{15-6i+3}{3} \quad 0 \leq i \leq 2$$

$$= 56$$

$$c) [x^k] \left( \frac{x - x^{d+1}}{1-x} \right)^m = [x^{k-m}] \sum_{i=0}^m \binom{m}{i} (-x)^{d+i} \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} x^n$$

$$\alpha_i = \binom{m}{i} (-1)^i \quad \omega(\alpha_i p_j) = k-m = d i + j \quad k-m-d i \geq 0 \rightarrow i \leq \frac{k-m}{d}$$

$$p_j = \binom{j+m-1}{m-1} \quad j = k-m - d i \geq 0, 0 \leq i \leq m$$

$$[x^k] \bar{\Phi}(x) = \sum_{i=0}^{\lfloor \frac{k-m}{d} \rfloor} \binom{m}{i} (-1)^i \binom{k-d i - 1}{m-1}$$

$$2.14 \sum_{k=0}^{\infty} \sum_{j=0}^k a_j x^k = \sum_{k=0}^{\infty} \sum_{j=0}^k a_j x^j \cdot x^{k-j} = \bar{\Phi}_A(x) \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \bar{\Phi}_A(x)$$

2.15