CS370 F2000 Midterm

1. For a stable computation, we must have

$$\lim_{n\to\infty} I_n = 0$$

We have that

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

$$\lim_{n\to\infty} 2 = 2 \neq 0$$

Hence this recursion is not stable.

2. a) $$L_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_N)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_N)}$$

b) ???

3. ① $S_i(x_i) = y_i$

② $S_i(x_{i+1}) = y_{i+1}$

③ $S'_i(x_{i+1}) = S'_{i+1}(x_{i+1})$

④ $S'(x_1) = S'(x_n)$

Let $\Delta x_i = x_{i+1} - x_i$

① $a_i(x_i - x_i) + b_i(x_i - x_i)(x - x_{i+1}) + c_i(x_i - x_{i+1}) = y_i$

$$\rightarrow c_i = \frac{y_i}{x_i - x_{i+1}} = \frac{-y_i}{\Delta x_i}$$

② $a_i(x_{i+1} - x_i) + b_i(x_{i+1} - x_i)(x_{i+1} - x_{i+1}) + c_i(x_{i+1} - x_i) = y_{i+1}$

$$\rightarrow a_i = \frac{y_{i+1}}{x_{i+1} - x_i} = \frac{y_{i+1}}{\Delta x_i}$$

③ $S_i'(x) = a_i + b_i(2x - x_{i+1} - x_i) + c_i$

$$S_{i+1}'(x) = a_{i+1} + b_{i+1}(2x - x_{i+2} - x_{i+1}) + c_{i+1}$$

$$a_i + b_i(2x_{i+1} - x_{i+1} - x_i) + c_i = a_{i+1} + b_{i+1}(2x_{i+1} - x_{i+2} - x_{i+1}) + c_{i+1}$$

$$a_i + c_i + b_i \Delta x_i = a_{i+1} + c_{i+1} - b_{i+1}\Delta x_{i+1}$$

$$b_i \Delta x_i + b_{i+1}\Delta x_{i+1} = a_{i+1} + c_{i+1} - a_i - c_i$$

④ $a_1 + c_1 - b_1 \Delta x_1 = a_{n-1} + c_{n-1} + b_{n-1}\Delta x_{n-1}$

$$b_{n-1}\Delta x_{n-1} + b_1 \Delta x_1 = a_{n-1} + c_{n-1} - a_1 - c_1$$

To solve for b, we solve the system

$$\begin{bmatrix} b_1 & b_2 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & b_3 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \\ 0 & 0 & \cdots & & b_{n-2} & b_{n-1} \\ b_1 & 0 & \cdots & & \cdots & b_{n-1} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_{n-1} \end{bmatrix} = \begin{bmatrix} a_2 + c_2 - a_1 - c_1 \\ a_3 + c_3 - a_2 - c_2 \\ \vdots \\ \vdots \\ a_{n-1} + c_{n-1} - a_1 - c_1 \end{bmatrix}$$

4. The Taylor series expansion of U(xi) gives us

$$U(x_{i+1}) = U(x_i) + U'(x_i)\Delta x_i + O(\Delta x_i^2)$$

$$U'(x_i) = \frac{U(x_{i+1}) - U(x_i)}{\Delta x_i} + O(\Delta x_i)$$

Using the same formula for U''(xi), we obtain

$$U''(x_i) = \frac{U'(x_{i+1}) - U'(x_i)}{\Delta x_i} + O(\Delta x_i)$$

Substituting in the approximations for U'(xi) and U'(xi+1), we obtain

$$U''(x_i) = \frac{\frac{U(x_{i+2}) - U(x_{i+1})}{\Delta x_{i+1}} - \frac{U(x_{i+1}) - U(x_i)}{\Delta x_i}}{\Delta x_i} + O(\Delta x_i)$$

$$= \frac{U(x_{i+2}) - 5U(x_{i+1}) + 4U(x_i)}{4\Delta x_i^2} + O(\Delta x_i)$$

5. [Not covered]