Assignment 6

1.

$$x[n] = a_0 + a_{-2}e^{-j\frac{4\pi}{5}n} + a_2 e^{j\frac{4\pi}{5}n} + a_{-4}e^{-j\frac{8\pi}{5}n} + a_4 e^{j\frac{8\pi}{5}n}$$

$$= 1 + \left(e^{-j\frac{\pi}{4}}e^{-j\frac{4\pi}{5}n} + e^{j\frac{\pi}{4}}e^{j\frac{4\pi}{5}n}\right) + \left(2e^{-j\frac{\pi}{3}}e^{-j\frac{8\pi}{5}n} + 2e^{j\frac{\pi}{3}}e^{j\frac{8\pi}{5}n}\right)$$

$$= 1 + 2\cos\left(\frac{4\pi}{5}n + \frac{\pi}{4}\right) + 4\cos\left(\frac{8\pi}{5}n + \frac{\pi}{3}\right)$$

$$= 1 + 2\sin\left(\frac{4\pi}{5}n + \frac{3\pi}{4}\right) + 4\sin\left(\frac{8\pi}{5}n + \frac{5\pi}{6}\right)$$

2. $a_{11} = a_1 = a_{-1} = 5$

Since the signal is real-valued, we have that $a_2 = a_{-2}^* = e^{j\pi/4}, a_4 = a_{-4}^* = 2e^{j\pi/3}$. We have that

By Parseval's theorem,

$$\sum_{n=0}^{9}|x[n]|^2 = 500 = 10\sum_{n=0}^{9}|a_n|^2$$

$$50 = \sum_{n=0}^{9}|a_n|^2 = \sum_{n=-1}^{8}|a_n|^2 = |a_{-1}|^2 + |a_1|^2 + a_0^2 + \sum_{n=2}^{8}|a_n|^2$$

$$= 25 + 25 + \underbrace{a_0^2 + \sum_{n=2}^{8}|a_n|^2}_{=0}$$

$$x[n] = 5e^{j\frac{2\pi}{10}n} + 5e^{-j\frac{2\pi}{10}n}$$

$$= 10\cos\left(\frac{\pi}{5}n\right)$$

3. Since x(t) is real,

$$even\{x(\text{t})\} = \frac{x(\text{t}) + x(\text{t})}{2} \xrightarrow{Fourier} Re\{X(\text{jw})\}$$

By point 3, the inverse Fourier transform of Re{X(jw)} is $|t|e^{-|t|}$.

Since x(t) = 0 for t $\leq$ 0, x(-t) = 0 for t > 0. Then we have that

$x(\text{t}) = 2|t|e^{-|t|}, t \geq 0$

Hence we have
$\boldsymbol{x(\text{t}) = 2|t|e^{-|t|}u_{-1}(\text{t})}$

4. a) $$\hat{x}(j\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{t} e^{-j\omega t} dt + \frac{1}{2\pi}\int_{0}^{\infty} e^{-t} e^{-j\omega t} dt$$

$$= \frac{j}{\omega + j} - \frac{j}{\omega - j}$$

$$= \frac{2}{1+\omega^2}$$

b) Differentiation in frequency property

$$j\frac{d}{d\omega}\hat{x}(j\omega) = -\frac{j4\omega}{(1+\omega^2)^2}$$

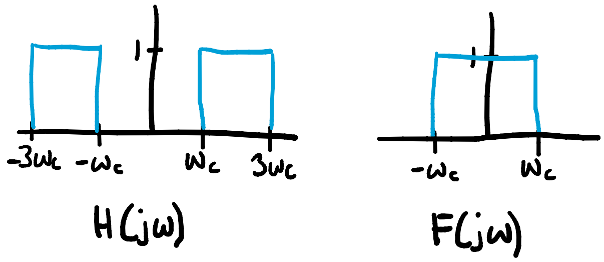
c) The duality property states that

$$x(t) \longleftrightarrow X(\omega)$$
$$X(t) \longleftrightarrow 2\pi x(\omega)$$

Hence we have that the Fourier transform is

$$j2\pi \omega e^{-|\omega|}$$

5. a) Let $f(t) = \frac{\sin w_c t}{\pi t}$



Since h(t) = f(t)g(t), we have that H(jw) = $\frac{1}{2\pi}F$(jw) * G(jw).
If G(jw) is 2 Dirac impulses, one at and one at, then convolving F(jw) with it will produce 2πH(jw).
Then we have that

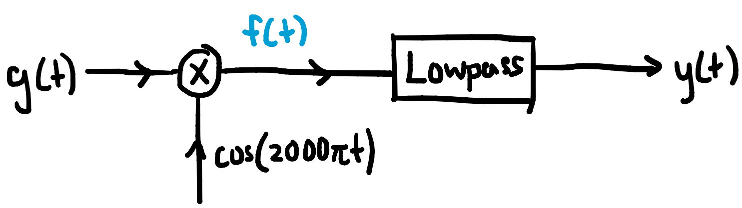
G(jw) = $2\pi(\delta(w-2w_c)+\delta(w+2w_c))$

Then we have that

g(t) = 2cos($2w_c t$)

b) As w_c increases, f(t) becomes more concentrated at the origin. Since h(t) = f(t)g(t), h(t) becomes **more concentrated** as well.

6. The block diagram is shown below.



We first take the Fourier transform of g(t), G(jw).

$$\hat{G}(jw) = \frac{1}{2j}\hat{X}(j(w-2000\pi)) - \frac{1}{2j}\hat{X}(j(w+2000\pi))$$

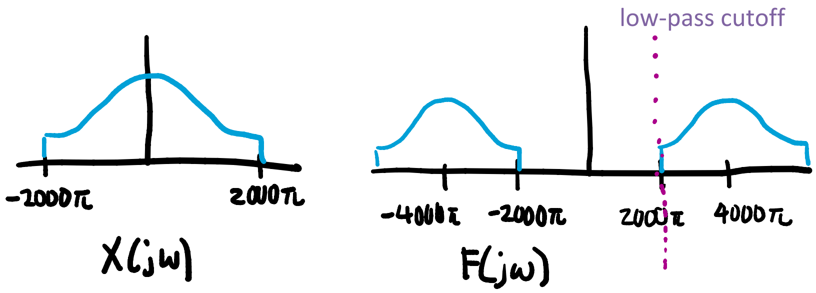
We take the Fourier transform of f(t), F(jw).

$$\hat{F}(jw) = \frac{1}{2}\hat{G}(j(w-2000\pi)) + \frac{1}{2}\hat{G}(j(w+2000\pi))$$

$$= \frac{1}{4j}\hat{X}(j(w-2000\pi-2000\pi)) - \frac{1}{4}\hat{X}(j(w+2000\pi-2000\pi))$$
$$+ \frac{1}{4j}\hat{X}(j(w-2000\pi+2000\pi)) - \frac{1}{4}\hat{X}(j(w+2000\pi+2000\pi))$$

$$= \frac{1}{4}j\hat{X}(j(w-4000\pi)) - \frac{1}{4}jX(j(w+4000\pi))$$

Diagrams for X(jw) and F(jw) are shown below.



We can see that nothing passes the low-pass cutoff. Hence we have that

y(t) = 0

7. [copy-pasta of official solution]
When the image undergoes Fourier transform, its low spatial frequency components will be concentrated near the optical axis, while the high spatial frequency components of the image will appear further away from the optical axis. Thus a high-pass filter (a) can be implemented by blocking the light on and near the optical axis in the plane containing the Fourier transform of the image, e.g. using a piece of transparent material with an opaque disk in the middle. A pinhole or an iris placed in the plane containing the Fourier transform of the image can serve as a low-pass filter (b) as these would let only on- and near-axis light pass through.