

# MATH 239 F2008 Final

1. Each part must not be equal to 1; the allowed parts are  $P = \{2, 3, 4, \dots\}$ . Hence the generating series for a part is

$$\Phi_p(x) = \frac{x^2}{1-x}$$

The generating function for compositions of  $n$  with an even number of parts is

$$\sum_{i=0}^{\infty} 2^i \Phi_p(x) = 2 \sum_{i=0}^{\infty} \Phi_p(x) = 2 \frac{1}{1-\frac{x^2}{1-x}} = \frac{2(1-x)}{1-x-x^2}$$

2. a) i. The expression  $(\epsilon - 0 - 0) ((1 - 11) (0 - 00))^* (\epsilon - 1 - 11)$  is the block decomposition for this set of strings.
- ii. The expression  $(\epsilon - 0^*000) ((1^*111) (0^*000))^* (\epsilon - 1^*111)$  is the block decomposition for this set of strings.

b) 010000

- c) The elements are not uniquely created. The string  $00.00.00 = 0.0.0.0.0$  can be created these two ways.

d)  $\Phi_s(x) = \frac{1}{1-(x+x^2+x^3)}$

3. By theorem 4.8,  $a_n$  satisfies the linear recurrence relation with initial conditions given by

$$a_n - 2a_{n-1} + a_{n-2} - a_{n-3} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 1, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

$$a_0, a_1, a_2 = 1$$

$$a_3 = 3$$

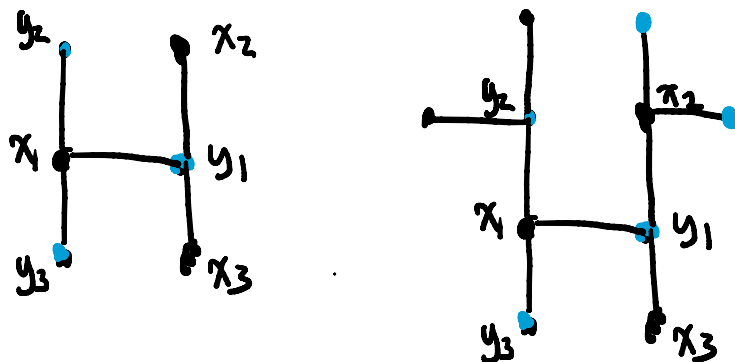
$$a_n = 2a_{n-1} - a_{n-2} + a_{n-3}, \quad n \geq 4$$

4. Let  $G$  be a connected graph. We prove that if an edge  $e$  is a bridge, it belongs to every spanning tree of  $G$ . Let  $e$  be an edge in  $G$  that is a bridge. Suppose that  $e$  does not belong to every spanning tree of  $G$ . Let  $T$  be a spanning tree not containing  $e$ . Then the tree  $T$  is a spanning subgraph of  $G \setminus e$ . Let  $u$  and  $v$  be any two vertices of  $G \setminus e$ . There must be a unique path from  $u$  to  $v$  in  $T$ , which is also a unique path from  $u$  to  $v$  in  $G \setminus e$ . Then  $G \setminus e$  must be connected, which is a

contradiction, since  $e$  is a bridge. Hence  $e$  must belong to every spanning tree of  $G$ .

We prove that if an edge  $e$  belongs to every spanning tree of  $G$ , it is a bridge. Let  $e$  be an edge belonging to every spanning tree of  $G$ . Suppose  $e$  is not a bridge. Then  $G \setminus e$  is connected, and  $G \setminus e$  must have a spanning tree  $T$ . Since  $V(G) = V(G \setminus e)$ ,  $T$  is also a spanning tree of  $G$ . This is a contradiction, since  $T$  does not contain  $e$  and  $e$  belongs to every spanning tree of  $G$ . Hence if  $e$  belongs to every spanning tree of  $G$ , it must be a bridge.

5. a) Since  $G$  is bipartite,  $G$  must not contain any odd cycles. Let  $X, Y$  be a bipartition of  $G$ . Let  $x_1$  be a vertex in  $G$  in  $X$ . It must have degree at least 3, so it must have 3 neighbours in  $Y$ . Let  $y_1, y_2, y_3$  be these neighbours.  $y_1$  must have 3 neighbours in  $X$ , one of which is  $x_1$ . Let the other two neighbours be  $x_2$  and  $x_3$ . If  $y_2$  is adjacent to  $x_2$  or  $x_3$ , we have a path of length 5. Otherwise,  $x_2$  and  $y_2$  must each have 2 distinct neighbours, and we have a path of length 5.



- b)  $K_{3,3}$  is a bipartite graph with degree at least 3 with no path of length 6.

- c) From part a,  $G$  must have a path of length at least 5. Suppose  $G$  does not contain a path of length 6. Let  $P = \{v_1, v_2, v_3, v_4, v_5\}$  be a path of length 5.  $P$  must alternate between vertices in bipartitions  $X$  and  $Y$ . Without loss of generality, let  $v_1$  be in  $X$ .  $v_1$  is adjacent to  $v_2$ , and since  $G$  has degree at least 3, it must be adjacent to 2 other vertices. These 2 other vertices must be in  $P$ , otherwise we could extend  $P$  to length 6 by adding the vertex to it. But this is a contradiction, since  $P$  is of length 5 and only has 2 vertices in  $Y$ , one of which is  $v_2$ . Hence  $G$  must contain a path of length at least 6.



6. a) Euler's formula states that  $p - q + f = 2$  for a connected planar graph. Suppose  $P$  has  $k$  components. Then we have

$$\sum_{i=1}^k (|V_i| - |E_i| + f_i) = 2k$$

$$|V| - |E| + f = 2k \geq 2 \quad (\text{since } k \geq 1)$$

$$102 - 300 + f \geq 2$$

$$f \geq 200$$

b) Suppose  $P$  does not contain any cycles. Then we have  $E = V - k$ , where  $k$  is the number of components in  $P$ . Hence we have  $E > V$ . This is a contradiction, so  $P$  must contain at least one cycle.

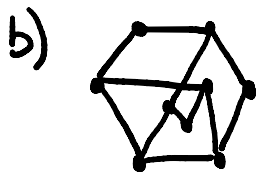
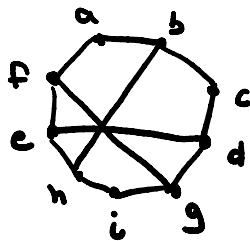
c) Since  $P$  contains cycles, the boundary of each face of  $P$  must contain a cycle. Hence there must be at least 3 edges in every face, so it must have degree at least 3.

d) From part a, we have that

$$102 - 300 + f = 2k$$

where  $k$  is the number of components. If  $k > 1$ , then we have  $f < 200$ , which is a contradiction. Hence there must only be one component, so  $P$  must be connected.

7. a)  $H$  is not planar. This subgraph of  $H$  contains an edge subdivision of  $K_{3,3}$ , so by Kuratowski's theorem, is nonplanar.



8. a) 1, 2, 4, 3, 5, 11, 7, 6, 8, 10, 12, 14, 9, 17, 13, 15, 16, 18

b)  $X_0: 7, 9$   
 $X: 7, 9$   
 $Y:$

$X: 7, 9$

$Y: 4, 6, 8$

$X: 7, 9, 1, 5, 17$

$Y: 4, 6, 8, 2, 14, 16, 18$

$X: 7, 9, 1, 5, 17, 3, 11, 13, 15$

$Y: 4, 6, 8, 2, 14, 16, 18, 10$

10 is unsaturated, we create new matching with augmenting path 7 8 17 14 11 1

X0: 9

X: 9

Y:

X: 9

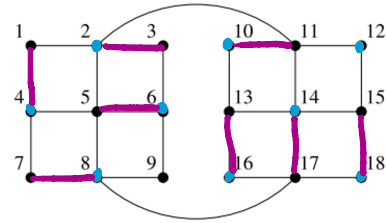
Y: 6, 8

X: 9, 5, 7

Y: 6, 8, 2, 4

X: 9, 5, 7, 3, 1

Added no vertices to Y, we terminate.



Our maximum matching is

$\{(1,4), (2,3), (5,6), (7,8), (10,11), (13,16), (14,17), (15,18)\}$ .

Our minimum cover is  $\{2,4,6,8,11,13,15,17\}$

9.

$D = \{1, 3\}$

$N(D) = \{a, c, f\}$

$|N(D)| = 2 < 3 = |D|$