CS370 Review Questions 2

1.

A) Catastrophic cancellation is when the relative error of a computation is very large, caused by the subtraction of two large numbers of similar magnitude and same sign.

B) i. We could have catastrophic cancellation when computing

$$-b-\sqrt{b^2-4ac} \quad \text{or} \quad -b+\sqrt{b^2-4ac}$$

since based on the sign of b, one of these will be the subtraction of 2 numbers of same sign, and they are of similar magnitude.

ii. Only one of the two computations above will lead to catastrophic cancellation. We could compute the one without catastrophic cancellation, then compute the other using the formula

$$r_1 = \frac{c}{a r_2}$$

2.

Let p̄_n be the floating point representation of p_n. Then we have

$$e_n = \bar{p}_n - p_n = 4\bar{p}_{n-1} - 3\bar{p}_{n-2} + 1 - 4p_{n-1} + 3\bar{p}_{n-2} - 1$$

$$= 4e_{n-1} - 3e_{n-2}$$

$$e_n - 4e_{n-1} + 3e_{n-2} = 0$$

With a=1, b=-4, c=3, we compute the roots of the equation ax^2+bx+c=0

$$x^2 - 4x + 3 = (x-1)(x-3)$$

Hence we have

$$e_n = c_1 3^n + c_2 1^n = c_1 3^n + c_2$$

Using the initial conditions, we compute c1 and c2.

$$e_0 = c_1 + c_2 = \bar{p}_0 - p_0 = 0$$

$$c_1 = -c_2$$

$$e_1 = 3c_1 + c_2 = \bar{p}_1 - p_1 = fl(\pi) - \pi$$

$$c_1 = \frac{fl(\pi) - \pi}{2}, \quad c_2 = -\frac{fl(\pi) - \pi}{2}$$

Hence we have

$$\lim_{n\to\infty} |e_n| = \lim_{n\to\infty} \left| \frac{f1(\pi)-\pi}{2} 3^n - \frac{f1(\pi)-\pi}{2} \right| = +\infty$$

The recursion is not stable.

3. a) We have that

$$r(-1) = L_1(-1) - 3L_2(-1) = 1 \longrightarrow a-b+c=1$$

$$r(0) = L_1(0) - 3L_2(0) = -3 \longrightarrow c=-3$$

$$r(1) = L_1(1) - 3L_2(1) = 0 \longrightarrow a+b+c=0$$

Solving the system of equations, we obtain a=7/2, b=-1/2, c=-3

$$r(x) = \frac{7}{2}x^2 - \frac{1}{2}x - 3$$

b) We have that

$$p(-1) = aL_1(-1) + bL_2(-1) + cL_3(-1) + dL_4(-1) = a$$

$$p(-1) = (-1)^2 + 1 = 2$$

$$a = 2$$

$$p(0) = b$$

$$b = 1$$

$$p(1) = c$$

$$c = 2$$

$$p(2) = d$$

$$d = 5$$

$$p(x) = 2L_1(x) + L_2(x) + 2L_3(x) + 5L_4(x)$$

4. a) We have that

$$S_1'(x) = \alpha + \frac{4}{3}x - x^2$$

$$S_2'(x) = 13 - \frac{32}{3}x + 3\beta x^2$$

The first two conditions for a spline are

① $S_1(2) = S_2(2)$

$$1 + 2\alpha + \frac{2}{3}(4) - \frac{1}{3}(8) = -7 + 13(2) - \frac{16}{3}(4) + 8\beta$$

$$\alpha - 4\beta = -5/3$$

② $S_1'(2) = S_2'(2)$

$$\alpha + \frac{4}{3}(2) - 4 = 13 - \frac{32}{3}(2) + 12\beta$$

$$\alpha - 12\beta = -7$$

Solving the system of equations, we obtain

$$\alpha = 1$$
$$\beta = 2/3$$

Hence we have

$$S'(0) = 1$$
$$S'(3) = 13 - \frac{32}{3}(3) + 3\left(\frac{2}{3}\right)(9)$$
$$= -1$$

b) We have that

$$S_1(1) = 1 \neq -1 = S_2(1)$$

Since S(x) is not continuous, it cannot be a spline.

c) We have 3(N-1) = 3N - 3 unknowns. We have 2(N-1) interpolation equations, and N-2 first derivative equations, giving us 3N-4 equations. Hence we need one more equation, so we need one boundary condition.

5. a) We write this higher-order system of differential equations as a first-order system.

Let $W_1 = w(t)$

$$W_1' = W_2$$
$$W_2' = W_3$$
$$W_3' = 1 - tW_1 z$$
$$z' = W_1 + t^2 - \sin(W_2 z)$$

$$\underbrace{\begin{bmatrix} w_1' \\ w_2' \\ w_3' \\ z' \end{bmatrix}}_{y'(t)} = \underbrace{\begin{bmatrix} w_2 \\ w_3 \\ 1 - t w_1 z \\ w_1 + t^2 - \sin(w_2 z) \end{bmatrix}}_{f(t, y(t))}$$

b) We write this as a system of first order differential equations.

$$\text{Let } Y = y(t)$$

$$Y' = X$$

$$X' = t - Y$$

Our initial conditions become Y(0) = 1, X(0) = 0. We estimate X1* and Y1* using Euler's method. We then proceed with Modified Euler.

$$Y_1^* = Y_0 + hX_0 = 1$$

$$X_1^* = X_0 + h(t_0 - Y_0) = -1$$

$$Y_1 = Y_0 + \frac{h}{2}(X_0 + X_1^*) =$$

$$= \tfrac{1}{2}$$

$$X_1 = X_0 + \frac{h}{2}(t_0 - Y_0 + (t_0 + h - Y_1^*)$$

$$= -\tfrac{1}{2}$$