MATH 239 Exercises 4

4.1 a) $G(x) = \dfrac{x-2x^2+x^3}{1-3x+3x^2-2x^3}$

By theorem 4.8, g_n satisfies the linear recurrence relation with initial conditions given by

$$g_n - 3g_{n-1} + 3g_{n-2} - 2g_{n-3} = \begin{cases} 1, & n=1 \\ -2, & n=2 \\ 1, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$$g_0 = 0 \qquad g_2 = 1$$
$$g_1 = 1$$
$$g_n = 3g_{n-1} - 3g_{n-2} + 2g_{n-3},\ n \geq 3$$

$$g_3 = 0 \qquad g_6 = 0 \qquad g_9 = 0$$
$$g_4 = -1 \qquad g_7 = 1$$
$$g_5 = -1 \qquad g_8 = 1$$

b) $G(x) = \dfrac{x^2-x^3}{1-3x+3x^2-2x^3}$

By theorem 4.8, g_n satisfies the linear recurrence relation with initial conditions given by

$$g_n - 3g_{n-1} + 3g_{n-2} - 2g_{n-3} = \begin{cases} 1, & n=2 \\ -3, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$$g_0, g_1 = 0$$
$$g_2 = 1$$
$$g_n = 3g_{n-1} - 3g_{n-2} + 2g_{n-3},\ n \geq 3$$

c) $G(x) = \dfrac{1}{1-x^2-2x^3-3x^4-2x^5-x^6}$

By theorem 4.8, g_n satisfies the linear recurrence relation with initial conditions given by

$$g_n - g_{n-2} - 2g_{n-3} - 3g_{n-4} - 2g_{n-5} - g_{n-6} = \begin{cases} 1, & n=0 \\ 0, & n\geq 0 \end{cases}$$

Hence we have

$g_0 = 1$ $\quad g_3 = 2$

$g_1 = 0$ $\quad g_4 = 4$

$g_2 = 1$ $\quad g_5 = 6$

$$g_n = g_{n-2} + 2g_{n-3} + 3g_{n-4} + 2g_{n-5} - g_{n-6}, \quad n \geq 6$$

d) $$G(x) = \frac{x + x^2 + x^3}{1 - x^2 - 2x^3 - 3x^4 - 2x^5 - x^6}$$

By theorem 4.8, g_n satisfies the linear recurrence relation with initial conditions given by

$$g_n - g_{n-2} - 2g_{n-3} - 3g_{n-4} - 2g_{n-5} - g_{n-6} = \begin{cases} 1, & 1 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$g_0 = 0$ $\quad g_3 = 2$

$g_1 = 1$ $\quad g_4 = 3$

$g_2 = 1$ $\quad g_5 = 7$

$$g_n = g_{n-2} + 2g_{n-3} + 3g_{n-4} + 2g_{n-5} - g_{n-6}, \quad n \geq 6$$

e) $$G(x) = \frac{1 + x - 2x^2 - x^3 + x^4}{1 - 2x^2 - x^3 + x^4}$$

By theorem 4.8, g_n satisfies the linear recurrence relation with initial conditions given by

$$g_n - 2g_{n-2} - g_{n-3} + g_{n-4} = \begin{cases} 1, & 0 \leq n \leq 1 \\ -2, & n=2 \\ -1, & n=3 \\ 1, & n=4 \\ 0, & n \geq 5 \end{cases}$$

Hence we have

$g_0, g_1 = 1$ $\quad g_3 = 2$

$g_2 = 0$

$$g_n = 2g_{n-2} + g_{n-3} - g_{n-4}, \quad n \geq 4$$

4.2 a) The first part must be odd; its allowed parts are P1 = {1, 3, 5…}. The rest of the parts have no restrictions - the allowed parts are P2 = {1, 2, 3,…}. Their generating series are

$$\Phi_{P_1}(x) = \sum_{i=0}^{\infty} x^{2i+1} = \frac{x}{1-x^2}$$

$$\Phi_{P_2}(x) = \frac{x}{1-x}$$

We have that there is always one part, where the first is odd, and the rest of the parts can be anything, so by the product and string lemmas we have

$$K(x) = \frac{x}{1-x^2}\left(\frac{1}{1-\left(\frac{x}{1-x}\right)}\right) = \frac{x}{(1+x)(1-2x)}$$

b) $$K(x) = \frac{x}{(1+x)(1-2x)} = \frac{A}{1+x} + \frac{B}{1-2x}$$

$$x = A(1-2x) + B(1+x)$$

$$K(x) = \frac{1}{3(1-2x)} - \frac{1}{3(1+x)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3} 2^n x^n - \sum_{n=0}^{\infty} \frac{1}{3}(-1)^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{3}(2^n - (-1)^n) x^n = \sum_{n=0}^{\infty} \left(\frac{2^n}{3} + \frac{(-1)^{n-1}}{3}\right) x^n$$

Using the coefficient [x^n] of our summation, we have that the probability is

$$\frac{\frac{2^n}{3} + \frac{(-1)^{n-1}}{3}}{2^{n-1}} = \frac{2}{3} + \frac{1}{3}\left(\frac{-1}{2}\right)^{n-1}$$

4.3 a) By theorem 4.8, c_n satisfies the linear recurrence relation with initial conditions given by

$$c_n - 5c_{n-1} + 8c_{n-2} - 4c_{n-3} = \begin{cases} 1, & n=0 \\ -2, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$c_0 = 0 \quad c_2 = 15$
$c_1 = 5$

$c_n = 5c_{n-1} - 8c_{n-2} + 4c_{n-3}, \; n \geq 3$

b) Using partial fractions, we obtain

$$\frac{1-2x^2}{(1-x)(-1+2x)^2} = \frac{A}{1-x} + \frac{B}{-1+2x} + \frac{C}{(-1+2x)^2}$$

$$1-2x^2 = A(-1+2x)^2 + B(1-x)(-1+2x) + C(1-x)$$

$$\frac{1-2x^2}{(1-x)(-1+2x)^2} = \frac{-1}{1-x} - \frac{1}{-1+2x} + \frac{1}{(-1+2x)^2}$$

$$= -\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} 2^n x^n (n+1)$$

$$= \sum_{n=0}^{\infty} \left(2^n (n+2) - 1\right) x^n$$

$$c_n = 2^n (n+2) - 1$$

4.4 a) By theorem 4.8, a_n satisfies the linear recurrence relation with initial conditions given by

$$a_n - 3a_{n-2} - 2a_{n-3} = \begin{cases} 1, & n=1 \\ 7, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$a_0 = 0 \quad a_2 = 7$
$a_1 = 1$

$a_n = 3a_{n-2} - 2a_{n-3}, \; n \geq 3$

b) Using partial fractions, we obtain

$$\frac{x+7x^2}{(1-2x)(1+x)^2} = \frac{A}{1-2x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

$$x+7x^2 = A(1+x)^2 + B(1-2x)(1+x) + C(1-2x)$$

$$\frac{x+7x^2}{(1-2x)(1+x)^2} = \frac{1}{1-2x} - \frac{3}{1+x} + \frac{2}{(1+x)^2}$$

$$= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} 3(-1)^n x^n + \sum_{n=0}^{\infty} 2(n+1)(-1)^n x^n$$

$$= \sum_{n=0}^{\infty} \left(2^n + (-1)^n(2n+5)\right) x^n$$

$$a_n = 2^n + (-1)^n(2n+5)$$

4.5 a) By theorem 4.8, c_n satisfies the linear recurrence relation with initial conditions given by

$$c_n - 4c_{n-1} + 5c_{n-2} - 2c_{n-3} = \begin{cases} 3, & n=0 \\ -11, & n=1 \\ 11, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$$c_0 = 3 \quad c_2 = 0 \quad c_4 = 6$$
$$c_1 = 1 \quad c_3 = 1 \quad c_5 = 19$$

$$c_n = 4c_{n-1} - 5c_{n-2} + 2c_{n-3}, \quad n \geq 6$$

b) We have that

$$1 - 4x + 5x^2 - 2x^3 = (1-2x)(1-x)^2$$

Theorem 4.14 implies that there are constants A, B, and C such that for n > 0,

$$c_n = A \cdot 2^n + (B + Cn)$$

Substituting c1 = 1, c2 = 0, and c3 = 1, we solve the 3 equations to obtain
A = 1, B = 2, C = -3. Hence we have

$$c_n = 2^n + 2 - 3n$$

4.6 a) $G(x) = \dfrac{1+x}{1-2x+x^2-2x^3}$

b) We have that

$$1 - 2x + x^2 - 2x^3 = (1-2x)(1+x^2) = (1-2x)(1-ix)(1+ix)$$

Theorem 4.14 implies that there are constants A, B, and C such that for n > 0,

$$g_n = A\cdot 2^n + B\cdot i^n + C(-i)^n$$

Substituting g1 = 2, g2 = 3, and g3 = 6, we solve the 3 equations to obtain
A = 4/5, B = 1/10 - i/5, C = 1/10 + i/5. Hence we have

$$g_n = \frac{4}{5}\cdot 2^n + \left(\frac{1}{10} - \frac{i}{5}\right)i^n + \left(\frac{1}{10} + \frac{i}{5}\right)(-i)^n$$

4.7 a) $C(x) = \dfrac{1+3x+3x^2}{1+x-2x^2-2x^3}$

b) $1+x-2x^2-2x^3 = (1+x)(1-\sqrt{2}x)(1+\sqrt{2}x)$

Theorem 4.14 implies that there are constants A, B, and C such that for n > 0,

$$c_n = A(-1)^n + B\sqrt{2}^n + C(-\sqrt{2})^n$$

Substituting c1 = 2, c2 = 3, and c3 = 3, we solve the 3 equations to obtain
A = -1, B = 1 + 1/(2sqrt(2)), C = 1 - 1/(2sqrt(2)). Hence we have

$$c_n = (-1)^{n+1} + \left(1+\frac{1}{2\sqrt{2}}\right)\sqrt{2}^n + \left(1-\frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^n$$

4.8 a) Using partial fractions, we have that

$$\frac{1+3x-x^2}{1-3x^2-2x^3} = \frac{1+3x-x^2}{(1-2x)(1+x)^2} = \frac{A}{1-2x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

$$1+3x-x^2 = A(1+x)^2 + B(1-2x)(1+x) + C(1+x)^2$$

$$\frac{1+3x-x^2}{1-3x^2-2x^3} = \frac{1}{1-2x} + \frac{1}{1+x} - \frac{1}{(1+x)^2}$$

$$= \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} (n+1)(-1)^n x^n$$

$$= \sum_{n=0}^{\infty} (2^n + (-1)^n n) x^n$$

$$b_n = 2^n + (-1)^n n$$

b) By theorem 4.8, b_n satisfies the linear recurrence relation with initial conditions given by

$$b_n - 3b_{n-2} - 2b_{n-3} = \begin{cases} 1, & n=0 \\ 3, & n=1 \\ -1, & n=2 \\ 0, & n \geq 3 \end{cases}$$

Hence we have

$$b_0 = 1 \qquad b_2 = 2$$

$$b_1 = 3$$

$$b_n = 3b_{n-2} + 2b_{n-3}, \quad n \geq 0$$

4.9 a) $H(x) = \dfrac{1+4x+3x^2-x^3}{1+2x-x^2-4x^3-2x^4}$

b) $1+2x-x^2-4x^3-2x^4 = (1+x)^2(1-\sqrt{2}x)(1+\sqrt{2}x)$

Theorem 4.14 implies that there are constants A, B, and C and D such that for n > 0,

$$h_n = (A+Bn)(-1)^n + C\sqrt{2}^n + D(-\sqrt{2})^n$$

Substituting h1 = 2, h2 = 0, and h3 = 5, and h4 = 0, we solve the 4 equations to obtain
A = 0, B = -1, C = 1/2 + 1/(2sqrt(2), D = 1/2 - 1/(2sqrt(2). Hence we have

$$h_n = -n(-1)^n + \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)\sqrt{2}^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^n$$

4.10 a) $n^2 = A\binom{n+2}{2} + B(n+1) + C$

$$= \frac{1}{2}An^2 + \frac{3}{2}An + A + Bn + B + C$$

$$1 = \frac{1}{2}A$$

$$0 = \frac{3}{2}A + B$$

$$0 = B + C$$

$A=2,\ B=-3,\ C=1$

b) $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$

Differentiating and multiplying both sides by x

$\frac{x}{(1-x)^2}=\sum_{n=0}^{\infty}nx^n$

Differentiating and multiplying both sides by x

$\frac{x+x^2}{(1-x)^3}=\sum_{n=0}^{\infty}n^2x^n$

Differentiating and multiplying both sides by x

c) $\frac{x+4x^2+x^3}{(1-x)^4}=\sum_{n=0}^{\infty}n^3x^n$

d) From part a, we have that F_0(x) = 1/(1-x).

$$F_d(x)=\sum_{n=0}^{\infty}n^dx^n=\sum_{n=1}^{\infty}n^dx^n=x\sum_{n=1}^{\infty}n^dx^{n-1}$$

$$=x\frac{d}{dx}\sum_{n=1}^{\infty}n^{d-1}x^n=x\frac{d}{dx}\sum_{n=0}^{\infty}n^{d-1}x^n$$

$$=x\frac{d}{dx}F_{d-1}(x)$$

e) $$F_d(x)=x\frac{d}{dx}\frac{P_{d-1}(x)}{(1-x)^d}=x\frac{(1-x)^d\frac{d}{dx}P_{d-1}(x)-P_{d-1}(x)(-1)(d)(1-x)^{d-1}}{(1-x)^{2d}}$$

$$=\frac{(1-x)\frac{d}{dx}P_{d-1}(x)+P_{d-1}(x)d}{(1-x)^{1+d}}$$

$$P_d(x)=(1-x)\frac{d}{dx}P_{d-1}(x)+P_{d-1}(x)\,d$$

4.11 ???

4.12 a) $\frac{1}{\sqrt{1-4x}}=\sum_{k=0}^{\infty}(-1)^k4^k\binom{-1/2}{k}x^k$

For k=0, the coefficient of x^0 is 1. For k >= 1,

$$(-1)^k 4^k \binom{-1/2}{k} = (-4)^k \frac{1}{k!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(\frac{1}{2}-k\right)$$

$$= 2^k \frac{1}{k!}(1)(3)(5)\cdots(2k-1)$$

$$= \frac{(1)(3)(5)\cdots(2k-1)}{k!} \cdot \frac{(2)(4)\cdots(2k)}{k!}$$

$$= \binom{2k}{k}$$

b) $$\sum_{j=0}^{n} \binom{2j}{j}\binom{2n-2j}{n-j} = \sum_{j=0}^{n} \frac{(2j)!}{j!\,j!} \cdot \frac{(2n-2j)!}{(n-j)!\,(n-j)!}$$