MATH 239 S2009 Final

1. a) $$[x^n]\frac{(1+2x)(1+x^2)^m}{(1-x)^3} = [x^n](1+2x)\left[\sum_{k=0}^{m}\binom{m}{k}x^{2k}\right]\left[\sum_{k=0}^{\infty}\binom{k+2}{2}x^k\right]$$

$$= [x^n](1+2x)\sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{m}{j/2}\binom{k-j+2}{2}x^k$$

$$= \sum_{j=0}^{n}\binom{m}{j/2}\binom{n-j+2}{2} + 2\sum_{j=0}^{n-1}\binom{m}{j/2}\binom{n-j+1}{2}$$

b) $$[x^n]\frac{1}{(1-x)(1-x^m)} = [x^n]\left[\sum_{k=0}^{\infty}x^k\right]\sum_{k=0}^{\infty}x^{mk}$$

$$= \left\lfloor \frac{n}{m} \right\rfloor + 1$$

2. $$[x^{5000}] = \left[\sum_{k=0}^{50}x^{20k}\right]\left[\sum_{k=0}^{40}x^{50k}\right]\sum_{k=0}^{70}x^{100k}$$

3. a) The set of all strings where every even block of 1s is followed by an even block of 0s, except for the suffix block of 1s.

b) This is the block decomposition for this set of strings.

c) $$\Phi(x) = \frac{1}{1-x}\left(\frac{1}{1-\left(\frac{x^2}{(1-x)^2}-\frac{x^3}{(1-x^2)^2}\right)}\right)\frac{1}{1-x} = \frac{1+2x+x^2}{1-3x^2-3x^3}$$

d) S' can be described by the recursive expression
T = 0 T 1 ⌣ 0*01
S' = 1* T*

$$\Phi_T(x) = x^2\Phi_T(x) + \frac{x^2}{1-x}$$

$$\Phi_T(x) = x^2\Phi_T(x) + \frac{x^2}{1-x}$$

$$\Phi_T(x) = \frac{x^2}{1-x-x^2+x^3}$$

$$\Phi_S(x) = \frac{1}{1-x}\left(\frac{1}{1-\Phi_T(x)}\right) = \frac{1+x}{1-x-2x^2+x^3}$$

e) $\Phi_S(x) = \sum_{n=0}^{\infty} s_n x^n$

By theorem 4.8, s_n satisfies the linear recurrence relation with initial conditions given by

$$s_n - s_{n-1} - 2s_{n-2} + s_{n-3} = \begin{cases} 1, & n=0 \\ 1, & n=1 \\ 0, & \text{otherwise} \end{cases}$$

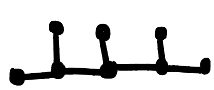
Hence we have

$$s_0 = 1$$
$$s_1 = 2$$
$$s_2 = 4$$
$$s_n = s_{n-1} + 2s_{n-2} - s_{n-3},\ n \geq 3$$

4. We prove that if G is a tree, the total number of vertices is p = 2n_3 + 2. If G has 1 or 2 vertices, it must have 2 vertices of degree 1. Otherwise, every vertex of degree 1 must be adjacent to a vertex of degree 3.

We prove this as follows: suppose there are more than 3 vertices and two vertices of degree 1 are adjacent to each other. Then these two vertices must not be connected to any other vertices, and hence G would not be connected, which is a contradiction.

Hence every vertex of degree 1 is connected to a vertex of degree 3. Then each vertex of degree 3 needs 2 more adjacent vertices. If G has 4 vertices, the other two vertices are degree 1. Otherwise, the other two vertices must be of degree 3, except 2 end-vertices, which are at the end and hence have 2 vertices of degree 1 adjacent. Hence G has p = 2n_3 + 2 vertices.



If G has p = 2n_3 + 2 vertices, then every vertex of degree 3 must be connected to two other vertices of degree 3 in a line, except for the two ends. The other neighbours must be of degree 1. This is a tree.

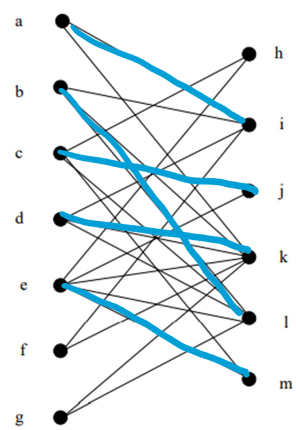
5. a) 1, 5, 7, 13, 4, 9, 10, 3, 6, 2, 12, 11, 8

b) 3, 4, 6, 9, 10

c) G is not bipartite. There is an odd cycle 1, 5, 4, 2, 8, 3, 7.

d) We can colour the vertices in each level of T all the same colour, and alternate the colouring of levels. The exception are the two vertices that are connected by the one same-level edge - we colour these two vertices different colours. Since no tree edges join vertices in the same level, and there is only one non-tree edge that joins vertices of the same level (which we addressed), this means each vertex has no adjacent neighbours of the same colour. Hence H is 3-colourable.

6. a) a, i, b, l, e, m



b) ai, bl, em, cj, dk

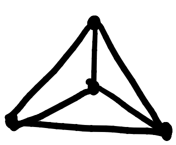
c) X0 = {f, g}
X = {}
Y = {}

d) C = {i, k, l, c, e}

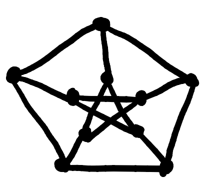
7. a) p - q + f = 2

b) G has vertices all of the same degree and faces all of the same degree. Hence it is a platonic solid. The platonic solid with vertices of degree 5 is the icosahedron, which has 20 faces.

c) i.



ii.



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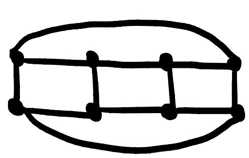
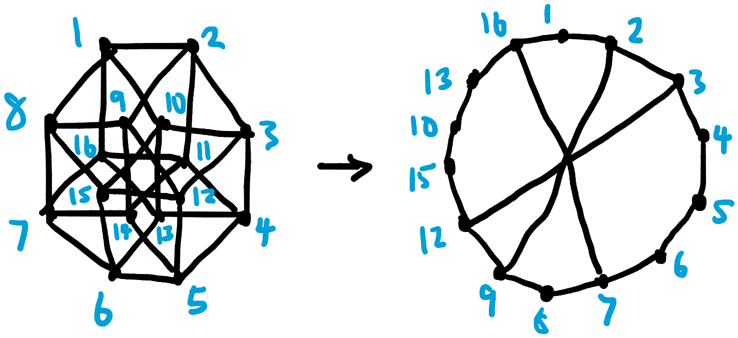


8. a) Not planar - it is an edge subdivision of K5.

b) Not planar - this subgraph is an edge subdivision of K3,3:



9. a) If N(x) is not a subset of V(P), then there exists some neighbour of x that is not in P. Then we can extend P to make a longer path by adding that neighbour to the end of the path, which is a contradiction. Hence N(x) is a subset of V(P). The same argument goes for y.

b) Let P be the longest path in G. Suppose P does not have length k. Let x be an endpoint of P. N(x) must be a subset of P. But P has less than k vertices, and x is adjacent to at least k other vertices. This is a contradiction. Hence P must have length at least k.