

MATH 239 Exercises 2

2.1 a) $[x^8] \sum_{n=0}^{\infty} \binom{n+7-1}{7-1} x^n = 3003$

b) $[x^{10}] x^6 \sum_{n=0}^{\infty} \binom{n+5-1}{5-1} (2x)^n = 1120$

c) $[x^5] (x^3 + 5x^4) \sum_{k=0}^{\infty} \binom{6}{k} (3x)^k = \binom{6}{5}(3^5) + \binom{6}{4}(3^4) = 2673$

d) $[x^9] (-4x)^5 + \sum_{n=0}^{\infty} \binom{n+2-1}{2-1} (3x)^n = 196830$

e) $[x^n] \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} (2+3)^j = \binom{n+k-1}{k-1} (2+3)^n$

f) $[x^{n+1}] x^k \sum_{j=0}^{\infty} \binom{j+1k-1}{2k-1} (4x)^j = \binom{n+k}{2k-1} 4^{n+1-k}$

g) $[x^n] x^k \sum_{j=0}^{\infty} \binom{j+m-1}{m-1} (x^2)^j = \binom{\frac{n-k}{2} + m - 1}{m-1}$

h) $[x^n] \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} (x^2)^j + \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} (7x^2)^j = \binom{\frac{n}{2} + k - 1}{k-1} + \binom{\frac{n}{2} + k - 1}{k-1} (7^{\frac{n}{2}})$

2.2 a) $\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} (-1)^n x^n = \sum_{n=0}^{\infty} (n+1)(-1)^n x^n$

b) $\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$

c) $\frac{1}{(1+x^3)} = \sum_{n=0}^{\infty} \binom{n}{0} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n x^{3n}$

d) $\frac{1}{(1-2x)} = \sum_{n=0}^{\infty} \binom{n}{0} (2x)^n = \sum_{n=0}^{\infty} (2x)^n$

e) $\frac{1}{(1+2x^2)} = \sum_{n=0}^{\infty} \binom{n+1}{1} (-2x^2)^n = \sum_{n=0}^{\infty} (n+1)(-2x^2)^n$

f) $\frac{1}{(1-2x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} (2x)^n$

2.3 a) $\sum_{k=0}^n \binom{n}{k} k = \sum_{k=0}^n \frac{n! \cdot k}{k!(n-k)!} = \sum_{k=0}^n \frac{n(n-1)!}{(k-1)!(n-1-(k-1))!}$

$$= \sum_{k=0}^n n \binom{n-1}{k-1} = n \binom{n-1}{0-1} + \sum_{k=1}^n n \binom{n-1}{k-1}$$

$$= 0 + n \sum_{k=0}^{n-1} \binom{n-1}{k} = n 2^{n-1}$$

$$\text{b) } \sum_{n=0}^n \binom{n}{k} k(k-1) = \sum_{k=0}^n \frac{n! k(k-1)}{k!(n-k)!} = \sum_{k=0}^n \frac{n(n-1)(n-2)!}{(k-2)!(n-2-(k-2))!}$$

$$= \sum_{k=0}^n n(n-1) \binom{n-2}{k-2} = n(n-1) \left[\cancel{\binom{n-2}{0-2}} + \cancel{\binom{n-1}{0-1}} + \sum_{k=2}^n \binom{n-2}{k-2} \right]$$

$$= n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} = n(n-1) 2^{n-2}$$

2.4 Let $\overline{\Phi}_A(x) = (1+x)^{-2} = \sum_{k=0}^{\infty} \binom{k+1}{1} (-x)^k = \sum_{k=0}^{\infty} (k+1)(-x)^k$

$$\overline{\Phi}_B(x) = (1-2x)^{-2} = \sum_{k=0}^{\infty} \binom{k+1}{1} (2x)^k = \sum_{k=0}^{\infty} (k+1)(2x)^k$$

By the product lemma, the weight function of the product of the two generating series is the sum of their individual ones. To find the coefficient of x^n , we find the pairs in the cross product of the terms of A and B where the exponents of x sum to n. Let w be the weight function of the product of the two generating series. Hence we find the pairs (i, j) where

$$a_i = (i+1)(-1)^i \quad w(a_i, p_j) = n = i+j$$

$$p_j = (j+1) 2^j$$

$$[x^n] (1+x)^{-2} (1-2x)^{-2} = \sum_{i=0}^n (i+1)(-1)^i (n-i+1) 2^{n-i}$$

$$= \frac{1}{27} (2^{n+2} (5+3n) + (-1)^n (7+3n))$$

2.5 a) Methodology is the same as 2.4 above.

$$[x^n] \frac{(1+x)^a}{(1-x^2)^a} = [x^n] \left[\sum_{k=0}^a \binom{a}{k} x^k \right] \left[\sum_{n=0}^{\infty} \binom{n+a-1}{a-1} x^{2n} \right] \rightarrow n = i+2j$$

$$= [x^n] \sum_{k=0}^{\lfloor \frac{n}{a} \rfloor} \binom{a}{n-2k} \binom{k+a-1}{a-1}$$

$$[x^n] \frac{1}{(1-x)^a} = [x^n] \sum_{k=0}^{\lfloor \frac{n}{a} \rfloor} \binom{k+a-1}{a-1}$$

$$= \binom{n+a-1}{a-1}$$

b) ?????

$$2.6 \quad \underline{\Phi}_B(x) = \sum_{r \in B} x^{w(r)} = \sum_{o \in A_1} x^{w(o)} + \sum_{o \in A_2} x^{w(o)} + \dots$$

$$= \sum_{j=0}^{\infty} \underline{\Phi}_{A_j}(x)$$

2.7 We can use induction. Base case: 2 sets (proved in Product Lemma 2.12). Suppose for k sets, we have

$$\underline{\Phi}_{A_1 \times A_2 \times \dots \times A_k}(x) = \prod_{j=1}^k \underline{\Phi}_{A_j}(x)$$

We prove that for k+1 sets, we have

$$\underline{\Phi}_{A_1 \times \dots \times A_{k+1}}(x) = \prod_{j=1}^{k+1} \underline{\Phi}_{A_j}(x)$$

We have already shown that for two sets, the generating series of $A \times B$ is the product of the two generating series. Hence we have that

$$\underline{\Phi}_{(A_1 \times A_k) \times A_{k+1}}(x) = \left[\prod_{j=1}^k \underline{\Phi}_{A_j}(x) \right] \cdot \underline{\Phi}_{A_{k+1}}(x) = \prod_{j=1}^{k+1} \underline{\Phi}_j(x)$$

Hence we have proven the statement.

2.8 ???

2.9 a) { aaaa, aaab, aaba, abaa, baaa, aabb, abab, abba, baba, baab, bbaa, abbb, babb, bbab, bbbba, bbbb }

Weight 0: 5

Weight 1: 10

Weight 2: 1

$$\Phi(x) = 5 + 10x + x^2$$

b) ???

2.10 a)

2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	5	4	3	2	1

Generating series = $x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$

$$= \sum_{i=1}^6 i \cdot x^{i+1} + \sum_{i=1}^5 (6-i) x^{i+7}$$

$$b) \Phi(x) = \sum_{i=1}^8 \frac{i(i+1)}{2} x^{i+2} + \sum_{i=1}^8 \frac{(9-i)(10-i)}{2} x^{i+10}$$

2.11

0	1	2	3	4	5
6	10	8	6	4	2

$$\Phi(x) = 6 + \sum_{i=1}^5 (10-2i) x^i$$

2.12 a) This is a weight function. For every natural number n, there are only a finite number of elements in S that are weight n, since there are only a finite number of pairs (a,b) where a=n, since $0 \leq |b| \leq a$.

$$\begin{aligned}
 \Phi_s(x) &= \sum_{(a,b) \in S} x^{w((a,b))} = \sum_{(a,b) \in S} x^a = \sum_{a=0}^{\infty} x^a \left[\sum_{b=-a}^a 1 \right] = \sum_{a=0}^{\infty} (2a+1) x^a \\
 &= \sum_{a=0}^{\infty} x^a + \sum_{a=0}^{\infty} 2a x^a = \sum_{a=0}^{\infty} \binom{a}{0} x^a + 2 \sum_{a=0}^{\infty} \binom{a}{1} x^a \\
 &= (1-x)^{-1} + 2(1-x)^{-2} = \frac{3-x}{(1-x)^2}
 \end{aligned}$$

- b) This is not a weight function. For every natural number n , there are an infinite number of pairs (a,b) where $a+b=n$.
- c) This is not a weight function. For every natural number n , there are an infinite number of pairs (a,b) where $2a+b=n$.

2.13 a) The generating function for the outcome of one dice is $(x + x^2 + x^3 + x^4 + x^5 + x^6)$. Hence the generating function for the sum of 4 dice is $(x + x^2 + x^3 + x^4 + x^5 + x^6)^4$.