

Exercises 2

1. a) Let $p_3(x) = ax^3 + bx^2 + cx + d$. Without loss of generality, suppose $x_0 < x_1 < x_2$. Then we can put our equations into a linear system:

$$\begin{aligned} ax_0^3 + bx_0^2 + cx_0 + d &= f(x_0) \\ ax_2^3 + bx_2^2 + cx_2 + d &= f(x_2) \\ \frac{d}{dx}(ax_1^3 + bx_1^2 + cx_1 + d) &= f'(x_1) \rightarrow 3ax_1^2 + 2bx_1 + c = f'(x_1) \\ \frac{d}{dx}\left(\frac{d}{dx}(ax_1^3 + bx_1^2 + cx_1 + d)\right) &= f''(x_1) \rightarrow 6ax_1 + 2b = f''(x_1) \\ &\downarrow \\ \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} &= \begin{bmatrix} f(x_0) \\ f(x_2) \\ f'(x_1) \\ f''(x_1) \end{bmatrix} \end{aligned}$$

Reducing the matrix to partial RREF gives us

$$\left[\begin{array}{cccc} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & 1 & x_2+x_0 & x_2^2+x_0x_2+x_0^2 \\ 0 & 0 & 2x_1-x_2-x_0 & 3x_1^2-(x_2^2+x_0x_2+x_0^2) \\ 0 & 0 & 2 & 6x_1 \end{array} \right]$$

We compute the determinant:

$$\begin{aligned} &6x_1^2 + 2x_2^2 + 2x_0^2 + 2x_0x_2 - 6x_1x_2 - 6x_0x_1 \\ &= 3(x_1 - x_0)^2 + 3(x_2 - x_1)^2 - (x_2 - x_0)^2 \\ &= 3(x_1 - x_0)^2 + 3(x_2 - x_1)^2 - ((x_1 - x_0) + (x_2 - x_1))^2 \\ &= 2(x_1 - x_0)^2 - 2(x_1 - x_0)(x_2 - x_1) + 2(x_2 - x_1)^2 \\ &= ((x_1 - x_0) - (x_2 - x_1))^2 + (x_1 - x_0)^2 + (x_2 - x_1)^2 > 0 \end{aligned}$$

Since the determinant is nonzero, the system must have a unique solution.

b) ???

2. a) The degree of each polynomial function is n-1

b) We have that the sum of the Lagrange basis functions is 1. This is because $g(x)$ represents the unique polynomial of degree at most $n-1$ passing through the points $(x_1, 1), \dots, (x_n, 1)$. This must be the function $g(x) = 1$.

3. We have that

$$S(-3) = -27 \quad S'(-1) \text{ continuous}$$

$$S(-1) = -1 \quad S''(-1) \text{ continuous}$$

$$S(1) = 1 \quad S''(-3) = S''(3) = 0$$

$$S(x) = \begin{cases} ax^3 + bx^2 + cx + d, & -3 \leq x \leq -1 \\ ex^3 + fx^2 + gx + h, & -1 \leq x \leq 1 \\ jx^3 + kx^2 + lx + m, & 1 \leq x \leq 3 \end{cases}$$

$$-27 = -27a + 9b - 3c + d$$

$$-1 = -a + b - c + d$$

$$a = -1$$

$$-1 = -e + f - g + h$$

$$b = -9$$

$$3a - 2b + c = 3e - 2f + g$$

$$c = -10$$

$$-6a + 2b = -6e + 2f$$

$$d = -3$$

$$1 = e + f + g + h$$

$$e = 2$$

$$1 = j + k + l + m$$

$$f = 0$$

$$27 = 27j + 9k + 3l + m$$

$$g = -1$$

$$3e + 2f + g = 3j + 2k + l$$

$$h = 0$$

$$6e + 2f = 6j + 2k$$

$$j = -1$$

$$-18a + 2b = 0$$

$$k = 9$$

$$18j + 2k = 0$$

$$l = -10$$

$$m = 3$$

$$S(0) = h = 0$$

<https://www.wolframalpha.com/input/?i=-27%3D-27a%2B9b-3c%2Bd%2C+-1%3D-a%2Bb-c%2Bd%2C+-1%3D-e%2Bf-g%2Bh%2C+3a-2b%2Bc%3D3e-2f%2Bg%2C+-6a%2B2b%3D-6e%2B2f%2C+1%3De%2Bf%2Bg%2Bh%2C+1%3Dj%2Bk%2Bl%2Bm%2C+27%3D27j%2B9k%2B3l%2Bm%2C+3e%2B2f%2Bg%3D3j%2B2k%2Bl%2C+6e%2B2f%3D6j%2B2k%2C+-18a%2B2b%3D0%2C+18j%2B2k%3D0>

$$4. \text{ a) } a_1 - 25 + 9 - 1 = 26 - a_2 + a_3 + 1$$

$$a_1 + a_2 - a_3 = 44$$

$$26 + 57 + 9a_4 + 27a_5 = -163 + 624 - 540 + 27a_6$$

$$a_4 + 3a_5 - 3a_6 = -18$$

$$25 - 18 + 3 = a_2 - 2a_3 - 3$$

$$a_2 - 2a_3 = 13$$

$$a_2 = 19$$

$$19 + 6a_4 + 27a_5 = 208 - 360 + 27a_6$$

$$2a_4 + 9a_5 - 9a_6 = -57$$

$$-120 + 24a_6 = 0$$

$$a_6 = 5$$

$$a_1 = 28, a_2 = 19, a_3 = 3, a_4 = 3, a_5 = -2, a_6 = 5$$

$$\text{b) } S(-3) = 7$$

$$S(-1) = 11$$

$$S(0) = 26$$

$$S(3) = 56$$

$$L_1(x) = \frac{(x+1)(x)(x-3)}{(-3+1)(-3)(-3-3)}$$

$$L_2(x) = \frac{(x+3)(x)(x-3)}{(-1+3)(-1)(-1-3)}$$

$$L_3(x) = \frac{(x+3)(x+1)(x-3)}{(3)(1)(-3)}$$

$$L_4(x) = \frac{(x+3)(x+1)(x)}{(3+3)(3+1)(3)}$$

$$p(x) = 7L_1(x) + 11L_2(x) + 26L_3(x) + 56L_4(x)$$

$$= 26 + \frac{397}{24}x + \frac{11}{18}x^2 - \frac{67}{72}x^3$$

5. Let $\Delta x_i = x_{i+1} - x_i$

① $S_i(x_i) = f_i$

$$\begin{aligned} f_i &= a_i(x_{i+1} - x_i) + b_i(x_i - x_i)(x_i - x_{i+1}) + c_i(x_i - x_i) \\ &= a_i(x_{i+1} - x_i) \end{aligned}$$

② $S_i(x_{i+1}) = f_{i+1}$

$$\begin{aligned} f_{i+1} &= a_i(x_{i+1} - x_i) + b_i(x_{i+1} - x_i)(x_{i+1} - x_{i+1}) + c_i(x_{i+1} - x_i) \\ &= c_i(x_{i+1} - x_i) \end{aligned}$$

$$\begin{aligned} S'_i(x) &= \frac{d}{dx} a_i x_{i+1} - a_i x + b_i(x^2 - (x_{i+1} + x_i)x + x_{i+1}x_i) + c_i x - c_i x_i \\ &= -a_i + b_i(2x - x_{i+1} - x_i) + c_i \end{aligned}$$

$$S'_{i+1}(x) = -a_{i+1} + b_{i+1}(2x - x_{i+2} - x_{i+1}) + c_{i+1}$$

③ $S'_i(x_{i+1}) = S'_{i+1}(x_{i+1})$

$$-a_i + b_i(2x_i - x_{i+1} - x_i) + c_i = -a_{i+1} + b_{i+1}(2x_{i+1} - x_{i+2} - x_{i+1}) + c_{i+1}$$

$$-a_i - b_i \Delta x_i + c_i = -a_{i+1} - b_{i+1} \Delta x_{i+1} + c_{i+1}$$

$$b_{i+1} \Delta x_{i+1} - b_i \Delta x_i = -a_{i+1} + c_{i+1} + a_i - c_i$$

④ $S'(x_0) = y$

$$-a_1 + b_1(2x_1 - x_2 - x_1) + c_1 = y$$

$$-a_1 - b_1 \Delta x_1 + c_1 = y$$

$$b_i \Delta x_i = y + a_i - c_i$$

$$a_i = \frac{f_i}{\Delta x_i}$$

$$c_i = \frac{f_{i+1}}{\Delta x_i}$$

There are $n-1 + n-1 + n-2 + 1 = 3n-3$ equations, and $3n-3$ unknowns. To solve for b , we solve the system

$$\begin{bmatrix} -b_1 & b_2 & 0 & 0 & \cdots & 0 \\ 0 & -b_2 & b_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -b_{n-2} & b_{n-1} & & \\ b_1 & \cdots & 0 & 0 & & \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_{n-1} \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} -a_2 + c_2 + a_1 - c_1 \\ -a_3 + c_3 + a_2 - c_2 \\ \vdots \\ -a_n + c_n + a_{n-1} - c_{n-1} \\ y + a_1 - c_1 \end{bmatrix}$$

$$6. \quad a-b+c-d=0 \quad \left. \right\} \text{continuity of } S(x)$$

$$a+b+c+d=0$$

$$3a-2b+c=1 \quad \left. \right\} \text{continuity of } S'(x)$$

$$3a+2b+c=1$$

$$-6a+2b=0 \quad \left. \right\} \text{continuity of } S''(x)$$

$$6a+2b=0$$

No solutions exist for this set of equations, so there does not exist a choice of coefficients for which $S(x)$ is a natural cubic spline or a clamped spline.