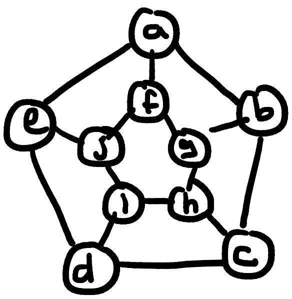
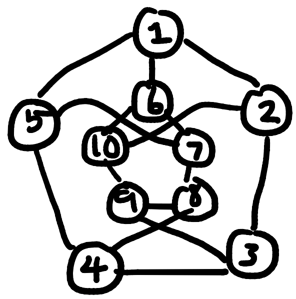
MATH 239 Exercises 4 (Graphs)

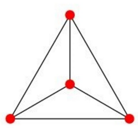
4.4.1 We see that the shortest cycle in G1 is of length 4, while the shortest cycle in G2 is of length 5. Hence they cannot be isomorphic. G1 contains 5 cycles of length 4, while G3 contains only 2. Hence G1 and G3 are not isomorphic, and neither are G2 and G3.

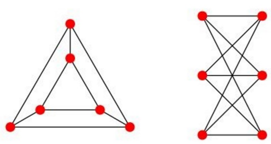
By labeling the nodes in G1 and H as follows, we can create a bijection between them.
f: V(G1) -> V(H)
f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4, f(e) = 5, f(f) = 6, f(g) = 10, f(h) = 9, f(i) = 8, f(j) = 7

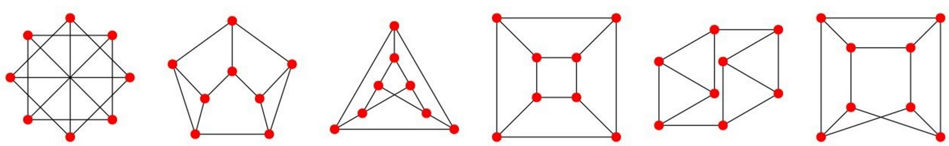




4.4.2



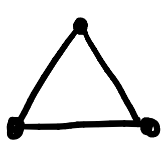




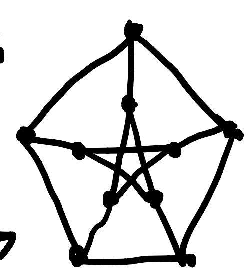
4.4.3 Since V(S_nk) is equal to the set of k-element subsets of {1,2…n}, there are (n choose k) vertices. Two subsets are adjacent if they have exactly (k-1) elements in common - for each subset, there are k other subsets with this property. Hence deg(v)=k for every vertex. Hence we have

$$|E(S_{n,k})| = \frac{1}{2}\sum_{v \in V} \deg(v) = \frac{1}{2}k\binom{n}{k}$$

4.4.4 a) O_1:



O_2:

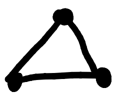


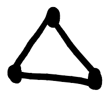
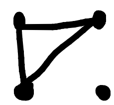
b)

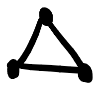
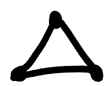
It's literally the same

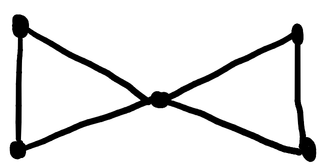
c) There are ((2n+1) choose n) vertices. For each n-subset of the (2n+1)-set, there are (2n+1-n=n+1) possible elements that can be in the subsets that are disjoint to it. Hence there are (n+1 choose n)=n+1 subsets that are disjoint to it - each vertex is of degree n+1. Hence we have

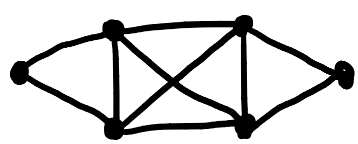
$$|E(G)| = \frac{1}{2}\sum_{v\in V} \deg(v) = \frac{1}{2}(n+1)\binom{2n+1}{n+1}$$

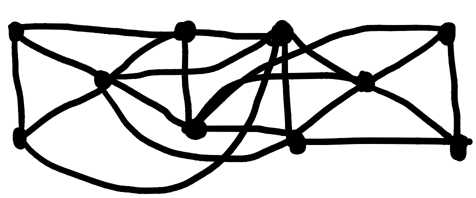
4.4.5 a) 

b) G:  G': 

L(G):  L(G'): 

c) L(G): 

L(L(G)): 

L(L(L(G))): 

4.4.6 a) The expression (0* 11* 0*)$\smile$0* represents this set of strings. The generating series is hence

$$\left(\frac{1}{1-x}\cdot\frac{x}{1-x}\cdot\frac{1}{1-x}\right)+\frac{1}{1-x} = \frac{1-x+x^2}{(1-x)^3}$$

Using partial fractions, we obtain

$$\frac{1-x+x^2}{(1-x)^3} = \frac{1}{1-x} - \frac{1}{(1-x)^2} + \frac{1}{(1-x)^3}$$

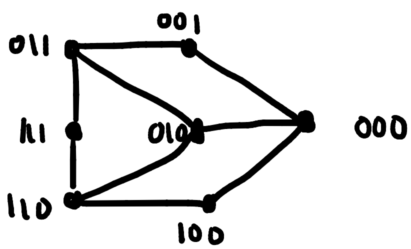
$$= \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+1}{1} x^n + \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$$

$$= \sum_{n=0}^{\infty} \left(1 + \frac{1}{2}n + \frac{1}{2}n^2\right) x^n$$

Hence we have the coefficient of x^n for the generating series, which is equal to the number of vertices.

$$|V(G_n)| = 1 + \frac{1}{2}n + \frac{1}{2}n^2$$

b) G_3:



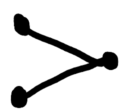
G_4: Too much work sorry :P

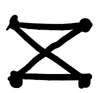
c) For each vertex, we can count its neighbours which have fewer 1s (either delete the first or the last 1). We iterate through the sizes of the block of ones.

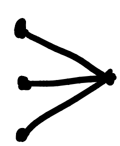
$$|E(G_n)| = n + \sum_{k=2}^{n} 2(n-k+1) = n + n^2 - n$$

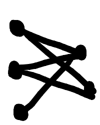
$$= n^2$$

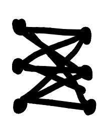
4.4.7. a) $K_{1,1}$: 

$K_{2,1}$: 

$K_{2,2}$: 

$K_{3,1}$: 

$K_{3,2}$: 

$K_{3,3}$: 

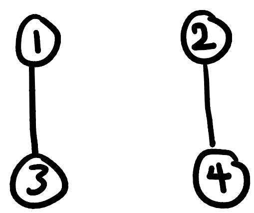
b) K_m,n has m_n vertices. We have that each of the m vertices are adjacent to each of the n vertices. Hence there are mn edges.

c) Let X and Y be the partition of K. Let x be the number of vertices in x. Hence the graph must have x(p-x) edges. Taking the derivative with respect to x, we get p-2x, which equals 0 when x=p/2. The second derivative with respect to x is -2. Hence x=p/2 is the value of x that maximizes the number of edges. Hence the maximum number of edges is (p/2)(p/2) = p^2/4.

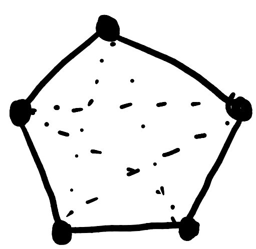
d) For any bipartite graph on p vertices, its maximum number of edges must be the complete bipartite graph with the maximal number of edges, which we have found the maximum edges for in part c above.

e) If k > 0, since G is a k-regular graph, every vertex must have k edges. Every edge in G must have one end in X and one in Y. Hence we have k|X| = k|Y|, and hence |X|=|Y|. This is not valid with k=0, since that is the graph with no edges that is trivially bipartite, and we have that |X|=|Y| for every combination of X and Y.

4.4.8 a)



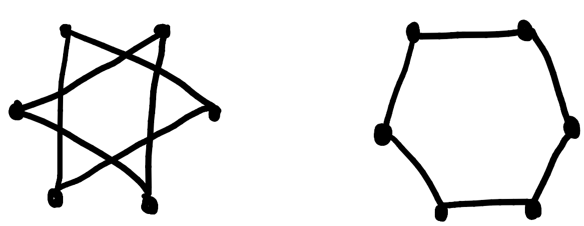
b)



c) For a graph to be isomorphic to its complement, it must have the same number of edges as its complement. Hence it must have half the number of edges as the complete graph with the same number of vertices. A n-vertex complete graph has n(n-1)/2 edges - hence a 6-vertex complete graph has 15 edges. This is not divisible by 2, so there are no 6-vertex graphs isomorphic to its complement.

d) Since G1 is isomorphic to G2, there exists a bijection f: V(G1)->V(G2), such that f(u) and f(v) are adjacent in G2 if and only if u and v are adjacent in G1. Equivalently, f(u) and f(v) are not adjacent in G2 if and only if u and v are not adjacent in G1. Since V(G1) and V(~G1) are the same and V(G2) and V(~G2) are the same, f is a bijection from V(~G1) -> V(~G2). Hence G1 is isomorphic to G2 if and only if ~G1 and ~G2 are isomorphic.

e)



Let a,b,c,d,e,f be the vertices in a 2-regular graph with 6 vertices. Let a be adjacent to b and c.

If b and c are adjacent, then a, b, and c form a triangle, and they must not be connected to any other vertices. Hence d, e, and f must form another triangle. Hence there is only one isomorphism where a is adjacent to b and c, and b and c are adjacent to each other.

If b and c are not adjacent, then they must each be adjacent to another vertex. If they are adjacent to the same vertex, we have a square, and two remaining vertices. This is not a 2-regular graph, since we have two remaining vertices, which cannot possibly have two vertices each. Hence b and c must be adjacent to different vertices - we arbitrarily pick d and e. d and e must both be adjacent to f, since they cannot be adjacent to a, b, or c (or each other, otherwise we have a pentagon and one lone vertex).

Hence these are the only two isomorphisms.

f) The complement of a 2-regular graph on 6 vertices is a 3-regular graph on 6 vertices. Since there are only two non-isomorphic 2-regular graphs on 6 vertices, there are only two non-isomorphic 3-regular graphs on 6 vertices.

4.4.9. Nope sorry :P

4.4.10. Also nope :P

4.4.11. We can partition the graph into vertices with even numbers and vertices with odd numbers. Two vertices with both even numbers cannot have an edge between them, since there are no even numbers greater than 2 that are prime. The same logic applies for vertices between odd numbers. Hence every edge that exists must be between a vertex with an odd number and one with an even number, and hence B_n is bipartite.

4.4.12. a) The two graphs are isomorphic (rotate one). We can create the bijection
1->G, 2->H, 3->A, 4->B, 5->C, 6->D, 7->E, 8->F

b) The two graphs are isomorphic. We can create the bijection
1->G, 2->C, 3->B, 4->D, 5->E, 6->F, 7->A

4.5.1 a)

	1	2	3	4	5	6	7
1	0	1	0	0	1	1	1
2	1	0	1	0	0	0	1
3	0	1	0	1	0	0	1
4	0	0	1	0	1	1	0
5	1	0	0	0	0	1	1
6	1	0	0	1	1	0	1
7	1	1	1	0	0	1	0

Adjacency

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1								
2				1			1					1
3										1	1	1
4								1	1	1		
5	1				1			1				
6		1			1	1			1			
7			1			1	1				1	

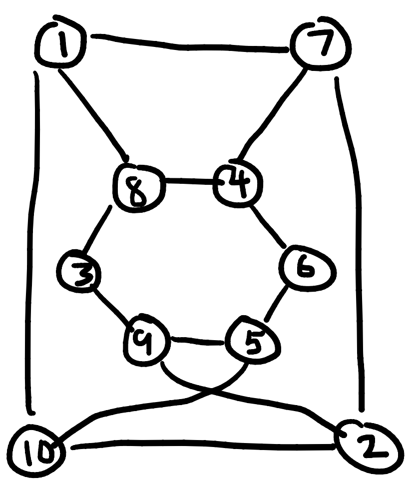
Incidence

b)

i. The diagonal elements of the matrix A^2 represent the number of walks of length 2 from a vertex to itself, which is the degree of that vertex.

ii. The off-diagonal elements of the matrix A^2 represent the number of paths of length 2 from a vertex i to a vertex j.

4.5.2 a)



X: 1-5

Y: 6-10

b) A bipartite graph has every edge with an endpoint in one partition, let's say X, and the other endpoint in the other, let's say Y. To create the adjacency matrix in this form, we order the vertices, with vertices in X first, then the vertices in Y. There are no edges between vertices of X, which creates zeros in the upper left hand corner. There are no edges between the vertices of Y, which creates zeros in the upper right hand corner. Hence the only nonzero entries correspond to edges between X and Y, represented by M, and its transpose in the lower left hand corner (since an edge between x->y is an edge y->x).

4.6.1 a) Let P=(v1,v2,...vl) be the path of maximum length in G. v1 cannot be adjacent to any vertices that are not in P. We prove this via contradiction: suppose there is a vertex u adjacent to v1 not in P. Then the walk P'=(u, v1, v2...vl) is a path in G longer than P, which contradicts our statement that P is the longest path in G.

Hence v1 can only be adjacent to the other vertices in P, so deg(v1) <= l-1. Since G has minimum degree k, we have deg(v1) >= k. Hence we have k <= l-1 = length(P), so P has length at least k.

b) Let P=(v1,v2,...vl) be the path of maximum length in G. We have that l >= k+1 from part a above. As from above, vertex v1 is only adjacent to the other vertices in P. v1 is adjacent to v2, as par the path, and also k-1 vertices in {v3,...vl}, since its degree is at least k. Hence there is an index k <= i <= l such that vi is adjacent to v1. Hence the walk (v1,...vi, v1) is a cycle of length at least k+1 in G.

4.6.2 a) We prove this using induction. Our base case, k=1, is true, since a walk of length 1 determines whether two vertices are adjacent. If they are adjacent, there is 1 walk from i to j; otherwise 0.

We find the (i, j)th entry of A^(k+1), given that the (i, j)th entry of A^k is the number of walks of length k from i to j. We have that A^(k+1) = A^k * A. Hence the (i, j)th entry of A^(k+1) is

$$\sum_{k=1}^{n} a_{ik} b_{kj}$$

Where a_ik is the (i, k)th entry of A^k, and b_kj is the (k, j)th entry of A. b_kj is 1 if there is an edge from k to j, and 0 otherwise. Hence the product = a_ik (which is the number of walks of length k from i to k) if and only if there is an edge between k and j, which is equivalent to having a walk of length k+1 between i and j.

b) We have that the diagonals of A^2+A each has a value of k, and all the off-diagonals have a value of 1. The diagonals of A^2+A have the value of A^2, since the diagonals of A are 0. Hence this means that every vertex of G has degree k.

We also have that every pair of vertices is either adjacent to each other, or has a path of length 2 between them. Hence G is a k-regular graph where every pair of vertices is either adjacent, or has a path of length 2 between them.

4.6.3 a) We have that the (i, j)th entry of A^2 is

$$\sum_{k=1}^{n} a_{ik} a_{kj}$$

where a_ik and a_kj are whether vertices i and k, and whether vertices k and j are adjacent, respectively. Hence this product is equal to 1 if and only if there i and k are adjacent and k and j are adjacent. The sum of all k iterates through every vertex. For i != j, this is thus the number of walks between vertices i and j with length 2, which is equivalent to the number of paths of length 2 between i and j, since there are no self-loops or multiple edges.

b) Looking at the equation in part a above, we can see that if i = j, this product is 1 if and only if i and k are adjacent. Hence the sum counts the number of vertices adjacent to i, which is the degree of vertex i.

4.6.4. Washington -> Oregon -> Nevada -> Arizona -> New Mexico -> Oklahoma -> Missouri -> Nebraska -> Wyoming -> Montana -> North Dakota -> Minnesota -> Wisconsin -> Michigan -> Ohio -> West Virginia -> Washington DC

4.6.5 a) math -> bath -> both -> moth -> math

b) pink -> pint -> pant -> part -> port -> pout -> gout -> glut -> glue -> blue

4.6.6 We prove this by induction. Our base case n=1 is trivial. For some k > 1, assume the k-cube contains a Hamilton cycle. We can see that a k+1-cube is constructed by taking two k-cubes, adding 0 to the end of each of the first, and a 1 to the end of each of the second. Hence we can take the Hamilton cycle for a k-cube and delete the last edge from the cycle, producing a path P. Let it start at vertex a and end at vertex b. Let the reverse of this path be P', starting at b' and ending at a'. We can follow P on one k-cube and P' on the second. Since there is an edge between a and a' and one between b and b', we can start at a, follow P to b through every point in k-cube 1, go to b', follow P' to a' through every point in k-cube 2, then back to a, forming a Hamilton cycle in the k+1-cube.

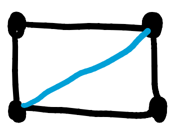
4.6.7 For a complete bipartite graph K_mn, we prove that it has a Hamilton cycle if m=n and m > 1. Let M = m1, m2…mm) and N = (n1, n2, ..nn) be the partitions of K_mn. We can create a path starting at m1. Since K_mn is a complete bipartite graph, there is a vertex from m1 to every vertex in N. We construct our path as follows: m1 -> n1 -> m2 -> …. The next element in the path is the next corresponding element in the other partition. Since m=n, we have that after traversing every vertex, our last element in the path will be nn. We can close the cycle by adding a vertex from nn to m1, and hence we have a Hamilton cycle.

For a complete bipartite graph K_mn, we prove that m=n and m > 1 if it has a Hamilton cycle. Suppose m <= 1. There cannot be a cycle, since we have less than 3 vertices. Hence we must have that m > 1. Suppose that m != n. We arbitrarily suppose m > n. Let M = m1, m2…mm) and N = (n1, n2, ..nn) be the partitions of K_mn. We can create a path starting at m1. Since K_mn is a complete bipartite graph, there is a vertex from m1 to every vertex in N. We construct our path as follows: m1 -> n1 -> m2 -> …. Since m > n, we will end at vertex nn with no more edges to the remaining elements in M. Hence no Hamilton cycle exists, and we have a contradiction. Hence we must have that m=n.

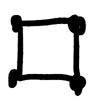
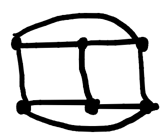
4.6.8 Let G be a graph with an odd closed walk. Let C = (v1,…vn) be a shortest odd closed walk. We prove that this is an odd cycle. Suppose for contradiction that there exists some 1 <= i < j < n such that vi = vj in C. Then (v0,…vi, vj+1,…vn) and (vi,…vj) are shorter closed walks than C. Then the length of one of these two walks is odd, since they must sum to an odd number, and it is a shorter odd walk than C, which contradicts our original statement. Hence C must be an odd cycle.

4.6.9 a) Let C = (v1,…vn) be the shortest cycle in a graph. Suppose for contradiction that there exists a diagonal between vertices i and j. Then (v0,…vi, vj+1,…vn) and (vi, vi+1,…vj) are shorter cycles than C, which contradicts our original statement. Hence C has no diagonals.

b) Let C = (v1,…vn) be the shortest odd cycle in a graph. Suppose for contradiction that there exists a diagonal between vertices i and j. Then (v0,…vi, vj+1,…vn) and (vi,…vj) are shorter cycles than C. Then the length of one of these two cycles is odd, since they must sum to an odd number, and it is a shorter cycle than C, which contradicts our original statement. Hence C has no diagonals.

c) 

4.6.10 a) Let G be a k-regular graph of girth 4. Let x and y be two adjacent vertices in G. They must not have a common neighbour, otherwise there would be a cycle of length 3. Hence they must each have k-1 new neighbours. Hence there must be at least 2 + 2(k-1) = 2k vertices.

b) $k=2$:  $k=3$: 

We can create a k-regular graph of girth 4 with precisely 2k vertices by connecting pairs of vertices into squares, and then connecting the top and bottom two ends of the chain of squares.

c) Let v be a vertex in this graph. It must have k neighbours, but these neighbours must not be connected, otherwise there would be a cycle of 3. None of these neighbours share a neighbour other than v, otherwise there would be a cycle of 4. Hence each of these k neighbours must have k-1 new neighbours. Hence there must be 1 + k + k(k-1) = k^2 + 1 vertices.

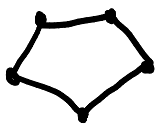
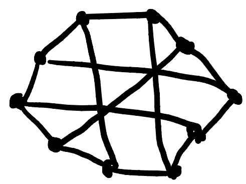
d) Let x and y be adjacent vertices in G. Let V1 = {x, y}. Each vertex in V1 must be adjacent to an addition k-1 vertices. These vertices must not be adjacent to one another, otherwise there would be a cycle of 3. Hence they must each have k-1 new neighbours. Let V2 be this set of new vertices. We generalize this as follows: Vd is the set of vertices adjacent to the vertices in Vd-1, that are not in Va where a < k-1. Each Vd has a cardinality of (k-1) times the number of vertices of Vk-1, and V1 has 2 vertices. Summing the number of elements in V0 through Vt, we obtain

$$|V(G)| \geq 2\sum_{i=0}^{t-1}(k-1)^i = \frac{2(k-1)^t - 2}{k-2}$$

e) Let v be a vertex of G. Let V0 be {v}. v must be adjacent to k vertices; let V1 be the set consisting of these vertices. None of the elements of V1 can be adjacent to one another, otherwise there would be a cycle of 3. Hence these k vertices must be adjacent to (k-1) new vertices; let these be in V2. None of these vertices can be adjacent to each other, otherwise there would be a cycle of 4. We generalize this as follows: Vd is the set of vertices adjacent to the vertices in Vd-1, that are not in Va where a < k-1. Summing the number of elements in V0 through Vt, we obtain

$$|V(G)| \geq 1 + \sum_{i=0}^{t-1} k(k-1)^i = 1 + \frac{k(k-1)^t - 1}{k-1-1}$$

$$= \frac{k(k-1)^t - 2}{k-2}$$

f) $k=2$:  $k=3$: 

4.10.1 We prove this statement by strong induction. Our base case, B_1, is trivially connected.

Observe B_n, where n > 1. We know that B_(n-1) is connected, which means that in B_n, there exists a path from every vertex in {1,…n-1} to the vertex 1. We observe n. There exists a prime number r such that n < r < 2n. Hence there must exist a natural number x where r = x + n, for some x < n, so vertices x and n are adjacent in B_n. Since x < n, there is a path from x to 1. Hence there is a path from n to 1, and hence B_n is connected.

4.10.2 Let G be a connected graph. Suppose there are two longest paths in G that do not have a vertex in common, lets call them P and Q. Since G is connected, there must be a path R from a vertex u in P to a vertex v in Q, where no vertex of R is in P or Q. We can divide P into P1uP2 at u, and same for Q: Q = Q1vQ2. Without loss of generality, we assume that |P1| > |P2| and |Q1| > |Q2|. Then we have a path P1uRvQ1 which is longer than both P and Q, which is a contradiction to our original statement. Hence if G is connected, any two longest paths in G have a vertex in common.

4.10.3