MATH 239 Exercises 1

1.1 a) There are $\binom{n+t-1}{t-1}$ n-element multisets with t types.

To find the number of multisets where every type element to occur at most once, we find the number of n-element subsets of a t-element set (the types), which is equal to

$$\binom{t}{n}$$

Hence the probability is

$$\frac{\binom{t}{n}}{\binom{n+t-1}{t-1}}$$

b) We have at least one element of each type - the rest of the elements in the multiset form a multiset of size n-t with elements of t types, so there are

$$\binom{n-t+t-1}{t-1} = \binom{n-1}{t-1}$$

different outcomes.
Hence the probability is

$$\frac{\binom{n-1}{t-1}}{\binom{n+t-1}{t-1}}$$

c) If n is odd, the probability is 0. If n is even, we can create a bijection between M(n,t) = the set of multisets of size n with t types, and B(n/2,t) = the set of multisets of size n/2 and t types. For each multiset (m1,m2…mt) of M(n,t) we can define a new multiset by saying for each 1 <= i <= n, mi_new = mi/2. This creates a new multiset of size n/2 with t types. Conversely, for a multiset of size n/2 and t types, we can create a new multiset by saying for each 1 <= i <= n, mi_new = mi*2. This creates a new multiset of size n with t types. Hence M(n,t) and B(n/2,t) are mutually inverse bijections. The number of multisets in B(n/2,t) is

$$\binom{n/2+t-1}{t-1}$$

Hence the probability is $\dfrac{\binom{n/2+t-1}{t-1}}{\binom{n+t-1}{t-1}}$

d) Since each element occurs an odd number of times, it must occur at least once. If t > n, the probability is 0. Otherwise, the rest of the elements in the multiset form a multiset of size n-t with elements of t types, where each element occurs an even number of times. Using the formula from part c, we have that the probability is

$$\frac{\binom{\frac{n-t}{2}+t-1}{t-1}}{\binom{n+t-1}{t-1}}$$

e) We first choose k elements from t types - there are tCk ways to do this. From part b, we have the formula for the number of multisets where each type of element occurs at least once - in this case we have k types now. Hence the probability is

$$\frac{\binom{n-1}{k-1}\binom{t}{k}}{\binom{n-t+1}{t-1}}$$

f) We first choose k elements from t types - there are tCk ways to do this. We have at least 2 elements of each type - the rest of the elements in the multiset form a multiset of size n-2k with elements of t types. Of the non-k types, there can be at most one of each element. We iterate through the number of those t-k types present. Hence the probability is

$$\frac{\binom{t}{k}\sum_{i=0}^{t-k}\binom{n-k-i-1}{k-1}\binom{t-k}{i}}{\binom{n+t-1}{t-1}}$$

1.2 A) For each number 1 <= i <= 6, there is one way to roll all i's. Hence there are 6 outcomes.

B) We have 6C1 choices for the number for the 5-of-a-kind and 6C5 ways to choose 5 dice for it. There are 5C1 choices for the number for the single, and 1C1 ways to choose one dice for it. Hence there are (6C1)(6C5)(5C1)(1C1) = 180 outcomes.

C) We have 6C1 choices for the number for the 4-of-a-kind and 6C4 ways to choose 4 dice for it. There are 5C1 choices for the number for the pair, and 2C2 ways to choose two dice for it. Hence there are (6C1)(6C4)(5C1)(2C2) = 450 outcomes.

D) We have 6C1 choices for the number for the 4-of-a-kind and 6C4 ways to choose 4 dice for it. There are 5C2 choices for the number for the two singles, and 2! ways to organize them. Hence there are (6C1)(6C4)(5C2)2! = 1800 outcomes

E) (6C2)(6C3)(3C3) = 300 outcomes

F) (6C1)(6C3)(5C1)(3C2)(4C1)(1C1) = 7200 outcomes

G) (6C1)(6C3)(5C3)3! = 7200 outcomes

H) (6C3)(6C2)(4C2)(2C2) = 1800 outcomes

I) (6C2)(6C2)(4C2)(4C2)2! = 16200 outcomes

J) (6C1)(6C2)(5C4)4! = 10800 outcomes

K) 6! = 720 outcomes

1.3 There would be dCk ways to choose k numbers for the k pairs, mC2k ways to choose the 2k dice for those pairs, and we can use the multinomial coefficient to organize them. There would be (d-k)C(m-2k) ways to choose m-2k numbers for the singles, and (m-2k)! ways to choose their organization.

$$\frac{\binom{d}{k}\binom{m}{2k}\frac{(2k)!}{2^k}\binom{d-k}{m-2k}(m-2k)!}{d^m}$$

1.4 a) Symmetry: A bijection is injective and surjective, and has an inverse which is also a bijection. We use a bijection f : A -> B. Since f is a bijection, then f^-1 exists and f^-1 : B -> A.

Reflexivity: We can define f : A -> A. This is a bijection, since every element of A will be the image of at least one a in A (since it maps to itself), and every element of A will be the image of exactly one element (since it maps to itself).

Transitivity: Let f : A -> B and g : B -> C by bijections. Then their composition f . G : A -> C is a bijection. Every element of C will be the image of at least one element in A, since every element of C has one image in B, which maps to an element in A. Every element in C will be the image of exactly one element in A, since it will be the image of exactly one element in B, which is the image of exactly one element in A.

b) Let f: A->B and g:B->A be functions between sets A and B. Assume that for every a in A, g(f(a)) = a, and for every b in B, f(g(b)) = b, as stated in the proposition.

Assume for the sake of contradiction that A and B are not bijective. Since A and B are bijective if and only if |A| = |B|, we must have |A| > |B| or |A| < |B|.
If |A| > |B| then there exists some distinct a1, a2 in A and b in B such that f(a1) = b and f(a2) = b. This means g(f(a1)) = g(b) = a1 and g(f(a2)) = g(b) = a2. However, a1 != a2, and we have a contradiction. The vice versa would be true for |A| < |B|. Hence it must be that A and B are bijective.

1.5 a) $g(b) = \begin{cases} b/2 & \text{if } b \text{ is even} \\ -\frac{b+1}{2} & \text{if } b \text{ is odd} \end{cases}$

b) For any a in Z, we have

$$f(a) = \begin{cases} 2a, & a \geq 0 \rightarrow f(a) \text{ is even} \\ -1-2a, & a < 0 \rightarrow f(a) \text{ is odd} \end{cases}$$

$$g(f(a)) = \left\{ \begin{array}{l} \frac{2a}{2} = a \\ \frac{-((-1-2a)+1)}{2} = a \end{array} \right\} = a$$

c) For any b in N, we have

$$g(b) = \begin{cases} b/2 & \text{if } b \text{ is even} \rightarrow g(b) \geq 0 \\ -\frac{b+1}{2} & \text{if } b \text{ is odd} \rightarrow g(b) < 0 \end{cases}$$

$$f(g(b)) = \left\{ \begin{array}{l} 2\left(\frac{b}{2}\right) = b \\ -1-2\left(\frac{b+1}{2}\right) = b \end{array} \right\} = b$$

1.6 Let $f: S \rightarrow M$

$$f(s_i) = s_i - s_{i-1} - 1$$

$$g: M \rightarrow S$$

$$g(m_i) = \left(\sum_{j=1}^{i} s_j\right) + i$$

For all $s_i \in S$, we have

$$g(f(s_i)) = g(s_i - s_{i-1} - 1) = \left(\sum_{j=1}^{i} s_j - s_{j-1} - 1\right) + i$$

$$= s_i - i + i = s_i$$

For all $m_i \in M$, we have

$$f(g(m_i)) = f\left(\left(\sum_{j=1}^{i} m_j\right) + i\right) = \left(\sum_{j=1}^{i} m_j\right) + i - \left(\sum_{j=1}^{i-1} m_j + i - 1\right) - 1$$

$$= m_i$$

1.7 a) The LHS of the equation is a description of the number of ways to choose a subset of any size from an n-element set, then one of which is 'special'. We start with the formula for a subset of k elements: there are nCk ways to choose a subset of k elements from an n-element set. From this subset of k, there are k ways to choose this 'special' element. Summing over all possible ks, we get

$$\sum_{k=0}^{n} \binom{n}{k} k$$

Alternatively, we can use a different method. There are n ways to choose the 'special' element from an n-element set. There are 2^(n-1) different subsets of the remaining (n-1)-element list:

$$n 2^{n-1}$$

Hence the LHS and RHS are equivalent.

b) The logic for this is similar to part a above. The LHS of the equation is a description of the number of ways to choose a subset of any size from an n-element set, then <u>two</u> of which are 'special'. We start with the formula for a subset of k elements: there are nCk ways to choose a subset of k elements from an n-element set. From this subset of k, there are k ways to choose the first 'special' element, and (k-1) ways to choose the second 'special' element. Summing over all possible ks, we get

$$\sum_{k=0}^{n} \binom{n}{k} k(k-1)$$

Alternatively, we can use a different method. There are n ways to choose the first 'special' element from the n-element set, and n-1 ways to choose the second 'special' element from the remaining. There are 2^(n-2) different subsets of the remaining (n-2)-element list:

$$n(n-1)2^{n-2}$$

Hence the LHS and RHS are equivalent.

1.8 Let A(n) be the set of subsets of even length of an n-element set. Hence the LHS describes |A(n)|. Let B(n) be the set of subsets of odd length of an n-element set. Hence the RHS describes |B(n)|.

There is a bijection between these two sets. If n is odd, that means for every even k, n-k is odd. Hence for every subset in A, we can take its complement, which will be a subset in B. This also works vice versa: for every subset in B, we take its complement to arrive at its corresponding subset in A.

If n is even, choose one element in A to be a_0. For every subset in A, if this subset contains a_0, remove a_0 from this subset to obtain a subset in B. If this subset does not contain a_0, add a_0 to this subset to obtain a subset in B. This works vice versa: for every subset in B, if this subset contains a_0, remove a_0 from this subset to obtain a subset in A. If this subset does not contain a_0, add a_ 0 to this subset to obtain a subset in A.

Since there is a bijection between these two sets, they must have the same number of elements. Hence the LHS equals the RHS.

1.9 a) |Tn| is the number of functions that map the set of numbers {1,2…n} to {1,2,3}. For each number {1,2,3}, there are n possibilities for which number in {1,…n} maps to it in each function. Hence, we have

$$|T_n| = 3^n$$

b) We define g: Sn -> Tn as follows: for every (A,B) in Sn, we return the function in Tn where the numbers in {1,…n} that appear neither in A or B map to 1, the numbers in B but not A map to 2, and the numbers in A map to 3. Since A is a subset of B is a subset of {1,…n}, they will cover every possible combination of mappings from {1,…n} to {1,2,3}, which corresponds to all the functions in Tn.

c) We define g^-1: Tn -> Sn as follows: for every function f in Tn, we return the pair (A,B) where A contains the numbers a where f(a) = 3 and B contains the numbers b where f(b)=2 or f(b)=3.

d) As described in part a, we have that

$$|T_n| = 3^n$$

To find |Sn|, we iterate through all the possible sizes of B in the pair (A,B). For each size 0<=k<=n, we have that the number of possible sets B of size k is (n choose k). The number of subsets of B (which has k elements) is 2^k, which is the number of possible sets A for this B. Hence we have

$$|S_n| = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Since g: Sn -> Tn is a bijection, we have |Sn| = |Tn|.

1.10 We construct f: A(n,k)->B(n,k). For a sequence (a1…ak) in A(n,k), we have that a1 is 0 mod 1, a2 is 0 mod 2,…ak is 0 mod k. We create a sequence (b1,…bk) where

$$b_k = \sum_{i=k}^{n} \frac{a_i}{i}$$

Hence we must have b1 >= b2 >= … bk, and since a1 + … _ ak = n, we must have b1 + … + bk = n.

We construct g: B(n,k) -> A(n,k). For a sequence (b1…bk) in B(n,k), we create a sequence (a1,…ak) where

$$a_k = k(b_k - b_{k+1})$$

For any sequence (a1,..ak) in A(n,k) we have for every aj where 0 <= j <= k:

$$g(f((\ldots a_j \ldots))) = g((\ldots \sum_{l=j}^{n} \frac{a_l}{l} \ldots)) = (\ldots j \left(\sum_{l=k}^{n} \frac{a_l}{l} - \sum_{l=k+1}^{n} \frac{a_l}{i}\right) \ldots)$$
$$= (\ldots a_j \ldots)$$

Hence we have g(f(a)) = a. For any sequence (b1,..bk) in B(n,k) we have for every bj where 0 <= j <= k:

$$f(g((\ldots b_j \ldots))) = f((\ldots j(b_j - b_{j+1}) \ldots)) = \sum_{l=j}^{n} \frac{l(b_l - b_{l+1})}{l}$$
$$= (\ldots b_j \ldots)$$

Hence we have a pair of mutually inverse bijections between A(n,k) and B(n,k).

Hence we have a pair of mutually inverse bijections between A(n,k) and B(n,k).

1.11

The LHS of the equation describes the number of n-element multisets with elements of t types.
We can also count the number of n-element multisets with elements of t types by fixing one of the types, we'll call this type K. If there are k elements of type K in the multiset, the rest of the elements form a multiset of size n - k, with t-1 types. Iterating through every possible k from 0-n, we obtain

$$\sum_{k=0}^{n} \binom{n-k+t-2}{t-2}$$

Since the LHS and RHS describe the same number, we have deduced the statement.

1.12

The LHS of the equation describes the number of n-element multisets with elements of t types.
We can also count this number by fixing the number of types in the multiset that occur at least once, let's call this k. There are tCk choices to choose these k types. The number of multisets where each of these k types occurs at least once in the multiset is (n-1)C(k-1). Iterating through all k from 0 to t:

$$\sum_{k=0}^{t} \binom{t}{k}\binom{n-1}{k-1}$$

1.13

1. If we 'group' the one and two together, there are 6! permutations of 6 different numbers. There are 2 ways to organize the numbers 1 and 2 with each other.
Hence the probability is 2*6!/7! = 2/7

2. Since all permutations are equally likely, and in each, 1 is either to the left or to the right of 2.
Hence the probability is 1/2

3. Since there are 4 odd numbers and 3 even numbers, the only permutations are those alternating even and odd numbers, starting and ending with odd. There are 4! ways to arrange these 4 odd numbers in the odd slots. There are 3! ways to arrange the 3 even numbers in the even slots.
Hence the probability is 4!3!

4. No thanks
5. Also nope

1.14

A) If s >= 5, there are r ways to get a five-of-a-kind.

B) There are r-5+2 = r-3 choices for the consecutive values of the cards. The cards must all be the same suit, so there are s choices for the choice of suit.
Hence the count is s(r-3)

C) If s >= 4, there are r choices for the value of the quad and sC4 ways to choose 4 suits for them.
There are (r-1) choices for value of the single and s ways to choose a suit for it.
Hence the count is rs(sC4)(r-1)

D) If s >= 3, there are r choices for the value of the triple and sC3 ways to choose 3 suits for it. There are (r-1) choices for the value of the pair and s ways to choose a suit for them.
Hence the count is rs(sC3)(r-1)

E) There are rC5 - (r-3) choices for the values of the cards that are not consecutive. There are s choices of suit for these cards.
Hence the count is s(rC5 - r + 3)

F) There are r-5+2 = r-3 choices for the consecutive values of the cards. There are s^5-s choices for the suits on the cards that don't give them the same suit.
Hence the count is (r-3)(s^5-s)

G) There are r choices for the value of the triple and sC3 ways to choose 3 suits for it. There are ((r-1)C2) choices for the values of the singles and s^2 ways to choose suits for them.
Hence the count is rs^2(sC3)((r-1)C2)

H) There are rC2 choices for the values of the two pairs and sC2 ways to choose 2 suits for each. There are r-2 choices for the value of the singles and s ways to choose suits for it.
Hence the count is (rC2)(sC2)(sC2)(r-2)s

I) There are r choices for the value of the pair and sC2 ways to choose 2 suits for them. There are rC4 choices for the values of the singles, and s^4 choices for suits for them.
Hence the count is r(sC2)(rC4)(s^4)

J) There are rC5 - (r-3) choices for the values of the cards that are not consecutive and are all different values. There are s^5 - s choices for the suits which do not give all 5 cards the same suit.
Hence the count is (rC5 - r + 3)(s^5 - s)

1.15 a)
$$E(x) = \binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)x + \binom{3}{2}\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)(2x) + \binom{3}{3}\left(\frac{1}{6}\right)^3(3x)$$
$$= \frac{1}{2}x$$

For every dollar the Player wages, the expected value they win is \$0.50. I would not play this game.

b)
$$E(x) = \sum_{i=1}^{m}\binom{n}{i}\left(\frac{1}{d}\right)^i\left(\frac{d-1}{d}\right)^{m-i} ix$$
$$= \sum_{i=1}^{m}\frac{m!}{i!(m-i)!}\left(\frac{1}{d}\right)^i\left(\frac{d-1}{d}\right)^{m-i} ix$$
$$= \sum_{i=1}^{m}\frac{m(m-1)!}{(i-1)!(m-1-(i-1))!}\left(\frac{1}{d}\right)^i\left(\frac{d-1}{d}\right)^{m-i} x$$
$$= \frac{m}{d}\sum_{i=1}^{m}\binom{m-1}{i-1}\left(\frac{1}{d}\right)^{i-1}\left(\frac{d-1}{d}\right)^{m-1-(i-1)} x$$
$$= \frac{m}{d}\sum_{i=1}^{m-1}\binom{m-1}{i}\left(\frac{1}{d}\right)^{i}\left(\frac{d-1}{d}\right)^{m-1-i} x$$

$$(\text{binomial}) = \frac{m}{d}\left(\frac{1}{d} + \frac{d-1}{d}\right)^{m-1} x$$

$$= \frac{m}{d} x$$

The Player wins if m > d. The House wins if m < d. The game is fair if m = d.