MATH 239 F2005 Midterm

1. a) $$[x^{10}]\frac{x^3}{(1+3x)^6} = [x^{10}]\, x^3 \sum_{n=0}^{\infty} \binom{n+5}{5}(-3)^n x^n$$

$$= [x^7] \sum_{n=0}^{\infty} \binom{n+5}{5}(-3)^n x^n$$

$$= \binom{12}{5}(-3)^7$$

b) $$(1-x)^{-(k+1)}(1-x)^{-1} = \sum_{n=0}^{\infty}\binom{n+k}{k}x^n \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{m+k}{k}x^n$$

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2. a) The generating function for compositions with even parts is

$$\Phi_E(x) = \frac{x^2}{1-x^2}$$

The generating function for compositions with odd parts is

$$\Phi_O(x) = \frac{x}{1-x^2}$$

We have that there are 8 parts, where the first and last parts are even, and all other parts are odd. Hence by the product lemma, the generating function for the number of compositions is

$$\Phi_E(x)\left[\Phi_O(x)\right]^6 \Phi_E(x) = \left(\frac{x^2}{1-x^2}\right)^2\left(\frac{x}{1-x^2}\right)^6$$

$$= \frac{x^{10}}{(1-x^2)^8}$$

The number of compositions of n is

$$[x^n]\frac{x^{10}}{(1-x^2)^8} = [x^{n-10}]\sum_{n=0}^{\infty}\binom{n-7}{7}x^{2n}$$

$$=\binom{\frac{n-10}{2}-7}{7}$$

b) The generating function for a composition with even parts where every part is greater than or equal to 4 is

$$\Phi_E(x) = \frac{x^4}{1-x^2}$$

By the string lemma, we have that the generating function for the number of compositions of n where every part is an even number greater than or equal to 4 is

$$\frac{1}{1-\frac{x^4}{1-x^2}} = \frac{1-x^2}{1-x^2-x^4}$$

3\. a) The expression 1* ((00)*0011* ⌣ (00)*011(11)*)* (ε ⌣ (00)*00) is the block decomposition for this set of strings.

b) The expression 0* (100* ⌣ 1100*⌣ 11*0)* 1* is the block decomposition for this set of strings.

c) $$\Phi(x) = \frac{1}{1-x}\left(\frac{1}{1-\left(\frac{x^2}{1-x^2}\cdot\frac{x^2+x^3}{1-x^3}\right)}\right)x$$

4\. a) $$a_n - 5a_{n-1} + 6a_{n-2} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 0, & n\geq 2 \end{cases}$$

$$a_0 = 1$$

$$a_1 = 4$$

$$a_n = 5a_{n-1} - 6a_{n-2},\ n\geq 2$$

b) $b_0 = 11$

$b_1 - 11b_0 = -92$

$b_2 - 11b_1 + 40b_0 = 188$

$$b_n = [x^n]\ \frac{11 - 92x + 188x^2}{1 - 11x + 40x^2 - 48x^3}$$

$$1 - 11x + 40x^2 - 48x^3 = (1-3x)(1-4x)^2$$

Theorem 4.14 implies there are constants A, B, C, D such that for sufficiently large n,

$$b_n = (A + Bn) + (C + Dn)2^n$$

We can fit the recurrence relation to b1=29, b2=67, b3=105, and b4=-133

$$29 = (A+B) + (C+D)\cdot 2 = A + B + 2C + 2D$$

$$67 = (A+2B) + (C+2D)\cdot 2^2 = A + 2B + 4C + 8D$$

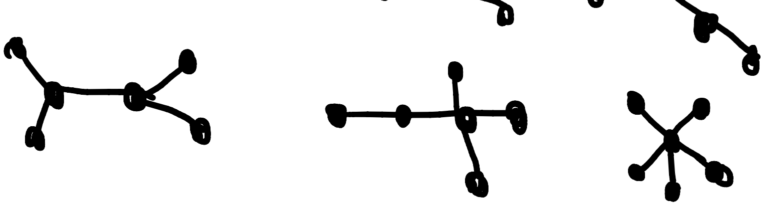
$$105 = (A+3B) + (C+3D)\cdot 2^3 = A + 3B + 8C + 24D$$

$$-133 = (A+4B) + (C+4D)\cdot 2^4 = A + 4B + 16C + 64D$$

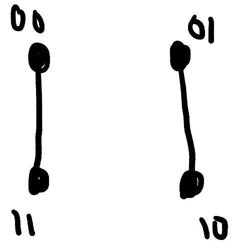
A=-285, B=-238, C=345, D=-69

$$b_n = -285 - 238n + 2^n(345 - 69n)$$

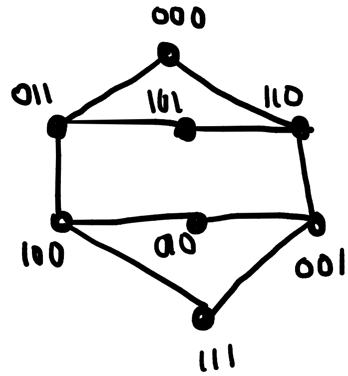
5. a)



b) i. A_2:



A_3:



ii. For each vertex string, there are n-1 strings of length n that differ in exactly 2 positions. Hence we have

$$|E(A_n)| = \frac{1}{2}\sum_{v \in V} \deg(v) = \frac{1}{2}n(n-1)$$

iii. Let x and y be two strings of length n that differ by 2 characters. There are 3 possibilities for the 2 differing characters:

We can have two 0s from x changed to 1 in y. In this case, the parity of 0s in both strings are the same.

We can have two 1s from x changed to 0 in y. This is symmetrical to the case above; the parity of 0s in both strings are the same.

We can change one 1 to a 0, and one 0 to 1 from x to y. In this case, the parity of 0s in both strings are the same.

Hence in all cases, the parity of 0s in x and y are the same.

We then have that in An, every string with an even parity of zeros is connected, and every string with an odd parity of zeros is connected.

Hence An has 2 components, and is not connected.

6.a) Let p be the number of vertices in the tree. Since the tree is cubic, there are (k + 3(p-k))/2 edges. Since it is a tree, it must have p-1 edges. Hence we have

$$p-1=\frac{k+3(p-k)}{2}$$

$$2p-2=k+3p-3k$$

$$p=2k-2=2(k-1)$$

b)

Since G has no cycles, none of its components can have cycles. Hence every component is a tree. For each component T_i where 1 <= i <= c, we have that

$$\sum_{i=1}^{c} |E(T_i)| = \sum_{i=1}^{c} (|V(T_i)|-1) = -c + \sum_{i=1}^{c} |V(T_i)| = p - c$$