Exercises 2

1. a) Let p3(x) = ax^3 + bx^2 + cx + d. Without loss of generality, suppose x0 < x1 < x2. Then we can put our equations into a linear system:

$$a x_0^3 + b x_0^2 + c x_0 + d = f(x_0)$$
$$a x_2^3 + b x_2^2 + c x_2 + d = f(x_2)$$
$$\frac{d}{dx}\left(a x_1^3 + b x_1^2 + c x_1 + d\right) = f'(x_1) \rightarrow 3a x_1^2 + 2b x_1 + c = f'(x_1)$$
$$\frac{d}{dx}\left(\frac{d}{dx}\left(a x_1^3 + b x_1^2 + c x_1 + d\right)\right) = f''(x_1) \rightarrow 6a x_1 + 2b = f''(x_1)$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_2) \\ f'(x_1) \\ f''(x_1) \end{bmatrix}$$

Reducing the matrix to partial RREF gives us

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & 1 & x_2 + x_0 & x_2^2 + x_0 x_2 + x_0^2 \\ 0 & 0 & 2x_1 - x_2 - x_0 & 3x_1^2 - (x_2^2 + x_0 x_2 + x_0^2) \\ 0 & 0 & 2 & 6x_1 \end{bmatrix}$$

We compute the determinant:

$$6x_1^2 + 2x_2^2 + 2x_0^2 + 2x_0 x_2 - 6x_1 x_2 - 6x_0 x_1$$
$$= 3(x_1 - x_0)^2 + 3(x_2 - x_1)^2 - (x_2 - x_0)^2$$
$$= 3(x_1 - x_0)^2 + 3(x_2 - x_1)^2 - \left((x_1 - x_0) + (x_2 - x_1)\right)^2$$
$$= 2(x_1 - x_0)^2 - 2(x_1 - x_0)(x_2 - x_1) + 2(x_2 - x_1)^2$$
$$= \left((x_1 - x_0) - (x_2 - x_1)\right)^2 + (x_1 - x_0)^2 + (x_2 - x_1)^2 > 0$$

Since the determinant is nonzero, the system must have a unique solution.

b) ???

2. a) The degree of each polynomial function is n-1

b) We have that the sum of the Lagrange basis functions is 1. This is because g(x) represents the unique polynomial of degree at most n-1 passing through the points (x1, 1),…(xn, 1). This must be the function g(x) = 1.

3. We have that

$S(-3) = -27 \qquad S'(-1)$ continous

$S(-1) = -1 \qquad S''(-1)$ continuous

$S(1) = 1 \qquad S''(-3) = S''(1) = 0$

$$S(x) = \begin{cases} ax^3 + bx^2 + cx + d, & -3 \le x \le -1 \\ ex^3 + fx^2 + gx + h, & -1 \le x \le 1 \end{cases}$$

$-27 = -27a + 9b - 3c + d$

$-1 = -a + b - c + d$

$-1 = -e + f - g + h$

$1 = e + f + g + h$

$3a - 2b + c = 3e - 2f + g$

$6a + b = 6e + f$

$-18a + b = 0$

$-18e + f = 0$

$S(0) = h = 18/5$

$a = -1/5$

$b = -18/5$

$c = 6/5$

$d = 18/5$

$e = -1/5$

$f = -18/5$

$g = 6/5$

$h = 18/5$

4. a) $a_1 - 25 + 9 - 1 = 26 - a_2 + a_3 + 1$

$a_1 + a_2 - a_3 = 44$

$26 + 57 + 9a_4 + 27a_5 = -163 + 624 - 540 + 27a_6$

$a_4+3a_5-3a_6=-18$

$25-18+3=a_2-2a_3-3$

$a_2-2a_3=13$

$a_2=19$

$19+6a_4+27a_5=208-360+27a_6$

$2a_4+9a_5-9a_6=-57$

$-120+24a_6=0$

$a_6=5$

$a_1=28,\ a_2=19,\ a_3=3,\ a_4=3,\ a_5=-2,\ a_6=5$

b) $S(-3)=7$

$S(-1)=11$

$S(0)=26$

$S(3)=56$

$$L_1(x)=\frac{(x+1)(x)(x-3)}{(-3+1)(-3)(-3-3)}$$

$$L_2(x)=\frac{(x+3)(x)(x-3)}{(-1+3)(-1)(-1-3)}$$

$$L_3(x)=\frac{(x+3)(x+1)(x-3)}{(3)(1)(-3)}$$

$$L_4(x)=\frac{(x+3)(x+1)(x)}{(3+3)(3+1)(3)}$$

$$p(x)=7L_1(x)+11L_2(x)+26L_3(x)+56L_4(x)$$

$$=26+\frac{397}{24}x+\frac{11}{18}x^2-\frac{67}{72}x^3$$

5. Let $\Delta x_i = x_{i+1} - x_i$

① $S_i(x_i) = f_i$

$$f_i = a_i(x_{i+1} - x_i) + b_i(x_i - x_i)(x_i - x_{i+1}) + c_i(x_i - x_i)$$

$$= a_i(x_{i+1} - x_i)$$

② $S_i(x_{i+1}) = f_{i+1}$

$$f_{i+1} = a_i(x_{i+1} - x_{i+1}) + b_i(x_{i+1} - x_i)(x_{i+1} - x_{i+1}) + c_i(x_{i+1} - x_i)$$

$$= c_i(x_{i+1} - x_i)$$

$$S_i'(x) = \frac{d}{dx} a_i x_{i+1} - a_i x + b_i(x^2 - (x_{i+1} + x_i)x + x_{i+1}x_i) + c_i x - c_i x_i$$

$$= -a_i + b_i(2x - x_{i+1} - x_i) + c_i$$

$$S_{i+1}'(x) = -a_{i+1} + b_{i+1}(2x - x_{i+2} - x_{i+1}) + c_{i+1}$$

③ $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$

$$-a_i + b_i(2x_i - x_{i+1} - x_i) + c_i = -a_{i+1} + b_{i+1}(2x_{i+1} - x_{i+2} - x_{i+1}) + c_{i+1}$$

$$-a_i - b_i \Delta x_i + c_i = -a_{i+1} - b_{i+1}\Delta x_{i+1} + c_{i+1}$$

$$b_{i+1}\Delta x_{i+1} - b_i \Delta x_i = -a_{i+1} + c_{i+1} + a_i - c_i$$

④ $S'(x_0) = y$

$$-a_1 + b_1(2x_1 - x_2 - x_1) + c_1 = y$$

$$-a_1 - b_1 \Delta x_1 + c_1 = y$$

$$b_1 \Delta x_1 = y + a_1 - c_1$$

$$a_i = \frac{f_i}{\Delta x_i}$$

$$c_i = \frac{f_{i+1}}{\Delta x_i}$$

There are n-1 + n-1 + n-2 + 1 = 3n-3 equations, and 3n-3 unknowns. To solve for b, we solve the system

$$\begin{bmatrix} -b_1 & b_2 & 0 & 0 & \cdots & 0 \\ 0 & -b_2 & b_3 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \\ 0 & \cdots & & & -b_{n-2} & b_{n-1} \\ b_1 & \cdots & & & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_{n-1} \\ \Delta x_1 \end{bmatrix} = \begin{bmatrix} -a_2 + c_2 + a_1 - c_1 \\ -a_3 + c_3 + a_2 - c_2 \\ \vdots \\ -a_n + c_n + a_{n-1} + c_{n-1} \\ y + a_1 - c_1 \end{bmatrix}$$

6.

$$\left.\begin{aligned} a - b + c - d &= 0 \\ a + b + c + d &= 0 \end{aligned}\right\} \text{continuity of } S(x)$$

$$\left.\begin{aligned} 3a - 2b + c &= 1 \\ 3a + 2b + c &= 1 \end{aligned}\right\} \text{continuity of } S'(x)$$

$$\left.\begin{aligned} -6a + 2b &= 0 \\ 6a + 2b &= 0 \end{aligned}\right\} \text{continuity of } S''(x)$$

No solutions exist for this set of equations, so there does not exist a choice of coefficients for which S(x) is a natural cubic spline or a clamped spline.