

Exercises 2

1. a) Let $p_3(x) = ax^3 + bx^2 + cx + d$. Without loss of generality, suppose $x_0 < x_1 < x_2$. Then we can put our equations into a linear system:

$$\begin{aligned} ax_0^3 + bx_0^2 + cx_0 + d &= f(x_0) \\ ax_2^3 + bx_2^2 + cx_2 + d &= f(x_2) \\ \frac{d}{dx}(ax_1^3 + bx_1^2 + cx_1 + d) &= f'(x_1) \rightarrow 3ax_1^2 + 2bx_1 + c = f'(x_1) \\ \frac{d}{dx}\left(\frac{d}{dx}(ax_1^3 + bx_1^2 + cx_1 + d)\right) &= f''(x_1) \rightarrow 6ax_1 + 2b = f''(x_1) \\ &\downarrow \\ \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} &= \begin{bmatrix} f(x_0) \\ f(x_2) \\ f'(x_1) \\ f''(x_1) \end{bmatrix} \end{aligned}$$

Reducing the matrix to partial RREF gives us

$$\left[\begin{array}{cccc} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & 1 & x_2+x_0 & x_2^2+x_0x_2+x_0^2 \\ 0 & 0 & 2x_1-x_2-x_0 & 3x_1^2-(x_2^2+x_0x_2+x_0^2) \\ 0 & 0 & 2 & 6x_1 \end{array} \right]$$

We compute the determinant:

$$\begin{aligned} &6x_1^2 + 2x_2^2 + 2x_0^2 + 2x_0x_2 - 6x_1x_2 - 6x_0x_1 \\ &= 3(x_1 - x_0)^2 + 3(x_2 - x_1)^2 - (x_2 - x_0)^2 \\ &= 3(x_1 - x_0)^2 + 3(x_2 - x_1)^2 - ((x_1 - x_0) + (x_2 - x_1))^2 \\ &= 2(x_1 - x_0)^2 - 2(x_1 - x_0)(x_2 - x_1) + 2(x_2 - x_1)^2 \\ &= ((x_1 - x_0) - (x_2 - x_1))^2 + (x_1 - x_0)^2 + (x_2 - x_1)^2 > 0 \end{aligned}$$

Since the determinant is nonzero, the system must have a unique solution.

b) ???

2. a) The degree of each polynomial function is n-1

b) We have that the sum of the Lagrange basis functions is 1. This is because $g(x)$ represents the unique polynomial of degree at most $n-1$ passing through the points $(x_1, 1), \dots, (x_n, 1)$. This must be the function $g(x) = 1$.

3. We have that

$$S(-3) = -27 \quad S'(-1) \text{ continuous}$$

$$S(-1) = -1 \quad S''(-1) \text{ continuous}$$

$$S(1) = 1 \quad S''(-3) = S''(1) = 0$$

$$S(x) = \begin{cases} ax^3 + bx^2 + cx + d & , -3 \leq x \leq -1 \\ ex^3 + fx^2 + gx + h & , -1 \leq x \leq 1 \end{cases}$$

$$-27 = -27a + 9b - 3c + d$$

$$-1 = -a + b - c + d \quad a = -1/5$$

$$-1 = -e + f - g + h \quad b = -13/5$$

$$1 = e + f + g + h \quad c = 6/5$$

$$3a - 2b + c = 3e - 2f + g \quad d = 18/5$$

$$6a + b = 6e + f \quad e = -1/5$$

$$-18a + b = 0 \quad f = -13/5$$

$$-18e + f = 0 \quad g = 6/5$$

$$S(0) = h = 18/5 \quad h = 18/5$$

$$4. a) a_1 - 25 + 9 - 1 = 2b - a_2 + a_3 + 1$$

$$a_1 + a_2 - a_3 = 44$$

$$2b + 57 + 9a_4 + 27a_5 = -163 + 624 - 540 + 27a_6$$

$$a_4 + 3a_5 - 3a_6 = -18$$

$$25 - 18 + 3 = a_1 - 2a_3 - 3$$

$$a_1 - 2a_3 = 13$$

$$a_1 = 19$$

$$19 + 6a_4 + 27a_5 = 208 - 360 + 27a_6$$

$$2a_4 + 9a_5 - 9a_6 = -57$$

$$-120 + 24a_6 = 0$$

$$a_6 = 5$$

$$a_1 = 28, a_2 = 19, a_3 = 3, a_4 = 3, a_5 = -2, a_6 = 5$$

b) $S(-3) = 7$

$$S(-1) = 11$$

$$S(0) = 26$$

$$S(3) = 56$$

$$L_1(x) = \frac{(x+1)(x)(x-3)}{(-3+1)(-3)(-3-3)}$$

$$L_2(x) = \frac{(x+3)(x)(x-3)}{(-1+3)(-1)(-1-3)}$$

$$L_3(x) = \frac{(x+3)(x+1)(x-3)}{(3)(1)(-3)}$$

$$L_4(x) = \frac{(x+3)(x+1)(x)}{(3+3)(3+1)(3)}$$

$$p(x) = 7L_1(x) + 11L_2(x) + 26L_3(x) + 56L_4(x)$$

$$= 26 + \frac{397}{24}x + \frac{11}{16}x^2 - \frac{67}{72}x^3$$

$$5. S_i(x_i) = f_i$$

$$\begin{aligned} f_i &= a_i(x_{i+1} - x_i) + b_i(x_i - x_i)(x_i - x_{i+1}) + c_i(x_i - x_i) \\ &= a_i(x_{i+1} - x_i) \end{aligned}$$

$$S_i(x_{i+1}) = f_{i+1}$$

$$\begin{aligned} f_{i+1} &= a_i(x_{i+1} - x_{i+1}) + b_i(x_{i+1} - x_i)(x_{i+1} - x_{i+1}) + c_i(x_{i+1} - x_i) \\ &= c_i(x_{i+1} - x_i) \end{aligned}$$

$$\begin{aligned} S'_i(x) &= \frac{d}{dx} a_i x_{i+1} - a_i x + b_i x^2 - (x_{i+1} + x_i) x + x_{i+1} x_i + c_i x - c_i x_i \\ &= -a_i + 2b_i x - x_{i+1} + x_i + c_i \end{aligned}$$

$$S'_{i+1}(x) = -a_{i+1} + 2b_{i+1}x - x_{i+2} + x_{i+1} + c_{i+1}$$

$$S'_i(x_{i+1}) = S'_{i+1}(x_{i+1})$$

$$-a_i + 2b_i x_{i+1} - x_{i+1} + x_i + c_i = -a_{i+1} + 2b_{i+1} x_{i+1} - x_{i+2} + x_{i+1} + c_{i+1}$$

$$S'(x_0) = \text{specified}$$

$$b_0 = \frac{\frac{f_0}{x_1 - x_0} + x_1 - x_0 - \frac{f_1}{x_2 - x_1} + \text{specified}}{2x_0}$$

$$a_L = \frac{f_L}{x_{L+1} - x_L} \quad b_i - b_{i+1} = \frac{a_L - a_{i+1} + 2x_{i+1} - x_i - x_{i+2} - c_i + c_{i+1}}{2x_{i+1}}$$

$$c_L = \frac{f_{L+1}}{x_{L+1} - x_L}$$

$$\begin{aligned}
 & 6. \quad a - b + c - d = 0 && \} \text{ continuity of } S(x) \\
 & a + b + c + d = 0 \\
 & 3a - 2b + c = 1 && \} \text{ continuity of } S'(x) \\
 & 3a + 2b + c = 1 \\
 & -6a + 2b = 0 && \} \text{ continuity of } S''(x) \\
 & 6a + 2b = 0
 \end{aligned}$$

No solutions exist for this set of equations, so there does not exist a choice of coefficients for which $S(x)$ is a natural cubic spline or a clamped spline.