MATH 239 Exercises 3

3.1 The expressions ε, 0, and 1 are unambiguous. This is trivial. They each only produce a set containing one string, so each string in the set is only produced once.

The expression R $\smile$ S is unambiguous if and only if *R* intersect *S* = empty. We have that R $\smile$ S produces *R* U *S*. Since R and S are both unambiguous, each string in R and S is produced exactly once. If *R* intersect *S* = empty, then we have that the sets of strings they produce are disjoint, and since each string is produced exactly once, each string in *R* U *S* will be produced exactly once. Conversely, if R $\smile$ S is unambiguous, each string in *R* U *S* be produced exactly once. Since each string in R and S respectively is produced exactly once, the sets *R* and *S* must be disjoint, because otherwise there would be strings produced twice in *R* U *S*.

The expression RS is unambiguous if and only if there is a bijection *RS* ⇌ *R* x *S*. If there is a bijection *RS* ⇌ *R* x *S*, we have that every string in RS maps to a string in *R* x *S*. By definition of the cross product, *R* x *S* produces a set where for every string s in *R* x *S*, there is exactly one way to write s = po with p in *R* and o in *S*. Conversely, if RS is unambiguous, every string in RS is produced exactly once by *RS*. Hence for every string s in *RS*, there is exactly one way to write s = po with p in *R* and o in *S*, so there must be a bijection between *RS* ⇌ *R* x *S*.

The expression R* is unambiguous if and only if each of the concatenation products R^k is unambiguous and the union Uk=0->inf R^k is a disjoint union of sets. If each of the concatenation products R^k is unambiguous and the union Uk=0->inf R^k is a disjoint union of sets, we have a repeated application of R $\smile$ S; since each R^k is unambiguous, each string in its language is produced exactly once, and since it is a union of disjoint sets, there are no overlapping strings and so each is produced only once. Conversely, if R* is unambiguous, each of the concatenation products R^k must be unambiguous; otherwise the union would produce strings that are produced multiple times. The union must be one of disjoint sets, otherwise there would be overlapping strings that are produced more than once.

3.2 AB = 10.001, 10.100, 10.1001, 101.001, 101.100, 101.1001
BA = 001.10, 100.10, 1001.10, 001.101, 100.101, 1001.101

AB is ambiguous: 10.1001 = 101.001 = 101001.
BA is unambiguous.

$$\Phi_A(x) = x^2 + x^3$$

$$\Phi_B(x) = x^3 + x^3 + x^4 = 2x^3 + x^4$$

$$\Phi_{AB}(x) = \Phi_A(x)\,\Phi_B(x) = (x^2+x^3)(2x^3+x^4)$$

$$= x^7 + 3x^6 + 2x^5$$

$$\Phi_{BA}(x) = x^7 + 3x^6 + 2x^5$$

3.3 A* is unambiguous. It produces either two 0s, two 1s, or a single 0 sandwiched by 1s at a time. Consecutive 0s can only be produced one way, and each single 0 can only be produced one way. Consecutive 1s can only be produced either sandwiching a single 0, or with double 1s.

B* is ambiguous: 001.10 = 00.110

$$\Phi_A(x) = 2x^2 + x^3$$

$$\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)} = \frac{1}{1-2x^2-x^3}$$

3.4 A) (ε ⌣ 0 ⌣ 00⌣(0000)0*) ((1 ⌣ (111)1*) (0 ⌣ 00⌣(0000)0*))* (ε ⌣ 1 ⌣ (111)1*)

B) Same as a?

C) 1* (0* 01)* 0*

D) (00)* ((11)*1 (00)*00)* (ε ⌣ (11)*1)

3.5 a) Binary strings that end with 0 and every block of 1s with odd length is succeeded by a block of 0s of even length, and every block of 1s with even length is succeeded by a block of 0s of odd length.

b) $$\Phi_G(x) = \frac{1}{1-x}\left(\frac{1}{1-\left(\left(\frac{x}{1-x^2}\cdot\frac{x^2}{1-x^2}\right)+\left(\frac{x^2}{1-x^2}\cdot\frac{x}{1-x^2}\right)\right)}\right)$$

$$= \frac{1-2x^2+x^4}{1-x-2x^2+x^4-x^5}$$

c) By theorem 4.8, g_n satisfies the linear recurrence relation with initial conditions given by

$$g_n - g_{n-1} - 2g_{n-2} + g_{n-4} - g_{n-5} = \begin{cases} 1, & n=0 \\ -2, & n=2 \\ 1, & n=4 \\ 0, & \text{otherwise} \end{cases}$$

Hence we have

$$g_0, g_1, g_2 = 1$$

$$g_3 = 3$$

$$g_4 = 5$$

$$g_n = g_{n-1} + 2g_{n-2} - g_{n-4} + g_{n-5}, \quad n \geq 5$$

3.6 a) The expression (ε ⌣ 0*00) ((1*111) (0*00))* (ε ⌣ 1*111) is the block decomposition for this set of strings. This leads to the generating series

$$\left(1+\frac{x^2}{1-x}\right)\left(\frac{1}{1-\left(\frac{x^3}{1-x}\cdot\frac{x^2}{1-x}\right)}\right)\left(1+\frac{x^3}{1-x}\right)=\frac{(1-x+x^3)(1-x+x^2)}{1-2x+x^2-x^5}$$

b) We have

$$\Phi(x)=\frac{1-2x+2x^2-x^4+x^5}{-2x+x^2-x^5}=\sum_{n=0}^{\infty}g_n x^n$$

By theorem 4.8, g_n satisfies the linear recurrence relation with initial conditions given by

$$g_n-2g_{n-1}+g_{n-2}-g_{n-5}=\begin{cases}1, & n=0\\ -2, & n=1\\ 2, & n=2\\ -1, & n=4\\ 1, & n=5\\ 0, & \text{otherwise}\end{cases}$$

Hence we have

$$g_0=1 \qquad g_3=2$$
$$g_1=0 \qquad g_4=2$$
$$g_2=1 \qquad g_5=4$$

$$g_n=2g_{n-1}-g_{n-2}+g_{n-5},\ n\geq 6$$

3.7 a) The expression 1* ((00)*001 ⌣ (00)*011)* is the block decomposition for this set of strings. This leads to the generating series

$$\frac{1}{1-x}\left(\frac{1}{1-\left(\frac{x^3}{1-x^2}+\frac{x^3}{1-x^2}\right)}\right)=\frac{1+x}{1-x^2-2x^3}$$

b) By theorem 4.8, h_n satisfies the linear recurrence relation with initial conditions given by

$$h_n-h_{n-2}-2h_{n-3}=\begin{cases}1, & n=0\\ 1, & n=1\\ 0, & n\geq 2\end{cases}$$

Hence we have

$$h_0, h_1, h_2=1$$

$$h_n=h_{n-2}+2h_{n-3},\ n\geq 3$$

3.8 a) K can be described by the recursive expression
S = ε ⌣ 0 S 1*1
K = 1* S* 0*

$$S(x) = 1 + \frac{x^2 S(x)}{1-x}$$

$$S(x) = \frac{1-x}{1-x-x^2}$$

$$K(x) = \frac{1}{1-x} \cdot \frac{1}{1-S(x)} \cdot \frac{1}{1-x} = \frac{1-x-x^2}{-x^2+2x^3-x^4}$$

b) $$K(x) = -\frac{1}{x^2}\left(\frac{1-x-x^2}{1-2x+x^4}\right) = -\frac{1}{x^2}\sum_{n=0}^{\infty} k_n x^n$$

By theorem 4.8, k_n satisfies the linear recurrence relation with initial conditions given by

$$k_n - 2k_{n-1} + k_{n-4} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ -1, & n=2 \\ 0, & n \geq 3 \end{cases}$$

Hence we have

$$k_0, k_1, k_2 = 1$$

$$k_3 = 2$$

$$k_4 = 4$$

$$k_n = 2k_{n-1} - k_{n-4}, \quad n \geq 5$$

$$[x^n]K(x) = k_{n+2}$$

3.9 a) The expression

$$\bigcup_{n=0}^{m} \{0\}^n \left(\bigcup_{n=1}^{m} \{1\}^n \bigcup_{n=1}^{m} \{0\}^n\right)^* \bigcup_{n=0}^{m} \{1\}^n$$

is the block decomposition for this set of strings. This leads to the generating series

$$\Phi(x) = \left[\sum_{n=0}^{m} x^n\right]\left(\frac{1}{1-\left(\left[\sum_{n=1}^{m} x^n\right]\left[\sum_{n=1}^{m} x^n\right]\right)}\right)\left[\sum_{n=0}^{m} x^n\right] = \frac{1-2x^{m+1}+x^{2m+2}}{1-2x+2x^{m+2}-x^{2m+2}}$$

b) The expression

$$\bigcup_{n=0}^{m} \{0\}^n \left(\bigcup_{n=1}^{k} \{1\}^n \bigcup_{n=1}^{m} \{0\}^n\right)^* \bigcup_{n=0}^{k} \{1\}^n$$

is the block decomposition for this set of strings. This leads to the generating series

$$\Phi(x) = \left[\sum_{n=0}^{m} x^n\right]\left(\frac{1}{1-\left(\left[\sum_{n=1}^{k} x^n\right]\left[\sum_{n=1}^{m} x^n\right]\right)}\right)\left[\sum_{n=0}^{k} x^n\right] = \frac{(1-x^{k+1})(1-x^{m+1})}{1-2x+x^{m+2}+x^{k+2}-x^{k+m+2}}$$

3.10 a) The expression (ε ⌣ (11)*1) ((0 ⌣ 00) (11)*1)* 0* is the block decomposition for this set of strings. It specifies the strings which start with an empty string or an odd number of 1s, followed by either one or two 0s then an odd number of ones, then followed by 0 or more occurrences of 0.

b) The generating series for the expression is

$$\left(1+\frac{x}{1-x^2}\right)\left(\frac{1}{1-\left((x+x^2)\frac{x}{1-x^2}\right)}\right)\left(\frac{1}{1-x}\right) = \frac{1-x+x^2}{1-x-2x^2+x^3+x^4}$$

3.11 a) The expression 1* (01*1 ⌣ 001 ⌣ 0011)* (ε ⌣ 0 ⌣ 00) is the block decomposition for this set of strings. It specifies the strings starting with any number of 1s, followed by blocks where there is either only one 0 or there are less than three 1s, and ending with two or less 0s.

b) $$\Phi(x) = \frac{1}{1-x}\left(\frac{1}{1-\left(\frac{x^2}{1-x}+x^3+x^4\right)}\right)(1+x+x^2)$$

$$= \frac{1+x+x^2}{1-x-x^2-x^3+x^5}$$

3.12 a) The expression (ε ⌣ (00)*0) ((1 ⌣ 11) (00)*0)* (ε ⌣ 1 ⌣ 11) is the block decomposition for this set of strings.

$$\Phi(x) = \left(1+\frac{x}{1-x^2}\right)\left(\frac{1}{1-\left((x+x^2)\frac{x}{1-x^2}\right)}\right)(1+x+x^2)$$

$$= \frac{1+2x+x^2-x^4}{1-2x^2-x^3}$$

b) $$|N_n| = [x^n]\ -2+x+\frac{3+x-3x^2}{1-2x^2-x^3} = [x^n]-2+x+\sum_{j=0}^{\infty} k_j x^j$$

By theorem 4.8, k_j satisfies the linear recurrence relation with initial conditions given by

$$k_j - k_{j-2} - k_{j-3} = \begin{cases} 3, & j=0 \\ 1, & j=1 \\ -3, & j=2 \\ 0, & j\geq 3 \end{cases}$$

Hence we have

$$k_0 = 3 \qquad k_2 = 0$$
$$k_1 = 1$$
$$k_j = k_{j-2} + k_{j-3}, \quad j \geq 3$$

$$|N_n| = \begin{cases} 1 & , n=0 \\ 2 & , n=1 \\ 0 & , n=2 \\ |N_{n-2}| + |N_{n-3}| & , n \geq 3 \end{cases}$$

3.13 a) The expression 1* ((00)*0(11)*1 ⌣ (00)*00(11)*11)* is the block decomposition for this set of strings.

$$P(x) = \frac{1}{1-x}\left(\frac{1}{-\left(\left(\frac{x}{1-x^2}\right)^2 + \left(\frac{x^2}{1-x^2}\right)^2\right)}\right) = \frac{1+x-x^2-x^3}{1-3x^2}$$

b) By theorem 4.8, p_n satisfies the linear recurrence relation with initial conditions given by

$$p_n - 3p_{n-2} = \begin{cases} 1, & 0 \leq n \leq 1 \\ -1, & 2 \leq n \leq 3 \\ 0, & n \geq 4 \end{cases}$$

Hence we have

$$p_0, p_1 = 1$$
$$p_2, p_3 = 2$$
$$p_n = 3p_{n-2}, \quad n \geq 4$$
$$\downarrow$$
$$p_4 = 2 \cdot 3^1$$
$$p_5 = 2 \cdot 3$$
$$p_6 = 2 \cdot 3^2$$
$$\vdots$$
$$p_n = 2 \cdot 3^{\lfloor n/2 \rfloor - 1}$$

3.14 a) There are no ways for 11000 to overlap itself nontrivially. Hence we have C(x) = 0.
By Theorem 3.26, we have

$$Q(x) = \frac{1}{1-2x+x^5}$$

b) The generating series for a single part is x + x^2 + x^3 + x^4. By the string lemma,

$$R(x) = \frac{1}{1-(x+x^2+x^3+x^4)} = \frac{1-x}{1-2x+x^5}$$

c) $Q(x) - xQ(x) = (1-x)Q(x) = R(x)$

d) ???

3.15 0110 overlaps itself, with the ending 110. Hence we have C(x) = x^3.

$$V(x) = \frac{1+x^3}{(1-2x)(1+x^3)+x^4}$$

3.16 a) 0101 overlaps itself, with the ending 01. Hence we have C(x) = x^2.

$$\Phi(x) = \frac{1+x^2}{(1-2x)(1+x^2)+x^4} = \frac{1+x^2}{1-2x+x^2-2x^3+x^4}$$

b) The string (01)^r will overlap itself, with (01), (01)^2,...(01)^(r-1). Hence we have

$$C(x) = \sum_{k=1}^{r-1} x^{2k} = \frac{x^{2r}-x^2}{x^2-1}$$

$$\Phi(x) = \frac{1-C(x)}{(1-2x)(1+C(x))+x^{2r}} = \frac{1-2x^2+x^{2r}}{1-2x-x^{2r+2}+2x^{2r+1}}$$

3.17 The recursion R produces either the string B, or the string A prefixing R. Hence, it either terminates the recursion with B, or prefixes the string with any number of As, which is equivalent to S = A*B.

$$S(x) = \frac{1}{1-A(x)} B(x) = \frac{B(x)}{1-A(x)}$$

$$R(x) = B(x) + A(x)R(x)$$

$$R(x) = \frac{B(x)}{1-A(x)}$$