

Exercises 5

$$\begin{aligned}
 1. a) \quad W^{-nk} + W^{-(N-n)k} &= W^{-nk} + W^{-Nk} W^{nk} \\
 &= W^{-nk} + W^{nk} = e^{\frac{-2\pi i nk}{N}} + e^{\frac{2\pi i nk}{N}} \\
 &= \cos\left(\frac{-2\pi nk}{N}\right) + i\sin\left(\frac{-2\pi nk}{N}\right) + \cos\left(\frac{2\pi nk}{N}\right) + i\sin\left(\frac{2\pi nk}{N}\right) \\
 &= \cos\left(\frac{2\pi nk}{N}\right) - i\sin\left(\frac{2\pi nk}{N}\right) + \cos\left(\frac{2\pi nk}{N}\right) + i\sin\left(\frac{2\pi nk}{N}\right) \\
 &= 2\cos\left(\frac{2\pi nk}{N}\right)
 \end{aligned}$$

b) [Assignment 4 question]

$$\begin{aligned}
 F_k &= \sum_{n=0}^{N-1} f_n W^{-nk} = \sum_{n=0}^{N/2-1} f_n W^{-nk} + \sum_{n=N/2}^{N-1} f_n W^{-nk} \\
 &= \sum_{n=0}^{N/2-1} f_n W^{-nk} + \sum_{n=1}^{N/2} f_{N-n} W^{-(N-n)k} \\
 &= f_0 + f_{\frac{N}{2}} (-1)^k + \sum_{n=1}^{N/2-1} f_n W^{-nk} + f_n W^{-(N-n)k} \\
 &= f_0 + f_{\frac{N}{2}} (-1)^k + \sum_{n=1}^{N/2-1} f_n (W^{-nk} + W^{-(N-n)k}) \\
 &= f_0 + f_{\frac{N}{2}} (-1)^k + \sum_{n=1}^{N/2-1} 2f_n \cos\left(\frac{2\pi nk}{N}\right)
 \end{aligned}$$

We can clearly see that F_k is real.

$$\begin{aligned}
 2. F_{2k} &= \sum_{n=0}^{N-1} f_n W^{-2nk} = \sum_{n=\frac{N}{4}}^{\frac{3N}{4}-1} W^{-2nk} \\
 &= \frac{W^{k(2-\frac{3N}{2})} (W^{Nk} - 1)}{W^{2k} - 1} \\
 &= 0
 \end{aligned}$$

3. We have that $W^0 = 1$, so F_0 is just equal to the sum of the f_i s.

4. Let F_k be the DFT of the original signal $f_0 \dots f_{N-1}$. Let G_k be the DFT of the new signal.

For $k = 0$:

$$G_0 = F_0 + c$$

For $1 \leq k \leq N-1$:

$$\begin{aligned}
 G_k &= \sum_{n=0}^{N-1} (f_n + c) W^{-nk} = F_k + c \sum_{n=0}^{N-1} W^{-nk} \\
 &= F_k + c \left(\frac{1 - W^{-Nk}}{1 - W^{-k}} \right) \\
 &= F_k
 \end{aligned}$$

- 5.
- i. $F[k] = (6, -1 - i, 0, -1 + i)$
 - ii. $F[k] = (8, -2, 0, -2)$
- I have no idea why it's real

6. $\frac{1}{N} \sum_{k=0}^{N-1} F_k \bar{F}_k = \sum_{n=0}^{N-1} f_n \bar{f}_n \leftarrow$ Correct version that our slides cover

$$\begin{aligned}
 \frac{1}{N} \sum_{k=0}^{N-1} F_k \bar{F}_k &= \frac{1}{N} \sum_{k=0}^{N-1} F_k \overline{\sum_{n=0}^{N-1} f_n W^{-nk}} \\
 &= \frac{1}{N} \sum_{k=0}^{N-1} F_k \sum_{n=0}^{N-1} \bar{f}_n W^{nk} \\
 &= \frac{1}{N} \sum_{n=0}^{N-1} \bar{f}_n \sum_{k=0}^{N-1} F_k W^{nk} \\
 &= \frac{1}{N} \sum_{n=0}^{N-1} \bar{f}_n N f_n \\
 &= \sum_{n=0}^{N-1} f_n \bar{f}_n
 \end{aligned}$$

7. Properties of Fourier transform:

- Signal that is purely real has $F_k = \overline{F_{N-k}}$
- Signal that is purely imaginary has $F_k = -\overline{F_{N-k}}$

Our algorithm works as follows. We multiply the second signal by j and add it to the first. Then we take the FFT of this sum.

The FFT of the first signal is $F'_k = \frac{F_k + \overline{F_{N-k}}}{2}$. The contributions of the first (real) signal will add, and the contributions of the second (imaginary) signal will cancel out.

The FFT of the second signal is $F''_k = j \frac{F_k - \overline{F_{N-k}}}{2}$. The contributions of the first (real) signal will cancel out, and the contributions of the second (imaginary) signal will add. We also need to multiply by j , since our FFT has real and imaginary parts swapped (since we multiplied our second signal by j originally).