MATH 239 Exercises 8

8.2.1 We prove this by induction on the number of vertices. Let T be a tree on n vertices.

Base cases: A tree on 1 vertex has no perfect matchings, and a tree on two vertices has one perfect matching.

Inductive hypothesis: Assume for all 2 < k < n, a tree on k vertices has at most one perfect matching. Consider a tree on n vertices. Let v be a leaf of T and u be its parent. The edge uv must be in any perfect matching, since the matching must include v and uv is the only edge incident to v. Hence the number of perfect matchings in T equals the number of perfect matchings in T \ {u, v}. The graph T \ {u, v} is a forest, where each component has less than n vertices. Hence each component is a tree with at most one perfect matching. The number of perfect matchings in a graph is the product of the number of perfect matchings in each component, and hence T \ {u, v} has at most one perfect matching, so T has at most one perfect matching.

8.2.2 For Kn, if n is odd, there are 0 perfect matchings. If n is even, there are (n!/(2^(n/2) (n/2)!) ways to partition the n vertices into pairs. Hence there are

$$\frac{n!}{2^{n/2}(n/2)!}$$ perfect matchings.

For Km,n, if m != n, there are 0 perfect matchings. If m = n, there are n! permutations of n, so there are n! perfect matchings.

8.2.3 We find a recurrence for a_n, where a_n denotes the number of perfect matchings of L_n. For n > 2, we have that we can add a pair of vertices to any perfect matching in a_(n-1). The only perfect matching we can create is to connect the two vertices with a vertical line, since every other vertex is already part of an edge in the perfect matching. We can also take any perfect matching in a_(n-2), and add two pairs of vertices. There are 2 perfect matchings in two pairs of vertices where they are not connected by vertical edges, so we have that for each perfect matching in a_(n-2), there are 2 ways to add a two pairs of vertices. Hence we have

$$a_n = a_{n-1} + 2a_{n-2}, \quad n \geq 3$$
$$a_1 = 1$$
$$a_2 = 3$$

The characteristic polynomial is

$$1 - x - 2x^2 = (1+x)(1-2x)$$

Hence the solution is of the form

$$A(-1)^n + B \cdot 2^n$$

Substituting n=1 and n=2, we obtain

$$1 = -A + 2B$$

$$3 = A + 4B$$

Solving, we obtain A=1/3, B=2/3. Hence we have

$$a_n = \frac{1}{3}(-1)^n + \frac{2}{3} \cdot 2^n$$

8.2.4 Each vertex in an n-cube is a bitstring (b1,..bn). We can construct a perfect matching by selecting the edge between every vertex (0, b2,..bn) and (1, b2,…bn). This covers every vertex of the n-cube, and each edge only has one end in any vertex.

8.2.5 Each row has 8 squares. We can cover each row with 4 dominos, lined up end to end. Repeat this for each row, and hence we have covered the board in 32 dominos.

8.2.6 Each domino on the board must cover exactly one white and one black square. Removing two opposite corner squares either removes two white or two black squares. Hence there is an uneven number of white and black squares, and we cannot cover them with 31 dominos.

8.2.7 Let G be a graph with an even number of vertices and a Hamilton cycle. We can create a perfect matching by selecting alternating edges in the cycle. Since there is an even number of vertices, every vertex will be paired with exactly one other vertex that is adjacent to it.

8.2.8 If u and v are adjacent to each other in the Hamilton cycle, we can create a perfect matching in G by selecting the alternating edges in the cycle where the edge uv is selected (so no other edges are incident with u or v). Then, H has the same perfect matching except without edge uv.

If u and v are not adjacent to each other in the Hamilton cycle, they must have an even number of vertices in both paths from u to v in the cycle, since adjacent vertices are in different partitions. Hence we can create a perfect matching in G by selecting alternating edges in each path, where neither endpoint u or v is incident with an edge. Since there are an even number of edges in each path, every vertex (except u and v) are in the matching.

8.2.9 The chessboard (without squares removed) can be modeled as a bipartite graph with an even number of vertices. Each square is a vertex and adjacent squares are connected by an edge. Hence we have a bipartition between black and white squares. The graph has a Hamilton cycle. Covering the board with dominos is equivalent to finding a perfect matching. Removing two squares with opposite colours from the chessboard is equivalent to removing two vertices in different partitions (and their edges) from the board. Hence the resulting graph has a perfect matching, and can be covered by 31 dominos.

8.2.10 We prove the statement by induction on n. Our base case, n=2, is true, since it consists of the vertices 1 and 2, which sum to 3 and hence the two are adjacent and that is our perfect matching.

Assume that the statement is true for any 2 < k < n. Observe the prime graph B_n. If n is odd, we are done. If n is even, there must exist a prime number p between n and 2n. We must have that p = n + k for some odd number 1 <= k <= n. Then the set of integers {k, k+ 1,...n-1, n} can be partitioned into pairs {k, 2n}, {k+1, 2n-1},.. and so on up to {n+floor(k/2), n+ceil(k/2)}. Each of the pairs sums to p=n+k. By the induction hypothesis, the remaining set of integers {1,2,..k-1} has a perfect matching. Thus, the B_n must have a perfect matching.

8.2.11 If C is a cover, every edge of G has at least one end in C. Hence G \ C must have no edges, since removing all edges in C removes all edges incident to a vertex in C, which is every edge. Hence V(G) \ C is a set of pairwise non-adjacent vertices.

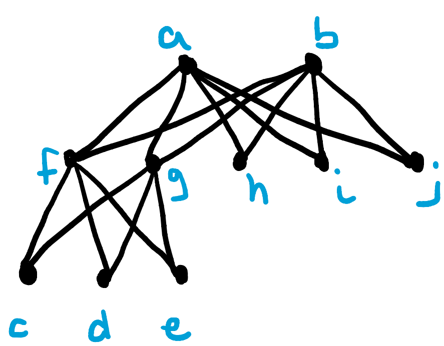
If V(G) \ C is a set of pairwise nonadjacent vertices, it must be that G \ C has no edges. Hence every edge in G must be incident with a vertex in C, and C must be a cover.

8.2.12 We provide a counterexample. Observe K4. It has matchings of size |M|=1 or |M|=2. However, every cover must be of size at least |C|=3. Hence there does not exist an M and C of the same size.

8.2.13 We can formulate this as a graph problem as follows. Let G be the graph representing this problem, with bipartition X, Y. Each non-zero entry in N is a vertex in G in partition X. Each row and column in N is a vertex in G in partition Y. A vertex in X is connected to a vertex in Y if and only if the entry is in the row (or column) of vertex Y. A matching is hence a set of pairs of entries and rows/columns such that each row/column is only matched with one entry. Hence to find the largest-set of non-zero entries with this property, we would like to find the maximum matching in G.

8.2.14 A cover of G is a set of vertices where every edge has at least one end in C, which represents the set of rows and columns that the non-zero entries in N cover.

8.2.15 This graph satisfies all the statements. It has no perfect matching because in a perfect matching, vertices i and j must be adjacent to either a and b. Hence edges f, g, and h must be adjacent to vertices c, d, and e in the perfect matching. But vertex h is only adjacent to vertices a and b, so there cannot be a perfect matching.



8.2.16 Let G be a graph with p=2n vertices and deg(v) >= n for every vertex v. Consider any matching M. If M is perfect, then we are done.

If M is not perfect, then there must exist distinct vertices u and v that are not saturated by M. If u and v are adjacent, then there is an augmenting path of length 1. We can connect these two vertices to create a larger matching. We can continue doing this until M is a matching with no pairs of adjacent vertices not saturated by M. If M is perfect, then we are done.

Otherwise, there must exist distinct vertices u and v are not adjacent and not saturated by M. They must each be adjacent to at least n other vertices. Since every vertex has deg(v) >= n, vertex u must be adjacent to some vertex x, and v adjacent to some vertex y, such that xy is in M. Hence there is an augmented path of length 3, and by Theorem 8.1.1, there is a M' that is larger than M. We can repeat this for all non-adjacent pairs to create a perfect matching.

8.2.17 Let x be the number of vertices incident to edges in M', and y be the number of vertices incident to edges in M. Suppose that |M'| > 2|M|. Then we must have x > 2y. Since at most 2y edges have at least one vertex in M and x > 2y, there exists at least one edge e in M' that have no vertices in M. But this means that e can be added to M to make a larger matching, contradicting the statement that M is not contained in any larger matchings. Hence we must have that |M'| <= 2|M|.