

MATH 239 F2008 Final

1. Each part must not be equal to 1; the allowed parts are $P = \{2, 3, 4, \dots\}$. Hence the generating series for a part is

$$\Phi_p(x) = \frac{x}{1-x}$$

The generating function for compositions of n with an even number of parts is

$$\sum_{i=0}^{\infty} 2^i \Phi_p(x) = 2 \sum_{i=0}^{\infty} \Phi_p(x) = 2 \frac{1}{1-\frac{x}{1-x}} = \frac{2(1-x)}{1-2x}$$

2. a) i. The expression $(\epsilon - 0 - 0) ((1 - 11) (0 - 00))^* (\epsilon - 1 - 11)$ is the block decomposition for this set of strings.
- ii. The expression $(\epsilon - 0^*000) ((1^*111) (0^*000))^* (\epsilon - 1^*111)$ is the block decomposition for this set of strings.

b) 010000

- c) The elements are not uniquely created. The string $00.00.00 = 0.0.0.0.0$ can be created these two ways.

d) $\Phi_3(x) = \frac{1}{1-(x+x^2+x^3)}$

3. By theorem 4.8, a_n satisfies the linear recurrence relation with initial conditions given by

$$a_n - 2a_{n-1} + a_{n-2} - a_{n-3} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 1, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

$$a_0, a_1, a_2 = 1$$

$$a_3 = 3$$

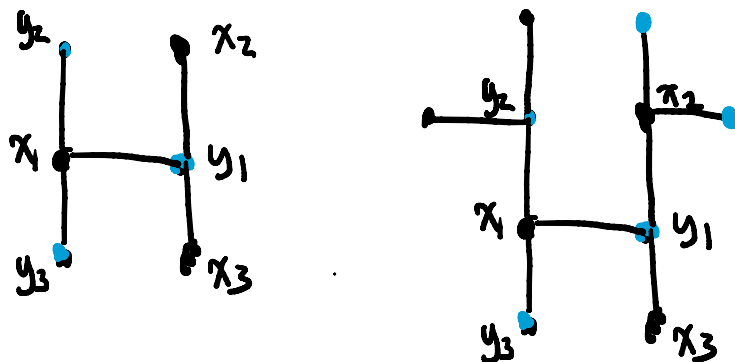
$$a_n = 2a_{n-1} - a_{n-2} + a_{n-3}, \quad n \geq 4$$

4. Let G be a connected graph. We prove that if an edge e is a bridge, it belongs to every spanning tree of G . Let e be an edge in G that is a bridge. Suppose that e does not belong to every spanning tree of G . Let T be a spanning tree not containing e . Then the tree T is a spanning subgraph of $G \setminus e$. Let u and v be any two vertices of $G \setminus e$. There must be a unique path from u to v in T , which is also a unique path from u to v in $G \setminus e$. Then $G \setminus e$ must be connected, which is a

contradiction, since e is a bridge. Hence e must belong to every spanning tree of G .

We prove that if an edge e belongs to every spanning tree of G , it is a bridge. Let e be an edge belonging to every spanning tree of G . Suppose e is not a bridge. Then $G \setminus e$ is connected, and $G \setminus e$ must have a spanning tree T . Since $V(G) = V(G \setminus e)$, T is also a spanning tree of G . This is a contradiction, since T does not contain e and e belongs to every spanning tree of G . Hence if e belongs to every spanning tree of G , it must be a bridge.

5. a) Since G is bipartite, G must not contain any odd cycles. Let X, Y be a bipartition of G . Let x_1 be a vertex in G in X . It must have degree at least 3, so it must have 3 neighbours in Y . Let y_1, y_2, y_3 be these neighbours. y_1 must have 3 neighbours in X , one of which is x_1 . Let the other two neighbours be x_2 and x_3 . If y_2 is adjacent to x_2 or x_3 , we have a path of length 5. Otherwise, x_2 and y_2 must each have 2 distinct neighbours, and we have a path of length 5.



- b) $K_{3,3}$ is a bipartite graph with degree at least 3 with no path of length 6.

- c) From part a, G must have a path of length at least 5. Suppose G does not contain a path of length 6. Let $P = \{v_1, v_2, v_3, v_4, v_5\}$ be a path of length 5. P must alternate between vertices in bipartitions X and Y . Without loss of generality, let v_1 be in X . v_1 is adjacent to v_2 , and since G has degree at least 3, it must be adjacent to 2 other vertices. These 2 other vertices must be in P , otherwise we could extend P to length 6 by adding the vertex to it. But this is a contradiction, since P is of length 5 and only has 2 vertices in Y , one of which is v_2 . Hence G must contain a path of length at least 6.



6. a) Euler's formula states that $p - q + f = 2$ for a connected planar graph. Suppose P has k components. Then we have

$$\sum_{i=1}^k (|V_i| - |E_i| + f_i) = 2k$$

$$|V| - |E| + f = 2k \geq 2 \quad (\text{since } k \geq 1)$$

$$102 - 300 + f \geq 2$$

$$f \geq 200$$

b) Suppose P does not contain any cycles. Then we have $E = V - k$, where k is the number of components in P . Hence we have $E > V$. This is a contradiction, so P must contain at least one cycle.

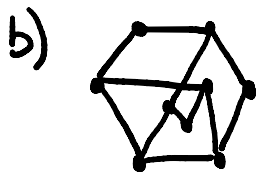
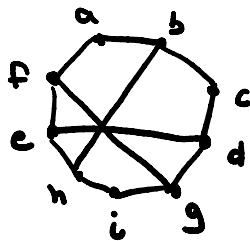
c) Since P contains cycles, the boundary of each face of P must contain a cycle. Hence there must be at least 3 edges in every face, so it must have degree at least 3.

d) From part a, we have that

$$102 - 300 + f = 2k$$

where k is the number of components. If $k > 1$, then we have $f < 200$, which is a contradiction. Hence there must only be one component, so P must be connected.

7. a) H is not planar. This subgraph of H contains an edge subdivision of $K_{3,3}$, so by Kuratowski's theorem, is nonplanar.



8. a) 1, 2, 4, 3, 5, 11, 7, 6, 8, 10, 12, 14, 9, 17, 13, 15, 16, 18

b) $X_0: 7, 9$
 $X: 7, 9$
 $Y:$

$X: 7, 9$

$Y: 4, 6, 8$

$X: 7, 9, 1, 5, 17$

$Y: 4, 6, 8, 2, 14, 16, 18$

$X: 7, 9, 1, 5, 17, 3, 11, 13, 15$

$Y: 4, 6, 8, 2, 14, 16, 18, 10$

10 is unsaturated, we create new matching with augmenting path 7 8 17 14 11 1

X0: 9

X: 9

Y:

X: 9

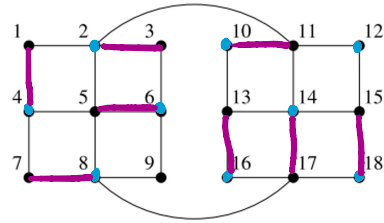
Y: 6, 8

X: 9, 5, 7

Y: 6, 8, 2, 4

X: 9, 5, 7, 3, 1

Added no vertices to Y, we terminate.



Our maximum matching is

$\{(1,4), (2,3), (5,6), (7,8), (10,11), (13,16), (14,17), (15,18)\}$.

Our minimum cover is $\{2,4,6,8,11,13,15,17\}$

9.

$D = \{1, 3\}$

$N(D) = \{a, c, f\}$

$|N(D)| = 2 < 3 = |D|$