MATH213 W2022 Midterm

1. $y = \sqrt{r^2 - x^2}$, $y' = -\frac{x}{\sqrt{r^2-x^2}}$, $y'' = -\frac{r^2}{(r^2-x^2)^{3/2}}$

Using the Taylor Series centered at x=0, we obtain

$$y(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2}$$

$$= r + 0 - \frac{x^2}{2r}$$

$$= r - \frac{1}{2r}x^2$$

$a = -\frac{1}{2r}$, $b = 0$, $c = r$

2. $0 = 3x(t\to\infty) - 5x^2(t\to\infty)$

$x(t\to\infty) = 3/5$

The steady state solution is 3/5, or 0.6

3. $\frac{dN}{dt} = -\lambda N(t)$

$$\int \frac{1}{N(t)}\,dN = \int -\lambda\,dt$$

$$\ln(N(t)) + C_1 = -\lambda t + C_2$$

$$\ln(N(t)) = -\lambda t + C_3$$

$$N(t) = e^{-\lambda t + C_3} = Ce^{-\lambda t}$$

Since we have that the half-life of I-131 is 8 days we have that after 8 days, a sample of 1 I-131 will become 1/2. The unit for λ is **days^-1**. Hence we have

$$N(0) = 1 = Ce^{-\lambda \cdot 0} = C \quad \rightarrow C = 1$$

$$N(8) = 1/2 = e^{-8\lambda}$$

$$\frac{1}{2} = e^{-8\lambda}$$

$$\lambda = \frac{\ln(2)}{8}$$

The number of days for I-131 atoms to decrease to 1/10 of their initial value is

$$\frac{1}{10} = e^{-\frac{\ln(2)}{8}t}$$

$$t = \frac{8\ln(10)}{\ln(2)} \approx 26.6 \text{ days}$$

4. $$\mathcal{L}^{-1}\left\{\frac{2(s+2)}{s^2+5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+3)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}$$

We know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = u_{-1}(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}u_{-1}(t)$$ By the exponential modulation property

Hence we have

$$\mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= 2e^{-3t}u_{-1}(t)$$

5. We have that

$$X(0) = \int_{-\infty}^{\infty} x(t)\, e^{-t(0)}\, dt = \int_{-\infty}^{\infty} x(t)\, dt$$

Hence **X(0) = 6**.

6. $\mathcal{L}\{x(t)\} = \mathcal{L}\{3e^{-2t}u_{-1}(t)\}$

$= 3\mathcal{L}\{e^{-2t}u_{-1}(t)\}$ By linearity

$= \dfrac{3}{s+2}, \ Re(s) > -2$ By exponential modulation

$\mathcal{L}\{x(-t)\} = \mathcal{L}\{3e^{2t}u_{-1}(-t)\}$

$= 3\int_{-\infty}^{\infty} e^{-st}e^{2t}u_{-1}(-t)dt$

$= 3\int_{-\infty}^{0} e^{(2-s)t}dt$

$= \dfrac{3}{2-s}, \ Re(s) < 2$

$\mathcal{L}\{g(t)\} = \mathcal{L}\{x(t) + 4x(-t)\}$

$= \mathcal{L}\{x(t)\} + 4\mathcal{L}\{x(-t)\}$

$= \dfrac{3}{s+2} + \dfrac{12}{2-s}, \ -2 < Re(s) < 2$

Since g(t) is a linear combination of x(t) and x(-t), the ROC of G(s) is **-2 < Re(s) < 2**.

7. $\mathcal{L}\{e^{-5t}u_{-1}(t-3)\} = \int_{-\infty}^{\infty} e^{-st}e^{-5t}u_{-1}(t-3)dt$

$= \int_{3}^{\infty} e^{-(s+5)t}dt$

$= \dfrac{e^{-3(s+5)}}{s+5}, \ Re(s) > -5$

8. $\mathcal{L}\left\{\frac{dx}{dt}\right\} = \mathcal{L}\left\{-3y(t) + 2\delta(t)\right\}$

$$sX(s) - x(0^-) = -3Y(s) + 2$$

1 $sX(s) + 3Y(s) = 3$

$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\left\{5x(t)\right\}$

$$sY(s) - y(0^-) = 5X(s)$$

2 $X(s) = \frac{sY(s)}{5}$

$$\frac{s^2Y(s)}{5} + 3Y(s) = 3$$

$$Y(s) = \frac{15}{s^2+15}$$

$$X(s) = \frac{3s}{s^2+15}$$