

$$1. a) [x^{10}] \frac{x^3}{(1+3x)^6} = [x^{10}] x^3 \sum_{n=0}^{\infty} \binom{n+5}{5} (-3)^n x^n$$

$$= [x^7] \sum_{n=0}^{\infty} \binom{n+5}{5} (-3)^n x^n$$

$$= \binom{12}{5} (-3)^7$$

$$b) (1-x)^{-(k+1)} (1-x)^{-1} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n+k}{k} x^n = \sum_{n=0}^{\infty} \binom{n+k+1}{k+1} x^n = (1-x)^{-(k+2)}$$

$$\sum_{m=0}^n \binom{n+k}{k} = \binom{n+k+1}{k+1} \quad \text{for } p=n+k$$

$$\sum_{m=k}^p \binom{m}{k} = \binom{p+1}{k+1}$$

2. a) The generating function for compositions with even parts is

$$\Phi_E(x) = \frac{x^2}{1-x^2}$$

The generating function for compositions with odd parts is

$$\Phi_O(x) = \frac{x}{1-x^2}$$

We have that there are 8 parts, where the first and last parts are even, and all other parts are odd. Hence by the product lemma, the generating function for the number of compositions is

$$\Phi_E(x) [\Phi_O(x)]^6 \Phi_E(x) = \left(\frac{x^2}{1-x^2} \right)^2 \left(\frac{x}{1-x^2} \right)^6$$

$$= \frac{x^{10}}{(1-x^2)^8}$$

The number of compositions of n is

$$[x^n] \frac{x^{10}}{(1-x^2)^8} = [x^{n-10}] \sum_{n=0}^{\infty} \binom{n-7}{7} x^{2n} \\ = \binom{\frac{n-10}{2}-7}{7}$$

- b) The generating function for a composition with even parts where every part is greater than or equal to 4 is

$$\Phi_E(x) = \frac{x^4}{1-x^2}$$

By the string lemma, we have that the generating function for the number of compositions of n where every part is an even number greater than or equal to 4 is

$$\frac{1}{1 - \frac{x^4}{1-x^2}} = \frac{1-x^2}{1-x^2-x^4}$$

3. a) The expression $1^* ((00)^*0011^* \cup (00)^*011(11)^*)^* (\epsilon \cup (00)^*00)$ is the block decomposition for this set of strings.

- b) The expression $0^* (100^* \cup 1100^* \cup 1111^*0)^* 1^*$ is the block decomposition for this set of strings.

$$c) \Phi(x) = \frac{1}{1-x} \left(\frac{1}{1 - \left(\frac{x^2}{1-x^2} \cdot \frac{x^2+x^3}{1-x^3} \right)} \right) x$$

$$4. a) a_n - 5a_{n-1} + 6a_{n-2} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 0, & n \geq 2 \end{cases}$$

$$a_0 = 1$$

$$a_1 = 4$$

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad n \geq 2$$

$$b) \quad b_0 = 11$$

$$b_1 - 11b_0 = -92$$

$$b_2 - 11b_1 + 40b_0 = 188$$

$$b_n = [x^n] \frac{11 - 92x + 188x^2}{1 - 11x + 40x^2 - 48x^3}$$

$$1 - 11x + 40x^2 - 48x^3 = (1 - 3x)(1 - 4x)^2$$

By partial fractions, we have

$$\frac{11 - 92x + 188x^2}{(1 - 3x)(1 - 4x)^2} = \frac{A}{1 - 3x} + \frac{B}{1 - 4x} + \frac{C}{(1 - 4x)^2}$$

$$11 - 92x + 188x^2 = A(1 - 4x)^2 + B(1 - 3x)(1 - 4x) + C(1 - 3x)$$

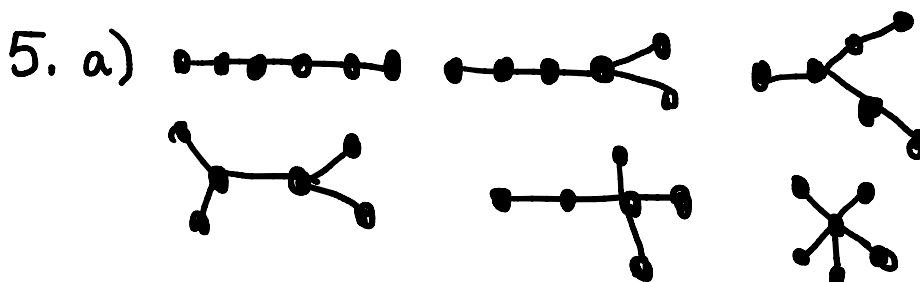
$$A = 11 \quad B = 1 \quad C = -1$$

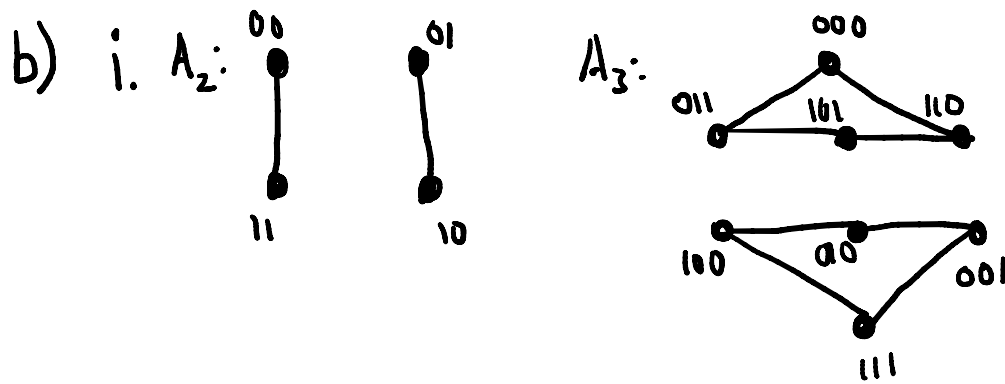
$$\frac{11 - 92x + 188x^2}{(1 - 3x)(1 - 4x)^2} = \frac{11}{1 - 3x} + \frac{1}{1 - 4x} - \frac{1}{(1 - 4x)^2}$$

$$= \sum_{n=0}^{\infty} 11 \cdot 3^n x^n + 4^n x^n - (n+1) 4^n$$

$$= \sum_{n=0}^{\infty} (11 \cdot 3^n + 4^n(-n)) x^n$$

$$b_n = 11 \cdot 3^n - 4^n n$$





ii. For each vertex string, there are $n-1$ strings of length n that differ in exactly 2 positions. Hence we have

$$|E(A_n)| = \frac{1}{2} \sum_{v \in V} \deg(v) = \frac{1}{2} 2^n (n-1) = 2^{n-1} (n-1)$$

iii. Let x and y be two strings of length n that differ by 2 characters. There are 3 possibilities for the 2 differing characters:

We can have two 0s from x changed to 1 in y . In this case, the parity of 0s in both strings are the same.

We can have two 1s from x changed to 0 in y . This is symmetrical to the case above; the parity of 0s in both strings are the same.

We can change one 1 to a 0, and one 0 to 1 from x to y . In this case, the parity of 0s in both strings are the same.

Hence in all cases, the parity of 0s in x and y are the same.

We then have that in A_n , every string with an even parity of zeros is connected, and every string with an odd parity of zeros is connected.

Hence A_n has 2 components, and is not connected.

6.a) Let p be the number of vertices in the tree. Since the tree is cubic, there are $(k + 3(p-k))/2$ edges. Since it is a tree, it must have $p-1$ edges. Hence we have

$$p-1 = \frac{k + 3(p-k)}{2}$$

$$2p-2 = k + 3p-3k$$

$$p = 2k-2 = 2(k-1)$$

b)

Since G has no cycles, none of its components can have cycles. Hence every component is a tree. For each component T_i where $1 \leq i \leq c$, we have that

$$\sum_{i=1}^c |E(T_i)| = \sum_{i=1}^c (|V(T_i)| - 1) = -c + \sum_{i=1}^c |V(T_i)| = p - c$$