Exercises 6

1. True. $||x||_1 = \sum_{i=1}^{n} |x_i| \geq \max|x_i| = ||x||_\infty$

2. True.

3. Let B = PAP, where P is the permutation matrix with 1s on the anti-diagonal (note: this means $P = P^T = P^{-1}$). B is orthogonally similar to A. We can find the LU factorization of B, B = LU. Then by similarity, we have

   $A = (PLP^T)(PUP^T)$

   PLP^T is an upper unit diagonal triangular matrix and PUP^T is a lower triangular matrix, so we have found the UL factorization of A.

4. We have that $LULUx = b$. We introduce variables z, c and d such that
   $Lz = b$
   $z = Uc$
   $c = Ld$
   $d = Ux$
   We have a system of equations representing $A^2x = b$. Each equation contains either an upper or lower triangular matrix, which can be solved with back substitution and forward solve, respectively. We start by solving for the first equation in our system, which gives us z, which we use to solve the second equation, and so on until we reach the last equation that when solved, gives us our solution x. Back substitution and forward solve both take O(n^2) flops, so our algorithm uses O(n^2) flops.

5. We have that $(LU^T)x = U^T L^T x = b$. Let $z = L^T x$. Then we have two equations

   $U^T z = b$
   $z = L^T x$

   Transposing a matrix takes O(n^2) flops, since we can just iterate each entry once. Thus it takes O(n^2) flops to calculate L^T and U^T.

   Since U is an upper-triangular matrix, then U^T is an lower-triangular matrix. Then we can use back substitution to solve for z in O(n^2) flops.

   Since L is a lower-triangular matrix, then L^T is an upper triangular matrix. Then we can use forward solve to solve for x in O(n^2) flops.

   Thus we have solved the system in O(n^2) flops.

6. A) The matrix is ill-conditioned. Because we have one very large singular value and one very small singular value, the ratio between these two is very large. This means the condition number will also be very large. Small perturbations in A may lead to relatively large perturbations in the solution.

B) The matrix is well-conditioned. We have two of the same large singular values in the matrix, so the ratio between the two is 1. The condition number will also be small.

C) The matrix is well-conditioned. Both singular values are equally small. This means the condition number is also small.

7. $$\|A\|_2 = \max_x \frac{\|Ax\|_2}{\|x\|_2}$$

$$\|A^{-1}\|_2 = \max_x \frac{\|A^{-1}x\|_2}{\|x\|_2} = \max_{Ax} \frac{\|A^{-1}Ax\|_2}{\|Ax\|_2} = \max_{Ax} \frac{\|x\|_2}{\|Ax\|_2}$$

$$= \max_x \frac{\|x\|_2}{\|Ax\|_2}$$ Since $x \neq 0$ if and only if $Ax \neq 0$, since A is non-singular

$$= \left(\left(\max_x \frac{\|x\|_2}{\|Ax\|_2}\right)^{-1}\right)^{-1}$$

$$= \left(\min_x \frac{\|Ax\|_2}{\|x\|_2}\right)^{-1}$$

Plugging these into our equation, we obtain

$$K(A) = \|A\|_2\|A^{-1}\|_2 = \frac{\max_x \frac{\|Ax\|_2}{\|x\|_2}}{\min_x \frac{\|Ax\|_2}{\|x\|_2}}$$

8. $l_2 = -1/2$
$l_i = -\frac{1}{u_{i-1}}$ for all $i > 2$

$u_1 = 2$
$u_i = 2 + l_i = 2 - \frac{1}{u_{i-1}}$ for all $i > 1$

This resolves to

$l_i = -\frac{i-1}{i}$ for all i

$u_i = \frac{i+1}{i}$ for all i

I have no idea what they want us to notice about the sequence l_i

9. Let the LU factorization be

$$L = \begin{bmatrix} 1 & & & & & \\ l_2 & 1 & & & & \\ & l_3 & \ddots & & & \\ & & \ddots & \ddots & & \\ t_1 & t_2 & & & l_n & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_1 & w_1 & & & v_1 \\ & u_2 & w_2 & & \vdots \\ & & \ddots & \ddots & v_{n-2} \\ & & & \ddots & w_{n-1} \\ & & & & u_n \end{bmatrix}$$

Let the entries of the original matrix be denoted

$$S_n = \begin{bmatrix} a_1 & b_1 & & & & x \\ c_2 & a_2 & b_2 & & & \\ & c_3 & a_3 & b_3 & & \\ & & \ddots & \ddots & \ddots & b_{n-1} \\ y & & & & c_n & a_n \end{bmatrix}$$

Then we have that

$t_1 = \dfrac{y}{a_1}$

$t_2 = -\dfrac{y}{a_2}$

$v_1 = x$

$v_i = -l_i v_{i-1}$ for $1 < i \leq n-2$

$w_i = b_i$ for $1 \leq i \leq n-2$

$w_{n-1} = b_{n-1} - l_i v_{n-2}$

$u_1 = a_1$

$u_i = a_i - l_i w_{i-1}$ for $1 < i \leq n$

$l_i = \dfrac{c_i}{u_{i-1}}$ for $1 < i < n$

It takes O(1) flops to calculate t_1 and t_2. Then for each row i, we can compute v_i, w_i,u_i, and l_i from previous values in O(1) flops. There is at most 4 variables for each row. Then we have that the LU factorizaton for this matrix can be calculated in O(n) flops.