MATH 239 W2009 Final

1. (n choose k) is the number of k-element subsets of an n-element set.

Let x be an element in the n-element set. We also can count the number of these subsets by first counting the number of these subsets with x, which is equivalent choosing (k-1) elements out of an (n-1)-element set, which is (n-1 choose k-1). We also want to add the count of the number of these subsets where x is not in the set, which is (n-1) choose k.

2. $$[x^n](1+x)^2\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}3^k x^{2k}$$

$$=\begin{cases}\binom{n/2+m-1}{m-1}3^{n/2}+\binom{\frac{n-2}{2}+m-1}{m-1}3^{\frac{n-2}{2}}, & n \text{ is even}\\[2ex] 2\binom{\frac{n-1}{2}+m-1}{m-1}3^{\frac{n-1}{2}}, & n \text{ is odd}\end{cases}$$

3. The first k parts must be even; their allowed values are P1={2,4,6…}. The last k parts must be at least 2; their allowed values are P2={2,3,4…}. The generating series for a single part is, respectively

$$\Phi_{P_1}(x)=\frac{x^2}{1-x^2}$$

$$\Phi_{P_2}(x)=\frac{x^2}{1-x}$$

By the Product lemma, the generating series for the compositions of n into these 2k parts is

$$\Phi(x)=\Phi_{P_1}(x)^k\,\Phi_{P_2}(x)^k=\left(\frac{x^2}{1-x^2}\right)^k\left(\frac{x^2}{1-x}\right)^k$$

4. The expression (ε ⌣ 0 ⌣ 00⌣ 0*0000) ((1 ⌣ 1*111) (0 ⌣ 00⌣ 0*0000))* (ε ⌣ 1 ⌣ 1*111) is the block decomposition for this set of strings.

5. a) 1100, 1111, 0000, 0111, 0011

b) 0

c) $$\Phi(x)=\frac{1}{1-x^2}\left(\frac{1}{1-\left(\frac{x^4}{(1-x^2)^2}+\frac{x^4}{(1-x^2)^2}\right)}\right)\frac{1}{1-x^2}=\frac{1}{1-2x^2-x^4}$$

d) By theorem 4.8, b_n satisfies the linear recurrence relation with initial conditions given by

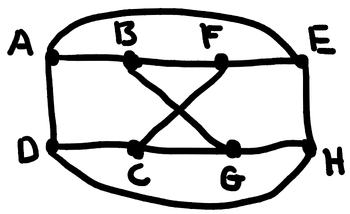
$$b_n - 2b_{n-2} - b_{n-4} = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases}$$

$b_0 = 1$ $b_2 = 2$

$b_1 = 0$ $b_3 = 0$

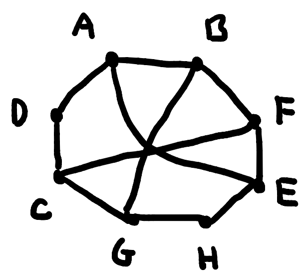
$$b_n = 2b_{n-2} + b_{n-4}, \quad n \geq 4$$

6. a) We have the isomorphism 1A, 2B, 3F, 4E, 5D, 6C, 7G, 8H



b) The shortest path between the two vertices with degree 1 in graph 1 is 5, while it is 4 in graph 2.

c) This subgraph is an edge subdivision of K3,3, so by Kuratowski's Theorem, the graph is non-planar.

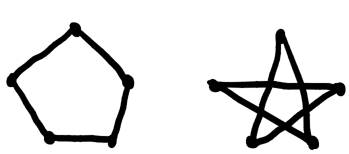


7. a) 1, 2, 10, 3, 11, 9, 4, 6, 8, 12, 5, 7

b) There is an odd cycle 4,11,10,9,12, so the graph is not bipartite.

c) C = {4, 9, 11, 1, 3, 6, 7}. This implies the maximum matching is less than or equal to 7.

8. a)



b) Assume G is planar and has p = 11 vertices and q edges. We must have

$$q \leq 3p-6$$

Its complement !G must have (p choose 2)-q edges. Suppose !G is also planar. Then we have

$$\frac{p(p-1)}{2} - q \leq 3p-6$$

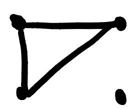
Combining the 2 equations, we obtain

$$\frac{p(p-1)}{2} \leq 6p-12$$

$$p \leq 10.772$$

That contradicts our statement that p = 11. Hence at least one of G and !G must be nonplanar.

9. a)



b) Suppose G is a graph with one face of odd degree and the rest being even. We have that

$$\sum_f \deg(f) = 2|E(G)|$$

So the sum of the degrees of all the faces must be even. But the sum of degrees of faces in G is odd, since there is exactly one face of odd degree. This is a contradiction. Hence there cannot be exactly one face of odd degree.

10. Let G be a planar graph with no cycles of length less than 6.

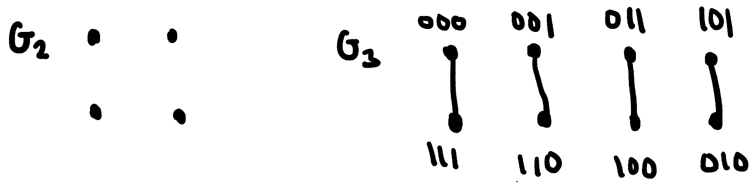
We prove the statement by induction on the number of vertices.

Base case: All graphs on one vertex are 3-colourable.

Inductive hypothesis: Assume the result is true for all planar graphs with no cycles of length less than 6 on p <= k vertices, where k >= 1.

Inductive step: Consider the planar graph G on p=k+1 vertices. We have that G must have a vertex of degree 2 or less, let's call it v. Suppose we remove vertex v, and all edges incident to v, from G, and call the resulting graph G'. Then G' has k vertices, and is a planar graph with no cycles less than 6, so must be 3-colourable. There are at most 2 vertices in G' adjacent to v in G, so we can assign these vertices 2 different colours in the 3-colouring of G'. Thus there is at least one of the 3 colours remaining. Assign one of these remaining colours to v, so that v has a different colour from all of its adjacent vertices in G. Thus we have a 3-colouring of G, and the result is true for p=k+1. Hence we have proved the result by induction.

11. a)



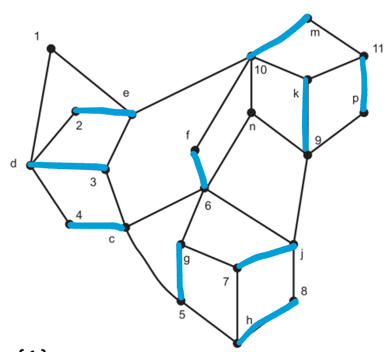
b) Gn has 2^n vertices. For each vertex string, there are (n choose 3) strings of length n that differ in exactly 3 positions. Hence each vertex is of degree (n choose 3), and we have

$$|E(G_n)| = \frac{1}{2}\sum \deg(v) = \frac{1}{2}\cdot 2^n \cdot \binom{n}{3} = 2^{n-1}\binom{n}{3}$$

12. a)
X0 = {1, 4}
X = {1, 4}
Y = {}
X = {1, 4}
Y = {d, e, c}
X = {1, 4, 3, 2, 5}
Y = {d, e, c, g, h}
X = {1, 4, 3, 2, 5, 7, 8}
Y = {d, e, c, g, h, j}

j is an unsaturated vertex in Y, so we have the augmenting path 4c5g7j

b)



c)
X0 = {1}
X = {1}
Y = {}
X = {1}
Y = {e, d}
X = {1, 2, 3}
Y = {e, d, c}
X = {1, 2, 3, 4}
Y = {e, d, c}

We have added no new vertices in Y, so we terminate with maximum matching (drawn above) M = {2e, 3d, 4c, 5g, 6f, 7j, 8h, 9k, 10m, p11} and minimum cover C = {e, d, c, 5, 6, 7, 8, 9, 10, 11}. M is a maximum cover since it is the same size as our cover; |M| = |C|.