MATH 239 W2012 Midterm

1. a) $$[x^n](1+3x^2)^{-5} = [x^n]\sum_{n=0}^{\infty}\binom{n+4}{4}(-3)^n x^{2n}$$

$$= \begin{cases} 0 & , n \text{ is odd} \\ \binom{\frac{n}{2}+4}{4}(-3)^{n/2} & , n \text{ is even} \end{cases}$$

b) $$[x^5](1+3x^3)^3(1-x)^{-2} = [x^5]\left[\sum_{k=0}^{3}\binom{3}{k}3^k x^{3k}\right]\sum_{j=0}^{\infty}(j+1)x^j$$

$$= \binom{3}{0}3^0(5+1) + \binom{3}{1}3^1(2+1) = 33$$

c) $$[x^n](1-5x)^{-3}(1+x)^m = [x^n]\left[\sum_{j=0}^{\infty}\binom{n+2}{2}\cdot 5^j x^j\right]\sum_{k=0}^{m}\binom{m}{k}x^k$$

$$= \sum_{j=0}^{n}\binom{j+2}{2}\cdot 5^j\binom{m}{n-j}$$

d) $$[x^{73}](1+x^2)^{10}(1-x^4)^2 = [x^{73}]\left[\sum_{k=0}^{10}\binom{10}{k}x^{2k}\right](1-2x^4+x^8)$$

$$= 0$$

2. a) The generating function for a part of at least 3 is

$$\Phi(x) = \frac{x^3}{1-x}$$

By the product lemma, the generating function for compositions of n of these parts is

$$(\Phi(x))^{100} = \left(\frac{x^3}{1-x}\right)^{100} = \frac{x^{300}}{(1-x)^{100}}$$

b) By the string lemma, the generating function is

$$\frac{1}{1-\Phi(x)} = \frac{1}{1-\frac{x^3}{1-x}} = \frac{1-x}{1-x-x^3}$$

c) $$\sum_{k=0}^{100} \Phi(x)^k = \sum_{k=0}^{\infty} \Phi(x)^k - \sum_{k=101}^{\infty} \Phi(x)^k = \frac{1}{1-\Phi(x)} - \frac{\Phi(x)^{101}}{1-\Phi(x)}$$

$$= \frac{1-\Phi(x)^{101}}{1-\Phi(x)} = \frac{1-\frac{x^{303}}{(1-x)^{101}}}{1-\frac{x^3}{1-x}}$$

$$= \frac{(1-x)^{101}-x^{303}}{(1-x)^{100}(1-x-x^3)}$$

3\. a) (ε ⌣ (00)*00 ⌣ 0 ⌣ 000) (((11)*11 ⌣ 1 ⌣ 111)((00)*00 ⌣ 0 ⌣ 000))* (ε ⌣ (11)*11 ⌣ 1 ⌣ 111)

$$\Phi(x) = \left(1+\frac{x^2}{1-x^2}+x+x^3\right)^2 \frac{1}{1-\left(\frac{x^2}{1-x^2}+x+x^3\right)^2}$$

$$= \frac{1+x-x^5}{1-x-2x^2+x^5}$$

b) 1* (000*1 ⌣ 01*1)* 0*

$$\Phi(x) = \frac{1}{1-x}\left(\frac{1}{1-\left(\frac{x^3}{1-x}+\frac{x^4}{1-x}\right)}\right)\frac{1}{1-x} = \frac{1}{1-2x+x^4}$$

4\. a) $$\left(1+x^2\right)\frac{1}{1-\left(x\left(1+\frac{x}{1-x^2}\right)\right)} = \frac{1-x^4}{1-x-2x^2+x^3}$$

b) It does not create 00111 - where an even block of zeros is followed by an odd block of 1s.

5\. a) The characteristic polynomial is

$$1+3x-4x^2 = (1-x)(1+4x)$$

The solution is hence of the form

$$b_n = A + B(-4)^n$$

b) To find a particular solution of this recurrence, we try a solution of the form b_n = alpha*n where alpha is some constant. Substituting into the recurrence, we get

$$\alpha n + 3\alpha(n-1) - 4\alpha(n-2) = 5$$

$$5\alpha = 5$$

$$\alpha = 1$$

Hence a particular solution is b_n = n.
The characteristic polynomial of the corresponding homogenous linear recurrence relation is

$$1 + 3x - 4x^2 = (1-x)(1+4x)$$

The general solution is hence of the form

$$b_n = A + B(-4)^n + n$$

Setting n=0 and n=1 respectively, we get

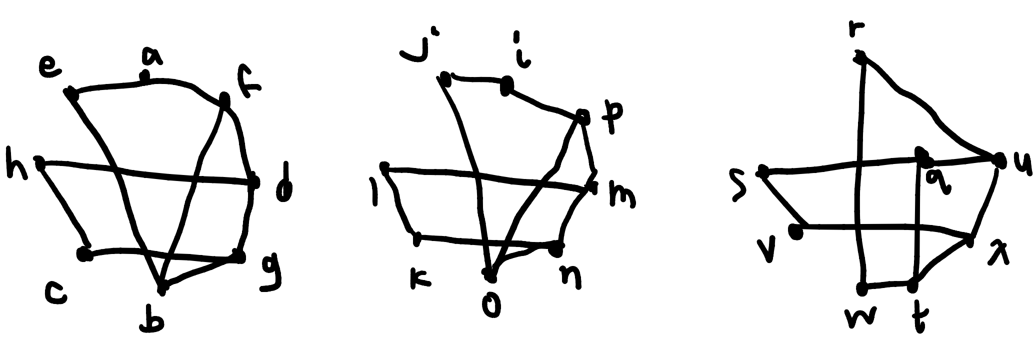
$$1 = A + B(-4)^0 + 0$$

$$2 = A - 4B + 1$$

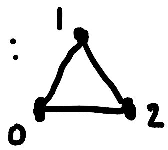
We get A=1 and B=0. Hence

$$b_n = 1 + n$$

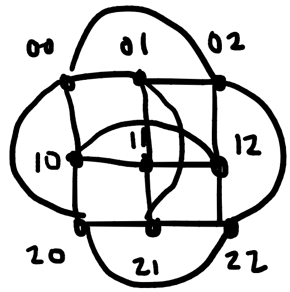
6. G and H are isomorphic. J is not isomorphic to G, since in G, there are only 2 nodes (d and g) that are each in two distinct 4-cycles, while in J, there are 3 (q, u, x).



7. a) H_1:



H_2:



b) For each character in the string, there are 3 possibilities for which character it could be. There are n characters. Hence there are 3^n vertices in H_n.

c) For each string of length n, there are two possibilities each character could change to, and there are n characters that we can flip one at a time. Hence each vertex has degree 2n.

$$|E(H_n)| = \frac{1}{2}\sum_{v\in V} 2n = \frac{1}{2}\cdot 3^n \cdot 2n = n3^n$$