From Centralized to Decentralized Control of Complex Systems

In the Pursuit of Efficient Control Approaches

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Fall semester 2019

Lecture 3

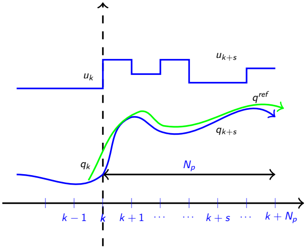
Outline

1. An optimization-based approach for control of complex systems
   - Generic prediction model
   - Set-theoretic methods
   - Mixed-integer techniques

Model Predictive Control (MPC)

Propoi (1963), Cutler et al. (2007), Richalet and O'Donovan (2009)

- Optimization-based control law
- Implicit (on-line) vs. explicit (off-line) implementation
- Constraints handling
- Can be implemented in a distributed fashion



model of the system

reference trajectory

$$\begin{aligned}&\arg\min_{\mathbf{u}}\; V_f(x(k+N_p), q(k+N_p)) + \sum_{s=1}^{N_p-1} V_n(x(k+s), u(k+s), q(k+s))\\ &\text{subject to: } \begin{cases} x(k+s+1) = f(x(k+s), u(k+s)), & s = 0 : N_p - 1,\\ h(x(k+s), u(k+s), q(k+s)) \leq 0, & s = 1 : N_p - 1.\end{cases}\end{aligned}$$

constraint sets representation

optimization solver

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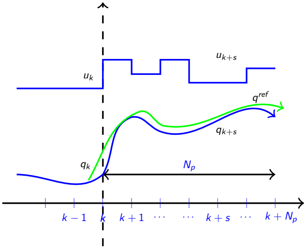

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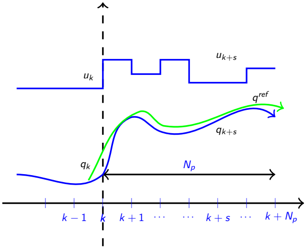

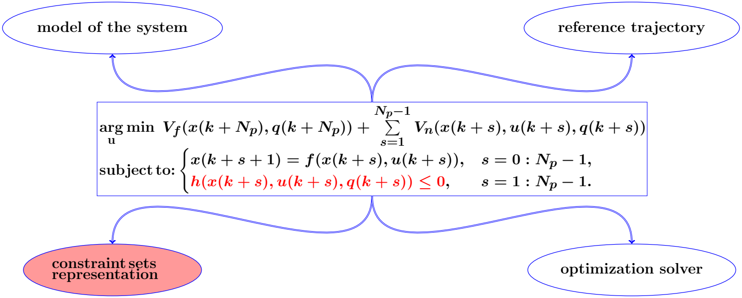

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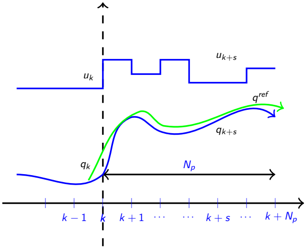

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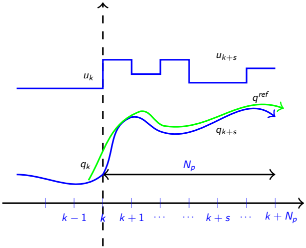

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constraint sets representation

optimization solver

A generic prediction model

Consider the following discrete-time autonomous system :

$$x(k+1) = f(x(k)), \quad x(k) \in \mathcal{S},$$

where $x(k) \in \mathbb{R}^n$ is the current state and the mapping $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous on $\mathbb{R}^n$ satisfying the condition $f(0) = 0$. The state constraint set $\mathcal{S}$ is a compact set containing the origin in its interior.

Consider also the following discrete-time invariant system :

$$\begin{aligned} x(k+1) &= f(x(k), u(k)), \quad (x(k), u(k)) \in \mathcal{S} \times \mathcal{U}, \\ y(k) &= g(x(k)), \quad y(k) \in \mathcal{Y}. \end{aligned}$$

where, in addition to the first system, $u(k) \in \mathbb{R}^m$ is the current control input, $y(k) \in \mathbb{R}^p$ is the output, the mappings $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g(\cdot) : \mathbb{R}^p \to \mathbb{R}^n$ are assumed to be continuous with $f(0,0) = 0$ and $g(0) = 0$. The control constraint set $\mathcal{U}$ is a compact sets containing the origin in its interior.

A generic prediction model

Consider the following discrete-time autonomous system affected by additive disturbances:

$$x(k+1) = f(x(k), w(k)), \quad (x(k), w(k)) \in \mathcal{S} \times \mathcal{W},$$

where $x(k) \in \mathbb{R}^n$ is the current state and the mapping $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous on $\mathbb{R}^n$ satisfying the condition $f(0) = 0$. The state constraint set $\mathcal{S}$ is a compact set containing the origin in its interior. The disturbance $w(k)$ is bounded, i.e. $w(k) \in \mathcal{W}$ and $\mathcal{W} \subset \mathbb{R}^w$ is a convex and compact set containing the origin.

Consider also the following discrete-time invariant system affected by additive disturbances:

$$\begin{aligned} & x(k+1) = f(x(k), u(k), w(k)), \quad (x(k), u(k), w(k)) \in \mathcal{S} \times \mathcal{U} \times \mathcal{W}, \\ & y(k) = g(x(k)), \quad y(k) \in \mathcal{Y}. \end{aligned}$$

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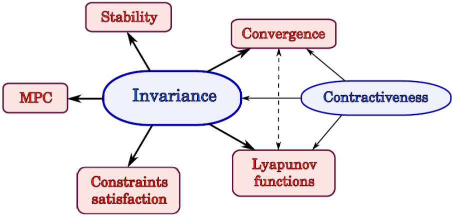
Set-theoretic methods – An Overview

Set-theoretic methods in control: deriving properties of dynamical systems by means of suitable sets in the state/input/output spaces.

Links between set-theoretic analysis and MPC:

- (robust) positively invariant sets
- reachable sets
- terminal sets
- feasible sets

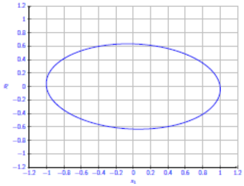


[0]Excellent textbook Blanchini and Miani (2008): *Set-theoretic methods in control*, Springer 2008.

Set-theoretic methods

Various families of sets in control:

- ellipsoids (Kurzhanskiĭ and Vályi (1997))
- polyhedra (Motzkin et al. (1959))
- (B)LMIs (Nesterov and Nemirovsky (1994))



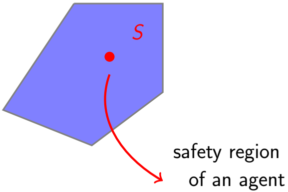
$$x^T Q x \leq \gamma$$

Issues to be considered:

- flexibility of representation
- numerical implementation

In the multi-agent context:

- obstacles
- safety regions
- feasible regions



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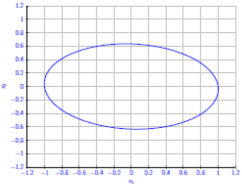

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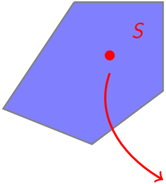


safety region
of an agent

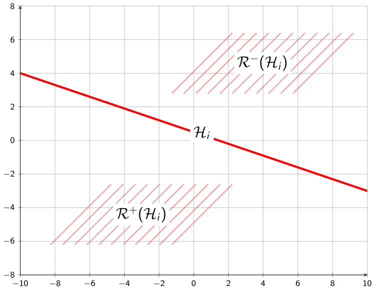
Set-theoretic methods – Polyhedral sets description

Let us define a collection of hyperplanes

$$\mathcal{H}_i = \left\{x:\ h_i x = k_i,\ (h_i, k_i) \in \mathbb{R}^{1\times n} \times \mathbb{R}\right\}$$

which partition the space in regions

$$\mathcal{R}^{+}(\mathcal{H}_i) = \{x:\quad h_i x \leq \quad k_i\}$$
$$\mathcal{R}^{-}(\mathcal{H}_i) = \{x:\ -h_i x \leq -k_i\}$$



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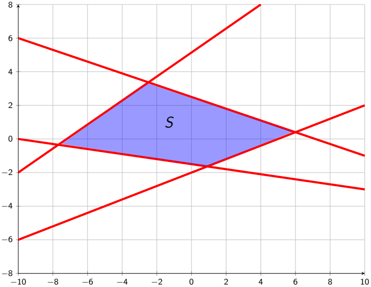
describing a bounded polyhedral set



$$S = \left\{ x \in \mathbb{R}^n : \bigcap_i \mathcal{R}^+(\mathcal{H}_i), \quad i = 1 : N \right\}$$

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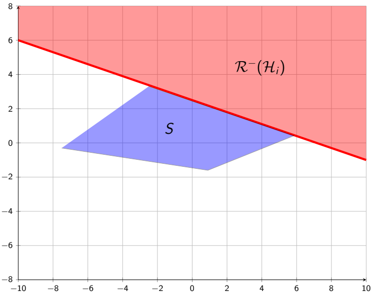

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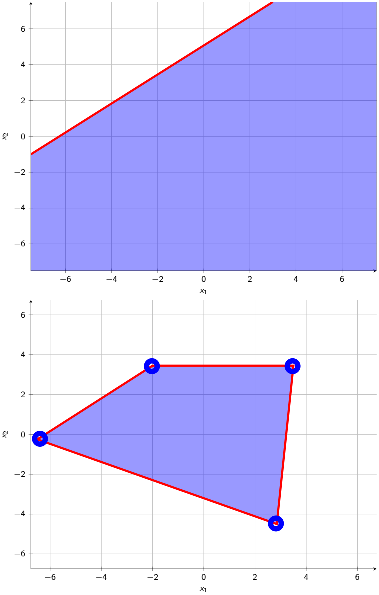
Families of sets – polyhedral sets

Best compromise: polytopic(zonotopic) sets



Polyhedral sets:

- dual representation
  - half-space:

$$h_i x \leq k_i, \; i = 1 \dots N_h$$

  - vertex:

$$\sum_i \alpha_i v_i, \; \alpha_i \geq 0, \; \sum_i \alpha_i = 1, \; i = 1 \dots N_v$$

- efficient algorithms for set containment problems Gritzmann and Klee (1994)
- can approximate any convex shape Bronstein (2008)

[0]See also other papers and lectures of F. Stoican (UPB, Romania), S. Olaru (CentraleSupélec, France) on set-theory and its various applications in control engineering.

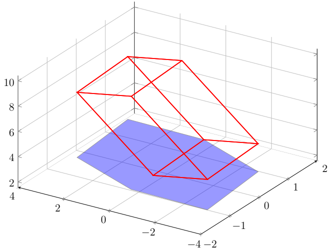
Families of sets – zonotopic sets

Zonotopic sets Ferrez et al. (2001):

- obtained as hypercube projection
- Minkowski sum of generators:

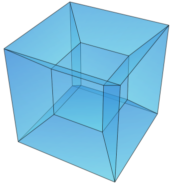
$$Z(c, G) = \left\{ c + \sum_i \lambda_i g_i,\ |\lambda_i| \leq 1,\ i = 1 \ldots N_g \right\}$$

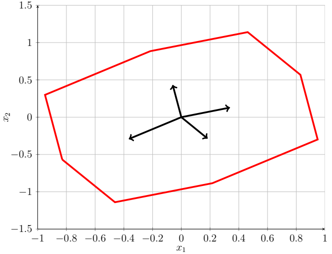
- limited to symmetric objects



Zonotope

A hypercube is an n-dimensional analogue of a square ($n = 2$) and a cube ($n = 3$).





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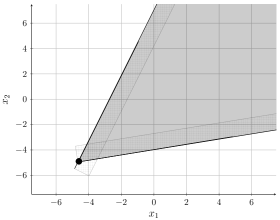
Set representations

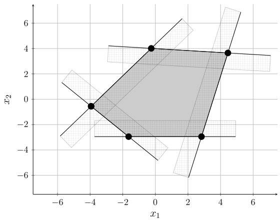
Definition (Cone)

For a finite collection of vectors $Y = \{y_1 \dots y_d\} \subseteq \mathbb{R}^n$, *the cone of* Y *is defined as* $cone(Y) \triangleq \{t_1 y_1 + \dots t_d y_d : \ t_i \in \mathbb{R}_+\} = \{Yt, \ t \in \mathbb{R}^n_+\}$.

Definition (Convex hull)

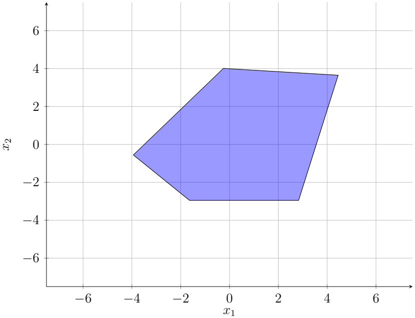
For a finite collection of points $V = \{v_1 \dots v_d\} \subseteq \mathbb{R}^n$, *the convex hull of* V *is defined as* $conv(V) \triangleq \{\alpha_1 v_1 + \dots \alpha_d v_d : \ \alpha_i \in \mathbb{R}_+, \sum_i \alpha_i = 1\} = \{V\alpha, \ \alpha \in \mathbb{R}^n_+, \mathbf{1}^T\alpha = 1\}$.



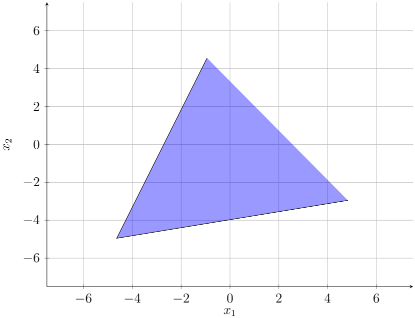


Constructions – exemplifications

bounded polyhedral set (polytope)



unbounded polyhedral set (pointed cone)



Set operations

- Projection along a sub-space
- The Minkowski sum of two sets $P, Q \subseteq \mathbb{R}^n$ is defined to be

$$P \oplus Q = \{x + y : \ x \in P, y \in Q\}$$

- The Pontryagin difference is defined as

$$P \ominus Q = \{x \in P : \ x + y \in P, \forall y \in Q\} .$$

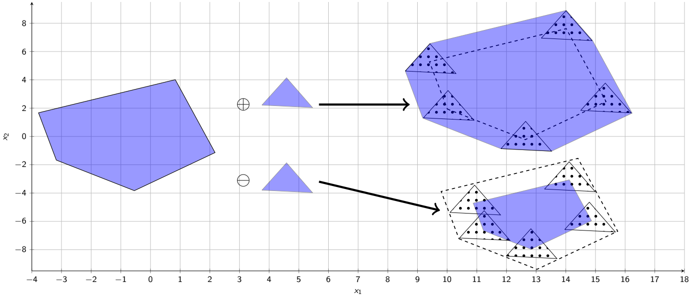
- Given two convex sets P, Q, the Hausdorff distance is defined as

$$d_H(P, Q) = max\left\{\bar{d}_H(P, Q), \bar{d}_H(Q, P)\right\}$$

where $\bar{d}_H(P, Q) = \underset{x \in P}{max} \ \underset{y \in Q}{min} \ d(x, y)$, and $d(x, y)$ is a distance measured in a given norm in the $\mathbb{R}^n$ space.

Set operations – exemplifications

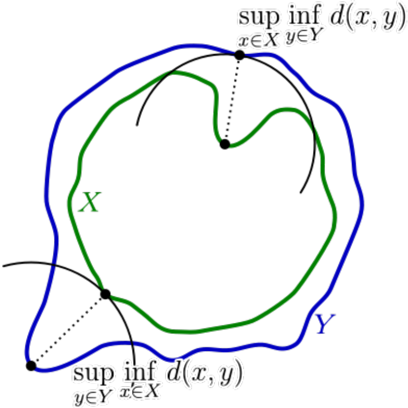
Minkowski sum / Pontryagin difference



Set operations – exemplifications

Hausdorff distance

The Hausdorff distance is the longest distance you can be forced to travel by an adversary who chooses a point in one of the two sets, from where you then must travel to the other set.
In other words, it is the greatest of all the distances from a point in one set to the closest point in the other set.



Set-theoretic methods – Invariance notions

Blanchini and Miani (2008)
Consider a discrete-time autonomous system in $\mathbb{R}^n$:

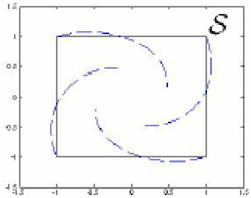
$$x(k+1) = f(x(k)), \text{ with } f(0) = 0 \text{ and } x(k) \in \mathcal{S}.$$

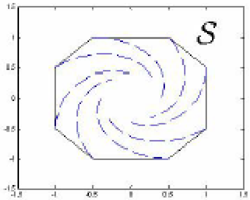
Definition (Positive invariance – Blanchini (1999))

A set $\mathcal{S} \in \mathbb{R}^n$ is positively invariant if for any $x_0 \in \mathcal{S}$, the solution $x(k, x_0)$ satisfies $x(k, x_0) \in \mathcal{S}$ for $k \in \mathbb{N}$.

Definition (Positive invariance (equivalent definition))

A set $\mathcal{S} \in \mathbb{R}^n$ is positively invariant if $f(\mathcal{S}) \subset \mathcal{S}$.





Set-theoretic methods – Invariance notions

Consider a discrete-time invariant system in $\mathbb{R}^n$ affected by bounded disturbances $w(k) \in \mathbb{W}$

$$x(k+1) = f(x(k), w(k)), \text{ with } f(0,0) = 0.$$

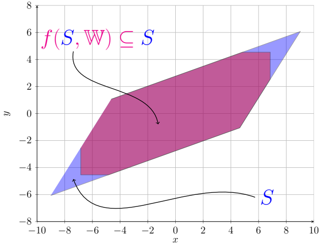
Definition (RPI set – Blanchini (1999))

A set $\mathcal{S}$ is called Robust Positively Invariant (RPI) iff $\forall x(0) \in \mathcal{S}$ and $\forall w(k) \in \mathbb{W}$ then $x(k) \in \mathcal{S}$ for $k > 0$.

Definition (mRPI set – Blanchini (1999))

A set Ω_∞ is called minimal Robust Positively Invariant (mRPI) iff it is a RPI set in $\mathbb{R}^n$ contained in every RPI set of the system.

RPI set



Links between set-theoretic analysis and MPC:

- (robust) positively invariant sets
- reachable sets
- terminal sets
- feasible sets

Set-theoretic methods – Invariance notions

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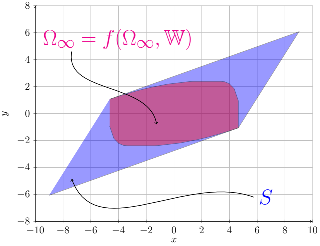
mRPI set



Links between set-theoretic analysis and MPC:

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Positive invariance conditions for linear systems

Objective: find invariance test for linear time invariant (LTI) dynamics.

Idea: exploit the definition and the generic set-theoretic condition:

$x(k+1) = Ax(k)$

A set $\mathcal{S}$ is Positively Invariant if one of the following holds:

- $\forall x(k) \in \mathcal{S}$ then $x(k+1) \in \mathcal{S}$
- $A\mathcal{S} \subseteq \mathcal{S}$.

$x(k+1) = Ax(k) + w(k), w(k) \in \mathcal{W}$

A set $\mathcal{S}$ is Robust Positively Invariant (RPI) if one of the following holds:

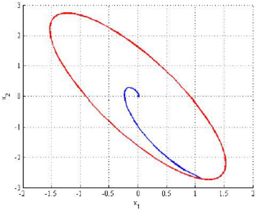
- $\forall x \in \mathcal{S}$ then $Ax + w \in \mathcal{S}, \forall w \in \mathcal{W}$
- $A\mathcal{S} + \mathcal{W} \subseteq \mathcal{S}$
- $A\mathcal{S} \subseteq \mathcal{S} \ominus \mathcal{W}$
- $\mathcal{W} \subseteq \mathcal{S} \ominus A\mathcal{S}$ (the right hand side represents the largest disturbance set for an invariant set $\mathcal{S}$)

[0] See also other papers and lectures of F. Stoican (UPB, Romania), S. Olaru (CentraleSupélec, France) on set-theory and its various applications in control engineering.

Positive invariance conditions for linear systems

Positive invariance conditions for LTI dynamics $x(k+1) = Ax(k)$

Ellipsoidal sets:
$\mathcal{E}_P = \{x^\top Px \leq 1\}$ is positive invariant if $\forall x(k) \in \mathcal{E}_P$ then $x(k+1) \in \mathcal{E}_P$



Applying the invariance condition

- $x^\top(k+1)Px(k+1) \leq 1$ when $x^\top(k)Px(k) \leq 1$
- $(Ax(k))^\top PAx(k) \leq 1$ when $x^\top(k)Px(k) \leq 1$
- This holds if the classical (quadratic) Lyapunov function stability condition holds $x^\top(k)A^\top PAx(k) \leq x^\top(k)Px(k) \leq 1$ and leads to

LMI test:
$A^T PA \leq P$ and $P = P^T > 0$

[0]See also other papers and lectures of F. Stoican (UPB, Romania), S. Olaru (CentraleSupélec, France) on set-theory and its various applications in control engineering.

Ultimate bounds

Theorem (Ultimate bounds set – Kofman et al. (2008))

Consider the stable system $x(k+1) = Ax(k) + w(k)$. Let there be the Jordan decomposition $A = V\Lambda V^{-1}$ and assume that $\left|w(k)\right| \leq \bar{w}, \forall k \geq 0$. Then there exists $l(\epsilon)$ such that for all $k \geq l(\epsilon)$:

$$\begin{aligned} |V^{-1}x(k)| &\leq (I - |\Lambda|)^{-1}|V^{-1}|\bar{w} + \epsilon \\ |x(k)| &\leq |V|(I - |\Lambda|)^{-1}|V^{-1}|\bar{w} + |V|\epsilon \end{aligned}$$

The set

$$\Psi = \left\{x : |V^{-1}x| \leq (I - |\Lambda|)^{-1}|V^{-1}|\bar{w}\right\}$$

represents a Robust Positive Invariant *approximation* of the mRPI.

$$\Omega_\infty \subset \Psi.$$

Mixed-Integer Programming (MIP)

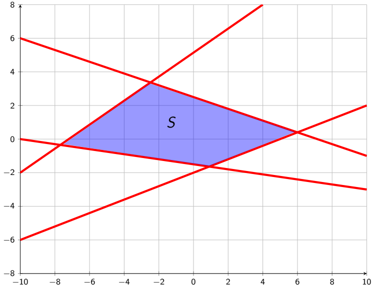
Grundel et al. (2007), Jünger et al. (2009)

- Flexible mathematical model for the formulation of decision and control problems based on optimization
  - combinatorial allocation problem
  - multicast routing problem
- Flexible mathematical model for the formulation of collision avoidance problems involving the control of Multi-Agent Systems
  - path following with obstacle and collision avoidance
  - formation control with collision avoidance
- Fast off-the-shelf solvers available
  - CPLEX, OSL, etc.
- Strong theoretical foundations
  - characterization of tractable special cases
  - NP-hard in general, but can also solve many large problems in practice

MIP – multi-agent context

Consider a bounded polyhedral set

$$S = \left\{x \in \mathbb{R}^n : h_i x \leq k_i,\ i = 1 : N\right\}$$



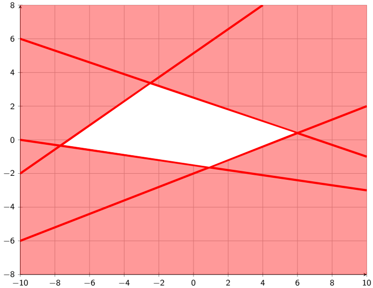
MIP – multi-agent context

Consider a bounded polyhedral set

$$S = \left\{x \in \mathbb{R}^n : h_i x \leq k_i,\ i = 1 : N\right\}$$

Consider the complement of S

$$\mathcal{C}(S) \triangleq cl(\mathbb{R}^n \setminus S) = \bigcup_i \mathcal{R}^-(\mathcal{H}_i), \quad i = 1 : N$$



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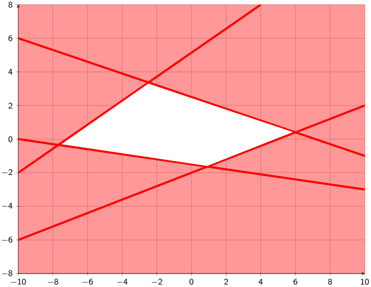
Define $\mathcal{C}(S)$ in a linear representation

$$-h_i x \le -k_i + M\alpha_i, \quad i = 1 :\ N$$

$$\sum_{i=1}^{i=N} \alpha_i \le N - 1$$

with $(\alpha_1, \dots, \alpha_N) \in \{0, 1\}^{N}$



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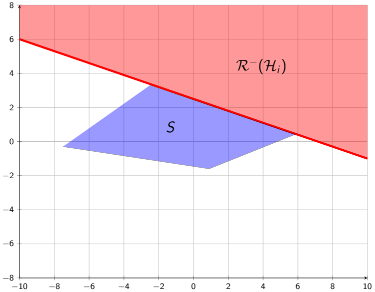

Any of the regions $\mathcal{R}^-(\mathcal{H}_i)$ of $\mathcal{C}(S)$ can be obtained by a suitable choice of binary variables

$$\mathcal{R}^-(\mathcal{H}_i) \longleftrightarrow (\alpha_1, \ldots, \alpha_N)^i \triangleq (1, \ldots, 1, \underbrace{0}_{i}, 1, \ldots, 1)$$

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