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Survey paper

Set invariance in control[☆]

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The paper provides a survey of the literature on invariant sets and their applications

Abstract

The properties of positively invariant sets are involved in many different problems in control theory, such as constrained control, robustness analysis, synthesis and optimization. In this paper we provide an overview of the literature concerning positively invariant sets and their application to the analysis and synthesis of control systems. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Invariant sets; Lyapunov functions; Stability; Control synthesis; Constrained control

1. Introduction

Since the theory of Lyapunov was introduced for ordinary differential equations the notion of invariant set has been involved in many problems concerning the analysis and the control of dynamical systems.

Given a dynamic system a subset of the state space is said to be *positively invariant* if it has the property that, if it contains the system state at some time, then it will contain it also in the future. A subset of the state space is said *invariant* if the inclusion of the state at some times implies the inclusion in both the future and the past.

Let us consider for instance a dumped and unforced pendulum in a neighborhood of its stable position. The mechanical energy of this system is a function of the state. Since the system is dissipative the set of all states whose energy does not exceed a fixed level is a positively invariant set. Clearly it is not invariant since the energy of the system shall decrease during its evolution. If the pendulum is undamped and thus it conserves its energy during the evolution, the constant energy surfaces are examples of invariant sets.

As it is known, the concept of "energy of a system" has been formalized by means of Lyapunov theory and the notion of positive invariance has its origin in that theory. Precisely, given a Lyapunov function, its level surfaces are the boundaries of positively invariant sets. However, the interpretation of a positively invariant set as a set of "limited energy" inherited from the Lyapunov theory may be quite restrictive. In fact positively invariant sets have played the basic role, motivated by many practical problems, of "confinement sets". This key notion has been widely exploited, especially in the case of systems subject to constraints.

Let us consider the case of a dynamic system whose state variables are subject to constraints that define an admissible set in the state space. Due to the system dynamics, in general, not all the trajectories originating from admissible initial states will remain in such a set. Conversely, for any initial condition which belongs to a positively invariant subset of the admissible domain, constraints violations are avoided. Thus the inclusion of the state in a positively invariant set provides fundamental a priori information about any trajectory originating from it.

This simple idea can be easily extended to the case in which a control input is present. In this case we say that a set is *controlled invariant* or *viable* if, for all initial conditions chosen among its elements, we can keep the trajectory inside the set *by means of a proper control action*. Again the existence of a controlled invariant set is a fundamental step in the solution of several control synthesis problems especially in presence of constraints for two fundamental reasons. First such a set includes initial states whose future trajectories *meet design specifications* such as constraints satisfaction and convergence

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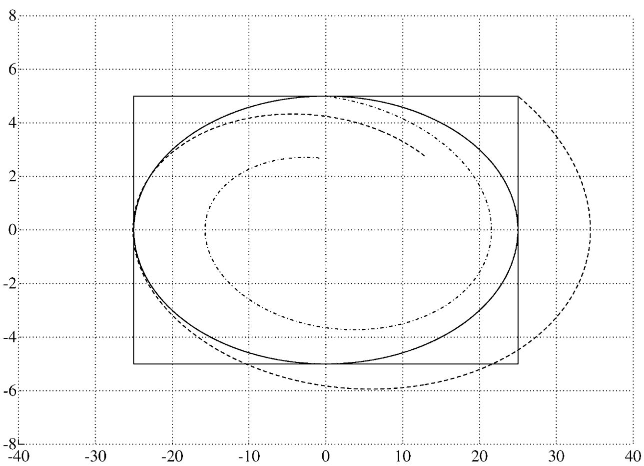

Fig. 1 . The constraint sets and the invariant ellipsoid.

to the desired equilibrium point. Second, as we will explain later, *the control law may be derived by means of the set*.

To better explain the previous concepts we introduce an example. Consider the following double-integrator continuous-time model (Gutman & Cwikel, 1987)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

and assume that its state variables are subject to the constraints $|x_1| \leq 25$ and $|x_2| \leq 5$. Such constraints form the rectangle represented in Fig. 1. Now take the stabilizing linear state feedback control law $u = -k(x_1 + x_2)$, with $k > 0$. If we take as initial state the vertex of the rectangle corresponding to $x_1(0) = 25$ and $x_2(0) = 5$, the resultant trajectory (dashed in Fig. 1) violates the constraints. For $k \geq 1/25$ the boundary rectangle includes the ellipsoid

$$\mathscr{S} = \{(x_1, x_2)\colon kx_1^2 + x_2^2 \leq 1/k\}$$

represented in the figure for $k = 1/25$ which is positively invariant for the closed loop system. This means that any trajectory starting from $\mathscr{S}$ (for instance the dot-dashed one in Fig. 1) remains inside $\mathscr{S}$ and converges to the origin. Thus $\mathscr{S}$ is not only a domain of attraction, but it is also a *safety region for the initial state*.

The term positively invariant for $\mathscr{S}$ is referred to the closed loop system. If we make reference to the original system, without any control action, the same set is called *controlled invariant*. This means that it can be rendered invariant by a proper control action, which is not necessarily the linear feedback considered above. For instance, other controllers can be derived by noticing that the ellipsoid is associated with the quadratic function $\Psi(x_1, x_2) = kx_1^2 + x_2^2$. Elementary computations show that the Lyapunov derivative is given by $\dot{\Psi}(x_1, x_2, u) = 2(kx_1x_2 + ux_2)$. Thus, any function $u(x_1, x_2)$ that renders $\dot{\Psi}$ non-positive inside $\mathscr{S}$ (for instance $u = -25\,k\,\mathrm{sgn}[x_2]$) is a possible control. In general a controlled invariant set may not admit linear controllers and *the existence of a linear controller* is one of the problems we deal with.

The constraints considered in the above example are defined in the state-space, but control constraints can be easily added. If we assume that the control must be such that $|u| \leq 1$, we introduce the additional condition $k|x_1 + x_2| \leq 1$. This new constraint slightly modifies the constraint set which now intersects the ellipsoid, rendering it useless. However, by slightly shrinking the said ellipsoid we can derive a new invariant domain which is included in the new constraint set (this property of preserving positive invariance under scaling will be analyzed later).

Another class of problems which can be handled by means of invariant sets are the analysis and synthesis of uncertain systems. In this case, we use the concept of "robust positive invariance". If we consider the previous example, and we assume that B may vary as $B = [0, 1 + \delta]^T$, then it can be seen that the considered ellipsoid does not remain positively invariant for the systems achieved by the control laws considered above. This is equivalent to the fact that the associated quadratic function $\Psi(x_1, x_2) = kx_1^2 + x_2^2$ is not a robust Lyapunov function. However, under proper bounds on the parameter variation δ, robustly invariant ellipsoids can be found.

The main goal of this paper is to present in a concise way the basic ideas, the main properties and the most successful applications of invariant sets in control engineering. We consider with special attention specific but important problems such as robust analysis and synthesis, control under constraints and disturbance rejection. The spirit of the paper is to summarize basic results in the simplest way. Thus this work deals mainly with special but important cases such as linear (possibly uncertain) systems and convex sets (mainly ellipsoids and polyhedra). This notwithstanding, the (few) cases in which the results can be significantly extended to more general classes of systems and sets are mentioned.

One fundamental problem we deal with is the tradeoff between the complexity of the description of a family of sets and its "optimality" properties. Indeed, the determination of invariant sets which are in some sense "the best", for instance finding the largest controlled invariant set inside a prescribed domain, *is often frustrated by the complexity of the representation*. This aspect concerns, for instance, ellipsoids and polytopes as candidate invariant sets: simple but conservative the former, non-conservative but arbitrarily complex the latter.

In view of the mentioned relationship with the Lyapunov theory the literature that somehow touches even marginally invariant sets is extremely wide. Therefore some necessary choices must be made in the selection of the material to stay within the limits of a journal paper. The results which have been considered are either classical results of wide generality, or recent engineering-oriented results. In particular, more attention has been devoted to the presentation of concrete results than to the description of theoretical aspects. This choice unavoidably led us to the exclusion of important recent developments in the theoretical analysis of set invariance. Nevertheless, the paper includes basic ideas and references we believe appropriate as starting point for further readings and research of theoretical nature.

In this paper we essentially deal with *positive invariance* rather than pure invariance, since the latter concept is less useful in an engineering context. In Section 2 we introduce the main definitions. In Section 3 we introduce some classical positive invariance conditions such as the Nagumo theorem and we describe in a formal way the relation between positively invariant sets and Lyapunov functions. In particular we show how a convex and compact invariant set containing the origin in its interior can "shape" a Lyapunov function. In Section 4 we present the main properties of the two most currently used families of candidate invariant sets: ellipsoids and polytopes. The problem of constructing positively invariant sets is faced in Section 5, where we will also consider the problem of the association of a feedback control to a controlled invariant set. In particular we will investigate when such a controller can be linear. In Section 6 we describe some applications of the positively invariant sets. In the end we will derive some conclusions and point out some current research directions in Section 7.

2. Basic definitions

In the following, given a set $\mathscr{S}$ we denote by $int\{\mathscr{S}\}$ its interior and by $\partial\mathscr{S}$ its boundary. If $\mathscr{S}$ is a polytope, then $vert\{\mathscr{S}\}$ is the set of its vertices. We call a C-set a convex and compact set including the origin in its interior.

We consider dynamic time-invariant, possibly uncertain, systems of the form

$$\Delta x(t) = f(x(t), u(t), w(t)), \tag{1}$$

$$y(t) = g(x(t)), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output, $w(t) \in \mathscr{W} \subset \mathbb{R}^q$ is an external input, and $\mathscr{W}$ is an assigned compact set. We denote by Δ the derivative operator in the continuous-time case and the shift forward operator in the discrete-time case (i.e. $\Delta x(t) = x(t+1)$). In the continuous-time case we assume that $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ and $g: \mathbb{R}^n \to \mathbb{R}^p$ are Lipschitz functions and that $u(t)$ and $w(t) \in \mathscr{W}$ are continuous. Note that under these assumptions the system admits a solution $x(t)$ which is uniquely defined on $\mathbb{R}^+$ for all $x(0) \in \mathbb{R}^n$ w and u. Although these assumptions may seem restrictive, they are quite reasonable for a tutorial purpose. Nevertheless, more general cases (such as the non-uniqueness of the solution) will be briefly discussed when necessary.

Definition 2.1. The set $\mathscr{S} \subset \mathbb{R}^n$ is said positively invariant for a system of the form

$$\Delta x(t) = f(x(t))$$

if for all $x(0) \in \mathscr{S}$ the solution $x(t) \in \mathscr{S}$ for $t > 0$. If $x(0) \in \mathscr{S}$ implies $x(t) \in \mathscr{S}$ for all $t \in \mathbb{R}$ then we say that $\mathscr{S}$ is invariant.

Definition 2.2. The set $\mathscr{S} \subset \mathbb{R}^n$ is said robustly positively invariant for the system

$$\Delta x(t) = f(x(t), w(t)) \tag{3}$$

if for all $x(0) \in \mathscr{S}$ and all $w(t) \in \mathscr{W}$ the solution is such that $x(t) \in \mathscr{S}$ for $t > 0$.

Definition 2.3. The set $\mathscr{S} \subset \mathbb{R}^n$ is said (robustly) controlled invariant for the system

$$\begin{aligned} &\Delta x(t) = f(x(t), u(t)), \\ &(\Delta x(t) = f(x(t), u(t), w(t)),\ w(t) \in \mathscr{W}), \\ &y(t) = g(x(t)) \end{aligned} \tag{4}$$

if there exists a continuous feedback control law

$$u(t) = \Phi(y(t))$$

which assures the existence and uniqueness of the solution on $\mathbb{R}^+$ and it is such that $\mathscr{S}$ is positively invariant for the closed loop system.

For simplicity sake, in the above definition it is explicitly required that the closed loop system admits a unique solution. This property is assured if, for instance, Φ is a Lipschitz function. This request may be dropped (see Aubin, 1991 Chapter 6).

Note that the type of control considered in the definition is static. As it is known, the case of a dynamic compensator of a given order n_c is equivalent to the static case for a suitably augmented plant. In this case $\mathscr{S}$ has to be thought as a subset of the *extended state space* of dimension $n + n_c$.

In the literature there are several definitions of concepts close to positive invariance (many of them being strictly equivalent at least under some proper assumptions). For the simple exposition, we limit ourself to the above definitions which are sufficient to describe the basic results as in the purpose of this paper.

We recall now the notion of Lyapunov function. In this case also there are several definitions suggested by the literature, each of them motivated by a specific problem. Thus we try to introduce a definition which covers most of the available ones. Given a continuous function $\Psi:\mathbb{R}^n \to \mathbb{R}$, for $k_2 \ge k_1 \ge 0$ let us define the (possibly empty) set $\mathcal{N}[\Psi, k_1, k_2]$ as

$$\mathcal{N}[\Psi, k_1, k_2] = \{x \in \mathbb{R}^n\colon\ k_1 \le \Psi(x) \le k_2\}$$

(with obvious meaning we allow for $k_2 = \infty$). In the sequel we use the same notation with a single argument

$$\mathcal{N}[\Psi, k] \doteq \mathcal{N}[\Psi, 0, k] = \{x\colon\ \Psi(x) \le k\}.$$

Definition 2.4. Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function such that

$$\Psi(0) = 0, \tag{5}$$

$$\Psi(x) > 0, \quad \text{if } x \neq 0. \tag{6}$$

Consider the system

$$\dot{x}(t) = f(x(t), w(t)).$$

If for every $x \in \mathcal{N}[\Psi, k_1, k_2]$ the Lyapunov derivative (Rouche, Habets & Laloy, 1977) is such that

$$D^+\Psi(x, w) \doteq \limsup_{h \to 0^+} \frac{\Psi(x + hf(x, w)) - \Psi(x)}{h} \le -\beta\Psi(x), \quad \text{for all } w \in \mathcal{W}, \tag{7}$$

for some $\beta > 0$, then we say that Ψ is a Lyapunov function in the strong sense for the system in the set $\mathcal{N}[\Psi, k_1, k_2]$. If the inequality (7) holds for $\beta = 0$ then we say that Ψ is a Lyapunov function in the weak sense.

In the above expression the existence of the superior limit is assured by the fact that Ψ is locally Lipschitz (see Yorke, 1968 or Rouche et al., 1977, Appendix I, Theorem 4.3). The "lim sup" can be replaced by the simple "lim" under additional assumptions on Ψ such as convexity or differentiability.

According to the previous definition we may have different cases. If $k_1 = 0$ and $k_2 = \infty$ the existence of such a function assures global (asymptotic) stability of the solution $x(t)$, while if $k_1 = 0$ and k_2 is finite, then we have local (asymptotic) stability (i.e. $x(t)$ converges to the origin for any $x(0) \in \mathcal{N}[\Psi, k_2]$). If $k_1 > 0$ and $k_2 = \infty$, global uniform ultimate boundedness in the set $\mathcal{N}[\Psi, k_1]$ is guaranteed (i.e. for all $x(0) \in \mathbb{R}^n$ there exists $T(x(0))$ s.t. $x(t) \in \mathcal{N}[\Psi, k_1]$ for $t \ge T(x(0))$. If $k_2 > k_1 > 0$ we have local uniform ultimate boundedness.

The definition of a strong Lyapunov function is easily restated for discrete-time systems

$$x(t + 1) = f(x(t), w(t)) \tag{8}$$

if we replace condition (7) by the condition that for every $x \in \mathcal{N}[\Psi, k_2]$ we have that

$$\Psi(f(x, w)) \le \max\{\lambda\Psi(x), k_1\} \quad \text{for all } w(t) \in \mathcal{W}, \tag{9}$$

for some positive $\lambda < 1$ (or $\lambda = 1$ in the weak case). The expression (9) is more complex than expression (7). The reason is very simple in view of the well established gap between discrete and continuous-time systems. The fact that the sequence $\Psi(x(t))$ is decreasing, i.e. $\Psi(x(t + 1)) < \Psi(x(t))$ as long as $k_1 \le \Psi(x(t)) \le k_2$, does not avoid possible "jumps" from $x(t) \in \mathcal{N}[\Psi, k_1]$ to $x(t + 1) \notin \mathcal{N}[\Psi, k_2]$.

Finally we give the following definition of a control Lyapunov function.

Definition 2.5. Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, as in Definition 2.4. We say that Ψ is a control Lyapunov function in the strong (weak) sense for the system (4) in the set $\mathcal{N}[\Psi, k_1, k_2]$ if there exists a continuous control $u = \Phi(y)$ assuring uniqueness of solution and such that Ψ is a Lyapunov function in the strong (weak sense) for the closed loop system.

All the introduced definitions could be extended to more general classes of abstract state space systems, including automata and distributed state space systems. However, in view of the purpose of the work, we do not extend the discussion in this sense.

3. Basic results

For easiness of presentation, in the following we will consider convex positively invariant sets. The simplifying assumption of convexity avoids unnecessary difficulties in the exposition. Nevertheless, some types of non-convex sets considered in literature will be mentioned. Furthermore we assume that the state is available for feedback control.

3.1. Invariance conditions for continuous-time systems

We introduce now the definition of tangent cone to a set, which will be very useful to characterize positively invariant sets. Consider any norm $\|\cdot\|$ in $\mathbb{R}^n$. Given a point $x \in \mathbb{R}^n$ and a set $\mathcal{S}$ let us define the distance of x from $\mathcal{S}$ as

$$dist(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} \|x - y\|.$$

Definition 3.1 (*Bouligand*, 1932). Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact set. Let $x \in \mathbb{R}^n$. The tangent cone (often referred to as contingent cone) to $\mathcal{S}$ in x is the set

$$\mathcal{C}_{\mathcal{S}}(x) = \left\{z \in \mathbb{R}^n\colon\ \liminf_{h \to 0} \frac{dist(x + hz, \mathcal{S})}{h} = 0\right\}. \tag{10}$$

Note that although the function $dist(x, \mathcal{S})$ depends on the considered norm, *the set* $\mathcal{C}_{\mathcal{S}}(x)$ *does not*. It is also easy

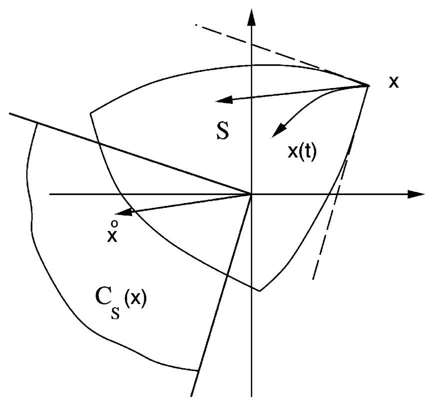

Fig. 2. The tangent cone.

to see that if $\mathcal{S}$ is convex so is $\mathcal{C}_{\mathcal{S}}(x)$ and "lim inf" can be replaced by "lim" in (10). Furthermore if $x \in int\{\mathcal{S}\}$ then $\mathcal{C}_{\mathcal{S}}(x) = \mathbb{R}^n$ and $x \notin \mathcal{S}$ then $\mathcal{C}_{\mathcal{S}}(x) = \emptyset$ (remember that $\mathcal{S}$ is compact). Thus the set $\mathcal{C}_{\mathcal{S}}(x)$ is non-trivial only on the boundary of $\mathcal{S}$. In geometric terms (see Fig. 2), the tangent cone for $x \in \partial\mathcal{S}$, is a cone having center in the origin which contains all the vectors (for instance the vector denoted by $\dot{x}$), whose directions point from x "inside" (or they are "tangent to") the set $\mathcal{S}$. If such a boundary is smooth in the point x, then the $\mathcal{C}_{\mathcal{S}}(x)$ is just the tangent halfspace shifted to the origin.

We are now able to state one basic result concerning positive invariance. This theorem was introduced for the first time by Nagumo (1942) and it was reconsidered later in different formulations (see for instance Brezis, 1970; Gard, 1980; Yorke, 1968). Here we use the standard version in terms of tangent cone (see also Aubin, 1991; Aubin & Cellina, 1988; Clarke, 1983; Feuer & Heymann, 1976) for details).

Theorem 3.1 (Nagumo, 1942). *Consider the system $\dot{x}(t) = f(x(t))$, and assume that, for each initial condition in a set $\mathcal{X}$, it admits a globally unique solution. Let $\mathcal{S} \subseteq \mathcal{X}$ be a closed and convex set. Then the set $\mathcal{S}$ is positively invariant for the system if and only if*

$$f(x) \in \mathcal{C}_{\mathcal{S}}(x) \quad \textit{for all } x \in \mathcal{S}. \tag{11}$$

The condition (11) of the theorem, often referred to as *sub-tangentiality condition*, is meaningful only for $x \in \partial\mathcal{S}$ since for $x \in int\{\mathcal{S}\}$, $\mathcal{C}_{\mathcal{S}}(x) = \mathbb{R}^n$. Thus the condition (11) can be replaced by

$$f(x) \in \mathcal{C}_{\mathcal{S}}(x) \quad \text{for all } x \in \partial\mathcal{S}.$$

The theorem has the following geometric interpretation (see Fig. 2). Roughly it says that if, for $x \in \partial\mathcal{S}$, the derivative $\dot{x}$ "points inside or it is tangent to $\mathcal{S}$", then the trajectory $x(t)$ remains in $\mathcal{S}$.

The above theorem holds for more general classes of sets which are not necessarily convex. Conversely, the requirement of the uniqueness of the solution is fundamental. Consider for instance the set $\mathcal{S} = \{0\}$ (i.e. the set including the origin only). Its tangent cone for $x = 0$ is $\{0\}$. The equation $\dot{x}(t) = \sqrt{x(t)}$ does fulfill the requirements of the theorem. However, for $x(0) = 0$ only the zero solution $x(t) = 0$, $t \geq 0$ remains inside $\mathcal{S}$. In fact, there are infinitely many non-zero solutions escaping from $\mathcal{S}$ each of them being of the form

$$x(t) = \begin{cases} 0 & \text{for } t \leq t_0, \\ (t - t_0)^2/4 & \text{for } t > t_0, \end{cases}$$

where t_0 is any non-negative real number. If uniqueness drops, the sub-tangentiality condition implies the *weak positive invariance*: for $x(0) \in \mathcal{S}$ there exists *at least one solution* such that $x(t) \in \mathcal{S}$ for $t \geq 0$ (actually this was the original Nagumo's formulation see also (Aubin & Cellina, 1988, Chapter 4, or Aubin, 1991, Chapter 1). The simple example shows that, if the solution is not unique, the *strong positive invariance* (i.e. the fact that all the solutions remain in the set) is not assured. For further details the reader is referred to Fernandes and Zanolin (1987).

The above theorem admits several extensions. For instance under the same assumptions of uniqueness of solution, the set $\mathcal{S}$ is robustly invariant for the system

$$\dot{x}(t) = f(x(t), w(t)), \quad w(t) \in \mathcal{W}$$

if and only if $\dot{x} = f(x, w) \in \mathcal{C}_{\mathcal{S}}(x)$, for all $x \in \mathcal{S}$ and $w \in \mathcal{W}$ (see again Aubin, 1991, Aubin & Cellina, 1988). Theorem 3.1, can be also used to characterize controlled invariance. Indeed the set $\mathcal{S}$ is controlled invariant if and only if there exists a continuous function $\Phi: \mathbb{R}^p \to \mathbb{R}^m$ (granting the existence and uniqueness of the solution) such that if $u = \Phi(g(x))$ for each x on the boundary of $\mathcal{S}$ the closed loop system satisfies the sub-tangentiality conditions (11).

3.2. Invariance conditions for discrete-time systems

The natural counterpart of the sub-tangentiality condition (11) for discrete-time systems is immediately written as

$$f(\mathcal{S}) \subseteq \mathcal{S}. \tag{12}$$

There is only one point worth a discussion here. As we have previously pointed out, Nagumo theorem can be formulated by saying that $\mathcal{S}$ is positively invariant if and only if (11) is satisfied just *on the boundary* of $\mathcal{S}$. This different formulation does not apply to discrete-time systems. It is very easy to provide examples of systems of the form $x(t+1) = f(x(t))$ such that $f(\partial\mathcal{S}) \subset \mathcal{S}$, but such that $f(\mathcal{S}) \not\subset \mathcal{S}$, just as the scalar system

$$x(t+1) = 2(x(t)^2 - 1), \quad \text{and} \quad \mathcal{S} = [-1, 1].$$

Nevertheless, for linear systems we have the following result.

Theorem 3.2. *Consider the LTI system*

$x(t+1) = Ax(t) + Bu(t),$

and let $\mathscr{S}$ be a C-set. Then $\mathscr{S}$ is controlled invariant if and only if for any $x \in \partial\mathscr{S}$ there exists u (which depends on x) such that

$Ax + Bu \in \mathscr{S}.$

Theorem 3.2 can be immediately extended to the case in which the control is constrained as $u \in \mathscr{U}$, where $\mathscr{U}$ is a C-set, or the system is uncertain

$x(t+1) = A(w(t))x(t) + B(w(t))u(t) + Ew(t),$

with $w(t) \in \mathscr{W}$. The condition has just to be modified as: for all $x \in \partial\mathscr{S}$ there exists $u \in \mathscr{U}$ such that $A(w)x + B(w)u + Ew \in \mathscr{S}$, for all $w \in \mathscr{W}$. It is not difficult to see that the condition may be limited to all x which are *extreme points* of the convex set $\mathscr{S}$. Conversely, the linearity of the system with respect to x and u is crucial in Theorem 3.2.

3.3. Invariant sets and Lyapunov functions

We now recall the fundamental connections between the notion of positive invariance and that of Lyapunov function. If $\Psi(x)$ is a Lyapunov function (or control Lyapunov function) in the set $\mathscr{N}[\Psi, k_1, k_2]$ for the system (3) (or (4)), then for each $k_1 \le k \le k_2$, the set $\mathscr{S} \doteq \mathscr{N}[\Psi, k]$ is invariant. To further investigate this point we introduce the following definition (see Luenberger, 1969).

Definition 3.2. Given a C-set $\mathscr{S}$ we define the Minkowski functional $\Psi_{\mathscr{S}}$ of $\mathscr{S}$ as

$\Psi_{\mathscr{S}}(x) \doteq \inf\{\mu \ge 0\colon\ x \in \mu\mathscr{S}\}.$

The function $\Psi_{\mathscr{S}}(x)$ is convex, positively homogeneous of order one (i.e. $\Psi_{\mathscr{S}}(\xi x) = \xi\Psi_{\mathscr{S}}(x)$ for $\xi \ge 0$). Note that $\mathscr{S} = \mathscr{N}[\Psi_{\mathscr{S}}, 1]$. Furthermore it is a norm if (and only if) $\mathscr{S}$ is 0-symmetric.[1] Its level surfaces are obtained by scaling the boundary of the set $\mathscr{S}$. Thus such boundary defines the shape of the function. In Fig. 3 we show the set $\mathscr{S}$ (the shadowed one), and the level surfaces corresponding to $\Psi = 0.5$, $\Psi = 1.5$ and $\Psi = 2$. By means of the previous definition we may introduce the notion of contractive set. In plain words, an invariant set $\mathscr{S}$ is contractive if it is invariant and, whenever the state is on the boundary, the control can "push it towards the interior".

[1] If $\mathscr{S}$ is convex closed and 0-symmetric, but not bounded, then $\Psi_{\mathscr{S}}$ is a seminorm.

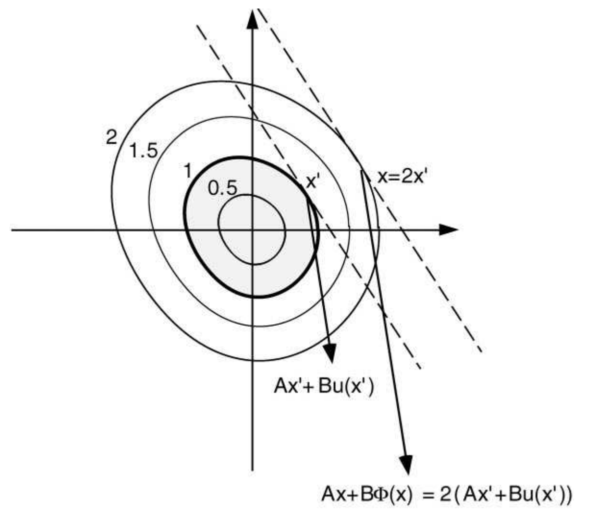

Fig. 3. The Minkowski function of $\mathscr{S}$ as a Lyapunov function.

Definition 3.3. A C-set $\mathscr{S}$ is contractive for a discrete-time system of the form (1) if there exists a control function $u(x)$ and a positive $\lambda < 1$, such that if $x(t)$ in $\mathscr{S}$ then $x(t+1) \in \lambda\mathscr{S}$ for all $w(t) \in \mathscr{W}$.

Definition 3.4. Let $\mathscr{S}$ be a C-set and let $\Psi(x)$ be its Minkowski functional. We say that $\mathscr{S}$ is contractive for a continuous-time system of the form (1) if there exists $\beta > 0$ and a control function $u(x)$ such that $D^+\Psi(x, u(x), w) \le -\beta$ (cf. Eq. (7)) for all $x \in \partial\mathscr{S}$ and $w \in \mathscr{W}$.

The above definitions are important because several techniques to derive Lyapunov functions *are based on the construction of contractive sets*. Assume that a certain contractive C-set is known. How can we derive a Lyapunov function and how can the interval $[k_1, k_2]$ be selected? Several answers to this question are available when we deal with linear systems.

Theorem 3.3. *Consider the LTI system (possibly with exogenous disturbance)*

$\Delta x(t) = Ax(t) + Bu(t) + Ew(t)$

with $w \in \mathscr{W}$, a given C-set. Assume that $\mathscr{S}$ is a contractive C-set. Then the following statements hold.

(i) *The Minkowski functional $\Psi_{\mathscr{S}}(x)$ of $\mathscr{S}$ is a control Lyapunov function in $\mathscr{N}[\Psi_{\mathscr{S}}, 1, \infty]$ and there exists a control $u = \Phi(x)$ such that the closed loop system state is globally ultimately bounded in $\mathscr{S}$.*
(ii) *If $E = 0$ then the Minkowski functional $\Psi_{\mathscr{S}}(x)$ is a global (i.e. on $\mathbb{R}^n$) control Lyapunov function for the system and there exists a control $u = \Phi(x)$ that globally asymptotically stabilizes the system.*

(iii) *If* $E=0$ *and* $u \in \mathscr{U}$, *a given C-set, then* $\Psi_{\mathscr{S}}(x)$ *is a control Lyapunov function for the system in the set* $\mathscr{N}[\Psi_{\mathscr{S}},1]$ *and there exists a control* $u=\Phi(x)$ *that locally asymptotically stabilizes the system* (*i.e.* $x(t)\to 0$ *for all* $x(0)\in\mathscr{N}[\Psi_{\mathscr{S}},1]$).

(iv) *If* $E=0$ *and* $B=0$ *then* $\Psi_{\mathscr{S}}(x)$ *is a Lyapunov function and the system* $\Delta x(t)=Ax(t)$ *is asymptotically stable.*

Given a contractive set, a control $\Phi(x)$ as in the theorem[2] is achieved by linearly scaling the values taken by $u(x)$ on the boundary of $\mathscr{S}$ as follows:

$$\Phi(x) \doteq \begin{cases} \Psi_{\mathscr{S}}(x)u(x') & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

$$\text{where } x' = x/\Psi_{\mathscr{S}}(x) \in \partial\mathscr{S}. \tag{13}$$

For instance consider the point $x=2x'$ in Fig. 3, where $x'\in\partial\mathscr{S}$. Then $\Psi(x')=1$ and $\Psi(x)=2$. The derivative in x' is $Ax'+Bu(x')$. Assume that it satisfies the sub-tangentiality conditions. The derivative in x is the same up to the scaling factor 2: $Ax+B\Phi(x)=2[Ax'+Bu(x')]$. Moreover, the tangent cone in both x and x' is the same, that is, the line passing through the origin parallel to the dashed lines. Thus the sub-tangentiality conditions are satisfied also in x. This property can also be shown by means of the Lyapunov derivative since it is very easy to see that the ratio $D^+\Psi/\Psi$ is equal in x' and x. Theorem 3.3 holds even if we assume that the state and input matrices are continuous functions $A(w(t))$ and $B(w(t))$ of the uncertain parameter $w(t)\in\mathscr{W}$.

3.4. Literature review

The proofs of the statements of this section are contained in several references such as Bitsoris (1988a), Bitsoris (1988b), Blanchini (1995), Feuer and Heymann (1976), Sznaier (1993), and Vassilaki and Bitsoris (1989).

The mentioned notion of tangent cone and its application to the characterization of positive invariance for differential inclusions is investigated in Aubin (1991) and Aubin and Cellina (1988). It is important to point out that there are further definitions of tangent cone due to Bony (1969) and Clarke (1983). However, all these definitions are equivalent under the assumptions that $\mathscr{S}$ is a convex set, as in our case, or that it has a smooth surface.

For a mathematical background concerning theoretical properties of invariant sets (such as their stability or attractivity) for ordinary differential equations the reader is referred to classical books concerning Lyapunov methods such as Hahn (1967), Lassalle and Lefschetz (1961), Lyapunov (1966), Rouche et al. (1977), Yoshizawa (1975) and Zubov (1964).

It is worthwhile mentioning the problem of *partial stability*, in which only part of the system state variables are required to be bounded (or to converge) (Vorotnikov, 1993; Vorotnikov, 1998; Oziraner, 1979). In this case, Lyapunov-like functions are considered which are not necessarily positive definite (they are only positive definite with respect to the considered subset of variables). These functions are associated to positively invariant sets that may be unbounded.

[2] Note that the control $u(x)$ in Definitions 3.3 and 3.4 is not necessarily stabilizing and it may be even not defined outside $\mathscr{S}$.

4. Special families of positively invariant sets

In this section we present some properties characterizing two important families of positively invariant sets, or controlled invariant sets, and their associated controllers which have been particularly successful in the solution of control engineering problems. These are the classes of ellipsoidal sets and the class of polyhedral sets. Other types of sets will be briefly considered at the end. Furthermore, we limit our attention mainly to linear systems. Nevertheless the few available results for non-linear differential/difference equations will be mentioned.

4.1. Ellipsoidal invariant sets

Ellipsoids are very popular as candidate invariant sets. An ellipsoidal set can be always represented as follows:

$$\mathscr{S} = \{x\in\mathbb{R}^n\colon x^{\mathrm{T}}Px \le 1\} \tag{14}$$

where P is a symmetric positive-definite matrix. The Minkowski function of $\mathscr{S}$ is the quadratic norm

$$\Psi_{\mathscr{S}}(x) = \|x\|_P \doteq (x^{\mathrm{T}}Px)^{1/2}$$

It is fundamental to note that the gradients of the functions $\Psi(x)$ and $\Psi(x)^2=(x^{\mathrm{T}}Px)$ have the same direction. Thus it makes no difference to take one or the other as Lyapunov function, since their Lyapunov derivatives have the same sign.

The tangent halfspace to $\mathscr{S}$ in $x\in\partial\mathscr{S}$ is $\mathscr{C}_{\mathscr{S}}(x)=\{y\colon 2x^{\mathrm{T}}Py\le 0\}$. Consider the case of a linear system $\dot{x}=Ax$. The sub-tangentiality conditions are given by $2x^{\mathrm{T}}PAx\le 0$, as long as $x^{\mathrm{T}}Px=1$. Such a condition can be written as

$$x^{\mathrm{T}}(A^{\mathrm{T}}P+PA)x\le 0$$

$\forall x\in\mathbb{R}^n$. Thus the Nagumo invariance condition leads to the Lyapunov inequality above. If $\mathscr{S}$ is contractive then there exists $\beta>0$ such that $x^{\mathrm{T}}(A^{\mathrm{T}}P+PA)x\le -2\beta x^{\mathrm{T}}Px<0$, for all $x\in\mathbb{R}^n$. This can be expressed, as it is known, by means of the Lyapunov equation

$$A^{\mathrm{T}}P+PA=-Q \quad \text{with } Q\succ 0, \tag{15}$$

where $Q \succ 0$ ($\prec 0$) means symmetric positive (negative) definite. Similarly it is immediate to see that the boundary contraction property for the discrete-time linear system $x(t+1) = Ax(t)$ leads to the discrete-time Lyapunov equation

$$A^{\mathrm{T}}PA - P = -Q \quad \text{with } Q \succ 0.$$

Let us consider now the case of an uncertain system. If A is a matrix polytope

$$A(w) = \sum_{i=1}^{r} A_i w_i \quad \text{where } \sum_{i=1}^{r} w_i = 1, \quad w_i \geq 0, \tag{16}$$

then the robust invariance condition for $\mathscr{S}$ can be expressed by means of a set of Lyapunov inequalities:

$$A_i^{\mathrm{T}}P + PA_i \prec 0, \quad i = 1, 2, \ldots, r.$$

This set of conditions is nice because it is convex in P. This means that if P_1 and P_2 satisfy the inequalities, also the convex combination $\alpha P_1 + (1-\alpha)P_2$ does. This property renders computationally tractable the problem of finding an invariant ellipsoid $\mathscr{S}$ for the polytope of matrices $A(w)$, (Boyd, Ghaoui, Feron & Balakrishnan 1994). Additional "convex" conditions such as $\mathscr{S}$ is included in a given polyhedron or that $\mathscr{S}$ includes a given polytope, can be easily incorporated (see again Boyd et al., 1994, Section 5.2).

An important property is that if $\mathscr{S}$ is contractive for the system $\dot{x} = Ax + Bu$ with some control $u(x)$ (see Definition 3.4), then $\mathscr{S}$ is contractive with a *linear* control which is of the form

$$u(t) = -\gamma B^{\mathrm{T}}Px(t), \tag{17}$$

where $\gamma > 0$ is a "sufficiently large" constant. Furthermore, if A is stable, and P is derived by means of (15), for some $Q \succ 0$, then any $\gamma > 0$ results in a stabilizing control (Gutman & Hagander, 1985).

Conversely, if $A(w)$ and $B(w)$ are uncertain, we cannot associate in general a linear controller to a controlled invariant ellipsoidal set (Petersen, 1985). A linear controller exists provided that B is certain or it satisfies some matching conditions (Barmish, Corless & Leitmann, 1983a; Barmish, Petersen & Feuer, 1983b; Corless & Leitmann, 1993). Consider now the case in which additive persistent disturbances are present

$$\dot{x}(t) = Ax(t) + Ew(t), \quad w(t) \in \mathscr{W}.$$

A known sufficient positive invariance condition holds in the special case $\mathscr{W} = \{w\colon w^{\mathrm{T}}w \leq 1\}$ (Schweppe, 1973), (Usoro, Schweppe, Gould & Wormley, 1982a). The ellipsoid (14) is robustly invariant if $Q = P^{-1}$ satisfies the condition

$$QA^{\mathrm{T}} + AQ + \alpha Q + \frac{1}{\alpha}EE^{\mathrm{T}} \preceq 0 \quad \text{for some } \alpha > 0, \tag{18}$$

where "$\preceq$" means negative semidefinite. For fixed $\alpha > 0$, the first inequality in (18) is a LMI (see Boyd et al., 1994, Section 6.1.3). Very recently the above condition has been proven to be also necessary if (A, E) is a reachable pair, (Brockman & Corless, 1998).

Further properties concerning positively invariant ellipsoidal sets, in particular their connections with the Riccati equation, can be found in textbooks of linear systems, for instance Schweppe (1973), Boyd et al. (1994), Zhou, Doyle and Glover (1994) and Sanchez Pena and Sznaier (1998).

4.2. Polyhedral invariant sets

Polyhedral sets have been involved in the solution of control problems starting from the 70's. Their importance is due to the fact that they are often natural expressions of physical constraints on state and control variables. Furthermore, their shape is in some sense "more flexible" than that of the ellipsoids, this fact leading to a better attitude in the approximation of reachability sets and domains of attraction of dynamic systems. The trade off of this flexibility is that they have in general a more complex representation.

A polyhedral C-set $\mathscr{S}$ can be represented in the following form (plane representation):

$$\mathscr{S} = \{x\colon Fx \leq \bar{1}\} \tag{19}$$

where F is a $r \times n$ matrix, and $\bar{1} \in \mathbb{R}^r$ denotes a vector of the form

$$\bar{1} = [1 \quad 1 \quad \ldots \quad 1]^{\mathrm{T}},$$

and the inequality has to be thought componentwise. Conversely $\mathscr{S}$ can be represented as

$$\mathscr{S} = \{x = X\alpha,\ \bar{1}^{\mathrm{T}}\alpha = 1,\ \alpha \geq 0\} \tag{20}$$

(vertex representation) where X is a $n \times s$ matrix, and $\alpha \in \mathbb{R}^s$ (note that $\bar{1}^{\mathrm{T}}\alpha = \sum_{i=1}^{s}\alpha_i$). In the representation (19), each row F_k of F is associated with the linear inequality $F_k x \leq 1$ while each column of the matrix X is a vertex of $\mathscr{S}$ in the representation (20). We state now some basic conditions for positive invariance of the considered sets.

Theorem 4.1. *The following conditions are equivalent.*

(i) *The polyhedral C-set $\mathscr{S}$ is positively invariant (contractive) for the system*

$$x(t+1) = Ax(t)$$

(ii) *There exists a positive $\lambda \leq 1$ ($\lambda < 1$), and an $r \times r$ nonnegative matrix H such that*

$$H\bar{1} \leq \lambda\bar{1},$$

$$HF = FA.$$

(iii) *There exists a positive* $\lambda \le 1$ ($\lambda < 1$), *and an* $s \times s$ *non-negative matrix P such that*

$$\bar{1}^T P \le \lambda \bar{1}^T,$$

$$XP = AX.$$

Condition (ii) is based on the following property (derived by Farkas' lemma, see for instance Hennet, 1989).

Lemma 4.1. *Given two polyhedra* $\mathscr{S}_1 = \{x\colon F_1 x \le g_1\}$ *and* $\mathscr{S}_2 = \{x\colon F_2 x \le g_2\}$ *then* $\mathscr{S}_1 \subseteq \mathscr{S}_2$ *if and only if there exists a non-negative matrix H such that* $HF_1 = F_2$ *and* $Hg_1 \le g_2$.

The set $\mathscr{S}$ is positively invariant iff $x \in \mathscr{S}$ implies $Ax \in \lambda\mathscr{S}$, say $FAx \le \bar{\lambda}1$. This means that the polytope $\mathscr{S}$ must be included in the polytope $\{x\colon FAx \le \bar{\lambda}1\}$. The application of the Lemma leads to (ii). Condition (iii) can also be easily explained. Each vector $x \in \mathscr{S}$ can be expressed as a convex combination of its vertices $x = X\alpha$. Then it is very easy to show that $\mathscr{S}$ is invariant iff $Ax_j \subseteq \lambda\mathscr{S}$, for each $x_j \in vert\{\mathscr{S}\}$ say if $Ax_j = Xp_j$, where p_j is a non-negative vector such that $\sum_{i=1}^{s} p_{ij} = \bar{1}^T p_j \le \lambda \le 1$, where p_{ij} is the ith component of p_j. By grouping together these equalities and inequalities, and setting $P = [p_{ij}]$, we get (iii).

To state the corresponding continuous-time result, we need the following definition.

Definition 4.1. The matrix $\hat{H}$ is called an M-matrix if all its non-diagonal entries are non-negative.

Theorem 4.2. *The following conditions are equivalent.*

(i) *The polyhedral* 0*-symmetric C-set* $\mathscr{S}$ *is positively invariant (contractive) for the system*

$$\dot{x}(t) = Ax(t). \tag{21}$$

(ii) *There exists* $\beta \ge 0$ ($\beta > 0$), *and an* $r \times r$ *M-matrix* $\hat{H}$ *such that*

$$\hat{H}\bar{1} \le -\beta\bar{1},$$

$$\hat{H}F = FA.$$

(iii) *There exists* $\beta \ge 0$ ($\beta > 0$), *and an* $s \times s$ *M-matrix* $\hat{P}$ *such that*

$$\bar{1}^T \hat{H} \le -\beta\bar{1}^T,$$

$$X\hat{H} = AX.$$

We explain the previous theorem, by means of the following property (Blanchini, 1991a).

Lemma 4.2. *The polytope* $\mathscr{S}$ *is contractive for system* (21) *if and only if there exists* $\bar{\tau} > 0$ *such that for* $0 < \tau \le \bar{\tau}$, $\mathscr{S}$ *is contractive for the following discrete-time Euler Approximating System* (*EAS*)

$$x(t+1) = [I + \tau A]x(t) \tag{22}$$

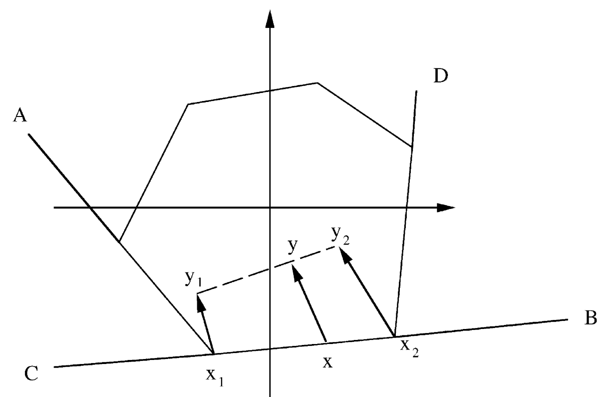

Fig. 4. The control at vertices.

The meaning of the lemma is immediate. The subtangentiality condition requires that for $x \in \partial\mathscr{S}$ the derivative $\dot{x} = Ax$ points "inside" the set $\mathscr{S}$. Thus, for a sufficiently small τ, $y = x + \tau Ax \in \mathscr{S}$ (see Fig. 4) which implies invariance of the EAS by Theorem 3.2. If we apply the conditions of Theorem 4.1 to the EAS we easily derive conditions (ii) and (iii) in Theorem 4.2 with $\hat{H} = (H - I)/\tau$, $\hat{P} = (P - I)/\tau$ and $\beta = (1 - \lambda)/\tau$.

In the literature there are further invariance conditions which are basically equivalent to those presented. Note also that the conditions (ii) of Theorems 4.1 and 4.2 apply to the cases in which $\mathscr{S}$ is not necessarily bounded, and the conditions (iii) of both theorems apply to the case in which $\mathscr{S}$ has empty interior. Theorems 4.1 and 4.2 show that

- if a polyhedral C-set $\mathscr{S}$ is given (say F or X are fixed) checking its positive invariance is a *linear programming problem*;
- If F or X are to be determined, the conditions become *bilinear*

The controlled-invariance of a polyhedral set $\mathscr{S}$ for a discrete-time linear system can be easily characterized as follows. The set $\mathscr{S}$ is controlled invariant for the system

$$x(t+1) = Ax(t) + Bu(t)$$

if and only if there exists a positive $\lambda \le 1$, a $s \times s$ matrix $P \ge 0$, and a $m \times s$ matrix U such that

$$AX + BU = XP, \tag{23}$$

and

$$\bar{1}^T P \le \lambda \bar{1}^T. \tag{24}$$

This is a compact formulation of the result in Gutman and Cwikel (1986a). The columns of the matrix U are the controls u_k associated to the the columns of X say the vertices x_k of $\mathscr{S}$. Expression (23) is equivalent to

$Ax_k + Bu_k = \sum_{k=1}^{s} p_{ik} x_i \in \lambda \mathscr{S}$. The inclusion is due to the fact that from (24) we get $\sum_{k=1}^{s} p_{ik} \leq \lambda$. Now, if we are able to associate to each vertex of $\mathscr{S}$ a control that "pushes" the state inside $\mathscr{S}$, then we may define, in view of the system linearity, a control that pushes each point of the boundary inside the set to apply Theorem 3.2. Consider the set in Fig. 4. If for the vertices x_1 and x_2 there exist controls u_1 and u_2 such that $Ax_i + Bu_i = y_i \in int[\mathscr{S}]$, $i = 1, 2$, and if we take a point x on the segment having extrema x_1 and x_2, $x = \xi x_1 + (1 - \xi) x_2$, $0 \leq \xi \leq 1$, then $u = \xi u_1 + (1 - \xi) u_2$ maps x in $y = \xi y_1 + (1 - \xi) y_2$, which is also in the interior of $\mathscr{S}$. This basic idea is easily generalizable to the n-dimensional case.

Conditions analogous to (23) and(24) hold in the continuous-time case (i.e. $AX + BU = X\hat{P}$, and $\bar{1}^T \hat{P} \leq -\beta \bar{1}^T$, with $\hat{P}$ M-matrix). They reflect the fact that the Nagumo sub-tangentiality condition can be assured by some continuous control on $\partial \mathscr{S}$ if and only if it can be assured in each vertex of $\mathscr{S}$ (Blanchini, 1995; Blanchini & Miani, 1996c).

The mentioned results have been presented in the literature by several contributions, although in slightly different (but strictly equivalent) forms. The positive invariance conditions for discrete-time systems are given in Benzaouia and Burgat (1988a), Bitsoris (1988a, b). The corresponding continuous-time conditions are in Vassilaki and Bitsoris (1989) and Bitsoris (1991). It is worth mentioning that very similar results where previously published in Molchanov and Pyatnitskii (1986, Part III) for polyhedral Lyapunov functions for uncertain systems. Invariance conditions of polytopes for systems with additive disturbances have been presented in Blanchini (1991a), De Santis (1994), Milani and Dorea (1996) and Tarbouriech and Burgat (1994). The case of systems with parametric uncertainties is considered (beside Molchanov & Pyatnitskii, 1986) in Rachid (1991), Blanchini (1991a), Sznaier (1993), Benzaouia and Mesquine (1994) and Milani and Carvalho (1995a). Extensions to singular systems have been presented in Georgiou and Krikelis (1991), Tarbouriech and Castelan (1993) and Tarbouriech and Castelan (1995). In Bitsoris and Gravalou (1995a) similar invariance results are given for sets represented by nonlinear constraints and nonlinear systems by means of the so called *comparison system* (see also Rouche et al., 1977 for a definition). The conditions (23) for controlled invariance have been given in Gutman and Cwikel (1986a). The extension to uncertain systems is in Blanchini (1995).

The final question we would like to deal with is the existence of positively invariant (contractive) polyhedral C-sets for linear systems. In the ellipsoidal case it is well-known that a sufficient and necessary condition is the marginal stability. This is not the case for polyhedral sets. It turns out that if a system is asymptotically stable then it admits polyhedral contractive C-sets (Benzaouia & Burgat, 1989b), although there is no upper bound for the complexity of their representation (i.e. the number of vertices or delimiting planes). Sufficient existence conditions for special classes of polytopes, with bounded complexity, have been presented in Bitsoris (1988b) and Hennet and Lasserre (1993) for discrete-time systems (Bitsoris, 1991) and in Castelan and Hennet (1993) for continuous-time systems. Necessary and sufficient existence conditions for marginally stable systems in terms of some phase conditions of the eigenvalues are found in Blanchini (1992). The problem of the existence of finitely determined invariant polyhedra (which have some special optimality or ε-optimality properties) under marginal stability assumptions is discussed in Gilbert and Tan (1991).

4.3. Invariant sets of other types

There are several further special families of sets that have been considered in literature beside ellipsoids and polyhedra. These sets and their associated Lyapunov functions have been involved mostly in stability analysis problems.

Piecewise-quadratic functions have been used by Xie, Shishkin and Fu (1995) for robustness analysis. They have been used in Rantzer and Johansson (1997b) for the robust stability analysis of hybrid systems. Polynomial Lyapunov functions (which are not necessarily convex) have been introduced in Zelentsowsky (1994). In Kiendl, Adamy and Stelzner (1992) vector norms are studied as candidate Lyapunov functions. Some conditions assuring the negativity of the derivative are given which generalize those presented here for polyhedral functions.

In the non-linear system case the determination of a (control) Lyapunov function is in general a hard task and this fact is a well-known practical limitation of the Lyapunov theory. It is natural that the same troubles are met in determination of an invariant (contractive) set. Fortunately, for special classes of non-linear systems of practical importance the "shape" of invariant sets can be suggested by the structure of the system. This is the case, for instance, of Hamiltonian systems (Fradkov, 1996; Fradkov, Makarov, Shiriaev & Tomchina, 1997; Rumiantsev, 1971; Shiriaev & Fradkov, 1998) which admit the (generalized) energy as natural Lyapunov function. Further techniques to construct Lyapunov functions and invariant sets for classes of non-linear systems are based on some integral expressions of the state variables. The reader is referred to Miller and Michel (1982) (see Sections 5.11 and 5.15) for details.

5. Construction of invariant sets and control synthesis

In this section we present some basic construction techniques and investigate the structure of the controllers that can be associated to controlled-invariant sets.

Essentially the construction of *ellipsoidal invariant sets* is based on the Lyapunov equation, on the Riccati equation or on linear matrix inequality techniques. We try to explain the success of these tools by considering a simple case. Let us suppose we wish to find a linear stabilizing gain K for a reachable pair (A, B) together with a quadratic control Lyapunov function $x^{\mathrm{T}}Px$. Then we have to consider the equation

$$(A + BK)^{\mathrm{T}}P + P(A + BK) \prec 0, \quad P \succ 0.$$

This equation is bilinear in P and K, thus not easy to handle as it is written. However, one can multiply both sides of the l.h.t. of the first inequality by $Q \doteq P^{-1}$ and parameterize K as $K = YP$ to achieve the following *linear matrix inequality*, which is much easier to handle

$$QA^{\mathrm{T}} + AQ + Y^{\mathrm{T}}B^{\mathrm{T}} + BY \prec 0, \quad Q \succ 0$$

(see Boyd et al. (1994) Section 7.2.1) The above property can be easily extended to the case of an uncertain pair (A, B). A quadratic control Lyapunov function and the associated linear control may be also determined by means of an algebraic Riccati equation (see for instance Zhou et al. (1994)).

In connection with the robust stabilization problem, pioneer papers for the construction of quadratic functions are Horisberger and Belanger (1976), Barmish et al. (1983a, b), Gutman (1979). The basic idea is to start with the construction of a quadratic function for the "nominal" system (assumed stable without loss of generality) and subsequently synthesize the controller which counteracts the perturbations. In this process the so called matching conditions for the uncertainties become fundamental. In later works, a Riccati equation approach has been proposed in Petersen and Hollot (1986), Rotea and Khargonekar (1989). These contributions led to important connections between the $\mathscr{H}_\infty$ theory and the quadratic stabilization theory (Khargonekar, Petersen & Zhou, 1990). A convex optimization approach to synthesize quadratic functions has been considered in Bernussou, Peres and Geromel (1991), and Gu, Chen, Zohdy and Loh (1991). A thorough review concerning the stabilization problem via quadratic functions can be found in Corless (1994). The Riccati equation has been used to provide ellipsoidal invariant domains in the solution of stabilization problems under control constraints (Lin, Saberi & Stoorvogel 1996a; Wredenhagen & Belanger, 1994; Kim & Bien, 1994). Constructive methods to determine invariant ellipsoids for systems with additive perturbations based on Eq. (18) are presented in Schweppe (1973), Usoro et al. (1982a), Boyd et al. (1994), Brockman and Corless (1997).

Constructing a *polyhedral invariant set* $\mathscr{S}$ is in principle harder that the computation of an ellipsoidal invariant set, because, as we have seen, the invariance conditions when $\mathscr{S}$ is not fixed are bilinear. Essentially the construction techniques are of two fundamental categories:

- Iterative methods leading to sets with some optimality properties but usually of high complexity;
- eigenstructure analysis/assignment methods leading to sets of low complexity.

The first procedures for the construction of invariant polyhedral sets appeared in the early 70's in the dynamic programming context, in particular as far it concerns the so called min–max infinite-time reachability problem (Bertsekas & Rhodes, 1971a, b; Bertsekas, 1972). Given a discrete-time system

$$x(t+1) = Ax(t) + Bu(t) + Ew(t),$$

$$w(t) \in \mathscr{W}, \qquad u(t) \in \mathscr{U}, \qquad x(t) \in \mathscr{X},$$

where $\mathscr{W}$, $\mathscr{X}$ and $\mathscr{U}$ are assigned constraint sets, the min–max reachability problem consists in finding a strategy $u(t) = \Phi(x(t))$ such that the constraints on u and x are satisfied for all $w(t) \in \mathscr{W}$. To provide necessary and sufficient conditions for the existence of such a strategy one has to construct the infinite-time reachability set which is the largest invariant subset of $\mathscr{X}$. This set can be computed by means of a backward procedure whose main idea is now explained.

Let $k = 0$ and $\mathscr{X}_0 = \mathscr{X}$. Denote by $\mathscr{X}_{-1}$ the set of all states $x \in \mathscr{X}$ for which there exists $u(x) \in \mathscr{U}$ such that the following state is included in $\mathscr{X}_0$ for all possible actions of the disturbance. Then repeat the argument by defining

$$\mathscr{X}_{-k-1} = \{x \in \mathscr{X}\colon \exists u(x) \in \mathscr{U}\colon \quad Ax + Bu(x) + Ew \in \mathscr{X}_{-k},\ \forall w \in \mathscr{W}\}. \tag{25}$$

The sequence $\mathscr{X}_{-k}$ is nested as $\mathscr{X}_{-k-1} \subseteq \mathscr{X}_{-k}$. The infinity-time reachability set is

$$\mathscr{X}_{-\infty} \doteq \bigcap_{i=0}^{\infty} \mathscr{X}_{-i}.$$

If $\mathscr{X}_{-i-1} = \mathscr{X}_{-i}$ for some $i \geq 0$, then $\mathscr{X}_{-i} = \mathscr{X}_{-\infty}$. The importance of the infinite-time reachability set lies in two basic facts. First, the desired (non-linear) strategy exists if and only if $\mathscr{X}_{-\infty}$ in non-empty. Second the control strategy Φ can be derived by means of $\mathscr{X}_{-\infty}$. The principles of this technique have been published in Bertsekas and Rhodes (1971a, b), Bertsekas (1972) and Glover and Schweppe (1971).

Techniques which are similar in spirit to the one sketched above have appeared later. In Morris and Brown (1976), Gutman and Cwikel (1986b), (1987), Keerthi and Gilbert (1987) and Blanchini, Mesquine and Miani (1995) recursive procedures for the backward construction of polytopic invariant sets under control and state constraints are presented. In particular, (Gutman & Cwikel, 1987 and Keerthi & Gilbert, 1987) considered the same example presented in the introduction of this paper so the reader can compare that kind of sets and the invariant ellipsoid provided here. In Lasserre (1993), the problem is solved under control constraints only. It is shown that, to reduce the computational burden, one can decompose the system in its unstable and stable part, neglecting the latter.

Procedures to derive polyhedral contractive sets and the associated Lyapunov functions for the robust stability analysis can be found in Brayton and Tong (1979), (1980), Michael, Nam and Vittal (1984), Molchanov and Pyatnitskii (1986), Ohta, Imanishi, Gong and Haneda (1993), Bhaya and Mota (1994), Blanchini and Miani (1996a), and in Blanchini (1994) and (1995) for the robust stabilization problem.

For what concerns the practical computation of the sequence in (25) it is important to notice the if $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{W}$ are polyhedral, so are the sets $\mathscr{X}_{-i}$ (this property does not hold if we consider ellipsoids (Glover & Schweppe, 1971; Bertsekas & Rhodes, 1971a). Unfortunately the intersection $\mathscr{X}_{-\infty}$ is not necessarily polyhedral (but it can be arbitrarily approximated by a polyhedron). For illustrative examples of the sequence $\mathscr{X}_{-k}$ the reader is referred to the papers by Gutman and Cwikel (1987), Keerthi and Gilbert (1987) and Blanchini (1994). The computation of $\mathscr{X}_{-k-1}$, given $\mathscr{X}_{-k}$, involves a projection procedure and the elimination of redundant constraints. Computational details are reported in Keerthi and Gilbert (1987) and Blanchini and Miani (1996c). In the absence of control (i.e. $B = 0$) the recursion is assured to converge in a finite number of steps (i.e. $\mathscr{X}_{-\infty} = \mathscr{X}_{-k}$ for some k), thus producing a finitely determined $\mathscr{X}_{-\infty}$, if A is asymptotically stable and $E = 0$ (Gilbert & Tan, 1991) or if A is asymptotically stable and $\mathscr{X}$ is "sufficiently large" with respect to $\mathscr{W}$ (Blanchini, 1992; Blanchini & Sznaier, 1995a; Kolmanovski & Gilbert, 1995).[3]

A fundamental difference with the ellipsoidal invariant sets is that, even in the case of linear systems, a controlled invariant polytope $\mathscr{S}$ cannot in general be associated to a linear feedback. Unfortunately, nonlinear compensators may be extremely complex for the implementation. So an important question is how to construct a controlled invariant polyhedral set which can be associated to a linear controller. There are quite different approaches to deal with this problem. The first one consists in fixing a polyhedral set $\mathscr{S}$ and characterizing the *family of all linear controllers* that renders this set invariant. This problem has been formulated and solved in Vassilaki, Hennet and Bitsoris (1988) and Mehdi and Benzaouia (1989) for discrete-time systems and in Vassilaki and Bitsoris (1989) for continuous-time systems, by means of linear programming. There are several extension to these results to uncertain systems (Blanchini, 1991a; Sznaier, 1993; Benvenuti & Farina, 1998), decentralized control (Bitsoris, 1988c), singular systems (Georgiou & Krikelis, 1991; Tarbouriech & Castelan, 1993, 1995), non-linear systems (Bitsoris & Gravalou, 1995a), and time-delay systems (Hennet & Tarbouriech, 1997; Hmamed, Benzaouia & Bensalah, 1995). This approach has the trouble that the problem may have no feasible solutions for unsuitable choices of $\mathscr{S}$. An algorithm based on pole placement that can be applied in the case of a failure of the LP problem is discussed in Bitsoris and Gravalou (1994).

The problem mentioned above can be faced *by means of analysis of invariant subspaces and eigenstructure assignment*. Structural conditions for the existence of stabilizing compensators that render invariant a (possibly unbounded) set of the form

$$\mathscr{P}(G,\rho) \doteq \{x\colon\ -\rho \le Gx \le \rho\} \tag{26}$$

where $G \in \mathbb{R}^{s\times n}$, for some integer $s \le rank\{B\}$, $rank\{GB\} = rank\{B\}$ and $\rho \in \mathbb{R}^s$, are given in Castelan and Hennet (1992) where it is shown that $ker\{G\}$ has to be an (A, B)-invariant subspace[4] and the triple (A, B, G) must satisfy some minimum phase conditions (which are fundamental to assure closed loop stability which is not guaranteed as $\mathscr{P}(G,\rho)$ is not a C-set as required in Theorem 3.3). Indeed, if these minimum phase conditions fail, we can assure only partial stability as defined in Vorotnikov (1993), (1998) and Oziraner (1979) with respect to the partial state variable $y = Gx$.

The eigenstructure assignment approach can be applied in the presence of control constraints of the form $-\bar{1} \le u \le \bar{1}$. If a linear gain K is considered, this produces state constraints we can write as $x \in \mathscr{P}(K, \bar{1})$ (cf. (26)). Thus an interesting problem is how to render invariant the set $\mathscr{P}(K, \bar{1})$. The problem is similar to the previous with the difference that $\mathscr{P}(K, \bar{1})$ is now a function of the compensator gain K. This problem has been considered in Benzaouia and Burgat (1988a) and Hennet and Beziat (1991). In Bitsoris and Vassilaki (1990), it is shown that there exists a stabilizing gain K assuring the invariance of $\mathscr{P}(K,\bar{1})$ if and only if the number of unstable pole does not exceed the number of control inputs (see also Bitsoris & Vassilaki, 1995b and Benzaouia & Hmamed, 1993). Extensions of these results are in Benzaouia and Mesquine (1994), where uncertain systems are considered, and in Dorea and Milani (1995a) where a modified LQ problem is investigated.

Some contributions in literature show how to determine invariant sets included in polyhedra of the form

[3] For instance if it includes in its interior the 0-reachable set of the system (A, E) with constrained input $w \in \mathscr{W}$.

[4] A subspace X is (A, B)-invariant if there exists K such that $(A + BK)X \subseteq X$.

$\mathscr{P}(G,\rho)$ a posteriori, i.e. when a stabilizing control law $u = Kx$ is assigned. Results in this direction are given in Gilbert and Tan (1991) where it is shown that given an asymptotically stable (closed loop) discrete-time system $x(t+1) = A_{cl}x(t)$ and a polyhedral set $\mathscr{P}(G,\rho)$, the largest invariant set in $\mathscr{P}(G,\rho)$ is polyhedral. Such a result admits extensions to the case in which additive disturbances and parametric uncertainties are present (Blanchini, 1992; Blanchini & Sznaier, 1995a; Blanchini, Miani & Sznaier, 1997; Kolmanovski & Gilbert, 1995). In the uncertain case, these very procedures serve as stability tests, analogous to those in Brayton and Tong (1979), (1980), Michael et al. (1984), Molchanov and Pyatnitskii (1986) and Ohta et al. (1993).

As it has been mentioned before, since it is not possible *in general* to associate a linear controller to a controlled-invariant polytope, it is often necessary to consider nonlinear control laws. In Gutman and Cwikel (1986a) it is shown that in the discrete-time case the feedback law that can be associated to a polytope can be inferred from the values of the control at the vertices. One of the controllers so obtained is piecewise linear. In Keerthi and Gilbert (1987) the discrete minimum-time problem is considered. The k-step controllability sets $\mathscr{R}_k$ to the origin (which are invariant) are computed by means of a backward procedure. The compensator expression is derived by imposing that the system state $x(t)\in\mathscr{R}_k$ moves to $\mathscr{R}_{k-1}$. Another possibility is that of providing a nonlinear controller by solving on-line an optimization problem. This technique originated in Bertsekas and Rhodes (1971a) and Glover and Schweppe (1971). More recent contributions on this type of control strategies are in Sznaier and Damborg (1990), (1992), (1993), Blanchini and Ukovich (1993) and Shamma (1996).

In the continuous-time case one of the basic principles to associate a control law to a controlled invariant set $\mathscr{S}$ *is based on the sub-tangentiality conditions*. For any $x\in\mathscr{S}$ the control u must be taken in such a way such that $Ax + Bu \in \mathscr{C}_{\mathscr{S}}(x)$. If the Minkowski function $\Psi_{\mathscr{S}}$ is continuously differentiable this condition can be written by means of its gradient (under weaker assumptions one can use the tangent cone, see for instance Maderner, 1992). It is very easy to show that, in the quadratic case this leads to the expression (17) Barmish et al. (1983a).

If u is constrained as $u\in\mathscr{U}$, then we can choose the one of minimum Euclidean norm assuring a prescribed degree of contractivity $\beta > 0$ (say the condition $\dot{\Psi} \leq -\beta\Psi$) (Peteresen & Barmish, 1987). If $\mathscr{S}$ and $\mathscr{U}$ have smooth surfaces, such a control is continuous. If the function $\Psi_{\mathscr{S}}$ is non-differentiable, as in the polyhedral case, these methods are more difficult to be implemented. It can be shown (Blanchini, 1995) that the piecewise-linear control of Gutman and Cwikel (1986a) applies to continuous-time systems. In the single-input case, a bang-bang stabilizing controller has been proposed in Blanchini and Miani (1996b).

The association of a proper control action to a nonlinear systems that renders a set $\mathscr{S}$ invariant is based on the same principles exposed above. The basic concept is the so called regulation map or feedback map (see Aubin, 1991, Definition 6.1.2 or Aubin & Cellina, 1988, Section 5.4). Given a continuous-time system $x'(t) = f(x,u)$ the regulation map is defined as the set of all control values which assure that the closed loop system satisfies the sub-tangentiality condition

$$R_{\mathscr{S}}(x) = \{u\in\mathscr{U}\colon f(x,u)\in\mathscr{C}_{\mathscr{S}}(x)\}$$

where $\mathscr{C}_{\mathscr{S}}(x)$ is the tangent cone and $\mathscr{U}$ is the set of all admissible controls. The non-emptiness of $R_{\mathscr{S}}(x)$ and the choice of the control as $u = \Phi(x)\in R_{\mathscr{S}}(x)$ is a necessary and sufficient condition for $\mathscr{S}$ to be positively invariant for the closed loop system. This property has been exploited in Lu and Packard (1996) for disturbance rejection. However, the problem is the determination of $\mathscr{S}$ such that $R_{\mathscr{S}}(x) \neq \emptyset$ for all $x\in\mathscr{S}$. There are special cases in which such a problem can be solved. In Rumiantsev (1971) a control scheme is proposed in which the Lyapunov function of the uncontrolled system (of a special class) is used as a closed loop Lyapunov function to assure asymptotic stability or to increase convergence. Contributions along this line are due to Fradkov (1996) and Fradkov et al. (1997) where the problem of reaching a prescribed energy level surface is solved for Hamiltonian systems by means of the so called speed-gradient control.

6. Applications

In this section we discuss some applications of set invariance proposed in the literature.

6.1. Invariant sets as theoretical tools

Invariant sets have been intensively used in mathematical literature. Their exploitation started with the Lyapunov theory for differential equations and in particular with the Lassalle invariance principle (see the books Hahn, 1967; Lassalle & Lefschetz, 1961; Lyapunov, 1966; Rouche et al., 1977; Zubov, 1964). Positively invariant sets have been shown to be very useful in the analysis of dynamical systems described by differential inclusions (Colombo, 1992; Aubin, 1991; Aubin & Cellina, 1988). Some extension theorems are based on the existence of positively invariant sets (Crandal, 1972). Furthermore, the existence of periodic solutions of periodic nonlinear systems can be proven by the determination of an invariant set (Yoshizawa, 1975). See (Zanolin, 1987; Fernandes & Zanolin, 1987 for recent developments on this topic.

The invariance principle has been used in the framework of adaptive control to prove convergence of some adaptation schemes (Byrnes & Martin, 1995; Ryan, 1998). To have a simple idea of such kind of applications the reader can look at the very simple but meaningful Example 3.10 in Khalil (1992).

One important problem in the analysis of dynamical systems worth to be mentioned for its importance in system engineering is the determination of the domain of attraction (or the stability domain) for a non-linear system of the form $\dot{x}(t) = f(x(t))$. Essentially, a stability domain is a set of the form $\mathcal{N}[\Psi, k]$, where $k > 0$ and Ψ is a Lyapunov function. The basic techniques to solve this problem are based on the Zubov's equation and La Salle's methods reported in the above mentioned books. Again, the literature on this topic is too wide to be adequately reported here. We refer to Genesio, Tartaglia and Vicino (1985) for a comprehensive survey.

Further applications of set invariance concern the qualitative analysis of biological systems, see Hutson and Schmitt (1992) for a recent review.

The invariant sets have served as theoretical tools in some specific control problems. For instance in the worst case peak-to-peak minimization (the so called l_1 theory) invariant sets have been used to provide some important properties of non-linear compensators (Shamma, 1994; Stoorvogel, 1996a).

6.2. *Invariance and control problems with time-domain constraints*

Undoubtedly the main applications of the invariant sets are the analysis and synthesis under time-domain constraints. The necessity of dealing with constraints is a serious matter in many real problems. In particular the case of control and state constraints of the form

$$x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U},$$

where $\mathcal{X}$ and $\mathcal{U}$ are assigned compact sets, has been deeply investigated, especially when the above constraints are linear, say $\mathcal{X}$ and $\mathcal{U}$ are polyhedral. The reason why invariant sets play a central role is that constraints violations can be avoided *if and only if the initial state belongs to a controlled invariant set* $\mathcal{S} \subseteq \mathcal{X}$, associated to a stabilizing control law such that $\Phi(\mathcal{S}) \subseteq \mathcal{U}$. Thus the determination of a controlled-invariant set is important for the following reasons:

- it provides a set of feasible initial states (domain of attraction);
- it characterizes the control law. This is accomplished by imposing that the corresponding control renders negative the derivative of the associated Lyapunov function.

These concepts were introduced very early in control literature (Bertsekas & Rhodes, 1971a; Bertsekas, 1972; Glover & Schweppe, 1971) in the more general context of pursuit-evasion games and a lot of works have appeared later.

Stabilization problems with control constraints only have been solved in Benzaouia and Burgat (1988a), Benzaouia and Hmamed (1993), Benzaouia and Mesquine (1994) and Bitsoris and Vassilaki (1995b) by means of linear controllers and in Lasserre (1993) and Blanchini et al. (1995) by means of non-linear compensators. Linear saturated compensators have been used in Benzaouia and Burgat (1988b), Dorea and Milani (1995a), Gutman and Hagander (1985) and Verriest and Pjunen (1996) and a piecewise linear compensator (obtained by means of a family of nested ellipsoidal sets) is presented in Wredenhagen and Belanger (1994). A continuous family of ellipsoidal invariant sets, depending on a positive parameter τ, is considered with the same goal i Suarez, Sols-Daun and Alvarez (1994), (1997). Uncertain systems with control constraints are considered in Corless and Leitmann (1993), Kim and Bien (1994) and Gutman and Gutman (1985).

Systems with both state and control constraints are considered in Gutman and Cwikel (1986a), (1986b), (1987), Keerthi and Gilbert (1987) and Blanchini and Miani (1996b) where nonlinear compensator and polyhedral invariant sets have been used. The fundamental advantage is that the polyhedral sets are more flexible to cope with control/state constraints than the ellipsoidal sets since *they are capable to approximate the largest domain of attraction with arbitrary precision*.

In Kolmanovski and Gilbert (1997) the computation of the largest invariant set under state and control constraints is used to improve the compensator performance. A family of linear compensators with "increasing gain" is computed, and each of them is associated to its largest domain of attraction. The system is switched from a certain gain to a new one when the state enters the domain of attraction of the latter. A similar idea is pursued in the above mentioned paper (Wredenhagen & Belanger, 1994). An interesting application is the reference governor problem. Essentially the tracking problem under constraints can be solved by a device which is used in addition to the controller loop, that possibly inhibits the reference signal when this is driving the system to a constraint violation. The inhibition is activated when the state reaches the boundary of a pre-computed invariant set included in the feasible region. Contributions in this sense are in Gilbert and Tan (1991), Graettinger and Krogh (1992), Bemporad, Casavola and Mosca (1997) and Gilbert, Kolmanovsky and Tan (1995). This control strategy can be considered as an alternative approach to the *override* control (Glattfelder & Schaufelberger 1988). Its main advantage is that, being based on an invariant set, it naturally provides stability conditions. However, the actual implementation of the control scheme requires state feedback and its complexity obviously grows with that of the representation of the invariant set.

An application to engine control is presented in Kolmanovski, Gilbert and Cook (1997). Contractive sets are employed to improve the convergence performance of a feedback loop by means of the so called "heuristically enhanced control" (Sznaier & Damborg, 1990, 1992, 1993). Further relevant references on the problem of control under constraints, including (but not specifically oriented to) the techniques based on invariant sets are Tarbouriech and Hennet (1997) and Bernstein and Michel (1995).

6.3. *Invariance and robustness*

The notion of invariant set is fundamental for the analysis and synthesis problem of uncertain time-varying systems. It is known that, under some general assumptions, a nonlinear system with time-varying uncertainties admits a Lyapunov function if (and obviously only if) it is stable (Meilakhs, 1979; Lin, Sontag & Wang, 1996b). In the case of a linear uncertain system of the form

$$\dot{x}(t) = A(w(t))x(t), \quad w(t) \in \mathscr{W} \tag{27}$$

then the robust stability of this system is equivalent to the existence of a Lyapunov function which is a norm, and therefore to the existence of robustly contractive C-sets (Molchanov & Pyatnitskii, 1986; Brayton & Tong, 1979, 1980; Michael et al. 1984). We stress that, if the uncertainty parameter w is constant, then the stability of each constant matrix $A(w)$ does not imply the existence of a single robust invariant set for the system (clearly, each system $\dot{x}(t) = A(w)x(t)$, for fixed w, admits an invariant set $\mathscr{S}_w$).

Stability analysis and stabilization by means of quadratic Lyapunov functions is a widely investigated problem. Among the relevant references we recall Kamenetsky (1984), Horisberger and Belanger (1976), Barmish et al. (1983a), (1983b) and Yedavalli and Liang (1986) (see Corless, 1994 for a review on the subject). As mentioned above, *quadratic stability and stabilization for systems with polytopic uncertainties can be recast as a convex optimization problem* (Bernussou et al., 1991; Gu et al., 1991). Similar analysis results for more general uncertainty structures are in Garofalo, Celentano and Glielmo (1993). For a complete exposition on this topic see Boyd et al. (1994).

The quadratic functions give sufficient conditions for robust stability of a linear uncertain system, but *they do not provide necessary and sufficient conditions*. It turns out that the ellipsoidal shape of the associated invariant sets is not sufficiently general to "describe" uncertain dynamics even in the linear case. Conversely, the following property holds.

The existence of polyhedral contractive sets and their associated Lyapunov functions is a necessary and sufficient condition for the robust stability of the system (27).

This fact was pointed out in Molchanov and Pyatnitskii (1986), Brayton and Tong (1979), (1980) and Michael et al. (1984) (see also Bhaya & Mota, 1994 for an interesting discussion on these references). Examples of the conservativeness of the conditions based on quadratic functions are given in Blanchini and Miani (1996b) and Zelentsowsky (1994). Furthermore it has been shown that the uncertain pair $(A(w(t)), B(w(t)))$, $w(t) \in \mathscr{W}$ is stabilizable if and only if there exists a polyhedral Lyapunov function (Blanchini, 1995).

Further types of Lyapunov functions have been involved in the robust stability analysis such as piecewise quadratic functions (Xie et al., 1995; Rantzer & Johansson, 1997b), polynomial functions (Zelentsowsky, 1994). It turns out that the derived stability conditions are less conservative than those based on quadratic functions.

6.4. *Disturbance rejection*

An important problem in system analysis is to characterize the effect of a persistent unknown-but-bounded disturbance on a dynamical system. This goal may be achieved by determining the 0-reachable set of the system, which turns out to be, for a globally stable linear system, *the smallest invariant set for the system*. Techniques for such a computation are presented in Pecsvaradi and Narendra (1971), Lasserre (1987), Senin and Soldunov (1990), Lasserre (1993), D'Alessandro and De Santis (1992), Graettinger and Krogh (1991), Zhu, Zhang and He (1992). A review is provided in Gayek (1991).

The disturbance rejection synthesis problem has been approached by means of invariant sets in the context of dynamic programming (Bertsekas & Rhodes 1971a; Bertsekas & Rhodes, 1971b; Glover & Schweppe, 1971; Bertsekas, 1972). Further work has been done in Usoro et al. (1982a). Ellipsoidal invariant sets have been subsequently applied to the control of a power plant (Parlos, Henry, Schweppe, Gould & Lanning, 1988). Further contributions along these lines are in Abedor, Nagpal and Poolla (1996), where the so called $*$-norm is introduced as a counterpart of the l_1 norm considered in Dahleh and Pearson (1987) and Vidyasagar (1986) for disturbance rejection (see also Venkatesh & Dahleh, 1995 for some comments about the conservatism of the approach).

The approach based on invariant sets for persistent disturbance rejection is receiving a renewed attention beside the more recent $\mathscr{L}_1$ theory (Blanchini & Ukovich, 1993; Shamma 1994, 1996; Blanchini & Sznaier, 1995a; Stoorvogel, 1996a; Fialho & Georgiou, 1997). Papers which deal with the disturbance rejection problem for non-linear systems by means of invariant sets are Lu (1995), Lu and Packard (1996).

6.5. *Performance analysis via invariant sets*

Invariant sets may be used in the performance evaluation of uncertain systems. It is known that certain performance bounds can be given in terms of ellipsoidal bounding sets, some of whom have already been discussed. For instance to find an overbound for the peak of the impulse response for a SISO linear system (A, b, c) one can compute an invariant ellipsoid for A including the initial state vector $x(0) = b$ and included in the set $\{x\colon |Cx| \le \mu\}$. Then μ turns out to be an overbound. Minimizing μ is a convex optimization problem. This result is particularly meaningful since it can be easily extended to the case of a polytopic uncertain $A(w)$. A rich summary of bounds of this kind is given in Boyd et al. (1994, Chapter 6).

In Barmish and Sankaran (1979) it is shown how to analyze the propagation effect of the uncertainty by means of polyhedral reachability (non-convex) sets. It is shown how the convex hull of these sets can be recursively determined. In Fialho and Georgiou (1995) it is shown how to compute the $\mathscr{L}_1$ norm of an uncertain system by approximating its 0-reachable set. In Blanchini, Miani and Sznaier (1997) the evaluation of the peak overshoot of the step response of an uncertain system, possibly under the effect of a persistent disturbance is performed via invariant sets. In De Santis (1994) it is shown how to maximize the disturbance, size while preserving the invariance of a given polyhedral set. The trade-off between the robust invariance margin and the convergence speed has been analyzed in Sznaier (1993).

6.6. The receding horizon control

A very popular method to control a dynamical system is the receding horizon technique. If we consider a dynamical discrete-time system such a strategy consists in fixing an horizon T and, for a given state $x(t)$, computing a sequence of T control vectors $u(t), u(t+1), \ldots, u(t+T-1)$, by means of a finite horizon optimization. The feedback strategy is then achieved by applying the first value $u(t)$ and then performing the computation again at $t+1$ and so on. This very popular technique was introduced several years ago (Propoi, 1963; De Vleger, Verbuggen & Bruijn 1982; Chang & Seborg, 1983; Keerthi & Gilbert, 1988) and it is also known as model-predictive control, see Garcia, Prett and Morari (1989) for further references and applications. Invariant sets may be successfully involved in this context in order to deal with constraints and to provide stability. One way to do this is to assume that *the initial state $x(0)$ is included in an appropriate invariant set* (Blanchini & Ukovich, 1993; Gutman & Cwikel, 1986a; Sznaier & Damborg, 1990; Bemporad et al., 1997; Bemporad & Mosca, 1998). It is worthwhile to mention the fact that, if linear/quadratic functionals are minimized on-line, then a computational advantage is achieved by involving polyhedral invariant sets instead of generic convex invariant set (even ellipsoids), since the former introduce linear constraints in the optimization procedure. A different possibility, which has been used to prove stability of the control scheme, *is to enforce, as a terminal constraint, the inclusion of the final state $x(t+T)$ in an invariant set including the origin*. This idea, suggested in Sznaier and Damborg (1987) and Sznaier and Damborg (1989), was considered in Mayne and Michalska (1990) where the receding horizon control for a nonlinear system is designed to reach a proper ellipsoidal invariant set associated to a linear control. A feature of the approach is that the set of feasible initial states, which can be very hard to compute, is defined implicitly by the set of all states $x(0)$ for which the corresponding on-line problem admits a feasible solution.

7. Concluding remarks

Techniques involving invariant sets have been considered in the literature of the last thirty years and they are still appealing as it is evidenced by the numerous recent contributions. However these techniques present several drawbacks.

The techniques based on ellipsoidal sets are *conservative*. This fact is well established in robustness analysis as well in the determination of domains of attraction under constraints. Polyhedral sets provide non-conservative solutions but *they lead to computationally intensive algorithms*. This is one of the most serious troubles although the fast improving computer performances alleviate the problem.

The synthesis techniques based on invariant sets *are best suited for state-feedback control problems* especially for uncertain systems. Clearly, if the system is certain and linear, the controller can be always associated to an observer for output feedback. However, since the initial state is unknown an observer-based control may fail to keep the actual state in the region. This problem may be not too crucial if the observer convergence is fast. Furthermore, in the discrete-time case, a dead-beat observer can be used which provides the exact value of the state in finite time. If the past input and output values can be assumed to be known, the initial estimation error is zero, so we have virtually no restrictions in considering state feedback.

This simple idea cannot be applied if there are non-negligible uncertainties or additive noise (as it is almost always the case). The measurement feedback problem under disturbances has been investigated, in a set-theoretic framework, in the past. Essentially the basic idea is to equip a state-feedback controller with a set-theoretic observer (Bertsekas & Rhodes, 1971a, b; Glover & Schweppe, 1971). This type of approach has very recently received a renewed attention (Stoorvogel, 1996b; Shamma & Tu, 1997).

On the other hand, beside the mentioned troubles, invariant sets have several outstanding properties from both a theoretical and practical point of view which have been evidenced in this paper.

We believe that there are still several open problems that are worth an investigation. For instance, we have seen that the only family of sets of practical use having a bounded complexity are the ellipsoids. For the reasons explained above it would be important to develop algorithms to find other classes of invariant sets to achieve a reasonable tradeoff between conservatism and complexity. It would also be important to synthesize invariant sets together with output-feedback (possibly dynamic) controllers. The synthesis and analysis of nonlinear systems, by means of invariant sets is also a challenging problem, which, needless to say, leads to strong mathematical difficulties. Finally we believe that the potentiality of the theory of invariant sets in real applications, which is still not completely explored, deserves more attention and we hope to see more research activity on this subject in the future.

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