

EFFECTFUL MEALY MACHINES: BISIMULATION AND TRACE

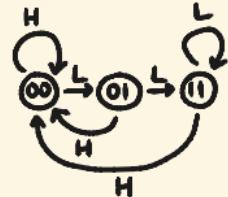
Filippo Bonchi, Elena Di Stefano, Mario Román

[ESL, de Felice, Román (LICS 2022)
Monoidal streams for dataflow programming]

[Bonchi, ESL, Román (preprint 2024)
Effectful Mealy machines: bisimulation and trace]

EFFECTFUL STREAMS: OVERVIEW

TEMPERATURE
CONTROLLER



RANDOM
WALK

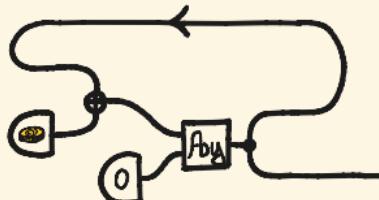


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construct a syntax for reactive programs

- effect agnostic
- with formal and compositional semantics



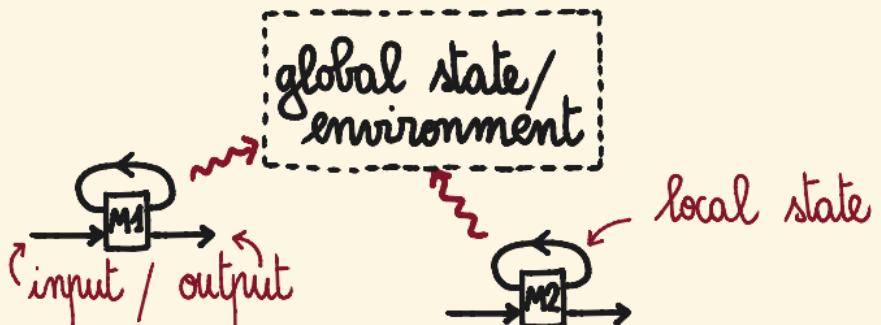
a random walk.

$$A = \boxed{g} \circ b := A^0 \boxed{g^0} \circ b^0 \quad \begin{matrix} \text{first computation} \\ \downarrow \end{matrix} \quad M \quad \begin{matrix} \text{local memory} \\ \nearrow \end{matrix} \quad A^1 = \boxed{g^+} \circ b^1 \quad \leftarrow \text{tail}$$

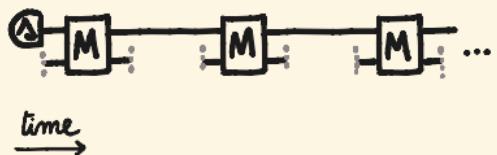
coalgebraic traces.

SEMANTICS OF EFFECTFUL MEALY MACHINES

- Mealy machines with global effects



- Trace semantics in streams



EXAMPLES OF MEALY MACHINES

A classical Mealy machine is

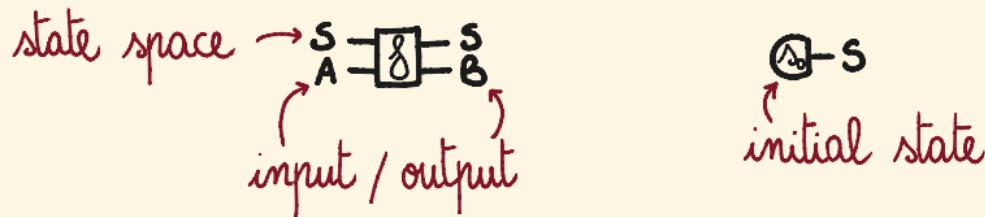
$$\begin{cases} t : S \times A \rightarrow P(S \times B) & \text{transition relation} \\ s_0 : I \rightarrow PS & \text{initial states} \end{cases}$$

A Markov decision process is

$$\begin{cases} t : S \times A \rightarrow DS & \text{transition probability} \\ r : S \times A \rightarrow DU & \text{reward} \\ s_0 : I \rightarrow DS & \text{initial distribution} \end{cases}$$

MEALY MACHINES

Systems are $f: S \otimes A \rightarrow S \otimes B$ with $s_0: I \rightarrow S$



- native sequential and parallel compositions
- parametric in the underlying process theory
- premonoidal categories for global effects

~ what is their behaviour?
when are two of them equivalent?

[cf. Katis, Sabadini, Walters 1997]

COALGEBRAIC SEMANTICS

Systems are coalgebras $f: S \rightarrow F(S)$

input/output

$$S \rightarrow (S \times B)^A$$

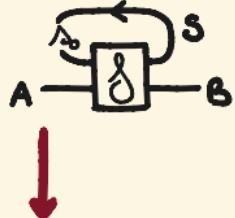
non-determinism

$$S \rightarrow P(S \times B)$$

- bisimulation is equality in the final coalgebra
 - ~ how do these compose?
 - how to change the underlying process theory?

OVERVIEW

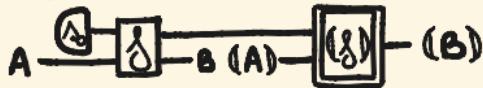
effectful Mealy machines



↓
bisimilarity quotient

↓

effectful streams



free construction
~ syntax



uniformity quotient

↓

coalgebraic construction
~ semantics

→ parametric in the choice of effect

STRING DIAGRAMS

MONOIDAL CATEGORIES

$$\begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} = \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} = \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array}$$

COPY-DISCARD CATEGORIES

$$\begin{array}{c} \text{---} \\ | \end{array} \quad \text{---} \cdot$$

$$\begin{array}{c} \text{---} \\ | \end{array} \cdot = \text{---} \quad \text{---} \circlearrowleft = \begin{array}{c} \text{---} \\ | \end{array}$$

$$\begin{array}{c} \text{---} \\ | \end{array} \circlearrowleft = \begin{array}{c} \text{---} \\ | \end{array} \cdot = \begin{array}{c} \text{---} \\ | \end{array} \cdot \circlearrowleft$$

CARTESIAN CATEGORIES

$$\begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} \cdot \circlearrowleft = \begin{array}{c} \text{---} \\ | \end{array} \cdot \circlearrowleft \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}}$$

$$\begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} \cdot = \text{---}$$

PREMONOIDAL CATEGORIES

$$\begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{h}} \begin{array}{c} \text{---} \\ | \end{array} \text{---} \circlearrowleft \neq \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{h}} \begin{array}{c} \text{---} \\ | \end{array} \text{---} \circlearrowleft$$

FEEDBACK PREMONOIDAL CATEGORIES

$$\begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} \text{---} \circlearrowleft$$

$$\begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{u}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{v}} = \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{u}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{v}} \text{---} \circlearrowleft$$

$$\begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{u}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{v}} = \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{u}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{v}}$$

~~$$\begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{u}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{g}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{v}}$$~~

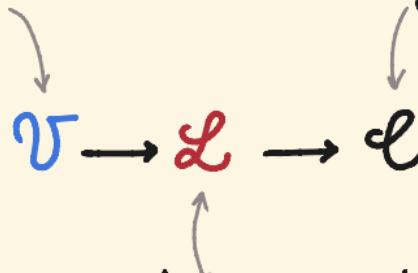
OUTLINE

- [• effectful copy-discard categories]
- effectful Mealy machines
- bisimulation
- effectful streams
- causal processes

EFFECTFUL COPY-DISCARD CATEGORIES

Values can be copied and discarded (cartesian)

$$\begin{array}{c} \text{---} \square \curvearrowleft = \text{---} \square \\ \text{---} \square \bullet = \text{---} \end{array}$$



Effectful computations may have global effects (premonoidal)

$$\begin{array}{c} \text{---} \square \bullet \quad \text{---} \square \\ \text{---} \square \quad \text{---} \square \end{array} \neq \begin{array}{c} \text{---} \square \quad \text{---} \square \\ \text{---} \square \bullet \end{array}$$

local computations interchange (monoidal)

$$\begin{array}{ccc} A - \boxed{\delta} - B & = & A - \boxed{\delta} - B \\ A' - \boxed{\delta} - B' & = & A' - \boxed{\delta'} - B' \\ \end{array} = \begin{array}{ccc} A - \boxed{\delta} - B & = & A - \boxed{\delta} - B \\ A' - \boxed{\delta'} - B' & = & A' - \boxed{\delta} - B' \end{array}$$

ex (Set , Stoch , State_S)

($\text{cart}(\mathcal{C})$, $\mathcal{Z}(\mathcal{C})$, \mathcal{C}) for a cd -premonoidal \mathcal{C}

OUTLINE

- effectful copy-discard categories

[• effectful Mealy machines]

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EFFECTFUL MEALY MACHINES

A Mealy machine $(f, S, s_0) : A \rightarrow B$ in $(\mathcal{U}, \mathcal{L}, \mathcal{C})$
is a morphism

$$f : S \otimes A \rightarrow S \otimes B$$

$$S_A = \boxed{f} = S_B$$

with an initial state

$$s_0 : I \rightarrow S$$

$$\textcircled{A} - S$$

A morphism of Mealy machines $u : (f, S, s_0) \rightarrow (g, T, t_0)$
is a value morphism $u : S \rightarrow T$ in \mathcal{U}

such that

$$S_A = \boxed{f} \xrightarrow{u} T_B = S_A = \boxed{g} \xrightarrow{T} T_B$$

$$\textcircled{A} \xrightarrow{u} T = \textcircled{t_0} - T$$

[cf. Katis, Sabadini, Walters 1997 ; EDL, Giamola, Román, Sabadini, Sobociński 2022]

EFFECTFUL CATEGORY OF MEALY MACHINES

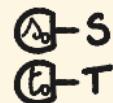
Mealy is an effectful category where

- objects are the objects of \mathcal{C}
- morphisms $(f, S, s) : A \rightarrow B$ are Mealy machines quotiented by value isomorphisms $u : S \xrightarrow{\cong} T$

$$\begin{array}{c} S \\ \text{---} \\ A \end{array} \xrightarrow{\quad f \quad} \begin{array}{c} T \\ \text{---} \\ B \end{array} = \begin{array}{c} S - u \\ \text{---} \\ A \end{array} \xrightarrow{\quad g \quad} \begin{array}{c} T \\ \text{---} \\ B \end{array}$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \xrightarrow{\quad u \quad} \begin{array}{c} T \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \xrightarrow{\quad t_0 \quad} T$$

- composition tensors the state spaces



MEALY MACHINES ARE FREE

$\mathcal{S} := \text{ptcl}_{\text{iso}}$

THEOREM

Mealy is the free pointed-feedback category over \mathcal{C} .

$$\text{Mealy}(A, B) = \int^{\mathbf{(A, S)} \in \text{ptcl}_{\text{iso}}} \mathcal{C}(S \circ A, S \circ B)$$



[cf. Katis, Sabadini, Walters 1997 ; EDL, Giamola, Román, Sabadini, Sobociński 2022]

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- effectful copy-discard categories
- effectful Mealy machines

[• bisimulation]

- effectful streams
- causal processes

COALGEBRAIC BISIMULATION

A bisimulation is a span of coalgebras.

$$\begin{array}{ccccc} S & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & T \\ \delta \downarrow & & \downarrow \alpha & & \downarrow g \\ F(S) & \xleftarrow[F(\pi_1)]{} & F(R) & \xrightarrow[F(\pi_2)]{} & F(T) \end{array}$$

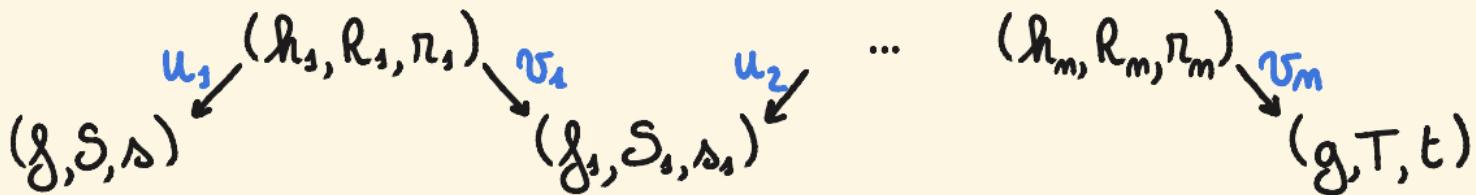
THEOREM [Rutten (2000)]

When $F: \text{Set} \rightarrow \text{Set}$ preserves weak pullbacks,
bisimilarity is an equivalence relation.

[Aczel & Mendler (1989), Rutten (2000)]

BISIMULATION

Two effectful Mealy machines $(f, S, s), (g, T, t) : A \rightarrow B$ are bisimilar if they belong to the same connected component in $\text{Mealy}(A, B)$:



THEOREM

For Mealy machines in $(\mathcal{V}, \mathcal{L}, \mathcal{C})$,
bisimulation implies trace equivalence.

PROOF: By coinduction. \square

COALGEBRAIC BISIMULATION

PROPOSITION

When $\mathcal{C} = \text{Kl}(M)$, for a commutative monad preserving weak pullbacks, effectful bisimulation coincides with coalgebraic bisimulation.

$$(f, S, s) \approx (g, T, t) \quad \text{iff} \quad \begin{array}{ccc} & (h, R, r) & \\ u \swarrow & & \searrow v \\ (f, S, s) & & (g, T, t) \end{array}$$

EXAMPLES

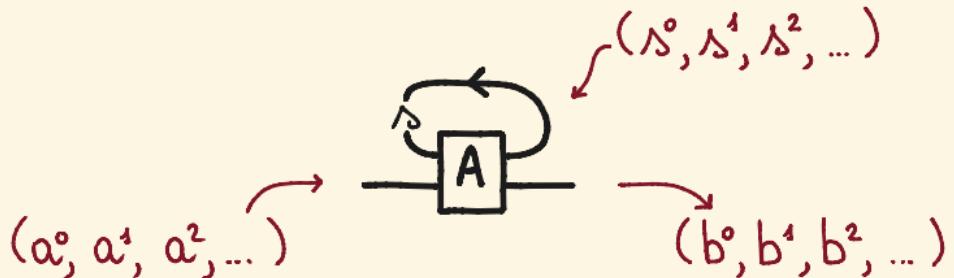
- Set
- Rel
- Stoch
- Par
- pStoch

OUTLINE

- effectful copy-discard categories
 - effectful Mealy machines
 - bisimulation

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- causal processes

EXECUTING MEALY MACHINES



~ what should the semantic universe be ?
when do two Mealy machines have the same executions ?

STREAMS ARE COINDUCTIVE

A stream of elements of A is

- an element $a^0 \in A$
- a stream a^+ of elements of A

↪ the set of streams is the final coalgebra of the functor

$$A \times (-) : \text{cSet} \rightarrow \text{cSet}$$

EFFECTFUL STREAMS

An effectful stream $f: A \rightarrow B$ on $(\mathcal{U}, \mathcal{L}, \mathcal{C})$ is

- a memory $M_g \in \mathcal{L}$
- a first action $\delta^\circ: A^\circ \rightarrow M_g \otimes B^\circ$ in \mathcal{C}
- the rest of the action $f^+: M_g \cdot A^+ \rightarrow B^+$

$$A - \boxed{f} - B = A^\circ - \boxed{\delta^\circ} - B^\circ \xrightarrow{M_g} A^+ - \boxed{f^+} - B^+$$

quotiented by sliding

$$\begin{cases} \delta^\circ; (\pi \otimes 1) \\ f^+ = \pi \cdot g^+ \end{cases} = g^\circ \quad \text{for } \pi: M_g \rightarrow M_g \text{ in } \mathcal{L}$$

$$-\boxed{\delta^\circ} - \boxed{f^+} - = -\boxed{\delta^\circ} - \boxed{\pi} - \boxed{f^+} - \sim -\boxed{\delta^\circ} - \boxed{\pi} - \boxed{f^+} - = -\boxed{\delta^\circ} - \boxed{g^+} -$$

EFFECTFUL STREAMS

The profunctor Stream : $\mathcal{C}^{N^{op}} \times \mathcal{C}^N \rightarrow \text{Set}$ is the final coalgebra of the functor

$$F : [\mathcal{C}^{N^{op}} \times \mathcal{C}^N, \text{Set}] \rightarrow [\mathcal{C}^{N^{op}} \times \mathcal{C}^N, \text{Set}]$$

$$F(Q)(A, B) := \int^{M \in \mathcal{C}} \mathcal{C}(A^\circ, M \otimes B^\circ) \times Q(M \cdot A^+, B^+)$$

quotient by
sliding on the memory

The diagram illustrates a sequence of operations: $A^\circ \xrightarrow{\text{go}} M_B \xrightarrow{f^+} B^+$. The label M_B is positioned above the arrow from go to f^+ . Red arrows point from the text "quotient by sliding on the memory" to the M_B label and the f^+ label.

COMPOSITIONAL STRUCTURE OF STREAMS

THEOREM

Effectful streams form an effectful category Stream.

- composition and monoidal actions are defined coinductively:
for $F: N_g \cdot A \rightarrow B$ and $g: N_g \cdot B \rightarrow C$,

$$\begin{cases} (F;_N g)^\circ := \begin{array}{c} Ng \\ \xrightarrow{\quad g \quad} \\ A^\circ \end{array} \quad \begin{array}{c} Ng \\ \xrightarrow{\quad g^\circ \quad} \\ B^\circ \end{array} \quad \begin{array}{c} Mg \\ \xrightarrow{\quad g \quad} \\ C^\circ \end{array} \\ (F;_N g)^+ := F^+;_M g^+ \end{cases}$$

$$\begin{cases} (\mathbb{X} \otimes_N F)^\circ := \begin{array}{c} N_\delta \\ \xrightarrow{\quad g \quad} \\ A^\circ \end{array} \quad \begin{array}{c} M_\delta \\ \xrightarrow{\quad g \quad} \\ B^\circ \end{array} \\ X^\circ \end{array} \\ (\mathbb{X} \otimes_N F)^+ := X^+ \otimes_M F^+ \end{cases}$$

FEEDBACK ON EFFECTFUL STREAMS

$\partial : \text{Stream} \rightarrow \text{Stream}$

$\partial(A) := (I, A^\circ, A^!, \dots)$

THEOREM

Stream has ∂ -feedback.

- feedback is defined coinductively

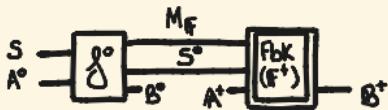
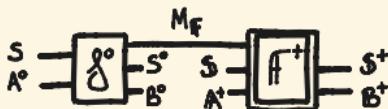
$$F : (S \cdot \partial S) \otimes A \rightarrow S \otimes B$$

$$\text{Fbk}_S F : S \cdot A \rightarrow B$$

$$M(\text{Fbk}_S^S F) := M(F) \otimes S^\circ$$

$$(\text{Fbk}_S^S F)^\circ := \emptyset^\circ$$

$$(\text{Fbk}_S^S F)^+ := \text{Fbk}_{S^+}^{S^\circ}(F^+)$$



OUTLINE

- effectful copy-discard categories
- effectful Mealy machines
- bisimulation
- effectful streams

[• causal processes]

STREAM COMPUTATIONS

- Sliding equivalence might be difficult to handle
- causal stream functions are old :

[Raney 1958] shows that they are
the executions of deterministic Mealy machines

⇒ is there a similar explicit form for effectful streams ?

STREAM COMPUTATIONS

CAUSAL STREAM FUNCTIONS

Stream computations $(p_m)_{m \in \mathbb{N}} : A \rightarrow B$ in a cartesian category
are families $p_m : A_0 \times \dots \times A_m \longrightarrow B_m$.

STOCHASTIC PROCESSES

Stochastic stream computations $(p_m)_{m \in \mathbb{N}} : A \rightarrow B$
are families $p_m : A_0 \times \dots \times A_m \longrightarrow \mathcal{D}(B_0 \times \dots \times B_m)$
such that $p_m(a_0, \dots, a_m) = \sum_{a \in A_{m+1}} p_{m+1}(a_0, \dots, a_m, a)$.

~ is there a monoidal version?

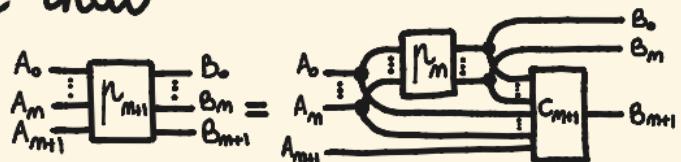
[Sprunger & Katsumata (2019), Mustalu & Vene (2008)]

CAUSAL PROCESSES

A causal process $p: A \rightarrow B$ in a copy-discard category \mathcal{C} is a family of morphisms

$$p_m : A_0 \otimes \cdots \otimes A_m \rightarrow B_0 \otimes \cdots \otimes B_m$$

such that



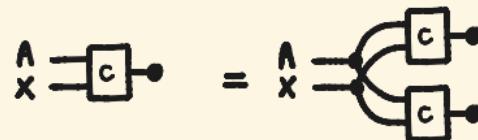
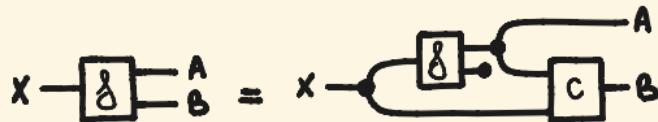
for some $C_{m+1}: B_0 \otimes \cdots \otimes B_m \otimes A_0 \otimes \cdots \otimes A_m \otimes A_{m+1} \rightarrow B_{m+1}$

COMPOSING CAUSAL PROCESSES

cl copy - discard

QUASI-TOTAL CONDITIONALS [Ello & Jacobs (2017), Brütz (2020), EDL & Román (2023)]

For all $f: X \rightarrow A \otimes B$ there is $c: A \otimes X \rightarrow B$ st



THEOREM

causal processes form a monoidal category Proc when cl has quasi-total conditionals.

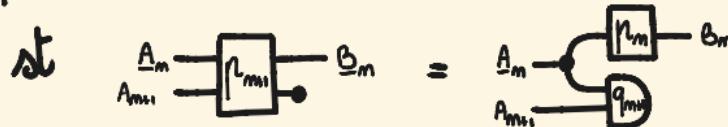
CAUSAL PROCESSES : EXAMPLES

Set

$$p_m : A_0 \times \dots \times A_m \rightarrow B_m$$

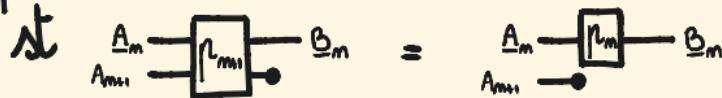
Par

$$p_m : A_0 \times \dots \times A_m \rightarrow (B_0 \times \dots \times B_m) + 1$$



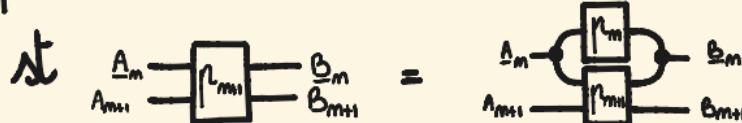
Stoch

$$p_m : A_0 \times \dots \times A_m \rightarrow \mathcal{D}(B_0 \times \dots \times B_m)$$



Rel

$$p_m : A_0 \times \dots \times A_m \rightarrow \mathcal{P}(B_0 \times \dots \times B_m)$$

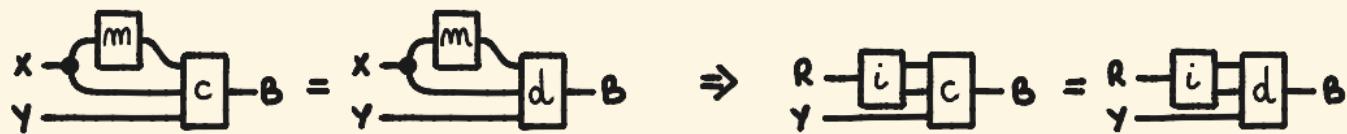
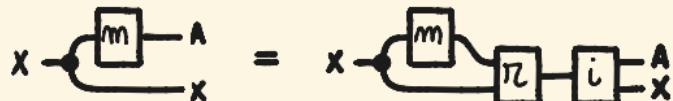


CAUSAL PROCESSES ARE STREAMS

• copy - discard

RANGES

For all $m: X \rightarrow A$ there are $\begin{cases} r: A \otimes X \rightarrow R & \text{deterministic} \\ i: R \rightarrow A \otimes X & \text{total} \end{cases}$



THEOREM

Consider $(\text{funcl}, \text{tot cl}, cl)$.

If cl has quasi-total conditionals and ranges,
 $\text{Proc} \simeq \text{Stream}$.

TRACES ARE EFFECTFUL TRACES

Compute the traces of a Mealy machine

$$(f, S, s) : A \rightarrow B$$

in some known cases.

(b_0, \dots, b_m) is a trace of (a_0, \dots, a_m)

Set if $s_0 = s$ and $\forall k \leq m \quad (s_{k+1}, b_k) = f(s_k, a_k)$

Rel if $\exists (s_0, \dots, s_{m+1}) \quad s_0 \in S$
and $\forall k \leq m \quad (s_{k+1}, b_k) \in f(s_k, a_k)$

pStoch with probability $\sum_{(s_0, \dots, s_{m+1})} s(s_0 | *) \cdot \prod_{k \leq m} f(s_{k+1}, b_k | s_k, a_k)$

SUMMARY

- formal compositional semantics for effectful stream computations
- trace equivalence and bisimulation of effectful Mealy machines
- characterisation as causal stream processes

FUTURE WORK

- coproducts and trace for choice and iteration
- linear temporal logic
- behavioural distances
- coinduction up-to dinaturality

$$\boxed{g^\circ} - \boxed{f^+} = \boxed{g^\circ} - \boxed{\textcolor{red}{R}} - \boxed{f^+} \sim \boxed{g^\circ} - \boxed{\textcolor{red}{R}} - \boxed{f^+} = \boxed{g^\circ} - \boxed{g^+}$$