

LHC seminar

4 June 2025

# EFFECTFUL MEALY MACHINES

## BISIMULATION & TRACES

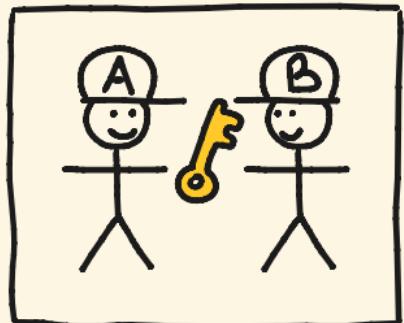
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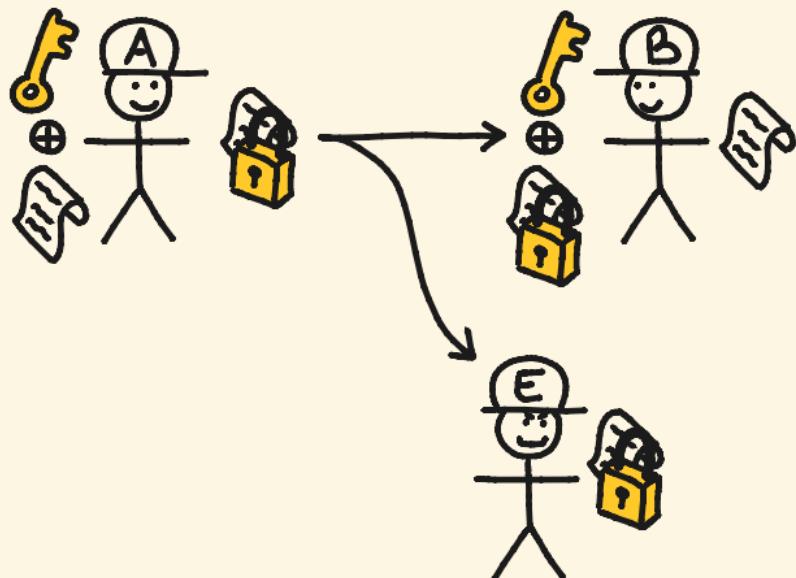
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# ONE-TIME PAD PROTOCOL

1. share a key through  
a secure channel



2. send an encrypted  
message through a  
public channel



# REPEATING THE ONE-TIME PAD

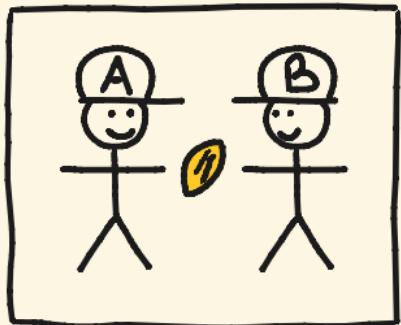
Sending  $n$  messages securely requires  $n$  private keys  
↳ not very useful

- ⇒ • privately share a seed 
- use identical pseudorandom number generator to obtain a new key for each message



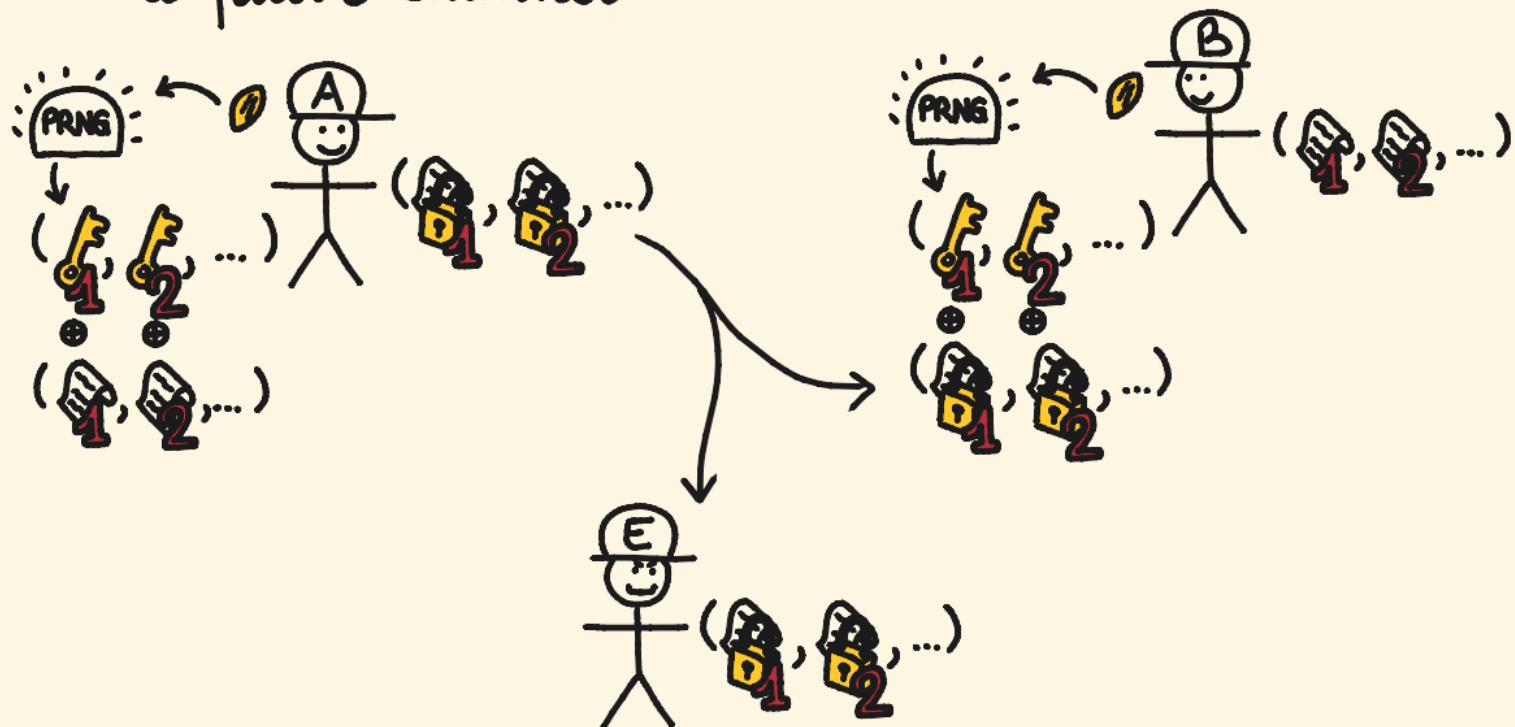
# STREAM CIPHER PROTOCOL (1)

1. share a seed through a secure channel
2. share a pseudorandom number generator



# STREAM CIPHER PROTOCOL (2)

3. send a stream of encrypted messages through a public channel

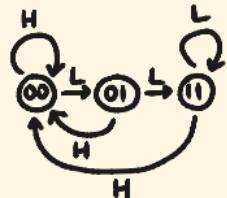


# MEALY MACHINES

A classical Mealy machine is

$$\begin{cases} t : S \times A \rightarrow P(S \times B) & \text{transition relation} \\ s_0 : 1 \rightarrow PS & \text{initial states} \end{cases}$$

TEMPERATURE  
CONTROLLER



A Markov decision process is

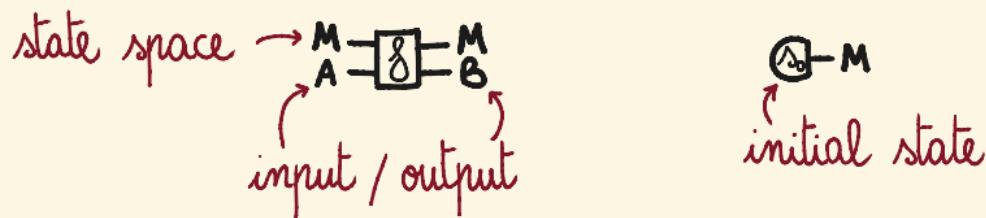
$$\begin{cases} t : S \times A \rightarrow DS & \text{transition probability} \\ r : S \times A \rightarrow DU & \text{reward} \\ s_0 : 1 \rightarrow DS & \text{initial distribution} \end{cases}$$

RANDOM  
WALK



# MEALY MACHINES

Systems are  $f: M \otimes A \rightarrow M \otimes B$  with  $s_0: I \rightarrow M$



- native sequential and parallel compositions
- parametric in the underlying process theory
- premonoidal categories for global effects

~ what is their behaviour?  
when are two of them equivalent?

[cf. Katis, Sabadini, Walters 1997]

# COALGEBRAIC SEMANTICS

Systems are coalgebras  $f : M \rightarrow F(M)$

input/output

$$M \rightarrow (M \times B)^A$$

non-determinism

$$M \rightarrow P(M \times B)$$

- bisimulation is equality in the final coalgebra

→ how do these compose?

how to change the underlying process theory?

# MEALY MACHINES AND COALGEBRAS

Two faces of Mealy machines

$$M \times A \rightarrow D(M \times B)$$

$$M \rightarrow D(M \times B)^A$$

$$M \times A \rightarrow P(M \times B)$$

$$M \rightarrow P(M \times B)^A$$

$$\begin{array}{c} M \\ A = \boxed{\delta} = B \end{array}$$



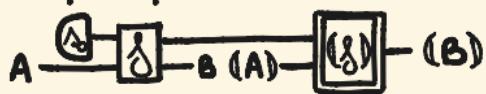
# OVERVIEW

effectful Mealy machines



bisimilarity

effectful streams



[ $\approx$  causal processes]

free construction  
~ syntax



uniformity

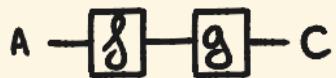


coalgebraic construction  
~ trace semantics

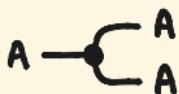
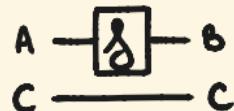
# OUTLINE

- [ • effectful copy-discard categories ]
- effectful Mealy machines & bisimulations
- coinductive traces
- clausal traces

# STRING DIAGRAMS



A — A



A →

$$\text{---} \cap \text{---} = \text{---} \cap \text{---}$$

$$\text{---} \cap \text{---} = \text{---}$$

$$\text{---} \cap \text{---} = \text{---} \cap \text{---}$$

# THE INTERCHANGE LAW

$$\begin{array}{c} A \xrightarrow{\quad g \quad} B \\ A' \xrightarrow{\quad g \quad} B' \end{array} = \begin{array}{c} A \xrightarrow{\quad g \quad} B \\ A' \xrightarrow{\quad g \quad} B' \end{array} = \begin{array}{c} A \xrightarrow{\quad g \quad} B \\ A' \xrightarrow{\quad g \quad} B' \end{array}$$

→ holds in monoidal categories

# VALUES

Values are both :

- deterministic



- total



ex  $(3 \cdot -) : \mathbb{R} \rightarrow \mathbb{R}$

non-ex Flip :  $\{1\} \rightarrow \mathcal{D}(\{\text{H, T}\})$        $\neq$

$(3/-) : \mathbb{R} \rightarrow \mathbb{R}$

$\neq$

# LOCAL COMPUTATIONS

Local computations interchange,

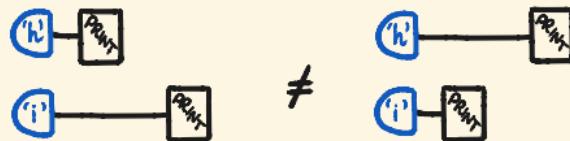
$$\begin{array}{c} A - \boxed{\delta} - B \\ A' - \boxed{\delta'} - B' \end{array} = \begin{array}{c} A - \boxed{\delta} - B \\ A' - \boxed{\delta'} - B' \end{array} = \begin{array}{c} A - \boxed{\delta} - B \\ A' - \boxed{\delta'} - B' \end{array}$$

ex Stoch

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

# EFFECTFUL COMPUTATIONS

Effectful computations may have global effects.



ex state promonads, 10 monad

[Power and Robinson (1997)]

# EFFECTFUL COPY-DISCARD CATEGORIES

Values can be copied and discarded (cartesian)

$$\begin{array}{c} \text{---} \square \text{---} \sqcap = \text{---} \square \text{---} \square \sqcap \\ \text{---} \square \text{---} \bullet = \text{---} \bullet \end{array}$$

$$\mathcal{V} \rightarrow \mathcal{L} \rightarrow \mathcal{C}$$

$$\begin{array}{c} \text{---} \square \text{---} \square \neq \text{---} \square \text{---} \square \\ \text{---} \square \text{---} \square \quad \text{---} \square \text{---} \square \\ \text{---} \square \text{---} \square \quad \text{---} \square \text{---} \square \end{array}$$

local computations interchange (monoidal)

$$\begin{array}{ccc} A - \boxed{\delta} - B & = & A - \boxed{\delta} - B \\ A' - \boxed{\delta} - B' & = & A' - \boxed{\delta'} - B' = A - \boxed{\delta} - B \\ & & A' - \boxed{\delta'} - B' \end{array}$$

[Jeffrey (1997), cf. Levy (2022), Power and Thielecke (1997)]

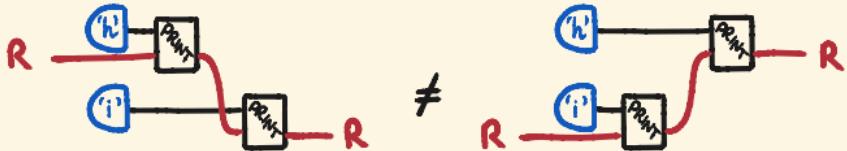
# EXAMPLES OF EFFECTFUL TRIPLES

- $(\text{cart}(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{C})$  for a  $\text{cd}$ -premonoidal  $\mathcal{C}$
- $(\text{cSet}, \text{Kl}(\mathcal{Z}(T)), \text{Kl}(T))$  for a  $\text{cSet}$ -monad  $T$
- $(\text{cSet}, \text{Set}, \text{Par})$
- $(\text{cSet}, \text{Rel}_{\text{TOT}}, \text{Rel})$
- $(\text{cSet}, \text{Stoch}, \text{Stoch}_{\leq})$
- $(\text{cSet}, \text{Stoch}, \text{States})$

# STRING DIAGRAMS FOR PREMONOIDAL CATEGORIES

Add a "runtime" wire to non-central morphisms

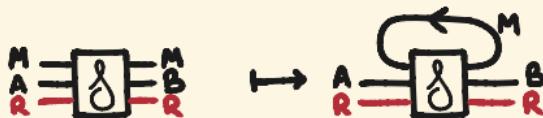
$$g: A \rightarrow B \quad \text{is} \quad \begin{array}{c} A \\ \text{---} \\ R \end{array} \xrightarrow{\quad g \quad} \boxed{g} \xrightarrow{\quad g \quad} \begin{array}{c} B \\ \text{---} \\ R \end{array}$$



[Jeffrey (1997); Román (2022); Román, Sobociński (2025)]

# FEEDBACK EFFECTFUL CATEGORIES

$$\text{FbK}_M : \mathcal{C}(M \otimes A, M \otimes B) \rightarrow \mathcal{C}(A, B)$$



$$A \xrightarrow[R]{\quad} B = A \xrightarrow[R]{\quad} B \circ M \otimes N$$

$$A \xrightarrow[R]{\quad} B = A \xrightarrow[R]{\quad} B$$

$$A \xrightarrow[R]{\quad} C = A \xrightarrow[R]{\quad} C \circ M$$

$$A' \xrightarrow[R]{\quad} B' = A' \xrightarrow[R]{\quad} B' \circ M$$

$$A \xrightarrow[R]{\quad} B = A \xrightarrow[R]{\quad} B \circ u$$

$u : N \rightarrow M$  in  $\mathcal{L}$



# OUTLINE

- effectful copy-discard categories
- [• effectful Mealy machines & bisimulations ]
- coinductive traces
- clausal traces

# EFFECTFUL MEALY MACHINES

a Mealy machine  $(f, M, s_0) : A \rightarrow B$  in  $(\mathcal{V}, \mathcal{L}, \mathcal{C})$   
is a morphism

$$f : M \otimes A \rightarrow M \otimes B \quad \text{in } \mathcal{C} \quad \begin{array}{c} M \\ \otimes \\ A \end{array} = \boxed{\delta} = \begin{array}{c} M \\ \otimes \\ B \end{array}$$

with an initial state

$$s_0 : I \rightarrow M \quad \text{in } \mathcal{L} \quad \begin{array}{c} I \\ \rightarrow \\ M \end{array}$$

ex  $(\text{Set}, \text{Rel}_{\text{tor}}, \text{Rel})$

$$\left\{ \begin{array}{l} f : M \times A \rightarrow P(M \times B) \\ s_0 \subseteq M \end{array} \right.$$

ex  $(\text{Set}, \text{Stoch}, \text{Stoch}_{\leq})$

$$\left\{ \begin{array}{l} f : M \times A \rightarrow \mathcal{D}(M \times B) \\ s_0 \in \mathcal{D}(M) \end{array} \right.$$

[cf. Katis, Sabadini, Walters (1997); EDL, Gjajola, Román, Sabadini, Sobociński (2022)]

# MORPHISMS OF MEALY MACHINES

A morphism of Mealy machines  $u: (f, M, s_0) \rightarrow (g, N, t_0)$   
is a value morphism  $u: M \rightarrow N$  in  $\mathcal{U}$

such that

$$\begin{array}{c} M \\ \xrightarrow{\quad u \quad} \\ A \end{array} = \begin{array}{c} N \\ \xrightarrow{\quad g \quad} \\ B \end{array}$$
$$\begin{array}{c} A \\ \xrightarrow{\quad u \quad} \\ N \end{array} = \begin{array}{c} t_0 \\ - \\ N \end{array}$$

ex ( $\text{Set}$ ,  $\text{Rel}_{\text{tor}}$ ,  $\text{Rel}$ )

$$\forall s \in M \quad \forall a \in A \quad \forall t \in N \quad \forall b \in B$$

$$(t, b) \in g(u(s), a) \Leftrightarrow \exists s' \in M \quad u(s') = t \wedge (s', b) \in f(s, a)$$

ex ( $\text{Set}$ ,  $\text{Stoch}$ ,  $\text{Stoch}$ )

$$\forall s \in M \quad \forall a \in A \quad \forall t \in N \quad \forall b \in B$$

$$g(t, b | u(s), a) = \sum_{s': u(s') = t} f(s', b | s, a)$$

# EFFECTFUL CATEGORY OF MEALY MACHINES

Mealy is an effectful category where

- objects are the objects of  $\mathcal{C}$
- morphisms  $(f, M, s) : A \rightarrow B$  are Mealy machines quotiented by value isomorphisms  $u : M \xrightarrow{\cong} N$

$$\begin{array}{c} M \\ A \\ R \end{array} = \begin{array}{c} M \\ A \\ R \end{array} \xrightarrow{u} \begin{array}{c} N \\ B \\ R \end{array} = \begin{array}{c} M \\ A \\ R \end{array} \xrightarrow{u} \begin{array}{c} N \\ B \\ R \end{array}$$

$$\begin{array}{c} \textcircled{A} \\ \textcircled{B} \end{array} \xrightarrow{u} N = \begin{array}{c} \textcircled{A} \\ \textcircled{B} \end{array} \xrightarrow{v} N$$

- composition tensors the state spaces  $\rightsquigarrow$  local states

$$\begin{array}{c} M \\ N \\ A \\ R \end{array} \xrightarrow{u} \begin{array}{c} M \\ N \\ C \\ R \end{array} = \begin{array}{c} \textcircled{A} \\ \textcircled{B} \end{array} \xrightarrow{u} M \quad \begin{array}{c} \textcircled{B} \\ \textcircled{C} \end{array} \xrightarrow{v} N$$

# MEALY MACHINES ARE FREE

## THEOREM

Mealy is the free pointed-feedback category over  $\mathcal{C}$ .

$$\text{Mealy}(A, B) = \int^{\rho(\lambda_0, M) \in \text{pt}\mathcal{C}_{\text{iso}}} \mathcal{C}(M \otimes A, M \otimes B)$$



[cf. Katis, Sabadini, Walters (1997); EDL, Gjaniola, Román, Sabadini, Sobociński (2022)]

# COALGEBRAIC MEALY MACHINES

$T: \text{Set} \rightarrow \text{Set}$  monad

$F_T: \text{Set} \rightarrow \text{Set}$   
 $X \mapsto T(X \times B)^A$

a  $T$ -Mealy machine is an  $F_T$ -coalgebra

$$f: X \rightarrow T(X \times B)^A$$

$$\Leftrightarrow \hat{f}: X \otimes A \rightarrow X \otimes B \quad \text{in } \text{Kl } T$$

$\Leftrightarrow$  effectful Mealy machine in  
 $(\text{Set}, \text{Kl}(Z(T)), \text{Kl}(T))$

# COALGEBRAIC BISIMULATION

A bisimulation is a span of coalgebras.

$$\begin{array}{ccccc} M & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & N \\ \delta \downarrow & & \downarrow \alpha & & \downarrow g \\ F_T(M) & \xleftarrow{F_T(\pi_1)} & F_T(R) & \xrightarrow{F_T(\pi_2)} & F_T(N) \end{array}$$

**THEOREM** [Rutten (2000)]

When  $F_T: \text{Set} \rightarrow \text{Set}$  preserves weak pullbacks,  
bisimilarity is an equivalence relation.

[Aczel & Mendler (1989), Rutten (2000)]

# BISIMULATION

For two effectful Mealy machines  $(f, M, s), (g, N, t) : A \rightarrow B$ ,  
a bisimulation is a sequence of spans of morphisms.

$$(f, M, s) \xleftarrow{u_1} (h_1, R_1, \pi_1) \xrightarrow{\pi_1} (f_1, M_1, s_1) \xleftarrow{u_2} \dots \xleftarrow{u_m} (h_m, R_m, \pi_m) \xrightarrow{\pi_m} (g, N, t)$$

## PROPOSITION

When  $\mathcal{C} = \text{Kl}(T)$ , for a commutative monad  $T$  preserving weak pullbacks, effectful bisimulation coincides with coalgebraic bisimulation.

ex cSet, Par, Rel, cStoch, ncStoch

# BISIMILARITY IS FREE

UNIFORM FEEDBACK [cf. Glăseanu (2002); ăzănescu, ăfănescu (1994)]

$$\left\{ \begin{array}{lcl} M \\ \text{---} \\ A & \xrightarrow{\delta} & B \\ R & \xrightarrow{u} & R \\ \text{---} \\ \text{---} & = & \text{---} \\ \text{---} & = & \text{---} \end{array} \right. = \begin{array}{lcl} M \\ \text{---} \\ A & \xrightarrow{u} & B \\ R & \xrightarrow{g} & R \\ \text{---} \\ \text{---} & = & \text{---} \\ \text{---} & = & \text{---} \end{array} \Rightarrow \begin{array}{lcl} M \\ \text{---} \\ A & \xrightarrow{\delta} & B \\ R & \xrightarrow{t_0} & R \\ \text{---} \\ \text{---} & = & \text{---} \\ \text{---} & = & \text{---} \end{array}$$

## THEOREM

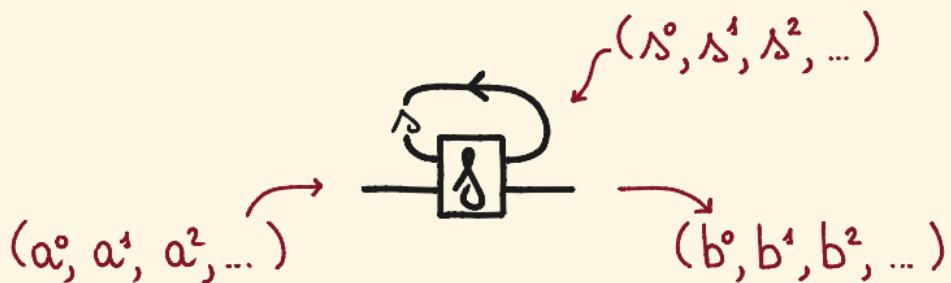
Mealy  $\xrightarrow{\text{bisim}}$  form the free uniform feedback structure over  $(\mathcal{U}, \mathcal{L}, \ell)$ .

# OUTLINE

- effectful copy-discard categories
- effectful Mealy machines & bisimulations
- [ • coinductive traces
- clausal traces

]

# EXECUTING MEALY MACHINES



~ what should the semantic universe be?  
when do two Mealy machines have the same executions?

# STREAMS ARE COINDUCTIVE

A stream of elements of  $A$  is

- an element  $a^0 \in A$
- a stream  $a^+$  of elements of  $A$

↪ the set of streams is the final coalgebra of the functor

$$A \times (-) : \text{cSet} \rightarrow \text{cSet}$$

# EFFECTFUL STREAMS

$A = (A^0, A^1, \dots)$ ,  $B = (B^0, B^1, \dots)$  streams of objects of  $(\mathcal{U}, \mathcal{L}, \mathcal{C})$ .

An effectful stream  $F: A \rightarrow B$  on  $(\mathcal{U}, \mathcal{L}, \mathcal{C})$  is

- a memory  $M_g \in \mathcal{L}$
- a first action  $\delta^\circ: A^0 \rightarrow M_g \otimes B^0$  in  $\mathcal{C}$
- the rest of the action  $F^+: M_g \cdot A^+ \rightarrow B^+$

$$A \xrightarrow[\text{IR}]{} \boxed{F} \xrightarrow[\text{IR}]{} B = A^0 \xrightarrow[\text{R}]{\delta^\circ} B^0 \xrightarrow[\text{R IR}]{M_g} A^+ \xrightarrow{\quad F^+ \quad} B^+$$

quotiented by sliding: for  $\pi: M_g \rightarrow M_g$  in  $\mathcal{L}$

$$\delta^\circ = g^\circ; (\pi \otimes 1) \qquad \qquad \qquad g^+ = \pi \cdot F^+$$

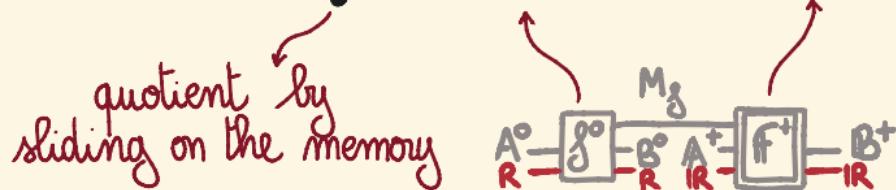
$$-\boxed{\delta^\circ} - \boxed{F^+} - = -\boxed{g^\circ} \boxed{\pi} - \boxed{F^+} - \sim -\boxed{g^\circ} - \boxed{\pi} \boxed{F^+} - = -\boxed{g^\circ} - \boxed{g^+} -$$

# EFFECTFUL STREAMS

The profunctor Stream :  $\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}$  → Set is the final coalgebra of the functor

$$F : [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \text{Set}] \rightarrow [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \text{Set}]$$

$$F(Q)(A, B) := \int^{M \in \mathcal{C}} \mathcal{C}(A^\circ, M \otimes B^\circ) \times Q(M \cdot A^+, B^+)$$



# COMPOSITIONAL STRUCTURE OF STREAMS

## THEOREM

Effectful streams form an effectful category Stream.

- composition and monoidal actions are defined coinductively:  
for  $f: N_g \cdot A \rightarrow B$  and  $g: N_g \cdot B \rightarrow C$ ,

$$\begin{cases} (\mathcal{F}_{j_N} g)^\circ := \text{Diagram } 1 \\ (\mathcal{F}_{j_N} g)^+ := \mathcal{F}_{j_M}^+ g^+ \end{cases}$$

$$\left\{ \begin{array}{l} (\mathbb{X} \otimes_N \mathbb{F})^\circ := \frac{N_\delta}{A^\circ} \xrightarrow{\quad g^\circ \quad} \frac{M_\delta}{B^\circ} \\ (\mathbb{X} \otimes_N \mathbb{F})^+ := \mathbb{X}^+ \otimes_M \mathbb{F}^+ \end{array} \right.$$

# FEEDBACK ON EFFECTFUL STREAMS

$\partial$ : Stream → Stream

$\partial(A) := (I, A^\circ, A^!, \dots)$

## THEOREM

Stream has  $\partial$ -feedback.

- feedback is defined coinductively

$$F : (S \cdot \partial S) \otimes A \rightarrow S \otimes B$$

$$Fbk_S F : S \cdot A \rightarrow B$$

$$M(Fbk_S^S F) := M(F) \otimes S^\circ$$

$$(Fbk_S^S F)^\circ := \emptyset^\circ$$

$$(Fbk_S^S F)^+ := Fbk_{S^+}^S (F^+)$$

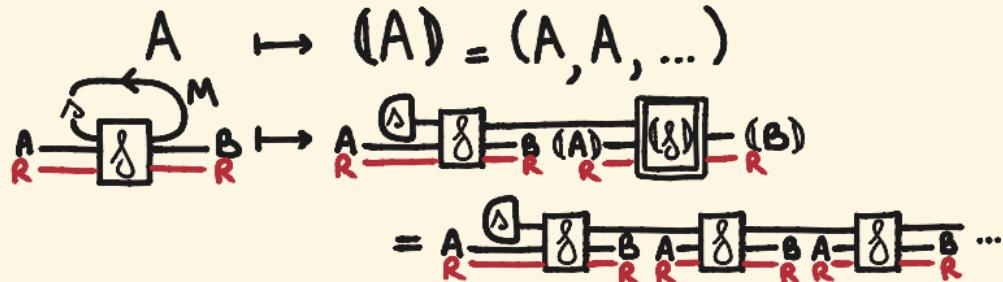


# COMPOSITIONAL TRACE SEMANTICS

## THEOREM

There is an effectful functor

$$\text{Tr} : \text{Mealy} \rightarrow \text{Stream}$$



# TRACE IS UNIVERSAL

## THEOREM

The trace functor  $\text{Tr} : \text{Mealy} \rightarrow \text{Stream}$   
is the unique feedback effectful functor  
determined by freeness of Mealy.

FBR $\mathbb{E}\text{ffCat}$

$$\text{Mealy}(-) \downarrow \begin{array}{c} \text{EffCat} \\ \parallel \\ \text{Mealy}_{(v,z,e)} / \text{bisim} \end{array} \Rightarrow \exists! \text{ Tr} : \text{Mealy}_{(v,z,e)} \rightarrow \text{Stream}_{(v,z,e)}$$

## COROLLARY

Bisimilarity implies trace equivalence.

# OUTLINE

- effectful copy-discard categories
- effectful Mealy machines & bisimulations
- coinductive traces

[ • clausal traces ]

# STREAM COMPUTATIONS

- Sliding equivalence might be difficult to handle
- causal stream functions are old :

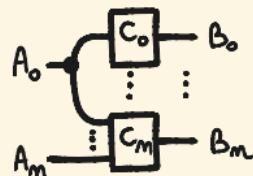
[Raney (1958)] shows that they are  
the executions of deterministic Mealy machines

⇒ is there a similar explicit form for effectful streams ?

# CAUSAL STREAM FUNCTIONS

Stream computations  $(c_m)_{m \in \mathbb{N}} : A \rightarrow B$  in a cartesian category  
are families  $c_m : A_0 \times \dots \times A_m \rightarrow B_m$ .

↪  $c_m(a_0, \dots, a_m) \in B_m$  is the output at time  $m$



reconstructs the outputs until time  $m$

# STOCHASTIC PROCESSES

Stochastic stream computations  $(p_m)_{m \in \mathbb{N}} : A \rightarrow B$

are families  $p_m : A_0 \times \dots \times A_m \longrightarrow \mathcal{D}(B_0 \times \dots \times B_m)$

such that  $p_m(a_0, \dots, a_m) = \sum_{a \in A_{m+1}} p_{m+1}(a_0, \dots, a_m, a)$ .

⇒  $p_m(a_0, \dots, a_m) \in \mathcal{D}(B_0 \times \dots \times B_m)$  is the distribution of  
the outputs until time  $m$   
↳ the outputs may be correlated

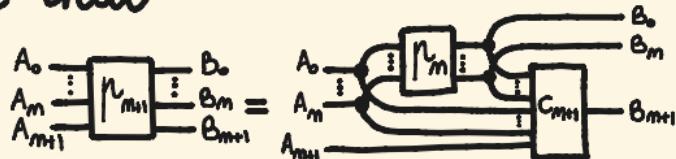
Is there a monoidal version of causal stream functions?

# CAUSAL PROCESSES

A causal process  $p: A \rightarrow B$  in a copy-discard category  $\mathcal{C}$  is a family of morphisms

$$p_m : A_0 \otimes \cdots \otimes A_m \rightarrow B_0 \otimes \cdots \otimes B_m$$

such that



for some  $C_{m+1}: B_0 \otimes \cdots \otimes B_m \otimes A_0 \otimes \cdots \otimes A_m \otimes A_{m+1} \rightarrow B_{m+1}$

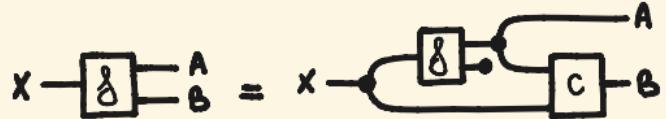
→  $p_m$  determines the input-output behaviour  
until time  $m$

# COMPOSING CAUSAL PROCESSES

• copy - discard

CONDITIONALS [Elho & Jacobs (2017), Bittz (2020), EDL & Román (2023)]

For all  $f: X \rightarrow A \otimes B$  there is  $c: A \otimes X \rightarrow B$  st

$$x - \boxed{\delta} \begin{matrix} A \\ B \end{matrix} = x - \begin{array}{c} \delta \\ \bullet \\ c \end{array} \begin{matrix} A \\ B \end{matrix}$$


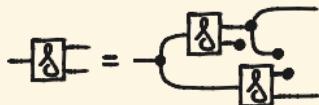
## THEOREM

causal processes form a monoidal category  $\text{Proc}$  when  $\mathcal{C}$  has conditionals with sharp domain.

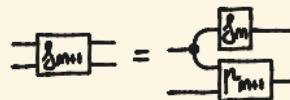
# CAUSAL PROCESSES : EXAMPLES

## CONDITIONALS

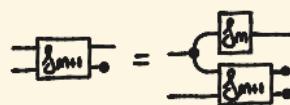
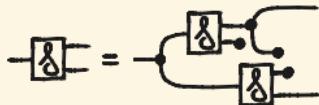
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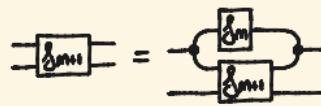
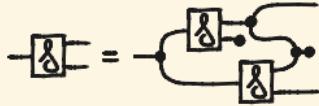
## CAUSALITY CONDITION



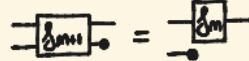
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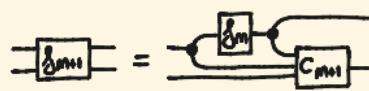
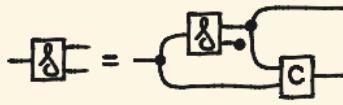
Rel



Stock



nStock

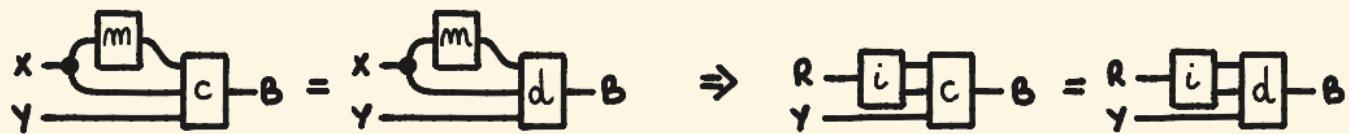
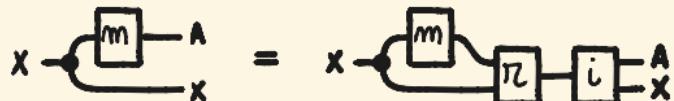


# CAUSAL PROCESSES ARE STREAMS

• copy - discard

## RANGES

For all  $m: X \rightarrow A$  there are  $\begin{cases} r: A \otimes X \rightarrow R \\ i: R \rightarrow A \otimes X \end{cases}$  deterministic total



## THEOREM

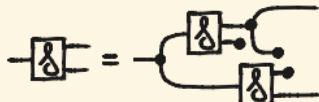
Consider  $(\text{funcl}, \text{tot cl}, cl)$ .

If  $cl$  has conditionals and ranges,  
 $\text{Proc} \simeq \text{Stream}$ .

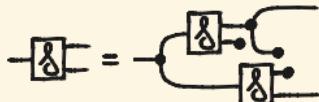
# TRACES ARE EFFECTFUL TRACES

## CONDITIONALS

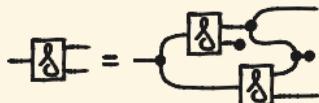
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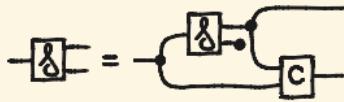
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Rel



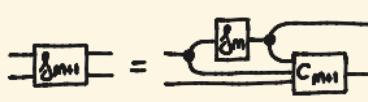
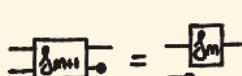
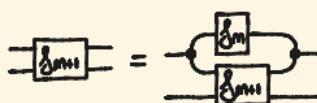
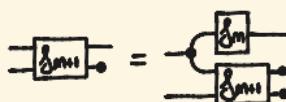
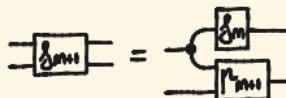
Stock



nStock



## CAUSALITY CONDITION



## TRACE PREDICATE

$$\Delta_0 = \Delta \wedge \forall i \leq n \\ (\delta_{i+1}, b_i) = f(\delta_i, a_i)$$

$$\Delta_0 = \Delta \wedge \forall i \leq n \\ (\delta_{i+1}, b_i) = f(\delta_i, a_i)$$

$$\exists \Delta_0, \dots, \Delta_m \Delta_0 \in \Delta \\ \wedge \forall i \leq n (\delta_{i+1}, b_i) \in f(\delta_i, a_i)$$

$$\sum_{\Delta_0, \dots, \Delta_m} s(\Delta_0) \\ \cdot \prod_{i \leq m} f(\delta_{i+1}, b_i | \Delta_i, a_i)$$

$$\sum_{\Delta_0, \dots, \Delta_m} s(\Delta_0) \\ \cdot \prod_{i \leq m} f(\delta_{i+1}, b_i | \Delta_i, a_i)$$

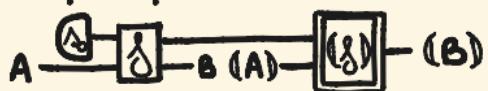
# SUMMARY

effectful Mealy machines



bisimilarity

effectful streams



[ $\approx$  causal processes]

free construction  
~ syntax



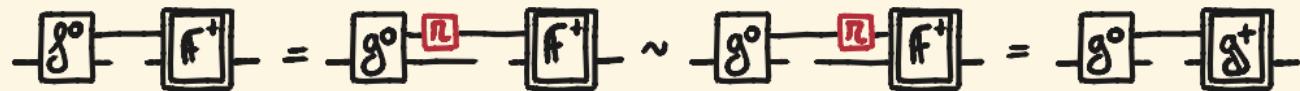
uniformity



coalgebraic construction  
~ trace semantics

# FUTURE WORK

- Adding choice, iteration and higher-order
- Behavioural metrics, metric enrichment
- cloinduction up-to dinaturality
- Tree-like behaviour via final coalgebra of an appropriate  $\wp$



[Filippo Bonchi, ESD, Mario Román

Effectful Mealy machines: bisimulation and trace (2025) LiCS ]