

Dagstuhl seminar

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# TRACE SEMANTICS OF EFFECTFUL MEALY MACHINES

Filippo Bonchi  
Università di Pisa

Elena Di Stefano  
University of Oxford

Mario Román  
University of Oxford

# TRANSITION SYSTEMS AS COALGEBRAS

$$t : M \rightarrow T(M \times B)^A$$



- behaviour given by final coalgebras

$$\begin{array}{ccc} M & \xrightarrow{\quad h \quad} & \Omega \\ t \downarrow & & \downarrow \omega \\ T(M \times B)^A & \rightarrow & T(\Omega \times B)^A \end{array}$$

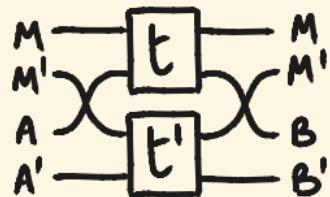
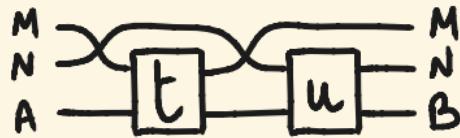
~~ how do they compose ?

# TRANSITION SYSTEMS AS MORPHISMS

$$t : M \otimes A \rightarrow M \otimes B$$



- native sequential and parallel composition



~ what's their behaviour?

[Katis, Sabadini, Walters 1997]

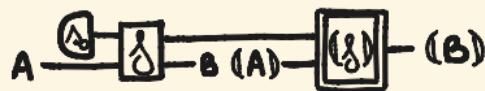
# OVERVIEW

effectful Mealy machines



bisimilarity

effectful streams



free construction  
~ syntax

uniformity

coalgebraic construction  
~ trace semantics

# EFFECTFUL TRIPLES

Values can be copied and discarded (cartesian)

$$\begin{array}{c} \text{---} \square \text{---} = \text{---} \square \text{---} \\ | \qquad \qquad \qquad | \qquad \qquad \qquad | \\ \text{---} \square \text{---} \bullet = \text{---} \bullet \end{array}$$

$$\mathcal{V} \rightarrow \mathcal{L} \rightarrow \mathcal{C}$$

Effectful computations may have global effects (premonoidal)

$$\begin{array}{c} \text{---} \square \text{---} \square \\ | \qquad \qquad | \\ \text{---} \square \text{---} \square \\ | \qquad \qquad | \\ \text{---} \square \text{---} \square \end{array} \neq \begin{array}{c} \text{---} \square \text{---} \square \\ | \qquad \qquad | \\ \text{---} \square \text{---} \square \end{array}$$

local computations interchange (monoidal)

$$\begin{array}{ccc} A - \boxed{\delta} - B & = & A - \boxed{\delta} - B \\ A' - \boxed{\delta} - B' & = & A' - \boxed{\delta'} - B' = A - \boxed{\delta} - B \\ & & A' - \boxed{\delta'} - B' \end{array}$$

[Jeffrey (1997), cf. Levy (2022), Power and Thielecke (1997)]

# EXAMPLES OF EFFECTFUL TRIPLES

- $(\text{cart}(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{C})$  for a  $\text{cd}$ -premonoidal  $\mathcal{C}$
- $(\text{cSet}, \text{Kl}(\mathcal{Z}(T)), \text{Kl}(T))$  for a  $\text{cSet}$ -monad  $T$
- $(\text{cSet}, \text{Set}, \text{Set})$
- $(\text{cSet}, \text{Rel}_{\text{TOT}}, \text{Rel})$
- $(\text{cSet}, \text{Kl } \mathfrak{D}, \text{Kl } \mathfrak{D}_{\leq})$
- $(\text{cSet}, \text{Kl } \mathfrak{D}, \text{cStates}_S)$

# OUTLINE

For an effectful triple  $(\mathcal{V}, \mathcal{L}, cl)$ , construct:

- [ • effectful Mealy machines ]
- bisimulations
- traces (effectful streams)

# EFFECTFUL MEALY MACHINES

a Mealy machine  $(f, M, s_0) : A \rightarrow B$  in  $(\mathcal{V}, \mathcal{L}, \mathcal{C})$   
is a morphism

$$f : M \otimes A \rightarrow M \otimes B$$

$$\begin{array}{c} M \\ A \xrightarrow{f} B \end{array}$$

with an initial state

$$s_0 : I \rightarrow M$$

$$\begin{array}{c} \otimes \\ I \xrightarrow{s_0} M \end{array}$$

ex  $(\text{cSet}, \text{Rel}_{\text{tor}}, \text{Rel})$

$$\left\{ \begin{array}{l} f : M \times A \rightarrow P(M \times B) \\ s_0 \subseteq M \end{array} \right.$$

ex  $(\text{cSet}, \text{Stoch}, \text{Stoch})$

$$\left\{ \begin{array}{l} f : M \times A \rightarrow \mathcal{D}(M \times B) \\ s_0 \in \mathcal{D}(M) \end{array} \right.$$

[cf. Katis, Sabadini, Walters (1997); EDL, Giamola, Román, Sabadini, Sobociński (2022)]

# MORPHISMS OF MEALY MACHINES

A morphism of Mealy machines  $u: (f, M, s_0) \rightarrow (g, N, t_0)$   
is a value morphism  $u: M \rightarrow N$  in  $\mathcal{U}$

such that

$$\begin{array}{c} M \\ \xrightarrow{\quad s \quad} \\ A \end{array} \xrightarrow{u} \begin{array}{c} N \\ \xrightarrow{\quad t \quad} \\ B \end{array} = \begin{array}{c} M \\ \xrightarrow{\quad u \quad} \\ A \end{array} \xrightarrow{g} \begin{array}{c} N \\ \xrightarrow{\quad t_0 \quad} \\ B \end{array}$$
$$\begin{array}{c} \xrightarrow{\quad s_0 \quad} \\ u \end{array} - N = \begin{array}{c} t_0 \\ - N \end{array}$$

ex ( $\text{Set}$ ,  $\text{Rel}_{\text{tor}}$ ,  $\text{Rel}$ )

$$\forall s \in M \quad \forall a \in A \quad \forall t \in N \quad \forall b \in B$$

$$(t, b) \in g(u(s), a) \Leftrightarrow \exists s' \in M \quad u(s') = t \wedge (s', b) \in f(s, a)$$

ex ( $\text{Set}$ ,  $\text{Stoch}$ ,  $\text{Stoch}$ )

$$\forall s \in M \quad \forall a \in A \quad \forall t \in N \quad \forall b \in B$$

$$g(t, b | u(s), a) = \sum_{s': u(s') = t} f(s', b | s, a)$$

# EFFECTFUL CATEGORY OF MEALY MACHINES

Mealy is an effectful category where

- objects are the objects of  $\mathcal{C}$
- morphisms  $(f, M, s) : A \rightarrow B$  are Mealy machines quotiented by value isomorphisms  $u : M \xrightarrow{\cong} N$

$$\begin{array}{c} M \\ \text{---} \\ A \end{array} \xrightarrow{\quad u \quad} \begin{array}{c} N \\ \text{---} \\ B \end{array} = \begin{array}{c} M \\ \text{---} \\ A \end{array} \xrightarrow{\quad u \quad} \begin{array}{c} g \\ \text{---} \\ B \end{array}$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \xrightarrow{\quad u \quad} \begin{array}{c} N \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \xrightarrow{\quad u \quad} \begin{array}{c} N \\ \text{---} \\ \text{---} \end{array}$$

- composition tensors the state spaces  $\rightsquigarrow$  local states

$$\begin{array}{c} M \\ \text{---} \\ N \\ \text{---} \\ A \end{array} \xrightarrow{\quad g \quad} \begin{array}{c} M \\ \text{---} \\ N \\ \text{---} \\ C \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \xrightarrow{\quad g \quad} \begin{array}{c} M \\ \text{---} \\ N \\ \text{---} \\ C \end{array}$$

# MEALY MACHINES ARE FREE

$\mathcal{S} := \text{ptcl}_{\text{iso}}$

## THEOREM

Mealy is the free pointed-feedback category over  $\text{cl}$ .

$$\text{Mealy}(A, B) = \int^{\mathbb{P}(\lambda_0, M) \in \text{ptcl}_{\text{iso}}} \text{cl}(M \otimes A, M \otimes B)$$



[cf. Katis, Sabadini, Walters (1997); EDL, Gianola, Román, Sabadini, Sobociński (2022)]

# OUTLINE

For an effectful triple  $(\mathcal{V}, \mathcal{L}, cl)$ , construct:

- effectful Mealy machines

[ • bisimulations ]

- traces (effectful streams)

# COALGEBRAIC BISIMULATION

A bisimulation is a span of coalgebras.

$$\begin{array}{ccccc} M & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & N \\ \delta \downarrow & & \downarrow \alpha & & \downarrow g \\ F(M) & \xleftarrow[F(\pi_1)]{} & F(R) & \xrightarrow[F(\pi_2)]{} & F(N) \end{array}$$

THEOREM [Rutten (2000)]

When  $F: \text{Set} \rightarrow \text{Set}$  preserves weak pullbacks,  
bisimilarity is an equivalence relation.

[Aczel & Mendler (1989), Rutten (2000)]

# BISIMULATION

For two effectful Mealy machines  $(f, M, s), (g, N, t) : A \rightarrow B$ ,  
a bisimulation is a sequence of spans of morphisms.

$$(f, M, s) \xleftarrow{u_1} (h_1, R_1, \pi_1) \xrightarrow{\pi_1} (f_1, M_1, s_1) \xleftarrow{u_2} \dots \xleftarrow{u_m} (h_m, R_m, \pi_m) \xrightarrow{\pi_m} (g, N, t)$$

## PROPOSITION

When  $\mathcal{C} = \text{Kl}(T)$ , for a commutative monad  $T$  preserving weak pullbacks, effectful bisimulation coincides with coalgebraic bisimulation.

ex cSet, Par, Rel, cStoch, ncStoch

# OUTLINE

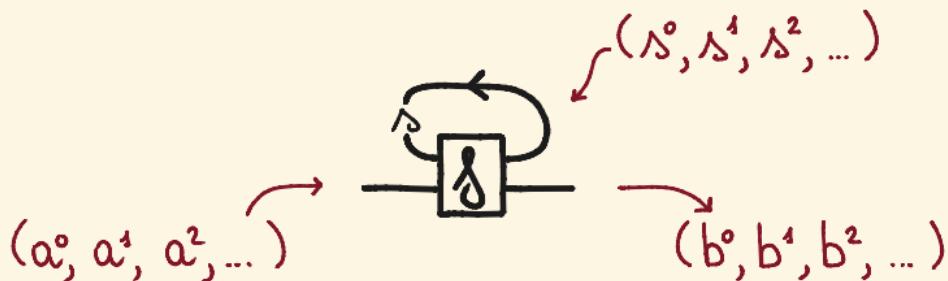
For an effectful triple  $(\mathcal{V}, \mathcal{L}, cl)$ , construct:

- effectful Mealy machines

- bisimulations

- traces (effectful streams) ]

# EXECUTING MEALY MACHINES



~ what should the semantic universe be?  
when do two Mealy machines have the same executions?

# EFFECTFUL STREAMS

An effectful stream  $f: A \rightarrow B$  on  $(\mathcal{U}, \mathcal{L}, \mathcal{C})$  is

- a memory  $M_g \in \mathcal{L}$
- a first action  $\delta^\circ: A^\circ \rightarrow M_g \otimes B^\circ$  in  $\mathcal{C}$
- the rest of the action  $f^+: M_g \cdot A^+ \rightarrow B^+$

$$A - \boxed{f} - B = A^\circ - \boxed{\delta^\circ} - B^\circ \xrightarrow{M_g} A^+ - \boxed{f^+} - B^+$$

quotiented by sliding

$$\left\{ \begin{array}{l} \delta^\circ; (\pi \otimes 1) \\ f^+ = \pi \cdot g^+ \end{array} \right. = g^\circ \quad \text{for } \pi: M_g \rightarrow M_g \text{ in } \mathcal{L}$$

$$\boxed{\delta^\circ} - \boxed{f^+} - = \boxed{\delta^\circ} - \boxed{\pi} - \boxed{f^+} - \sim \boxed{\delta^\circ} - \boxed{\pi} - \boxed{f^+} - = \boxed{\delta^\circ} - \boxed{g^+} -$$

# EFFECTFUL STREAMS

The profunctor Stream :  $\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}$  → Set is the final coalgebra of the functor

$$F : [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \text{Set}] \rightarrow [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \text{Set}]$$

$$F(Q)(A, B) := \int^{M \in \mathcal{C}} \mathcal{C}(A^\circ, M \otimes B^\circ) \times Q(M \cdot A^+, B^+)$$

quotient by  
sliding on the memory



# COMPOSITIONAL STRUCTURE OF STREAMS

## THEOREM

Effectful streams form an effectful category Stream.

- composition and monoidal actions are defined coinductively:  
for  $F: Ng \cdot A \rightarrow B$  and  $g: Ng \cdot B \rightarrow C$ ,

$$\begin{cases} (F;_N g)^\circ := \begin{array}{c} Ng \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} & \text{with } \begin{array}{c} Ng \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} \\ (F;_N g)^+ := F^+;_M g^+ & \end{cases}$$

$$\begin{cases} (\mathbb{X} \otimes_N F)^\circ := \begin{array}{c} Ng \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} & \text{with } \begin{array}{c} Ng \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} \\ (\mathbb{X} \otimes_N F)^+ := \mathbb{X}^+ \otimes_M F^+ & \end{cases}$$

# FEEDBACK ON EFFECTFUL STREAMS

$\partial$ : Stream → Stream

$\partial(A) := (I, A^\circ, A^!, \dots)$

## THEOREM

Stream has  $\partial$ -feedback.

- feedback is defined coinductively

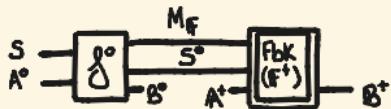
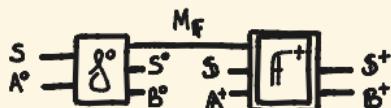
$$F : (S \cdot \partial S) \otimes A \rightarrow S \otimes B$$

$$Fbk_S F : S \cdot A \rightarrow B$$

$$M(Fbk_S^S F) := M(F) \otimes S^\circ$$

$$(Fbk_S^S F)^\circ := \emptyset^\circ$$

$$(Fbk_S^S F)^+ := Fbk_{S^+}^S (F^+)$$

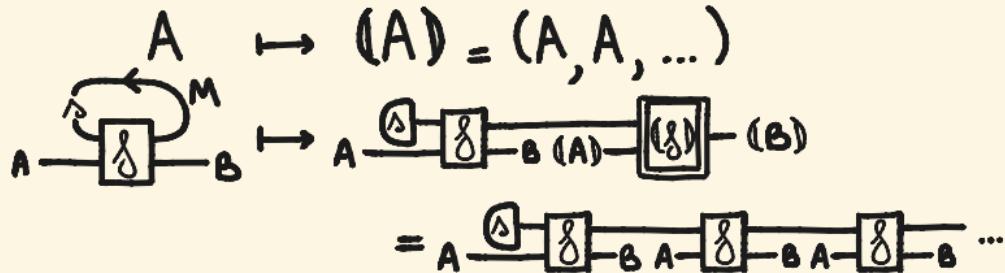


# COMPOSITIONAL TRACE SEMANTICS

## THEOREM

There is an effectful functor

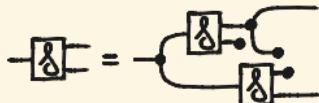
$$\text{Tr} : \text{Mealy} \rightarrow \text{Stream}$$



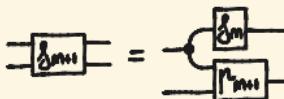
# TRACES ARE EFFECTFUL TRACES

## CONDITIONALS

Set



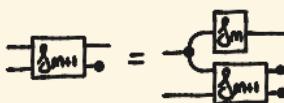
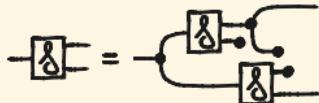
## CAUSALITY CONDITION



## TRACE PREDICATE

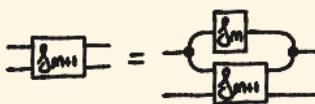
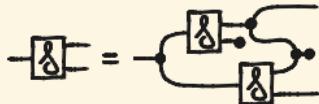
$$\Delta_0 = \Delta \wedge \forall i \leq n \\ (\delta_{\text{init}}, b_i) = f(\delta_i, a_i)$$

Par



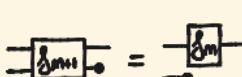
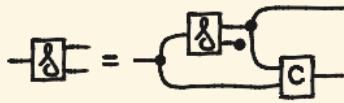
$$\Delta_0 = \Delta \wedge \forall i \leq n \\ (\delta_{\text{init}}, b_i) = f(\delta_i, a_i)$$

Rel



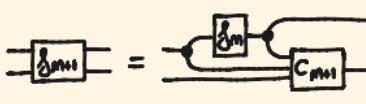
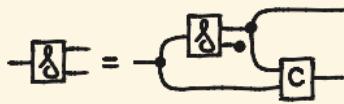
$$\exists \Delta_0, \dots, \Delta_{m+1} \Delta_0 \in \Delta \\ \wedge \forall i \leq n (\delta_{\text{init}}, b_i) \in f(\delta_i, a_i)$$

Stock



$$\sum_{\Delta_0, \dots, \Delta_m} s(\Delta_0) \\ \cdot \prod_{i \leq m} f(\delta_{\text{init}}, b_i | \Delta_i, a_i)$$

nStock



$$\sum_{\Delta_0, \dots, \Delta_m} s(\Delta_0) \\ \cdot \prod_{i \leq m} f(\delta_{\text{init}}, b_i | \Delta_i, a_i)$$

# TRACE IS UNIVERSAL

## THEOREM

The trace functor  $\text{Tr} : \text{Mealy} \rightarrow \text{Stream}$   
is the unique feedback effectful functor  
determined by freeness of Mealy.

FBR<sup>EffCat</sup>

$$\begin{array}{ccc} \text{Mealy}(-) & \Rightarrow \exists! \text{ Tr} : \text{Mealy}_{(v,z,e)} & \xrightarrow{\quad} \text{Stream}_{(v,z,e)} \\ \downarrow & & \downarrow \\ \text{EffCat} & & \text{Mealy}_{(v,z,e)} / \text{bisim} \end{array}$$

## COROLLARY

Bisimilarity implies trace equivalence.

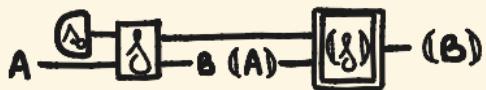
# SUMMARY

effectful Mealy machines



↓ trace

effectful streams



≈ causal processes

free construction  
~ syntax

↓  $\exists!$

coalgebraic construction  
~ semantics

# FUTURE WORK

- Adding choice, iteration and higher-order
- Behavioural metrics, metric enrichment
- cloinduction up-to dinaturality

$$-\boxed{g^\circ} - \boxed{f^+} - = -\boxed{g^\circ} - \boxed{n} - \boxed{f^+} - \sim -\boxed{g^\circ} - \boxed{n} - \boxed{f^+} - = -\boxed{g^\circ} - \boxed{g^+} -$$