TallCat Seminar 9 March 2023

EVIDENTIAL DECISION THEORY VIA PARTIAL MARKOV CATEGORIES

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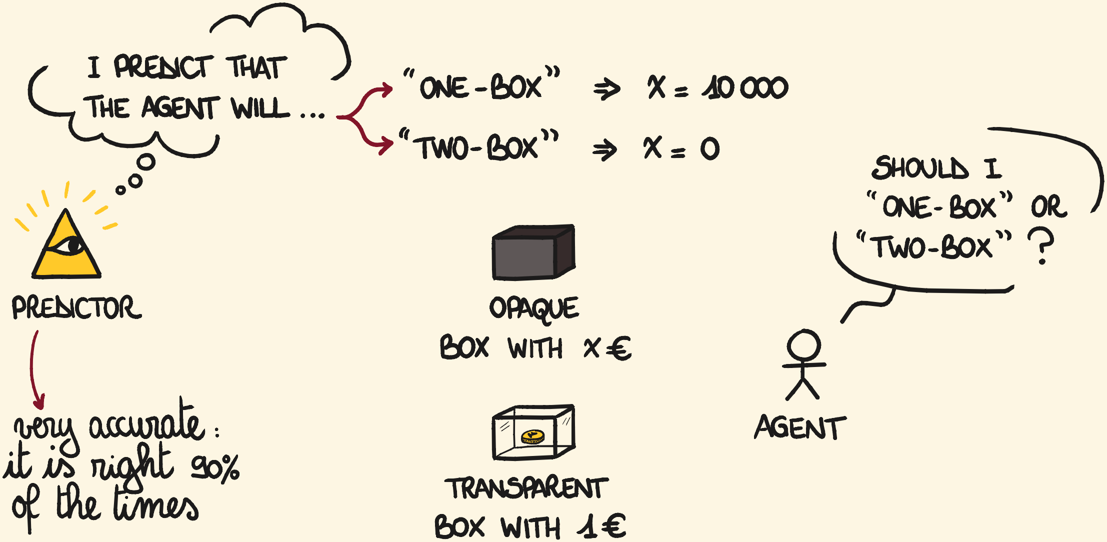
NEWCOMB'S PROBLEM



(cute drawing inspired by Mario's slides for NWPT '22)

CAUSAL DECISION THEORY

<u>Causal</u> decision theory answers :

"Which action would <u>cause</u> the best-case scenario ?"

Whatever the predictor did,
I get 1€ extra if I two-box
⇒ I will <u>two-box</u>

BEST !

PREDICTOR \ AGENT	ONE-BOX	TWO-BOX
ONE-BOX	10 000 €	10 001€
TWO-BOX	0 €	1 €

EVIDENTIAL DECISION THEORY

Evidential decision theory answers :

"Which action would be evidence for the best-case scenario ?"

My action is evidence for the prediction :
if I one-box I expect 10 000 €,
if I two-box I expect 1 € .
⇒ I will one-box

PREDICTOR \ AGENT	ONE-BOX	TWO-BOX
ONE-BOX	10 000 €	10 001 €
TWO-BOX	0 €	1 €

MOST LIKELY

EVIDENTIAL VS CAUSAL DECISION THEORY

	 CAUSAL DECISION THEORIST	 EVIDENTIAL DECISION THEORIST
EXPECTED UTILITY	0.9 × 1 € + 0.1 × 10 001 € = 1 001 €	0.9 × 10 000 € + 0.1 × 0 € = 9 000 €

MOTIVATION

- lack of a <u>clear syntax</u> to specify the (implicit) assumptions of decision problems
- lack of a <u>calculus</u> for solving them (according to Evidential Decision Theory)

DESIDERATA

1. express probabilistic processes → Markov
2. express constraints → partiality & discreteness
3. explicitly capture implicit assumptions → category theory

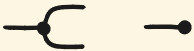
(DISCRETE) PARTIAL MARKOV CATEGORIES

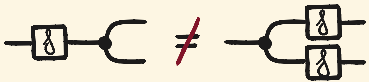
- add partiality to Markov categories

OR

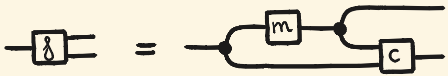
- add probability to (discrete) cartesian restriction categories

COPY - DISCARD STRUCTURE

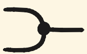


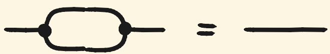


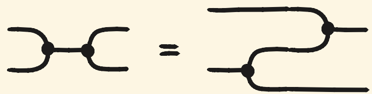
CONDITIONALS



PARTIAL FROBENIUS STRUCTURE







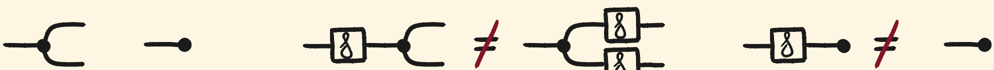
(DISCRETE) PARTIAL MARKOV CATEGORIES

- REMOVE TOTALITY FROM ~~add partiality to~~ Markov categories

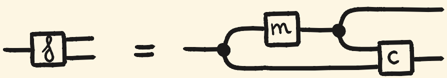
OR

- REMOVE DETERMINISM FROM ~~add probability to~~ (discrete) cartesian restriction categories

COPY-DISCARD STRUCTURE



CONDITIONALS



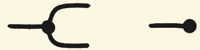
PARTIAL FROBENIUS STRUCTURE

OUTLINE

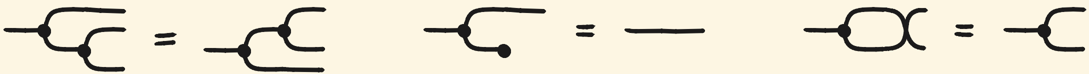
- Partial Markov categories
- Discrete partial Markov categories
- Bayes, Pearl and Jeffrey
- Evidential decision theory
- Exact observations

COPY-DISCARD CATEGORIES

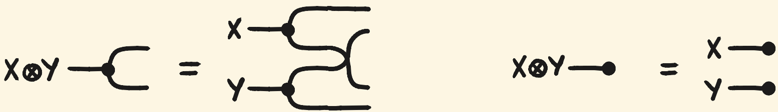
A copy-discard category is a symmetric monoidal category where every object is a uniform cocommutative comonoid.



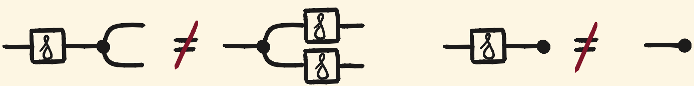
COCOMMUTATIVE COMONOID



UNIFORMITY



NO NATURALITY REQUIRED

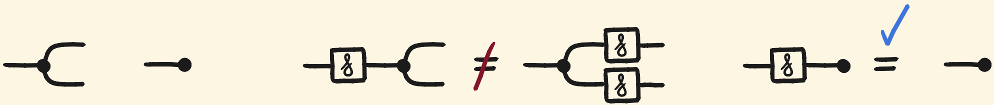


[Corradini & Gadducci 1999]

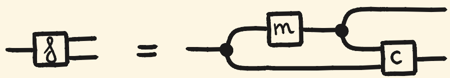
MARKOV CATEGORIES & CONDITIONALS

A Markov category (with conditionals) is a copy-discard category with conditionals where all morphisms are total.

COPY-DISCARD STRUCTURE



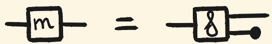
CONDITIONALS



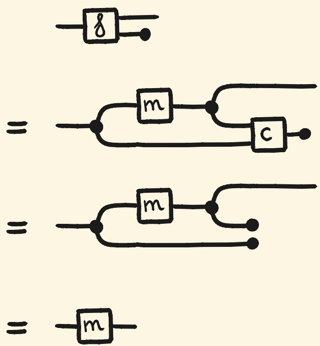
[Fritz 2020]

MARGINALS IN MARKOV CATEGORIES

Marginals in Markov categories are as expected :

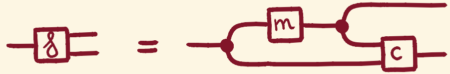


PROOF



conditionals :

$\rightsquigarrow$



$\rightsquigarrow$ totality

□

DROPPING TOTALITY

We want to keep the nice marginals of Markov categories.

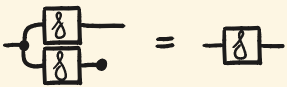
Should we ask conditionals to be total? ✗ NO

⇝ too strong: total conditionals fail to exist in $\mathrm{Kleisli}(\mathcal{D}_{\leq 1})$.

Can we ask conditionals to be quasi-total? ✓ YES

⇝ sweet spot: quasi-total conditionals usually exist and give nice marginals.

QUASI-TOTAL MORPHISM (in a copy-discard category)

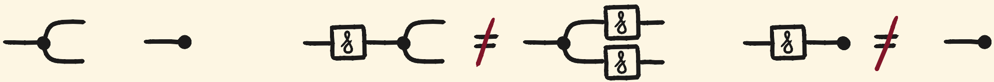


⇝ failure is deterministic

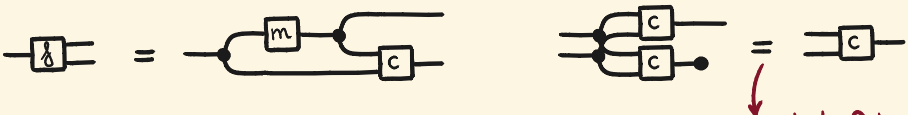
PARTIAL MARKOV CATEGORIES

A partial Markov category is a copy-discard category with quasi-total conditionals.

COPY-DISCARD STRUCTURE



CONDITIONALS



quasi-totality

EXAMPLES : PARTIAL STOCHASTIC PROCESSES

A <u>partial stochastic process</u> is a stochastic process that may fail.

↳ Maybe monad

your favourite Markov category

PROPOSITION

Partial stochastic processes form a partial Markov category.

$\begin{cases} \mathcal{C} \text{ Markov category with coproducts} \\ \text{Some ugly technical conditions} \end{cases}$

$\Rightarrow \mathrm{Kl}(\cdot + 1)$ is a partial Markov category.

EXAMPLES

- $\mathrm{Kl}(\mathcal{D}(\cdot + 1))$ ⇝ finitary subdistributions
- $\mathrm{Kl}(\mathcal{G}iry_B(\cdot + 1))$ ⇝ subdistributions on standard Borel spaces

SUBDISTRIBUTIONS

A <u>subdistribution</u> σ on A is a distribution on $A+1$:

$\sigma \in \mathcal{D}_{\leq 1}(A)$ is a function $\sigma : A \to [0,1]$ such that

- its support, $\mathrm{supp}(\sigma) := \{a \in A \mid \sigma(a) > 0\}$, is finite, and
- its total probability mass is at most 1, $\sum_{a \in A} \sigma(a) \leq 1$.

A morphism $X - \boxed{f} - A$ in $\mathrm{Kl}\mathcal{D}_{\leq 1}$ is a function $X \to \mathcal{D}_{\leq 1}(A)$

$f(a|x)$ = "probability of a given x"

$f(\perp|x)$ = "probability of failure"

Composition is

$$X - \boxed{f} - \boxed{g} - B \; (b|x) := \sum_{a \in A} f(a|x) \cdot g(b|a)$$

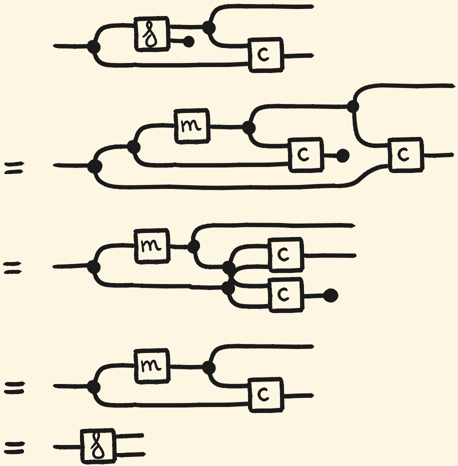
$$X - \boxed{f} - \boxed{g} - B \; (\perp|x) := \sum_{a \in A} f(a|x) \cdot g(\perp|a) + f(\perp|x)$$

MARGINALS OF QUASI-TOTAL CONDITIONALS

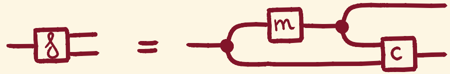
Marginals in partial Markov categories are as expected :

is always a marginal .

PROOF

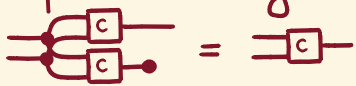


⇝ conditionals



⇝ associativity , commutativity

⇝ quasi-totality



□

CONDITIONALS IN SUBDISTRIBUTIONS

A quasi-total morphism $g : X \to B$ is a function $g : X \to \mathcal{D}B + 1$.

The marginal of $f : X \to A \otimes B$ is

$$X - \boxed{m} - A\,(a|x) = X - \boxed{f} \begin{smallmatrix} - A \\ - \bullet \end{smallmatrix}\,(a|x) = \sum_{b \in B} f(a, b|x)$$

$$X - \boxed{m} - A\,(\perp|x) = X - \boxed{f} \begin{smallmatrix} - A \\ - \bullet \end{smallmatrix}\,(\perp|x) = f(\perp|x)$$

A conditional of f is:

$$\begin{smallmatrix} A \\ X \end{smallmatrix} \boxed{c} - B\,(b|a,x) = \begin{cases} \dfrac{f(a,b|x)}{m(a|x)} & m(a|x) \neq 0 \\ 0 & m(a|x) = 0 \end{cases}$$

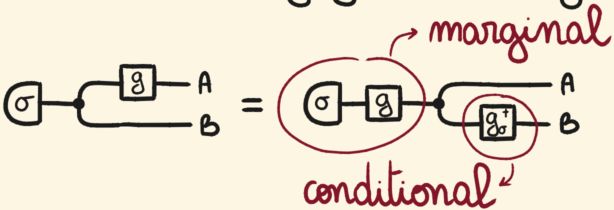
$$\begin{smallmatrix} A \\ X \end{smallmatrix} \boxed{c} - B\,(\perp|a,x) = \begin{cases} 0 & m(a|x) \neq 0 \\ 1 & m(a|x) = 0 \end{cases}$$

BAYES INVERSION

The Bayes inversion of a channel $g : B \to A$ with respect to a distribution $\sigma : I \to B$ is classically defined as

$$g^{\dagger}_{\sigma}(b|a) := \frac{g(a|b)\,\sigma(b)}{\sum_{b' \in B} g(a|b')\,\sigma(b')}$$

In a partial Markov category, it is a $g^{\dagger}_{\sigma} : A \to B$ such that



Bayes inversions are instances of quasi-total conditionals.

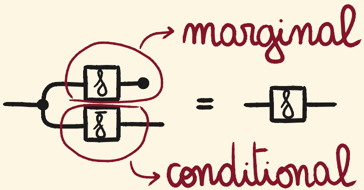
[Cho & Jacobs 2019]

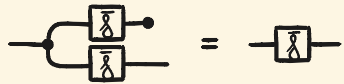
NORMALISATION

The normalisation of a partial channel $f : X \to A$ is classically defined as

$$\bar{f}(a|x) := \frac{f(x|a)}{1 - f(\perp|a)}$$

In a partial Markov category, it is a $\bar{f} : X \to A$ such that





Normalisations are instances of quasi-total conditionals.

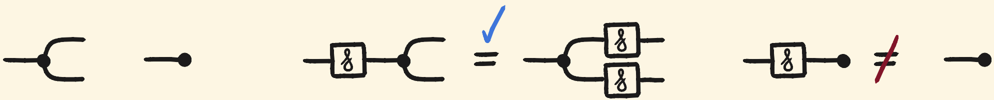
OUTLINE

- Partial Markov categories
- [Discrete partial Markov categories]
- Bayes, Pearl and Jeffrey
- Evidential decision theory
- Exact observations

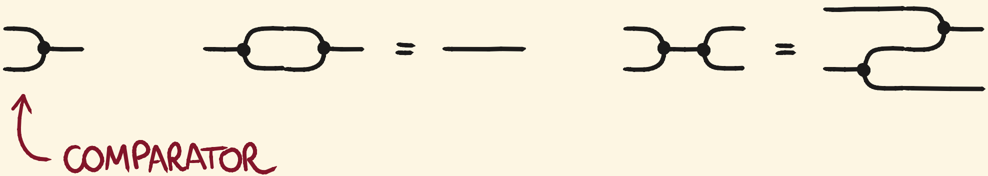
CONSTRAINTS VIA PARTIAL FROBENIUS

A discrete cartesian restriction category is a copy-discard category with comparators where all morphisms are deterministic.

COPY-DISCARD STRUCTURE



PARTIAL FROBENIUS STRUCTURE

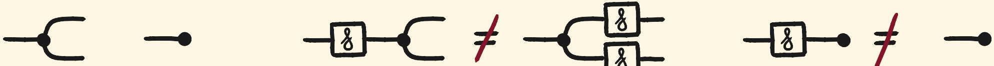


[Cockett & Lack 2003, Cockett, Guo & Hofstra 2012]

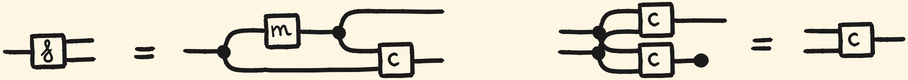
DISCRETE PARTIAL MARKOV CATEGORIES

A discrete partial Markov category is a copy-discard category with quasi-total conditionals and comparators.

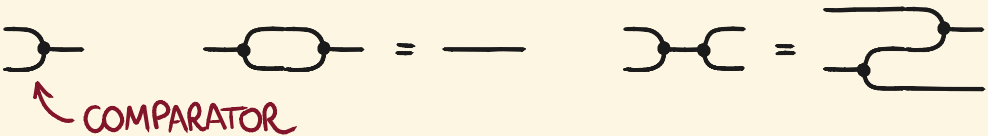
COPY-DISCARD STRUCTURE



CONDITIONALS



PARTIAL FROBENIUS STRUCTURE



EXAMPLES

FINITARY SUBDISTRIBUTIONS

$$\supset\!\!\bullet\!\!-\,(x \mid x_1, x_2) := \begin{cases} 1 & x = x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\supset\!\!\bullet\!\!-\,(\perp \mid x_1, x_2) := \begin{cases} 0 & x_1 = x_2 \\ 1 & \text{otherwise} \end{cases}$$

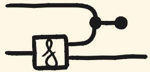
STANDARD BOREL SUBDISTRIBUTIONS

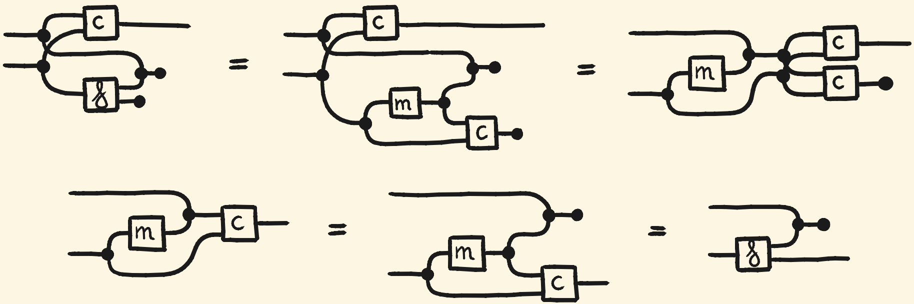
$$\supset\!\!\bullet\!\!-\,(x \mid x_1, x_2) := \begin{cases} 1 & x = x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$

⇝ not computable?
bad behaviour?

$$\supset\!\!\bullet\!\!-\,(\perp \mid x_1, x_2) := \begin{cases} 0 & x_1 = x_2 \\ 1 & \text{otherwise} \end{cases}$$

CONDITIONALS BY BENDING WIRES

A quasi-total conditional of  is a normalisation of  .

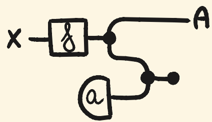


OUTLINE

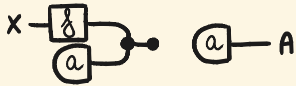
- Partial Markov categories
- Discrete partial Markov categories
- [Bayes, Pearl and Jeffrey]
- Evidential decision theory
- Exact observations

OBSERVATIONS AS CONSTRAINTS

Observing that $f : X \to A$ produced $a \in A$ means keeping only the scenarios in which f produces a.



When the observation is deterministic,

$=$ 

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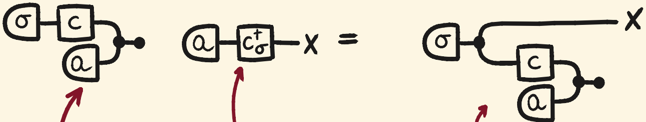
SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^{\dagger}_{\sigma}$ evaluated on a.

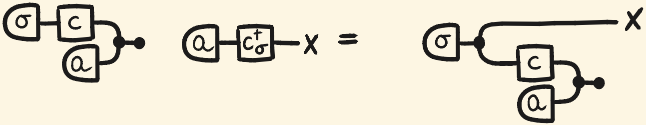


$$P(X = x \mid A = a) = \frac{P(A = a \mid X = x) \cdot P(X = x)}{\sum_{y \in X} P(A = a \mid X = y) \cdot P(X = y)}$$

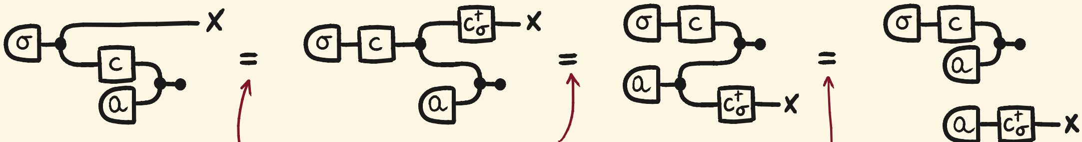
classical formula
for Bayes theorem

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^{\dagger}_{\sigma}$ evaluated on a.



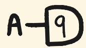
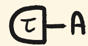
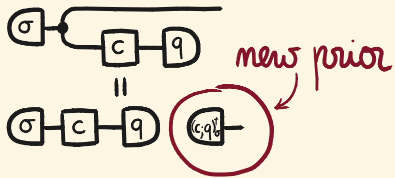
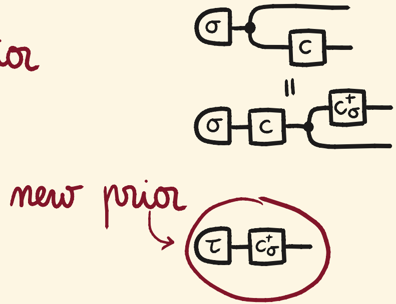
PROOF



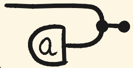
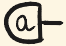
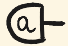
conditionals Frobenius determinism

$\square$

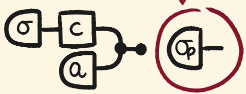
PEARL'S VS JEFFREY'S UPDATES

	PEARL	JEFFREY
PRIOR		
CHANNEL		
EVIDENCE		
UPDATE RULE		

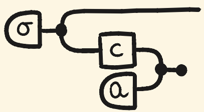
PEARL'S VS JEFFREY'S UPDATES

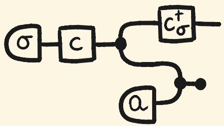
Pearl's update on  coincides with Jeffrey's update on , whenever  is deterministic.

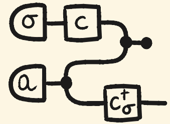
PROOF

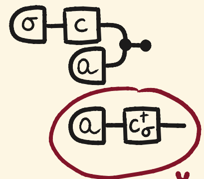


Pearl's updated prior

=  (Pearl's rule)

=  (conditionals)

=  (Frobenius)

=  (determinism)

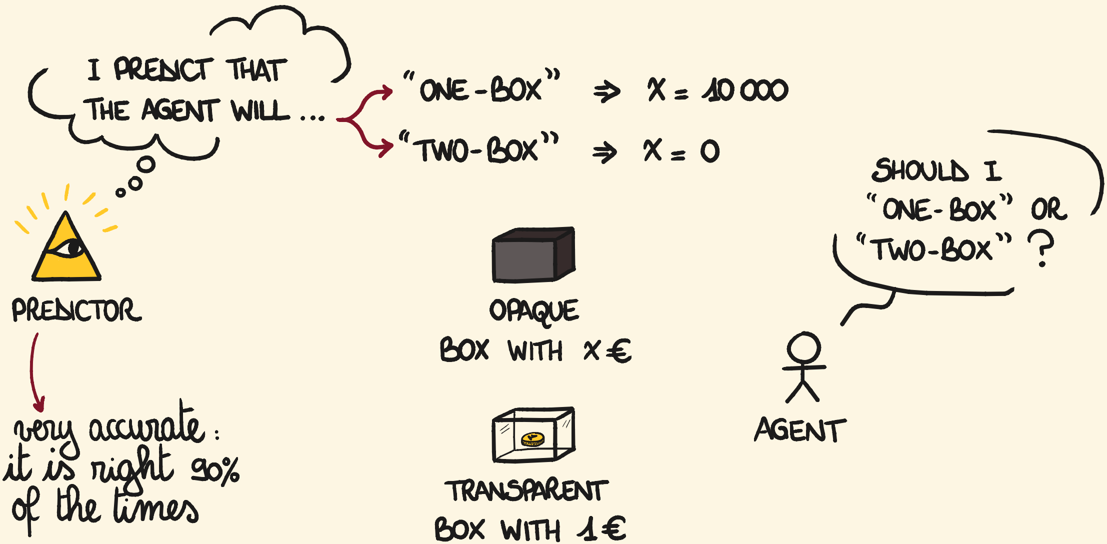
Jeffrey's updated prior

□

OUTLINE

- Partial Markov categories
- Discrete partial Markov categories
- Bayes, Pearl and Jeffrey
- [Evidential decision theory]
- Exact observations

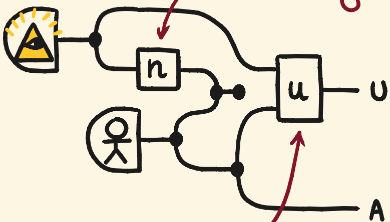
NEWCOMB'S PROBLEM



(cute drawing inspired by Mario's slides for NWPT '22)

NEWCOMB'S PROBLEM CATEGORICALLY

noise: the predictor
is right 90% of the times

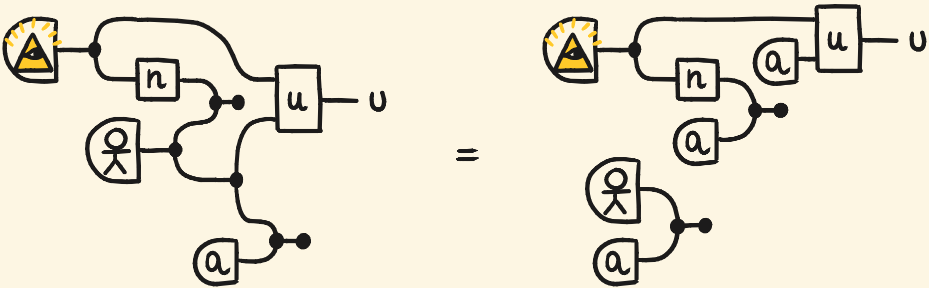


utility function

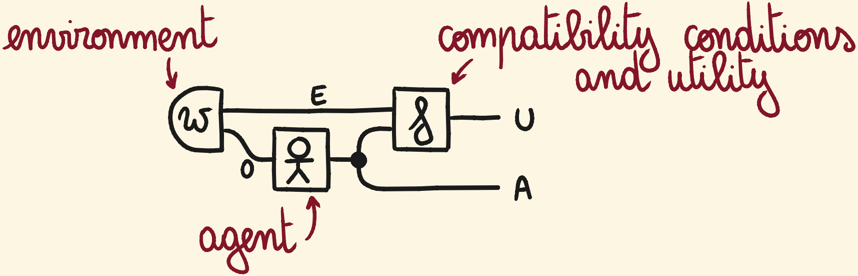
AGENT / PREDICTOR	ONE-BOX	TWO-BOX
ONE-BOX	10 000 €	10 001 €
TWO-BOX	0 €	1 €

SOLVING NEWCOMB'S PROBLEM

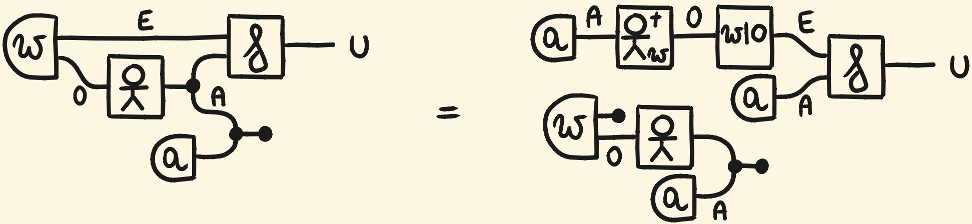
Evidential decision theory asks:
"Which action would be evidence for the best-case scenario?"
i.e. "Which action maximises the average of the state below?"



A TEMPLATE FOR DECISION PROBLEMS



We want to maximise the average of



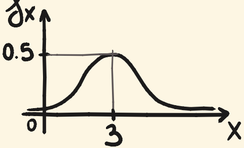
OUTLINE

- Partial Markov categories
- Discrete partial Markov categories
- Bayes, Pearl and Jeffrey
- Evidential decision theory
- [Exact observations]

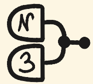
FREELY ADDING EXACT OBSERVATIONS

Comparators may

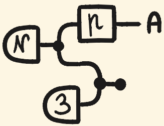
- not exist ⇝ equality need not be computable
- behave badly ⇝ probability 0 events may occur



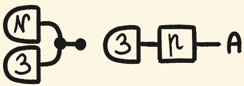
$P(X = 3) = 0$



$= 0$



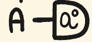
$(a) =$

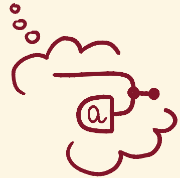


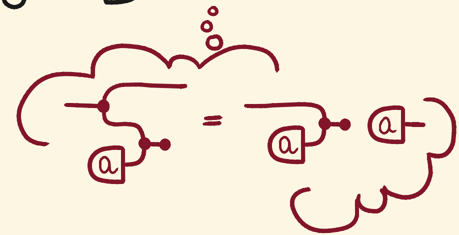
$(a) = 0 \cdot p(a|3) = 0$

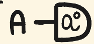
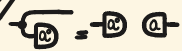
⇒ we add exact observations syntactically

PROCESSES WITH EXACT OBSERVATIONS

We construct a partial Markov category exOb($\mathcal{C}$) on top of a Markov category $\mathcal{C}$ by freely adding, for every deterministic state  in $\mathcal{C}$, a costate  and quotienting by  .



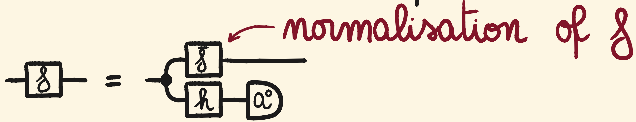


exOb($\mathcal{C}$) = ($\mathcal{C}$ + {  |  deterministic}) / 

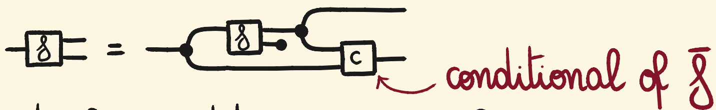
embeds faithfully into ($\mathcal{C}$ + ) / partial Frobenius

COMPUTING PROCESSES WITH EXACT OBSERVATIONS

Morphisms in $\mathrm{exOb}(\mathcal{C})$ have a normal form



that can be computed by conditioning in $\mathcal{C}$,
and they have conditionals



that can be computed by conditioning in $\mathcal{C}$.

SUMMARY

- Partial Markov = Markov − totality
- Discrete partial Markov = partial Markov + equality check
- Decision problems are easier with string diagrams (and also Bayes, Pearl & Jeffrey)
- Problems with equality checks? Add them freely

THANKS FOR LISTENING!