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TRACE SEMANTICS OF EFFECTFUL MEALY MACHINES

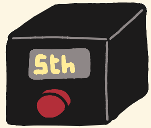
Filippo Bonchi
Università di Pisa

Elena Di Lavore
University of Oxford

Mario Román
University of Oxford

TRANSITION SYSTEMS AS COALGEBRAS

$$t : M \longrightarrow T(M \times B)^A$$



- behaviour given by final coalgebras

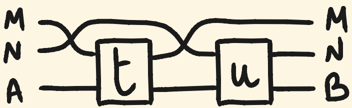
$$\begin{tikzcd} M \arrow[r, dashed, "h"] \arrow[d, "t"'] & \Omega \arrow[d, "\omega"] \\ T(M \times B)^A \arrow[r] & T(\Omega \times B)^A \end{tikzcd}$$

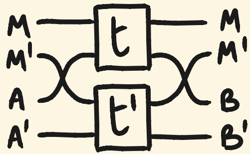
⇝ how do they compose ?

TRANSITION SYSTEMS AS MORPHISMS

$$t : M \otimes A \longrightarrow M \otimes B$$

- native sequential and parallel composition



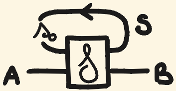


⇝ what's their behaviour ?

[Katis, Sabadini, Walters 1997]

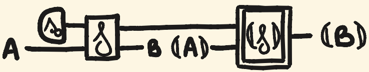
OVERVIEW

effectful Mealy machines



bisimilarity

effectful streams



free construction

~ syntax

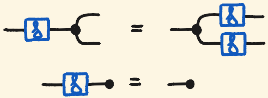
uniformity

coalgebraic construction

~ trace semantics

EFFECTFUL TRIPLES

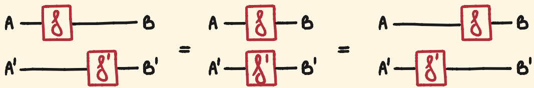
Values can be copied and discarded (cartesian)



Effectful computations may have global effects (premonoidal)

$$\mathcal{V} \longrightarrow \mathcal{L} \longrightarrow \mathcal{C}$$

Local computations interchange (monoidal)



[Jeffrey (1997), cf. Levy (2022), Power and Thielecke (1997)]

EXAMPLES OF EFFECTFUL TRIPLES

- $(\mathrm{cart}(\mathcal{C}), Z(\mathcal{C}), \mathcal{C})$ for a cs-premonoidal $\mathcal{C}$
- $(\mathcal{S}et, \mathrm{Kl}(Z(T)), \mathrm{Kl}(T))$ for a $\mathcal{S}et$-monad T
- $(\mathcal{S}et, \mathcal{S}et, \mathcal{S}et)$
- $(\mathcal{S}et, \mathcal{R}el_{TOT}, \mathcal{R}el)$
- $(\mathcal{S}et, \mathrm{Kl}\,\mathcal{D}, \mathrm{Kl}\,\mathcal{D}_{\leq})$
- $(\mathcal{S}et, \mathrm{Kl}\,\mathcal{D}, \mathcal{S}tate_{S})$

OUTLINE

For an effectful triple $(\mathcal{V}, \mathcal{L}, \mathcal{C})$, construct:

[• effectful Mealy machines]

• bisimulations

• traces (effectful streams)

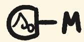
EFFECTFUL MEALY MACHINES

a <u>Mealy machine</u> $(f, M, s_0) : A \to B$ in $(\mathcal{V}, \mathcal{L}, \mathcal{C})$
is a morphism

$$f : M \otimes A \to M \otimes B$$

with an initial state

$$s_0 : I \to M$$



<u>ex</u> $(\mathsf{Set}, \mathsf{Rel}_{\mathsf{TOT}}, \mathsf{Rel})$

$$\begin{cases} f : M \times A \to \mathcal{P}(M \times B) \\ s_0 \subseteq M \end{cases}$$

<u>ex</u> $(\mathsf{Set}, \mathsf{Stoch}, \mathsf{Stoch})$

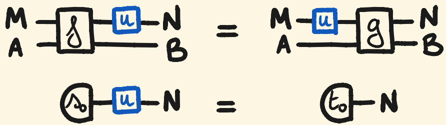
$$\begin{cases} f : M \times A \to \mathcal{D}(M \times B) \\ s_0 \in \mathcal{D}(M) \end{cases}$$

[cf. Katis, Sabadini, Walters (1997); EDL, Gianola, Román, Sabadini, Sobociński (2022)]

MORPHISMS OF MEALY MACHINES

a **morphism** of Mealy machines $u : (f, M, s_0) \to (g, N, t_0)$
is a value morphism $u : M \to N$ in $\mathcal{V}$

such that



ex ($\mathbf{Set}$, $\mathbf{Rel}_{TOT}$, $\mathbf{Rel}$)

$\forall s \in M \; \forall a \in A \; \forall t \in N \; \forall b \in B$

$(t, b) \in g(u(s), a) \iff \exists s' \in M \;\; u(s') = t \wedge (s', b) \in f(s, a)$

ex ($\mathbf{Set}$, $\mathbf{Stoch}$, $\mathbf{Stoch}$)

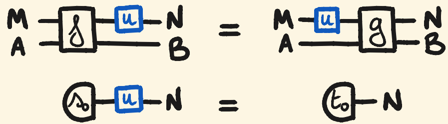
$\forall s \in M \; \forall a \in A \; \forall t \in N \; \forall b \in B$

$$g(t, b \mid u(s), a) = \sum_{s' : u(s') = t} f(s', b \mid s, a)$$

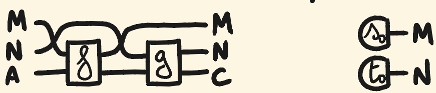
EFFECTFUL CATEGORY OF MEALY MACHINES

Mealy is an effectful category where

- objects are the objects of $\mathcal{C}$
- morphisms $(f, M, s_0) : A \to B$ are Mealy machines quotiented by value isomorphisms $u : M \xrightarrow{\cong} N$



- composition tensors the state spaces $\rightsquigarrow$ local states



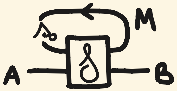
MEALY MACHINES ARE FREE

$\mathcal{S} := pt\mathcal{C}_{iso}$

THEOREM

Mealy is the free pointed-feedback category over $\mathcal{C}$.

$$\mathrm{Mealy}(A,B) = \int^{(s_0,M)\in pt\mathcal{C}_{iso}} \mathcal{C}(M\otimes A, M\otimes B)$$



[cf. Katis, Sabadini, Walters (1997); EDL, Gianola, Román, Sabadini, Sobociński (2022)]

OUTLINE

For an effectful triple $(\mathcal{V}, \mathcal{L}, \mathcal{C})$, construct:

- effectful Mealy machines
- [bisimulations]
- traces (effectful streams)

COALGEBRAIC BISIMULATION

A bisimulation is a span of coalgebras.

$$\begin{tikzcd} M \arrow[d, "f"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\alpha"] & N \arrow[d, "g"] \\ F(M) & F(R) \arrow[l, "F(\pi_1)"] \arrow[r, "F(\pi_2)"'] & F(N) \end{tikzcd}$$

THEOREM [Rutten (2000)]

When $F : \mathbf{Set} \to \mathbf{Set}$ preserves weak pullbacks, bisimularity is an equivalence relation.

[Aczel & Mendler (1989), Rutten (2000)]

BISIMULATION

For two effectful Mealy machines $(f, M, s), (g, N, t) : A \to B$, a <u>bisimulation</u> is a sequence of spans of morphisms.

$$\begin{tikzcd}[column sep=small] & (h_1, R_1, r_1) \arrow[dl, "u_1"'] \arrow[dr, "v_1"] & & \arrow[dl, "u_2"'] \cdots & (h_n, R_n, r_n) \arrow[dr, "v_n"] & \\ (f, M, s) & & (f_1, M_1, s_1) & & & (g, N, t) \end{tikzcd}$$

PROPOSITION

When $\mathcal{C} = Kl(T)$, for a commutative monad T preserving weak pullbacks, effectful bisimulation coincides with coalgebraic bisimulation.

<u>ex</u> Set, Par, Rel, Stoch, pStoch

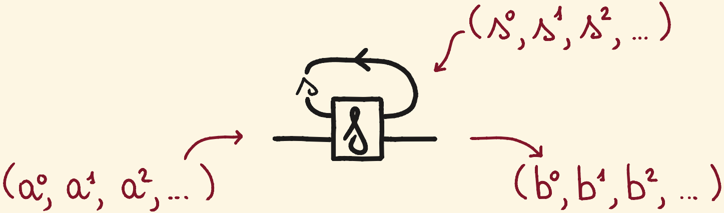
OUTLINE

For an effectful triple $(\mathcal{V}, \mathcal{L}, \mathcal{C})$, construct:

- effectful Mealy machines
- bisimulations

[- traces (effectful streams)]

EXECUTING MEALY MACHINES

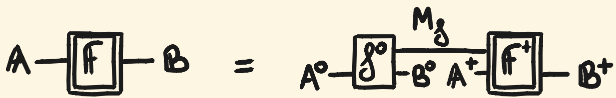


⇝ what should the semantic universe be?
when do two Mealy machines have the same executions?

EFFECTFUL STREAMS

An *effectful stream* $F : A \to B$ on $(\mathcal{V}, \mathcal{L}, \mathcal{C})$ is

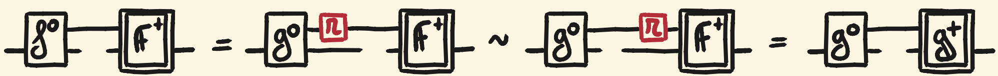
- a memory $M_f \in \mathcal{L}$
- a first action $f^\circ : A^\circ \to M_f \otimes B^\circ$ in $\mathcal{C}$
- the rest of the action $F^+ : M_f \cdot A^+ \to B^+$



quotiented by sliding

$$\begin{cases} f^\circ ; (r \otimes 1) = g^\circ \\ g^+ = r \cdot F^+ \end{cases}$$

for $r : M_f \to M_g$ in $\mathcal{L}$



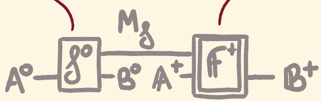
EFFECTFUL STREAMS

The profunctor Stream : $\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}} \longrightarrow \mathbf{Set}$ is the final coalgebra of the functor

$$F : [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \mathbf{Set}] \longrightarrow [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \mathbf{Set}]$$

$$F(Q)(A,B) := \int^{M \in \mathcal{C}} \mathcal{C}(A^{\circ}, M \otimes B^{\circ}) \times Q(M \cdot A^{+}, B^{+})$$

quotient by sliding on the memory

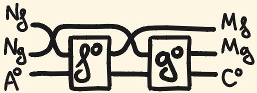


COMPOSITIONAL STRUCTURE OF STREAMS

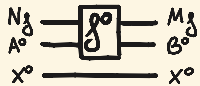
THEOREM

Effectful streams form an effectful category Stream.

- composition and monoidal actions are defined coinductively:
  for $\mathbb{F} : N_f \cdot \mathbb{A} \to \mathbb{B}$ and $g : N_g \cdot \mathbb{B} \to \mathbb{C}$,

$(\mathbb{F} ;_N g)^\circ :=$ 

$(\mathbb{F} ;_N g)^+ := \mathbb{F}^+ ;_M g^+$

$(\mathbb{X} \otimes_N \mathbb{F})^\circ :=$ 

$(\mathbb{X} \otimes_N \mathbb{F})^+ := \mathbb{X}^+ \otimes_M \mathbb{F}^+$

FEEDBACK ON EFFECTFUL STREAMS

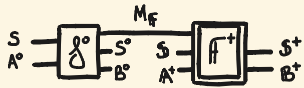
$\partial : \mathsf{Stream} \longrightarrow \mathsf{Stream}$ $\qquad$ $\partial(A) := (I, A^0, A^1, \dots)$

THEOREM

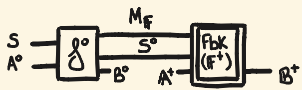
Stream has ∂-feedback.

- feedback is defined coinductively

$$\mathbb{F} : (S \cdot \partial \mathbb{S}) \otimes A \longrightarrow \mathbb{S} \otimes B$$



$$\mathrm{Fbk}_{\mathbb{S}}\, \mathbb{F} : S \cdot A \longrightarrow B$$



$$M(\mathrm{Fbk}^S_{\mathbb{S}}\, \mathbb{F}) := M(\mathbb{F}) \otimes \mathbb{S}^0$$
$$(\mathrm{Fbk}^S_{\mathbb{S}}\, \mathbb{F})^0 := f^0$$
$$(\mathrm{Fbk}^S_{\mathbb{S}}\, \mathbb{F})^+ := \mathrm{Fbk}^{S^0}_{\mathbb{S}^+}(\mathbb{F}^+)$$

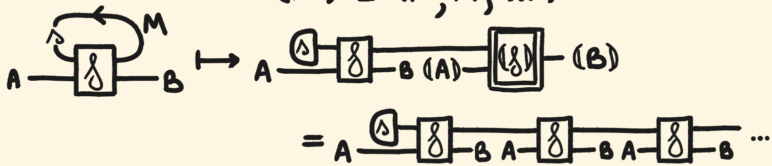
COMPOSITIONAL TRACE SEMANTICS

THEOREM

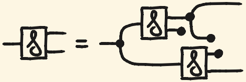
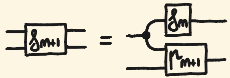
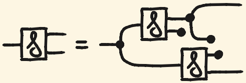
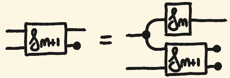
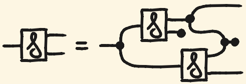
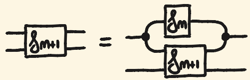
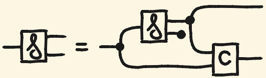
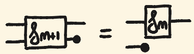
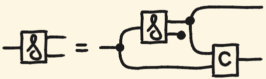
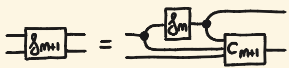
There is an effectful functor

$$\mathrm{Tr} : \mathrm{Mealy} \to \mathrm{Stream}$$

$$A \mapsto (\!(A)\!) = (A, A, \dots)$$



TRACES ARE EFFECTFUL TRACES

	CONDITIONALS	CAUSALITY CONDITION	TRACE PREDICATE
Set			$s_0 = s \wedge \forall i \leq n \; (s_{i+1}, b_i) = f(s_i, a_i)$
Par			$s_0 = s \wedge \forall i \leq n \; (s_{i+1}, b_i) = f(s_i, a_i)$
Rel			$\exists s_0, \dots, s_{n+1} \; s_0 \in s \wedge \forall i \leq n \; (s_{i+1}, b_i) \in f(s_i, a_i)$
Stoch			$\sum_{s_0, \dots, s_{n+1}} s(s_0) \cdot \prod_{i \leq n} f(s_{i+1}, b_i \mid s_i, a_i)$
pStoch			$\sum_{s_0, \dots, s_{n+1}} s(s_0) \cdot \prod_{i \leq n} f(s_{i+1}, b_i \mid s_i, a_i)$

TRACE IS UNIVERSAL

THEOREM

The trace functor $\mathsf{Tr} : \mathsf{Mealy} \to \mathsf{Stream}$ is the unique feedback effectful functor determined by freeness of Mealy.

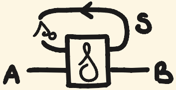
$$\begin{tikzcd} \mathbb{FbkEffCat} \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\ \mathbb{EffCat} \arrow[u, bend left, "\mathsf{Mealy}"] \end{tikzcd} \qquad \Rightarrow \exists!\ \mathsf{Tr} : \begin{tikzcd} \mathsf{Mealy}_{(\mathcal{V},\mathcal{L},\mathcal{C})} \arrow[rr] \arrow[dr] & \arrow[d, phantom, "="] & \mathsf{Stream}_{(\mathcal{V},\mathcal{L},\mathcal{C})} \\ & \mathsf{Mealy}_{(\mathcal{V},\mathcal{L},\mathcal{C})}/\mathsf{bisim} \arrow[ur] & \end{tikzcd}$$

COROLLARY

Bisimilarity implies trace equivalence.

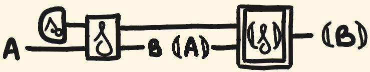
SUMMARY

effectful Mealy machines



$\downarrow$ trace

effectful streams



$\simeq$ causal processes

free construction

~ syntax

$\downarrow$ $\exists!$

coalgebraic construction

~ semantics

FUTURE WORK

- Adding choice, iteration and higher-order
- Behavioural metrics, metric enrichment
- Coinduction up-to dinaturality