

CALCO

17 June 2025

EFFECTFUL MEALY MACHINES

BISIMULATION & TRACES

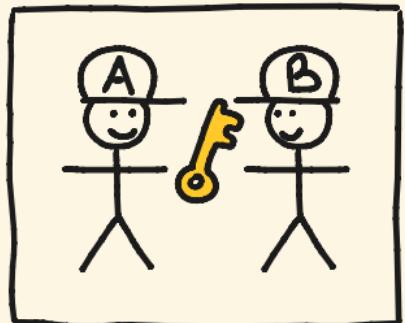
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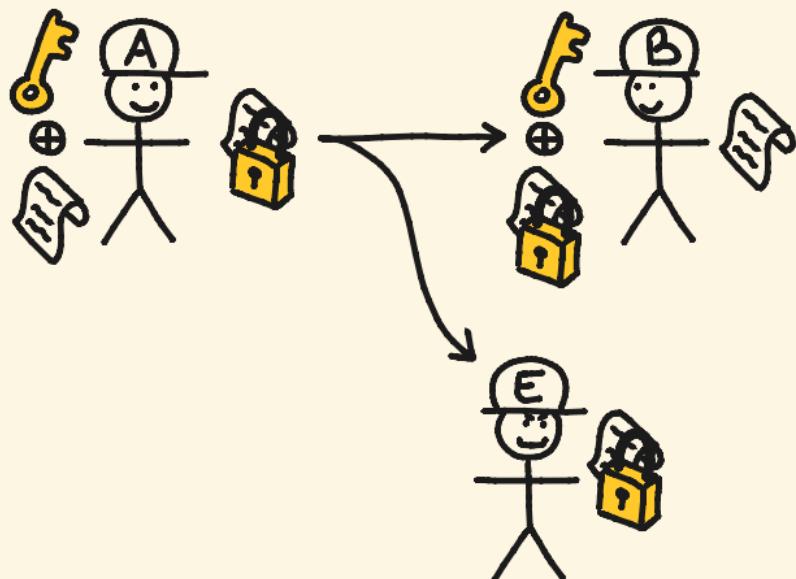
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ONE-TIME PAD PROTOCOL

1. share a key through
a secure channel



2. send an encrypted
message through a
public channel



REPEATING THE ONE-TIME PAD

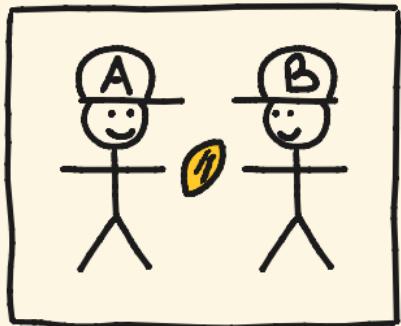
Sending n messages securely requires n private keys
↳ not very useful

- ⇒ • privately share a seed 
- use identical pseudorandom number generator to obtain a new key for each message



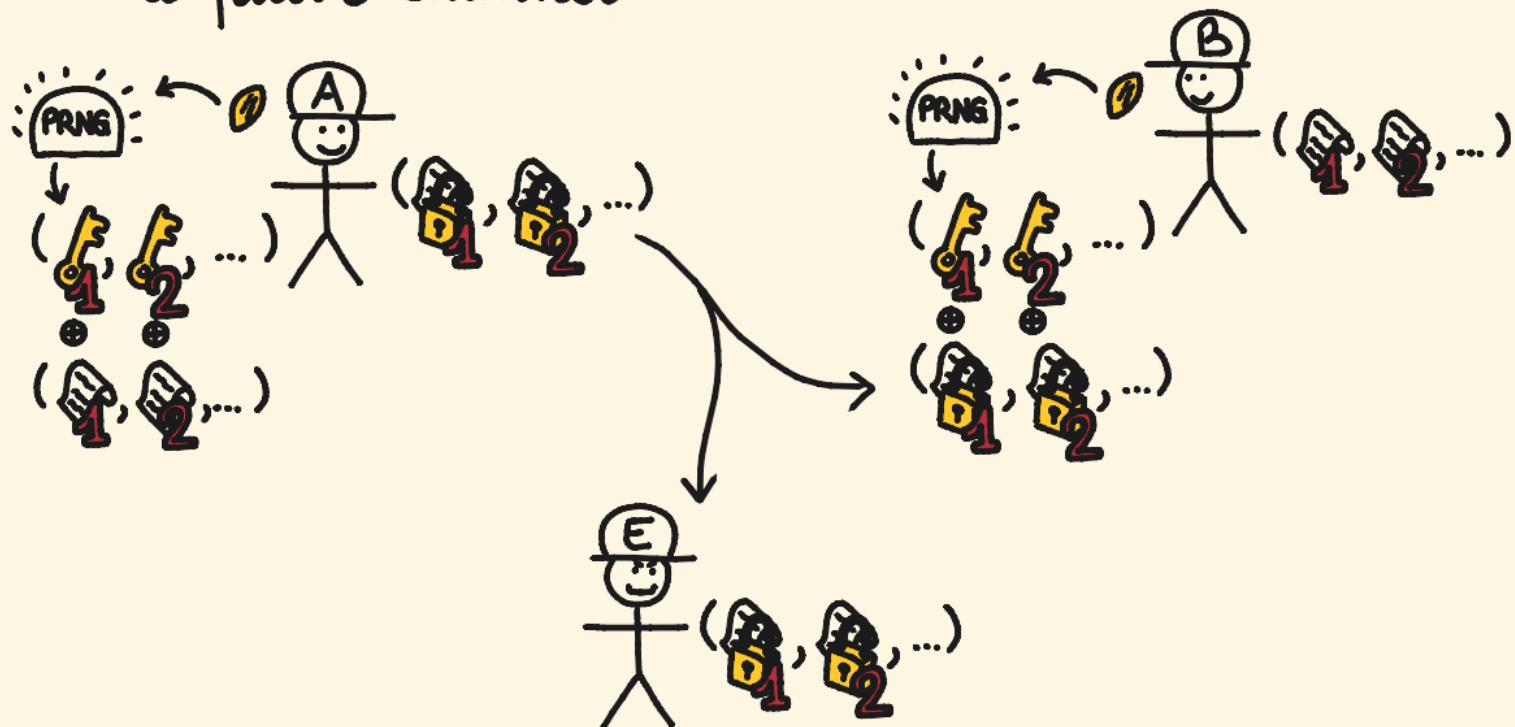
STREAM CIPHER PROTOCOL (1)

1. share a seed through a secure channel
2. share a pseudorandom number generator



STREAM CIPHER PROTOCOL (2)

3. send a stream of encrypted messages through a public channel

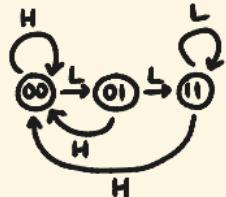


MEALY MACHINES

A classical Mealy machine is

$$\begin{cases} t : S \times A \rightarrow P(S \times B) & \text{transition relation} \\ s_0 : 1 \rightarrow PS & \text{initial states} \end{cases}$$

TEMPERATURE
CONTROLLER



A Markov decision process is

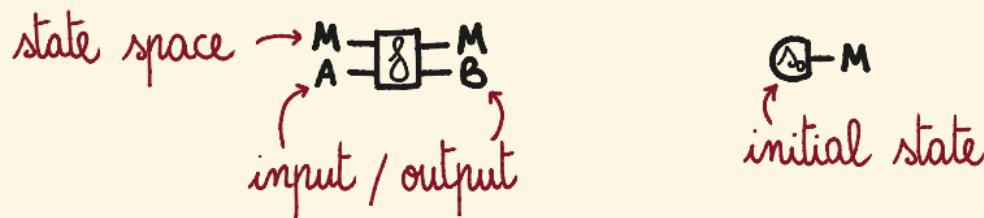
$$\begin{cases} t : S \times A \rightarrow DS & \text{transition probability} \\ r : S \times A \rightarrow DU & \text{reward} \\ s_0 : 1 \rightarrow DS & \text{initial distribution} \end{cases}$$

RANDOM
WALK



MEALY MACHINES

Systems are $f: M \otimes A \rightarrow M \otimes B$ with $s_0: I \rightarrow M$



- native sequential and parallel compositions
- parametric in the underlying process theory
- premonoidal categories for global effects

~ what is their behaviour?
when are two of them equivalent?

[cf. Katis, Sabadini, Walters 1997]

COALGEBRAIC SEMANTICS

Systems are coalgebras $f : M \rightarrow F(M)$

input/output

$$M \rightarrow (M \times B)^A$$

non-determinism

$$M \rightarrow P(M \times B)$$

- bisimulation is equality in the final coalgebra

→ how do these compose?

how to change the underlying process theory?

MEALY MACHINES AND COALGEBRAS

Two faces of Mealy machines

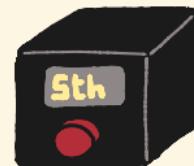
$$M \times A \rightarrow D(M \times B)$$

$$M \rightarrow D(M \times B)^A$$

$$M \times A \rightarrow P(M \times B)$$

$$M \rightarrow P(M \times B)^A$$

$$\begin{array}{c} M \\ A = \boxed{\delta} = B \end{array}$$



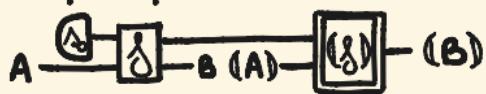
OVERVIEW

effectful Mealy machines



bisimilarity

effectful streams



[≈ causal processes]

free construction
~ syntax



uniformity

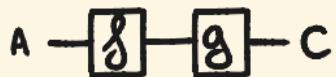


coalgebraic construction
~ trace semantics

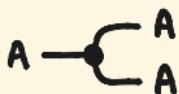
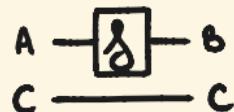
OUTLINE

- [• effectful copy-discard categories]
- effectful Mealy machines & bisimulations
- coinductive traces
- clausal traces

STRING DIAGRAMS



A — A



A →

$$\text{---} \cap \text{---} = \text{---} \cap \text{---}$$

$$\text{---} \cap \text{---} = \text{---}$$

$$\text{---} \cap \text{---} = \text{---} \cap \text{---}$$

THE INTERCHANGE LAW

$$\begin{array}{c} A \xrightarrow{\quad g \quad} B \\ A' \xrightarrow{\quad g \quad} B' \end{array} = \begin{array}{c} A \xrightarrow{\quad g \quad} B \\ A' \xrightarrow{\quad g \quad} B' \end{array} = \begin{array}{c} A \xrightarrow{\quad g \quad} B \\ A' \xrightarrow{\quad g \quad} B' \end{array}$$

→ holds in monoidal categories

VALUES

Values are both :

- deterministic



- total



ex $(3 \cdot -) : \mathbb{R} \rightarrow \mathbb{R}$

non-ex Flip : $\{1\} \rightarrow \mathcal{D}(\{\text{H, T}\})$ \neq

$(3/-) : \mathbb{R} \rightarrow \mathbb{R}$

\neq

LOCAL COMPUTATIONS

Local computations interchange,

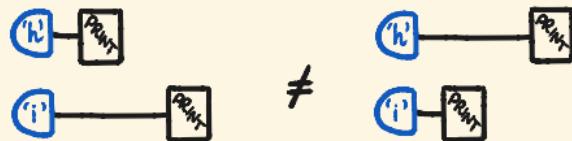
$$\begin{array}{c} A - \boxed{\delta} - B \\ A' - \boxed{\delta'} - B' \end{array} = \begin{array}{c} A - \boxed{\delta} - B \\ A' - \boxed{\delta'} - B' \end{array} = \begin{array}{c} A - \boxed{\delta} - B \\ A' - \boxed{\delta'} - B' \end{array}$$

ex Stoch

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

EFFECTFUL COMPUTATIONS

Effectful computations may have global effects.



ex state promonads, 10 monad

[Power and Robinson (1997)]

EFFECTFUL COPY-DISCARD CATEGORIES

Values can be copied and discarded (cartesian)

$$\begin{array}{c} \text{---} \square \text{---} \sqcap = \text{---} \square \text{---} \square \sqcap \\ \text{---} \square \text{---} \bullet = \text{---} \bullet \end{array}$$

$$\mathcal{V} \rightarrow \mathcal{L} \rightarrow \mathcal{C}$$

Effectful computations may have global effects (premonoidal)

$$\begin{array}{c} \text{---} \square \text{---} \square \neq \text{---} \square \text{---} \square \\ \text{---} \square \text{---} \square \\ \text{---} \square \text{---} \square \end{array}$$

local computations interchange (monoidal)

$$\begin{array}{ccc} A - \boxed{\delta} - B & = & A - \boxed{\delta} - B \\ A' - \boxed{\delta} - B' & = & A' - \boxed{\delta'} - B' = A - \boxed{\delta} - B \\ & & A' - \boxed{\delta'} - B' \end{array}$$

[Jeffrey (1997), cf. Levy (2022), Power and Thielecke (1997)]

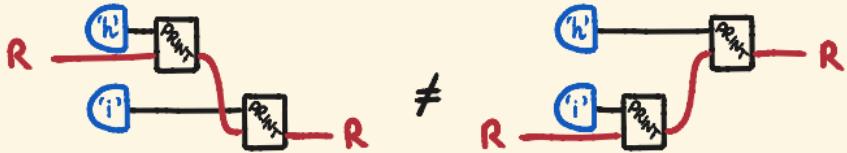
EXAMPLES OF EFFECTFUL TRIPLES

- $(\text{cart}(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{C})$ for a cd -premonoidal \mathcal{C}
- $(\text{cSet}, \text{Kl}(\mathcal{Z}(T)), \text{Kl}(T))$ for a cSet -monad T
- $(\text{cSet}, \text{Set}, \text{Par})$
- $(\text{cSet}, \text{Rel}_{\text{TOT}}, \text{Rel})$
- $(\text{cSet}, \text{Stoch}, \text{Stoch}_{\leq})$
- $(\text{cSet}, \text{Stoch}, \text{States})$

STRING DIAGRAMS FOR PREMONOIDAL CATEGORIES

Add a "runtime" wire to non-central morphisms

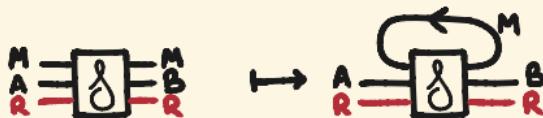
$$g: A \rightarrow B \quad \text{is} \quad \begin{array}{c} A \\ \text{---} \\ R \end{array} \xrightarrow{\quad g \quad} \boxed{g} \xrightarrow{\quad g \quad} \begin{array}{c} B \\ \text{---} \\ R \end{array}$$



[Jeffrey (1997); Román (2022); Román, Sobociński (2025)]

FEEDBACK EFFECTFUL CATEGORIES

$$\text{FbK}_M : \mathcal{C}(M \otimes A, M \otimes B) \rightarrow \mathcal{C}(A, B)$$



$$A \xrightarrow[R]{\quad} B = A \xrightarrow[R]{\quad} B \circ M \otimes N$$

$$A \xrightarrow[R]{\quad} B = A \xrightarrow[R]{\quad} B$$

$$A \xrightarrow[R]{\quad} C = A \xrightarrow[R]{\quad} C \circ M$$

$$A' \xrightarrow[R]{\quad} B' = A' \xrightarrow[R]{\quad} B' \circ M$$

$$A \xrightarrow[R]{\quad} B = A \xrightarrow[R]{\quad} B \circ u$$

$u : N \rightarrow M$ in \mathcal{L}



OUTLINE

- effectful copy-discard categories
- [• effectful Mealy machines & bisimulations]
- coinductive traces
- clausal traces

EFFECTFUL MEALY MACHINES

a Mealy machine $(f, M, s_0) : A \rightarrow B$ in $(\mathcal{V}, \mathcal{L}, \mathcal{C})$
is a morphism

$$f : M \otimes A \rightarrow M \otimes B \quad \text{in } \mathcal{C} \quad \begin{array}{c} M \\ \otimes \\ A \end{array} = \boxed{f} = \begin{array}{c} M \\ \otimes \\ B \end{array}$$

with an initial state

$$s_0 : I \rightarrow M \quad \text{in } \mathcal{L} \quad \begin{array}{c} \oplus \\ I \end{array} - M$$

ex $(\text{Set}, \text{Rel}_{\text{tor}}, \text{Rel})$

$$\left\{ \begin{array}{l} f : M \times A \rightarrow P(M \times B) \\ s_0 \subseteq M \end{array} \right.$$

ex $(\text{Set}, \text{Stoch}, \text{Stoch}_{\leq})$

$$\left\{ \begin{array}{l} f : M \times A \rightarrow \mathcal{D}(M \times B) \\ s_0 \in \mathcal{D}(M) \end{array} \right.$$

[cf. Katis, Sabadini, Walters (1997); EDL, Gjajola, Román, Sabadini, Sobociński (2022)]

MORPHISMS OF MEALY MACHINES

A morphism of Mealy machines $u: (f, M, s_0) \rightarrow (g, N, t_0)$
is a value morphism $u: M \rightarrow N$ in \mathcal{U}

such that

$$\begin{array}{c} M \\ \xrightarrow{\quad u \quad} \\ A \end{array} = \begin{array}{c} N \\ \xrightarrow{\quad g \quad} \\ B \end{array}$$
$$\begin{array}{c} A \\ \xrightarrow{\quad u \quad} \\ N \end{array} = \begin{array}{c} t_0 \\ - \\ N \end{array}$$

ex (Set , Rel_{tor} , Rel)

$$\forall s \in M \quad \forall a \in A \quad \forall t \in N \quad \forall b \in B$$

$$(t, b) \in g(u(s), a) \Leftrightarrow \exists s' \in M \quad u(s') = t \wedge (s', b) \in f(s, a)$$

ex (Set , Stoch , Stoch)

$$\forall s \in M \quad \forall a \in A \quad \forall t \in N \quad \forall b \in B$$

$$g(t, b | u(s), a) = \sum_{s': u(s') = t} f(s', b | s, a)$$

EFFECTFUL CATEGORY OF MEALY MACHINES

Mealy is an effectful category where

- objects are the objects of \mathcal{C}
- morphisms $(f, M, s) : A \rightarrow B$ are Mealy machines quotiented by value isomorphisms $u : M \xrightarrow{\cong} N$

$$\begin{array}{c} M \\ A \\ R \end{array} = \begin{array}{c} M \\ A \\ R \end{array} \xrightarrow{u} \begin{array}{c} N \\ B \\ R \end{array} = \begin{array}{c} M \\ A \\ R \end{array} \xrightarrow{u} \begin{array}{c} N \\ B \\ R \end{array}$$

$$\begin{array}{c} \textcircled{A} \\ \textcircled{B} \end{array} \xrightarrow{u} N = \begin{array}{c} \textcircled{A} \\ \textcircled{B} \end{array} \xrightarrow{v} N$$

- composition tensors the state spaces \rightsquigarrow local states

$$\begin{array}{c} M \\ N \\ A \\ R \end{array} \xrightarrow{u} \begin{array}{c} M \\ N \\ C \\ R \end{array} = \begin{array}{c} \textcircled{A} \\ \textcircled{B} \end{array} \xrightarrow{u} M \quad \begin{array}{c} \textcircled{B} \\ \textcircled{C} \end{array} \xrightarrow{v} N$$

MEALY MACHINES ARE FREE

THEOREM

Mealy is the free pointed-feedback category over \mathcal{C} .

$$\text{Mealy}(A, B) = \int^{\rho(\lambda_0, M) \in \text{pt}\mathcal{C}_{\text{iso}}} \mathcal{C}(M \otimes A, M \otimes B)$$



[cf. Katis, Sabadini, Walters (1997); EDL, Gjaniola, Román, Sabadini, Sobociński (2022)]

COALGEBRAIC MEALY MACHINES

$T: \text{Set} \rightarrow \text{Set}$ monad

$F_T: \text{Set} \rightarrow \text{Set}$
 $X \mapsto T(X \times B)^A$

a T -Mealy machine is an F_T -coalgebra

$$f: X \rightarrow T(X \times B)^A$$

$$\Leftrightarrow \hat{f}: X \otimes A \rightarrow X \otimes B \quad \text{in } \text{Kl } T$$

\Leftrightarrow effectful Mealy machine in
 $(\text{Set}, \text{Kl}(Z(T)), \text{Kl}(T))$

COALGEBRAIC BISIMULATION

A bisimulation is a span of coalgebras.

$$\begin{array}{ccccc} M & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & N \\ \delta \downarrow & & \downarrow \alpha & & \downarrow g \\ F_T(M) & \xleftarrow{F_T(\pi_1)} & F_T(R) & \xrightarrow{F_T(\pi_2)} & F_T(N) \end{array}$$

THEOREM [Rutten (2000)]

When $F_T: \text{Set} \rightarrow \text{Set}$ preserves weak pullbacks,
bisimilarity is an equivalence relation.

[Aczel & Mendler (1989), Rutten (2000)]

BISIMULATION

For two effectful Mealy machines $(f, M, s), (g, N, t) : A \rightarrow B$,
a bisimulation is a sequence of spans of morphisms.

$$(f, M, s) \xleftarrow{u_1} (h_1, R_1, \pi_1) \xrightarrow{\pi_1} (f_1, M_1, s_1) \xleftarrow{u_2} \dots \xleftarrow{u_m} (h_m, R_m, \pi_m) \xrightarrow{\pi_m} (g, N, t)$$

PROPOSITION

When $\mathcal{C} = \text{Kl}(T)$, for a commutative monad T preserving weak pullbacks, effectful bisimulation coincides with coalgebraic bisimulation.

ex cSet, Par, Rel, cStoch, ncStoch

BISIMILARITY IS FREE

UNIFORM FEEDBACK [cf. Glăseanu (2002); ăzănescu, ăzănescu (1994)]

$$\left\{ \begin{array}{lcl} M \\ \text{---} \\ A & \xrightarrow{\delta} & B \\ R & \xrightarrow{u} & R \\ \text{---} \\ \text{---} & = & \text{---} \\ \text{---} & = & \text{---} \end{array} \right. = \begin{array}{lcl} M \\ \text{---} \\ A & \xrightarrow{u} & B \\ R & \xrightarrow{g} & R \\ \text{---} \\ \text{---} & = & \text{---} \\ \text{---} & = & \text{---} \end{array} \Rightarrow \begin{array}{lcl} M \\ \text{---} \\ A & \xrightarrow{\delta} & B \\ R & \xrightarrow{t_0} & R \\ \text{---} \\ \text{---} & = & \text{---} \\ \text{---} & = & \text{---} \end{array}$$

THEOREM

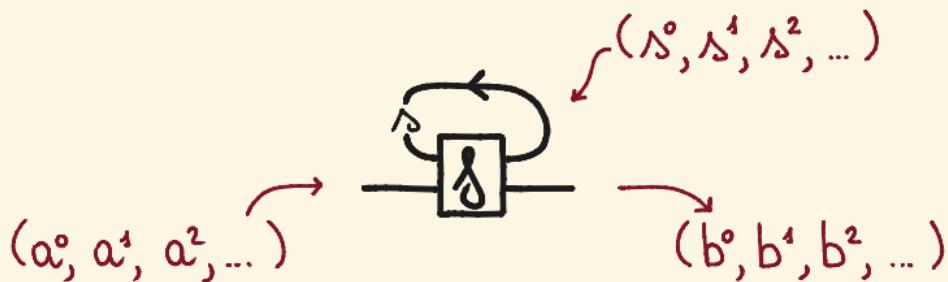
Mealy $\xrightarrow{\text{bisim}}$ form the free uniform feedback structure over $(\mathcal{U}, \mathcal{L}, \ell)$.

OUTLINE

- effectful copy-discard categories
- effectful Mealy machines & bisimulations
- [• coinductive traces
- clausal traces

]

EXECUTING MEALY MACHINES



~ what should the semantic universe be?
when do two Mealy machines have the same executions?

STREAMS ARE CONDUCTIVE

A stream (a^0, a^1, a^2, \dots) of elements of A is

- an element $a^0 \in A$
- a stream a^+ of elements of A

⇒ the set of streams is the final coalgebra of the functor

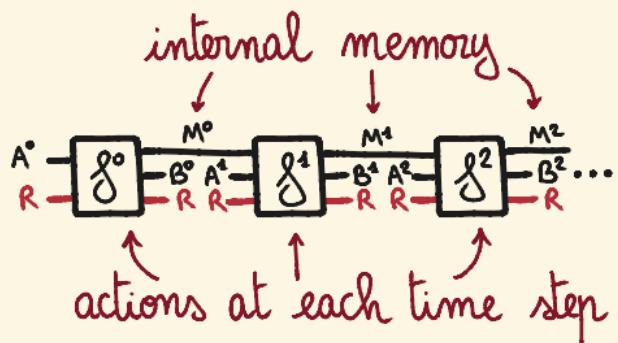
$$A \times (-) : \text{cSet} \rightarrow \text{cSet}$$

STREAM PROCESSES SHOULD BE CONDUCTIVE TOO

Objects are streams of objects of $(\mathcal{V}, \mathcal{L}, \mathcal{C})$

$$A = (A^0, A^1, A^2, \dots), \quad B = (B^0, B^1, B^2, \dots)$$

A stream process (= effectful stream) $f: A \rightarrow B$ is



[cf. EDL, de Felice, Román (2022)]

EFFECTFUL STREAMS

$A = (A^0, A^1, \dots)$, $B = (B^0, B^1, \dots)$ streams of objects of $(\mathcal{U}, \mathcal{L}, \mathcal{C})$.

An effectful stream $F: A \rightarrow B$ on $(\mathcal{U}, \mathcal{L}, \mathcal{C})$ is

- a memory $M_g \in \mathcal{L}$
- a first action $\delta^\circ: A^0 \rightarrow M_g \otimes B^0$ in \mathcal{C}
- the rest of the action $F^+: M_g \cdot A^+ \rightarrow B^+$

$$A \xrightarrow[\text{IR}]{} \boxed{F} \xrightarrow[\text{IR}]{} B = A^0 \xrightarrow[\text{R}]{\delta^\circ} B^0 \xrightarrow[\text{R IR}]{M_g} A^+ \xrightarrow{\quad F^+ \quad} B^+$$

quotiented by sliding: for $\pi: M_g \rightarrow M_g$ in \mathcal{L}

$$\delta^\circ = g^\circ; (\pi \otimes 1) \qquad \qquad \qquad g^+ = \pi \cdot F^+$$

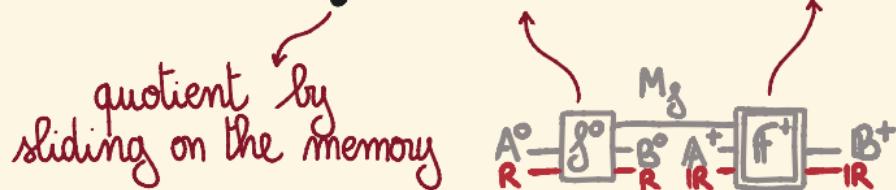
$$-\boxed{\delta^\circ} - \boxed{F^+} - = -\boxed{g^\circ} \boxed{\pi} - \boxed{F^+} - \sim -\boxed{g^\circ} - \boxed{\pi} \boxed{F^+} - = -\boxed{g^\circ} - \boxed{g^+} -$$

EFFECTFUL STREAMS

The profunctor Stream : $\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}$ → Set is the final coalgebra of the functor

$$F : [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \text{Set}] \rightarrow [\mathcal{C}^{\mathbb{N}^{op}} \times \mathcal{C}^{\mathbb{N}}, \text{Set}]$$

$$F(Q)(A, B) := \int^{M \in \mathcal{C}} \mathcal{C}(A^\circ, M \otimes B^\circ) \times Q(M \cdot A^+, B^+)$$



[cf. EDL, de Felice, Román (2022)]

COMPOSITIONAL STRUCTURE OF STREAMS

THEOREM

Effectful streams form an effectful category Stream.

- composition and monoidal actions are defined coinductively:
for $f: N_g \cdot A \rightarrow B$ and $g: N_g \cdot B \rightarrow C$,

$$\begin{cases} (\mathcal{F}_{j_N} g)^\circ := \text{Diagram } 1 \\ (\mathcal{F}_{j_N} g)^+ := \mathcal{F}_{j_M}^+ g^+ \end{cases}$$

$$\left\{ \begin{array}{l} (\mathbb{X} \otimes_N \mathbb{F})^\circ := N_{\mathcal{S}} \\ (\mathbb{X} \otimes_N \mathbb{F})^+ := \mathbb{X}^+ \otimes_M \mathbb{F}^+ \end{array} \right.$$

FEEDBACK ON EFFECTFUL STREAMS

∂ : Stream → Stream

$$\partial(A) := (I, A^\circ, A^+, \dots)$$

THEOREM

Stream has ∂ -feedback.

- feedback is defined coinductively

$$F : (S \cdot \partial S) \otimes A \rightarrow S \otimes B$$

$$Fbk_S F : S \cdot A \rightarrow B$$

$$M(Fbk_S^S F) := M(F) \otimes S^\circ$$

$$(Fbk_S^S F)^\circ := \emptyset^\circ$$

$$(Fbk_S^S F)^+ := Fbk_{S^+}^S (F^+)$$

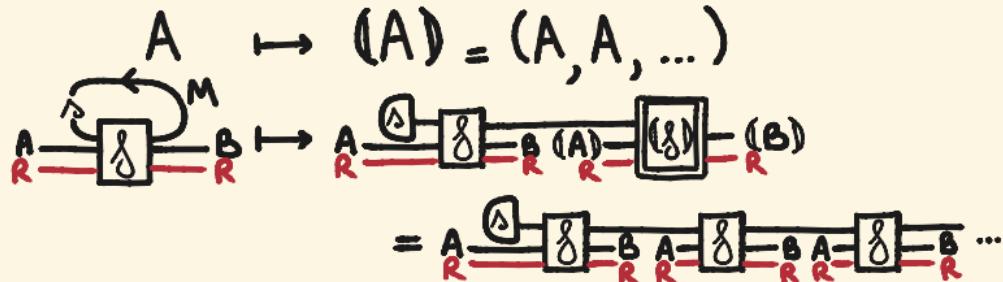


COMPOSITIONAL TRACE SEMANTICS

THEOREM

There is an effectful functor

$$\text{Tr} : \text{Mealy} \rightarrow \text{Stream}$$



TRACE IS UNIVERSAL

THEOREM

The trace functor $\text{Tr} : \text{Mealy} \rightarrow \text{Stream}$
is the unique feedback effectful functor
determined by freeness of Mealy.

FBR \mathbf{EffCat}

$$\text{Mealy}(-) \downarrow \begin{array}{c} \Rightarrow \exists! \text{ Tr} : \text{Mealy}_{(v,z,e)} \longrightarrow \text{Stream}_{(v,z,e)} \\ \text{EffCat} \\ \parallel \\ \text{Mealy}_{(v,z,e)} / \text{bisim} \end{array}$$

COROLLARY

Bisimilarity implies trace equivalence.

OUTLINE

- effectful copy-discard categories
- effectful Mealy machines & bisimulations
- coinductive traces

[• clausal traces]

STREAM COMPUTATIONS

- Sliding equivalence might be difficult to handle
- causal stream functions are old :

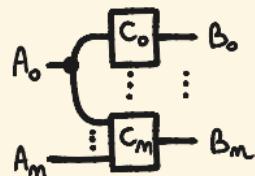
[Raney (1958)] shows that they are
the executions of deterministic Mealy machines

⇒ is there a similar explicit form for effectful streams ?

CAUSAL STREAM FUNCTIONS

Stream computations $(c_m)_{m \in \mathbb{N}} : A \rightarrow B$ in a cartesian category
are families $c_m : A_0 \times \dots \times A_m \rightarrow B_m$.

↪ $c_m(a_0, \dots, a_m) \in B_m$ is the output at time m



reconstructs the outputs until time m

STOCHASTIC PROCESSES

Stochastic stream computations $(p_m)_{m \in \mathbb{N}} : A \rightarrow B$

are families $p_m : A_0 \times \dots \times A_m \longrightarrow \mathcal{D}(B_0 \times \dots \times B_m)$

such that $p_m(a_0, \dots, a_m) = \sum_{a \in A_{m+1}} p_{m+1}(a_0, \dots, a_m, a)$.

⇒ $p_m(a_0, \dots, a_m) \in \mathcal{D}(B_0 \times \dots \times B_m)$ is the distribution of
the outputs until time m
↳ the outputs may be correlated

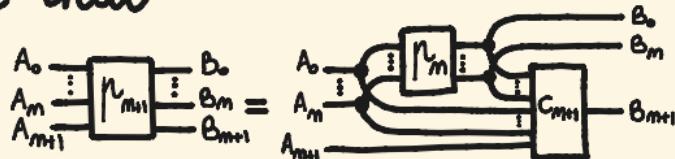
Is there a monoidal version of causal stream functions?

CAUSAL PROCESSES

A causal process $p: A \rightarrow B$ in a copy-discard category \mathcal{C} is a family of morphisms

$$p_m : A_0 \otimes \cdots \otimes A_m \rightarrow B_0 \otimes \cdots \otimes B_m$$

such that



for some $C_{m+1}: B_0 \otimes \cdots \otimes B_m \otimes A_0 \otimes \cdots \otimes A_m \otimes A_{m+1} \rightarrow B_{m+1}$

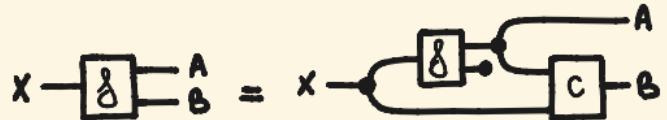
→ p_m determines the input-output behaviour
until time m

COMPOSING CAUSAL PROCESSES

• copy - discard

CONDITIONALS [Elho & Jacobs (2017), Bittz (2020), EDL & Román (2023)]

For all $f: X \rightarrow A \otimes B$ there is $c: A \otimes X \rightarrow B$ st

$$x - \boxed{\delta} \begin{matrix} A \\ B \end{matrix} = x - \begin{array}{c} \delta \\ \bullet \\ c \end{array} \begin{matrix} A \\ B \end{matrix}$$


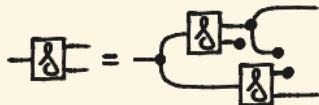
THEOREM

causal processes form a monoidal category Proc when \mathcal{C} has conditionals with sharp domain.

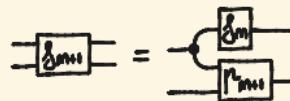
CAUSAL PROCESSES : EXAMPLES

CONDITIONALS

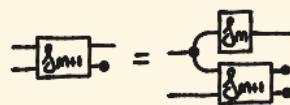
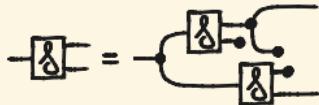
Set



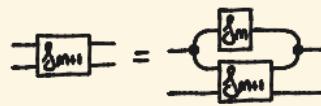
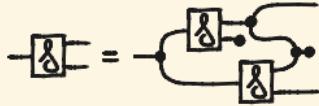
CAUSALITY CONDITION



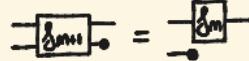
Par



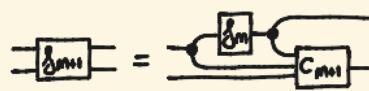
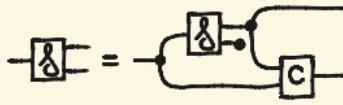
Rel



Stock



nStock

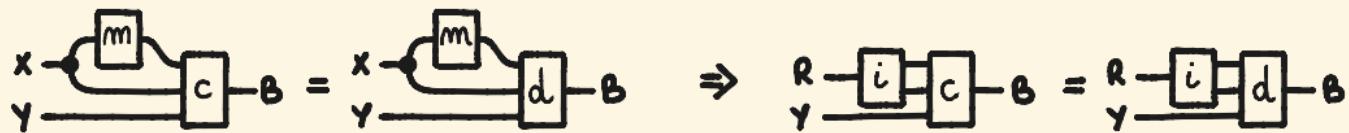
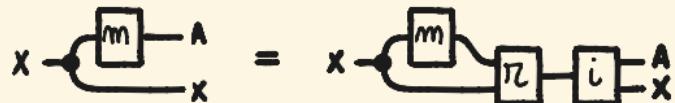


CAUSAL PROCESSES ARE STREAMS

• copy - discard

RANGES

For all $m: X \rightarrow A$ there are $\begin{cases} r: A \otimes X \rightarrow R \\ i: R \rightarrow A \otimes X \end{cases}$ deterministic total



THEOREM

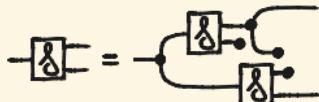
Consider $(\text{funcl}, \text{tot cl}, cl)$.

If cl has conditionals and ranges,
 $\text{Proc} \simeq \text{Stream}$.

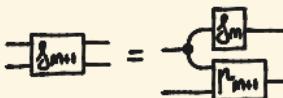
TRACES ARE EFFECTFUL TRACES

CONDITIONALS

Set



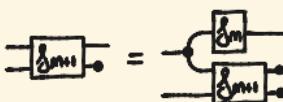
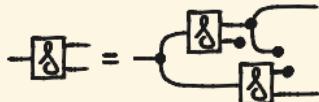
CAUSALITY CONDITION



TRACE PREDICATE

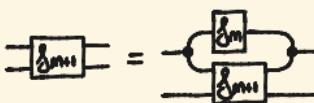
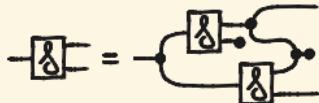
$$\Delta_0 = \Delta \wedge \forall i \leq n \\ (\delta_{\text{init}}, b_i) = f(\delta_i, a_i)$$

Par



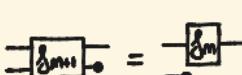
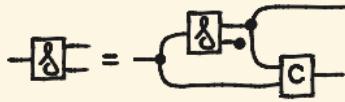
$$\Delta_0 = \Delta \wedge \forall i \leq n \\ (\delta_{\text{init}}, b_i) = f(\delta_i, a_i)$$

Rel



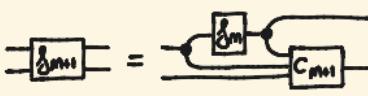
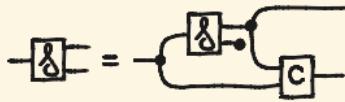
$$\exists \Delta_0, \dots, \Delta_{m+1} \Delta_0 \in \Delta \\ \wedge \forall i \leq n (\delta_{\text{init}}, b_i) \in f(\delta_i, a_i)$$

Stock



$$\sum_{\Delta_0, \dots, \Delta_m} s(\Delta_0) \\ \cdot \prod_{i \leq m} f(\delta_{\text{init}}, b_i | \Delta_i, a_i)$$

nStock



$$\sum_{\Delta_0, \dots, \Delta_m} s(\Delta_0) \\ \cdot \prod_{i \leq m} f(\delta_{\text{init}}, b_i | \Delta_i, a_i)$$

SUMMARY

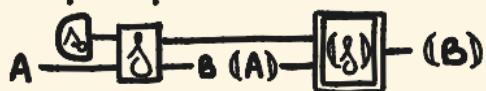
effectful Mealy machines



bisimilarity



effectful streams



[\approx causal processes]

free construction
~ syntax



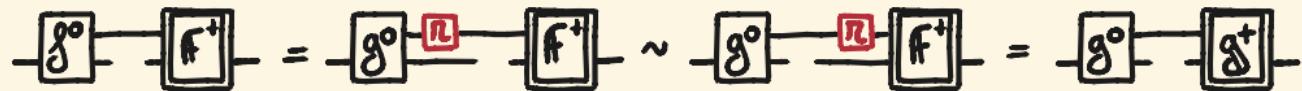
uniformity



coalgebraic construction
~ trace semantics

FUTURE WORK

- Adding choice, iteration and higher-order
- Behavioural metrics, metric enrichment
- cloinduction up-to dinaturality
- Tree-like behaviour via final coalgebra of an appropriate \wp



[Filippo Bonchi, ESD, Mario Román

Effectful Mealy machines: bisimulation and trace (2025) LiCS]