EFFECTFUL MEALY MACHINES:

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EFFECTFUL TRIPLES.

$$\mathcal{V} \longrightarrow \mathcal{L} \longrightarrow \mathcal{C}$$

cartesian → monoidal → premonoidal

values → local computations → effectful computations

BISIMULATION AND TRACE

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Ex. $\mathbf{Set} \to \mathbf{TRel} \to \mathbf{Rel}$

$\mathbf{Set} \to \mathbf{Stoch} \to \mathbf{Substoch}$

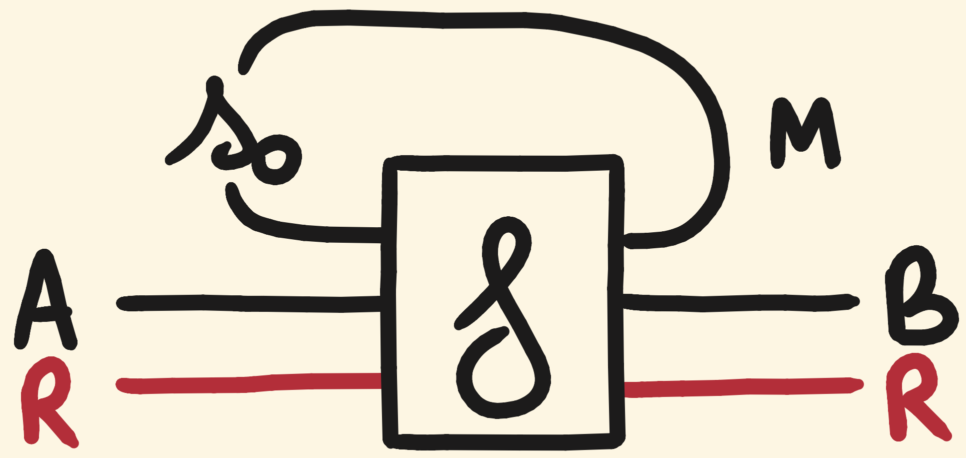
$\mathbf{Set} \to \mathbf{Stoch} \to \mathbf{State}(\mathbf{Stoch})$

$\mathbf{Set} \to \mathcal{Kl}(\mathcal{Z}(T)) \to \mathcal{Kl}T$

$\mathrm{cart}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$

MEALY MACHINES.

- $\delta : M \otimes A \to M \otimes B$
  ⇝ transition morphism
- $s_0 : I \to M$
  ⇝ initial state



THM. Mealy machines form a feedback effectful triple $\mathrm{Mealy}(\mathcal{V}, \mathcal{L}, \mathcal{C})$.

EX. coalgebraic Mealy machines :

$$M \to T(M \otimes B)^A \quad \leftrightsquigarrow \quad M \otimes A \to M \otimes B$$

BISIMULATION. $(f,s) \equiv (g,t)$ if there is

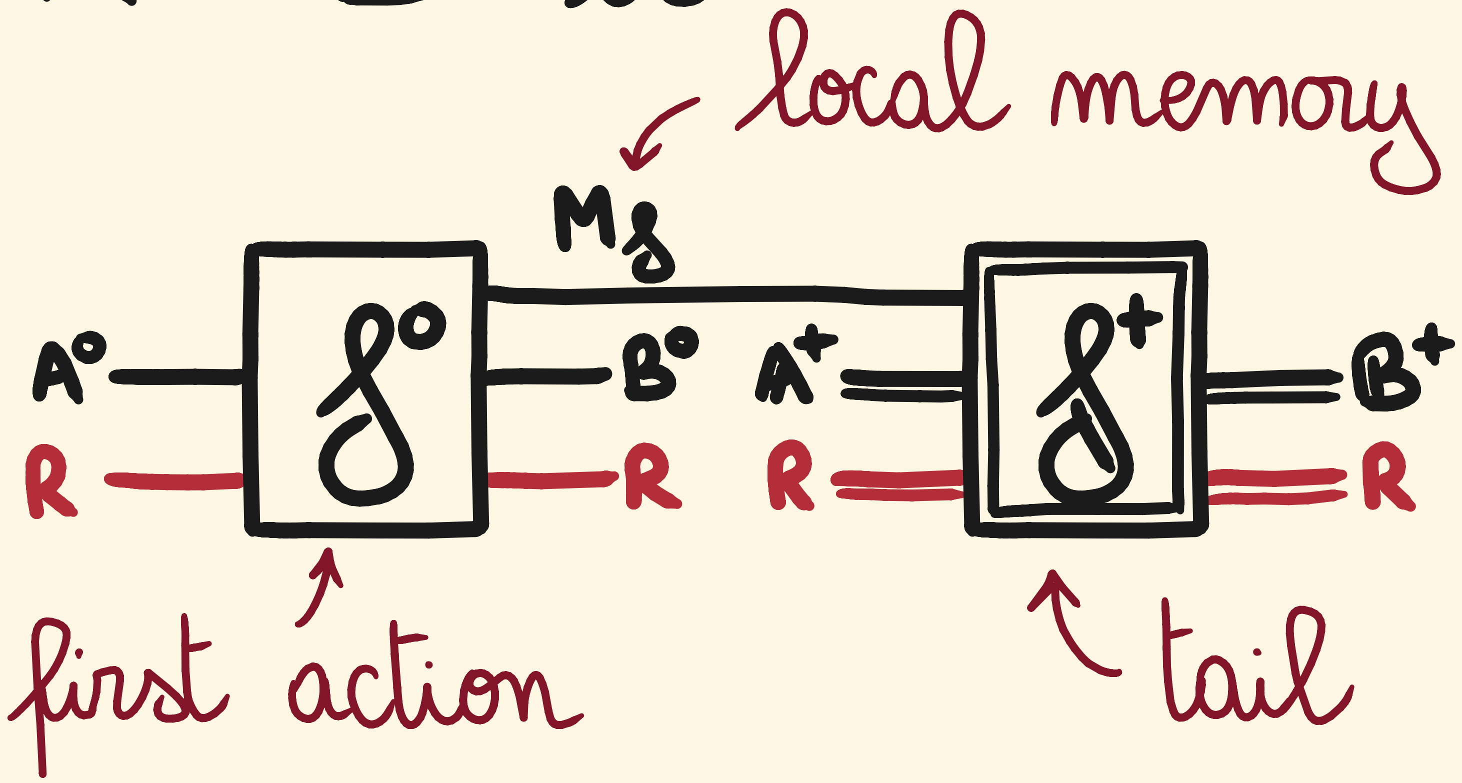
$$
\begin{tikzcd}[column sep=small]
(f,s) \arrow[dr, Rightarrow, "u_1"] & & (q_1,y_1) \arrow[dl, Rightarrow, "v_1"'] \arrow[dr, Rightarrow, "u_2"] & \dots & & (g,t) \arrow[dl, Rightarrow, "v_n"'] \\
& (p_1,x_1) & & & (p_n,x_n) &
\end{tikzcd}
$$

where morphisms $(f,s) \overset{u}{\Rightarrow} (h,r)$ are

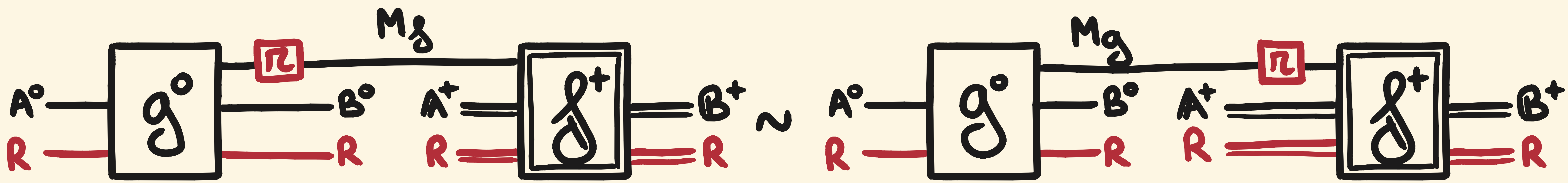
PROP. For coalgebraic Mealy machines, effectful bisimilarity is coalgebraic bisimilarity (for T preserving weak pullbacks).

EFFECTFUL STREAMS. $f : A \to B$ is

- $M_f \in \mathrm{obj}\,\mathcal{C}$
- $f^\circ : A^\circ \to M_f \otimes B^\circ$
- $f^+ : M_f \cdot A^+ \to B^+$



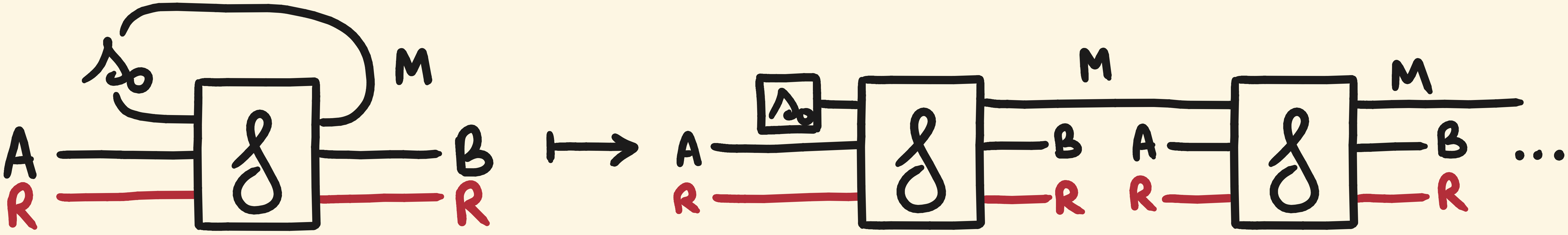
quotiented by sliding local computations:



THM. Effectful streams form a feedback effectful triple $\mathrm{Stream}(\mathcal{V}, \mathcal{L}, \mathcal{C})$.

EFFECTFUL TRACES. Execution of Mealy machines

$\mathsf{Tr} : \mathsf{Mealy}(\mathcal{V}, \mathcal{L}, \mathcal{C}) \longrightarrow \mathsf{Stream}(\mathcal{V}, \mathcal{L}, \mathcal{C})$



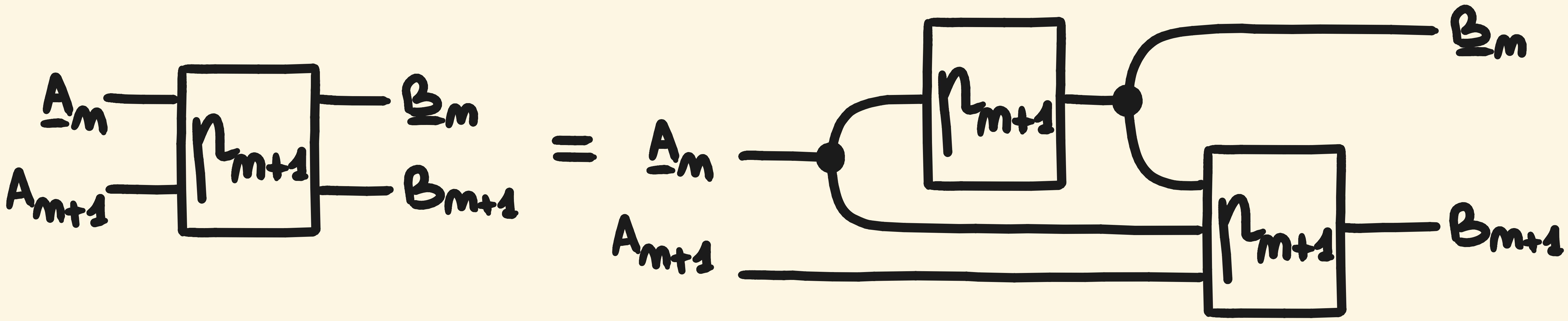
THM. Trace is universal: $\mathsf{Eff} \overset{\mathsf{Mealy}}{\underset{\perp}{\rightleftarrows}} \mathsf{FbkEff} \overset{\mathsf{Unif}}{\underset{\perp}{\rightleftarrows}} \mathsf{UFbkEff}$

$$\Rightarrow \begin{tikzcd}
\mathsf{Mealy}(\mathcal{V}, \mathcal{L}, \mathcal{C}) \arrow[rr, dashed, "\mathsf{Tr}"] \arrow[dr, dashed, "\mathsf{Bisim}"'] & & \mathsf{Stream}(\mathcal{V}, \mathcal{L}, \mathcal{C}) \\
& \mathsf{Mealy}_{\equiv}(\mathcal{V}, \mathcal{L}, \mathcal{C}) \arrow[ur, dashed, "\mathsf{Tr}"'] \arrow[u, phantom, "="] &
\end{tikzcd}$$

COR. Bisimilarity implies trace equivalence.

CAUSAL PROCESSES. $p : \mathbb{A} \to \mathbb{B}$ is

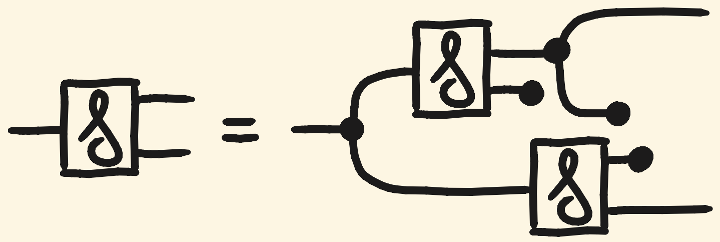
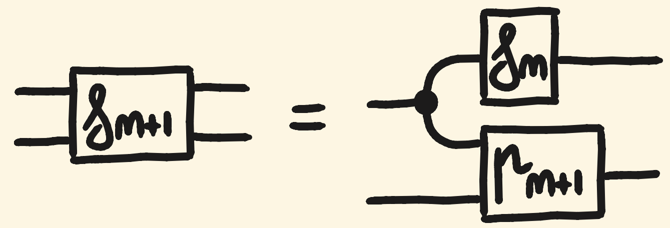
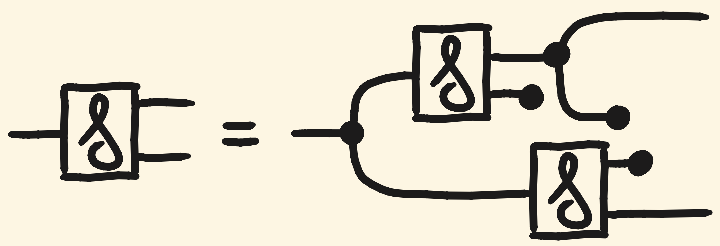
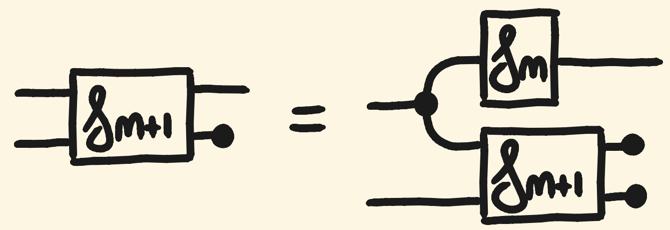
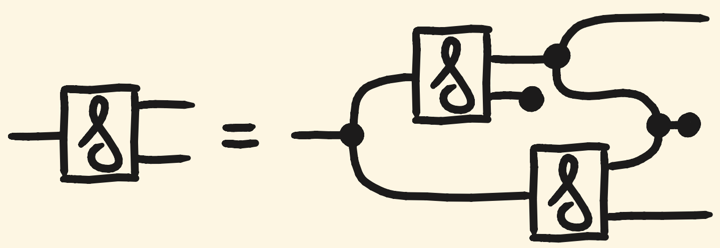
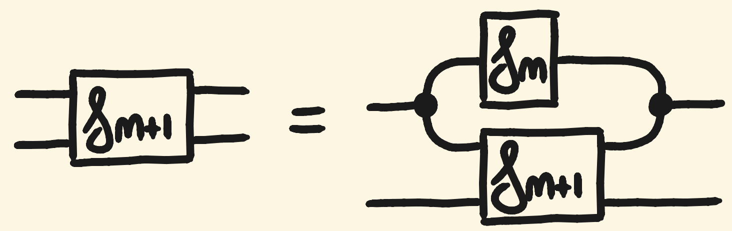
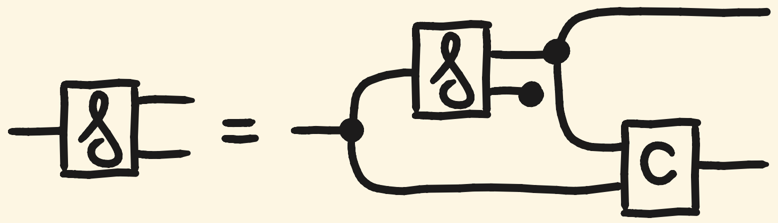
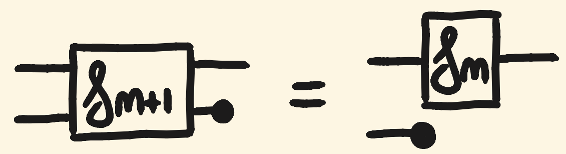
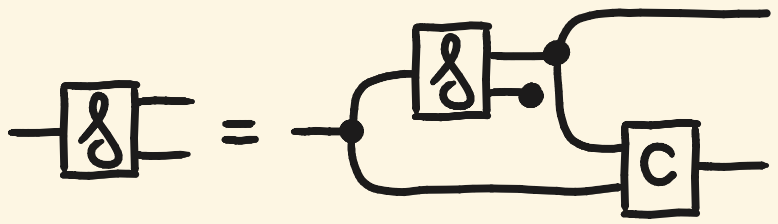
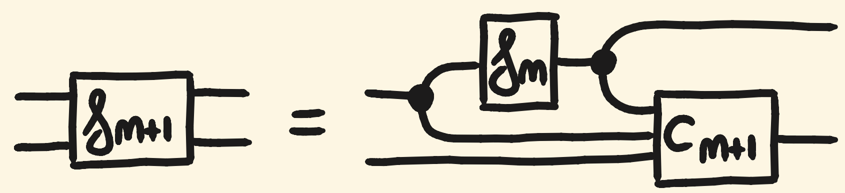
$$\{p_n : A_0 \otimes \cdots \otimes A_n \to B_0 \otimes \cdots \otimes B_n\}$$ such that



THM. $\mathscr{C}$ copy-discard with conditionals
$\Rightarrow \mathsf{Proc}(\mathscr{C})$ copy-discard.

THM. $\mathscr{C}$ copy-discard with conditionals and ranges
$\Rightarrow \mathsf{Proc}(\mathscr{C}) \cong \mathsf{Stream}(\mathsf{cart}(\mathscr{C}), \mathsf{tot}(\mathscr{C}), \mathscr{C})$.

TRACES ARE EFFECTFUL TRACES.

	CONDITIONALS	CAUSALITY CONDITION	TRACE PREDICATE
Set			$s_0 = s \wedge \forall i \leq n \ (s_{i+1}, b_i) = f(s_i, a_i)$
Par			$s_0 = s \wedge \forall i \leq n \ (s_{i+1}, b_i) = f(s_i, a_i)$
Rel			$\exists s_0, \dots, s_{n+1} \ s_0 \in s \wedge \forall i \leq n \ (s_{i+1}, b_i) \in f(s_i, a_i)$
Stoch			$\sum_{s_0, \dots, s_{n+1}} s(s_0) \cdot \prod_{i \leq n} f(s_{i+1}, b_i \mid s_i, a_i)$
Substoch			$\sum_{s_0, \dots, s_{n+1}} s(s_0) \cdot \prod_{i \leq n} f(s_{i+1}, b_i \mid s_i, a_i)$