ISOMORPHISM OF CATEGORIES

$\mathcal{C} \overset{\text{iso}}{\cong} \mathcal{D}$ iff $\begin{cases} \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd} \\ F ; G = 1_{\mathcal{C}} \ , \ G ; F = 1_{\mathcal{D}} \end{cases}$

⇝ F, G are not identities but isomorphisms

<u>ex</u> The functor category $[2, \mathcal{C}]$ is isomorphic to the product category $\mathcal{C} \times \mathcal{C}$.

⇝ There are two ways of relaxing this condition : equivalence, adjunction

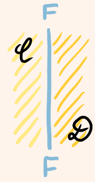
DIAGRAMS FOR CATEGORIES, FUNCTORS AND NATURAL TRANSFORMATIONS

categories

$\mathcal{C}$

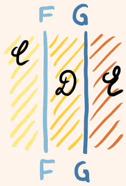
functors

$F : \mathcal{C} \longrightarrow \mathcal{D}$



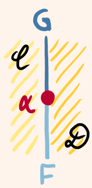
composition of functors

$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$



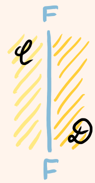
natural transformations

$\alpha : F \longrightarrow G$



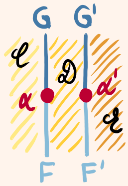
identity natural transformation

$\mathbb{1}_F : F \longrightarrow F$



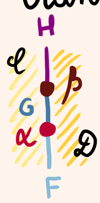
horizontal composition of natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & \mathcal{E} \end{tikzcd}$$



vertical composition of natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{D} \end{tikzcd}$$



OBJECTS AND MORPHISMS IN DIAGRAMS

c object of $\mathcal{C}$ can be represented as a functor

$$\begin{aligned} !_c : \mathbb{1} &\longrightarrow \mathcal{C} \\ * &\longmapsto c \\ 1_* &\longmapsto 1_c \end{aligned}$$

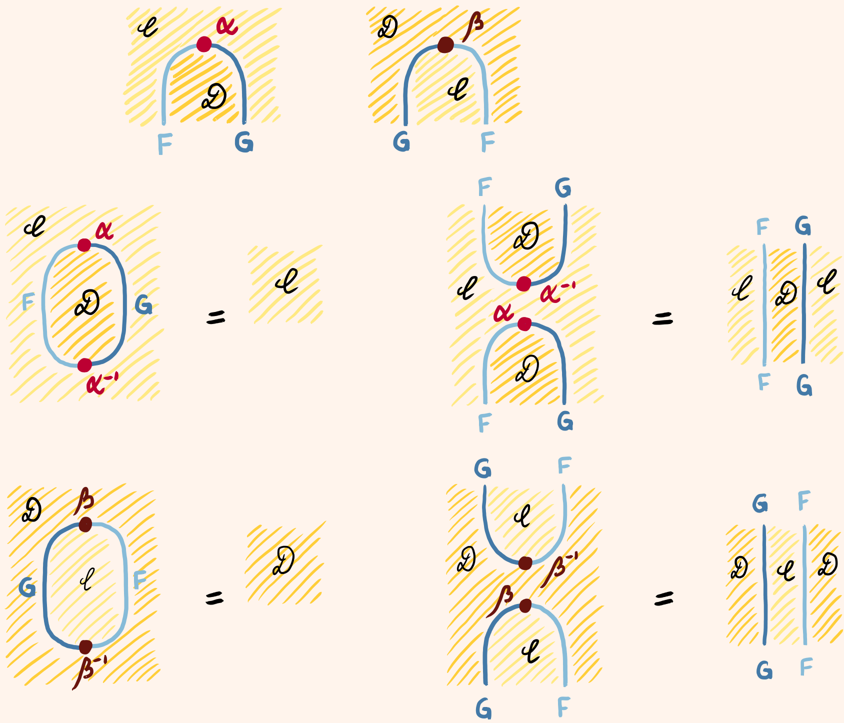
$f : c \longrightarrow c'$ in $\mathcal{C}$ can be represented as a natural transformation $!_f : !_c \longrightarrow !_{c'}$ whose only component is f

EQUIVALENCE OF CATEGORIES

$\mathcal{C} \overset{EQ}{\simeq} \mathcal{D}$ iff $\begin{cases} \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \\ F ; G \xrightarrow[\simeq]{\alpha} 1\!\!1_{\mathcal{C}} \;,\; G ; F \xrightarrow[\simeq]{\beta} 1\!\!1_{\mathcal{D}} \end{cases}$

natural isomophisms

With diagrams :



<u>ex</u> The category of finite ordinals FOrd is equivalent to the category finite sets and functions FinSet.

ADJUNCTIONS

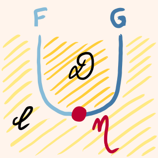
There are several equivalent definitions.

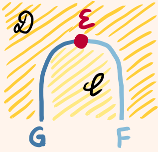
① ADJUNCTION BETWEEN CATEGORIES (♥)

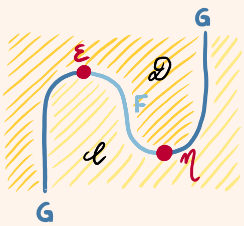
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd} \quad \text{iff} \quad \begin{cases} \eta : 1_{\mathcal{C}} \to F;G & \rightsquigarrow \text{unit} \\ \varepsilon : G;F \to 1_{\mathcal{D}} & \rightsquigarrow \text{counit} \\ \eta_G ; G\varepsilon = 1_G \\ F\eta ; \varepsilon_F = 1_F \end{cases}$$

$\rightsquigarrow$ also written $\underbrace{F}_{\text{left adjoint}} \dashv \overbrace{G}^{\text{right adjoint}}$

With diagrams:







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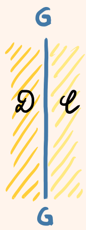
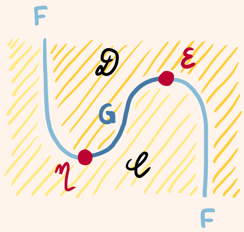


$\eta_G ; G\varepsilon = 1_G$

$$\begin{tikzcd} G(D) \arrow[r, "\eta_{GD}"] \arrow[dr, "1_{G(D)}"'] & GFG(D) \arrow[d, "G\varepsilon_D"] \\ & G(D) \end{tikzcd}$$



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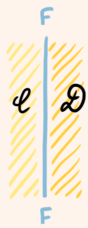


$F\eta ; \varepsilon_F = 1_F$

$$\begin{tikzcd} F(C) \arrow[r, "F\eta_C"] \arrow[dr, "1_{F(C)}"'] & FGF(C) \arrow[d, "\varepsilon_{F(C)}"] \\ & F(C) \end{tikzcd}$$

⚠ equivalence $\not\Rightarrow$ adjunction

In fact, equivalences that are also adjunctions are called adjoint equivalences.

ex The list functor $\mathrm{List} : \mathbf{Set} \to \mathbf{Mon}$
has a right adjoint $U : \mathbf{Mon} \to \mathbf{Set}$.

① Define $\varepsilon : U ; \mathrm{List} \to \mathbb{1}_{\mathbf{Mon}}$ and $\eta : \mathbb{1}_{\mathbf{Set}} \to \mathrm{List} ; U$.

② Check that ε, η are well defined, i.e. $\varepsilon_{(A,*,e)}$ is a monoid homomorphism and η_A is a function.

③ Check that ε, η are natural transformations.

④ Check that ε, η satisfy the zig-zag equations.

① $\mathbf{Set}$ ⇝ category of sets and functions

$\mathbf{Mon}$ ⇝ category of monoids and monoid homomorphisms

$$\begin{aligned} \mathrm{List} : \mathbf{Set} &\longrightarrow \mathbf{Mon} \\ A &\longmapsto (A^*, ::, ()) \end{aligned}$$

$::$ → concatenation; $()$ → empty list

⇝ $A^* := \{ (a_1, \dots, a_n) \mid a_i \in A,\ n \in \mathbb{N} \}$

$$\begin{aligned} f : A \to B &\longmapsto f^* : A^* \longrightarrow B^* \\ &\qquad (a_1, \dots, a_n) \longmapsto (f(a_1), \dots, f(a_n)) \end{aligned}$$

$$\begin{aligned} U : \mathbf{Mon} &\longrightarrow \mathbf{Set} \\ (A, *, e) &\longmapsto A \\ f &\longmapsto f \end{aligned}$$

⇝ 'forgets' the monoid structure

$$\begin{aligned} \varepsilon : U ; \mathrm{List} &\longrightarrow \mathbb{1}_{\mathbf{Mon}} \\ \varepsilon_{(A,*,e)} : \mathrm{List}(U(A,*,e)) \overset{(A^*, ::, ())}{=} &\longrightarrow (A, *, e) \\ (a_1, \dots, a_n) &\longmapsto a_1 * \dots * a_n \end{aligned}$$

$$\begin{aligned} \eta : \mathbb{1}_{\mathbf{Set}} &\longrightarrow \mathrm{List} ; U \\ \eta_A : A &\longrightarrow U(\mathrm{List}(A)) \overset{A^*}{=} \\ a &\longmapsto a \end{aligned}$$

② $\varepsilon_{(A,*,e)}$ is a monoid homomorphism because

$$\begin{cases} \varepsilon_{(A,*,e)}((a_1, \dots, a_n) :: (b_1, \dots, b_m)) = (a_1 * \dots * a_n) * (b_1 * \dots * b_m) \\ \varepsilon_{(A,*,e)}(()) = e \end{cases}$$

η_A is a function because $A \subseteq A^*$.

③ ε is a natural transformation

$$\begin{tikzcd} (a_1, \dots, a_n) \arrow[r, mapsto] & (f(a_1), \dots, f(a_n)) \\ \mathrm{List}(U(A,*,e)) \arrow[r, "\mathrm{List}(U(f))"] \arrow[d, "\varepsilon_{(A,*,e)}"'] & \mathrm{List}(U(B,\star,u)) \arrow[d, "\varepsilon_{(B,\star,u)}"] \\ (A,*,e) \arrow[r, "f"'] & (B,\star,u) \\ a_1 * \dots * a_n \arrow[r, mapsto] & f(a_1 * \dots * a_n) \end{tikzcd}$$

$\mathrm{List}(U(A,*,e)) = (A^*, ::, ())$, $\mathrm{List}(U(B,\star,u)) = (B^*, ::, ())$, $f(a_1 * \dots * a_n) = f(a_1) \star \dots \star f(a_n)$

η is a natural transformation

$$\begin{tikzcd} a \arrow[r, mapsto] & f(a) \\ A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ U(\mathrm{List}(A)) \arrow[r, "U(\mathrm{List}(f))"'] & U(\mathrm{List}(B)) \\ a \arrow[r, mapsto] & f(a) \end{tikzcd}$$

$U(\mathrm{List}(A)) = A^*$, $U(\mathrm{List}(B)) = B^*$

④ we need to show

$$\begin{tikzcd} b \arrow[r, mapsto] & b \\ U(B,\star,u) \arrow[r, "\eta_B"] \arrow[dr, "\mathbb{1}_B"'] & U(\mathrm{List}(U(B,\star,u))) \arrow[d, "U\varepsilon_{(B,\star,u)}"] \\ & U(B,\star,u) \\ & b \end{tikzcd}$$

$U(B,\star,u) = B$, $U(\mathrm{List}(U(B,\star,u))) = B^*$

$$\begin{tikzcd} (a_1, \dots, a_n) \arrow[r, mapsto] & ((a_1, \dots, a_n)) \\ \mathrm{List}(A) \arrow[r, "\eta^*_A"] \arrow[dr, "\mathbb{1}_{(A^*, ::, ())}"'] & \mathrm{List}(U(\mathrm{List}(A))) \arrow[d, "\varepsilon_{\mathrm{List}(A)}"] \\ & \mathrm{List}(A) \\ & (a_1, \dots, a_n) \end{tikzcd}$$

$\mathrm{List}(A) = (A^*, ::, ())$, $\mathrm{List}(U(\mathrm{List}(A))) = (A^{**}, ::, ())$

□

⇝ there are many examples like this one, where we can see the right adjoint as a 'forgetful' functor and the left adjoint as a 'free' functor.

ex The cartesian product is left adjoint to the exponential in Set.

① Define $\varepsilon : (-)^B ; -\times B \to 1_{Set}$ and $\eta : 1_{Set} \to -\times B ; (-)^B$.

② Check that ε, η are well defined, i.e. ε_A, η_A are functions.

③ Check that ε, η are natural transformations.

④ Check that ε, η satisfy the zig-zag equations.

① $$\begin{aligned} -\times B : Set &\longrightarrow Set \\ A &\longmapsto A\times B \\ f &\longmapsto f\times 1_B \end{aligned}$$

$$\begin{aligned} (-)^B : Set &\longrightarrow Set \\ A &\longmapsto A^B := \{ g : B \to A \} \\ f : A \to C &\longmapsto f^B : A^B \longrightarrow C^B \\ & \qquad\quad g \longmapsto g;f \end{aligned}$$

$$\varepsilon : (-)^B ; -\times B \longrightarrow 1_{Set}$$

$$\begin{aligned} \varepsilon_A : A^B \times B &\longrightarrow A \\ (g, b) &\longmapsto g(b) \end{aligned} \qquad \rightsquigarrow \; g(b) = ev(g, b)$$

$$\eta : 1_{Set} \longrightarrow -\times B ; (-)^B$$

$$\begin{aligned} \eta_A : A &\longrightarrow (A\times B)^B \\ a &\longmapsto f_a : B \to A\times B \\ & \qquad\quad b \mapsto (a, b) \end{aligned} \qquad \rightsquigarrow \; f_a = \langle !_a, 1_B \rangle$$

② ε_A, η_A are functions.

③ ε, η are natural

$$\begin{tikzcd}
A^B\times B \arrow[r, "f^B\times B"] \arrow[d, "\varepsilon_A"'] \arrow[dr, phantom, "\circlearrowleft"] & C^B\times B \arrow[d, "\varepsilon_C"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
(g, b) \arrow[r, mapsto] \arrow[d, mapsto] & (g;f, b) \arrow[d, mapsto] \\
g(b) \arrow[r, mapsto] & f(g(b)) = (g;f)(b)
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "\eta_C"] \\
(A\times B)^B \arrow[r, "(g\times B)^B"'] & (C\times B)^B
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d, mapsto] & g(a) \arrow[d, mapsto] \\
f_a \arrow[r, mapsto] & (g\times B); f_a = f_{g(a)}
\end{tikzcd}$$

$$\begin{aligned} (g\times B); f_a = f_{g(a)} : B &\longrightarrow C\times B \\ b &\longmapsto (g(a), b) \end{aligned}$$

④ zig-zag equations

$$\begin{tikzcd}
A^B \arrow[r, "\eta_{A^B}"] \arrow[dr, "1_{A^B}"'] & (A^B\times B)^B \arrow[d, "(\varepsilon_A)^B"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A^B
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, mapsto] \arrow[d, mapsto] & f_g : B \to A^B\times B,\; b \mapsto (g, b) \arrow[d, mapsto] \\
g = f_g ; \varepsilon_A &
\end{tikzcd}$$

$$\begin{aligned} g = f_g ; \varepsilon_A : B &\longrightarrow A^B\times B \longrightarrow A \\ b &\longmapsto (g, b) \longmapsto g(b) \end{aligned}$$

$$\begin{aligned} \rightsquigarrow (\varepsilon_A)^B : (A^B\times B)^B &\longrightarrow A^B \\ g &\longmapsto g ; \varepsilon_A \end{aligned}$$

$$\begin{tikzcd}
A\times B \arrow[r, "\eta_A\times B"] \arrow[dr, "1_{A\times B}"'] & (A\times B)^B\times B \arrow[d, "\varepsilon_{A\times B}"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A\times B
\end{tikzcd}
\qquad
\begin{tikzcd}
(a, b) \arrow[r, mapsto] \arrow[d, mapsto] & (f_a, b) \arrow[d, mapsto] \\
(a, b) = f_a(b) &
\end{tikzcd}$$

$$\begin{aligned} \rightsquigarrow \eta_A\times B : A\times B &\longrightarrow (A\times B)^B\times B \\ (a, b) &\longmapsto (f_a, b) \end{aligned}$$

$$f_a = \langle !_a, 1_B \rangle$$

$\square$

CARTESIAN CLOSED CATEGORY

$\mathcal{C}$ cartesian category such that for every object B of $\mathcal{C}$ the product functor

$$\begin{aligned} - \times B : \mathcal{C} &\longrightarrow \mathcal{C} \\ A &\longmapsto A \times B \\ f &\longmapsto f \times 1_B \end{aligned}$$

has a right adjoint.

⇝ the right adjoint is often denoted by

$(-)^B : \mathcal{C} \longrightarrow \mathcal{C}$ or $[B, -] : \mathcal{C} \longrightarrow \mathcal{C}$

and called exponential or internal-hom.

② ADJUNCTION BETWEEN CATEGORIES (classical)

$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ ($F \dashv G$) iff there is a bijection

$$\mathcal{C}(C, G(\Delta)) \overset{\alpha}{\simeq} \mathcal{D}(F(C), \Delta)$$

that is natural in $C \in \mathcal{C}_0$ and $\Delta \in \mathcal{D}_0$

i.e. the two functors

$$\begin{tikzcd}
 & \mathcal{C}^{op} \times \mathcal{C} \arrow[dr, "\mathcal{H}om_{\mathcal{C}}"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\
\mathcal{C}^{op} \times \mathcal{D} \arrow[ur, "\mathbb{1} \times G"] \arrow[dr, "F^{op} \times \mathbb{1}"'] & & \mathcal{S}et \\
 & \mathcal{D}^{op} \times \mathcal{D} \arrow[ur, "\mathcal{H}om_{\mathcal{D}}"'] &
\end{tikzcd}$$

are isomorphic.

Explicitly, naturality of α is

$\forall\, h : C' \to C$ in $\mathcal{C}$
$\forall\, k : \Delta \to \Delta'$ in $\mathcal{D}$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & h; f; G(k) \arrow[d, mapsto] \\
\mathcal{C}(C, G(\Delta)) \arrow[r, "h;(-);G(k)"] \arrow[d, "\alpha_{C,\Delta}"'] \arrow[dr, phantom, "//"] & \mathcal{C}(C', G(\Delta')) \arrow[d, "\alpha_{C',\Delta'}"] \\
\mathcal{D}(F(C), \Delta) \arrow[r, "F(h);(-);k"'] & \mathcal{D}(F(C'), \Delta') \\
\alpha_{C,\Delta}(f) \arrow[r, mapsto] & F(h); \alpha_{C,\Delta}(f); k
\end{tikzcd}$$

$$\alpha_{C',\Delta'}(h; f; G(k)) \;/\!/\; F(h); \alpha_{C,\Delta}(f); k$$

So, for every $f : C \to G(\Delta)$ in $\mathcal{C}$, $h : C' \to C$ in $\mathcal{C}$ and $k : \Delta \to \Delta'$ in $\mathcal{D}$, we have

$$F(h); \alpha_{C,\Delta}(f); k = \alpha_{C',\Delta'}(h; f; G(k)) \qquad \text{(NAT)}$$

Recall that the hom functor $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}et$ (HOM) is defined as:

for $C, \Delta \in \mathcal{C}_0$, $\mathcal{C}(C, \Delta) := \{ f : C \to \Delta \text{ in } \mathcal{C} \}$

for $h : C' \to C$, $k : \Delta \to \Delta'$ and $f : C \to \Delta$ in $\mathcal{C}$,

$\mathcal{C}(h, k)(f) := h; f; k \quad : C' \to \Delta'$

So, the composition $(\mathbb{1}_{\mathcal{C}^{op}} \times G); \mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}et$ is

for $C \in \mathcal{C}_0$ and $\Delta \in \mathcal{D}_0$, $\mathcal{C}(C, G(\Delta)) := \{ f : C \to G(\Delta) \text{ in } \mathcal{C} \}$

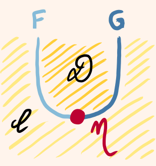
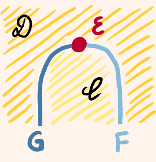
for $h : C' \to C$ and $f : C \to \Delta$ in $\mathcal{C}$, and $k : \Delta \to \Delta'$ in $\mathcal{D}$,

$\mathcal{C}(h, G(k))(f) := h; f; G(k) \quad : C' \to G(\Delta')$

PROPOSITION

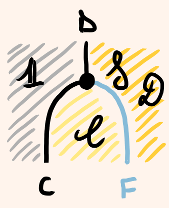
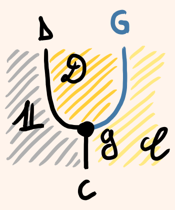
Definitions 1 and 2 of adjunction are equivalent.

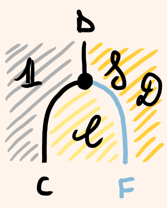
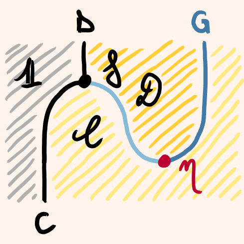
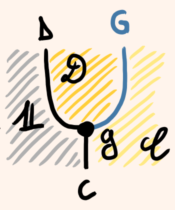
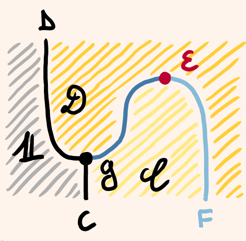
PROOF

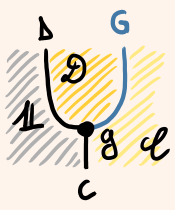
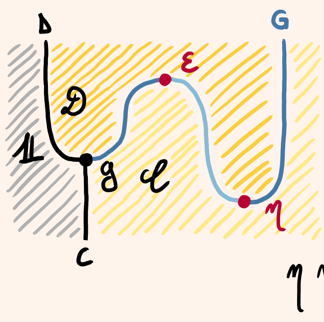
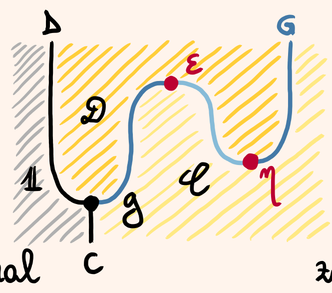
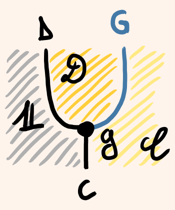
$\underline{1 \Rightarrow 2}$ Let $\mathcal{C} \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{D}$ ($\perp$) with  

satisfying the zig-zag equations.

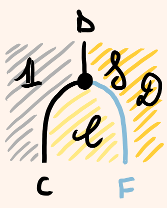
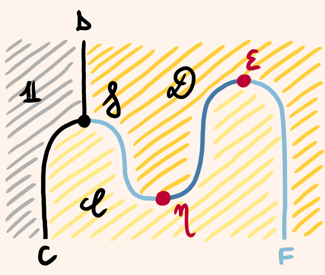
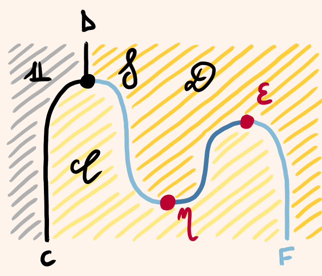
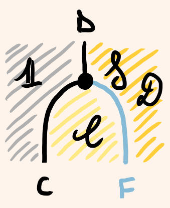
We need to find an isomorphism between the sets

$\{$  $\}$ $\overset{\alpha_{c,D}}{\simeq}$ $\{$  $\}$

We can define $\alpha_{c,D}($  $) :=$  and $\alpha^{-1}_{c,D}($  $) :=$  .

$\Rightarrow$ $\alpha_{c,D}(\alpha^{-1}_{c,D}($  $)) =$  $\overset{\uparrow}{=}$  $\overset{\uparrow}{=}$ 

η natural   zig-zag eq.

$\alpha^{-1}_{c,D}(\alpha_{c,D}($  $)) =$  $\overset{\uparrow}{=}$  $\overset{\uparrow}{=}$ 

ε natural   zig-zag eq.

We need to show that $\alpha : \mathcal{C}(-, G(-)) \longrightarrow \mathcal{D}(F(-), -)$

and $\alpha^{-1} : \mathcal{D}(F(-), -) \longrightarrow \mathcal{C}(-, G(-))$ are natural

$$\begin{tikzcd}
\mathcal{C}(C, G(D)) \arrow[r, "{\mathcal{C}(c, G(d))}"] \arrow[d, "\alpha_{C,D}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C}(C', G(D')) \arrow[d, "\alpha_{C',D'}"] \\
\mathcal{D}(F(C), D) \arrow[r, "{\mathcal{D}(F(c), d)}"'] & \mathcal{D}(F(C'), D')
\end{tikzcd}$$

$g \longmapsto c;g;G(d)$

$Fg;\varepsilon_D \longmapsto Fc;Fg;\varepsilon_D;d \overset{\uparrow}{=} F(c;g;Gd);\varepsilon_{D'}$

{F functor, ε natural

$$\begin{tikzcd}
\mathcal{D}(F(C), D) \arrow[r, "{\mathcal{D}(F(c), d)}"] \arrow[d, "\alpha^{-1}_{C,D}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D}(F(C'), D') \arrow[d, "\alpha^{-1}_{C',D'}"] \\
\mathcal{C}(C, G(D)) \arrow[r, "{\mathcal{C}(c, G(d))}"'] & \mathcal{C}(C', G(D'))
\end{tikzcd}$$

$f \longmapsto F(c);f;d$

$\eta_c;Gf \longmapsto c;\eta_c;Gf;Gd \overset{\uparrow}{=} \eta_{c'};G(Fc;f;d)$

{G functor, η natural

$\underline{2 \Rightarrow 1}$ Let $\mathcal{C} \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{D}$ ($\perp$) with $\alpha : \mathcal{C}(-, G(-)) \longrightarrow \mathcal{D}(F(-), -)$ natural isomorphism.

We need to define $\varepsilon : G;F \longrightarrow \mathbb{1}_{\mathcal{D}}$, $\eta : \mathbb{1}_{\mathcal{C}} \longrightarrow F;G$.

Our candidates are $\begin{cases} \varepsilon_D := \alpha_{GD,D}(\mathbb{1}_{GD}) & : FGD \longrightarrow D \\ \eta_C := \alpha^{-1}_{C,FC}(\mathbb{1}_{FC}) & : C \longrightarrow GFC \end{cases}$

We check naturality.

$$\begin{tikzcd}
FGD \arrow[r, "FGd"] \arrow[d, "\varepsilon_D"'] & FGD' \arrow[d, "\varepsilon_{D'}"] \\
D \arrow[r, "d"'] & D'
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "\eta_c"'] & C' \arrow[d, "\eta_{c'}"] \\
GFC \arrow[r, "GFc"'] & GFC'
\end{tikzcd}$$

$\varepsilon_D;d$	$\eta_c;GFc$	
$= \alpha_{GD,D}(\mathbb{1}_{GD});d$	$= \alpha^{-1}_{C,FC}(\mathbb{1}_{FC});GFc$	by def of ε, η
$= F\mathbb{1}_{GD};\alpha_{GD,D}(\mathbb{1}_{GD});d$	$= \mathbb{1}_c;\alpha^{-1}_{C,FC}(\mathbb{1}_{FC});GFc$	by unitality, functoriality of F
$= \alpha_{GD,D'}(\mathbb{1}_{GD};\mathbb{1}_{GD};Gd)$	$= \alpha^{-1}_{C,FC'}(F\mathbb{1}_c;\mathbb{1}_{FC};Fc)$	by NAT
$= \alpha_{GD,D'}(Gd;\mathbb{1}_{GD'};\mathbb{1}_{GD'})$	$= \alpha^{-1}_{C,FC'}(Fc;\mathbb{1}_{FC'};\mathbb{1}_{FC'})$	by unitality, functoriality of F
$= FGd;\alpha_{GD',D'}(\mathbb{1}_{GD'});\mathbb{1}_{D'}$	$= c;\alpha^{-1}_{C',FC'}(\mathbb{1}_{FC'});G\mathbb{1}_{FC'}$	by NAT
$= FGd;\alpha_{GD',D'}(\mathbb{1}_{GD'})$	$= c;\alpha^{-1}_{C',FC'}(\mathbb{1}_{FC'})$	by unitality, functoriality of G
$= FGd;\varepsilon_{D'}$	$= c;\eta_{c'}$	by def of ε, η

$\Rightarrow$ ε, η are natural

We check the zig-zag equations.

$\eta_{GD};G\varepsilon_D$	$F\eta_c;\varepsilon_{FC}$	
$:= \alpha^{-1}_{GD,FGD}(\mathbb{1}_{FGD});G\alpha_{GD,D}(\mathbb{1}_{GD})$	$:= F(\alpha^{-1}_{C,FC}(\mathbb{1}_{FC}));\alpha_{GFC,FC}(\mathbb{1}_{FC})$	by def of ε, η
$= \mathbb{1}_{GD};\alpha^{-1}_{GD,FGD}(\mathbb{1}_{FGD});G\alpha_{GD,D}(\mathbb{1}_{GD})$	$= F(\alpha^{-1}_{C,FC}(\mathbb{1}_{FC}));\alpha_{GFC,FC}(\mathbb{1}_{FC});\mathbb{1}_{FC}$	by unitality
$= \alpha^{-1}_{GD,D}(F\mathbb{1}_{GD};\mathbb{1}_{FGD};\alpha_{GD,D}(\mathbb{1}_{GD}))$	$= \alpha_{C,FC}(\alpha^{-1}_{C,FC}(\mathbb{1}_{FC});\mathbb{1}_{FC};\mathbb{1}_{FC})$	by NAT
$= \alpha^{-1}_{GD,D}(\alpha_{GD,D}(\mathbb{1}_{GD}))$	$= \alpha_{C,FC}(\alpha^{-1}_{C,FC}(\mathbb{1}_{FC}))$	by unitality
$= \mathbb{1}_{GD}$	$= \mathbb{1}_{FC}$	because α is an isomorphism

$\square$

PROPOSITION

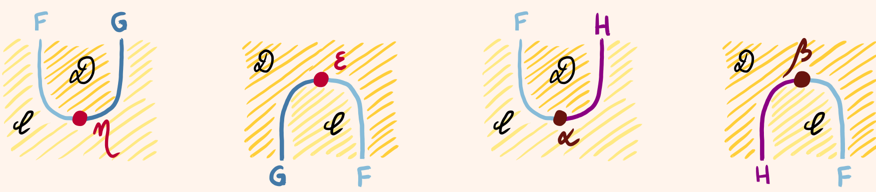
Right (left) adjoints are unique up to isomorphism.

⇝ I can talk about <u>the</u> internal-hom or <u>the</u> free monoid on X.

PROOF

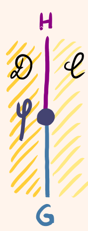
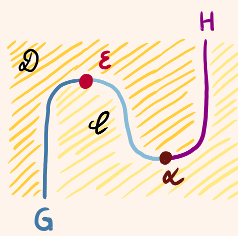
Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and suppose we have two adjunctions $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ and $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "H"] \end{tikzcd}$

with units and counits

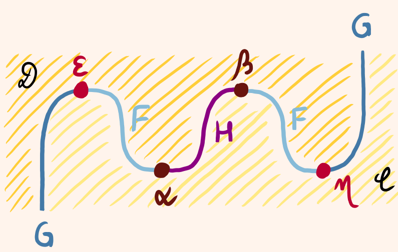
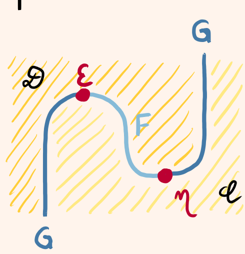


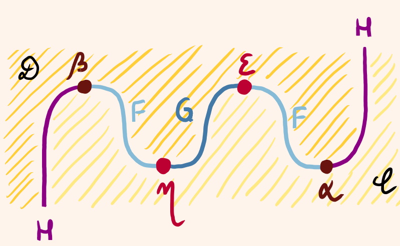
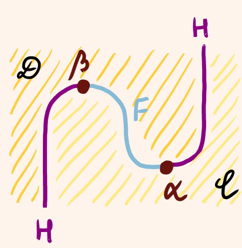
.

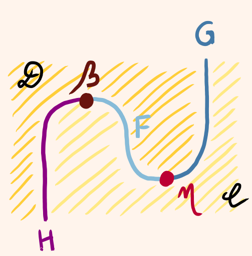
We want to show that there is a natural isomorphism $\varphi : G \xrightarrow{\simeq} H$.

Define φ as  := 

We need to show that φ has an inverse.

 $\underset{F \dashv H}{\overset{=}{\uparrow}}$  $\underset{F \dashv G}{\overset{=}{\uparrow}}$ 

 $\underset{F \dashv G}{\overset{=}{\uparrow}}$  $\underset{F \dashv H}{\overset{=}{\uparrow}}$ 

⇒ the inverse of φ is 

□

③ ADJUNCTION BETWEEN CATEGORIES (useful)

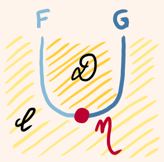
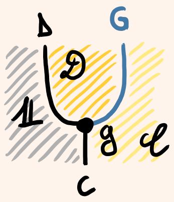
$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ ($F \dashv G$) iff there is a natural transformation $\eta : \mathbb{1}_{\mathcal{C}} \longrightarrow F ; G$

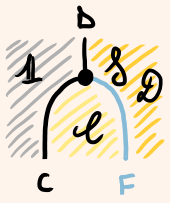
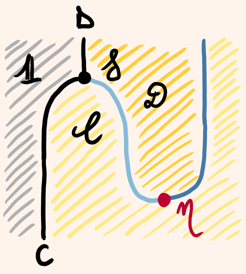
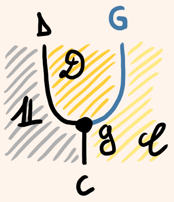
such that $\forall\, g : C \longrightarrow G(D)$

$\exists !\, f : F(C) \longrightarrow D$

$$\begin{tikzcd} GF(C) \arrow[r, dashed, "G(f)"] & G(D) \\ C \arrow[u, "\eta_C"] \arrow[ur, "g"'] & \end{tikzcd}$$

⇝ we have the unit and we can construct the counit

 such that $\forall$ 

$\exists !$  :  = 

④ ADJUNCTION BETWEEN CATEGORIES (useful)

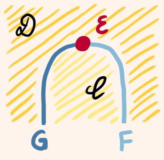
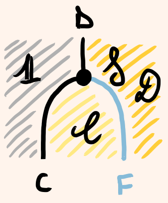
$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ ($F \dashv G$) iff there is a natural transformation $\varepsilon : G ; F \longrightarrow \mathbb{1}_{\mathcal{D}}$

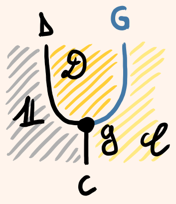
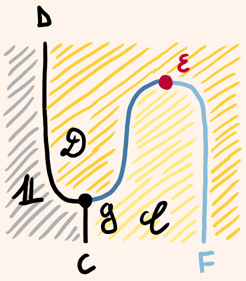
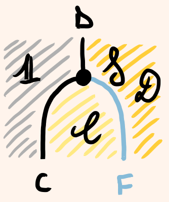
such that $\forall\, f : F(C) \longrightarrow D$

$\exists !\, g : C \longrightarrow G(D)$

$$\begin{tikzcd} F(C) \arrow[r, dashed, "F(g)"] \arrow[dr, "f"'] & FG(D) \arrow[d, "\varepsilon_D"] \\ & D \end{tikzcd}$$

⇝ we have the counit and we can construct the unit

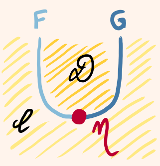
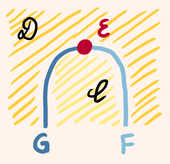
 such that $\forall$ 

$\exists !$  :  = 

PROPOSITION

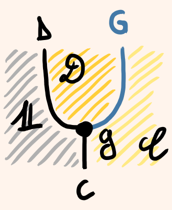
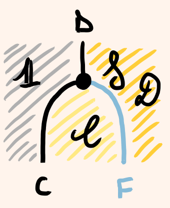
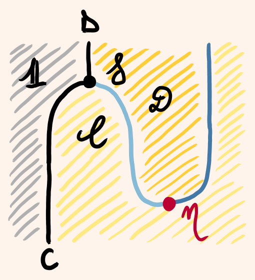
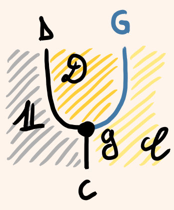
Definitions 1 and 3 of adjunction are equivalent.

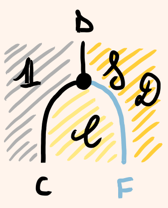
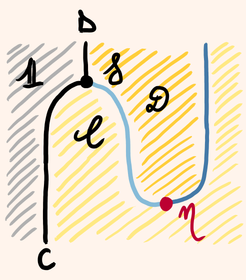
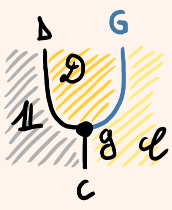
PROOF

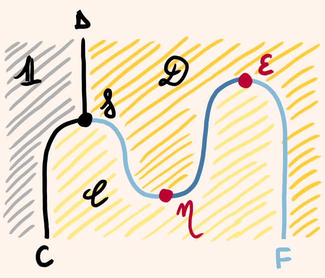
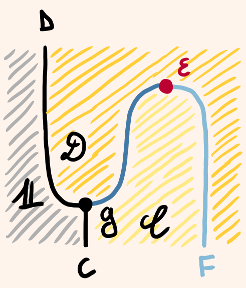
1⇒3| Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ with  

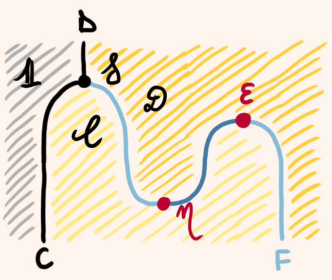
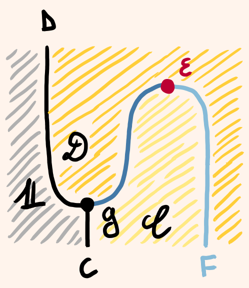
satisfying the zig-zag equations.

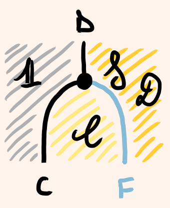
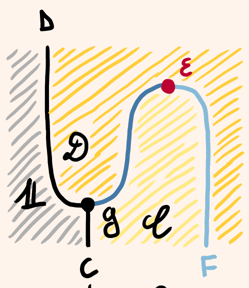
We need to show that

∀  ∃!  :  =  ⊛

Suppose there is such  :  = 

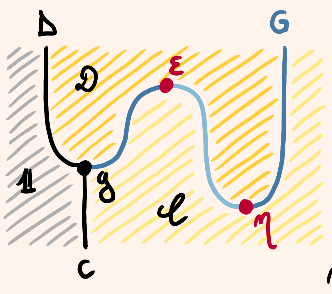
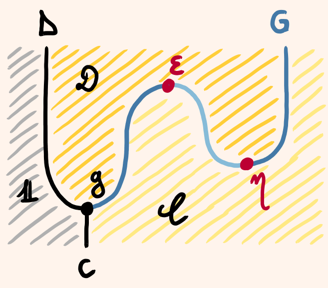
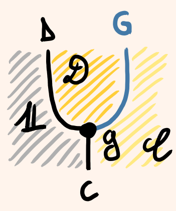
⇒  = 

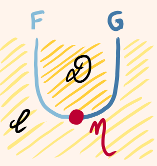
naturality of ε ⇒  = 

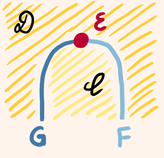
zig-zag eq. ⇒  = 

So, if f exists, it must be $f = F(g) ; \varepsilon_D$.

Now, we need to show that this f satisfies ⊛.

 = (naturality of η)  = (zig-zag equation) 

3⇒1| Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ with  such that ⊛.

We need to define  and show the zig-zag equations.

Let $g = 1_{G(D)}$ ⇒ $\exists!\ \varepsilon_D : FG(D) \longrightarrow D$ such that

$$\eta_{G(D)} ; G\varepsilon_D = 1_{G(D)}$$

We need to show that ε is natural and $F\eta_C ; \varepsilon_{F(C)} = 1_{F(C)}$.

$$\begin{tikzcd} FG(D) \arrow[r, "FG(g)"] \arrow[d, "\varepsilon_D"'] & FG(D') \arrow[d, "\varepsilon_{D'}"] \\ D \arrow[r, "g"'] & D' \end{tikzcd} \qquad \Rightarrow \quad Gg : G(D) \longrightarrow G(D')$$

$$\Rightarrow \exists!\ f : FG(D) \longrightarrow D' \qquad \eta_{G(D)} ; Gf = Gg$$

$$\begin{cases} \eta_{G(D)} ; GFGg ; G\varepsilon_{D'} \overset{\eta \text{ natural}}{=} Gg ; \eta_{G(D')} ; G\varepsilon_{D'} = Gg \\ \eta_{G(D)} ; G\varepsilon_D ; Gg = Gg \end{cases}$$

$$\Rightarrow FGg ; \varepsilon_{D'} = f = \varepsilon_D ; g$$

$$\eta_C : C \longrightarrow GF(C) \quad \Rightarrow \exists!\ f : F(C) \longrightarrow F(C) \qquad \eta_C ; Gf = \eta_C$$

$$\begin{cases} \eta_C ; G1_{F(C)} = \eta_C ; 1_{GF(C)} = \eta_C \\ \eta_C ; GF\eta_C ; G\varepsilon_{F(C)} \underset{\eta \text{ natural}}{=} \eta_C ; \eta_{GF(C)} ; G\varepsilon_{F(C)} = \eta_C \end{cases}$$

$$\Rightarrow 1_{F(C)} = f = F\eta_C ; \varepsilon_{F(C)}$$

□

RECAP DEFINITIONS OF ADJUNCTION

4

ε natural α^{-1} injective

zig-zag eq. natural 2

1

η natural α injective

3

THEOREM (RAPL = right adjoints preserve limits)

$$\begin{cases} \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \ (\perp) & \text{adjunction} \\ J : \mathcal{J} \to \mathcal{D} & \text{diagram in } \mathcal{D} \end{cases}$$

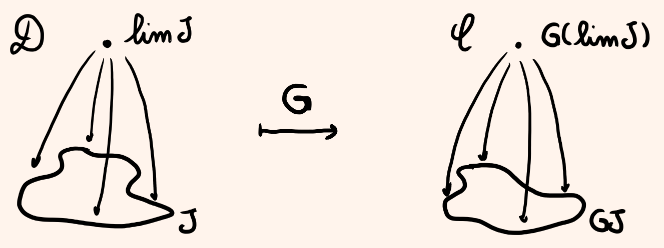
$\Rightarrow G(\lim J) \simeq \lim(G(J))$

PROOF

$(\lim J, d)$ is a cone over J by hypothesis

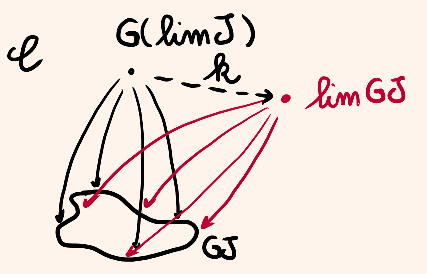
$\Rightarrow (G(\lim J), Gd)$ is a cone over GJ

$$\mathcal{D} \quad \begin{tikzcd} & \lim J \arrow[dl, "d_i"'] \arrow[dr, "d_j"] \arrow[d, phantom, "\circlearrowleft"] & \\ J(i) \arrow[rr, "J(f)"'] & & J(j) \end{tikzcd} \quad \overset{G}{\longmapsto} \quad \mathcal{C} \quad \begin{tikzcd} & G(\lim J) \arrow[dl, "Gd_i"'] \arrow[dr, "Gd_j"] \arrow[d, phantom, "\circlearrowleft"] & \\ GJ(i) \arrow[rr, "GJ(f)"'] & & GJ(j) \end{tikzcd}$$



$\Rightarrow$ by def of limit, $\exists!\ k : G(\lim J) \longrightarrow \lim GJ$ cone morphism

$$\begin{tikzcd} G(\lim J) \arrow[rr, dashed, "k"] \arrow[dr, "Gd_i"'] & \circlearrowleft & \lim GJ \arrow[dl, "c_i"] \\ & GJ(i) & \end{tikzcd}$$



If we find $h : \lim GJ \longrightarrow G(\lim J)$ cone morphism

$\Rightarrow h ; k = 1_{\lim GJ}$ by definition of limit

$$\mathcal{C} \quad \begin{tikzcd} & \lim GJ \arrow[dl, "c_i"'] \arrow[dr, "c_j"] \arrow[d, phantom, "\circlearrowleft"] & \\ GJ(i) \arrow[rr, "GJ(f)"'] & & GJ(j) \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \mathcal{D} \quad \begin{tikzcd} & F\lim GJ \arrow[dl, "Fc_i"'] \arrow[dr, "Fc_j"] \arrow[d, phantom, "\circlearrowleft"] & \\ FGJ(i) \arrow[rr, "FGJ(f)"'] & & FGJ(j) \end{tikzcd}$$

cone over FGJ

$$\overset{\varepsilon}{\Rightarrow} \quad \mathcal{D} \quad \begin{tikzcd} & F\lim GJ \arrow[dl, "Fc_i"'] \arrow[dr, "Fc_j"] \arrow[d, phantom, "\circlearrowleft"] & \\ FGJ(i) \arrow[rr, "FGJ(f)"] \arrow[d, "\varepsilon_{J(i)}"'] & \circlearrowleft & FGJ(j) \arrow[d, "\varepsilon_{J(j)}"] \\ J(i) \arrow[rr, "J(f)"'] & & J(j) \end{tikzcd}$$

cone over J

$$\overset{\lim J}{\Rightarrow} \quad \begin{tikzcd} F\lim GJ \arrow[rr, dashed, "g"] \arrow[dr, "Fc_i ; \varepsilon_{J(i)}"'] & \circlearrowleft & \lim J \arrow[dl, "d_i"] \\ & J(i) & \end{tikzcd}$$

$\exists!\ g : F\lim GJ \to \lim J$ cone morphism

$$\overset{G}{\longmapsto} \quad \begin{tikzcd} GF\lim GJ \arrow[rr, dashed, "Gg"] \arrow[dr, "GFc_i ; G\varepsilon_{J(i)}"'] & \circlearrowleft & G\lim J \arrow[dl, "Gd_i"] \\ & GJ(i) & \end{tikzcd}$$

Gg cone morphism

$$\begin{tikzcd} \lim GJ \arrow[r, "\eta_{\lim GJ}"] \arrow[d, "c_i"'] \arrow[dr, phantom, "\circlearrowright"] & GF\lim GJ \arrow[d, "GFc_i"] \\ GJ(i) \arrow[r, "\eta_{GJ(i)}"] \arrow[dr, "1_{GJ(i)}"'] & GFGJ(i) \arrow[d, "G\varepsilon_{J(i)}"] \\ & GJ(i) \end{tikzcd}$$

$\eta_{\lim GJ}$ cone morphism because η natural and zig-zag eq.

$$\Rightarrow \quad \begin{tikzcd} \lim GJ \arrow[rr, "\eta_{\lim GJ} ; Gg"] \arrow[dr, "c_i"'] & \circlearrowleft & G\lim J \arrow[dl, "Gd_i"] \\ & GJ(i) & \end{tikzcd}$$

$h = \eta_{\lim GJ} ; Gg : \lim GJ \longrightarrow G\lim J$ cone morphism

$\Rightarrow h ; k = 1_{\lim GJ}$ because $h ; k : \lim GJ \longrightarrow \lim GJ$ is a cone morphism

We are left to prove that $k ; h = 1_{G\lim J}$

$$\begin{tikzcd} & FG\lim J \arrow[dl, dashed, "\varepsilon_{\lim J}"'] \arrow[d, "FGd_i"] \\ \lim J \arrow[d, "d_i"'] \arrow[r, phantom, "\circlearrowleft"] & FGJ(i) \arrow[dl, "\varepsilon_{J(i)}"] \\ J(i) & \end{tikzcd}$$

$FG\lim J$ cone over J with unique morphism $\varepsilon_{\lim J}$

$$\begin{tikzcd} FG\lim J \arrow[r, "Fk"] \arrow[dr, "FGd_i"'] & F\lim GJ \arrow[r, dashed, "g"] \arrow[d, "Fc_i"] \arrow[dr, phantom, "\circlearrowleft"] & \lim J \arrow[d, "d_i"] \\ & FGJ(i) \arrow[r, "\varepsilon_{J(i)}"'] & J(i) \end{tikzcd}$$

$Fk ; g : FG\lim J \longrightarrow \lim J$ cone morphism

$$\Rightarrow \quad \begin{tikzcd} F\lim GJ \arrow[dr, dashed, "g"] \arrow[d, "Fc_i"'] & & FG\lim J \arrow[ll, "Fk"'] \arrow[dl, dashed, "\varepsilon_{\lim J}"'] \arrow[d, "FGd_i"] \\ FGJ(i) \arrow[dr, "\varepsilon_{J(i)}"'] & \lim J \arrow[d, "d_i"] & FGJ(i) \arrow[dl, "\varepsilon_{J(i)}"] \\ & J(i) & \end{tikzcd}$$

$Fk ; g = \varepsilon_{\lim J}$

$\Rightarrow GFk ; Gg = G\varepsilon_{\lim J}$

$\eta_{G\lim J} ; GFk ; Gg = \eta_{G\lim J} ; G\varepsilon_{\lim J}$

$k ; \eta_{\lim GJ} ; Gg = 1_{G\lim J}$ $\quad$ η natural

$k ; h = 1_{G\lim J}$

$\square$