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EVIDENTIAL DECISION THEORY VIA PARTIAL MARKOV CATEGORIES

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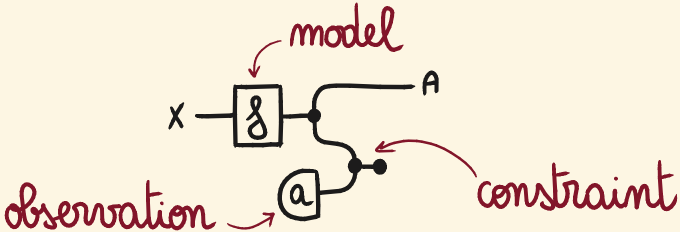
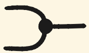
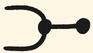
PARTIALITY FOR OBSERVATIONS

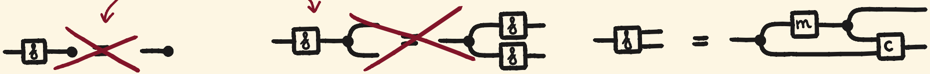
Updating a model on an observation means restricting the model to scenarios that are compatible with this observation.



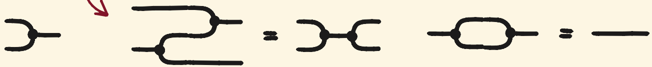
constraints  cannot be total computations because  $\neq$  .

OVERVIEW

Combine Markov and cartesian restriction categories to express <u>partial</u> <u>stochastic</u> processes.



Add the <u>discrete structure</u> to express equality checking.



Morphisms $X - \boxed{f} - A$ are <u>partial stochastic channels</u>

$f(a|x)$ = "probability of a given x"

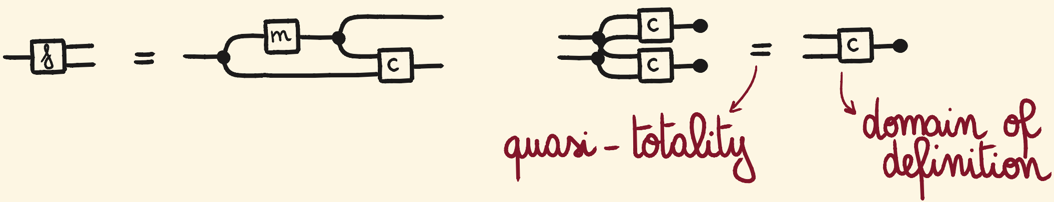
$f(\perp|x)$ = "probability of failure"

PARTIAL MARKOV CATEGORIES

A partial Markov category is a copy-discard category with quasi-total conditionals.

COPY-DISCARD STRUCTURE

CONDITIONALS



EXAMPLES : PARTIAL STOCHASTIC PROCESSES

Partial stochastic processes form a partial Markov category.

maybe monad on a Markov category

THEOREM

$\left\{ \begin{array}{l} \mathcal{C} \text{ Markov category with conditionals and coproducts} \\ \text{Some ugly technical conditions} \end{array} \right.$

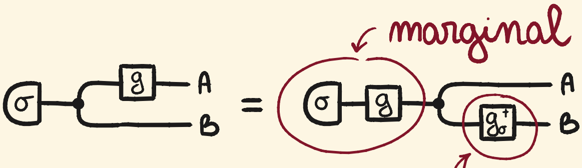
$\Rightarrow \mathrm{Kl}(\cdot + 1)$ is a partial Markov category.

EXAMPLES

- $\mathrm{Kl}(\mathcal{D}(\cdot + 1))$ $\rightsquigarrow$ finitary subdistributions
- $\mathrm{Kl}(\mathrm{Giry}_B(\cdot + 1))$ $\rightsquigarrow$ subdistributions on standard Borel spaces

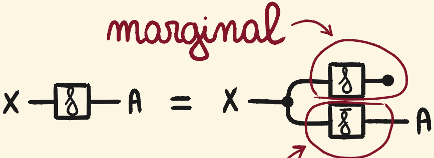
BAYES INVERSION & NORMALISATION

Bayes inversions and normalisations are particular cases of quasi-total conditionals:



$$\rightsquigarrow \quad g^\dagger_\sigma(b|a) := \frac{g(a|b)\,\sigma(b)}{\sum_{b'\in B} g(a|b')\,\sigma(b')}$$

$g^\dagger_\sigma$ is a Bayes inversion of g w.r.t. σ



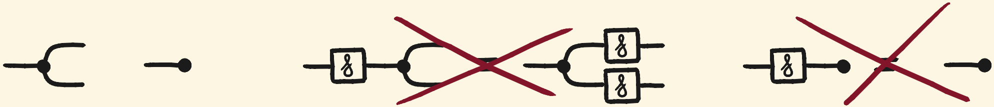
$$\rightsquigarrow \quad \bar{f}(a|x) := \frac{f(x|a)}{1 - f(\perp|a)}$$

$\bar{f}$ is a normalisation of f

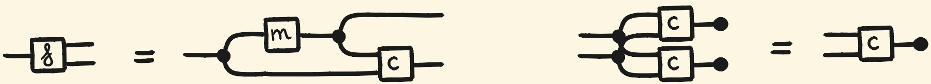
DISCRETE PARTIAL MARKOV CATEGORIES

A discrete partial Markov category is a copy-discard category with quasi-total conditionals and comparators.

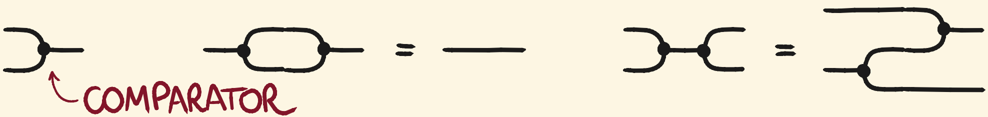
COPY-DISCARD STRUCTURE



CONDITIONALS

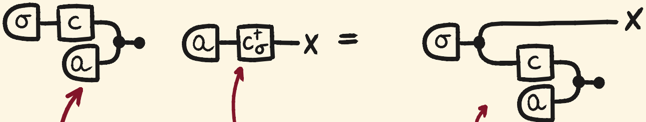


PARTIAL FROBENIUS STRUCTURE



SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^\dagger_\sigma$ evaluated on a.

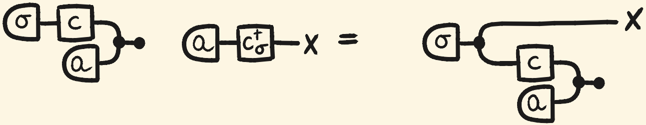


$$P(X=x|A=a) = \frac{P(A=a|X=x)\cdot P(X=x)}{\sum_{y\in X} P(A=a|X=y)\cdot P(X=y)}$$

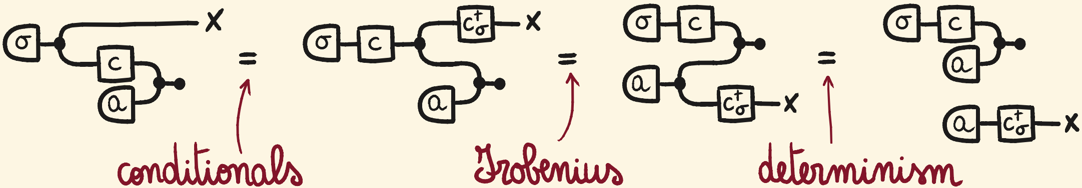
classical formula for Bayes theorem

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^\dagger_\sigma$ evaluated on a.



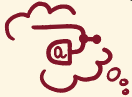
PROOF



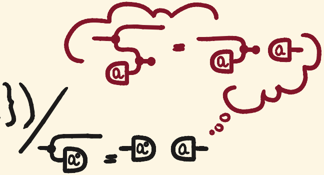
□

PROCESSES WITH EXACT OBSERVATIONS

For a Markov category $\mathcal{C}$ with conditionals, we construct a partial Markov category $\mathrm{exOb}(\mathcal{C})$:



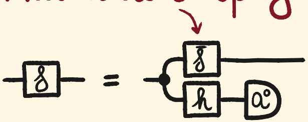
$\mathrm{exOb}(\mathcal{C}) = (\mathcal{C} + \{ A \text{—} a^{\circ} \mid a \text{—} A \text{ deterministic} \}) /$



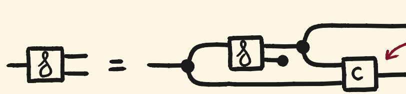
embeds faithfully into $(\mathcal{C} + \ \ )/$ partial Frobenius

conditionals and normalisations are computed in $\mathcal{C}$

normalisation of f

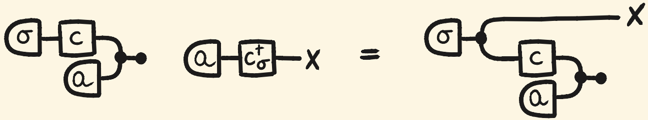


conditional of $\bar{f}$



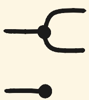
SUMMARY

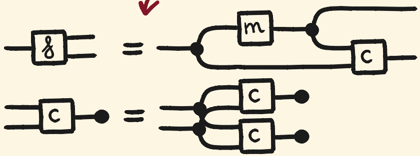
Discrete partial Markov categories express stochastic processes with observations and updates.

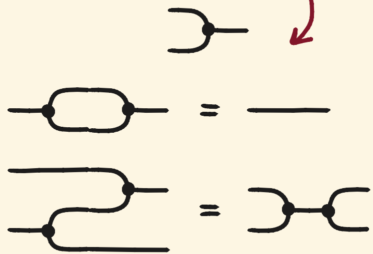


Synthetic Bayes theorem

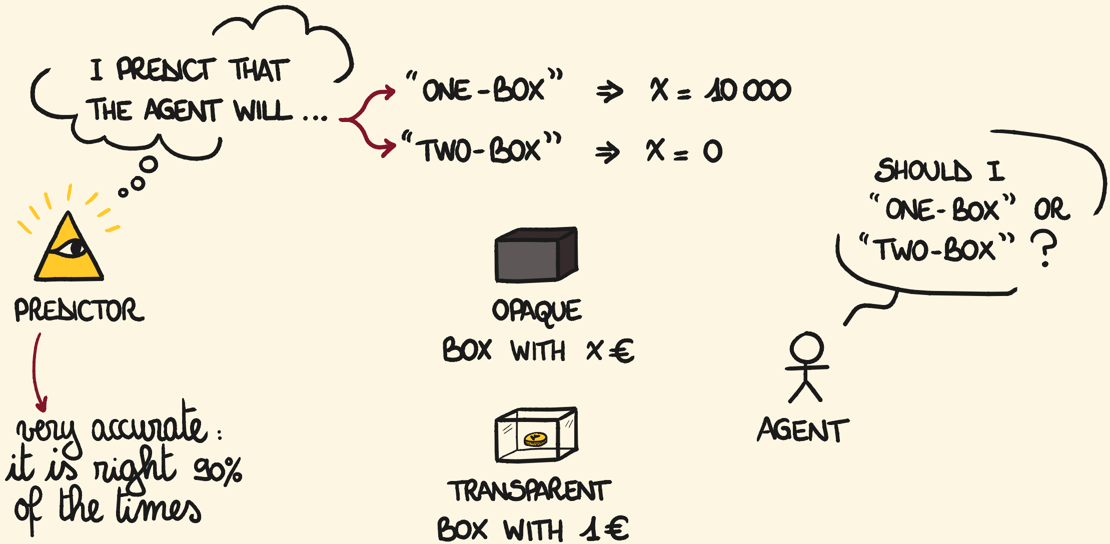
They are copy-discard categories with conditionals and comparators.







NEWCOMB'S PROBLEM



EVIDENTIAL DECISION THEORY

Evidential decision theory answers :

"Which action would be evidence for the best-case scenario ?"

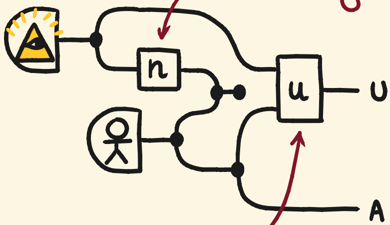
My action is evidence for the prediction :
if I one-box I expect 10 000 €,
if I two-box I expect 1 € .
⇒ I will one-box

PREDICTOR \ AGENT	ONE-BOX	TWO-BOX
ONE-BOX	10 000 €	10 001 €
TWO-BOX	0 €	1 €

MOST LIKELY

NEWCOMB'S PROBLEM CATEGORICALLY

noise: the predictor
is right 90% of the times



utility function

AGENT / PREDICTOR	ONE-BOX	TWO-BOX
ONE-BOX	10 000 €	10 001 €
TWO-BOX	0 €	1 €

SOLVING NEWCOMB'S PROBLEM

Evidential decision theory asks:
"Which action would be evidence for the best-case scenario?"
i.e. "Which action maximises the average of the state below?"