Bob Walters memorial 17 July 2023

TOWARDS MARKOV BICATEGORIES

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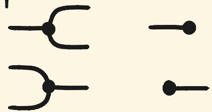
joint work with Mario Román

OVERVIEW

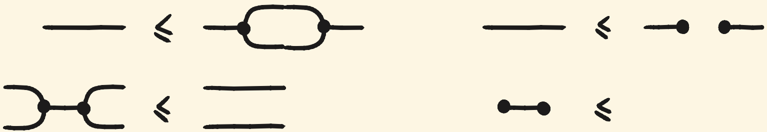
- Cartesian bicategories of relations are a syntax for relations
- Some facts hint at bicategories of stochastic relations
  - ↳ Minimal conditionals in partial Markov categories
  - ↳ Convex sets of distributions monad

CARTESIAN BICATEGORIES OF RELATIONS

A cartesian bicategory is a locally posetal monoidal bicategory
with a cocommutative comonoid
and a commutative monoid

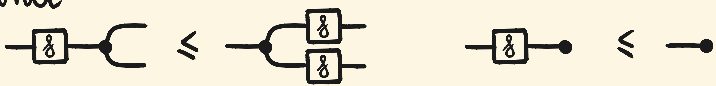


structures satisfying

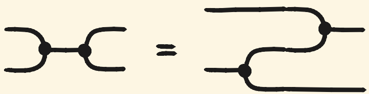


(adjointness)

and



(lax naturality)



(Frobenius)

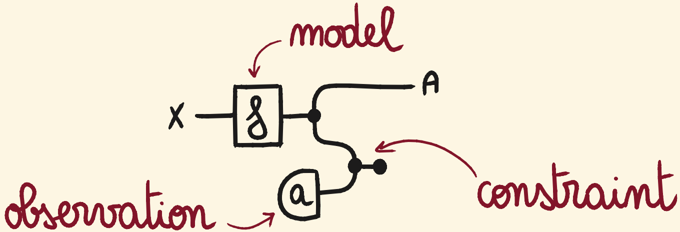
[Carboni & Walters, 1987]

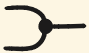
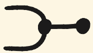
OUTLINE

- Observations & updates
- Canonical conditionals and inequalities

PARTIALITY FOR OBSERVATIONS

Updating a model on an observation means restricting the model to scenarios that are compatible with this observation.



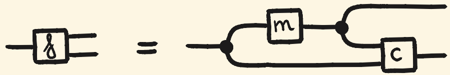
constraints  cannot be total computations because  ≠  .

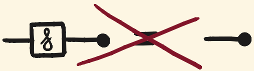
OVERVIEW

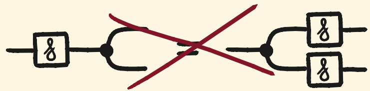
Combine Markov and cartesian restriction categories to express partial stochastic processes.

cartesian restriction

Markov with conditionals

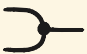


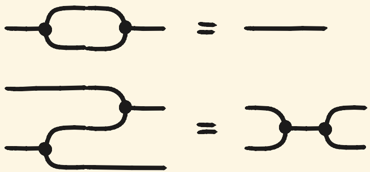




Add the discrete structure to express equality checking.

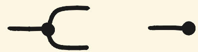
discrete cartesian restriction



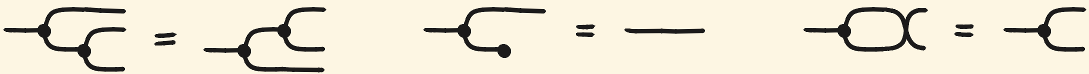


COPY-DISCARD CATEGORIES

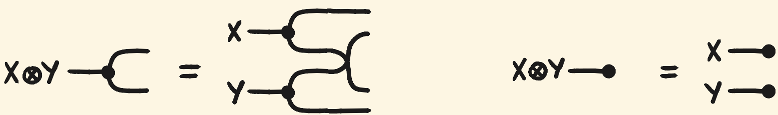
A copy-discard category is a symmetric monoidal category where every object is a uniform cocommutative comonoid.



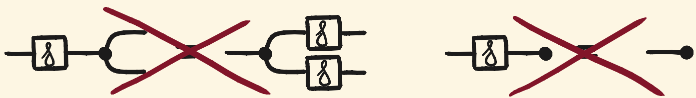
COCOMMUTATIVE COMONOID



UNIFORMITY



NO NATURALITY REQUIRED

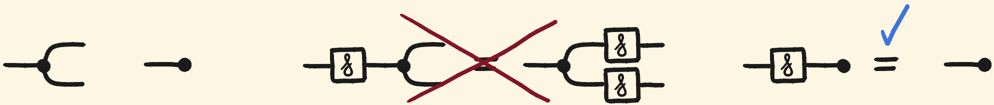


[Corradini & Gadducci 1999]

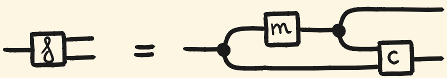
MARKOV CATEGORIES & CONDITIONALS

A Markov category with conditionals is a copy-discard category with conditionals where all morphisms are total.

COPY - DISCARD STRUCTURE



CONDITIONALS



[Fritz 2020]

FINITARY DISTRIBUTIONS

A <u>finitary distribution</u> $\sigma \in \mathcal{D}(A)$ is a function $\sigma : A \to [0,1]$ such that

- its support, $\mathrm{supp}(\sigma) := \{a \in A \mid \sigma(a) > 0\}$, is finite, and
- its total probability mass is 1, $\sum_{a\in A} \sigma(a) = 1$.

A morphism $X - \boxed{f} - A$ in $\mathrm{Kl}\mathcal{D}$ is a function $X \to \mathcal{D}(A)$

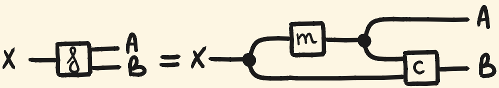
$f(a|x) =$ "probability of a given x"

Composition is

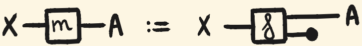
$$X - \boxed{f} - \boxed{g} - B \; (b|x) := \sum_{a\in A} f(a|x) \cdot g(b|a)$$

CONDITIONALS

$\mathrm{Kl}\mathcal{D}$ has conditionals.



$$m(a|x) := \sum_{b \in B} f(a,b|x)$$



$$c(b|a,x) := \begin{cases} \dfrac{f(a,b|x)}{m(a|x)} & \text{if } m(a|x) \neq 0 \\ \sigma(b) & \text{if } m(a|x) = 0 \end{cases}$$

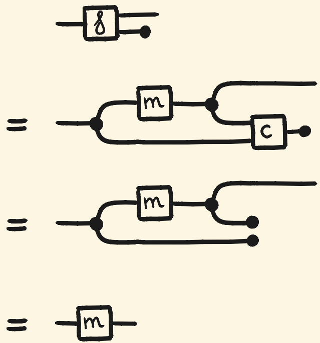
any distribution on B

$\rightsquigarrow$ conditionals are not unique and they cannot be

MARGINALS IN MARKOV CATEGORIES

Marginals in Markov categories are as expected :

PROOF



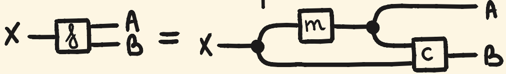
conditionals :

⇝ totality

□

DROPPING TOTALITY

We want to keep the nice marginals of Markov categories.



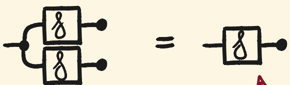
Should we ask conditionals to be total? ✗ NO

⇝ too strong: total conditionals fail to exist in $\mathrm{Kl}(\mathcal{D}_{\leq 1})$.

Can we ask conditionals to be quasi-total? ✓ YES

⇝ sweet spot: quasi-total conditionals usually exist and give nice marginals.

QUASI-TOTAL MORPHISM (in a copy-discard category)



⇝ failure is deterministic

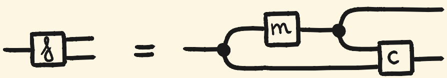
domain of definition

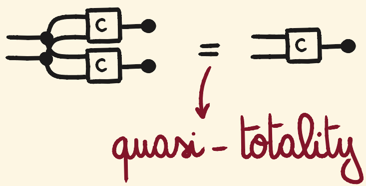
PARTIAL MARKOV CATEGORIES

A partial Markov category is a copy-discard category with quasi-total conditionals.

COPY - DISCARD STRUCTURE

CONDITIONALS





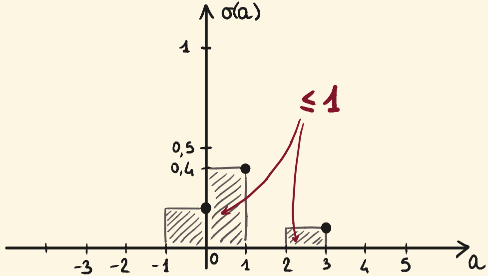
SUBDISTRIBUTIONS

A <u>subdistribution</u> σ on A is a distribution on $A+1$:

$\sigma \in \mathcal{D}_{\leq 1}(A)$ is a function $\sigma : A \to [0,1]$ such that

- its support, $\mathrm{supp}(\sigma) := \{a \in A \mid \sigma(a) > 0\}$, is finite, and
- its total probability mass is at most 1, $\sum_{a \in A} \sigma(a) \leq 1$.

<u>ex</u> $A = \mathbb{N}$, $\sigma =$



Subdistributions give a monad $\mathcal{D}_{\leq 1} : \mathbf{Set} \to \mathbf{Set}$

(there's a distributive law $\mathcal{D}(-)+1 \longrightarrow \mathcal{D}(-+1)$)

SUBDISTRIBUTIONS

a morphism $X - \boxed{f} - A$ in $\mathrm{Kl}\mathcal{D}_{\leq 1}$ is a function $X \to \mathcal{D}_{\leq 1}(A)$

$f(a|x)$ = "probability of a given x"

$f(\perp|x)$ = "probability of failure"

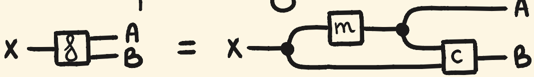
Composition is:

$$X - \boxed{f} - \boxed{g} - B \ (b|x) := \sum_{a \in A} f(a|x) \cdot g(b|a)$$

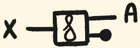
$$X - \boxed{f} - \boxed{g} - B \ (\perp|x) := f(\perp|x) + \sum_{a \in A} f(a|x) \cdot g(\perp|a)$$

CONDITIONALS IN SUBDISTRIBUTIONS

A quasi-total morphism $g : X \to B$ is a function $g : X \to \mathcal{D}B + 1$.



The marginal of f is:

 $(a|x)$ =  $(a|x) = \sum_{b \in B} f(a, b|x)$

 $(\perp|x)$ =  $(\perp|x) = f(\perp|x)$

A conditional of f is:

 $(b|a,x) = \begin{cases} \dfrac{f(a,b|x)}{m(a|x)} & m(a|x) \neq 0 \\ 0 & m(a|x) = 0 \end{cases}$

 $(\perp|a,x) = \begin{cases} 0 & m(a|x) \neq 0 \\ 1 & m(a|x) = 0 \end{cases}$

EXAMPLES : PARTIAL STOCHASTIC PROCESSES

A <u>partial stochastic process</u> is a stochastic process that may fail.

↳ Maybe monad

a Markov category with conditionals

Partial stochastic processes form a partial Markov category.

PROPOSITION

$\begin{cases} \mathcal{C} \text{ Markov category with conditionals and coproducts} \\ \text{Some ugly technical conditions} \end{cases}$

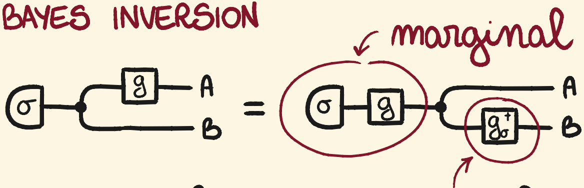
$\Rightarrow \mathrm{Kl}(\cdot + 1)$ is a partial Markov category.

EXAMPLES

- $\mathrm{Kl}(\mathcal{D}(\cdot + 1))$ ⇝ finitary subdistributions
- $\mathrm{Kl}(\mathcal{G}iry_B(\cdot + 1))$ ⇝ subdistributions on standard Borel spaces

BAYES INVERSION & NORMALISATION

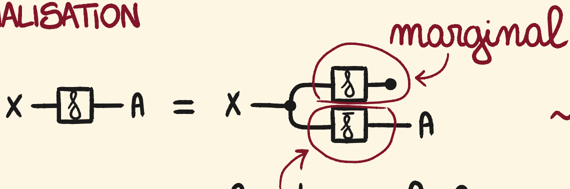
BAYES INVERSION



$$\rightsquigarrow \quad g^\dagger_\sigma(b|a) := \frac{g(a|b)\,\sigma(b)}{\sum_{b'\in B} g(a|b')\,\sigma(b')}$$

$g^\dagger_\sigma$ is a Bayes inversion of g w.r.t. σ

NORMALISATION



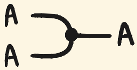
$$\rightsquigarrow \quad \bar{f}(a|x) := \frac{f(x|a)}{1-f(\perp|a)}$$

$\bar{f}$ is a normalisation of f

→ Both are particular cases of quasi-total conditionals.

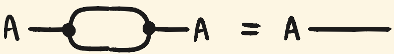
EQUALITY CHECK

$\mathrm{Kl}\,\mathcal{D}_{\leq 1}$ has equality checks.

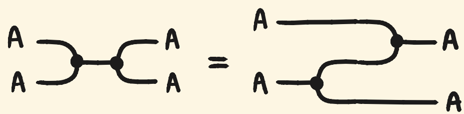


$$(a, a') := \begin{cases} \delta_a & \text{if } a = a' \\ \delta_\perp & \text{if } a \neq a' \end{cases}$$

Equality checks interact with the comonoid structure.



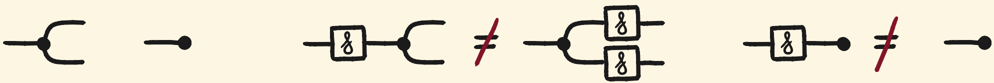
and



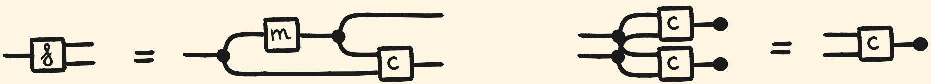
DISCRETE PARTIAL MARKOV CATEGORIES

A discrete partial Markov category is a copy-discard category with quasi-total conditionals and comparators.

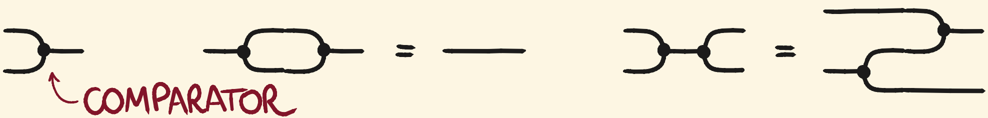
COPY-DISCARD STRUCTURE



CONDITIONALS

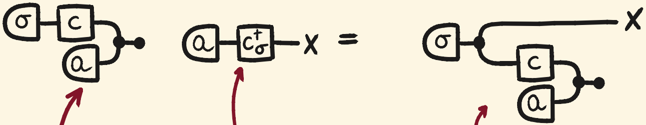


PARTIAL FROBENIUS STRUCTURE



SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^{\dagger}_{\sigma}$ evaluated on a.

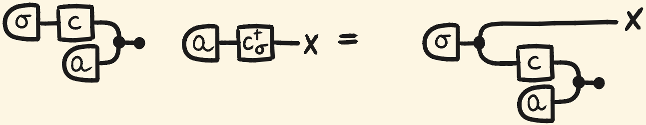


$$P(X = x \mid A = a) = \frac{P(A = a \mid X = x) \cdot P(X = x)}{\sum_{y \in X} P(A = a \mid X = y) \cdot P(X = y)}$$

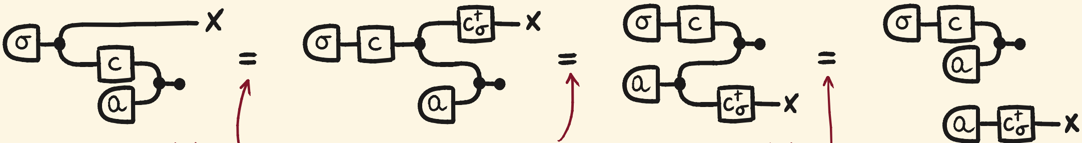
classical formula for Bayes theorem

SYNTHETIC BAYES THEOREM

A deterministic observation $a : I \to A$ from a prior $\sigma : I \to X$ through a channel $c : X \to A$ determines an update proportional to the Bayes inversion $c^\dagger_\sigma$ evaluated on a.



PROOF



conditionals

Frobenius

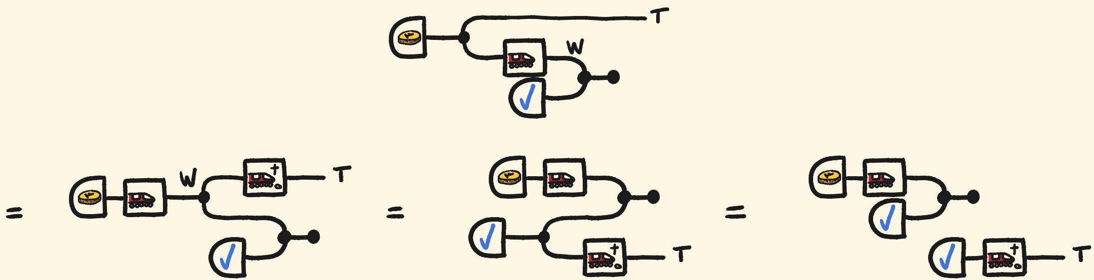
determinism

□

EXAMPLE : THE SLEEPING BEAUTY

$$= \begin{cases} H \Rightarrow \text{to Rome} \rightarrow \text{wake up once} \\ T \Rightarrow \text{to Naples} \rightarrow \text{wake up 3 times} \end{cases}$$

If Sleeping Beauty wakes up, how should she update her belief?



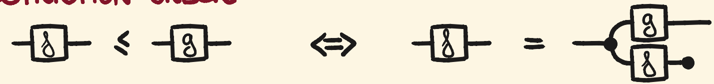
OUTLINE

- Observations & updates
- Canonical conditionals and inequalities

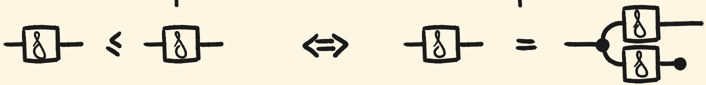
MINIMAL CONDITIONALS

$\mathcal{C}$ partial Markov category

RESTRICTION ORDER



→ this is a partial order on quasi-total morphisms



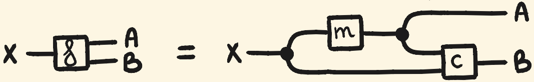
(reflexivity) (quasi-totality)

MINIMAL CONDITIONALS

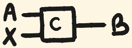
$\mathcal{C}$ has minimal conditionals if, for every , the partial order on its quasi-total conditionals has a minimal element.

MINIMAL CONDITIONALS IN SUBDISTRIBUTIONS

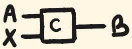
A quasi-total morphism $g : X \to B$ is a function $g : X \to \mathcal{D}B + 1$.



The conditionals of f are:



$$(b \mid a, x) = \begin{cases} \dfrac{f(a, b \mid x)}{m(a \mid x)} & m(a \mid x) \neq 0 \\ \sigma(b) & m(a \mid x) = 0 \end{cases}$$



$$(\perp \mid a, x) = \begin{cases} 0 & m(a \mid x) \neq 0 \\ \sigma(\perp) & m(a \mid x) = 0 \end{cases}$$

for some $\sigma \in \mathcal{D}(B) + 1$.

$\Rightarrow$ The minimal choice for σ is $\begin{cases} \sigma(b) = 0 & \text{for } b \in B \\ \sigma(\perp) = 1 \end{cases}$.

$\rightsquigarrow$ fail on unexpected observations

CONVEX SETS OF DISTRIBUTIONS

There's a monad $\mathcal{C} : \mathsf{Set} \to \mathsf{Set}$:

$$\mathcal{C}(X) := \{ A \subseteq \mathcal{D}(X) \mid A \neq \emptyset,\ A \text{ convex} \}$$

Composition in $\mathcal{K}\ell(\mathcal{C})$ is

$$\left(A \xrightarrow{\boxed{f}} B \xrightarrow{\boxed{g}} C\right)(a) := \bigcup_{\sigma \in f(a)} \sum_{b \in \operatorname{supp} \sigma} \sigma(b) \cdot g(b)$$

$$= \bigcup_{\sigma \in f(a)} \left\{ \sum_{b \in \operatorname{supp} \sigma} \sigma(b)\, \tau_b \;\middle|\; \tau_b \in g(b) \right\}$$

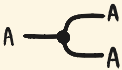
There's a partial order on morphisms

$$X \xrightarrow{\boxed{f}} A \;\leq\; X \xrightarrow{\boxed{g}} A \iff \forall_{x \in X}\ f(x) \subseteq g(x)$$

[Jacobs 2008]

CONVEX SETS OF DISTRIBUTIONS

COPY, DISCARD, UNIFORM

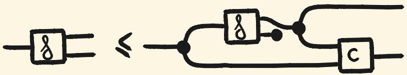


$(a) := \{(\delta_a, \delta_a)\}$

$(a) := \{\delta_*\}$

$(*) := \mathcal{D}(A)$

LAX CONDITIONALS



CONVEX SETS OF SUBDISTRIBUTIONS

There's a monad $\mathcal{C} : \mathbf{Set} \to \mathbf{Set}$:

$$\mathcal{C}_{\leq 1}(X) := \{ A \subseteq \mathcal{D}(X+1) \mid A \neq \emptyset ,\ A \text{ convex} \}$$

Composition in $\mathrm{Kl}(\mathcal{C}_{\leq 1})$ is

$$\left(A \xrightarrow{f} B \xrightarrow{g} C\right)(a) := \bigcup_{\sigma \in f(a)} \sum_{b \in \mathrm{supp}\,\sigma} \sigma(b) \cdot g(b)$$

$$= \bigcup_{\sigma \in f(a)} \Big\{ \sum_{b \in \mathrm{supp}\,\sigma} \sigma(b)\, \tau_b \;\Big|\; \tau_b \in g(b) \Big\}$$

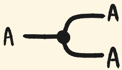
There's a partial order on morphisms

$$X \xrightarrow{f} A \;\leq\; X \xrightarrow{g} A \iff \forall x \in X \;\; f(x) \subseteq g(x)$$

[Bonchi, Sokolova & Vignudelli 2019]

CONVEX SETS OF SUBDISTRIBUTIONS

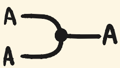
COPY, DISCARD, UNIFORM



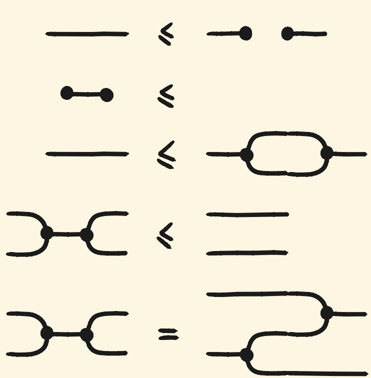
$(a) := \{\delta_{(a,a)}\}$

$(a) := \{\delta_{*}\}$

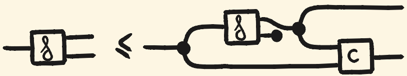
$(*) := \mathcal{D}(A)$



$$(a,a') := \begin{cases} \{\delta_a\} & a = a' \\ \{\delta_\perp\} & a \neq a' \end{cases}$$

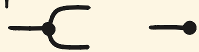


LAX CONDITIONALS

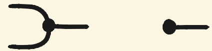


MARKOV BICATEGORIES (TENTATIVE)

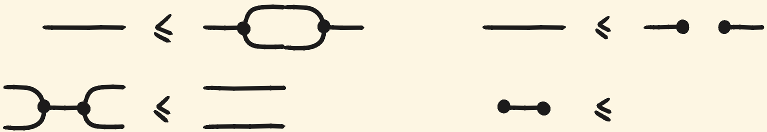
A Markov bicategory is a locally posetal monoidal bicategory with a cocommutative comonoid



and a commutative monoid

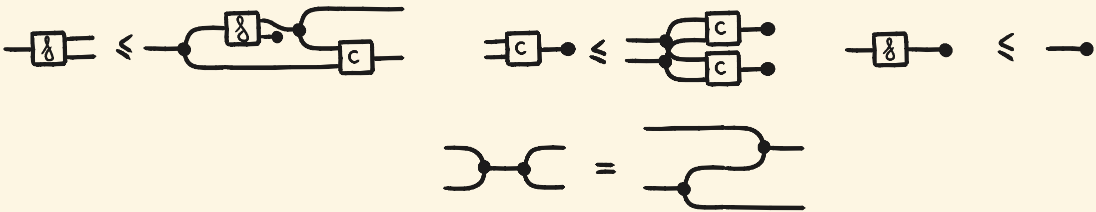


structures satisfying



(adjointness)

and



CONCLUSIONS & FUTURE WORK

- Partial Markov categories express updates
- Canonical conditionals via the restriction order
- The Kleisli of convex sets of (sub)distributions has a partial order

⇝ What is the 'right' structure to capture $\mathrm{Kl}(\mathcal{C})$ and $\mathrm{Kl}(\mathcal{C}_{\leq 1})$?

⇝ Would this give some interesting promonoidal structure?

$$S(X, A \triangleleft B) := \mathcal{C}(X, A) \times \mathcal{C}(A \otimes X, B)$$