Open games on categories with feedback

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November 13, 2020

1 Introduction

We want to extend the construction of open games to any category with feedback C.

2 Definition

2.1 The category Game(C)

We consider a symmetric monoidal category C with feedback $\circlearrowleft\colon \mathsf{C}(A \otimes \partial S, B \otimes S) \to \mathsf{C}(A, B)$, where $\partial\colon \mathsf{C} \to \mathsf{C}$ is a faithful strict monoidal endofunctor on C that preserves feedback (it is a feedback functor). We think of C as the category of 'actions' or 'things that happen'. The category $\mathsf{Game}(\mathsf{C})$ then gives the actions that can happen depending on a choice of a parameter (which we think of as strategy or policy).

Intuitively, morphisms are (equivalence relations of) families of morphisms in C parametrized by a set Σ together with a best response function.

Definition 2.1 (The category $\mathsf{Game}(\mathsf{C})$)**.** Objects are the objects of C. A morphism $g\colon A \xrightarrow{\Sigma} B$ in $\mathsf{Game}(\mathsf{C})$ is given by:

- A *play function* $\mathsf{P}_g\colon \Sigma \to \mathsf{C}(A, B)$.
- A *best response function* $\mathsf{B}_g\colon \mathsf{C}(\partial(B), A) \to Rel(\Sigma)$.

Two morphisms, $g\colon A \xrightarrow{\Sigma} B$ and $g'\colon A \xrightarrow{\Sigma'} B$, are considered equal whenever there is an isomorphism of sets $s\colon \Sigma \cong \Sigma'$ such that, for every $\sigma, \sigma' \in \Sigma$,

- $\mathsf{P}_g(\sigma) = \mathsf{P}_{g'}(s(\sigma))$
- $(\sigma, \sigma') \in \mathsf{B}_g(\kappa) \Leftrightarrow (s(\sigma), s(\sigma')) \in \mathsf{B}_{g'}(\kappa)$

We think of Σ as the set of possible strategies for a game. Choosing the strategy gives a way of playing the game, which is represented with a morphism in C. The strategy is chosen depending on the context of the game: for every context the best response function returns a relation of preference over the set of strategies Σ. The fixpoint of this relation gives, given the context, the preferred strategy.

Definition 2.2 (Categorical structure)**.** Let $g\colon A \xrightarrow{\Sigma}\mathrel{\mkern-14mu}\mapsto B$ and $h\colon B \xrightarrow{T}\mathrel{\mkern-14mu}\mapsto C$. Their composition $g\,;h\colon A \xrightarrow{\Sigma\times T}\mathrel{\mkern-14mu}\mapsto C$ is given by:

- $\mathsf{P}_{g;h}(\sigma,\tau) \coloneqq \mathsf{P}_g(\sigma)\,;\mathsf{P}_h(\tau)$.
- $\mathsf{B}_{g;h}(\kappa) \coloneqq \{(\sigma,\sigma',\tau,\tau') \in (\Sigma\times T)^2 : (\sigma,\sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_h(\tau))\,;\kappa) \wedge (\tau,\tau') \in \mathsf{B}_h(\kappa\,;\mathsf{P}_g(\sigma))\}$

$\mathbb{1}_A\colon A \xrightarrow{1}\mathrel{\mkern-14mu}\mapsto A$ is given by:

- $\mathsf{P}_{\mathbb{1}_A}(*) \coloneqq \mathbb{1}_A$.
- $\mathsf{B}_{\mathbb{1}_A}(\kappa) \coloneqq \{(*,*)\}$.

2.2 Monoidal structure

On objects, the monoidal product coincides with the monoidal product in C. On morphisms, it is lifted from C. Let $g\colon A \xrightarrow{\Sigma}\mathrel{\mkern-14mu}\mapsto B$ and $h\colon C \xrightarrow{T}\mathrel{\mkern-14mu}\mapsto D$ be two morphisms in $\mathsf{Game}(\mathsf{C})$. Their monoidal product $g\otimes h\colon A\otimes C \xrightarrow{\Sigma\times T}\mathrel{\mkern-14mu}\mapsto B\otimes D$ is given by:

- $\mathsf{P}_{g\otimes h}(\sigma,\tau) \coloneqq \mathsf{P}_g(\sigma)\otimes \mathsf{P}_h(\tau)$.
- $\mathsf{B}_{g\otimes h}(\kappa) \coloneqq \{(\sigma,\sigma',\tau,\tau') \in (\Sigma\times T)^2 : (\sigma,\sigma') \in \mathsf{B}_g(\circlearrowleft_D(\kappa\,;(\mathbb{1}_A\otimes \mathsf{P}_h(\tau)))) \wedge (\tau,\tau') \in \mathsf{B}_h(\circlearrowleft_B(\kappa\,;(\mathsf{P}_g(\sigma)\otimes \mathbb{1}_C)))\}$

2.3 Feedback operator

Definition 2.3 (Delay functor)**.** On objects, the delay functor coincides with the delay functor in C. On morphisms, it is lifted from C. Let $g\colon A \xrightarrow{\Sigma}\mathrel{\mkern-14mu}\mapsto B$ be a morphism in $\mathsf{Game}(\mathsf{C})$. Then $\partial(g)\colon \partial(A) \xrightarrow{\Sigma}\mathrel{\mkern-14mu}\mapsto \partial(B)$ is given by:

- $\mathsf{P}_{\partial(g)}(\sigma) \coloneqq \partial(\mathsf{P}_g(\sigma))$.
- $\mathsf{B}_{\partial(g)}(\partial(\kappa)) \coloneqq \mathsf{B}_g(\kappa)$. This gives the definition of the best response function because ∂ is faithful.

Definition 2.4 (Feedback operator)**.** Let $g\colon \partial S\otimes A \xrightarrow{\Sigma}\mathrel{\mkern-14mu}\mapsto S\otimes B$ be a morphism in $\mathsf{Game}(\mathsf{C})$. Then $\circlearrowleft_S g\colon A \xrightarrow{\Sigma}\mathrel{\mkern-14mu}\mapsto B$ is given by:

- $\mathsf{P}_{\circlearrowleft_S g}(\sigma) \coloneqq \circlearrowleft_S(\mathsf{P}_g(\sigma))$.
- $\mathsf{B}_{\circlearrowleft_S g}(\kappa) \coloneqq \mathsf{B}_g(\kappa\otimes \mathbb{1}_{\partial S})$.

A Proofs

Lemma A.1. *Composition is associative*

Proof. Let $g\colon A \xrightarrow{\Sigma} B$, $h\colon B \xrightarrow{T} C$ and $l\colon C \xrightarrow{R} D$ be morphisms in $\mathsf{Game}(\mathsf{C})$. For the play function, associativity follows from associativity in C.

$$\begin{aligned}
\mathsf{P}_{g;(h;l)}(\sigma,\tau,\rho) &\coloneqq \mathsf{P}_g(\sigma)\,;(\mathsf{P}_h(\tau)\,;\mathsf{P}_l(\rho))\\
&= (\mathsf{P}_g(\sigma)\,;\mathsf{P}_h(\tau))\,;\mathsf{P}_l(\rho)\\
&=: \mathsf{P}_{(g;h);l}(\sigma,\tau,\rho)
\end{aligned}$$

For the best response function, associativity follows from functoriality of ∂ and associativity in C.

$$\begin{aligned}
(\sigma,\sigma',\tau,\tau',\rho,\rho') \in \mathsf{B}_{g;(h;l)}(\kappa) \Leftrightarrow \quad & (\sigma,\sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_{h;l}(\tau,\rho))\,;\kappa)\\
& \wedge(\tau,\tau',\rho,\rho') \in \mathsf{B}_{h;l}(\kappa\,;\mathsf{P}_g(\sigma))\\
\Leftrightarrow \quad & (\sigma,\sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_h(\tau)\,;\mathsf{P}_l(\rho))\,;\kappa)\\
& \wedge(\tau,\tau') \in \mathsf{B}_h(\partial(\mathsf{P}_l(\rho))\,;\kappa\,;\mathsf{P}_g(\sigma))\\
& \wedge(\rho,\rho') \in \mathsf{B}_l(\kappa\,;\mathsf{P}_g(\sigma)\,;\mathsf{P}_h(\tau))\\
\Leftrightarrow \quad & (\sigma,\sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_h(\tau))\,;\partial(\mathsf{P}_l(\rho))\,;\kappa)\\
& \wedge(\tau,\tau') \in \mathsf{B}_h(\partial(\mathsf{P}_l(\rho))\,;\kappa\,;\mathsf{P}_g(\sigma))\\
& \wedge(\rho,\rho') \in \mathsf{B}_l(\kappa\,;\mathsf{P}_{g;h}(\sigma,\tau))\\
\Leftrightarrow \quad & (\sigma,\sigma',\tau,\tau') \in \mathsf{B}_{g;h}(\partial(\mathsf{P}_l(\rho))\,;\kappa)\\
& \wedge(\rho,\rho') \in \mathsf{B}_l(\kappa\,;\mathsf{P}_{g;h}(\sigma,\tau))\\
\Leftrightarrow \quad & (\sigma,\sigma',\tau,\tau',\rho,\rho') \in \mathsf{B}_{(g;h);l}(\kappa)
\end{aligned}$$

□

Lemma A.2. *Composition is unital.*

Proof. Let $g\colon A \xrightarrow{\Sigma} B$ be a morphism in $\mathsf{Game}(\mathsf{C})$. For the play function, unitality follows from unitality in C.

$$\mathsf{P}_{\mathbb{1}_A;g}(*,\sigma) \coloneqq \mathsf{P}_{\mathbb{1}_A}(*)\,;\mathsf{P}_g(\sigma) = \mathbb{1}_A\,;\mathsf{P}_g(\sigma) = \mathsf{P}_g(\sigma)$$

$$\mathsf{P}_{g;\mathbb{1}_B}(\sigma,*) \coloneqq \mathsf{P}_g(\sigma)\,;\mathsf{P}_{\mathbb{1}_B}(*) = \mathsf{P}_g(\sigma)\,;\mathbb{1}_B = \mathsf{P}_g(\sigma)$$

For the best response function, unitality follows from functoriality of ∂ and unitality in C.

$$\begin{aligned}
(*,*,\sigma,\sigma') \in \mathsf{B}_{\mathbb{1}_A;g}(\kappa) \Leftrightarrow \quad & (*,*) \in \mathsf{B}_{\mathbb{1}_A}(\partial(\mathsf{P}_g(\sigma))\,;\kappa)\\
& \wedge(\sigma,\sigma') \in \mathsf{B}_g(\kappa\,;\mathsf{P}_{\mathbb{1}_A}(*))\\
\Leftrightarrow \quad & (\sigma,\sigma') \in \mathsf{B}_g(\kappa\,;\mathbb{1}_A)\\
\Leftrightarrow \quad & (\sigma,\sigma') \in \mathsf{B}_g(\kappa)
\end{aligned}$$

$$
\begin{aligned}
(\sigma, \sigma', *, *) \in \mathsf{B}_{g;\mathbb{1}_B}(\kappa) \Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_{\mathbb{1}_B}(*))\,;\kappa) \\
& \wedge (*, *) \in \mathsf{B}_{\mathbb{1}_B}(\kappa\,;\mathsf{P}_g(\sigma)) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\partial(\mathbb{1}_B)\,;\kappa) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\mathbb{1}_{\partial(B)}\,;\kappa) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\kappa)
\end{aligned}
$$

□

Proposition A.3. *It is a category.*

Proof. Composition clearly preserves equivalence classes. Composition is associative and unital. □

Lemma A.4. *The monoidal product is a bifunctor.*

Proof. Let $g\colon A \xrightarrow{\Sigma} B$, $h\colon C \xrightarrow{T} D$, $g_1\colon A_1 \xrightarrow{\Sigma_1} B_1$ and $h_1\colon C_1 \xrightarrow{T_1} D_1$ be morphisms in $\mathsf{Game}(\mathsf{C})$.

For the play function, functoriality follows from functoriality of the monoidal product in $\mathsf{Game}(\mathsf{C})$.

$$
\begin{aligned}
\mathsf{P}_{(g;h)\otimes(g_1;h_1)}(\sigma, \tau, \sigma_1, \tau_1) &:= \mathsf{P}_{g;h}(\sigma, \tau) \otimes \mathsf{P}_{g_1;h_1}(\sigma_1, \tau_1) \\
&:= (\mathsf{P}_g(\sigma)\,;\mathsf{P}_h(\tau)) \otimes (\mathsf{P}_{g_1}(\sigma_1)\,;\mathsf{P}_{h_1}(\tau_1)) \\
&= (\mathsf{P}_g(\sigma) \otimes \mathsf{P}_{g_1}(\sigma_1))\,;(\mathsf{P}_h(\tau) \otimes \mathsf{P}_{h_1}(\tau_1)) \\
&=: \mathsf{P}_{g\otimes g_1}(\sigma, \sigma_1)\,;\mathsf{P}_{h\otimes h_1}(\tau, \tau_1) \\
&=: \mathsf{P}_{(g\otimes g_1);(h\otimes h_1)}(\sigma, \sigma_1, \tau, \tau_1)
\end{aligned}
$$

For the best response function, functoriality follows from the feedback axioms, functoriality of ∂ and functoriality of the monoidal product in $\mathsf{Game}(\mathsf{C})$.

$$
\begin{aligned}
& (\sigma,\sigma',\tau,\tau',\sigma_1,\sigma_1',\tau_1,\tau_1') \in \mathsf{B}_{(g;h)\otimes(g_1;h_1)}(\kappa) \\
\Leftrightarrow\quad & (\sigma,\sigma',\tau,\tau') \in \mathsf{B}_{g;h}(\circlearrowleft_{C_1} (\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1;h_1}(\sigma_1,\tau_1)))) \\
& \wedge(\sigma_1,\sigma_1',\tau_1,\tau_1') \in \mathsf{B}_{g_1;h_1}(\circlearrowleft_{C} (\kappa\,;(\mathsf{P}_{g;h}(\sigma,\tau)\otimes \mathbb{1}_{A_1}))) \\
\Leftrightarrow\quad & (\sigma,\sigma') \in \mathsf{B}_{g}(\partial(\mathsf{P}_h(\tau));\circlearrowleft_{C_1} (\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1;h_1}(\sigma_1,\tau_1)))) \\
& \wedge(\tau,\tau') \in \mathsf{B}_{h}(\circlearrowleft_{C_1} (\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1;h_1}(\sigma_1,\tau_1)))\,;\mathsf{P}_g(\sigma)) \\
& \wedge(\sigma_1,\sigma_1') \in \mathsf{B}_{g_1}(\partial(\mathsf{P}_{h_1}(\tau_1));\circlearrowleft_{C} (\kappa\,;(\mathsf{P}_{g;h}(\sigma,\tau)\otimes \mathbb{1}_{A_1}))) \\
& \wedge(\tau_1,\tau_1') \in \mathsf{B}_{h_1}(\circlearrowleft_{C} (\kappa\,;(\mathsf{P}_{g;h}(\sigma,\tau)\otimes \mathbb{1}_{A_1}))\,;\mathsf{P}_{g_1}(\sigma_1)) \\
\Leftrightarrow\quad & (\sigma,\sigma') \in \mathsf{B}_{g}(\circlearrowleft_{B_1} ((\partial(\mathsf{P}_h(\tau))\otimes\partial(\mathsf{P}_{h_1}(\tau_1)))\,;\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1}(\sigma_1)))) \\
& \wedge(\tau,\tau') \in \mathsf{B}_{h}(\circlearrowleft_{C_1} (\kappa\,;(\mathsf{P}_g(\sigma)\otimes \mathsf{P}_{g_1}(\sigma_1))\,;(\mathbb{1}_B \otimes \mathsf{P}_{h_1}(\tau_1)))) \\
& \wedge(\sigma_1,\sigma_1') \in \mathsf{B}_{g_1}(\circlearrowleft_{B} ((\partial(\mathsf{P}_h(\tau))\otimes\partial(\mathsf{P}_{h_1}(\tau_1)))\,;\kappa\,;(\mathsf{P}_g(\sigma)\otimes \mathbb{1}_{A_1}))) \\
& \wedge(\tau_1,\tau_1') \in \mathsf{B}_{h_1}(\circlearrowleft_{C} ((\kappa\,;(\mathsf{P}_g(\sigma)\otimes \mathsf{P}_{g_1}(\sigma_1))\,;(\mathsf{P}_h(\tau)\otimes \mathbb{1}_{B_1}))) \\
\Leftrightarrow\quad & (\sigma,\sigma') \in \mathsf{B}_{g}(\circlearrowleft_{B_1} (\partial(\mathsf{P}_{h\otimes h_1}(\tau,\tau_1))\,;\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1}(\sigma_1)))) \\
& \wedge(\tau,\tau') \in \mathsf{B}_{h}(\circlearrowleft_{C_1} (\kappa\,;\mathsf{P}_{g\otimes g_1}(\sigma,\sigma_1)\,;(\mathbb{1}_B \otimes \mathsf{P}_{h_1}(\tau_1)))) \\
& \wedge(\sigma_1,\sigma_1') \in \mathsf{B}_{g_1}(\circlearrowleft_{B} (\partial(\mathsf{P}_{h\otimes h_1}(\tau,\tau_1))\,;\kappa\,;(\mathsf{P}_g(\sigma)\otimes \mathbb{1}_{A_1}))) \\
& \wedge(\tau_1,\tau_1') \in \mathsf{B}_{h_1}(\circlearrowleft_{C} ((\kappa\,;\mathsf{P}_{g\otimes g_1}(\sigma,\sigma_1)\,;(\mathsf{P}_h(\tau)\otimes \mathbb{1}_{B_1}))) \\
\Leftrightarrow\quad & (\sigma,\sigma',\sigma_1,\sigma_1') \in \mathsf{B}_{g\otimes g_1}(\partial(\mathsf{P}_{h\otimes h_1}(\tau,\tau_1))\,;\kappa) \\
& \wedge(\tau,\tau',\tau_1,\tau_1') \in \mathsf{B}_{h\otimes h_1}(\kappa\,;\mathsf{P}_{g\otimes g_1}(\sigma,\sigma_1)) \\
\Leftrightarrow\quad & (\sigma,\sigma',\sigma_1,\sigma_1',\tau,\tau',\tau_1,\tau_1') \in \mathsf{B}_{(g\otimes g_1);(h\otimes h_1)}(\kappa)
\end{aligned}
$$

□

Lemma A.5. *The monoidal product is associative.*

Proof. The associator $\mathbb{a}$ in $\mathsf{Game}(\mathsf{C})$ is lifted from the associator $\mathbf{a}$ in C.

$$
\mathbb{a}_{A,C,E}\colon A\otimes(C\otimes E) \xrightarrow{1} (A\otimes C)\otimes E
$$
$$
\begin{cases} \mathsf{P}_{\mathbb{a}_{A,C,E}}(*) \coloneqq \mathbf{a}_{A,C,E} \\ \mathsf{B}_{\mathbb{a}_{A,C,E}}(\kappa) \coloneqq \{(*,*)\} \end{cases}
$$

It is an isomorphism because $\mathbf{a}$ is. Let $g\colon A \xrightarrow{\Sigma} B$, $h\colon C \xrightarrow{T} D$ and $l\colon E \xrightarrow{R} F$ be morphisms in $\mathsf{Game}(\mathsf{C})$. In order to prove naturality, we need to check that $(g\otimes(h\otimes l))\,;\mathbb{a}_{B,D,F} = \mathbb{a}_{A,C,E}\,;((g\otimes h)\otimes l)$. For the play function, associativity follows from associativity of the monoidal product in $\mathsf{Game}(\mathsf{C})$.

$$
\begin{aligned}
\mathsf{P}_{(g\otimes(h\otimes l));\mathbb{a}_{B,D,F}}((\sigma,(\tau,\rho)),*) \coloneqq & \mathsf{P}_{g\otimes(h\otimes l)}(\sigma,(\tau,\rho))\,;\mathbf{a}_{B,D,F} \\
\coloneqq & (\mathsf{P}_g(\sigma)\otimes(\mathsf{P}_h(\tau)\otimes \mathsf{P}_l(\rho)))\,;\mathbf{a}_{B,D,F} \\
= & \mathbf{a}_{A,C,E}\,;((\mathsf{P}_g(\sigma)\otimes \mathsf{P}_h(\tau))\otimes \mathsf{P}_l(\rho)) \\
=: & \mathsf{P}_{\mathbb{a}_{A,C,E};((g\otimes h)\otimes l)}(*,((\sigma,\tau),\rho))
\end{aligned}
$$

For the best response function, associativity follows from the feedback axioms, the fact that ∂ us strict monoidal and associativity in C.

$$
\begin{aligned}
& (\sigma, \sigma', \tau, \tau', \rho, \rho', *, *) \in \mathsf{B}_{(g \otimes (h \otimes l)) ; \mathbb{a}_{B,D,F}}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{g \otimes (h \otimes l)}(\partial(\mathbf{a}_{B,D,F}) ; \mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa) \\
& \wedge (*, *) \in \mathsf{B}_{\mathbb{a}_{B,D,F}}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathsf{P}_{g \otimes (h \otimes l)}(\sigma, \tau, \rho)) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{g \otimes (h \otimes l)}(\kappa) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\circlearrowleft_{D \otimes F} (\kappa ; (\mathbb{1}_A \otimes \mathsf{P}_{h \otimes l}(\tau, \rho)))) \\
& \wedge (\tau, \tau', \rho, \rho') \in \mathsf{B}_{h \otimes l}(\circlearrowleft_B (\kappa ; (\mathsf{P}_g(\sigma) \otimes \mathbb{1}_{C \otimes E}))) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\circlearrowleft_D (\circlearrowleft_F (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; (\mathbb{1}_A \otimes (\mathbb{1}_C \otimes \mathsf{P}_l(\rho))) ; \mathbf{a}_{A,C,F}) ; (\mathbb{1}_A \otimes \mathsf{P}_h(\tau)))) \\
& \wedge (\tau, \tau') \in \mathsf{B}_h(\circlearrowleft_F (\circlearrowleft_B (\kappa ; (\mathsf{P}_g(\sigma) \otimes \mathbb{1}_{C \otimes E})) ; (\mathbb{1}_C \otimes \mathsf{P}_l(\rho)))) \\
& \wedge (\rho, \rho') \in \mathsf{B}_l(\circlearrowleft_D (\circlearrowleft_B (\kappa ; (\mathsf{P}_g(\sigma) \otimes \mathbb{1}_{C \otimes E})) ; (\mathsf{P}_h(\tau) \otimes \mathbb{1}_E))) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\circlearrowleft_D (\circlearrowleft_F (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; (\mathbb{1}_{A \otimes C} \otimes \mathsf{P}_l(\rho))) ; (\mathbb{1}_A \otimes \mathsf{P}_h(\tau)))) \\
& \wedge (\tau, \tau') \in \mathsf{B}_h(\circlearrowleft_B (\circlearrowleft_F (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; (\mathbb{1}_{A \otimes C} \otimes \mathsf{P}_l(\rho))) ; (\mathsf{P}_g(\sigma) \otimes \mathbb{1}_C))) \\
& \wedge (\rho, \rho') \in \mathsf{B}_l(\circlearrowleft_{B \otimes D} (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; ((\mathsf{P}_g(\sigma) \otimes \mathsf{P}_h(\tau)) \otimes \mathbb{1}_E))) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau') \in \mathsf{B}_{g \otimes h}(\circlearrowleft_F (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; (\mathbb{1}_{A \otimes C} \otimes \mathsf{P}_l(\rho)))) \\
& \wedge (\rho, \rho') \in \mathsf{B}_l(\circlearrowleft_{B \otimes D} (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; (\mathsf{P}_{g \otimes h}(\sigma, \tau) \otimes \mathbb{1}_E))) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{(g \otimes h) \otimes l}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E}) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{(g \otimes h) \otimes l}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E}) \\
& \wedge (*, *) \in \mathsf{B}_{\mathbb{a}_{A,C,E}}(\partial(\mathsf{P}_{(g \otimes h) \otimes l}(\sigma, tau, \rho)) ; \mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa) \\
\Leftrightarrow \quad & (*, *, \sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{\mathbb{a}_{A,C,E} ; ((g \otimes h) \otimes l)}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa)
\end{aligned}
$$

□

Lemma A.6. *The monoidal product is unital.*

Proof. □

Lemma A.7. *The triangle and pentagon equations hold.*

Proof. □