Open games on categories with feedback

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1 Introduction

Disclaimer. This is a very preliminary draft and all the references are missing (as well as some proofs). This work is based on Jules Hedges' open games and categories with feedback as in Katis, Sabadini and Walters (as well as our previous work on them).

We want to extend the construction of open games to any category with feedback C.

2 Definition

2.1 The category Game(C)

We consider a symmetric monoidal category C with feedback $\circlearrowleft\colon \mathsf{C}(A \otimes \partial S, B \otimes S) \to \mathsf{C}(A, B)$, where $\partial\colon \mathsf{C} \to \mathsf{C}$ is a faithful strict monoidal endofunctor on C that preserves feedback (it is a feedback functor). We think of C as the category of 'actions' or 'things that happen'. The category $\mathsf{Game}(\mathsf{C})$ then gives the actions that can happen depending on a choice of a parameter (which we think of as strategy or policy).

Intuitively, morphisms are (equivalence relations of) families of morphisms in C parametrized by a set Σ together with a best response function.

Definition 2.1 (The category $\mathsf{Game}(\mathsf{C})$)**.** Objects are the objects of C. A morphism $g\colon A \xrightarrow{\Sigma} B$ in $\mathsf{Game}(\mathsf{C})$ is given by:

- A *play function* $\mathsf{P}_g\colon \Sigma \to \mathsf{C}(A, B)$.
- A *best response function* $\mathsf{B}_g\colon \mathsf{C}(\partial(B), A) \to \mathit{Rel}(\Sigma)$.

Two morphisms, $g\colon A \xrightarrow{\Sigma} B$ and $g'\colon A \xrightarrow{\Sigma'} B$, are considered equal whenever there is an isomorphism of sets $s\colon \Sigma \cong \Sigma'$ such that, for every $\sigma, \sigma' \in \Sigma$,

- $\mathsf{P}_g(\sigma) = \mathsf{P}_{g'}(s(\sigma))$
- $(\sigma, \sigma') \in \mathsf{B}_g(\kappa) \Leftrightarrow (s(\sigma), s(\sigma')) \in \mathsf{B}_{g'}(\kappa)$

We think of Σ as the set of possible strategies for a game. Choosing the strategy gives a way of playing the game, which is represented with a morphism in C. The strategy is chosen depending on the context of the game: for every context the best response function returns a relation of preference over the set of strategies Σ. The fixpoint of this relation gives, given the context, the preferred strategy.

Definition 2.2 (Categorical structure)**.** Let $g\colon A \xrightarrow{\Sigma} B$ and $h\colon B \xrightarrow{T} C$. Their composition $g\,;h\colon A \xrightarrow{\Sigma\times T} C$ is given by:

- $\mathsf{P}_{g;h}(\sigma,\tau) \coloneqq \mathsf{P}_g(\sigma)\,;\mathsf{P}_h(\tau)$.
- $\mathsf{B}_{g;h}(\kappa) \coloneqq \{(\sigma,\sigma',\tau,\tau') \in (\Sigma\times T)^2 : (\sigma,\sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_h(\tau))\,;\kappa) \wedge (\tau,\tau') \in \mathsf{B}_h(\kappa\,;\mathsf{P}_g(\sigma))\}$

$\mathbb{1}_A\colon A \xrightarrow{1} A$ is given by:

- $\mathsf{P}_{\mathbb{1}_A}(*) \coloneqq \mathbb{1}_A$.
- $\mathsf{B}_{\mathbb{1}_A}(\kappa) \coloneqq \{(*,*)\}$.

2.2 Monoidal structure

On objects, the monoidal product coincides with the monoidal product in C. On morphisms, it is lifted from C. Let $g\colon A \xrightarrow{\Sigma} B$ and $h\colon C \xrightarrow{T} D$ be two morphisms in $\mathsf{Game}(\mathsf{C})$. Their monoidal product $g\otimes h\colon A\otimes C \xrightarrow{\Sigma\times T} B\otimes D$ is given by:

- $\mathsf{P}_{g\otimes h}(\sigma,\tau) \coloneqq \mathsf{P}_g(\sigma)\otimes\mathsf{P}_h(\tau)$.
- $\mathsf{B}_{g\otimes h}(\kappa) \coloneqq \{(\sigma,\sigma',\tau,\tau') \in (\Sigma\times T)^2 : (\sigma,\sigma') \in \mathsf{B}_g(\circlearrowleft_D (\kappa\,;(\mathbb{1}_A\otimes\mathsf{P}_h(\tau)))) \wedge (\tau,\tau') \in \mathsf{B}_h(\circlearrowleft_B (\kappa\,;(\mathsf{P}_g(\sigma)\otimes\mathbb{1}_C)))\}$

2.3 Feedback operator

Definition 2.3 (Delay functor)**.** On objects, the delay functor coincides with the delay functor in C. On morphisms, it is lifted from C. Let $g\colon A \xrightarrow{\Sigma} B$ be a morphism in $\mathsf{Game}(\mathsf{C})$. Then $\partial(g)\colon \partial(A) \xrightarrow{\Sigma} \partial(B)$ is given by:

- $\mathsf{P}_{\partial(g)}(\sigma) \coloneqq \partial(\mathsf{P}_g(\sigma))$.
- $\mathsf{B}_{\partial(g)}(\partial(\kappa)) \coloneqq \mathsf{B}_g(\kappa)$. This gives the definition of the best response function because ∂ is faithful.

Definition 2.4 (Feedback operator)**.** Let $g\colon \partial S\otimes A \xrightarrow{\Sigma} S\otimes B$ be a morphism in $\mathsf{Game}(\mathsf{C})$. Then $\circlearrowleft_S g\colon A \xrightarrow{\Sigma} B$ is given by:

- $\mathsf{P}_{\circlearrowleft_S g}(\sigma) \coloneqq \circlearrowleft_S (\mathsf{P}_g(\sigma))$.
- $\mathsf{B}_{\circlearrowleft_S g}(\kappa) \coloneqq \mathsf{B}_g(\kappa\otimes\mathbb{1}_{\partial S})$.

A Proofs

Lemma A.1. *Composition is associative*

Proof. Let $g\colon A \xrightarrow{\Sigma} B$, $h\colon B \xrightarrow{T} C$ and $l\colon C \xrightarrow{R} D$ be morphisms in $\mathsf{Game}(\mathsf{C})$. For the play function, associativity follows from associativity in C.

$$\begin{aligned}
\mathsf{P}_{g;(h;l)}(\sigma,\tau,\rho) &:= \mathsf{P}_g(\sigma)\,;(\mathsf{P}_h(\tau)\,;\mathsf{P}_l(\rho))\\
&= (\mathsf{P}_g(\sigma)\,;\mathsf{P}_h(\tau))\,;\mathsf{P}_l(\rho)\\
&=: \mathsf{P}_{(g;h);l}(\sigma,\tau,\rho)
\end{aligned}$$

For the best response function, associativity follows from functoriality of ∂ and associativity in C.

$$\begin{aligned}
(\sigma,\sigma',\tau,\tau',\rho,\rho') \in \mathsf{B}_{g;(h;l)}(\kappa) \Leftrightarrow\ & \quad (\sigma,\sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_{h;l}(\tau,\rho))\,;\kappa)\\
& \wedge(\tau,\tau',\rho,\rho') \in \mathsf{B}_{h;l}(\kappa\,;\mathsf{P}_g(\sigma))\\
\Leftrightarrow\ & \quad (\sigma,\sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_h(\tau)\,;\mathsf{P}_l(\rho))\,;\kappa)\\
& \wedge(\tau,\tau') \in \mathsf{B}_h(\partial(\mathsf{P}_l(\rho))\,;\kappa\,;\mathsf{P}_g(\sigma))\\
& \wedge(\rho,\rho') \in \mathsf{B}_l(\kappa\,;\mathsf{P}_g(\sigma)\,;\mathsf{P}_h(\tau))\\
\Leftrightarrow\ & \quad (\sigma,\sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_h(\tau))\,;\partial(\mathsf{P}_l(\rho))\,;\kappa)\\
& \wedge(\tau,\tau') \in \mathsf{B}_h(\partial(\mathsf{P}_l(\rho))\,;\kappa\,;\mathsf{P}_g(\sigma))\\
& \wedge(\rho,\rho') \in \mathsf{B}_l(\kappa\,;\mathsf{P}_{g;h}(\sigma,\tau))\\
\Leftrightarrow\ & \quad (\sigma,\sigma',\tau,\tau') \in \mathsf{B}_{g;h}(\partial(\mathsf{P}_l(\rho))\,;\kappa)\\
& \wedge(\rho,\rho') \in \mathsf{B}_l(\kappa\,;\mathsf{P}_{g;h}(\sigma,\tau))\\
\Leftrightarrow\ & \quad (\sigma,\sigma',\tau,\tau',\rho,\rho') \in \mathsf{B}_{(g;h);l}(\kappa)
\end{aligned}$$

□

Lemma A.2. *Composition is unital.*

Proof. Let $g\colon A \xrightarrow{\Sigma} B$ be a morphism in $\mathsf{Game}(\mathsf{C})$. For the play function, unitality follows from unitality in C.

$$\mathsf{P}_{\mathbb{1}_A;g}(*,\sigma) := \mathsf{P}_{\mathbb{1}_A}(*)\,;\mathsf{P}_g(\sigma) = \mathbb{1}_A\,;\mathsf{P}_g(\sigma) = \mathsf{P}_g(\sigma)$$

$$\mathsf{P}_{g;\mathbb{1}_B}(\sigma,*) := \mathsf{P}_g(\sigma)\,;\mathsf{P}_{\mathbb{1}_B}(*) = \mathsf{P}_g(\sigma)\,;\mathbb{1}_B = \mathsf{P}_g(\sigma)$$

For the best response function, unitality follows from functoriality of ∂ and unitality in C.

$$\begin{aligned}
(*,*,\sigma,\sigma') \in \mathsf{B}_{\mathbb{1}_A;g}(\kappa) \Leftrightarrow\ & \quad (*,*) \in \mathsf{B}_{\mathbb{1}_A}(\partial(\mathsf{P}_g(\sigma))\,;\kappa)\\
& \wedge(\sigma,\sigma') \in \mathsf{B}_g(\kappa\,;\mathsf{P}_{\mathbb{1}_A}(*))\\
\Leftrightarrow\ & \quad (\sigma,\sigma') \in \mathsf{B}_g(\kappa\,;\mathbb{1}_A)\\
\Leftrightarrow\ & \quad (\sigma,\sigma') \in \mathsf{B}_g(\kappa)
\end{aligned}$$

$$
\begin{aligned}
(\sigma, \sigma', *, *) \in \mathsf{B}_{g;\mathbb{1}_B}(\kappa) \Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\partial(\mathsf{P}_{\mathbb{1}_B}(*))\,;\kappa) \\
& \wedge (*, *) \in \mathsf{B}_{\mathbb{1}_B}(\kappa\,;\mathsf{P}_g(\sigma)) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\partial(\mathbb{1}_B)\,;\kappa) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\mathbb{1}_{\partial(B)}\,;\kappa) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\kappa)
\end{aligned}
$$

□

Proposition A.3. *It is a category.*

Proof. Composition clearly preserves equivalence classes. Composition is associative and unital. □

Lemma A.4. *The monoidal product is a functor.*

Proof. Let $g\colon A \xrightarrow{\Sigma} B$, $h\colon C \xrightarrow{T} D$, $g_1\colon A_1 \xrightarrow{\Sigma_1} B_1$ and $h_1\colon C_1 \xrightarrow{T_1} D_1$ be morphisms in $\mathsf{Game}(\mathsf{C})$.

For the play function, functoriality follows from functoriality of the monoidal product in $\mathsf{Game}(\mathsf{C})$.

$$
\begin{aligned}
\mathsf{P}_{(g;h)\otimes(g_1;h_1)}(\sigma, \tau, \sigma_1, \tau_1) &:= \mathsf{P}_{g;h}(\sigma, \tau) \otimes \mathsf{P}_{g_1;h_1}(\sigma_1, \tau_1) \\
&:= (\mathsf{P}_g(\sigma)\,;\mathsf{P}_h(\tau)) \otimes (\mathsf{P}_{g_1}(\sigma_1)\,;\mathsf{P}_{h_1}(\tau_1)) \\
&= (\mathsf{P}_g(\sigma) \otimes \mathsf{P}_{g_1}(\sigma_1))\,;(\mathsf{P}_h(\tau) \otimes \mathsf{P}_{h_1}(\tau_1)) \\
&=: \mathsf{P}_{g\otimes g_1}(\sigma, \sigma_1)\,;\mathsf{P}_{h\otimes h_1}(\tau, \tau_1) \\
&=: \mathsf{P}_{(g\otimes g_1);(h\otimes h_1)}(\sigma, \sigma_1, \tau, \tau_1)
\end{aligned}
$$

For the best response function, functoriality follows from the feedback axioms, functoriality of ∂ and functoriality of the monoidal product in $\mathsf{Game}(\mathsf{C})$.

$$\begin{aligned}
& (\sigma,\sigma',\tau,\tau',\sigma_1,\sigma_1',\tau_1,\tau_1') \in \mathsf{B}_{(g;h)\otimes(g_1;h_1)}(\kappa) \\
\Leftrightarrow \quad & (\sigma,\sigma',\tau,\tau') \in \mathsf{B}_{g;h}(\circlearrowleft_{C_1} (\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1;h_1}(\sigma_1,\tau_1)))) \\
& \wedge(\sigma_1,\sigma_1',\tau_1,\tau_1') \in \mathsf{B}_{g_1;h_1}(\circlearrowleft_{C} (\kappa\,;(\mathsf{P}_{g;h}(\sigma,\tau) \otimes \mathbb{1}_{A_1}))) \\
\Leftrightarrow \quad & (\sigma,\sigma') \in \mathsf{B}_{g}(\partial(\mathsf{P}_h(\tau));\circlearrowleft_{C_1} (\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1;h_1}(\sigma_1,\tau_1)))) \\
& \wedge(\tau,\tau') \in \mathsf{B}_{h}(\circlearrowleft_{C_1} (\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1;h_1}(\sigma_1,\tau_1)))\,;\mathsf{P}_g(\sigma)) \\
& \wedge(\sigma_1,\sigma_1') \in \mathsf{B}_{g_1}(\partial(\mathsf{P}_{h_1}(\tau_1));\circlearrowleft_{C} (\kappa\,;(\mathsf{P}_{g;h}(\sigma,\tau) \otimes \mathbb{1}_{A_1}))) \\
& \wedge(\tau_1,\tau_1') \in \mathsf{B}_{h_1}(\circlearrowleft_{C} (\kappa\,;(\mathsf{P}_{g;h}(\sigma,\tau) \otimes \mathbb{1}_{A_1}))\,;\mathsf{P}_{g_1}(\sigma_1)) \\
\Leftrightarrow \quad & (\sigma,\sigma') \in \mathsf{B}_{g}(\circlearrowleft_{B_1} ((\partial(\mathsf{P}_h(\tau)) \otimes \partial(\mathsf{P}_{h_1}(\tau_1)))\,;\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1}(\sigma_1)))) \\
& \wedge(\tau,\tau') \in \mathsf{B}_{h}(\circlearrowleft_{C_1} (\kappa\,;(\mathsf{P}_g(\sigma) \otimes \mathsf{P}_{g_1}(\sigma_1))\,;(\mathbb{1}_B \otimes \mathsf{P}_{h_1}(\tau_1)))) \\
& \wedge(\sigma_1,\sigma_1') \in \mathsf{B}_{g_1}(\circlearrowleft_{B} ((\partial(\mathsf{P}_h(\tau)) \otimes \partial(\mathsf{P}_{h_1}(\tau_1)))\,;\kappa\,;(\mathsf{P}_g(\sigma) \otimes \mathbb{1}_{A_1}))) \\
& \wedge(\tau_1,\tau_1') \in \mathsf{B}_{h_1}(\circlearrowleft_{C} ((\kappa\,;(\mathsf{P}_g(\sigma) \otimes \mathsf{P}_{g_1}(\sigma_1))\,;(\mathsf{P}_h(\tau) \otimes \mathbb{1}_{B_1}))) \\
\Leftrightarrow \quad & (\sigma,\sigma') \in \mathsf{B}_{g}(\circlearrowleft_{B_1} (\partial(\mathsf{P}_{h\otimes h_1}(\tau,\tau_1))\,;\kappa\,;(\mathbb{1}_A \otimes \mathsf{P}_{g_1}(\sigma_1)))) \\
& \wedge(\tau,\tau') \in \mathsf{B}_{h}(\circlearrowleft_{C_1} (\kappa\,;\mathsf{P}_{g\otimes g_1}(\sigma,\sigma_1)\,;(\mathbb{1}_B \otimes \mathsf{P}_{h_1}(\tau_1)))) \\
& \wedge(\sigma_1,\sigma_1') \in \mathsf{B}_{g_1}(\circlearrowleft_{B} (\partial(\mathsf{P}_{h\otimes h_1}(\tau,\tau_1))\,;\kappa\,;(\mathsf{P}_g(\sigma) \otimes \mathbb{1}_{A_1}))) \\
& \wedge(\tau_1,\tau_1') \in \mathsf{B}_{h_1}(\circlearrowleft_{C} ((\kappa\,;\mathsf{P}_{g\otimes g_1}(\sigma,\sigma_1)\,;(\mathsf{P}_h(\tau) \otimes \mathbb{1}_{B_1}))) \\
\Leftrightarrow \quad & (\sigma,\sigma',\sigma_1,\sigma_1') \in \mathsf{B}_{g\otimes g_1}(\partial(\mathsf{P}_{h\otimes h_1}(\tau,\tau_1))\,;\kappa) \\
& \wedge(\tau,\tau',\tau_1,\tau_1') \in \mathsf{B}_{h\otimes h_1}(\kappa\,;\mathsf{P}_{g\otimes g_1}(\sigma,\sigma_1)) \\
\Leftrightarrow \quad & (\sigma,\sigma',\sigma_1,\sigma_1',\tau,\tau',\tau_1,\tau_1') \in \mathsf{B}_{(g\otimes g_1);(h\otimes h_1)}(\kappa)
\end{aligned}$$

$\square$

Lemma A.5. *The monoidal product is associative.*

Proof. The associator $\mathbb{a}$ in $\mathsf{Game}(\mathsf{C})$ is lifted from the associator $\mathbf{a}$ in C.

$$\mathbb{a}_{A,C,E} \colon A \otimes (C \otimes E) \xrightarrow{1} (A \otimes C) \otimes E$$
$$\begin{cases} \mathsf{P}_{\mathbb{a}_{A,C,E}}(*) \coloneqq \mathbf{a}_{A,C,E} \\ \mathsf{B}_{\mathbb{a}_{A,C,E}}(\kappa) \coloneqq \{(*,*)\} \end{cases}$$

It is an isomorphism because $\mathbf{a}$ is. Let $g\colon A \xrightarrow{\Sigma} B$, $h\colon C \xrightarrow{T} D$ and $l\colon E \xrightarrow{R} F$ be morphisms in $\mathsf{Game}(\mathsf{C})$. In order to prove naturality, we need to check that $(g\otimes(h\otimes l))\,;\mathbb{a}_{B,D,F} = \mathbb{a}_{A,C,E}\,;((g\otimes h)\otimes l)$. For the play function, associativity follows from associativity of the monoidal product in $\mathsf{Game}(\mathsf{C})$.

$$\begin{aligned}
\mathsf{P}_{(g\otimes(h\otimes l));\mathbb{a}_{B,D,F}}((\sigma,(\tau,\rho)),*) &\coloneqq \mathsf{P}_{g\otimes(h\otimes l)}(\sigma,(\tau,\rho))\,;\mathbf{a}_{B,D,F} \\
&\coloneqq (\mathsf{P}_g(\sigma) \otimes (\mathsf{P}_h(\tau) \otimes \mathsf{P}_l(\rho)))\,;\mathbf{a}_{B,D,F} \\
&= \mathbf{a}_{A,C,E}\,;((\mathsf{P}_g(\sigma) \otimes \mathsf{P}_h(\tau)) \otimes \mathsf{P}_l(\rho)) \\
&=: \mathsf{P}_{\mathbb{a}_{A,C,E};((g\otimes h)\otimes l)}(*,((\sigma,\tau),\rho))
\end{aligned}$$

For the best response function, associativity follows from the feedback axioms, the fact that ∂ us strict monoidal and associativity in C.

$$
\begin{aligned}
& (\sigma, \sigma', \tau, \tau', \rho, \rho', *, *) \in \mathsf{B}_{(g \otimes (h \otimes l)) ; \mathbb{a}_{B,D,F}}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{g \otimes (h \otimes l)}(\partial(\mathbf{a}_{B,D,F}) ; \mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa) \\
& \wedge (*, *) \in \mathsf{B}_{\mathbb{a}_{B,D,F}}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathsf{P}_{g \otimes (h \otimes l)}(\sigma, \tau, \rho)) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{g \otimes (h \otimes l)}(\kappa) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\circlearrowleft_{D \otimes F} (\kappa ; (\mathbb{1}_A \otimes \mathsf{P}_{h \otimes l}(\tau, \rho)))) \\
& \wedge (\tau, \tau', \rho, \rho') \in \mathsf{B}_{h \otimes l}(\circlearrowleft_B (\kappa ; (\mathsf{P}_g(\sigma) \otimes \mathbb{1}_{C \otimes E}))) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\circlearrowleft_D (\circlearrowleft_F (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; (\mathbb{1}_A \otimes (\mathbb{1}_C \otimes \mathsf{P}_l(\rho))) ; \mathbf{a}_{A,C,F}) ; (\mathbb{1}_A \otimes \mathsf{P}_h(\tau)))) \\
& \wedge (\tau, \tau') \in \mathsf{B}_h(\circlearrowleft_F (\circlearrowleft_B (\kappa ; (\mathsf{P}_g(\sigma) \otimes \mathbb{1}_{C \otimes E})) ; (\mathbb{1}_C \otimes \mathsf{P}_l(\rho)))) \\
& \wedge (\rho, \rho') \in \mathsf{B}_l(\circlearrowleft_D (\circlearrowleft_B (\kappa ; (\mathsf{P}_g(\sigma) \otimes \mathbb{1}_{C \otimes E})) ; (\mathsf{P}_h(\tau) \otimes \mathbb{1}_E))) \\
\Leftrightarrow \quad & (\sigma, \sigma') \in \mathsf{B}_g(\circlearrowleft_D (\circlearrowleft_F (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; (\mathbb{1}_{A \otimes C} \otimes \mathsf{P}_l(\rho))) ; (\mathbb{1}_A \otimes \mathsf{P}_h(\tau)))) \\
& \wedge (\tau, \tau') \in \mathsf{B}_h(\circlearrowleft_B (\circlearrowleft_F (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; (\mathbb{1}_{A \otimes C} \otimes \mathsf{P}_l(\rho))) ; (\mathsf{P}_g(\sigma) \otimes \mathbb{1}_C))) \\
& \wedge (\rho, \rho') \in \mathsf{B}_l(\circlearrowleft_{B \otimes D} (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; ((\mathsf{P}_g(\sigma) \otimes \mathsf{P}_h(\tau)) \otimes \mathbb{1}_E))) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau') \in \mathsf{B}_{g \otimes h}(\circlearrowleft_F (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; (\mathbb{1}_{A \otimes C} \otimes \mathsf{P}_l(\rho)))) \\
& \wedge (\rho, \rho') \in \mathsf{B}_l(\circlearrowleft_{B \otimes D} (\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E} ; (\mathsf{P}_{g \otimes h}(\sigma, \tau) \otimes \mathbb{1}_E))) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{(g \otimes h) \otimes l}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E}) \\
\Leftrightarrow \quad & (\sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{(g \otimes h) \otimes l}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa ; \mathbf{a}_{A,C,E}) \\
& \wedge (*, *) \in \mathsf{B}_{\mathbb{a}_{A,C,E}}(\partial(\mathsf{P}_{(g \otimes h) \otimes l}(\sigma, tau, \rho)) ; \mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa) \\
\Leftrightarrow \quad & (*, *, \sigma, \sigma', \tau, \tau', \rho, \rho') \in \mathsf{B}_{\mathbb{a}_{A,C,E} ; ((g \otimes h) \otimes l)}(\mathbf{a}^{-1}_{\partial(B), \partial(D), \partial(F)} ; \kappa)
\end{aligned}
$$

□

Lemma A.6. *The monoidal product is unital.*

Proof. □

Lemma A.7. *The triangle and pentagon equations hold.*

Proof. □