Chapter 3
Axiomatic Value Theory

So far we have focussed our discussion on the Core and related set theoretic solution concepts as concepts that address the fundamental problem of cooperative game theory—the identification of stable binding agreements between the participating players. As such these set theoretic solution concepts consist of allocations that satisfy some fundamental properties of negotiating power of coalitions and, consequently, are founded on a description of the bargaining power of the coalitions of players.

In set theoretic solution concepts such as the Core and the Weber set a collection of allocations is identified that is in many ways "acceptable" from certain points of view. In this regard these allocations represent a certain *standard of behavior*.[1]

The development of the Core and related concepts as standards of behavior can be viewed as an attempt to simplify or reduce the fundamental problem of cooperative game theory. The next step in this reductionist process is to introduce multiple standards of behavior that a solution has to satisfy. This was seminally pursued by Shapley (1953) in his ground breaking contribution, establishing the field of *value theory*. He introduced three axioms, each essentially describing a simple behavioral rule or property. Subsequently he showed that the collection of allocations satisfying these three axioms forms a singleton for *every* cooperative game. Thus, he arrived at the notion of an "axiomatic value"; a single-valued solution concept that is guaranteed to exist for every game and which is completely characterized by a given set of behavioral axioms.

Since then cooperative game theory has developed very rapidly as an axiomatic theory of single valued solution concepts for situations represented by games in characteristic function form. The Shapley value in particular has been the subject of

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[1] The notion of a standard of behavior was already introduced by von Neumann and Morgenstern (1953) and resulted into the notion of the *vNM solution*. The vNM solution is a set of allocations that are stable with regard to the standard of behavior introduced by von Neumann and Morgenstern (1953). A standard of behavior can be used as the foundation of an alternative approach to interactive decisions in general. Greenberg (1990) pursued this further and used the foundations of the vNM solution as the basis of his theory. I will not further go into this matter and the vNM solution concept. I refer to Greenberg (1990) and Owen (1995, Chapter XI) for an elaborate discussion.

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extensive debate and a wide variety of axiomatizations of the Shapley value have been developed during the past fifty years.

The fundamental notion of a value is introduced through the next definition as a function that assigns to every game in characteristic function form exactly one allocation.

Definition 3.1 A *value* is a function $\phi\colon \mathcal{G}^N \to \mathbb{R}^N$ that assigns to every cooperative game v a single allocation $\phi(v) \in \mathbb{R}^N$.

In this chapter we consider in particular so-called axiomatic values. A value is axiomatic if there exist a finite set of well-defined properties or *axioms* that fully characterize it. This means that the given axioms are independent in the sense that they do not imply each other and that the set of chosen axioms exactly generates the formulated value for every cooperative game. Such sets of characterizing axioms for a particular value are also known as *axiomatizations* of that value.

Subsequently the literature on cooperative game theory developed more axiomatic values by selecting different sets of axioms. Next game theorists developed axiomatic values on certain specific classes of cooperative games $\mathcal{H} \subset \mathcal{G}^N$. If restrictions on the domain of a value are applied, one usually indicates these solutions as "indices" rather than values. Here I mention two examples of indices on the class of simple cooperative games: The Shapley-Shubik index (Shapley and Shubik, 1954) and the Banzhaf power index (Banzhaf, 1965, 1968).

3.1 Equivalent Formulations of the Shapley Value

The seminal axiomatic value introduced by Shapley (1953) has had a very special appeal. First, Shapley characterized this value through three very straightforward and appealing axioms. Second, the value's computation is rather straightforward and can be done in multiple ways, including some rather appealing computational methods. Third, this value has a wide ranging applicability as shown by subsequent work. These three reasons make the Shapley value a lasting contribution to the theory of cooperation in interactive decision situations.

There have been developed numerous formulations of the Shapley value. In this short section I introduce the main formulations and present them independently from the axiomatizations that have been proposed by different authors.

I first discuss computationally the most convenient formulation to describe the Shapley value.

Definition 3.2 The *Shapley value* is the value function $\varphi\colon \mathcal{G}^N \to \mathbb{R}^N$, which for every player $i \in N$ is given by

$$\varphi_i(v) = \sum_{S\subset N\colon\, i\in S} \frac{\Delta_v(S)}{|S|} \tag{3.1}$$

If one considers a Harsanyi dividend to be the true productive value that a coalition generates, then the Shapley value is just the fair division of all these dividends among the members of the value-generating coalitions.

From this formulation it is already clear that the Shapley value combines power, feasibility and fairness. Indeed, a coalition is assigned its dividend, which represents its power as well as the collective value that is feasible for a coalition to generate. The equal division of the dividend over the members of the value-generating coalition refers to the fairness component in the Shapley value.

To illustrate the computational ease of this formulation we look at a simple example.

Example 3.3 Consider $N = \{1,2,3\}$. Consider that these three players are countries in a negotiation procedure. Country 1 has 5 votes, Country 2 has 4 votes, and Country 3 has 1 vote. To "win" a coalition of countries needs 6 votes. We give winning a value of 100, but consensus gives an additional payoff of 40. How should the honor points of a total $100 + 40 = 140$ that are attained in consensus be divided?

Formally we let v on N be given in the following table. This table also includes the Harsanyi dividends for every coalition in this game.

S	$\varnothing$	1	2	3	12	13	23	123
$v(S)$	0	0	0	0	100	100	0	140
$\Delta_v(S)$	0	0	0	0	100	100	0	-60

The Shapley value for this game is now given by

$$\varphi_1(v) = \frac{100}{2} + \frac{100}{2} + \frac{-60}{3} = 80$$
$$\varphi_2(v) = \frac{100}{2} + \frac{-60}{3} = 30$$
$$\varphi_2(v) = \frac{100}{2} + \frac{-60}{3} = 30$$

It is clear that Country 1 is the most powerful country in this negotiation game. However, surprisingly Country 2 is apparently as weak as Country 3 according to their respective Shapley values. This is due to the problem that Country 2 does not have enough votes to block Country 1 by cooperating with Country 3.

As a remark, I mention that the Core of this voting situation is given by a convex polyhedron with four corner points

$$C(v) = \text{Conv}\,\{(140,0,0),\ (100,40,0),\ (100,0,40),\ (60,40,40)\}.$$

In the Core, the powerful position of Country 1 is clearly depicted. In this case the Shapley value is in the Core, although it is not equal to its center given by (100,20,20). ■

Next consider the trade example that I developed in Examples 1.2, 1.5, and 1.9.

Example 3.4 I refer to Example 1.9 for the following description of the three person utilitarian trade game v_u:

S	$\varnothing$	1	2	3	12	13	23	123
$v_u(S)$	0	0	0	0	100	150	0	150
$\Delta_{v_u}(S)$	0	0	0	0	100	150	0	-100

The Shapley value for this game v_u is now given by

$$\begin{aligned}\varphi_1(v_u) &= \frac{100}{2} + \frac{150}{2} + \frac{-100}{3} = 91\tfrac{2}{3}\\ \varphi_2(v_u) &= \frac{100}{2} + \frac{-100}{3} = 16\tfrac{2}{3}\\ \varphi_2(v_u) &= \frac{150}{2} + \frac{-100}{3} = 41\tfrac{2}{3}\end{aligned}$$

In this trade example it is interesting to see that player 2 is assigned a positive payoff in the Shapley value. This is due to the fact that player 1 can trade with player 2 and generate a positive surplus. This gives player 2 a fair claim on the dividend of the trade coalition 12. This results in the Shapley value computed.

As a comparison, the Core of v_u can be determined as

$$C(v_u) = \left\{ (x_1, x_2, x_3) \,\middle|\, 100 \leqq x_1 \leqq 150,\; x_2 = 0,\; x_3 = 150 - x_1 \right\}$$

Certainly $\varphi(v_u) \notin C(v_u)$. The Core essentially reflects that player 2 is completely powerless, while as pointed out above, player 2 is still recognized in the Shapley value. ■

The relationship between the Shapley value and the Core of a cooperative game is a rather interesting point of discussion. Above I discussed examples in which the Shapley value is in the Core (Example 3.3) and the Shapley value is outside a non-empty Core (Example 3.4). The next example discusses a game with an empty Core. In that case, the Shapley value is well-defined, but is situated separately from the Core.

Example 3.5 Again consider $N = \{1, 2, 3\}$. The generated values of the various coalitions in this three-player game are depicted in the next table:

S	$\varnothing$	1	2	3	12	13	23	123
$v(S)$	0	6	0	0	10	10	10	15
$\Delta_v(S)$	0	6	0	0	4	4	10	-9

From the given coalitional values, it is clear that the Core is indeed empty, i.e., $C(v) = \varnothing$. This is due to the singular, individual claim of 6 units that player 1 can hold out for in conjunction with the claim of coalition 23 of 10 units; both cannot be satisfied from the total generated 15 units. On the other hand, the Shapley value of this particular game is given by $\varphi(v) = (7, 4, 4)$.

The difference between the Core and the Shapley value here is that the Core only considers the negotiation power of the various coalitions. In this game, the 2-player coalitions 12, 13, and 23 have conflicting powers, resulting in the emptiness of the Core. The Shapley value divides the generated Harsanyi dividends regardless of the bargaining power of these coalitions, resulting into a plausible imputation. ■

The formulation of the Shapley value that I give in (3.1) is directly related to the selectors and the allocations based on these selectors, underlying the Selectope. The following result is a straightforward consequence of the definitions of selector allocations and the Shapley value.

Corollary 3.6 *The Shapley value $\varphi(v)$ of $v \in \mathcal{G}^N$ is the average of all selector allocations of v, i.e.,*

$$\varphi_i(v) = \frac{1}{|A_N|} \sum_{\alpha \in A_N} m_i^{\alpha}(v), \tag{3.2}$$

where $A_N = \{\alpha \mid \alpha \colon 2^N \to N\}$ is the family of all selectors on N.

The following theorem provides an overview of the other main formulations of the Shapley value. These formulations are widely adopted and used in applications of this theory.

Theorem 3.7 *Let $v \in \mathcal{G}^N$ and $i \in N$. Then the following statements are equivalent:*

(i) $\varphi_i(v) \in \mathbb{R}$ *is the Shapley value of player i in game v.*

(ii) **The standard formulation**

$$\varphi_i(v) = \sum_{S \subset N \colon i \in S} \frac{(|S|-1)!\,(n-|S|)!}{n!} \left[\, v(S) - v(S \setminus \{i\}) \,\right] \tag{3.3}$$

(iii) **The probabilistic formulation**

$$\varphi_i(v) = \frac{1}{n!} \sum_{\rho \in \Upsilon(2^N)} x^{\rho} \tag{3.4}$$

where $\Upsilon(2^N)$ is the collection of Weber strings in 2^N, $n! = |\Upsilon(2^N)|$ is the number of Weber strings in 2^N, and x^{ρ} is the marginal allocation corresponding to the Weber string $\rho \in \Upsilon(2^N)$.

(iv) **The MLE formulation**

$$\varphi_i(v) = \int \frac{\partial E_v}{\partial x_i}(t, \ldots, t)\, dt \tag{3.5}$$

For a proof of Theorem 3.7 I refer to the appendix of this chapter.

3.2 Three Axiomatizations of the Shapley Value

One of the main goals of value theory is to identify a set of axioms that uniquely determines the value under consideration. Furthermore, it is the ultimate goal of this theory to identify axioms that are independent of each other. Independence can be shown by identifying other values that satisfy all axioms except one under consideration. Independence can be interpreted as the property that the collection of axioms is "minimal"; there are no unnecessary properties within the class of stated axioms. In the case of independent axioms one says that this class of axioms characterizes the value under consideration. In that case the axioms form an *axiomatization* of that particular value.

I discuss three fundamentally different axiomatizations of the Shapley value φ on $\mathcal{G}^N$. The first axiomatization is the original axiomatization developed by Shapley (1953). It considers four axioms: Efficiency, the null-player property, symmetry[2] and additivity. These four axioms are widely recognized as the most fundamental ones in cooperative game theory. These four properties provide a proper axiomatization of the Shapley value.

Shapley's axiomatization instigated an extensive debate on the additivity property. This was considered the weakest point in Shapley's axiomatization. From its inception it has been considered a very strong property and revised axiomatizations were called for. The main axiomatization that does not use additivity is the second one that we consider here, developed by Young (1985). He considered a monotonicity property in which higher individual marginal contributions imply a higher payoff to the player in question. Young showed that his *strong monotonicity* property replaces both the dummy and additivity properties. Therefore, it avoids the use of the additivity property.

Our final axiomatization is a relatively recent contribution to the literature on cooperative game theory, filling a void in our understanding of the Shapley value, which is nevertheless very intuitive. Indeed, van den Brink (2001) bases his axiomatization on a very natural notion of fairness in the allocation of the generated wealth. His fairness axiom replaces additivity as well as symmetry. This axiomatization therefore gives an explicit formulation to the fairness that is present in the Shapley value as an allocation rule.

Before proceeding into the detailed discussion of these axiomatizations I introduce some auxiliary notions. These notions help us formulate the various properties.

- A player $i \in N$ is a *null-player* in the game $v \in \mathcal{G}^N$ if for every coalition $S \subset N$: $v(S) = v(S \setminus \{i\})$. Hence, a null-player is a non-contributor in a cooperative game.

[2] Many sources in cooperative game theory also indicate the symmetry axiom as *anonymity*. Symmetry refers to an equal treatment property, while the term anonymity refers to the notion that the name of a player is not important; just what values each player generates determine the payoff to that player.

- Let $\rho \colon N \rightleftarrows N$ be any permutation on N. Then for every game $v \in \mathcal{G}^N$ by the *permutated game* $\rho v \in \mathcal{G}^N$ we mean the game such that for every coalition $S = \{i_1, \dots, i_K\}$

$$\rho v(\rho S) = v(S), \tag{3.6}$$

where $\rho S = \{\rho(i_1), \dots, \rho(i_K)\}$.
- Two players $i, j \in N$ are *equipoised* in the game $v \in \mathcal{G}^N$ if for every $S \subset N \setminus \{i, j\}$:

$$v(S \cup \{i\}) = v(S \cup \{j\}). \tag{3.7}$$

Equipoised players can replace each other without affecting the productive value generated by any coalition in question. Thus, for all cooperatively game theoretic purposes these players are completely equal.

3.2.1 Shapley's Axiomatization

Although Shapley's original axiomatization was formulated for a "universe" of players that contained any finite player set, we can easily reformulate his axiomatization for a given player set N. We first introduce the four axioms at the foundation of the axiomatization developed by Shapley (1953):

Efficiency: A value $\phi \colon \mathcal{G}^N \to \mathbb{R}^N$ is efficient if for every game $v \in \mathcal{G}^N$:

$$\sum_{i \in N} \phi_i(v) = v(N) \tag{3.8}$$

Null-player Property: A value $\phi \colon \mathcal{G}^N \to \mathbb{R}^N$ satisfies the null-player property if for every game $v \in \mathcal{G}^N$ it holds that $\phi_i(v) = 0$ for every null-player $i \in N$ in the game v.

Symmetry: A value $\phi \colon \mathcal{G}^N \to \mathbb{R}^N$ is symmetric if for every permutation $\rho \colon N \rightleftarrows N$:

$$\phi_{\rho(i)}(\rho v) = \phi_i(v) \tag{3.9}$$

Additivity: A value $\phi \colon \mathcal{G}^N \to \mathbb{R}^N$ is additive if for all games $v, w \in \mathcal{G}^N$ and every player $i \in N$:

$$\phi_i(v + w) = \phi_i(v) + \phi_i(w) \tag{3.10}$$

I first investigate the independence of these four fundamental properties. We use several examples of values to do this.

Example 3.8 For each of the four properties introduced above I discuss an example that satisfies the three other properties, but *not* the property in question.

Efficiency: Consider the value ψ^1 defined by

$$\psi_i^1(v) = \sum_{S \subset N\colon i \in S} \Delta_v(S) \tag{3.11}$$

This value is certainly not efficient for every game. We just state here that ψ^1 satisfies the null-player property, symmetry, as well as additivity.

The null-player property: Consider the *egalitarian value* ψ^2 given by

$$\psi_i^2(v) = \frac{v(N)}{n} \tag{3.12}$$

The egalitarian value satisfies efficiency, symmetry and additivity. However, it does not satisfy the null-player property.

Symmetry: Consider a selector $\alpha\colon 2^N \to N$ with $\alpha(S) \in S$ for all $S \neq \varnothing$. Now we define the value ψ^3 as the selector allocation m^α corresponding to α defined by

$$\psi_i^3(v) = m_i^\alpha(v) = \sum_{S\colon i=\alpha(S)} \Delta_v(S), \tag{3.13}$$

This value satisfies efficiency, the null-player property as well as additivity. With regard to the null-player property I remark that any player $i \in S$ is *not* a null-player if $\Delta_v(S) \neq 0$. Hence, $\psi_i^3(v) \neq 0$ only if i is a selector in a coalition S with $\Delta_v(S) \neq 0$, in turn implying that i is not a null-player in v.

Furthermore, the value ψ^3 does not satisfy the symmetry property. This is due to the effect of the selector α in its definition.

Additivity: A value that satisfies efficiency, the null-player property as well as symmetry, but does not satisfy the additivity property, is the *Nucleolus*. This concept was introduced by Schmeidler (1969) and is based on a model of a bargaining process. The bargaining conditions identify a unique allocation.

The nucleolus of a cooperative game is a rather complex concept. Consequently I will not discuss this concept in detail. Instead I limit myself to pointing out its existence and the fact that it is a value that shows that additivity is an independent requirement from the other three Shapley axioms in the standard axiomatization of the Shapley value.

The discussion provided in this example shows that these four listed properties are indeed independent from each other. ■

The main characterization of the Shapley value is provided in the following theorem. It states essentially that the four properties formulated in this section provide a proper axiomatization of the Shapley value. Since the proof of this fundamental insight is essential, we provide a constructive proof after the statement of the assertion.

Theorem 3.9 (Shapley, 1953) *The Shapley value φ is the unique value on $\mathcal{G}^N$ that satisfies efficiency, the null-player property, symmetry and additivity.*

Proof It is easy to check that the Shapley value φ satisfies the four properties that are listed in the assertion. I leave it to the reader to go through the analysis for this.

To show the reverse, recall first that, using the unanimity basis $\mathcal{U}$ of $\mathcal{G}^N$, every game $v \in \mathcal{G}^N$ can be written as

$$v = \sum_{S \subset N} \Delta_v(S)\, u_S. \tag{3.14}$$

Now let $\phi \colon \mathcal{G}^N \to \mathbb{R}^N$ be a value satisfying the four properties listed.

Let $S \subset N$ and $C \in \mathbb{R}$. Consider the game $w = C\, u_S$. Then all players $j \notin S$ are null-players in w, implying that $\phi_j(w) = 0$ for all $j \notin S$. On the other hand all players in S are equipoised and by symmetry this implies that $\phi_i(w) = \phi_j(w) = \bar{c}$ for all $i, j \in S$. Finally, by efficiency this now implies that

$$\sum_{i \in N} \phi_i(w) = \sum_{j \notin S} \phi_j(w) + \sum_{i \in S} \phi_i(w) = |S| \cdot \bar{c} \equiv w(N) = C. \tag{3.15}$$

Hence, we have derived that

$$\phi_i(w) = \phi_i\left(C\, u_S\right) = \begin{cases} \frac{C}{|S|} & \text{for } i \in S \\ 0 & \text{for } i \notin S \end{cases} \tag{3.16}$$

Thus we derive that for an arbitrary game v and arbitrary player $i \in N$ by the additivity property that

$$\begin{aligned} \phi_i(v) &= \sum_{S \subset N} \phi_i\left(\Delta_v(S)\, u_S\right) \\ &= \sum_{S \subset N \colon i \in S} \phi_i\left(\Delta_v(S)\, u_S\right) \\ &= \sum_{S \subset N \colon i \in S} \frac{\Delta_v(S)}{|S|} = \varphi_i(v) \end{aligned}$$

This indeed shows that ϕ necessarily is the Shapley value φ. ■

The axiomatization developed in this section can be called the "standard" axiomatization of the Shapley value. Although the original axiomatization of Shapley (1953) was developed in a slightly different fashion, the four axioms of efficiency, the null-player property, symmetry and additivity are recognized as the foundational axioms that have to be attributed to Shapley.

An alternative development of a closely related axiomatization was pursued by Weber (1988). He introduced a step-wise development of the axioms and the resulting values. In this approach the Shapley value is developed as a very special case of a *probabilistic value*. Weber also relates in that paper the class of probabilistic values with random-order values, based on the Weber strings discussed in the previous chapter. I do not pursue this approach here, but refer to Weber (1988) instead.

3.2.2 Young's Axiomatization

Next I turn to the question whether the Shapley value can be axiomatized using a monotonicity property. There are several formulations of monotonicity possible. I explore the two most important ones here.

First we introduce Shubik's coalitional monotonicity condition (Shubik, 1962). This is the most intuitive and straightforward expression of the idea that higher generated wealth leads to higher payoffs for the players.

Definition 3.10 A value ϕ satisfies *coalitional monotonicity* if for all games $v, w \in \mathcal{G}^N$ with $v(T) \geqq w(T)$ for some coalition $T \subset N$ and $v(S) = w(S)$ for all $S \neq T$, then it holds that

$$\phi_i(v) \geqq \phi_i(w) \quad \text{for all } i \in T \tag{3.17}$$

It should be clear that the Shapley value indeed satisfies this coalitional monotonicity property. (For a discussion on this, I refer to the problem section of this chapter.) Unfortunately, the coalitional monotonicity property does not characterize the Shapley value. Worse, no "Core" value satisfies this property as was shown by Young (1985).

Proposition 3.11 (Young, 1985, Theorem 1) *If $n \geqq 5$, then there exists no value ϕ such that $\phi(v) \in C(v)$ for every game $v \in \mathcal{G}^N$ with $C(v) \neq \varnothing$ and that satisfies coalitional monotonicity.*

Proof Let $n = 5$, i.e., $N = \{1, 2, 3, 4, 5\}$. We construct now two games $v, w \in \mathcal{G}^N$ that show the desired property. Define the following coalitions:

$$\begin{array}{lll} S_1 = \{3,5\} & S_2 = \{1,2,3\} & S_3 = \{1,2,4,5\} \\ S_4 = \{1,3,4\} & S_5 = \{2,4,5\} & \end{array}$$

Now construct $v \in \mathcal{G}^N$ such that

$$\begin{array}{lll} v(S_1) = 3 & v(S_2) = 3 & v(S_3) = 9 \\ v(S_4) = 9 & v(S_5) = 9 & v(N) = 11 \end{array}$$

and for all other coalitions S we define

$$v(S) = \begin{cases} \max_{S_k \subset S} v(S_k) & \text{if } S_K \subset S \text{ for some } k \in \{1,2,3,4,5\} \\ 0 & \text{otherwise} \end{cases}$$

Now $x \in C(v)$ if and only if

$$\sum_{i \in S_k} x_i \geqq v(S_k) \text{ for } k \in \{1,2,3,4,5\}.$$

Adding these inequalities results into $\sum_k \sum_{S_k} x_i = 3 \sum_N x_i \geqq 33$, implying that $\sum_N x_i \geqq 11$. But $\sum_N x_i = v(N) = 11$ by definition, which leads to the conclusions that all inequalities $\sum_{S_k} x_i \geqq v(S_k)$ must be equalities. These have a unique solution, $\bar{x} = (0, 1, 2, 7, 1)$. Now, $C(v) = \{\bar{x}\}$.

Now compare the game $w \in \mathcal{G}^N$ which is identical to v except that $w(S_3) = w(N) = 12$. A similar argument shows that there is a unique Core imputation given by $\hat{x} = (3, 0, 0, 6, 3)$. Hence, the allocation to players 2 and 4 has decreased, even though the value of the coalitions containing these two players has monotonically increased.

Now any Core value ϕ has to select the unique Core imputation for v and w. But then this value ϕ does not satisfy coalitional monotonicity by definition.[3] This easily extends to cases with $n \geqq 5$. ■

The problem with coalitional monotonicity is that it considers *absolute* changes in the wealth generated by the various coalitions. Instead we have to turn to a consideration of the *relative* changes in the wealth generated by coalitions. This is formulated in the notion of strong monotonicity.

Definition 3.12 A value ϕ satisfies *strong monotonicity* if for all games $v, w \in \mathcal{G}^N$ it holds that

$$D_i v(S) \geqq D_i w(S) \text{ for all } S \subset N \text{ implies } \phi_i(v) \geqq \phi_i(w) \tag{3.18}$$

where

$$D_i v(S) = \begin{cases} v(S) - v(S \setminus \{i\}) & \text{if } i \in S \\ v(S \cup \{i\}) - v(S) & \text{if } i \notin S \end{cases}$$

denotes the marginal contribution of a player to an arbitrary coalition $S \subset N$.

The following result is the main insight with regard to monotonicity and the Shapley value. It constitutes a very powerful axiomatization of the Shapley value.

[3] I remark here that any monotonic modification of a game can be decomposed into a number of one-step modifications in which the value of a single coalition is monotonically increased. Coalitional monotonicity can be applied to each of these one-step modifications to arrive the conclusion that under monotonic modification the values of all players should monotonically increase as well.

Theorem 3.13 (Young, 1985, Theorem 2) *The Shapley value φ is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry and strong monotonicity.*

For a proof of this theorem I refer to the appendix of this chapter.

3.2.3 van den Brink's Axiomatization

In this subsection I consider so-called "fairness" considerations with regard to the Shapley value. Explicit fairness axioms were first made in the context of allocating the benefits of a cooperative game among players that interact through a communication network. Myerson (1977) seminally introduced a fairness property to characterize his Myerson value for cooperative games with an exogenously given communication network. This was further developed by Myerson (1980) for non-transferable utility games with arbitrary cooperation structures.[4] Myerson imposed that the removal of a link affects both constituting players of that link in equal fashion. Hence, the marginal payoff to the deletion of a link is equal for the two players making up that particular link.

In his seminal contribution, van den Brink (2001) applied the same reasoning to any pair of equipoised players in an arbitrary game. Formally we recall the definition of two equipoised players. van den Brink's fairness property now requires that equipoised players are treated equal.

Definition 3.14 Let ϕ be some value on $\mathcal{G}^N$.

(a) The value ϕ satisfies the *equal treatment property* if for all players $i,j \in N$ and all games $v \in \mathcal{G}^N$:

$$i \text{ and } j \text{ are equipoised in } v \text{ implies } \phi_i(v) = \phi_j(v).$$

(b) The value ϕ is *fair* if for all players $i,j \in N$ and all games $v,w \in \mathcal{G}^N$:

$$i \text{ and } j \text{ are equipoised in } w \text{ implies } \phi_i(v+w) - \phi_i(v) = \phi_j(v+w) - \phi_j(v).$$

Given that equipoised players make equal contributions to coalitions, the equal treatment property requires the allocation of exactly the same payoff to either of these players. The fairness property on the other hand requires that, if a game in which two players are equipoised, is added to any other game, then these two players receive exactly the same marginal payoff from that addition.

The next proposition summarizes the relationships between these two fairness properties and the well-established Shapley axioms.

Proposition 3.15 *Let ϕ be some value on $\mathcal{G}^N$.*

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[4] For details I refer to the discussion of the Myerson value for arbitrary cooperation structures in Section 3.4 of this chapter.

(a) *ϕ satisfies the equal treatment property if and only if ϕ is symmetric.*
(b) *If ϕ satisfies symmetry and additivity, then ϕ also satisfies fairness.*
(c) *If ϕ satisfies the null-player property and fairness, then ϕ also satisfies symmetry.*

Proof

(a) This is rather straightforward and left as an exercise to the reader.
(b) Let ϕ satisfies symmetry and additivity. If $i, j \in N$ are equipoised in w, then for every v it holds that

$$\begin{aligned}\phi_i(v+w) - \phi_i(v) &= \phi_i(v) + \phi_i(w) - \phi_i(v) = \\ &= \phi_i(w) = \phi_j(w) = \\ &= \phi_j(v) + \phi_j(w) - \phi_j(v) = \phi_j(v+w) - \phi_j(v)\end{aligned}$$

thus ϕ indeed satisfies fairness.
(c) Let ϕ satisfies the null-player property and fairness. For the null game η the null-player property now implies that $\phi_i(\eta) = 0$ for all $i \in N$. If i and j are equipoised, then by fairness and the above we get

$$\phi_i(v) = \phi_i(v+\eta) - \phi_i(\eta) = \phi_j(v+\eta) - \phi_j(\eta) = \phi_j(v).$$

This indeed implies that ϕ is symmetric. ■

The fairness axiom can be implemented into a proper axiomatization of the Shapley value. The proof of the next theorem is relegated to the appendix of this chapter.

Theorem 3.16 (van den Brink, 2001, Theorem 2.5) *The Shapley value φ is the unique value on $\mathcal{G}^N$ that satisfies efficiency, the null-player property and fairness.*

Without discussion I also mention here that the fairness hypothesis is independent of the two other axioms in this axiomatization.[5]

3.3 The Shapley Value as a Utility Function

Until now I have considered a cooperative game strictly as a description of potentially attainable values by coalitions in some cooperative interactive decision situation. Then we proceeded to ask the fundamental question of cooperative game theory: "How do we allocate the wealth generated over the various individuals in the

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[5] As van den Brink (2001) reports, the normalized Banzhaf value satisfies efficiency as well as the null-player property, but is not fair. For a definition of the normalized Banzhaf value I refer to van den Brink (2001).

situation?" These allocations are mainly determined by negotiating power (the Core) or combination of attainability and fairness considerations (the Shapley value).

Here we develop a very different perspective on value theory. Roth introduced a utility perspective on the Shapley value. His theory is developed in the seminal contributions Roth (1977a, 1977b) and further discussed in Roth (1988). Roth's perspective is that a value represents a valuation or utility of a player regarding a cooperative interactive decision situation in which this player participates. Hence, a cooperative game describes the potential values that can be generated by the various coalitions of which the player is a member. The player evaluates this situation using a utility function. A value—in particular, the Shapley value—could act as such a utility function.

Roth found that the Shapley value can be interpreted as such a utility function with very interesting properties. In fact he showed that the Shapley value is a vNM utility function satisfying several risk neutrality hypotheses. I first summarize the general concept of a vNM utility function, before proceeding to investigate the Shapley value from this perspective.

As a preliminary, consider an arbitrary set X. A *binary relation* on X is defined as a subset R of the collection of all ordered pairs in X, i.e., $R \subset X \times X$. An ordered pair $(x, y) \in R$ is usually denoted by xRy.

A binary relation R on X is *complete* if for every pair $x, y \in X$ it holds that xRy or yRx. Furthermore, a binary relation R on X is *transitive* if for every triple $x, y, z \in X$ it holds that xRy and yRz imply that xRz. A binary relation R on X is denoted as a *preference relation* or simply as a "preference" if R is complete as well as transitive.

Now I turn to the development of Roth's approach to the Shapley value. Throughout we let $\mathbb{L}$ be a collection of *lotteries*, where a lottery is defined as a probability distribution ℓ on some finite set of events E with $|E| < \infty$. Thus, $\ell \colon E \to [0,1]$ such that $\sum_{e \in E} \ell(e) = 1$.

If $\ell_1, \ell_2 \in \mathbb{L}$ and $p \in [0,1]$, then the compound lottery is given by $[p\,\ell_1; (1-p)\,\ell_2] = p\,\ell_1 + (1-p)\,\ell_2 \in \mathbb{L}$.

Definition 3.17 Let $\succcurlyeq$ be some preference relation on the class of lotteries $\mathbb{L}$. Then a function $u \colon \mathbb{L} \to \mathbb{R}$ is an *expected utility function* for preference $\succcurlyeq$ on $\mathbb{L}$ if

$$\ell_1 \succcurlyeq \ell_2 \text{ if and only if } u(\ell_1) \geqq u(\ell_2) \tag{3.19}$$

and

$$u\,(p\,\ell_1 + (1-p)\,\ell_2) = p\,u(\ell_1) + (1-p)\,u(\ell_2) \tag{3.20}$$

An expected utility function represents the preference over a lottery by taking the expected utility resulting from some evaluation of the underlying events. This evaluation of the underlying events is usually denoted as the vNM utility function that represents the preference relation under consideration.[6] The main problem in vNM utility theory is the existence of such expected utility representations of preference relation on collections of lotteries.

Before we proceed, it is necessary to introduce some auxiliary notation. Let $\ell_1, \ell_2 \in \mathbb{L}$. Now $\ell_1 \sim \ell_2$ if and only if $\ell_1 \succcurlyeq \ell_2$ as well as $\ell_2 \succcurlyeq \ell_1$. Also, $\ell_1 \succ \ell_2$ if and only if $\ell_1 \succcurlyeq \ell_2$ and not $\ell_2 \succcurlyeq \ell_1$.

The following lemma summarizes the main insight from expected utility theory on the existence of vNM expected utility functions. We give this insight without proof.

Lemma 3.18 *Let $\succcurlyeq$ be given on $\mathbb{L}$. There exists an expected utility function u for $\succcurlyeq$ on $\mathbb{L}$ if the preference relation $\succcurlyeq$ satisfies the two following conditions:*

Continuity *For all lotteries $\ell_1, \ell_2 \in \mathbb{L}$, both the sets $\{p \mid [p\,\ell_1; (1-p)\,\ell_2] \succcurlyeq \ell\}$ and $\{p \mid [p\,\ell_1; (1-p)\,\ell_2] \preccurlyeq \ell\}$ are closed for all $\ell \in \mathbb{L}$.*

Substitutability *If $\ell_1 \sim \ell_2$, then for any $\ell \in \mathbb{L}$*

$$\left[\tfrac{1}{2}\,\ell_1; \tfrac{1}{2}\,\ell\right] \sim \left[\tfrac{1}{2}\,\ell_2; \tfrac{1}{2}\,\ell\right].$$

Furthermore, the expected utility function u is unique up to affine transformations.

Next in the development of Roth's theory of the Shapley value, I turn to the modeling of preferences over so-called "game positions" as probabilistic events rather than deterministic constructions. For that we define $\mathbb{E} = N \times \mathcal{G}^N$ as the collection of game positions. Here, we interpret a pair $(i, v) \in \mathbb{E}$ as the position of player i in cooperative game v. Or, for short, (i, v) is a *game position*. One can now consider a game position to be a probabilistic event: The fact that one has the position of player i in game v is an outcome of a complex process that is subject to many probabilistic influences.

Now I consider the preference $\succcurlyeq$ on $\mathbb{E}$. Here, $(i, v) \succcurlyeq (j, w)$ means that it is preferable to have position i in game v over having position j in game w. These preferences are assumed to be subjective, i.e., these preferences are individualistic. This latter assumption distinguishes this theory from standard value theory.

We also consider probabilistic mixtures of positions. Here $[p\,(i, v); (1-p)\,(j, w)]$ means that one has a game position (i, v) with probability p and game position (j, w) with probability $1-p$. Now we re-define $\mathbb{L}$ as the set of all lotteries over $\mathbb{E}$. Hence, the set $\mathbb{L}$ contains exactly all lotteries over game positions. Clearly the preference relation $\succcurlyeq$ can be applied to $\mathbb{L}$ with the method described above.

Modeling Hypothesis 3.19 *The preference $\succcurlyeq$ over the space of lotteries over game positions $\mathbb{L}$ satisfies continuity and substitutability.*

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[6] It is interesting to note here that von Neumann and Morgenstern (1953) provided the first treatment of expected utility functions. von Neumann and Morgenstern were the first to consider evaluating lotteries by the expected utility that can be obtained based on some utility function on the underlying events. In their honor we talk about vNM utility functions ever since.

From the hypothesis above we can immediately conclude that there exists some vNM expected utility function $\psi : \mathbb{E} \to \mathbb{R}$ such that for every lottery $\ell \in \mathbb{L}$:

$$\psi(\ell) = \sum_{(i,v)\in\mathbb{E}} \ell(i,v) \cdot \psi_i(v), \tag{3.21}$$

where $\psi_i(v) = \psi(i,v)$.

Regarding the preference relation $\succcurlyeq$ we can make certain regularity assumptions. I will develop these hypotheses sequentially and state some intermediary results as we discuss these hypotheses.

Axiom 3.20 *Let $\mathcal{G}^N_{-i}$ be the class of games in which player i is a null player and let η be the null game as introduced in Chapter 1 with $\eta(S) = 0$ for all $S \subset N$.*

If $v \in \mathcal{G}^N_{-i}$, then $(i,v) \sim (i,\eta)$.

Axiom 3.20 states that a player is indifferent between being a null player in an arbitrary game and having a game position in the null game η itself. Both positions are equally undesirable for the player under consideration.

Axiom 3.21 *For any permutation $\rho : N \rightleftarrows N$ it holds that $(i,v) \sim (\rho(i), \rho\, v)$.*

Axiom 3.21 is an anonymity hypothesis: The name of a position is irrelevant for the evaluation of the player's position in the game. In other words, the evaluation of a game position is only based on the values generated rather than personal information.

From Axioms 3.20 and 3.21 we derive two properties of the vNM expected utility function ψ:

(a) The expected utility function is symmetric in the sense that for any permutation ρ we have that $\psi_i(v) = \psi_{\rho(i)}(\rho\, v)$.
(b) We can normalize ψ by choosing $\psi_i(u_{\{i\}}) = 1$ and $\psi_i(\eta) = 0$.

The third axiom that we introduce links scaling of payoffs and the introduction of risk. Indeed, it states that scaling the rewards in a game is equivalent to the introduction of risk.

Axiom 3.22 *For any number $C > 1$ and game position $(i,v) \in \mathbb{E}$ it holds that*

$$(i,v) \sim \left[\, \tfrac{1}{C}\,(i, C\,v); \left(1 - \tfrac{1}{C}\right)\,(i,\eta) \right] \tag{3.22}$$

From the three axioms introduced thus far we can derive another familiar property. Namely, it states that the three axioms imply that the expected utility function is linear.

Lemma 3.23 *Under Axioms 3.20, 3.21 and 3.22 it holds that for every* $C \geqq 0$ *and game position* $(i, v) \in \mathbb{E}$: $\psi_i(C\,v) = C\,\psi_i(v)$.

Proof If $C = 0$ then by Axiom 3.20 it follows that $\psi_i(C\,v) = \psi_i(\eta) = 0$. Hence, we only investigate $C > 0$.

Without loss of generality we may assume that $C \geqq 1$.[7] Then by Axiom 3.22 we may apply (3.22) to arrive at

$$\begin{aligned}\psi_i(v) &= \psi_i\left[\tfrac{1}{C}\,(i, C\,v); \left(1 - \tfrac{1}{C}\right)\,(i, \eta)\right] = \\ &= \tfrac{1}{C}\,\psi_i(C\,v) + \left(1 - \tfrac{1}{C}\right)\,\psi_i(\eta) = \\ &= \tfrac{1}{C}\,\psi_i(C\,v).\end{aligned}$$

This shows the assertion. ■

Axiom 3.24 (Ordinary risk neutrality) *For every player i we have that*

$$(\,i, (q\,w + (1-q)\,v)\,) \sim \left[\,q(i, w);(1-q)(i, v)\,\right],$$

where $v, w \in \mathcal{G}^N$ *and* $q \in [0, 1]$.

Axiom 3.24 introduces additional properties into the discussion. The following properties reflect the consequences of this hypothesis for the expected utility function ψ.

Proposition 3.25 *The following two properties hold.*

(a) *Axiom 3.24 implies Axiom 3.22.*
(b) *Under Axioms 3.20, 3.21 and 3.22 it holds that* $\succcurlyeq$ *satisfies Axiom 3.24 if and only if for all games* $v, w \in \mathcal{G}^N$ *and every player* $i \in N$:

$$\psi_i(v + w) = \psi_i(v) + \psi_i(w)$$

Proof Assertion (a) is rather trivial. Therefore, we limit ourselves to the proof of assertion (b).

For all games $v, w \in \mathcal{G}^N$ and every player $i \in N$ we have

$$\psi_i(v + w) = \psi_i\left(2\left(\tfrac{1}{2}\,v + \tfrac{1}{2}\,w\right)\right) = 2\,\psi_i\left(\tfrac{1}{2}\,v + \tfrac{1}{2}\,w\right)$$

which follows from the axioms formulated. Now by Axiom 3.24 it follows that

$$\psi_i\left(\tfrac{1}{2}\,v + \tfrac{1}{2}\,w\right) = \psi\,\left[\,\tfrac{1}{2}\,(i, v);\tfrac{1}{2}\,(i, w)\right] = \tfrac{1}{2}\,\psi_i(v) + \tfrac{1}{2}\,\psi_i(w)$$

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[7] Indeed, for $0 < C < 1$ we can select $C' = \frac{1}{C}$.

using the expected utility formulation. ■

Assertion (b) in Proposition 3.25 links ordinary risk neutrality to additivity. It is clear that this in conjunction with Axioms 3.20, 3.21 and 3.22 brings us a step closer to characterizing the Shapley value as a vNM expected utility function on the class of game positions.

Until now we limited our discussion to some rather standard axioms of expected utility theory. The only exception being the neutrality with respect to ordinary risk (Axiom 3.24). Now we turn to the introduction of one additional essential axiom of risk neutrality, namely the neutrality with respect to *strategic* risk.

We first note that the unanimity game u_S for $S \subset N$ can be interpreted as a *pure bargaining game* in its simplest form. Indeed, it essentially describes the strategic bargaining of $|S|$ players over one unit of wealth. This leads to the following concept.

Definition 3.26 The *certainty equivalent* of a strategic position in the unanimity game u_S is a number $\bar{\gamma}(s)$, where $s = |S|$, such that for every position $i \in S$:

$$(i, u_S) \sim \left(i, \bar{\gamma}(s)\, u_{\{i\}} \right) \tag{3.23}$$

Remark that $\bar{\gamma}(1) = 1$ and that $\bar{\gamma}(s)$ is a measure of a player's opinion of her own bargaining ability in a pure bargaining game of the size s.

Under standard assumptions such as continuity of the preference relation, one can show that certainty equivalents exist. Here, we assume that there is no problem with the existence of these certainty equivalents and we can use this concept in the definition of certain properties. With reference to Roth (1977a) we introduce the following terminology:

- The preference relation $\succcurlyeq$ reflects an *aversion to strategic risk* if for all $s = 1, \dots, n$ it holds that $\bar{\gamma}(s) \leqq \frac{1}{s}$.
- The preference $\succcurlyeq$ is *neutral to strategic risk* if for all $s = 1, \dots, n$ it holds that $\bar{\gamma}(s) = \frac{1}{s}$.
- The preference $\succcurlyeq$ is *strategic risk loving* if for all $s = 1, \dots, n$ it holds that $\bar{\gamma}(s) \geqq \frac{1}{s}$.

The following result, which was seminally formulated in Roth (1977a), is given without a detailed proof. It follows quite straightforwardly from the results already formulated and discussed.

Theorem 3.27 *The Shapley value φ is the unique expected utility function on $\mathbb{E} = N \times \mathcal{G}^N$ that satisfies Axioms 3.20, 3.21, 3.22 and 3.24 and is neutral to strategic risk.*

The theorem states that the Shapley value is a very specific vNM expected utility function. It is exactly the one that is neutral to ordinary risk as well as neutral to strategic risk. In this respect, therefore, the Shapley value is neutral to the main forms of risk.

Beyond the realm of vNM utility theory, this insight is useful, since it provides a foundation for the use of the Shapley value as a utilitarian benchmark in the analysis of situations in which cooperative games describe certain value-generating situations. This is exactly the case for the analysis of the exercise of authority in hierarchical firms, developed by van den Brink and Gilles (2009). There a firm is a combination of an authority hierarchy and a production process. The first is described by a hierarchical network of authority relations and the second by a cooperative game that assigns to every coalition of productive workers a team production value. Every worker is now assumed to have a preference over the various positions that he can assume in this firm. Application of the Shapley value now exactly applies a risk neutral evaluation of these positions. It, thus, functions as a benchmark in this analysis.

3.4 The Myerson Value

Myerson (1977) and Myerson (1980) introduced a generalization of the Shapley value for decision situations with constraints on coalition formation. Myerson (1977) originally introduced network-based constraints on coalition formation in which context he discussed the resulting Shapley value.[8] In this section I provide an overview of how more general constraints on coalition formation affect the resulting Shapley value.

With reference to the discussion in Chapter 2, I recall that $\Omega \subset 2^N$ is an (institutional) coalitional structure on N if $\varnothing \in \Omega$. As in my discussion of the restricted Core $\mathcal{C}(\Omega, v)$ of a cooperative game $v \in \mathcal{G}^N$ in Chapter 2, I now turn to the question whether we can define an axiomatic value for such a situation in which there are constraints on coalition formation. Furthermore, I require this value to be closely related to the Shapley value in the sense that for the case without restrictions on coalition formation it is equivalent to the Shapley value. Myerson introduced the most plausible extension of the Shapley value for such situations, called the *Myerson value*.

The key assumption of the Myerson value is that only institutional coalitions can truly generate dividends that should be divided equally among the members of these coalitions. The definition of the Myerson value is constructed for the most general case in the next definition. There are some technical problems with defining the appropriate institutional coalitional structures for that. Next, we first introduce the technical preliminaries from Algaba, Bilbao, Borm, and López (2001) for the definition of the Myerson value.

Let $\Omega \subset 2^N$ such that $\varnothing \in \Omega$. We define the following concepts based on Ω:

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[8] Extensions of Myerson's analysis were pursued by Jackson and Wolinsky (1996), who extended the Myerson value to arbitrary network situations. For an elaborate discussion of network-based constraints on coalition formation in this sense I also refer to Jackson (2008, Chapter 12).

A basis of coalitional structure Ω. We define the collection of *supportable coalitions* in the coalitional structure Ω by

$$N(\Omega) = \{S \in \Omega \mid \exists\, T, R \in \Omega \setminus \{\varnothing, S\} \colon\; S = T \cup R \text{ and } T \cap R \neq \varnothing\} \quad (3.24)$$

The *basis* of Ω is now defined by

$$B(\Omega) = \Omega \setminus N(\Omega). \quad (3.25)$$

Hence, the basis of Ω consists exactly of the non-supportable coalitions. For the case that $\Omega = 2^N$ is the collection of *all* potential coalitions, it is clear that $B(2^N) = \{\{i,j\} \mid i \neq j \text{ and } i,j \in N\} \cup \{\{i\} \mid i \in N\} \cup \{\varnothing\}$.

Following (Algaba et al. 2001) the coalitions in $B(\Omega)$ are called the *supports* of Ω in the sense that every institutional coalition $S \in \Omega$ can be written as a union of supports in $B(\Omega)$.

The union stable cover of Ω. The *union stable cover* of the coalitional structure Ω is the smallest collection $\overline{\Omega} \subset 2^N$ such that $\Omega \subset \overline{\Omega}$ and for all $S, T \in \overline{\Omega}$ with $S \cap T \neq \varnothing$ it holds that $S \cup T \in \overline{\Omega}$.

It is easy to see that the union stable cover $\overline{\Omega}$ of Ω is equal to the union stable cover $\overline{B(\Omega)}$ of its basis $B(\Omega)$.

Ω*-Components.* Let $S \subset N$ be an arbitrary coalition. The family of Ω-*components* of S is defined by

$$C_\Omega(S) = \left\{T \in \overline{\Omega} \mid T \subset S \text{ and there is no } R \in \Omega \colon\; T \subsetneq R \subset S\right\} \quad (3.26)$$

A coalition $S \in C_\Omega(N)$ is called a *global* component here if it forms an Ω-component of the grand coalition N. We remark here that following Lemma 3.28 below the global components in the player set N are pairwise disjoint.

The grand Ω*-component.* The *grand* Ω-component is defined by

$$\widehat{C}_\Omega = \cup C_\Omega(N) \subset N \quad (3.27)$$

Hence, the grand Ω-component is the union of all global Ω-components of the grand coalition N. Players $i \notin \widehat{C}_\Omega$ that are not a member of the global Ω-component are denoted as Ω-*isolated* players—or simply as isolated players if the context is clear. These isolated players are not really involved in the building of any institutional or formable coalition $S \in \Omega$.

We require certain regularity conditions to be satisfied to define the Myerson value for a game with a coalitional structure. Crucial is the following property that $C_\Omega(S)$ is a collection of pairwise disjoint coalitions.

Lemma 3.28 *For every coalition $S \subset N$ with $C_\Omega(S) \neq \varnothing$, the Ω-components of S form a collection of pairwise disjoint subcoalitions of S.*

Proof Let $S \subset N$ be such that $C_\Omega(S) \neq \varnothing$. Assume that $T, R \in \overline{\Omega}$ are two components of S. If $T \cap R \neq \varnothing$, then $T \cup R \in \overline{\Omega}$. Hence, $T \cup R \in C_\Omega(S)$, which contradicts the maximality of T and R. Hence, $T \cap R = \varnothing$. ∎

The previous discussion of these concepts allows us to define the Myerson restriction of a cooperative game and allow the introduction of the Myerson value.

Definition 3.29 Let $\Omega \subset 2^N$ be a coalitional structure such that $\varnothing \in \Omega$ and let $v \in \mathcal{G}^N$.

(a) The Ω-*restriction* of the game v is given by $v_\Omega \colon 2^N \to \mathbb{R}$ with

$$v_\Omega(S) = \sum_{T \in C_\Omega(S)} v(T). \tag{3.28}$$

(b) Let $\mathfrak{W}^N = \{\Omega \mid \Omega \subset 2^N\}$ be the collection of all coalitional structures on N. The *Myerson value* is a function $\mu \colon \mathcal{G}^N \times \mathfrak{W}^N \to \mathbb{R}^N$ such that for every game v and every player $i \in N$ the Myerson value is the Shapley value of its Ω-restriction, i.e.,

$$\mu_i(v, \Omega) = \varphi_i\left(v_\Omega\right). \tag{3.29}$$

The Ω-restriction v_Ω of an arbitrary game $v \in \mathcal{G}^N$ is similar to the Kaneko–Wooders partitioning game introduced in the previous chapter. Indeed, if Ω is a partitioning of N, the two notions are the same. In the current context, however, the restricted game is based on the restricted game as introduced for communication network situations in Myerson (1977) and for conference structures in Myerson (1980).

The Myerson value is usually characterized by three axioms. Two of these axioms—component efficiency and the balanced payoff property[9]—have seminally been introduced by Myerson (1977). In this case these axioms are generalized to the setting of arbitrary coalitional structures rather than the coalitional generated by a communication network discussed in the seminal work by Myerson.

Let $\phi \colon \mathcal{G}^N \times \mathfrak{W}^N \to \mathbb{R}^N$ be some allocation rule on the class of games with a coalitional structure. Then we introduce three fundamental properties:

Component-efficiency property. For every global component $S \in C_\Omega(N)$:

$$\sum_{i \in S} \phi_i(v, \Omega) = v(S). \tag{3.30}$$

[9] The balanced payoff property was introduced under the nomen "fairness" in Myerson's seminal discussion of value allocation in a communication network. Later this was recognized as the wrong indication; Jackson modified the nomenclature to the balanced payoff property.

Isolated player property. For every Ω-isolated player $i \in N \setminus \widehat{C}_\Omega$ and for every $v \in \mathcal{G}^N$ it holds that $\mu_i(v, \Omega) = 0$.

Balanced payoff property. For every support coalition $S \in B(\Omega)$ and for all players $i, j \in S$:

$$\phi_i(v, \Omega) - \phi_i(v, \Omega') = \phi_j(v, \Omega) - \phi_j(v, \Omega') \tag{3.31}$$

where $\Omega' = \overline{B(\Omega) \setminus \{S\}}$ the union stable cover of the coalitional structure excluding support S.

The following theorem generalizes the results stated in Myerson (1977) and Myerson (1980). The result as stated below was developed in Algaba et al. (2001).

Theorem 3.30 *The Myerson value μ is the unique allocation rule on the class of games with a coalitional structure $\mathcal{G}^N \times \mathfrak{W}^N$ that satisfies the component-efficiency, isolated player and balanced payoff properties.*

For a proof of Theorem 3.30 I refer to the appendix of this chapter.

3.5 Appendix: Proofs of the Main Theorems

Proof of Theorem 3.7

Proof of (3.3)
Implement the definition of the Harsanyi dividend for any coalition S in the game v, given by

$$\Delta_v(S) = \sum_{T \subset S} (-1)^{|S|-|T|} v(T),$$

into the definition of the Shapley value:

$$\varphi_i(v) = \sum_{S \subset N:\, i \in S} \frac{1}{|S|} \left[\sum_{T \subset S} (-1)^{|S|-|T|} v(T) \right] = \sum_{T \subset N} \left[\sum_{S \subset N:\, T \cup \{i\} \subset S} (-1)^{|S|-|T|} \frac{v(T)}{|S|} \right]$$

Write

$$\delta_i(T) = \sum_{S \subset N:\, T \cup \{i\} \subset S} (-1)^{|S|-|T|} \frac{1}{|S|} \tag{3.32}$$

It is easy to see that if $i \notin T'$ and $T = T' \cup \{i\}$, then $\delta_i(T) = -\delta_i(T')$. All the terms on the right hand side of (3.32) are the same in both cases, except that $|T| = |T'| + 1$. Thus, a sign change can be applied throughout to arrive at

$$\varphi_i(v) = \sum_{T \subset N:\, i \in T} \delta_i(T) \left[v(T) - v(T \setminus \{i\}) \right]$$

Now if $i \in T$ and T has t members, there are exactly $\binom{n-t}{s-t}$ coalitions S with s members such that $T \subset S$. Thus,

$$\begin{aligned}
\delta_i(T) &= \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} \frac{1}{s} = \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} \int_0^1 x^{s-1}\, dx = \\
&= \int_0^1 \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} x^{s-1}\, dx \\
&= \int_0^1 x^{t-1} \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} x^{s-t}\, dx = \int_0^1 x^{t-1}(1-x)^{n-t}\, dx.
\end{aligned}$$

This is a well-known integral, leading to

$$\delta_i(T) = \frac{(t-1)!\,(n-t)!}{n!}.$$

This implies (3.3) as required.

Proof of (3.4)
This is the direct consequence of (3.3) and the definition of Weber strings and marginal values based on these Weber strings. The proof is therefore omitted.

Proof of (3.5)
Recall that $E_v(x) = \sum_{S \subset N} \Delta_v(S) \prod_{i \in S} x_i$. This implies that

$$\frac{\partial E_v}{\partial x_i}(x) = \sum_{S \subset N:\, i \in S} \Delta_v(S) \prod_{j \in S \setminus \{i\}} x_j$$

and, so,

$$\frac{\partial E_v}{\partial x_i}(t, \ldots, t) = \sum_{S \subset N:\, i \in S} \Delta_v(S) t^{|S|-1}.$$

This in turn implies that

$$\begin{aligned}
\int_0^1 \frac{\partial E_v}{\partial x_i}(t, \ldots, t)\, dt &= \int_0^1 \sum_{S \subset N:\, i \in S} \Delta_v(S) t^{|S|-1}\, dt = \\
&= \sum_{S \subset N:\, i \in S} \frac{\Delta_v(S)}{|S|} t^{|S|} \Bigg|_0^1 = \sum_{S \subset N:\, i \in S} \frac{\Delta_v(S)}{|S|} = \varphi_i(v)
\end{aligned}$$

This shows (3.5).

Proof of Theorem 3.13

Consider an arbitrary value ϕ on $\mathcal{G}^N$.

First we show that strong monotonicity implies the null-player property. Indeed, note that if ϕ is strongly monotone then for all games $v, w \in \mathcal{G}^N$:

$$D_i v(S) = D_i w(S) \text{ for all } S \subset N \text{ implies } \phi_i(v) = \phi_i(w). \tag{3.33}$$

Next consider the null game η defined by $\eta(S) = 0$ for all $S \subset N$. Then by symmetry we have that $\phi_i(\eta) = \phi_j(\eta)$ for all $i, j \in N$. By efficiency $\sum_N \phi_i(\eta) = \eta$(N)=0, and therefore $\phi_i(\eta) = 0$ for all $i \in N$.

Let $v \in \mathcal{G}^N$. Now by (3.33) it follows for any $i \in N$:

$$D_i v(S) = 0 = D_i \eta(S) \text{ for all } S \subset N \text{ implies } \phi_i(v) = 0 \tag{3.34}$$

Hence we have show the null-player property.

Again write the game v in its unanimity basis form $v = \sum_S \Delta_v(S)\, u_S$. Define the index I as the minimum number of non-zero terms in the unanimity decomposition of the game v. We prove the main assertion by induction on the index I.

$I = 0$ Then every player is a null-player, namely $v = \eta$. Hence, $\phi_i(v) = \phi_i(\eta) = 0 = \varphi_i(v)$.

$I = 1$ Then $v = C\, u_S$ for some $S \subset N$. For $i \notin S$ we have that $D_i(T) = 0$ for all $T \subset N$, and, thus, by (3.34), $\phi_i(v) = 0$. For all $i, j \in S$ symmetry implies that $\phi_i(v) = \phi_j(v)$. Combined with efficiency this implies that $\phi_i(v) = \frac{C}{|S|} = \varphi_i(v)$ for all $i \in S$.

Assume the assertion holds for I; prove it holds for $I + 1$. Assume that $\phi(w) = \varphi(w)$ for any game w with an index less or equal to I. Let v have index $I+1$. Then we can write

$$v = \sum_{k=1}^{I+1} \Delta_v(S_k)\, u_{S_k} \text{ with } \Delta_v(S_k) \neq 0.$$

Define $S = \cap_{k=1}^{I+1} S_k$ and suppose that $i \notin S$. Define the game

$$w = \sum_{k:\ i \in S_k} \Delta_v(S_k)\, u_{S_k}$$

Then w has an index of at most I with regard to $i \notin S$. Furthermore, $D_i w(T) = D_i v(T)$ for all $T \subset N$. Thus, by the induction hypothesis and strong monotonicity, it can be concluded that

$$\phi_i(v) = \phi_i(w) = \sum_{k:\ i \in S_k} \frac{\Delta_v(S_k)}{|S_k|} = \varphi_i(v)$$

Next suppose that $i \in S$. By symmetry, $\phi_i(v)$ is a constant c for all players in S. Likewise the Shapley value $\varphi_i(v)$ is a constant c' for all members of S. Since both values ϕ and φ satisfy efficiency and are equal for all i not in S, it has to be concluded that $c = c'$.

This completes the proof of Theorem 3.13.

Proof of Theorem 3.16

The proof of van den Brink's theorem is rather involved. The proof is based on the application of graph-theoretic concepts and techniques.

Suppose that ϕ satisfies efficiency, the null-player property and fairness. For every $v \in \mathcal{G}^N$ define its support by

$$S(v) = \{S \subset N \mid \Delta_v(S) \neq 0\}$$

and $I(v) = |S(v)|$ as the number of coalitions in its support, also denoted as the index of the game v.[10] As in the proof of Young's axiomatization we apply induction on the index $I(v)$.

$$I(v) = 0, \text{i.e.}, v = \eta.$$

In this case the null-player property implies that $\phi_i(v) = 0 = \varphi_i(v)$.

$$I(v) = 1, \text{i.e.}, v = C\, u_S \text{for some} S \subset N.$$

Since the null-player property and fairness imply symmetry, we can apply the same reasoning as the one used in the proof of Theorem 3.13. Hence we immediately conclude that

$$\phi_i(C\, u_S) = \varphi_i(C\, u_S) = \frac{C}{|S|}.$$

Proceed by induction: Assume that $\phi(v') = \varphi(v')$ for all games v' with $I(v') \leqq I$.

Now let $v \in \mathcal{G}^N$ be such that $I(v) = I + 1 \geqq 2$. Now we introduce a graph $g_v \subset \{ij \mid i, j \in N\}$ on N, where $ij = \{i, j\}$ represents an undirected link between i and j.

In particular, define $ij \in g_v$ if and only if $i \neq j$ and there exists some coalition $S \in S(v)$ with either $ij \subset S$ or $S \cap ij = \varnothing$.

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[10] Here I also refer to the proof of Theorem 3.13. There this index was simply introduced as I.

A *component* of g_v is a coalition $S \subset N$ that is maximally connected[11] in g_v, i.e., S is connected in g_v and it does not contain a strict superset that is connected in g_v as well.

We distinguish two cases for the graph g_v:
Case A: *N is a component of g_v.*
Now define

$$N'(v) = \{i \in N \mid i \in T \text{ for some } T \in S(v)\} \tag{3.35}$$

as the set of non-null players in v. Take $j \in N'(v)$ and let $T_0 = \{j\}$. For k we recursively define

$$T_k = \left\{ i \in N \setminus \left(\cup_{m=0}^{k-1} T_m \right) \,\middle|\, \text{There exists some } j \in T_{k-1} \text{ with } ij \in g_v \right\} \tag{3.36}$$

The collection $T = (T_0, \dots, T_K)$ is a partition of N consisting on non-empty coalitions only. Indeed, since N is a component of g_v the given procedure yields non-empty coalitions T_k.

Recall that $T_0 = \{j\}$. Now suppose that $\phi_j(v) = F$ for some $F \in \mathbb{R}$ and define $c_j = 0$. We now recursively will determine $\phi_i(v)$ for all $i \in T_k$ with $k = 1, \dots, m$.

Indeed, for $i \in T_k$ with $k \in \{1, \dots, m\}$ there exists some $h \in T_{k-1}$ and $T \in S(v)$ with either $ih \subset T$ or $T \cap ih = \varnothing$. Fairness now implies that

$$\phi_i(v) - \phi_i(v - \Delta_v(T)\, u_T) = \phi_h(v) - \phi_h(v - \Delta_v(T)\, u_T)$$

But then

$$\begin{aligned} \phi_i(v) &= \phi_h(v) - \phi_h(v - \Delta_v(T)\, u_T) + \phi_i(v - \Delta_v(T)\, u_T) = \\ &= F + c_h - \phi_h(v - \Delta_v(T)\, u_T) + \phi_i(v - \Delta_v(T)\, u_T) = F + c_i, \end{aligned} \tag{3.37}$$

where $c_i = c_h - \phi_h(v - \Delta_v(T)\, u_T) + \phi_i(v - \Delta_v(T)\, u_T)$. The value of c_i is determined since by the induction hypothesis both $\phi_h(v - \Delta_v(T)\, u_T)$ and $\phi_i(v - \Delta_v(T)\, u_T)$ are fully determined.

Efficiency now implies that

$$\sum_{i \in N} \phi_i(v) = n \cdot F + \sum_{i \in N} c_i = v(N).$$

From this it follows that $F = \frac{1}{n}\left(v(N) - \sum_N c_i\right)$ is uniquely determined. With $\phi_j(v) = F$ and (3.37) it can be concluded that all values $\phi_i(v)$, $i \in N$ are fully determined.

[11] Coalition S is *connected* in g_v if for all $i, j \in S$ with $i \neq j$ there exists a sequence $i_1, \dots, i_K \in S$ with $i_1 = i$, $i_K = j$ and $i_k i_{k+1} \in g_v$ for all $k = 1, \dots, K-1$.

Case B: N is not a component of g_v.
Case B is equivalent to the property that g_v is not a connected graph or network. Thus, there exist at least two components in g_v, denoted by B_1 and B_2 with $B_1 \cap B_2 = \varnothing$.[12]

We first show that g_v can at most have two components. Suppose to the contrary that there is a third component, B_3. Since $I(v) \geqq 2$ we have that $S(v) \neq \varnothing$. Assume without loss of generality that there exists $T \in S(v)$ with $T \subset B_3$. If $i \in B_!$ and $j \in B_2$, then $ij \cap R = \varnothing$. But then by definition $ij \in g_v$, which would be a contradiction to B_1 and B_2 being components of g_v.

So, g_v has exactly two components, say $V = B_1$ and $W = B_2$. Since $I(v) \geqq 2$ v can be written as the sum of two unanimity games, i.e., $v = C_V\, u_V + C_W\, u_W$ with $C_V, C_W \neq 0$. Also, $V \cup W = N$.[13]

We can now distinguish two cases regarding the size of N:

1. Suppose $n \geqq 3$.
   Suppose without loss of generality that $|V| \geqq 2$. Take $j \in V$ and $h \in W$. Define $w \in \mathcal{G}^N$ by[14]

   $$w = v + C_V\, u_{V-j+h} = C_V\, u_V + C_V\, u_{V-j+h} + C_W\, u_W.$$

   We first determine $\phi(w)$. Let $\phi_h(w) = F$. Fairness now implies that $\phi_j(w) - \phi_j(c_W\, u_W) = \phi_h(w) - \phi_h(C_W\, u_W)$. Since $I(C_W\, u_W) = 1$, we already determined that $\phi_j(C_W\, u_W) = 0$ and $\phi_h(C_W\, u_W) = \frac{C_W}{|W|}$. So,

   $$\phi_j(w) = \phi_h(w) - \phi_h(C_W\, u_W) + \phi_h(C_W\, u_W) = F - \frac{C_W}{|W|}.$$

   The null-player property implies that $\phi_i(C_V\, u_V + C_V\, u_{V-j+h}) = 0$ for $i \in W \setminus \{h\}$. Fairness and the fact that $I(C_V\, u_{V-j+h}) = 1$ implies for $i \in W \setminus \{h\}$ that

   $$\begin{aligned}\phi_h(C_V\, u_V + C_V\, u_{V-j+h}) &= \phi_i(C_V\, u_V + C_V\, u_{V-j+h}) - \phi_i(C_V\, u_{V-j+h}) + \\ &+ \phi_h(C_V\, u_{V-j+h}) = \frac{C_V}{|V|}.\end{aligned}$$

   For every $i \in W \setminus \{h\}$, fairness then implies that

   $$\phi_i(w) = \phi_h(w) - \phi_h(C_V\, u_V + C_V\, u_{T-j+h}) + \phi_i(C_V\, u_V + C_V\, u_{T-j+h}) = F - \frac{C_V}{|V|}.$$

   For $i \in V \setminus \{j\}$ fairness in a similar fashion implies that

   $$\phi_i(w) = \phi_j(w) - \phi_j(C_V\, u_{V-j+h}) + \phi_i(C_V\, u_{V-j+h}) = F - \frac{C_W}{|W|} + \frac{C_V}{|V|}.$$

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[12] It follows from this as well that $n \geqq 2$.

[13] If $V \cup W \neq N$, then for $i \in V, j \in W$ and $h \in N \setminus (V \cup W)$ it holds that $ih, jh \in g_v$. This is a contradiction to V and W being components of g_v.

[14] Here I use the notation $V - j + h = (V \setminus \{j\}) \cup \{h\}$.

We can summarize the conclusions now as follows:

$$\phi_i(w) = \begin{cases} F & \text{if } i = h \\ F - \frac{C_W}{|W|} & \text{if } i = j \\ F - \frac{C_V}{|V|} & \text{if } i \in W \setminus \{h\} \\ F - \frac{C_W}{|W|} + \frac{C_V}{|V|} & \text{if } i \in V \setminus \{j\} \end{cases} \tag{3.38}$$

With efficiency it now has to hold that

$$\sum_{i \in N} \phi_i(w) = n\,F - \frac{|V|}{|W|} C_W + \frac{|V| - |W|}{|V|} C_V \equiv 2C_V + C_W.$$

This in turn implies that

$$F = \frac{2|V| - |V| + |W|}{|V| \cdot n} C_V + \frac{|W| + |V|}{|W| \cdot n} C_W = \tfrac{C_V}{|V|} + \tfrac{C_W}{|W|}.$$

With the above this then determines that

$$\phi_i(w) = \begin{cases} \frac{C_V}{|V|} + \frac{C_W}{|W|} & \text{if } i = h \\ \frac{C_V}{|V|} & \text{if } i = j \\ \frac{C_W}{|W|} & \text{if } i \in W \setminus \{h\} \\ 2\frac{C_V}{|V|} & \text{if } i \in V \setminus \{j\} \end{cases} \tag{3.39}$$

We can finally determine the values $\phi_i(v)$, $i \in N$. Let $\phi_h(v) = F'$ be the anchor value.

Fairness and the null-player property together with Proposition 3.15(c) implies that $\phi_i(v) = F'$ for all $i \in W$.
For every $i \in V \setminus \{j\}$ fairness in turn implies that

$$\phi_i(v) = \phi_h(v) - \phi_h(w) + \phi_i(w) = F' - \tfrac{C_W}{|W|} + \tfrac{C_V}{|V|}.$$

Since Proposition 3.15(c) also implies that $\phi_j(v) = \phi_i(v)$ for all $i \in T \setminus \{j\}$, it can be concluded that $\phi_j(v) = F' - \frac{C_W}{|W|} + \frac{C_V}{|V|}$. Thus,

$$\phi_i(v) = \begin{cases} F' & \text{for } i \in W \\ F' - \frac{C_W}{|W|} + \frac{C_V}{|V|} & \text{for } i \in V \end{cases} \tag{3.40}$$

Efficiency determines that

$$\sum_{i \in N} \phi_i(v) = n\,F' - \frac{|V|}{|W|} C_W + C_V \equiv C_V + C_W$$

and, thus, $F' = \frac{|W|+|V|}{|V|\,n} C_W = \frac{C_W}{|W|}$. Hence,

$$\phi_i(v) = \begin{cases} \frac{C_W}{|W|} & \text{for } i \in W \\ \frac{C_V}{|V|} & \text{for } i \in V \end{cases} \tag{3.41}$$

2. Suppose that $n = 2$.
   Then $N = \{i,j\}$, $i \neq j$, and $v = C_i\,u_i + C_j\,u_j$. Without loss of generality we may assume that $C_i \geqq C_j$.
   Let $\phi_j(v) = F$. The null-player property implies that $\phi_j\left((C_i - C_j)\,u_i\right) = 0$. With efficiency this in turn implies that $\phi_i\left((C_i - C_j)\,u_i\right) = C_i - C_j$. Fairness now implies that

   $$\phi_i(v) = \phi_j(v) - \phi_j\left((C_i - C_j)\,u_i\right) + \phi_i\left((C_i - C_j)\,u_i\right) = F + C_i - C_j.$$

   Efficiency imposes that

   $$\phi_i(v) + \phi_j(v) = 2F + C_i + C_j \equiv C_i + C_j$$

   yielding that $F = C_j$ and

   $$\begin{aligned} \phi_i(v) &= F + C_i - C_j = C_i \\ \phi_j(v) &= F = C_j \end{aligned}$$

   We may conclude that in this case all values $\phi(v)$ are uniquely determined.

All cases discussed above imply that $\phi(v)$ is uniquely determined by the three properties imposed for all games $v \in \mathcal{G}^N$. Since the Shapley value φ satisfies these three properties, it is clear that $\phi = \varphi$. This proves the assertion.

Proof of Theorem 3.30

We first show that the Myerson value $\mu\colon \mathcal{G}^N \times \mathfrak{W}^N \to \mathbb{R}^N$ indeed satisfies the three stated properties in the assertion. For that purpose let $(v, \Omega) \in \mathcal{G}^N \times \mathfrak{W}^N$ be given.

1. *μ satisfies component-efficiency.*
   If $N \in \Omega$, then $C_\Omega(N) = \{N\}$ and

   $$\sum_{i \in N} \mu_i(v, \Omega) = \sum_{i \in N} \varphi_i(v_\Omega) = v_\Omega(N) = v(N).$$

   Suppose that $N \notin \Omega$ and consider $M \in C_\Omega(N)$. We now define $u^M \in \mathcal{G}^N$ by for every $S \in 2^N$

$$u^M(S) = v_\Omega(S \cap M) = \sum_{T \in C_\Omega(S \cap M)} v(T).$$

Furthermore, we note that $C_\Omega(S) = \cup_{T \in C_\Omega(N)} C_\Omega(S \cap T)$. This in turn implies that $v_\Omega = \sum_{T \in C_\Omega(N)} u^T$. This then leads to

$$\begin{aligned}
\sum_{i \in M} \mu_i(v, \Omega) &= \sum_{i \in M} \varphi_i \left(\sum_{T \in C_\Omega(N)} u^T \right) = \\
&= \sum_{i \in M} \varphi_i(u^M) + \sum_{T \in C_\Omega(N) \colon T \neq M} \left[\sum_{i \in M} \varphi_i(u^T) \right] = \\
&= v_\Omega(M) + \sum_{T \in C_\Omega(N) \colon T \neq M} \left[\sum_{i \in M} 0 \right] = v(M),
\end{aligned}$$

in which the last equation follows from Lemma 3.28 implying that $u^T = 0$ and, so, $\varphi_i(u^T) = 0$ for all $T \neq M$.

2. *μ satisfies the isolated player property.*
   For $i \notin \cup C_\Omega(N)$ we have that $C_\Omega(S) = C_\Omega(S \setminus \{i\})$ for all $S \in \Omega$. This implies that $v_\Omega(S) = v_\Omega(S \setminus \{i\})$ and, thus, $\mu_i(v, \Omega) = 0$.
3. *μ satisfies the balanced payoff property.*
   Let $B \in B(\Omega)$, $\Omega' = \overline{B(\Omega) \setminus \{B\}}$, and consider the game $w \in \mathcal{G}^N$ given by $w = v_\Omega - v_{\Omega'}$.

   First, note that $w(S) = 0$ for all $S \not\supseteq B$. Second, for every $j \in B\colon\ w(S \setminus \{j\}) = 0$, since $B \not\subseteq S \setminus \{j\}$. Hence,

$$\varphi_j(w) = \sum_{S \colon B \subset S} \frac{(s-1)!\,(n-s)!}{n!} w(S)$$

   where $s = |S|$ and $n = |N|$. From this it follows that $\varphi_j(w) = \varphi_k(w)$ for all $k \in B$. Hence, $\mu_j(v_\Omega) - \mu_j(v_{\Omega'}) = \mu_k(v_\Omega) - \mu_k(v_{\Omega'})$. This implies that μ indeed satisfies the balanced payoff property.

This leaves it to be shown that μ is the unique allocation rule on $\mathcal{G}^N \times \mathfrak{W}^N$ that satisfies these three properties.

Let $(v, \Omega) \in \mathcal{G}^N \times \mathfrak{W}^N$ and assume that there are two allocation rules γ^1 and γ^2 that satisfy these three properties. Let $\mathfrak{B}(\Omega) = \{B \in B(\Omega) \mid |B| \geqq 2\}$ be the class of non-unitary supports in Ω. The proof of the assertion is conducted by induction on the number of sets in $\mathfrak{B}(\Omega)$.

Now suppose that there are two allocation rules γ^1 and γ^2 on $\mathcal{G}^N \times \mathfrak{W}^N$ that satisfy these three desired properties. We will show that $\gamma^1 = \gamma^2$.

If $|\mathfrak{B}(\Omega)| = 0$, then $C_\Omega(N) = \{\, \{i\} \mid \{i\} \in \Omega\}$. Applying component efficiency and the isolated player property we get that $\gamma^1 = \gamma^2$.

Now assume as the induction hypothesis that $\gamma^1(v,\Omega') = \gamma^2(v,\Omega')$ for all Ω' with $|\mathfrak{B}(\Omega')| \leqq k-1$ and let $|\mathfrak{B}(\Omega)| = k$.

Consider $C \in \mathfrak{B}(\Omega)$. Balanced payoffs now implies that there exist numbers $c, d \in \mathbb{R}$ such that for all $j \in C$:

$$\gamma_j^1(v,\Omega) - \gamma_j^1(v, \overline{B(\Omega) \setminus \{C\}}) = c, \tag{3.42}$$

$$\gamma_j^2(v,\Omega) - \gamma_j^2(v, \overline{B(\Omega) \setminus \{C\}}) = d. \tag{3.43}$$

Now by the induction hypothesis for $j \in C$:

$$\gamma_j^1(v, \overline{B(\Omega) \setminus \{C\}}) = \gamma_j^2(v, \overline{B(\Omega) \setminus \{C\}})$$

So there is a constant $a = c - d$ such that

$$\gamma_j^1(v,\Omega) - \gamma_j^2(v,\Omega) = a \;\text{ for all } j \in C. \tag{3.44}$$

Given $M \in C_\Omega(N)$, by component efficiency of γ^1 and γ^2 we arrive at

$$\sum_{i \in M} \left[\gamma_j^1(v,\Omega) - \gamma_j^2(v,\Omega) \right] = 0$$

On the other hand, it is easy to see that for every $i, j \in M$ there exists a non-unitary supports sequence $B_1, B_2, \ldots, B_p$ in $\mathfrak{B}(\Omega)$, contained in M such that $i \in B_1, j \in B_p$ and $B_q \cap B_{q+1} \neq \varnothing$ for $q = 1, \ldots, p-1$.

Using equality (3.44) we have $\gamma_k^1(v,\Omega) - \gamma_k^2(v,\Omega) = a$ for $k \in B_1$. As $B_1 \cap B_2 \neq \varnothing$, there exists $h \in B_1 \cap B_2$ such that $\gamma_h^1(v,\Omega) - \gamma_h^2(v,\Omega) = a$. Thus, applying the property above of the constructed sequence as well as Equation (3.44) recursively for all elements in the sequence $B_1, B_2, \ldots, B_p$ we get that

$$\gamma_i^1(v,\Omega) - \gamma_i^2(v,\Omega) = \gamma_j^1(v,\Omega) - \gamma_j^2(v,\Omega).$$

Hence, $\gamma_i^1(v,\Omega) - \gamma_i^2(v,\Omega) = a$ for all $i \in M$, $M \in C_\Omega(N)$. This in turn implies that

$$\sum_{i \in M} \left[\gamma_j^1(v,\Omega) - \gamma_j^2(v,\Omega) \right] = |M|\, a.$$

Therefore, $|M|\, a = 0$ and hence $\gamma_j^1(v,\Omega) = \gamma_j^2(v,\Omega)$ for all $j \in M$.

This completes the proof of Theorem 3.30.

3.6 Problems

Problem 3.1 Provide a detailed proof of the probabilistic formulation (3.4) by deriving it from the standard formulation (3.3).

Problem 3.2 Consider the value ψ^1 introduced in Example 3.8. Show that this value indeed satisfies the null-player property, symmetry and additivity. Use a counter-example to show that this value indeed is not efficient as claimed.

Problem 3.3 Provide a proof of the following properties:

(a) Show that if a value ϕ on $\mathcal{G}^N$ satisfies Young's strong monotonicity property, then it also has to satisfy Shubik's coalitional monotonicity property.
(b) Show that the Shapley value φ satisfies strong monotonicity.
(c) Consider the monotonicity property introduced by van den Brink (2007). A value ϕ is *B-monotone* if for every player $i \in N$ and every pair of games $v, w \in \mathcal{G}^N$ with $v(S) \geqq w(S)$ for all $S \subset N$ with $i \in S$ it holds that $\phi_i(v) \geqq \phi_i(w)$.

  (i) Show that B-monotonicity implies Shubik's coalitional monotonicity.
  (ii) Also construct two counter examples that show that B-monotonicity and strong monotonicity do not imply one another.

Problem 3.4 (*Nullifying player properties*) Recall that the *egalitarian value* is given by $\psi^e \colon \mathcal{G}^N \to \mathbb{R}^N$ with

$$\psi_i^e(v) = \frac{v(N)}{n} \quad \text{for all } i \in N. \tag{3.45}$$

A variation of the egalitarian solution is the *equal surplus division rule* given by $\psi^{es} \colon \mathcal{G}^N \to \mathbb{R}^N$ with

$$\psi_i^{es}(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n} \quad \text{for all } i \in N. \tag{3.46}$$

Now, van den Brink (2007) introduces a property concerning the notion of a "nullifying" player. These players reduce the value of a coalition to zero. More precisely, a player $i \in N$ is a *nullifying player* in the game v if $v(S) = 0$ for every $S \subset N$ with $i \in S$.

Now a value ϕ satisfies the *nullifying player property* if $\phi_i(v) = 0$ for every nullifying player $i \in N$ in the cooperative game $v \in \mathcal{G}^N$.

(a) As van den Brink (2007) first showed, if one replaces the null player property in Shapley's seminal axiomatization of his value, one arrives at an axiomatization of the egalitarian solution. Now prove that a value ϕ is equal to the egalitarian solution ψ^e if and only if ϕ satisfies efficiency, symmetry, additivity and the nullifying player property.
(b) We can simplify this axiomatization further by replacing additivity and the nullifying player property by a single axiom: A value ϕ satisfies *null additivity* if for every pair of games $v, w \in \mathcal{G}^N$ and every player $i \in N$ it holds that $\phi_i(v + w) = \phi_i(v)$ if i is a nullifying player in w.

Show that the value ϕ is equal to the egalitarian solution ψ^e if and only if ϕ satisfies efficiency, symmetry and null-additivity.

(c) Next consider the additional property of invariance. A value ϕ is *invariant* if for every game $v \in \mathcal{G}^N$, real number $\alpha \in \mathbb{R}$ and vector $\beta \in \mathbb{R}^N$ it holds that $\phi(\alpha v + \beta) = \alpha\, \phi(v) + \beta$, where $\alpha v + \beta \in \mathcal{G}^N$ is defined by $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{j \in S} \beta_j$ for every $S \subset N$.

Prove that a value ϕ is equal to the equal surplus division rule ψ^{es} if and only if ϕ satisfies efficiency, symmetry, additivity, the nullifying player property for the class of zero-normalized games, as well as invariance.

Problem 3.5 Let N be some player set. Show that the union stable cover of some coalitional structure $\Omega \subset 2^N$ is equal to the union stable cover of its basis $B(\Omega)$.

Problem 3.6 Let Ω be some coalitional structure and let $S \in C_\Omega(N)$ be some Ω-component of the player set N. Show that for every $i,j \in S$ with $i \neq j$ there exists some sequence of non-unitary supports $B_1, B_2, \ldots, B_p$ in $\mathfrak{B}(\Omega)$, contained in S such that $i \in B_1, j \in B_p$ and $B_q \cap B_{q+1} \neq \varnothing$ for $q = 1, \ldots, p-1$.

Problem 3.7 Consider a coalition structure in the sense of Aumann and Drèze (1974) denoted by $\Omega_a = \{\varnothing, N_1, \ldots, N_m\}$ on player set N. Here, for every $k = 1, \ldots, m$ we assume that $N_k \neq \varnothing$ and that Ω_a forms a partitioning of N.

(a) Determine the support basis $B(\Omega_a)$ of the given coalition structure Ω_a. Furthermore, for this particular case determine the union stable cover $\overline{\Omega_a}$, the global components $C_{\Omega_a}(N)$, and the Ω_a-restriction of any arbitrary game v.
(b) Check Myerson's Characterization Theorem 3.30 for the given coalitional structure Ω_a.

Problem 3.8 We recall from the discussion in the previous chapter that a communication network on N is a set of communication links $g \subset \{ij \mid i,j \in N\}$, where $ij = \{i,j\}$ is a binary set representing a communication link between players i and j. Hence, if $ij \in g$, then it is assumed that players i and j are able to communicate with each other.

Two players i and j are now connected in the network g if these two players are connected by a path in the network, i.e., there exist $i_1, \ldots, j_K \in N$ such that $i_1 = i$, $i_K = j$ and $i_k i_{k+1} \in g$ for all $k = 1, \ldots, K-1$. Now a group of players $S \subset N$ is connected in the network g if all members $i,j \in S$ are connected in g. Now Myerson (1977) introduced

$$\Omega_g = \{S \subset N \mid S \text{ is connected in } g\}.$$

The coalitional structure Ω_g is the class of connected coalitions in the communication network g. A *network situation* is now given by (v, g), where $v \in \mathcal{G}^N$ and g is a communication network on N.

The Myerson value for communication situations is now defined by $\mu(v,g) = \mu(v, \Omega_g)$.

(a) Show that for this given communication situation, the class of supports of Ω_g is given by

$$B(\Omega_g) = \{\varnothing\} \cup \{\,\{i\} \mid i \in N\} \cup g. \tag{3.47}$$

(b) Consider some allocation rule ϕ on the class of communication situations. Restate component-efficiency, the isolated player property as well as the balanced payoff property for such communication situations.
(c) Show that the Myerson value μ defined above is the unique allocation rule on the class of communication situations that satisfies these three re-stated properties.[15]

Problem 3.9 (*The Proper Shapley Value*) Vorob'ev and Liapounov (1998) introduced the so-called "proper" Shapley value. This value is the fixed point of a Shapley correspondence. In order to define this construction, let N be some player set and let $w \in \mathbb{R}^N_+$ be an N-dimensional weight vector. Now the weight of player i is given by $w_i \geqq 0$ and the weight of coalition $S \subset N$ is computed as $w(S) = \sum_{i \in S} w_i$. The w-weighted Shapley value of the game $v \in \mathcal{G}^N$ is now for every player $i \in N$ defined by

$$\varphi_i^w(v) = \sum_{S \subset N\colon i \in S} \frac{w_i}{w(S)} \cdot \Delta_v(S). \tag{3.48}$$

We can obtain the (regular) Shapley value by taking equal weights. Of course, the weight vector w can be normalized to satisfy $\sum_{i \in N} w_i = 1$, i.e., w can be selected from the $(|N| - 1)$-dimensional unit simplex

$$\mathcal{S}^{|N|-1} = \left\{ w \in R^N_+ \;\middle|\; \sum_{i \in N} w_i = 1 \right\}$$

Let $v \in \mathcal{G}^N$ be a given game. Then the *Shapley mapping* $\Phi^v\colon \mathcal{S}^{|N|-1} \to \mathbb{R}^N$ assigns to every weight vector w in the corresponding simplex the Shapley value $\Phi^v(w) = \varphi^w(v)$.

(a) Show that for every game $v \in \mathcal{G}^N$ with $v(N) = 1$ and $\Delta_v(S) \geqq 0$ for every coalition $S \subset N$, there exists a unique fixed point of the Shapley mapping Φ^v, i.e., there exists a unique weight vector $w^* \in \mathbb{R}^N\colon \Phi^v(w^*) = w^*$.
This fixed point is denoted as the *proper Shapley value* of the game v.
(b) Show that the proper Shapley value w^* for the game $v \in \mathcal{G}^N$ with $v(N) = 1$ and $\Delta_v(S) \geqq 0$ for every coalition $S \subset N$, is exactly the solution to

[15] This is the original characterization of the Myerson value developed in Myerson (1977).

$$\max_{x \in \mathcal{S}^{|N|-1}} \prod_{S \subset N} \left(\sum_{j \in S} x_j \right)^{\Delta_v(S)} \tag{3.49}$$

It should be clear that this connects the proper Shapley value to the Nash bargaining solution.

Problem 3.10 (*The Kikuta-Milnor Value*) An alternative to the Shapley value is the family of so-called "compromise" values. These values are based on a reasonable upper and lower bound for the payoffs to the players in the game. The value that is generated by these upper and lower bounds is exactly the convex combination of these bounds that is efficient. Here I consider the Kikuta-Milnor compromise value based on a minimal and maximal marginal value for all players as these respective bounds.

Let $v \in \mathcal{G}^N$ and define for every $i \in N$ her upper bound as the maximal marginal contribution given by

$$\mu_i^u(v) = \max_{S \subset N \colon i \in S} v(S) - v(S - i) \tag{3.50}$$

and her lower bound as the minimal marginal contribution given by

$$\mu_i^l(v) = \min_{S \subset N \colon i \in S} v(S) - v(S - i) \tag{3.51}$$

The Kikuta-Milnor value $\kappa \colon \mathcal{G}^N \to \mathbb{R}^N$ is now defined by

$$\kappa(v) = \lambda\, \mu^u(v) + (1 - \lambda)\, \mu^l(v) \tag{3.52}$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \kappa_i(v) = v(N)$. Prove the following statements with regard to this compromise value.

(a) For every $v \in \mathcal{G}^N \colon\ \mu^u(v) \geqq \mu^l(v)$.
(b) The Kikuta-Milnor index is well defined on the space of all games $\mathcal{G}^N$ on player set N. Provide a well-defined expression of the Kikuta-Milnor value function κ.
(c) Under which conditions on v does it hold that $\lambda \in [0, 1]$? Be complete in your answer.
(d) If v is superadditive, then $\kappa(v) \in I(v)$. Is the reverse also true? If it is, then provide a proof. Otherwise, construct a counter example.
(e) If $v \in \mathcal{G}^N$ with $n = 3$ is superadditive, then

$$\kappa(v) \in C(v) \text{ implies } \varphi(v) \in C(v).$$

(f) Recall the notion of the dual game of $v \in \mathcal{G}^N$ as the game $v^\star \in \mathcal{G}^N$ defined by $v^\star(S) = v(N) - v(N \setminus S)$, $S \subset N$. It holds that $\kappa(v) = \kappa(v^\star)$.

(g) Consider a convex game $v \in \mathcal{G}^N$. What can be said about the relationship of the Kukita-Milnor value $\kappa(v)$ and the Core $C(v)$? Explain your conclusions in detail.

Problem 3.11 (*The CIS value*) Consider another set of upper and lower bounds for a game to define another compromise value. For the game $v \in \mathcal{G}^N$ I select this time for every player $i \in N$ the following obvious lower and upper bounds:

$$m_i^l(v) = v(i) \tag{3.53}$$

and

$$m_i^u(v) = v(N) - \sum_{i \in N} m_i^l(v) = v(N) - \sum_{i \in N} v(i) \tag{3.54}$$

The CIS compromise value is now defined by

$$\gamma(v) = \lambda\, m^u(v) + (1 - \lambda)\, m^l(v) \tag{3.55}$$

where $\lambda \in [0, 1]$ is such that $\sum_{i \in N} \gamma_i(v) = v(N)$. Prove the following statements with regard to the CIS compromise value.

(a) The CIS value is well-defined for the class of weakly essential games, i.e., those games $v \in \mathcal{G}^N$ such that $\sum_N v(i) \leqq v(N)$. Compute an exact expression for the CIS compromise value γ on the class of all weakly essential games.
(b) If $v \in \mathcal{G}^N$ with $n = 3$ is superadditive, then

$$\gamma(v) \in C(v) \text{ implies } \varphi(v) \in C(v).$$

(c) Compute $\gamma(v^\star)$ for any weakly essential game v, where $v^\star$ is the dual of v.