Chapter 2
The Core of a Cooperative Game

The main fundamental question in cooperative game theory is the question how to allocate the total generated wealth by the collective of all players—the player set N itself—over the different players in the game. In other words, what binding contract between the players in N has to be written? Various criteria have been developed.

In this chapter I investigate contracts that are stable against deviations by selfish coalitions of players. Here the power of the various coalitions in the negotiation process is central. Hence, the central consideration is that coalitions will abandon the negotiation process if they are not satisfied in their legitimate demands. This is at the foundation of the concept of the *Core* of a cooperative game and subject of the present chapter.

The Core stands in contrast with (Shapley) value theory, which is explored in the next chapter. Value theory aims at balancing criteria describing negotiation power with fairness considerations. This approach leads to the fair allocation of assigned values to the different coalitions.

In certain respects these two approaches to the fundamental question of cooperative game theory conflict with each other. Core allocations are founded on pure power play in the negotiation process to write a binding agreement. It is assumed that coalitions will not give up on their demands under any circumstance. This is very different in comparison with the Value. In Value theory it is assumed that coalitions will forego demands based on the exercise of pure power, but instead abide by a given set of behavioral principles including a certain reflection of power and fairness. This leads to situations in which the Value is not a Core allocation, while in other situations the Value is a central allocation in the Core. The first case refers to a conflict between fairness and power considerations, while the second case refers essentially to the alignment of the fairness and power aspects in these two approaches.

On the other hand there is also a very interesting overlap or agreement between the two approaches based on power (the Core) and balancing power and fairness (the Value). The common feature here is that power is a common feature of these two solution concepts. Therefore, it is crucial to understand the presence and exercise of power in allocation processes in a cooperative game. I will explore these aspects in this and the next chapter as well.

R.P. Gilles, *The Cooperative Game Theory of Networks and Hierarchies*,
Theory and Decision Library C 44, DOI 10.1007/978-3-642-05282-8_2,

After defining Core allocations and discussing the Core's basic properties, I introduce the existence problem and the well-known Bondareva–Shapley Theorem. The main condition for existence identified is that of *balancedness* of the underlying cooperative game.

An important extension of the Core concept captures the fact that not all groups of individuals actually can form as functional coalitions. This results into the consideration of constraints on coalition formation and collections of so-called "institutional" coalitions which have the required governance structure to make binding agreements. The equivalent existence theorem for these partitioning games is stated as the Kaneko–Wooders theorem. The main condition for existence of the Core under constraints on coalition formation in the context of arbitrary cooperative games is that of *strong balancedness* of the underlying collection of these institutional coalitions.

Subsequently I look at the specific structure of the Core if the institutional coalitions form a lattice. This results in a rather interesting set of applications, which includes the exercise of authority in hierarchical organization structures. Finally, I discuss a number of so-called "Core covers"—collections of allocations that include all Core allocations—including the Weber set and the Selectope.

2.1 Basic Properties of the Core

The fundamental idea of the Core is that an agreement among the players in N can only be binding if every coalition $S \subset N$ receives collectively at least the value that it can generate or claim within the characteristic function form game, which is actually the generated value given by $v(S)$. This leads to the following definition.

Definition 2.1 A vector $x \in \mathbb{R}^N$ is a *Core allocation* of the cooperative game $v \in \mathcal{G}^N$ if x satisfies the efficiency requirement

$$\sum_{i \in N} x_i = v(N) \tag{2.1}$$

and for every coalition $S \subset N$

$$\sum_{i \in S} x_i \geqq v(S). \tag{2.2}$$

Notation: $x \in C(v)$.

We recall the definition of the set of imputations $I(v)$ of the game $v \in \mathcal{G}^N$ given in Definition 1.14. As introduced there, an imputation is an individually rational and efficient allocation in the cooperative game v. This is subsumed in the definition of a Core allocation. Hence, $C(v) \subset I(v)$. Throughout we will therefore also use "Core imputation" for these allocations in $C(v)$.

The notion of the Core of a cooperative game has a long history. The basic idea was already formulated by Edgeworth (1881) in his discussion of trading processes between economic subjects in a non-market trading environment. These trading processes were based on the formulation of a collective trade contract. An allocation of economic commodities has to be stable against re-barter processes within coalitions of such economic actors. Allocations that satisfied this fundamental re-trade immunity are indicated as *Edgeworthian equilibrium* allocations. The set of such Edgeworthian equilibrium allocations is also indicated as the "contract curve" by Edgeworth himself or the "Core of an economy" since the full development of this theory in the 1960s. A full account of this theory is the subject of Gilles (1996).

Upon the development of game theory, and cooperative game theory in particular, the fundamental idea of blocking was formulated by Gillies (1953) in his Ph.D. dissertation. This idea was developed further in Gillies (1959) and linked with the work of Edgeworth (1881) by Shubik (1959). Since Shubik's unification, the theory of the Core—both in cooperative game theory as well as economic general equilibrium theory—took great flight.[1]

The main appeal and strength of the Core concept is that the notion of blocking is very intuitive. It is a proper formalization of the basic power that the various groups of players have in bargaining processes. A coalition is a group of players that has the institutional structure to plan and execute actions, including the allocation of generated value over the members of that coalition. (I discussed this also in Chapter 1.) In this respect, the value $v(S)$ that coalition S can generate is fully attainable. The blocking process now allows coalitions to fully access this attainable value and allocate it to its constituting members.

2.1.1 Representing the Core of a Three Player Game

I develop the representation of the Core of a three player game in detail. Assume that $N = \{1, 2, 3\}$ and that $v \in \mathcal{G}^N$ is a $(0, 1)$-normal game. Note that under these assumptions, there are only three non-trivial coalitions with potential blocking power to be considered, $12 = \{1, 2\}$, $13 = \{1, 3\}$, and $23 = \{2, 3\}$. Each of these three coalitions might execute its power to object against a proposed imputation. Indeed, other coalitions have no potentiality to block: the singleton coalitions only attain zero, while the grand coalition evidently is excluded from potentially blocking a proposed imputation.

We can summarize these observations by applying the definition of the Core to obtain the following set of inequalities that fully describe the Core of the game v:

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[1] For more details on the history of the Core equilibrium concept I also refer to Weintraub (1985), Hildenbrand and Kirman (1988), and Gilles (1996).

$$x_1, x_2, x_3 \geqq 0 \tag{2.3}$$
$$x_1 + x_2 + x_3 = 1 \tag{2.4}$$
$$x_1 + x_2 \geqq v(12) \tag{2.5}$$
$$x_1 + x_3 \geqq v(13) \tag{2.6}$$
$$x_2 + x_3 \geqq v(23) \tag{2.7}$$

Recalling that

$$I(v) = \mathcal{S}^2 \equiv \{x \in \mathbb{R}^3_+ \mid x_1 + x_2 + x_3 = 1\},$$

it is easy to see that the Core of the game v is given by

$$C(v) = \{x \in I(v) \mid \text{ Inequalities (2.5)–(2.7) hold }\}.$$

We can therefore conclude that generically the Core of a three player game v in $(0,1)$-normalization is fully determined by the three inequalities (2.5)–(2.7) in relation to the two-dimensional simplex. This is depicted in Fig. 2.1. The three inequalities (2.5)–(2.7) are also known as the *coalitional incentive constraints* imposed by the Core.

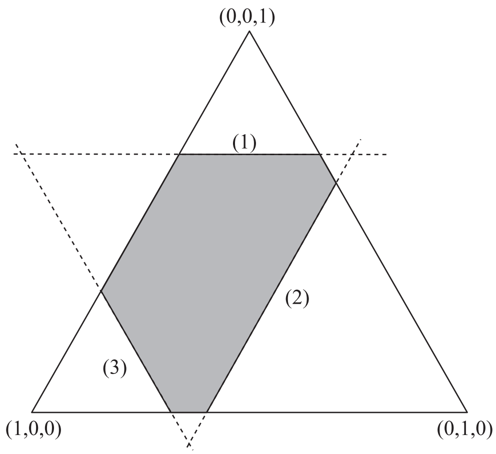

Fig. 2.1 Constructing the Core of a three player $(0,1)$-normal game

In Fig. 2.1 inequality (2.5) is depicted as (1), inequality (2.6) as (2), and inequality (2.7) as (3). The area between these three lines in the two-dimensional simplex is the set of allocations satisfying the coalitional incentive constraints for this game. (Outside the unit simplex of imputations, this area is enclosed by dashed lines.) The Core $C(v)$ is now the set of imputations satisfying these three coalitional incentive constraints; thus, $C(v)$ is the intersection of the depicted triangular area and the unit simplex $I(v)$ of imputations.

In Fig. 2.1 a situation is depicted in which $C(v) \neq \varnothing$. This is the case when there is an inverse triangle created between the three lines (1), (2) and (3). The intersection of this inverse triangle and the two-dimensional unit simplex is now the Core of the $(0, 1)$-normal game under consideration.

Next consider the case depicted in Fig. 2.2 on page 33. In this situation the three inequalities (2.5)–(2.7) do not form an inverse triangle, but rather a regular triangle. This indicates that the three inequalities are exclusive and that there are no imputations that satisfy them. In other words, this situation depicts the case of an empty Core, $C(v) = \varnothing$.

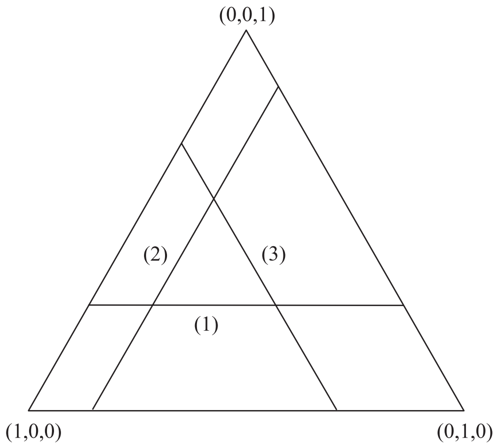

Fig. 2.2 A game with an empty Core

An example of a $(0, 1)$-normal game with three players that satisfies the conditions depicted in Fig. 2.2 is one in which

$$v(12) = v(13) = v(23) = \tfrac{3}{4}.$$

It is easy to verify that, indeed, $C(v) = \varnothing$.

Next we can elaborate on the various situations and shapes of the Core that might result in a three player game when it is not empty. First, we consider the case of a non-empty Core fully within the interior of the space of imputations. Formally, the Core of a game v is called "interior" if

$$\varnothing \neq C(v) \subset \operatorname{Int} I(v), \text{ where } \operatorname{Int} I(v) = \{x \in I(v) \mid x \gg 0\}.$$

An example of such an interior Core is given in Fig. 2.3 on page 34. The Core is well bounded away from the three facets of the two-dimensional unit simplex.

Example 2.2 Consider a $(0, 1)$-normal game with player set $N = \{1, 2, 3\}$ given by $v_1(12) = \frac{1}{2}$ and $v_1(13) = v_1(23) = \frac{3}{4}$. Now the Core is a single, interior imputation:

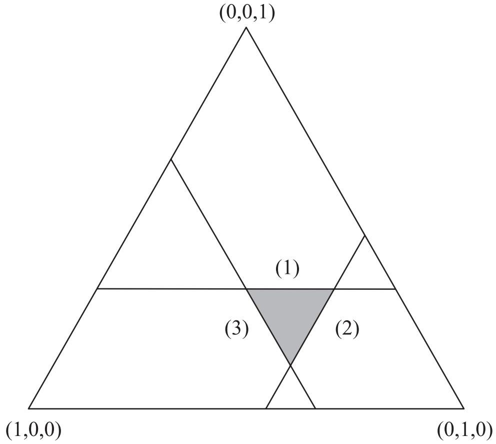

Fig. 2.3 An interior Core

$C(v_1) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$. A simple modification modification of the game v_1 given by $v_2(12) = \frac{3}{8}$ and $v_2(13) = v_2(23) = \frac{3}{4}$ results into an interior Core given by

$$C(v_2) = \text{Conv}\left\{\left(\frac{1}{8}, \frac{1}{4}, \frac{5}{8}\right), \left(\frac{1}{4}, \frac{1}{8}, \frac{5}{8}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)\right\}$$

The three corner points in this Core are determined by the intersection of two of the three coalitional incentive constraints (1), (2), and (3). ■

Second, we can consider cases in which the Core is not in the interior of the imputation space, but rather has imputations common with the boundary of the imputation space. The boundary of the imputation space $I(v)$ is defined by

$$\partial I(v) = \{x \in I(v) \mid x_i = 0 \text{ for some } i \in \{1, \ldots, n\}\}.$$

The Core of a game v is called “anchored” if

$$C(v) \cap \partial I(v) \neq \varnothing.$$

Examples of an anchored Core are given in the original depiction of the Core of a three player game in Fig. 2.1 as well as the one given in Fig. 2.4 on page 35.

The extreme case of an anchored Core is the one in which the set of imputations itself is equal to the Core. The simplest case with this property is subject of the next example.

Example 2.3 Consider the simple game $v \in \mathcal{G}^N$ with $N = \{1, 2, 3\}$ given by $v(S) = 0$ for all $S \subsetneq N$ and $v(N) = 1$. It is immediately clear that the game v is indeed simple and that the Core is equal to the imputation set, i.e., $C(v) = I(v)$. ■

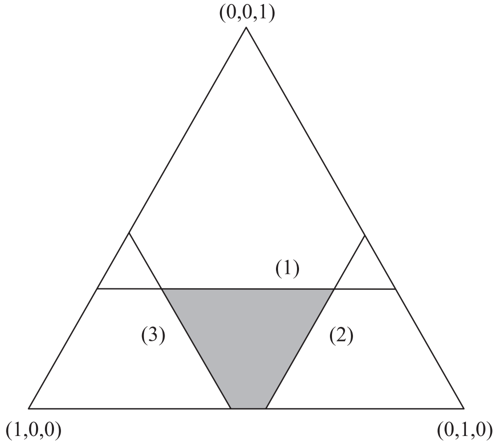

Fig. 2.4 An anchored Core

2.1.2 The Core and Domination

There is an alternative approach to the definition of the Core of a cooperative game. This approach is based on the domination of one imputation by another. The notion of domination has its roots in the seminal work on bargaining processes by Edgeworth (1881). Edgeworth considered coalitions of traders exchanging quantities of commodities in an economic bargaining process that is not based on explicit price formation, also known as Edgeworthian barter processes. Later this was interpreted as the "Core of an economy" (Hildenbrand and Kirman, 1988).

The concept of domination is well-known[2] in game theory in general, in particular within the context of normal form non-cooperative games. For cooperative games as well as non-cooperative games, the notion of dominance is essentially equivalent; the payoffs under the various situations are compared and one situation dominates the other if these payoffs are simply higher.

Definition 2.4 An imputation $x \in I(v)$ *dominates* another imputation $y \in I(v)$ with $x \neq y$ if there exists a coalition $S \subset N$ with $S \neq \varnothing$ such that the following two conditions hold

(D.1) $\sum_{i \in S} x_i \leqq v(S)$
(D.2) $x_i > y_i$ for every $i \in S$

An imputation $x \in I(v)$ is *undominated* if there is no alternative imputation $y \in I(v)$ such that y dominates x through some coalition S.

[2] In many texts, including Owen (1995), the definition of the notion of the Core of a cooperative game is simply based on the domination of imputations. The definition employed here—based on the satisfaction of all coalitions in Core allocations—is then treated as a derivative notion. Equivalence theorems are used to express under which conditions these two different approaches lead to the same sets of Core allocations.

The notion of domination is very closely related to the definition of the Core. Indeed, if $x \notin C(v)$, then there exists at least one coalition $S \subset N$ and a partial allocation $y_S \in \mathbb{R}^S_+$ for that particular coalition such that

$$\begin{aligned}&\sum_{i \in S} y_i = v(S) \\ &y_i > x_i \text{ and } y_i \geqq v(\{i\}) \text{ for all } i \in S\end{aligned} \tag{2.8}$$

In the literature this alternative definition of the Core is described by the notion of "blocking". Indeed, the pair (S, y_S) *blocks* the imputation x. Similarly, we say that a coalition $S \subset N$ can block the imputation x if there exists a partial allocation y_S for S such that the conditions (2.8) hold. The Core of the game v is now exactly the set of imputations that cannot be blocked by any coalition $S \subset N$.

Blocking and domination seem rather closely related. However, there is a subtle difference in the sense that domination is defined to be a relational property on the imputation set $I(v)$, while blocking is limited to distinguish Core imputations from non-Core imputations. This subtle difference is sufficient to arrive at different sets of imputations:

Theorem 2.5 *Let $v \in \mathcal{G}^N$ be a cooperative game. Then*

(a) *Every Core allocation $x \in C(v)$ is an undominated imputation.*
(b) *If v is superadditive, then $x \in C(v)$ if and only if x is undominated.*

For a proof of this theorem I refer to the Appendix of this chapter.

To show that the reverse of Theorem 2.5(a) is not valid unless the game is superadditive, I construct the next example.

Example 2.6 Consider the three-player game v given in the following table:

S	$\varnothing$	1	2	3	12	13	23	123
$v(S)$	0	2	2	2	5	5	5	6

Then the imputation set is a singleton given by $I(v) = \{(2,2,2)\} = \{x\}$. By definition this single imputation is undominated. (Indeed, there are no other imputations to dominate it.) However, this single imputation is not in the Core of the game v. Indeed for the coalition $S = \{1,2\}$ it holds that

$$x_1 + x_2 = 4 < 5 = v(S).$$

This implies that coalition S will object to the imputation x even though it is the only feasible allocation of the total generated wealth $v(N) = 6$. ■

2.1.3 Existence of Core Imputations

One of the main questions in the theory of the Core is the problem of the existence of Core imputations. I start the discussion with a simple investigation of games with an empty Core. The following proposition states a sufficient condition under which a cooperative game has an empty Core.

Proposition 2.7 *If $v \in \mathcal{G}^N$ is an essential constant-sum game, then $C(v) = \varnothing$.*

Proof Suppose to the contrary that the Core of v is not empty. Now select $x \in C(v)$. Then for every player $i \in N$

$$\sum_{j \in N \setminus \{i\}} x_j \geqq v(N \setminus \{i\}) \text{ and } x_i \geqq v(\{i\}).$$

Since v is a constant-sum game, it is the case that

$$v(N \setminus \{i\}) + v(\{i\}) = v(N).$$

Hence, from the above,

$$v(N) = \sum_{j \in N} x_j \geqq x_i + v(N \setminus \{i\})$$

implying that $x_i \leqq v(\{i\})$. This in turn implies that $x_i = v(\{i\})$. Therefore, by essentiality,

$$v(N) = \sum_{i \in N} x_i = \sum_{i \in N} v(\{i\}) < v(N).$$

This is a contradiction, proving the assertion. ■

The main problem regarding the existence of Core imputations was seminally solved by Bondareva (1963). However, her contribution was written in Russian and appeared in a rather obscure source. It therefore remained hidden for the game theorists in the West. Independently Shapley (1967) found the same fundamental existence theorem. Since information reached the West that Bondareva (1963) solved the problem first, this fundamental existence theorem has been known as the Bondareva–Shapley Theorem. If information had reached researchers in the West earlier, it might be known now simply as the Bondareva Theorem.

The main concept to understanding the exact requirements for a non-empty Core is that of a *balanced* collection of coalitions.

Definition 2.8 Let $\mathcal{B} \subset 2^N \setminus \{\varnothing\}$ be a collection of non-empty coalitions in the player set N. The collection $\mathcal{B}$ is *balanced* if there exist numbers $\lambda_S > 0$ for $S \in \mathcal{B}$ such that for every player $i \in N$:

$$\sum_{S \in \mathcal{B}:\, i \in S} \lambda_S = 1 \tag{2.9}$$

The numbers $\{\lambda_S \mid S \in \mathcal{B}\}$ in (2.9) are known as the *balancing coefficients* of the collection $\mathcal{B}$.

A balanced collection of coalitions B is *minimal* if it does not contain a proper subcollection that is balanced.

One can interpret balanced collections of coalitions as generalizations of partitions, using the notion of membership weights. Indeed, let $\mathcal{B} = \{S_1, \ldots, S_k\}$ be some partitioning of N, i.e., $\cup_{m=1}^{k} S_m = N$ and $S_m \cap S_l = \varnothing$ for all $m, l \in \{1, \ldots, k\}$. Then the collection $\mathcal{B}$ is balanced for the balancing coefficients $\lambda_S = 1$ for all $S \in \mathcal{B}$.

The next example gives some other cases of balanced collections, which are not partitions of the player set.

Example 2.9 Consider $N = \{1, 2, 3\}$. Then the collection $\mathcal{B} = \{12, 13, 23\}$ is balanced with balancing coefficients given by $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Indeed, every player is member of exactly two coalitions in $\mathcal{B}$, implying that the sum of the balancing coefficients for each player is exactly equal to unity.

We remark that for the three player case the collection $\mathcal{B}$ is in fact the *unique* balanced collection of coalitions that is not a partition of the set N.

Next let $N' = \{1, 2, 3, 4\}$ and consider $\mathcal{B}' = \{12, 13, 14, 234\}$. Then $\mathcal{B}'$ is balanced with balancing coefficients given by $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$. To verify this, we note that player 1 is member of the first three coalitions in $\mathcal{B}$, each having a weight of $\frac{1}{3}$. The three other players are member of one two-player coalition with player 1 and of the three-player coalition 234. So, the balancing coefficients add up to unity for these three players as well. ■

We derive some basic properties of balanced collections and state them in the next theorem. The proof of this theorem is relegated to the appendix of this chapter.

Theorem 2.10 *Let N be some player set.*

(a) *The union of balanced collections on N is balanced.*
(b) *A balanced collection is minimal if and only if it has a unique set of balancing coefficients.*
(c) *Any balanced collection is the union of minimal balanced collections.* ■

Theorem 2.10(b) immediately implies the following characterization of minimal balanced collections.

Corollary 2.11 *A minimal balanced collection on player set N consists of at most $|N|$ coalitions.*

The proof is the subject of a problem in the problem section of this chapter and is, therefore, omitted.

The fundamental Bondareva–Shapley Theorem on the existence of Core allocations now can be stated as follows:

Theorem 2.12 (Bondareva–Shapley Theorem) *For any game $v \in \mathcal{G}^N$ we have that $C(v) \neq \varnothing$ if and only if for every balanced collection $\mathcal{B} \subset 2^N$ with balancing coefficients $\{\lambda_S \mid S \in \mathcal{B}\}$ it holds that*

$$\sum_{S \in \mathcal{B}} \lambda_S \, v(S) \leqq v(N) \tag{2.10}$$

For a proof of the Bondareva–Shapley Theorem I again refer to the appendix of this chapter.

From Theorem 2.10 the Bondareva–Shapley Theorem can be restated using only *minimal* balanced collections of coalitions rather than all balanced collections.

Corollary 2.13 *Let $v \in \mathcal{G}^N$. Then $C(v) \neq \varnothing$ if and only if for every minimal balanced collection $\mathcal{B} \subset 2^N$ with balancing coefficients $\{\lambda_S \mid S \in \mathcal{B}\}$ it holds that*

$$\sum_{S \in \mathcal{B}} \lambda_S \, v(S) \leqq v(N)$$

For three player games the Bondareva–Shapley Theorem can be stated in a very intuitive fashion:

Corollary 2.14 *Let v be a superadditive three player game. Then $C(v) \neq \varnothing$ if and only if*

$$v(12) + v(13) + v(23) \leqq 2\, v(123) \tag{2.11}$$

Proof As stated before $N = \{1, 2, 3\}$ has exactly one non-trivial minimal balanced collection, namely $\mathcal{B} = \{12, 13, 23\}$ with balancing coefficients $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The assertion now follows immediately from the application of Corollary 2.13 to this particular minimal balanced collection. ■

2.2 The Core Based on a Collection of Coalitions

Until now we worked under the fundamental hypothesis that *every* group of players is a coalition in the sense that it can establish collective, purposeful behavior. Hence, *every* group of players is assumed to have an internal structure that allows its members to cooperate with each other through interaction or communication and make collective decisions. Hence, the coalition can act like a “club” and has the necessary internal communication structure. Therefore, one might say that such a coalition has a “constitution”—an institutional collective conscience.[3] This seems a rather strong assumption.

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[3] Myerson (1980) refers to such institutionally endowed coalitions as “conferences”. This seems less appropriate an indication since a conference usually refers to a formal meeting for discussion. In this regard the term of “institutional coalition” seems more to the point in an allocation process.

It is more plausible to assume that only a certain family of groups of players have a sufficient internal communication structure to operate as a coalition in cooperative game theoretic sense. Such a group of players can also be denoted as an *institutional coalition*. Formally, we introduce a collection $\Omega \subset 2^N$ of such institutional coalitions. Such a collection Ω is also called an *institutional coalitional structure* on N that describes certain institutional constraints on communication and cooperation between the various players in the game. Myerson (1980) was the first to consider the consequences of the introduction of such a limited class of institutional coalitions. He denoted the collection of institutional coalitions Ω as a "conference structure" on N.

Throughout we make the technical assumption that Ω is such that $\varnothing \in \Omega$. However, we do not assume that necessarily $N \in \Omega$, since the grand coalition N does not have to possess the appropriate structure to sustain internal communication, i.e., the grand coalition N does not have to be "institutional".

Example 2.15 Let N be an arbitrary player set. Then there are several well-investigated coalitional structures in the literature.

(a) If we choose Ω to be a partitioning of N, then Ω is called a *coalition structure*. In a seminal contribution Aumann and Drèze (1974) investigate the Core that results if only coalitions form within such a partitioning. I also refer to Greenberg (1994) for an elaborate discussion of the concept of coalition structure and its applications.

Usually these partitionings of the player set N are based on some social space or ordering. The coalitions that are feasible in such a social space are based on social neighborhoods, and are therefore indicated as *neighboring* coalitions. Consider $N = \{1, 2, 3, 4, 5\}$ where we place the players into a one-dimensional social space in the order of 1 through 5. Then an explicit example of a coalition structure is $\Omega = \{12, 3, 45\}$, where 12, 3 and 45 are the constituting neighboring coalitions.

Another example is given in a two-dimensional social space depicted in Fig. 2.5 for player set $N = \{1, 2, 3, 4, 5, 6, 7\}$. Here players are located in a two-dimensional social space and only neighboring players form institutional coalitions. In the given figure the neighborhoods are depicted as the red circles and the resulting coalition structure is $\Omega = \{123, 4, 5, 67\}$.

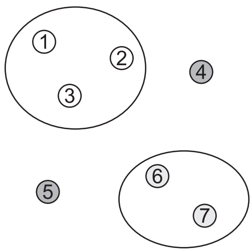

Fig. 2.5 Coalition structure of neighboring coalitions

(b) Greenberg and Weber (1986) introduced the notion of *consecutive* coalitions in the context of a public goods provision game. This refers to a specific type of neighboring coalitions based on an ordering of the players. In particular, consider the standard linear order of the player set $N = \{1, \ldots, n\}$. Now a coalition $S \subset N$ is a consecutive coalition if $S = [i,j] = \{k \in N \mid i \leqq k \leqq j\}$ for some pair of players $i,j \in N$. Hence, in a consecutive coalition every "intermediate" player is also a member of that coalition. The collection of consecutive coalitions is now given by

$$\Omega^C = \{[i,j] \mid i < j,\ i,j \in N\}.$$

In particular, $\{i\} \in \Omega^C$ for every player $i \in N$.

(c) Myerson (1977) went one step further than the case of a coalition structure and considered cooperation under explicit communication restrictions imposed by a communication network among the players. Formally, we define a *(communication) network* on N by a set of communication links

$$g \subset \{ij \mid i,j \in N\} \tag{2.12}$$

where $ij = \{i,j\}$ is a binary set. If $ij \in g$, then it is assumed that players i and j are able to communicate and, thus, negotiate with each other.

Two players i and j are *connected* in the network g if these two players are connected by a path in the network, i.e., there exist $i_1, \ldots, j_K \in N$ such that $i_1 = i$, $i_K = j$ and $i_k i_{k+1} \in g$ for all $k = 1, \ldots, K-1$. Now a group of players $S \subset N$ is connected in the network g if all members $i,j \in S$ are connected in g. Thus, Myerson (1977) introduced

$$\Omega_g = \{S \subset N \mid S \text{ is connected in } g\} \tag{2.13}$$

In Myerson (1977) the main institutional feature of an institutional coalition is therefore that its members can communicate, either directly or indirectly, with each other. This rather weak requirement would suffice to establish some constitutional structure on the coalition.

Consider the communication network depicted in Fig. 2.6. The coalitions that can form in this communication network are exactly those that are connected in this network. Hence, 146 is a coalition since its members can communicate. However, 16 is not a formable coalition since players 1 and 6 require the assistance of player 4 to communicate.

In this chapter I further explore these cases in a limited fashion. In particular, I investigate the non-emptiness of the Core for these types of restrictions on coalition formation in the discussion of applications of the Kaneko–Wooders existence theorem.

We limit our investigations to the Core allocations that are generated within the context of a coalitional structure $\Omega \subset 2^N$.

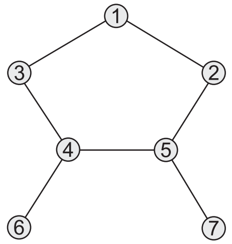

Fig. 2.6 A Myerson communication network

Definition 2.16 Let $\Omega \subset 2^N$. A vector $x \in \mathbb{R}^N$ is an Ω-*Core allocation* of the cooperative game $v \in \mathcal{G}^N$ if x satisfies the efficiency requirement

$$\sum_{i \in N} x_i = v(N) \tag{2.14}$$

and for every coalition $S \in \Omega$

$$\sum_{i \in S} x_i \geqq v(S). \tag{2.15}$$

Notation: $x \in C(\Omega, v)$.

We emphasize first that Ω-Core allocations are in general not satisfying the conditions of imputations. This is a consequence of the possibility that individual players might not be able to act independently from other players. This is particularly the case if permission is required for productive activities of such a player. Only if an individual player can disengage from such permission, she will be able to act independently.

I first investigate the geometric structure of the Ω-Core of a cooperative game for arbitrary collections of institutional coalitions $\Omega \subset 2^N$. In particular, it is clear that exploitation in Core allocations emerges if there are dependencies between different players in the game, i.e., if one player cannot act independently from another player as is the case in hierarchical authority situations.

In fact, only if the collection of institutional coalitions Ω is equal to the class of all possible groups of players 2^N it is guaranteed that the Core is in the imputation set. Indeed, $C(2^N, v) = C(v) \subset I(v)$. On the other hand if $\Omega \neq 2^N$, it is certainly not guaranteed that the resulting Core allocations are imputations. In particular, it has to hold that $\{i\} \in \Omega$ for every $i \in N$ in order that all allocations in $C(\Omega, v) \subset I(v)$.

Suppose that $x, y \in \mathbb{R}^N$ with $x \neq y$ are two allocations of $v(N)$. The difference $x - y \in \mathbb{R}^N$ is now obviously a transfer of side payments between the players in N. Such transfers are subject to the same considerations as Core allocations. Indeed, coalitions can block the execution of such transfers if they have the opportunity to provide its members with better side payments. For a class of institutional coalitions $\Omega \neq 2^N$ such stable side payments form a non-trivial class.

Definition 2.17 Let $\Omega \subset 2^N$. The set of *stable side payments* for Ω is defined by

$$\Gamma_\Omega = \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{l} \sum_N x_i = 0 \\ \sum_S x_i \geqq 0 \;\text{ for } S \in \Omega \end{array} \right\} \tag{2.16}$$

We remark first that for any $\Omega \subset 2^N$ it holds that $\Gamma_\Omega \neq \varnothing$. Indeed, $0 \in \Gamma_\Omega$ for arbitrary Ω.

For the original situation that every group of players is formable as an institutional coalition, we arrive at a trivial collection of stable side payments. Indeed, $\Gamma_{2^N} = \{0\}$. This implies that *any* proposed transfer can be blocked. Hence, there is no justified exploitation of some players by other players. This is in fact a baseline case.

The other extreme case is $\Omega = \varnothing$. In that case it is easy to see that

$$\Gamma_\varnothing = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = 0 \right\} \sim \mathbb{R}^{n-1}.$$

The above states that the set of stable side payments for the empty class of institutional coalitions is isomorph to the $(n-1)$-dimensional Euclidean space, where $n = |N|$ is the number of players. This establishes that $\Gamma_\varnothing$ is the largest set of stable side payments. This can be interpreted as that the absence of any institutional coalitions allows for arbitrary exploitation of players by others. In this regard, $\Gamma_\varnothing$ reflects a complete absence of any organization of the players.

It is straightforward to see that if one adds a stable side payment vector to a Core allocation that one obtains again a Core allocation. A formal proof of the next proposition is therefore left as an exercise to the reader.

Proposition 2.18 *Let $\Omega \subset 2^N$ and $v \in \mathcal{G}^N$. Then*

$$C(\Omega, v) = C(\Omega, v) + \Gamma_\Omega \tag{2.17}$$

To illustrate these definitions I introduce a simple example.

Example 2.19 Let $N = \{1, 2, 3\}$ and let $\Omega = \{1, 12, 123\}$. Now the set of stable side payments is given by

$$\begin{aligned} \Gamma_\Omega &= \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0,\ x_1 + x_2 \geqq 0 \,\text{and}\, x_1 \geqq 0\} = \\ &= \{(\lambda, \mu - \lambda, -\mu) \mid \lambda, \mu \geqq 0\} = \\ &= \text{Cone}\,\{(1, -1, 0), (0, 1, -1)\}. \end{aligned}$$

From the properties of the collection Γ_Ω it follows immediately that player 1 has extraordinary power. She is able to get arbitrary transfers made to her from the other players. in this regard, player 1 has a *middleman position* in the interaction network among the three players. The only limitation is that player 2 has to be kept reasonably satisfied as well and should be treated better than player 3. However, player

1 can extract significant surpluses from her middleman position in the interaction network.

Another interpretation of the coalitional structure Ω is that the institutional coalitions are in fact generated through the implementation of an authority hierarchy: Player 1 is the top player, then follows player 2, and finally player 3 is at the bottom of this simple hierarchy. Hence, the middleman position of player 1 is equivalent to being the superior in an authority situation in which she exercises veto power over the actions considered by the other two players. ■

The discussion in these lecture notes will concentrate on the question under which conditions the set of stable side payments is the singleton $\{0\}$, describing the case of no exploitation. Subsequently I discuss the case of hierarchical authority structures and the resulting constraints on coalition formation and the structure of the Core.

I will not address the problem of the existence of restricted Core allocations. Faigle (1989) investigated the precise conditions on the collection of coalitions Ω and the cooperative game $v \in \mathcal{G}^N$ such that $C(\Omega, v)$ is non-empty.

A Reformulation Restricting the ability of certain groups of players to form a proper institutional coalition in the sense of a value-generating organized entity has consequences for what constitutes a Core allocation. If only certain groups of players can form as value-generating entities, then the generated values in the cooperative game should be based on these institutional coalitions only. This is exactly the line of reasoning pursued by Kaneko and Wooders (1982). They define a corresponding cooperative game that takes into account that only certain groups of players can form institutional coalitions.

Definition 2.20 Let $\Omega \subset 2^N$ be a collection of institutional coalitions such that $\{i\} \in \Omega$ for every $i \in N$.

(i) Let $S \subset N$ be an arbitrary coalition. A collection $P(S) \subset \Omega$ is called a Ω-*partition* of S if $P(S)$ is a partitioning of S consisting of institutional coalitions only. The family of all Ω-partitions of S is denoted by $\mathcal{P}_\Omega(S)$.
(ii) Let $v \in \mathcal{G}^N$ be an arbitrary cooperative game. The Ω-*partitioning game* corresponding to v is the cooperative game $v_\Omega \in \mathcal{G}^N$ given by

$$v_\Omega(S) = \max_{P(S) \in \mathcal{P}_\Omega(S)} \sum_{T \in P(S)} v(T) \tag{2.18}$$

for all coalitions $S \subset N$.

The following proposition is rather straightforward and, therefore, stated without a proof, which is left to the reader.

Proposition 2.21 (Kaneko and Wooders, 1982) *Let $\Omega \subset 2^N$ be a collection of institutional coalitions such that $\{i\} \in \Omega$ for every $i \in N$ and let $v \in \mathcal{G}^N$ be an arbitrary cooperative game. Then the Ω-Core of the game v is equivalent to the (regular) Core of the corresponding Ω-partitioning game:*

$$C(\Omega, v) = C\,(v_\Omega)\,.$$

For the standard case $\Omega = 2^N$ this translates into the following corollary:

Corollary 2.22 *The Core of an arbitrary cooperative game $v \in \mathcal{G}^N$ is equal to the Core of its* ***superadditive cover*** *$\hat{v} \in \mathcal{G}^N$ given by*

$$\hat{v}(S) = \max_{P_S \in \mathcal{P}(S)} \sum_{T \in P_S} v(T),$$

where $\mathcal{P}(S)$ is the collection of all finite partitions of the coalition $S \subset N$.

2.2.1 Balanced Collections

This section is based on Derks and Reijnierse (1998). They investigate the properties of the set of stable side payments Γ_Ω in case the collection of institutional coalitions Ω is non-degenerate and/or balanced. They show that balancedness again plays a central role with regard to exploitation within such coalitionally structured games.

A collection $\Omega \subset 2^N$ has a *span* given by all coalitions that can be generated by Ω. Formally,

$$\mathrm{Span}\,(\Omega) = \left\{ S \subset N \;\middle|\; \chi_S = \sum_{R \in \Omega} \lambda_R \chi_R \ \text{ for some } \lambda_R \in \mathbb{R},\ R \in \Omega \right\} \tag{2.19}$$

where we recall that $\chi_S \in \{0,1\}^N$ is the indicator vector of coalition $S \subset N$ with $\chi_S(i) = 1$ if and only if $i \in S$.

A collection of institutional coalitions $\Omega \subset 2^N$ is *non-degenerate* if $\mathrm{Span}\,(\Omega) = 2^N$. This implies that the indicator vectors $\{\chi_R \mid R \in \Omega\}$ span the whole allocation space $\mathbb{R}^N$, i.e., these indicator vectors form a basis of $\mathbb{R}^N$.

Proposition 2.23 (Derks and Reijnierse, 1998, Theorem 4) *Let $\Omega \subset 2^N$. The set of stable side payments Γ_Ω is a pointed cone if and only if Ω is non-degenerate.*

Proof Remark that Γ_Ω is a pointed cone if and only if for any $x \in \mathbb{R}^N$ with $\sum_S x_i = 0$ for all $S \in \Omega$ we must have $x = 0$. This in turn is equivalent to the requirement that $\{\chi_S \mid S \in \Omega\}$ spans the Euclidean space $\mathbb{R}^N$. The latter is by definition the property that Ω is non-degenerate. ■

To illustrate this result and the definition of a non-degenerate collection of institutional coalitions we return to Example 2.19.

Example 2.24 Consider the Ω described in Example 2.19. We claim that this collection is non-degenerate. Indeed,

$$\{\chi_S \mid S \in \Omega\} = \{(1,0,0), (1,1,0), (1,1,1)\}\,,$$

which is a basis for $\mathbb{R}^3$. This implies immediately that Ω is in fact spanning 2^N.

Also, it is immediately clear that Γ_Ω is a pointed cone. In fact we can write

$$\Gamma_\Omega = \text{Cone }\{(1,-1,0),(1,0,-1),(0,1,-1)\}$$

which lists the directional vectors of exploitative relationships within the hierarchy described in Example 2.19. Again it is clear from these descriptions that the middleman (Player 1) has power to exploit the situation to the detriment of the two other players 2 and 3. ∎

We note that the non-degeneracy requirement is rather closely related to the balancedness condition of the collection Ω. For a balanced collection it is required that the balancing coefficients are strictly positive, while for non-degeneracy these coefficients can be arbitrary. A similar insight as Proposition 2.23 is therefore to be expected.

Proposition 2.25 (Derks and Reijnierse, 1998, Theorem 5) *Let $\Omega \subset 2^N$. The set of stable side payments Γ_Ω is a linear subspace of $\mathbb{R}^N$ if and only if Ω is balanced.*

Proof Suppose first that Ω is balanced with balancing coefficients λ. Thus, for short, $\chi_N = \sum_{S\in\Omega} \lambda_S \chi_S$.

Let $x \in \Gamma_\Omega$ and $S \in \Omega$. Then

$$-\sum_{i\in S} x_i = x \cdot \frac{1}{\lambda_S}\left(\sum_{T\in\Omega:\, T\neq S} \lambda_T \chi_T - \chi_N\right) = \frac{1}{\lambda_S}\sum_{T\in\Omega:\, T\neq S} \lambda_T \sum_{i\in T} x_i \geqq 0.$$

Since $S \in \Omega$ is arbitrary this shows that $-x \in \Gamma_\Omega$. Hence, Γ_Ω is a linear subspace of $\mathbb{R}^N$.

To show the converse, assume that Γ_Ω is a linear subspace of $\mathbb{R}^N$. The inequalities $\sum_N x_i \leqq 0$ and $\sum_S x_i \geqq 0$ for $S \in \Omega$ now imply that $\sum_T x_i \leqq 0$ for arbitrary $T \in \Omega$.

Using Farkas' Lemma (Rockafellar, 1970, Corollary 22.3.1) for linear spaces we can write

$$-\chi_T = \sum_{S\in\Omega} \mu_T(S)\chi_S - \nu_T\chi_N \tag{2.20}$$

for some well chosen coefficients $\mu_T \geqq 0$ and $\nu_T \geqq 0$. Adding (2.20) over all $T \in \Omega$ leads to

$$\nu\, \chi_N = \sum_{S\in\Omega}\left(1 + \sum_{T\in\Omega} \mu_T(S)\right)\chi_S \text{ with } \nu = \sum_{T\in\Omega} \nu_T.$$

Without loss of generality we may assume that $\nu > 0$. This in turn implies that Ω is in fact balanced. ∎

From Propositions 2.23 and 2.25 we immediately conclude the following:

Corollary 2.26 *Let $\Omega \subset 2^N$. There is no exploitation, i.e., $\Gamma_\Omega = \{0\}$, if and only if Ω is balanced as well as non-degenerate.*

In many ways this Corollary is the main conclusion from the analysis in this section. It provides a full characterization of the conditions under which there is no exploitation in stable side payments.

2.2.2 Strongly Balanced Collections

In the previous discussion I limited myself to the identification of some descriptive properties of an Ω-Core. In particular I identified the properties of the collection Ω of institutional coalitions that induce the resulting Ω-Core to be a linear space or a pointed cone. Thus far we did not discuss yet the non-emptiness of such Cores in the tradition of the Bondareva–Shapley Theorem 2.12.

The Core based on a collection Ω of institutional coalitions has the remarkable property that certain properties of the collection Ω imply that $C(\Omega, v) \neq \varnothing$ for *every* game $v \in \mathcal{G}^N$. This property is referred to as *strong balancedness* and was seminally explored in Kaneko and Wooders (1982). The main existence result of Kaneko and Wooders (1982) is therefore an extension of the Bondareva–Shapley Theorem 2.12.

I follow the formulation of the strong balancedness condition developed in Le Breton, Owen, and Weber (1992).

Definition 2.27 A collection $\Omega \subset 2^N$ of institutional coalitions is *strongly balanced* if $\{i\} \in \Omega$ for every player $i \in N$ and every balanced sub-collection $\mathcal{B} \subset \Omega$ contains a partitioning of N.

The remarkable extension of the Bondareva–Shapley Theorem can now be stated as follows. For a proof of this existence result, the Kaneko–Wooders theorem, we refer to the appendix of this chapter.

Theorem 2.28 (Kaneko and Wooders, 1982) *Let $\Omega \subset 2^N$ be some collection of institutional coalitions. For every cooperative game $v \in \mathcal{G}^N$ its Ω-Core is non-empty, i.e., $C(\Omega, v) \neq \varnothing$ for all $v \in \mathcal{G}^N$, if and only if $\Omega \subset 2^N$ is strongly balanced.*

Le Breton et al. (1992) discuss the strong balancedness property in the context of the two well-known and -explored examples of consecutive coalitions and communication networks introduced in Example 2.15.

First, it is immediately clear that the collection of consecutive coalitions Ω^C is obviously strongly balanced. Indeed, the players are ordered using the standard number ordering and all consecutive coalitions based on this ordering are in the collection Ω^C.

Second, in Myerson's communication network model one can ask the question when the collection of connected coalitions is strongly balanced. For the discussion of this we introduce some auxiliary concepts from graph or network theory.

Let $g \subset \{ij \mid i,j \in N,\ i \neq j\}$ be some network on the player set N, where $ij = \{i,j\}$ is an unordered pair of players. A *path* in the network g is a sequence of players $i_1, \ldots, i_K \in N$ such that $i_k i_{k+1} \in g$ for every $k \in \{1, \ldots, K-1\}$. A path consisting of the players $i_1, \ldots, i_K \in N$ in g is a *cycle* if $K \geqq 3$ and $i_1 = i_K$. Thus, a cycle is a communication path from a player to herself that consists of more than two players. Now a network g on N is *acyclic* if it does not contain any cycles.

Using the notion of an acyclic network we now arrive at the following characterization of strongly balanced collections in the Myerson network model.

Proposition 2.29 (Le Breton, Owen, and Weber, 1992) *Let g be a communication network on the player set N and let $\Omega_g \subset 2^N$ be the corresponding collection of connected coalitions in g. Then the collection Ω_g is strongly balanced if and only if the communication network g is acyclic.*

For a proof of this proposition I refer to the seminal source, Le Breton et al. (1992, pp. 421–422).

This main insight from Le Breton et al. (1992) shows that cycles in networks have a very important role in the allocation of values among the players. Communication cycles usually result in ambiguities about how players are connected and offer players multiple paths to link to other players to form coalitions. This results into auxiliary blocking opportunities, thus causing the Core to be possibly empty. A similar insight was derived by Demange (1994) for a slightly more general setting.

2.2.3 Lattices and Hierarchies

We continue with the discussion of a very common economic and social phenomenon, namely the consequences of the exercise of authority in hierarchically structured organizations.[4] We limit ourselves here to the study of the consequences of authority on coalition formation and the resulting Core.

The seminal contribution to cooperative game theory introducing hierarchical authority constraints on coalition formation was made by Gilles, Owen, and van den Brink (1992), who discussed so-called conjunctive authority structures. The next definition presents the formal introduction of such authority structures.

Definition 2.30 A map $H \colon N \to 2^N$ is a *permission structure* on player set N if it is irreflexive, i.e., $i \notin H(i)$ for all $i \in N$.

A permission structure $H \colon N \to 2^N$ is *strict* if it is acyclic, i.e., there is no sequence of players $i_1, \ldots, i_K \in N$ such that $i_{k+1} \in H(i_k)$ for all $k \in \{1, \ldots, K-1\}$ and $i_1 = i_K$.

The players $j \in H(i)$ are called the *subordinates* of player i in H. Similarly, the players $j \in H^{-1}(i) = \{j \in N \mid i \in S(j)\}$ are called player i's *superiors* in H. In certain permission structures all players might have subordinates as well as superiors.

[4] Such hierarchically structured organizations are very common in our society. Most production activities in the capitalist society are organized in hierarchical firms. Authority of superior over subordinates is very common and well accepted. The study of these organization structures has resulted into a very sizeable literature.

However, in a strict permission structure there are players that have no superiors and there are players that have no subordinates. In that respect a strict permission structure represents a true hierarchy.

Coalition formation in permission structures can be described in various different ways. The most straightforward description is developed in Gilles et al. (1992). We extend the notation slightly and denote by $H(S) = \cup_{i \in S} H(i)$ the subordinates of the coalition $S \subset N$.

Definition 2.31 Let H be a permission structure on N. A coalition $S \subset N$ is *conjunctively autonomous* in H if $S \cap H(N \setminus S) = \varnothing$. The collection of conjunctively autonomous coalitions is denoted by

$$\Omega_H = \{S \subset N \mid S \cap H(N \setminus S) = \varnothing\}.$$

Under the conjunctive approach[5] a coalition is autonomous if it does not have a member that has a superior outside the coalition. Hence, an autonomous coalition is hierarchically "self-contained" in the sense that all superiors of its members are members as well.

It is clear that $\Omega_H \subset 2^N$ is a rather plausible description of institutional coalitions under an authority structure H. Indeed, a coalition has to be autonomous under the authority structure H to be acting independently. It is our goal in this section to study the Core of such a collection of conjunctively autonomous coalitions.

We will first show that the introduction of a conjunctive authority structure on coalition formation is equivalent to putting lattice constraints on the collection of institutional coalitions. The material presented here is taken from Derks and Gilles (1995), who address the properties a these particular collections of coalitions.

Definition 2.32 A collection of coalitions $\Omega \subset 2^N$ is a *lattice* if $\varnothing, N \in \Omega$ and $S \cap T$, $S \cup T \in \Omega$ for all $S, T \in \Omega$.

A lattice is a collection that is closed for taking unions and intersections. Since N is finite, it therefore is equivalent to a topology on N.

We introduce some auxiliary notation. For every $i \in N$ we let

$$\partial_i \Omega = \cap\{S \in \Omega \mid i \in S\} \in \Omega \tag{2.21}$$

The map ∂_i simply assigns to player i the smallest institutional coalition that player i is a member of. We observe that $\partial_j \Omega \subset \partial_i \Omega$ if $j \in \partial_i \Omega$. Finally, we remark that

$$\partial \Omega = \{\partial_i \Omega \mid i \in N\}$$

---

[5] The term "conjunctive" refers to the veto power that can be exercised within the hierarchy introduced. Indeed each superior is in fact assumed to have full veto power over her subordinates. Only if all superiors of a player sign off in unison—or conjunctively—this player is allowed to go ahead with her planned actions.

is a basis of the lattice Ω, i.e., every coalition in Ω can be written as a union of $\partial\Omega$-elements. Also, $\partial\Omega$ is the unique minimal—or "smallest"—basis of the lattice Ω.

The link of lattices of institutional coalitions is made in the following result. For a proof I refer to the appendix of this chapter.

Theorem 2.33 *Let $\Omega \subset 2^N$ be any collection of coalitions. The following statements are equivalent:*

(i) *Ω is a lattice.*
(ii) *There is a permission structure H on N such that $\Omega = \Omega_H$.*

It is rather remarkable that hierarchical authority structures generate lattices and that each lattice corresponds to some authority structure. For such situations we can provide a complete characterization of the set of stable side payments. The proof of the next theorem is relegated to the appendix of this chapter.

Theorem 2.34 (Derks and Gilles, 1995, Theorem 2.3) *If $\Omega \subset 2^N$ is a lattice on N, then*

$$\Gamma_\Omega = \text{Cone}\left\{e_j - e_i \mid i \in N \text{ and} j \in \partial_i\Omega\right\} \tag{2.22}$$

where $e_h = \chi_{\{h\}} \in \mathbb{R}^N_+$ is the h-th unit vector in $\mathbb{R}^N$, $h \in N$.

Next we restrict our attention to a special class of lattices or authority structures. This concerns the class of lattices that is generated by strictly hierarchical authority structures. The main property that characterizes this particular class is that each individual player can be fully identified or determined through the family of coalition that she is a member of.

Definition 2.35 A collection of coalitions $\Omega \subset 2^N$ on N is *discerning* if for all $i,j \in N$ there exists some coalition $S \in \Omega$ such that either $i \in S$ and $j \notin S$ or $i \notin S$ and $j \in S$.

The properties that a discerning lattice satisfies is listed in the following theorem.[6] For a proof we again refer to the appendix of this chapter.

Theorem 2.36 *Let $\Omega \subset 2^N$ be a lattice on N. Then the following statements are equivalent:*

(i) *Ω is a discerning lattice.*
(ii) *For all $i,j \in N$: $\partial_i\Omega \neq \partial_j\Omega$.*
(iii) *For every $S \in \Omega \setminus \{\varnothing\}$ there exists a player $i \in S$ with $S \setminus \{i\} \in \Omega$.*
(iv) *There exists a strict permission structure H such that $\Omega = \Omega_H$.*

---

[6] A property that is not listed in this theorem is that a discerning lattice on N corresponds to a topology that satisfies the T_0-separation property.

(v) *For every player* $i \in N$: $\partial_i \Omega \setminus \{i\} \in \Omega$.
(vi) Ω *is a non-degenerate lattice.*

From this list of properties the following two conclusions can be drawn:

Corollary 2.37

(a) *The set of stable side payments* Γ_Ω *for a lattice* Ω *is a pointed cone if and only if* Ω *is discerning.*
(b) *The set of all coalitions* 2^N *is the only lattice* Ω *for which* $\Gamma_\Omega = \{0\}$.

We further explore some properties of the Core under lattice constraints on coalition formation in the next section on so-called Core catchers or Core covers.

2.3 Core Covers and Convex Games

We complete our discussion of the Core of a cooperative game by reviewing some supersets of the Core, also called "Core covers" or "Core catchers". Formally, a *Core cover* is a map $\mathcal{K} \colon \mathcal{G}^N \to 2^{\mathbb{R}^N}$ such that $C(v) \subset \mathcal{K}(v)$. I limit the discussion to two recently introduced and well known Core cover concepts, the Weber set (Weber, 1988) and the Selectope (Derks, Haller, and Peters, 2000).

The *Weber set* is founded on the principle that players enter the cooperative game in random order and collect their marginal contribution upon entering. The random order of entry is also called a "Weber string" and the generated payoff vector is called the corresponding "marginal (payoff) vector". Arbitrary convex combinations of such marginal vectors now represent players entering the game and collecting payoff with arbitrary probability distributions. Hence, the Weber set—defined as the convex hull of the set of these marginal vectors—is now the set of all these expected payoffs. In general the Core is a subset of the Weber set.

The *Selectope* is based on a closely related principle. Coalitions are assigned their Harsanyi dividend in the cooperative game that they participate in. A single member of each coalition is now selected to collect this dividend. These selections are in principle arbitrary. Now the Selectope—being the convex hull of all corresponding payoff vectors to such selected players—can be shown to be usually a superset of the Weber set.

2.3.1 The Weber Set

Throughout this section we develop the theory of the Weber set for a *discerning lattice* of coalitions $\Omega \subset 2^N$.[7] The main results regarding the Weber set and its rela-

[7] Equivalently, we may assume that the player set N is endowed with an acyclic or strict permission structure H with $\Omega_H = \Omega$. Throughout this section I prefer to develop the theory of the Weber set for the concept of a discerning lattice.

tion to the Core are developed for general discerning lattices of feasible institutional coalitions. We state results for the regular Core as corollaries of these main theorems for $\Omega = 2^N$.

Definition 2.38 Let $v \in \mathcal{G}^N$.

(i) A *Weber string* in Ω is a permutation $\rho\colon N \rightleftarrows N$ such that $\{\rho(1), \dots, \rho(k)\} \in \Omega$ for every $k \in N$.
(ii) The *Weber set* for v is given by

$$\mathcal{W}(\Omega, v) = Conv\,\{x^\rho \in \mathbb{R}^N \mid \rho \text{ is a Weber string in } \omega\}, \tag{2.23}$$

where $Conv\,(X)$ is the convex hull[8] of $X \subset \mathbb{R}^N$ and for every Weber string ρ in Ω we define

$$x_i^\rho = v(R_i) - v(R_i \setminus \{i\}) \tag{2.24}$$

with $R_i = \{\rho(1), \dots, \rho(j)\}$, where $j \in N$ is such that $\rho(j) = i$.

The existence of Weber strings have to be established in order to guarantee the non-emptiness of the Weber set.

Lemma 2.39

(a) *For a lattice Ω it holds that there exists a Weber string in Ω if and only if Ω is discerning.*
(b) *Every permutation is a Weber string in 2^N implying that the regular Weber set is given by*

$$\mathcal{W}(v) = \mathcal{W}(2^N, v) = \mathrm{Conv}\,\{x^\rho \mid \rho \textit{ is a permutation on } N\}.$$

The proof of Lemma 2.39(a) is relegated to the problem section of this chapter. Lemma 2.39(b) is straightforward.

The Weber set $\mathcal{W}(\Omega, v)$ on a discerning lattice Ω has some appealing properties. Let H be an acyclic permission structure such that $\Omega = \Omega_H$. Let $S \in \Omega$ and consider the corresponding unanimity game $u_S \in \mathcal{G}^N$. Then

$$\mathcal{W}(\Omega, u_S) = \mathrm{Conv}\,\{e_i \mid i \in D_H(S)\}\ \text{with} D_H(S) = \{i \in S \mid H(i) \cap S = \varnothing\}.$$

[8] The convex hull of a set X is defined as

$$\mathrm{Conv}\,(X) = \left\{ \sum_{k=1}^{M} \lambda_k \cdot x_k \;\middle|\; x_1, \dots, x_m \in X \text{ and } \sum_{k=1}^{m} \lambda_k = 1 \right\}.$$

This means that in the Weber set of a unanimity game the coalition's wealth is fully distributed over the players in the lowest level of the hierarchical structure H restricted to the coalition S.

The main theorem that determines the relationship of the Core with the Weber set is stated for convex games. The next definition generalizes the convexity notion seminally introduced by Shapley (1971).

Definition 2.40 A game $v \in \mathcal{G}^N$ is *convex* on the discerning lattice Ω if for all $S, T \in \Omega$:

$$v(S) + v(T) \leqq v(S \cup T) + v(S \cap T). \tag{2.25}$$

The main theorem can now be stated. For a proof I refer to the appendix of this chapter.

Theorem 2.41 (Derks and Gilles, 1995) *Let $\Omega \subset 2^N$ be a discerning lattice and $v \in \mathcal{G}^N$.*

(a) $C(\Omega, v) \subset \mathcal{W}(\Omega, v) + \Gamma_\Omega$.
(b) *The game v is convex on Ω if and only if*

$$C(\Omega, v) = \mathcal{W}(\Omega, v) + \Gamma_\Omega.$$

From the fact that $\Gamma_{2^N} = \{0\}$ the original seminal results from Shapley (1971) and Ichiishi (1981) can be recovered from Theorem 2.41.

Corollary 2.42 *Let $v \in \mathcal{G}^N$. Then*

(a) $C(v) \subset \mathcal{W}(v)$ *and*
(b) $C(v) = \mathcal{W}(v)$ *if and only if v is convex on 2^N.*

2.3.2 The Selectope

The Selectope has seminally been introduced by Hammer, Peled, and Sorensen (1977) and was further developed by Derks, Haller, and Peters (2000). In this section I will develop the theory of the Selectope through the analysis presented in Derks et al. (2000). Throughout this section I refer for detailed proofs of the main propositions to that paper. It would demand too much to develop these elaborate proofs here in full detail. In the previous section we developed the theory of the Weber set for discerning lattices of coalitions; for the Selectope we return to the standard case considering all feasible coalitions, i.e., $\Omega = 2^N$.

The Selectope is a cousin of the Weber set in the sense that both concepts consist of convex combinations of marginal value allocations. In the Weber set these marginal values are based on Weber strings. In the Selectope these marginal values are based on selectors.

Definition 2.43 Let $v \in \mathcal{G}^N$.

A *selector* is a function $\alpha \colon 2^N \to N$ with $\alpha(S) \in S$ for all $S \neq \varnothing$.

The *selector allocation* corresponding to the selector α is $m^\alpha \in \mathbb{R}^N$ defined by

$$m_i^\alpha(v) = \sum_{S \colon i = \alpha(S)} \Delta_v(S), \tag{2.26}$$

where $\Delta_v(S)$ is the Harsanyi dividend of S in the game v.

The *Selectope* of the game v is now defined as

$$\mathcal{S}(v) = Conv \left\{ m^\alpha(v) \in \mathbb{R}^N \;\middle|\; \alpha \colon 2^N \to N \text{ is a selector} \right\} \tag{2.27}$$

The Selectope of a game is the collection of all reasonable allocations based on the coalitional dividends that are generated in this game. Our goal is to provide a characterization of the Selectope of any game. For that we have to develop some more notions.

Recall that we can decompose a game v using the unanimity basis by $s = \sum_{S \subset N} \Delta_v(S) u_S$. Now we introduce a positive and negative part of the game v by

$$\begin{aligned} v^+ &= \sum_{S \colon \Delta_v(S) > 0} \Delta_v(S)\, u_S \\ v^- &= \sum_{S \colon \Delta_v(S) < 0} -\Delta_v(S)\, u_S \end{aligned}$$

It immediately follows that $v = v^+ - v^-$. We also mention here that a game is called *positive* if $v = v^+$, *negative* if $v = -v^-$, and *almost positive* if $\Delta_v(S) \geqq 0$ for all $S \in 2^N$ with $|S| \geqq 2$.[9]

Definition 2.44 The *dual* of a game $v \in \mathcal{G}^N$ is defined by the game $v^* \in \mathcal{G}^N$ with

$$v^*(S) = v(N) - v(N \setminus S) \text{ for all } S \subset N.$$

The following proposition combines Lemma 3 and Theorems 1 and 2 of Derks et al. (2000). For a proof of the statements in this proposition I refer to that paper.

Proposition 2.45 *Let* $v \in \mathcal{G}^N$.

(a) $\mathcal{S}(v) = C(v^+) - C(v^-)$.
(b) *The Selectope* $\mathcal{S}(v)$ *of* v *is equal to the Core of the convex game* $\tilde{v} = v^+ + \left(-v^-\right)^*$, *which in turn implies that* $\mathcal{S}(v) = \mathcal{W}(\tilde{v})$.
(c) *The following statements are equivalent:*

[9] A game is therefore almost positive if it can be written as the sum $v = v^+ + \sum_{i \in N^-} v(\{i\})\, u_i$, where $N^- = \{i \in N \mid v(\{i\}) < 0\}$.

(i) *$C(v) = \mathcal{S}(v)$;*
(ii) *$\mathcal{S}(v) \subset I(v)$, and*
(iii) *v is almost positive.*

Furthermore, $\mathcal{S}(v) = I(v)$ if and only if v is additive.

Since the Selectope is usually strictly larger than the Weber set as well as the Core of a game, it is a concept that is of less interest than the Core of a game. In that regard the Selectope is indeed a true Core "catcher".

This completes our discussion of the Selectope of a game. For further analysis of the Selectope I refer to Derks et al. (2000).

2.4 Appendix: Proofs of the Main Theorems

Proof of Theorem 2.5

Proof of (a)
Let $x \in C(v) \subset I(v)$ and suppose that there exists some $y \in I(v)$ that dominates x through the coalition S. Then it has to hold that $x_i < y_i$ for every player $i \in S$ and, additionally, that $\sum_{i \in S} y_i \leqq v(S)$. Therefore,

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leqq v(S),$$

contradicting the Core requirement that $\sum_{i \in S} x_i \geqq v(S)$.

Proof of (b)
Let v be superadditive. Since assertion (a) has been shown above, it only remains to be proven that every undominated imputation is in the Core of v.

Let $x \in I(v)$ be such that $x \notin C(v)$. Then it has to follow that $\sum_N x_i \neq v(N)$ and/or $\sum_S x_i > v(S)$ for some coalition $S \subset N$.

By definition of an imputation the first is impossible; therefore, the latter holds, i.e., $\sum_S x_i > v(S)$ for some coalition $S \subset N$.

Now let $\varepsilon = v(S) - \sum_{i \in S} x_i > 0$. Also define

$$\alpha = v(N) - v(S) - \sum_{j \in N \setminus S} v(\{j\}).$$

Then by superadditivity it holds that $\alpha \geqq 0$.

Finally, we introduce the following imputation:

$$y_j = \begin{cases} x_j + \frac{\varepsilon}{|S|} & \text{if} j \in S \\ v(\{j\}) + \frac{\alpha}{n - |S|} & \text{if} j \notin S \end{cases}$$

First, we remark that $y_j \geqq v(\{j\})$ for all $j \in N$, since $x_i \geqq v(\{i\})$ for all $i \in S$ and the definition of y given above. Furthermore, it follows from the definition of y that

$$\begin{aligned}\sum_{j \in N} y_j &= \sum_S x_i + \varepsilon + \sum_{N \setminus S} v(\{j\}) + \alpha = \\ &= \sum_S x_i + \varepsilon + v(N) - v(S) = v(N).\end{aligned}$$

Hence, we conclude that $y \in I(v)$. Moreover, from the definition we have that $y_i > x_i$ for all $i \in S$. Together this in turn implies that y dominates x through coalition S. This is a contradiction.

Proof of Theorem 2.10

This proof is based on the structure presented in Owen (1995, pp. 226–228).

Let N be some player set. We first show an intermediate result that is necessary for the proof of the three assertions stated in Theorem 2.10.

Lemma 2.46 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two balanced collections on N such that $\mathcal{B}_1 \subsetneq \mathcal{B}_2$. Then there exists a balanced collection $\mathcal{C}$ on N such that $\mathcal{B}_1 \cup \mathcal{C} = \mathcal{B}_2$ with $\mathcal{C} \neq \mathcal{B}_2$. Furthermore, the balancing coefficients for $\mathcal{B}_2$ are not unique.*

Proof Without loss of generality we may write

$$\begin{aligned}\mathcal{B}_1 &= \{S_1, \dots, S_k\} \\ \mathcal{B}_2 &= \{S_1, \dots, S_k, S_{k+1}, \dots, S_m\}\end{aligned}$$

with balancing coefficients $(\lambda_1, \dots, \lambda_k)$ and $(\mu_1, \dots, \mu_m)$ respectively.

Let $t > 0$ be any number. Now define

$$\nu_h = \begin{cases} (1+t)\mu_h - t\lambda_h & \text{if } h \in \{1, \dots, k\} \\ (1+t)\mu_h & \text{if } h \in \{k+1, \dots, m\} \end{cases}$$

For small enough t it is clear that $\nu_h > 0$ for all $h = 1, \dots, m$. In that case it also holds for every player $i \in N$ that

$$\sum_{h \colon i \in S_h \in \mathcal{B}_2} \nu_h = (1+t) \sum_{h \colon i \in S_h \in \mathcal{B}_2} \mu_h - t \sum_{h \colon i \in S_h \in \mathcal{B}_1} \lambda_h = (1+t) - t = 1,$$

implying that ν is a vector of balancing coefficients for $\mathcal{B}_2$. It follows that ν is not unique.

Now, there must be at least one $j \in \{1, \dots, k\}$ with $\lambda_j > \mu_j$. Indeed, otherwise for player $i \in S_{k+1}$ it has to hold that

$$1 = \sum_{h\colon i \in S_h \in \mathcal{B}_1} \lambda_h \leqq \sum_{h\colon i \in S_h \in \mathcal{B}_1} \mu_h < \sum_{h\colon i \in S_h \in \mathcal{B}_2} \mu_h = 1.$$

This would be a contradiction.

Now let

$$\hat{t} = \min \left\{ \frac{\mu_h}{\lambda_h - \mu_h} \,\middle|\, \lambda_h > \mu_h \right\}.$$

We introduce the collection

$$\mathcal{C}' = \{S_h \mid S_h \in \mathcal{B}_1,\ (1+\hat{t})\mu_h = \hat{t}\lambda_h\} \neq \varnothing$$

and let $\mathcal{C} = \mathcal{B}_2 \backslash \mathcal{C}'$. Obviously this collection satisfies the requirements formulated in the lemma. Finally, we complete the proof by showing that the defined coefficients ν for $\hat{t}$ is a set of balancing coefficients for the collection $\mathcal{C}$.

Indeed, we can write for any $i \in N$:

$$\begin{aligned}
\sum_{S \in \mathcal{C}\colon i \in S} \nu_S = \sum_{S \in \mathcal{C}\colon i \in S} & \left[(1+\hat{t})\mu_S - \hat{t}\lambda_S\right] \\
&= (1+\hat{t}) \sum_{S \in \mathcal{C}\colon i \in S} \mu_S - \hat{t} \sum_{S \in \mathcal{C}\colon i \in S} \lambda_S \\
&= (1+\hat{t}) \left(1 - \sum_{S \in \mathcal{C}'\colon i \in S} \mu_S\right) - \hat{t} \left(1 - \sum_{S \in \mathcal{C}'\colon i \in S} \lambda_S\right) \\
&= 1 - \sum_{S \in \mathcal{C}'\colon i \in S} \left[(1+\hat{t})\mu_S - \hat{t}\lambda_S\right]
\end{aligned}$$

Now for every coalition $S \in \mathcal{C}'$ we have that

$$\frac{\mu_S}{\lambda_S - \mu_S} = \hat{t},$$

and, therefore, for every $S \in \mathcal{C}'$:

$$(1+\hat{t})\mu_S - \hat{t}\lambda_S = \mu_S + \hat{t}(\mu_s - \lambda_S) = 0.$$

Together with the above this implies indeed that $\sum_{S \in \mathcal{C}\colon i \in S} \nu_S = 1$ for every $i \in N$. This in turn completes the proof that $\mathcal{C}$ is a balanced collection. ■

I now turn to the proof of the assertions stated in Theorem 2.10 using the formulated lemma.

Proof of (a)
We let $\mathcal{B}_1 = \{S_1, \ldots, S_k\}$ and $\mathcal{B}_2 = \{T_1, \ldots, T_m\}$ be two balanced collections on N with balancing coefficients $(\lambda_1, \ldots, \lambda_k)$ and $(\mu_1, \ldots, \mu_m)$ respectively. Now

$$\mathcal{C} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{U_1, \ldots, U_p\}$$

where $p \leqq k + m$. Let $0 < t < 1$. Define

$$\nu_j = \begin{cases} t\lambda_i & \text{if } U_j = S_i \in \mathcal{B}_1 \setminus \mathcal{B}_2 \\ (1-t)\mu_h & \text{if } U_j = T_h \in \mathcal{B}_2 \setminus \mathcal{B}_1 \\ t\lambda_i + (1-t)\mu_h & \text{if } U_j = S_i = T_h \in \mathcal{B}_1 \cap \mathcal{B}_2 \end{cases}$$

As one can verify, $(\nu_1, \ldots, \nu_p)$ are balancing coefficients for the collection $\mathcal{C}$. Hence, the union of two balanced collections is indeed balanced.

Now by induction on the number of collections, the union of any family of balanced collections has to be balanced as well. This completes the proof of (a).

Proof of (b)
Lemma 2.46 implies that the balancing coefficients will only be unique for minimally balanced collections. This leaves the converse to be shown.

Let $\mathcal{B} = \{S_1, \ldots, S_m\}$. Suppose that $\mathcal{B}$ has two distinct sets of balancing coefficients, λ and μ with $\lambda \neq \mu$. We may assume that $\lambda_j > \mu_j$ for at least one $j \in \{1, \ldots, m\}$. Again define $\nu = (1 + \hat{t})\, \mu - \hat{t}\, \lambda$ where

$$\hat{t} = \min \left\{ \frac{\mu_h}{\lambda_h - \mu_h} \;\middle|\; \lambda_h > \mu_h \right\}.$$

Now—following the same construction as in the proof of Lemma 2.46—ν is a set of balancing coefficients for the collection

$$\mathcal{C} = \{S_h \mid (1 + \hat{t})\, \mu_h \neq \hat{t}\, \lambda_h\}.$$

Since $\mathcal{C} \subsetneqq \mathcal{B}$ we have shown that $\mathcal{B}$ is not minimal.

Proof of (c)
Let $\mathcal{B}$ be some balanced collection on N. We prove Assertion (c) by induction on the number B of coalitions in the collection $\mathcal{B}$.

For $B = 1$ there is only one balanced collection, namely $\mathcal{B} = \{N\}$. This is clearly a minimal balanced collection.

Next suppose the assertion is true for $B - 1$ and lower. Suppose that $|\mathcal{B}| = B$.

If $\mathcal{B}$ is minimal itself, then the assertion is trivially satisfied. If $\mathcal{B}$ is not minimal, then it has a proper subcollection $\mathcal{C} \subsetneqq \mathcal{B}$ that is balanced. By Lemma 2.46 there exists another proper subcollection $\mathcal{D} \subsetneqq \mathcal{B}$ such that $\mathcal{C} \cup \mathcal{D} = \mathcal{B}$. Since $\mathcal{C}$ and $\mathcal{D}$ are proper subcollections, they each have $B - 1$ or fewer coalitions. By the induction hypothesis each can be expressed as the union of minimally balanced subcollections. Hence, $\mathcal{B}$ itself is the union of minimally balanced subcollections.

Proof of Theorem 2.12

The proof of the Bondareva–Shapley Theorem is based on duality theory in linear programming. Let $v \in \mathcal{G}^N$. First, it has to be observed that $C(v) \neq \varnothing$ if and only if the following linear programming problem has a solution:

$$\begin{aligned} \text{Minimize} \quad & P = \textstyle\sum_{i=1}^{n} x_i && (2.28) \\ \text{subject to} \quad & \textstyle\sum_{i \in S} x_i \geqq v(S) \text{ for all } S \subset N && (2.29) \end{aligned}$$

We now claim that $x \in C(v)$ if and only if it is a solution to this linear program such that $P = \sum_i x_i \leqq v(N)$.

First observe that indeed a solution $x^\star \in \mathbb{R}^N$ of this linear program such that the generated optimum $P^\star = \sum_{i=1}^{n} x_i^\star \leqq v(N)$ is indeed a Core imputation of v.

Conversely, if $x \in C(v)$, then it obviously satisfies the conditions for the linear program.

Hence, $\sum_i x_i = v(N)$ implying that $P = \sum_i x_i = v(N)$ is indeed a solution to the stated linear program.

Next we construct the dual linear program of the one formulated above. This is the linear program described by

$$\begin{aligned} \text{Maximize} \quad & Q = \textstyle\sum_{S \subset N} y_s\, v(S) && && (2.30) \\ \text{subject to} \quad & \textstyle\sum_{S:\, i \in S} y_S = 1 && \text{for all } i \in N && (2.31) \\ & y_S \geqq 0 && \text{for all } S \subset N && (2.32) \end{aligned}$$

Both linear programs as stated, are feasible. Hence, the minimum $P^\star$ has to be equal to the maximum $Q^\star$. This in turn implies that $C(v) \neq \varnothing$ if and only if $Q^\star = P^\star \leqq v(N)$.

But the stated maximization problem exactly requires the identification of balanced collections of coalitions over which is maximized. A simple reformulation immediately leads to the stated conditions in the Bondareva–Shapley Theorem 2.12.

Proof of Theorem 2.28

Only if: Let $\Omega \subset 2^N$ be a collection that is not strongly balanced. Then there exists a sub-collection $\Gamma \subset \Omega$ that is minimally balanced and is *not* a partitioning of N. (The latter statement is equivalent to the negation of the definition of strong balancedness.)

For Γ we now define the game $w \in \mathcal{G}^N$ given by

$$w(S) = \begin{cases} |S| & \text{if } S \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

We show that $C(w, \Omega) = C(w_\Omega) = \varnothing$ by application of the Bondareva–Shapley Theorem.

Let the balancing coefficients of Γ be given by $\{\lambda_S \mid S \in \Gamma\}$. Then

$$\sum_{S\in\Gamma} \lambda_S w_\Omega(S) = \sum_{S\in\Gamma} \lambda_S w(S) = \sum_{i\in N} \sum_{S\in\Gamma:\, i\in S} \lambda_S = |N| = n.$$

But since Γ does not contain a partititioning of N, we arrive at

$$w_\Omega(N) = \max_{P(N)\in\mathcal{P}_\Omega(N)} \sum_{T\in P(N)} w(T) < |N| = n.$$

Thus, with the above it follows that

$$\sum_{S\in\Gamma} \lambda_S w_\Omega(S) = n > w_\Omega(N).$$

From the Bondareva–Shapley Theorem it thus follows that $C(w_\Omega) = \varnothing$, which implies the desired assertion.

If: Let $\Omega \subset 2^N$ be a strongly balanced collection. Consider an arbitrary game $v \in \mathcal{G}^N$. We now check the Bondareva–Shapley conditions for v_Ω to show that $C(v, \Omega) = C(v_\Omega) \neq \varnothing$.

(A) First, let $\Gamma \subset \Omega$ be some minimally balanced collection. Since, by assumption, Γ contains a partitioning P^* of N, this implies that $\Gamma = P^*$. Thus, the balancing coefficients for Γ are unique and given by $\lambda_S = 1$ for all $S \in \Gamma$. Furthermore, since $S \in \Gamma \subset \Omega$, it follows immediately that $v_\Omega(S) = v(S)$. This implies that

$$\sum_{S\in\Gamma} \lambda_S v_\Omega(S) = \sum_{T\in P^*} v(T) \leqq \max_{P\in\mathcal{P}_\Omega(N)} \sum_{T\in P} v(T) = v_\Omega(N)$$

Hence, the Bondareva–Shapley condition is satisfied for any minimally balanced sub-collection of Ω.

(B) Next, let Γ be a minimally balanced collection with $\Gamma \setminus \Omega \neq \varnothing$. Let $\{\lambda_S \mid S \in \Gamma\}$ be the balancing coefficients for Γ.

Now, by definition, for every $S \in \Gamma$ with $S \notin \Omega$ there exists some Ω-partitioning $P^*(S) \in \mathcal{P}_\Omega(S)$ such that $v_\Omega(S) = \sum_{T\in P^*(S)} v(T)$. For every $T \in \Omega$ we now define

$$\Gamma_T = \{S \in \Gamma \mid S \notin \Omega \,\text{and} T \in P^*(S)\}.$$

We now introduce a modified collection $\widehat{\Gamma} \subset \Omega$ given by

$$\widehat{\Gamma} = \{T \mid T \in \Gamma \cap \Omega\} \cup \left(\bigcup_{S\in\Gamma:\, S\notin\Omega} P^*(S) \right).$$

We also introduce modified balancing coefficients for the modified collection $\widehat{\Gamma}$ given by

$$\delta_T = \begin{cases} \lambda_T + \sum_{S \in \Gamma_T} \lambda_S & \text{if } T \in \Gamma \cap \Omega \\ \sum_{S \in \Gamma_T} \lambda_S & \text{if } T \notin \Omega \text{ and } T \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

It can now be verified rather easily that $\{\delta_T \mid T \in \widehat{\Gamma}\}$ is a family of balancing coefficients for the collection $\widehat{\Gamma}$. This implies that $\widehat{\Gamma}$ is a balanced Ω-collection.

Now from assertion (A) shown above applied to the balanced Ω-collection $\widehat{\Gamma}$ it follows that

$$\sum_{T \in \widehat{\Gamma}} \delta_T v_\Omega(T) \leqq v_\Omega(N).$$

We conclude from (A) and (B) that indeed the game v_Ω satisfies the Bondareva-Shapley conditions for both minimal balanced Ω-collections as well as balanced non-Ω-collections, implying that $C(v, \Omega) = C(v_\Omega) \neq \varnothing$. This proves the assertion.

Proof of Theorem 2.33

(i) *implies* (ii)
Let Ω be a lattice on N. Define $H \colon N \to 2^N$ by

$$H(i) = \{j \in N \mid i \in \partial_j \Omega\} \setminus \{i\} \tag{2.33}$$

Evidently H is a permission structure on N. We proceed by showing that $\Omega = \Omega_H$.

Recall that $\partial\Omega$ is the smallest basis of Ω. We show that $\partial\Omega$ is also a basis of Ω_H implying the desired assertion.

Let $i \in N$. Now for $j \notin \partial_i \Omega$ it holds that $\partial_i \Omega \cap H(j) = \varnothing$, since otherwise for $h \in \partial_i \Omega \cap H(j)\colon\ j \in \partial_h \Omega \subset \partial_i \Omega$.

Next let $S \in \Omega_H$. Clearly for every $j \in S$ and $i \in \partial_j \Omega$ with $i \neq j$ we have that $j \in H(i)$. Thus, $S \cap H(i) \neq \varnothing$, and, therefore, $i \in S$. As a consequence, $\partial_j \Omega \subset S$ for every $j \in S$. Therefore, $S = \cup_{j \in S} \partial_j \Omega$. This indeed shows that $\partial\Omega$ is a basis of Ω_H.

(ii) *implies* (i)
It is easy to see that $\varnothing \in \Omega_H$ as well as $N \in \Omega_H$.

Next let $S, T \in \Omega_H$. Regarding $S \cup T$ we remark that $N \setminus (S \cup T) \subset N \setminus S$. Hence,

$$(S \cup T) \cap H(N \setminus (S \cup T)) \subset T \cup \left[S \cap H(N \setminus S)\right] = T \cap \varnothing = \varnothing$$

showing that $S \cup T \in \Omega_H$.

Next consider $S \cap T$. Since $N \setminus (S \cap T) = (N \setminus S) \cup (N \setminus T)$ it easily follows that

$$H(N \setminus (S \cap T)) = H(N \setminus S) \cup H(N \setminus T).$$

This in turn implies that

$$\begin{aligned}(S \cap T) \cap \left[H(N \setminus (S \cap T))\right] &= (S \cap T) \cap \left[H(N \setminus S) \cup H(N \setminus T)\right] \\ &= \left[(S \cap T) \cap H(N \setminus S)\right] \cup \left[(S \cap T) \cap H(N \setminus T)\right] \\ &\subset \left[S \cap H(N \setminus S)\right] \cup \left[T \cap H(N \setminus T)\right] \\ &= \varnothing \cup \varnothing = \varnothing.\end{aligned}$$

This shows that $S \cap T \in \Omega_H$.

Proof of Theorem 2.34

Let Ω be a lattice on N. Obviously, for every $i \in N$ and $j \in \partial_i \Omega$ the payoff vector $e_i - e_J$ is in Γ_Ω. Furthermore Γ_Ω is a cone. This leaves us to prove that

$$\Gamma_\Omega \subset \text{ Cone } \left\{e_j - e_i \mid i \in N \,\text{and} j \in \partial_i \Omega \right\}.$$

Let $x \in \Gamma_\Omega$ with $x \neq 0$. Select $j \in N$ to be such that $x_j > 0$.

Claim: There exists a player $i \in N$ with $x_i < 0$ and $j \in \partial_i \Omega$ such that there is some $\varepsilon > 0$ with

$$x' = x - \varepsilon(e_j - e_i) \in \Gamma_\Omega.$$

Proof Notice that from this it follows that $x'(S) = x(S) - \varepsilon$ for every $S \in \Omega$ with $i \notin S$ and $j \in S$.[10] Hence, whenever $x(S) = 0$ the situation $i \notin S$ and $j \in S$ is not allowed to occur. This implies that i has to be chosen in

$$T = \bigcap \left\{ S \in \Omega \;\middle|\; j \in S \,\text{and} \sum_{i \in S} x_i = 0 \right\} \in \Omega.$$

Since $N \in \Omega$ and $x(N) = 0$, T is well defined and non-empty. Furthermore, $x(T) = 0$ since for any $V, W \in \Omega$ with $x(V) = x(W) = 0$ we have that $V \cap W,\ G \cup W \in \Omega$. Thus, $x(V \cap W) \geqq 0$ as well as $x(V \cup W) \geqq 0$. Together with

$$x(V \cap W) + x(V \cup W) = x(V) + x(W) = 0$$

this implies that $x(V \cap W) = x(V \cup W) = 0$.[11]

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[10] For convenience we define $x(S) = \sum_S x_i$ for any vector $x \in \mathbb{R}^N$ and coalition $S \in \Omega$.

[11] Actually we have shown that the collection $\{V \in \Omega \mid x(V) = 0\}$ is a lattice on N as well.

Since $j \in T$, $x_j > 0$ and $x(T) = 0$, there is a player $i \in T$ with $x_i < 0$. Suppose now that for any $i \in T$ with $x_i < 0$: $j \notin \partial_i\Omega$. Then

$$T' = \cup \{\partial_i\Omega \mid i \in T \text{ and} x_i < 0\} \in \Omega$$

and $T' \subset T$. Furthermore, $x(T') \geqq 0$ and, hence,

$$0 < x_j \leqq x(T \setminus T') = -x(T') \leqq 0,$$

which is a contradiction. Therefore we conclude that there exists a player $i \in T$ with $x_i < 0$ and $j \in \partial_i\Omega$.

Consider for i the payoff vector given by $x' = x - \varepsilon(e_j - e_i)$ with

$$\varepsilon = \min \left[\{x_j, -x_j\} \cup \{x(S) \mid S \in \Omega \text{ with} j \in S \text{ and} i \notin S\} \right]$$

By definition of T and the choice of $i \in T$ it is clear that $x(S) > 0$ for all $S \in \Omega$ with $j \in S$ and $i \notin S$. Therefore, $\varepsilon > 0$.

Next we prove that $x' \in \Gamma_\Omega$. First we observe that $x'(N) = 0$. Now for every $S \in \Omega$ we distinguish the following cases:

(i) $i \in S$ and $j \in S$: Then $x'(S) = x(S) \geqq 0$.
(ii) $i \in S$ and $j \notin S$: This case cannot occur since $j \in \partial_i\Omega$.
(iii) $i \notin S, j \in S$ and $x(S) = 0$: This case cannot occur either since $i \in T$.
(iv) $i \notin S, j \in S$ and $x(S) > 0$: Then $x'(S) = x(S) - \varepsilon \geqq 0$ by choice of ε.
(v) $i \notin S$ and $j \notin S$: Then $x'(S) = x(S) \geqq 0$.

We may also conclude for arbitrary $S \in \Omega$ that $x'(S) \leqq x(S)$ since the case that $i \in S$ and $j \notin S$ does not occur. Thus,

$$\#\{S \in \Omega \mid x'(S) = 0\} \geqq \#\{S \in \Omega \mid x(S) = 0\} \tag{2.34}$$

and

$$\#\{h \in N \mid x'_h = 0\} \geqq \#\{h \in N \mid x_h = 0\}. \tag{2.35}$$

The variable ε now can be chosen such that one of the inequalities (2.34) and (2.35) has to be strict.

This proves the claim.

We may proceed with the proof of the assertion by repeatedly applying the claim to generate a finite sequence of payoff vectors $x_0 = x, x_1, \ldots, x_K$ with for every $k \in \{0, \ldots, K-1\}$

$$x_{k+1} = x_k - \varepsilon_k \left(e_{j_k} - e_{i_k}\right) \text{ where } j_k \in \partial_{i_k}\Omega.$$

The sequence is terminated whenever $x_K = 0$. Now we conclude

$$x = \sum_{k=0}^{K-1} \varepsilon_k \left(e_{j_k} - e_{i_k} \right) \in \text{Cone} \left\{ e_j - e_i \mid i \in N \, \text{and} \, j \in \partial_i \Omega \right\}$$

This completes the proof of the assertion stated in Theorem 2.34.

Proof of Theorem 2.36

(i) *implies* (ii)
Let $i, j \in N$ with $i \in S$ and $j \notin S$ for some $S \in \Omega$. Then $\partial_i \Omega \subset S$ and $\partial_j \Omega \not\subseteq S$. This implies that $\partial_i \Omega \neq \partial_j \Omega$.

(ii) *implies* (iii)
Suppose by contradiction that there exists $S \in \Omega$ with for every $i \in S$: $S \setminus \{i\} \notin \Omega$. Then $\cup_{j \in S:\, j \neq i} \partial_j \Omega \in \Omega$ contains $S \setminus \{i\}$ and is contained in S. So, $S = \cup_{j \in S:\, j \neq i} \partial_j \Omega$. Hence, for every $i \in S$ there exists some $j \in S \setminus \{i\}$ with $i \in \partial_j \Omega$.

Since S is finite there has to exist a sequence $i_1, \ldots, i_K$ in S with $i_K \in \partial_{i_1} \Omega$ and $i_{k+1} \in \partial_{i_k} \Omega$ for all $k \in \{1, \ldots, K-1\}$. By definition, therefore, $\partial_{i_1} \Omega = \partial_{i_2} \Omega = \cdots = \partial_{i_K} \Omega$. This is a contradiction to (ii).

(iii) *implies* (i)
This follows by definition by repeated application of (iii) to a coalition $S \in \Omega$ containing player i.

(ii) *is equivalent to* (iv)
By Theorem 2.33 there exists a permission structure H such that $\Omega = \Omega_H$. It is easy to see that H is acyclic if and only if $\partial_i \Omega_H \neq \partial_j \Omega_H$ for all $i \neq j$. This implies the equivalence.

(ii) *implies* (v)
First, $j \in \partial_i \Omega \setminus \{i\}$ implies that $\partial_j \Omega \subset \partial_i \Omega$. Now $i \notin \partial_j \Omega$ since otherwise $\partial_i \Omega \subset \partial_j \Omega$ implying equality in turn, and, hence, contradicting (ii). Therefore

$$\partial_i \Omega \setminus \{i\} = \bigcup_{j \in \partial_i \Omega:\; i \neq j} \partial_j \Omega \in \Omega.$$

(v) *implies* (vi)
We show that (v) implies that Γ_Ω is a pointed cone, thereby, through application of Proposition 2.23 showing that Ω is non-degenerate.

Suppose there exists $y \in \mathbb{R}^N$ with $\{y, -y\} \subset \Gamma_\Omega$. Then $y(N) = -y(N) = 0$ and $y(S) \geqq 0$ as well as $-y(S) \geqq 0$ for all $S \in \Omega$. Thus for any $i \in N$: $y(\partial_i \Omega) = 0$ and by (v) it follows that $y(\partial_i \Omega \setminus \{i\}) = 0$. Thus,

$$y_i = y(\partial_i \Omega) - y(\partial_i \Omega \setminus \{i\}) = 0$$

(vi) *implies* (ii)
Let $i, j \in N$. If $\partial_i\Omega = \partial_j\Omega$, then by Theorem 2.34 both $e_i - e_j$ and $e_j - e_i$ are in Γ_Ω. Hence, Γ_Ω is not a pointed cone. This would contradict that (vi) implies that Γ_Ω is a pointed cone.

Proof of Theorem 2.41

Before we develop the proofs of the two assertions stated in Theorem 2.41 we state and proof two intermediate insights. Recall first that Ω is a discerning lattice on N. Also, let H be an acyclic permission structure such that $\Omega = \Omega_H$.

Lemma 2.47 *Let $S \in \Omega$, $S \neq \varnothing$. Then $\Omega(S) = \Omega \cap 2^S$ and $\Omega' = \{N \setminus T \mid T \in \Omega\}$ are both discerning lattices on S and N, respectively.*

Proof Both $\Omega(S)$ and Ω' are lattices. Clearly $\Omega(S)$ contains a Weber string, implying that $\Omega(S)$ is discerning.

Also, for all $i, j \in N$ we may assume without loss of generality that there exists a coalition $S \in \Omega$ with $i \in S$ and $j \notin S$. Hence, $i \notin N \setminus S$ and $j \in N \setminus S$. Thus, Ω' is discerning. ■

Lemma 2.48 *For each coalition $S \in \Omega$ there is a Weber string in Ω of which S is one of its members.*

Proof Let $S \in \Omega$. By Lemma 2.47 there exist Weber strings in $\Omega(S)$ as well as $\Omega'(N \setminus S)$, say $(R'_i)_{i\in S}$ and $(R''_j)_{j\in N\setminus S}$, respectively. Then $(R_i)_{i\in N}$ is a Weber string in Ω where

$$R_i = \begin{cases} R'_i & \text{for } i \in S \\ \{i\} \cup (N \setminus R''_i) & \text{for } i \notin S \end{cases}$$

Clearly if $i \in S$, $R_i \in \Omega$; furthermore, $i \in R''_i = R_i$ and for i such that $R_i = R'_i \neq \{i\}$ there is a player $j \in S$ with $R - i \setminus \{i\} = R''_i \setminus \{i\} = R'_j = R_j$. Now if $i \notin S$ either $R_i = \{i\} \cup N \setminus R''_i = N \setminus (R''_i \setminus \{i\}) = N \setminus R''_j$ for a player $j \notin S$, or $R_i = N \setminus \varnothing = N$ and, since $(\Lambda')' = \Lambda$ for every lattice Λ, we conclude that $R_i \in \Omega$.

Furthermore, $i \in R_i$ and

- if $R''_i = N \setminus S$, then $R_i \setminus \{i\} = N \setminus R''_i = S = R'_j = R_j$ for some $j \in S$;
- if $R''_i \neq N \setminus S$ then there is a player $k \notin S$ with $R''_i = R''_k \setminus \{k\}$. Hence, $R_i \setminus \{i\} = N \setminus R''_i = N \setminus (R''_k \setminus \{k\}) = \{k\} \cup N \setminus R''_k = R_k$.

We conclude that for all players $i \in N$ we have $R_i \in \Omega$ and if $R_i \neq \{i\}$ there is a player j such that $R_i \setminus \{i\} = R_j$. ■

The following property now follows immediately from Lemmas 2.47 and 2.48:

Corollary 2.49 *Let $S_1, \dots, S_K$ be elements in Ω such that $S_1 \subset S_2 \subset \cdots \subset S_K$. Then there exists a Weber string in Ω of which $S_1, \dots, S_K$ are members.*

Proof of 2.41(a)
Since Ω is discerning by Lemma 2.39 there exists at least one Weber string in Ω.

Now suppose to the contrary that there exists a Core allocation $x \in C(\Omega, v)$ such that $x \notin \mathcal{W}(\Omega, v) + \Gamma_\Omega$. By Theorem 2.34 the set $\mathcal{W}(\Omega, v) + \Gamma_\Omega$ is polyhedral and since Ω is discerning/non-degenerate, it does not contain a nontrivial linear subspace. Thus x is an extreme point of

$$\operatorname{Conv}\left[\{x\} \cup (\mathcal{W}(\Omega, v) + \Gamma_\Omega)\right].$$

The normals of all supporting hyperplanes in an extreme point of a polyhedral set are well known to form a full dimensional cone. Therefore, there exists a normal vector, say $p \in \mathbb{R}^N$, with non-equal coefficients such that for each $y \in \mathcal{W}(\Omega, v)$ and $y' \in \Gamma_\Omega$:

$$p \cdot x < p \cdot (y + y').$$

Now by Theorem 2.34 for every $i \in N$, $j \in \partial_i\Omega$ and $M \geqq 0$ it holds that for every $y \in \mathcal{W}(\Omega, v)$:

$$p \cdot x < p \cdot y + M\, p \cdot (e_j - e_i) = p \cdot y + M\,(p_j - p_i) \tag{2.36}$$

implying that for every $i \in N$ and $j \in \partial_i\Omega$: $p_j \geqq p_i$. Now label the players in N such that $p_1 > p_2 > \cdots > p_n$. Consider for every $k \in N$ the coalition $S_k = \{1, \dots, k\}$. If $i \in S_k$, then $p_i \geqq p_k$, in turn implying that $p_j \geqq p_i \geqq p_k$ for all $j \in \partial_i\Omega$. Hence, $\partial_i\Omega \subset S_k$ for all $i \in S_k$ implying that

$$S_k = \bigcup_{i \in S_k} \partial_i\Omega \in \Omega.$$

Hence, $(S_k)_{k \in N}$ determines a Weber string in Ω. The corresponding marginal vector y for v now belongs to $\mathcal{W}(\Omega, v)$. Thus,

$$\begin{aligned} p \cdot x < p \cdot y &= \sum_{k \in N} \left[v(S_k) - v(S_{k-1})\right] \\ &= v(N)\, p_n + \sum_{k \in N} v(S_k)\,(p_k - p_{k+1}). \end{aligned}$$

Since $x \in C(\Omega, v)$ and $S_k \in \Omega$ $(k \in N)$, we conclude that $x(N) = v(N)$ and $x(S_k) \geqq v(S_k)$, $k \neq n$. Furthermore, $p_k - p_{k+1} \geqq 0$ for $k \neq n$. Therefore, the above may be rewritten as

$$\begin{aligned} p \cdot x &< v(N)\, p_n + \sum_{k \in N} v(S_k)\,(p_k - p_{k+1}) \\ &\leqq x(N)\, p_n + \sum_{k=1}^{n-1} \sum_{j=1}^{k} x_j\,(p_k - p_{k+1}) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{k} x_j\, p_k - \sum_{k=2}^{n} \sum_{j=1}^{k-1} x_j\, p_k \\ &= x_1\, p_1 + \sum_{k=2}^{n} p_k\, x_k = p \cdot x. \end{aligned}$$

This constitutes a contradiction, proving the desired inclusion.

Proof of 2.41(b)
Only if:
Suppose that $v \in \mathcal{G}^N$. Given the already established facts, we only have to show that every Weber allocation $x \in \mathcal{W}(\Omega, v)$ belongs to $C(\Omega, v)$. So, let $x \in \mathcal{W}(\Omega, v)$ correspond to some Weber string $(R_i)_{i \in N}$ with for every $i \in N$

$$x_i = v(R_i) - v(R_i \setminus \{i\}).$$

Since v is convex on Ω for every $i \in S \in \Omega$ we have that

$$v(R_i) + v(S \cap (R_i \setminus \{i\})) \geqq v(S \cap R_i) + v(R_i \setminus \{i\}).$$

Thus, for every $i \in S$:

$$x_i \geqq v(S \cap R_i) - v(S \cap R_i \setminus \{i\}).$$

Without loss of generality we may assume that N is labelled such that the players in S are labelled first and $\{1, \ldots, k\} \subset R_k$ for all $k \in S$. Then

$$\begin{aligned} \sum_{i \in S} x_i \geqq \sum_{k=1}^{|S|} x_k \geqq \sum_{k=1}^{|S|} & v(S \cap R_k) - v(S \cap R_k \setminus \{k\}) \\ &= \sum_{k=1}^{|S|} v(\{1, \ldots, k\}) - \sum_{k=1}^{|S|-1} v(\{1, \ldots, k\}) \\ &= v(\{1, \ldots, |S|\}) = v(S). \end{aligned}$$

Since S is chosen arbitrarily, we may conclude that $x \in C(\Omega, v)$.

If:

Suppose that $C(\Omega, v) = \mathcal{W}(\Omega, v) + \Gamma_\Omega$. Take $S, T \in \Omega$. Since Ω is discerning by Corollary 2.49 there exists a Weber string in Ω, say $(R_i)_{i \in N}$, such that it contains the coalitions $S \cup T$ and $S \cap T$.

Let $x \in \mathcal{W}(\Omega, v)$ be the corresponding Weber allocation—or marginal payoff vector. It is evident that $x(S \cap T) = v(S \cap T)$ as well as $x(S \cup T) = v(S \cup T)$. By hypothesis $x \in C(\Omega, v)$, which leads to the conclusion that

$$\begin{aligned} v(S) + v(T) &\leqq x(S) + x(T) = \\ &= x(S \cup T) + x(S \cap T) = \\ &= v(S \cup T) + v(S \cap T). \end{aligned}$$

This proves the convexity of v.

2.5 Problems

Problem 2.1 Let $N = \{1, 2, 3\}$ and consider the game $v \in \mathcal{G}^N$ given by the following table:

S	$\varnothing$	1	2	3	12	13	23	123
$v(S)$	0	0	5	6	15	0	10	20

(a) Give the $(0, 1)$-normalization v' of this game v.

(b) Compute the Core $C(v) \subset I(v)$ of the game v and of its $(0, 1)$-normalization v'. Show that these two Cores are essentially identical. Formulate this identity precisely.

(c) Draw the Core $C(v')$ of the $(0, 1)$-normalization v' of the game v in the two-dimensional simplex $\mathcal{S}^2$.

Problem 2.2 Consider the so-called *bridge game*, introduced by Kaneko and Wooders (1982, Example 2.3). Let $N = \{1, \ldots, n\}$ and define $v \in \mathcal{G}^N$ by

$$v(S) = \begin{cases} 1 & \text{if } |S| = 4 \\ 0 & \text{otherwise.} \end{cases}$$

In the bridge game only quartets can generate benefits.

Show that $C(v) \neq \varnothing$ if and only if $n = 4m$ for some $m \in \mathbb{N}$.

Problem 2.3 Construct a *non-essential* three player game that has an *empty* Core. Check the emptiness by depicting the three inequalities (2.5)–(2.7) for the $(0, 1)$-normalization of this game in the two-dimensional set of imputations of this game.

Problem 2.4 Construct an essential three player game that has a Core consisting of exactly *one* imputation. Check again the required uniqueness property by depicting the three inequalities (2.5)–(2.7) for the $(0, 1)$-normalization of this game in the two-dimensional simplex of imputations $\mathcal{S}^2$.

Problem 2.5 Construct a cooperative game that is essential—but not superadditive—such that its Core does not coincide with the set of undominated imputations, i.e., such that the Core is a strict subset of the set of undominated imputations.

Problem 2.6 Construct a detailed proof of Corollary 2.11 based on the original Bondareva-Shapley Theorem rather than Corollary 2.13.

Problem 2.7 Construct a detailed proof of Corollary 2.13.

Problem 2.8 Let v be an essential three player game $v \in \mathcal{G}^N$ with $N = \{1, 2, 3\}$. Show that $C(v)$ is a singleton—i.e., $\#C(v) = 1$—if and only if $v(12)+v(13)+v(23) = 2v(N)$.

Problem 2.9 One can also consider the case of "costly" blocking by a coalition. Let $v \in \mathcal{G}^N$. We introduce $\varepsilon \in \mathbb{R}$ as the cost for a coalition $S \subset N$ to block any imputation $x \in I(v)$.[12] This leads to the introduction of the *ε-Core* $C_\varepsilon(v) \subset I(v)$ defined by $x \in C_\varepsilon(v)$ if and only if

$$\sum_{i \in S} x_i \geqq v(S) - \varepsilon$$

for every non-empty coalition $S \subset N$.

(a) Show that for every game $v \in \mathcal{G}^N$ if $\varepsilon_1 < \varepsilon_2$, then $C_{\varepsilon_1}(v) \subset C_{\varepsilon_2}(v)$.
(b) Show that there exists a minimal cost parameter $\varepsilon \in \mathbb{R}$ such that $C_\varepsilon(v) \neq \varnothing$. This particular $C_\varepsilon(v)$ is known as the *least Core* of v indicated by $C_L(v)$.
(c) Show that the interior of $C_L(v)$ is empty. In particular draw the least Core of the $(0, 1)$-normalization of the trade game discussed in Examples 1.9 and 1.19.
(d) let $v, w \in \mathcal{G}^N$. Suppose that for ε_v and ε_w it holds that

$$C_{\varepsilon_v}(v) = C_{\varepsilon_w}(w) \neq \varnothing.$$

Show that for any $\delta > 0$ it now has to hold that

$$C_{\varepsilon_v - \delta}(v) = C_{\varepsilon_w - \delta}(w).$$

In particular show that $C_L(v) = C_L(w)$.

[12] Remark that $\varepsilon < 0$ is possible. This implies that the cost of blocking is negative, hence there are synergy effects or other benefits to the act of blocking.

Problem 2.10 Consider the communication network on $N = \{1,2,3,4,5,6,7\}$ depicted in Fig. 2.6 on page 42. Let $\Omega \subset 2^N$ be the collection of formable coalitions in this network in the sense of Myerson (1977). Compute the Ω-Core of the unanimity game u_{17}.

Problem 2.11 Give a proof of Proposition 2.21.

Problem 2.12 Consider the proof of Proposition 2.23. There it is stated that "Γ_Ω is a pointed cone if and only if for any $x \in \mathbb{R}^N$ with $\sum_S x_i = 0$ for all $S \in \Omega$ we must have $x = 0$." Show this property in detail.

Problem 2.13 Construct a proof of Lemma 2.39(a) by using the properties stated in Theorem 2.36.

Problem 2.14 A vector $x \in \mathbb{R}^N$ is a *Dual-Core allocation* of the cooperative game $v \in \mathcal{G}^N$ if x satisfies the efficiency requirement

$$\sum_{i \in N} x_i = v(N) \tag{2.37}$$

and for every coalition $S \subset N$

$$\sum_{i \in S} x_i \leqq v(S). \tag{2.38}$$

Notation: $x \in C^\star(v)$.[13]

(a) We define the *dual game* of v by $v^\star \in \mathcal{G}^N$ with $v^\star(S) = v(N) - v(N \setminus S)$. Show that $C^\star(v) = C(v^\star)$.
(b) Define the Dual-Weber set in the same fashion as the Dual-Core. Be precise in your formulation. What is the relationship between the Dual-Weber set of a game and the Weber set of its dual game?

[13] The Dual-Core is also called the "Anti-Core" by some authors. Since the definition is really based on a duality argument, I think it is more appropriate to refer to $C^\star$ as the Dual-Core.

http://www.springer.com/978-3-642-05281-1

The Cooperative Game Theory of Networks and Hierarchies
Gilles, R.P.
2010, XI, 270 p. 20 illus., Hardcover
ISBN: 978-3-642-05281-1