Introduction to the Theory of Coalition Games

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Foundations

Foundations laid in:
J. von Neumann, O. Morgenstern – *Theory of Games and Economic Behavior* (1944)

- mathematical models of cooperation in a given social environment
- coalition formation goes hand in hand with payoff negotiation
- analysis of coalition games is usually based only on payoff opportunities available to each coalition

Coalition Game: Assumptions

Players can form coalitions

Coalition is a collective decision-maker

Worth of each coalition is the total amount that the players from the coalition can jointly guarantee themselves, it is measured in abstract units of utility

Two Fundamental Questions

1. Which coalitions are likely to form?
2. How will the coalitions redistribute the payoffs among the players?

- We leave the behavioral aspects aside...
- ...and the attempt to answer the second question is in that follows!

Games with transferable utilities: every coalition can divide its worth in any possible way among its members

Examples

Example (Horse market)

Player 1 (a **seller**) has a horse which is worthless to him (unless he can sell it). Players 2 and 3 (**buyers**) value the horse at 90 and 100, respectively.

Which contract will be accepted by all the players?

Example (Small business)

Player 1 (an *owner*) runs the business. Each of the other players $2, \ldots, 11$ (**employees**) contributes an amount 10 000 to the total profit. No profits are generated without the owner.

What is a “fair” distribution of profit?

Examples (ctnd.)

Example (UNSC voting)

The United Nations Security Council consists of 5 permanent members and 10 other members. Every decision must be approved by 9 members including all the permanent members.

What is a "voting" power of the individual members?

Example (Cost allocation game)

The players are potential customers of a public service or a public facility. The **cost function** determines the cost of serving any group of customers by the most efficient means.

Which cost allocation will be accepted by all the customers?

Mathematical Model of Coalition Game

Definition

Let $N = \{1, \ldots, n\}$ be a finite set of players. A coalition is any subset of N. The set of all coalitions is denoted by 2^N.
A (coalition) game is a mapping

$$v : 2^N \to \mathbb{R}$$

such that $v(\emptyset) = 0$.

For any coalition $A \subseteq N$, the number $v(A)$ is called the worth of A.

Properties of Games

Definitions

A game v is

- superadditive if $A \cap B = \emptyset \quad \Rightarrow \quad v(A \cup B) \geqslant v(A) + v(B)$
- convex if $v(A \cup B) + v(A \cap B) \geqslant v(A) + v(B)$
- monotone if $A \subseteq B \quad \Rightarrow \quad v(A) \leqslant v(B)$
- constant-sum if $v(A) + v(N \setminus A) = v(N)$
- symmetric if $v(\pi(A)) = v(A)$ for every permutation π of N
- inessential if it is additive: $v(A) = \sum_{i \in A} v(\{i\})$

for every $A, B \subseteq N$

Properties of Games (ctnd.)

Fact

A game v is convex iff for every $i \in N$ and every $A \subseteq B \subseteq N \setminus \{i\}$

$$v(A \cup \{i\}) - v(A) \leqslant v(B \cup \{i\}) - v(B)$$

Fact

A game v is symmetric iff for every $A, B \subseteq N$

$$|A| = |B| \ \Rightarrow \ v(A) = v(B)$$

Inessential games are trivial from a game-theoretic viewpoint: if every player $i \in N$ demands at least $v(\{i\})$, then the distribution of $v(N)$ is uniquely determined

Examples

Example (Horse market)

$N = \{1, 2, 3\}$

If 1 sells the horse to 2 for the price x, he will effectively make a profit x, while 2's profit is $90 - x$. The total profit of the coalition $\{1, 2\}$ is thus 90. Similarly for $\{1, 3\}$. The grand coalition N should assign the horse to 3 who can eventually give side payments to 2.

$v(\{1, 2\}) = 90, \quad v(\{1, 3\}) = v(N) = 100$
$v(\{i\}) = v(\{2, 3\}) = 0, \quad i = 1, 2, 3$

The game v is **monotone, superadditive,** but **not convex**:

$$v(N) + v(\{1\}) < v(\{1, 2\}) + v(\{1, 3\})$$

Solution of Games

Definition

A payoff vector in a game v with a set of players $N = \{1, \ldots, n\}$ is any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Let $x(A) = \sum_{i \in A} x_i$, for every payoff vector x and $A \subseteq N$

Definition

A set $X_v = \{x \in \mathbb{R} \mid x(N) \leqslant v(N)\}$ is the set of feasible payoff vectors in a game v.

Definition

Let Γ be a set of games with a player set N. A solution is a function σ that associates with each game $v \in \Gamma$ a set $\sigma(v) \subseteq X_v$.

Solution of Games (ctnd.)

Questions

- Which properties should a solution σ satisfy?
- Is σ a unique solution satisfying the given axioms?
- For which class of games Γ holds true $\sigma(v) \neq \emptyset$ for each $v \in \Gamma$?
- Is $\sigma(v)$ computationally tractable?
- What is the mathematical content of **rationality, stability** and **fairness** of a solution σ?

Two main solution concepts

1. core
2. Shapley value

Properties of Solutions

Definitions

Let σ be a solution on Γ. We say that σ is

- nonempty on Γ if $\sigma(v) \neq \emptyset$
- Pareto optimal if $\sigma(v) \subseteq \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$
- individually rational if $x_i \geqslant v(\{i\})$ for every $i \in N$
- coalitionally rational if $x(A) \geqslant v(A)$ for every $A \subseteq N$
- treating players equally if the following condition is satisfied:

  $$v(A \cup \{i\}) = v(A \cup \{j\}) \quad \Rightarrow \quad x_i = x_j,$$

  for each $A \subseteq N \setminus \{i, j\}$
- additive if $\sigma(v_1 + v_2) = \sigma(v_1) + \sigma(v_2)$, whenever $v_i, v_1 + v_2 \in \Gamma$

The above properties must be satisfied for each $v \in \Gamma, x \in \sigma(v)$.

Core of Game

Definition (Shapley)

Let Γ be the set of all games with the player set $N = \{1, \ldots, n\}$. For any $v \in \Gamma$, put

$$\mathcal{C}(v) = \left\{x \in \mathbb{R}^n \mid x(N) = v(N),\ x(A) \geqslant v(A),\ \text{for every } A \subseteq N\right\}.$$

The core of a game is a (possibly empty) convex polytope in $\mathbb{R}^n$ given by the intersection of an affine hyperplane with $2^n - 2$ closed halfspaces.

Properties of Core

- a payoff vector $x \in \mathbb{R}^n$ belongs to the core if and only if no coalition can improve upon x
- members of core are thus highly stable payoff vectors
- the core solution is
  - *Pareto optimal*
  - *individually rational*
  - *coalitionally rational*
- the core solution needn't be
  - *nonempty*
  - *treating players equally*
  - *additive*

Example of Core

Example (Horse market)

$N = \{1, 2, 3\}$
$v(\{1, 2\}) = 90, \quad v(\{1, 3\}) = v(N) = 100$
$v(\{i\}) = v(\{2, 3\}) = 0, \quad i = 1, 2, 3$

$\mathcal{C}(v) = \{(t, 0, 100 - t) \in \mathbb{R}^3 \mid 90 \leqslant t \leqslant 100\}$
Each payoff vector $x \in \mathcal{C}(v)$ must satisfy

$$
\begin{aligned}
&x_i \geqslant 0, \quad i = 1, 2, 3 \\
&x_1 + x_2 \geqslant 90 \\
&x_1 + x_3 \geqslant 100 \\
&x_1 + x_2 + x_3 = 100
\end{aligned}
$$

Player 3 will purchase the horse at a price at least 90, player 2 is priced out of the market after bidding up the price to 90.

Superadditive Game with Empty Core

Example (Voting)

Three friends want to decide whether to go for a joint dinner. The decision is made by a simple majority of votes.

$N = \{1, 2, 3\}$
$v(\{i\}) = 0, \quad i = 1, 2, 3$
$v(A) = 1$, whenever $|A| \geqslant 2$

The game v is superadditive, yet $\mathcal{C}(v) = \emptyset$.

Core Questions

- Which class of games Γ possess nonemptiness of core?
- Can nonemptiness of core be efficiently decided?
- Provided the core of a game is nonempty
  - find its representation
  - recover at least one core element

Balanced Games

Notation:

$$\chi_A(i) = \begin{cases} 1, & i \in A, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for each } A \subseteq N.$$

Definition

A collection of coalitions $\mathcal{B} \subseteq 2^N$ with $\emptyset \notin \mathcal{B}$ is balanced if there are balancing weights $\delta_A \in (0,1]$, $A \in \mathcal{B}$ such that $\sum_{A \in \mathcal{B}} \delta_A \chi_A = \chi_N$.

Example

Any partition $A_1, \ldots, A_k$ of N with $A_i \neq \emptyset$ is a balanced collection: put $\delta_{A_i} = 1$.

Balanced Games (ctnd.)

Example

Let $N = \{1, 2, 3\}$ and $\delta_{\{1\}} = 1, \delta_{\{2,3\}} = \frac{2}{3}$, $\delta_{\{2\}} = \delta_{\{3\}} = \frac{1}{3}$. Then the set $\big\{\{1\}, \{2, 3\}, \{2\}, \{3\}\big\}$ is balanced.

Every balanced collection can be viewed as a generalized partition: each $i \in N$ devotes the fraction δ_A of his time to each coalition $A \in \mathcal{B}$ that contains him. This means $\sum_{A \in \mathcal{B}: i \in A} \delta_A = 1$.

Definition

A game with nonempty core is called balanced.

Balanced Games (ctnd.)

Theorem (Bondareva-Shapley)

A game v is balanced if and only if for each balanced collection $\mathcal{B}$ and each system of balancing weights δ_A, $A \in \mathcal{B}$ we have

$$v(N) \geqslant \sum_{A \in \mathcal{B}} \delta_A v(A).$$

Proof Use duality theorem in LP.

Balancedness of a game can be effectively tested:

- find all "minimal" balanced collections (Peleg, 1965)
- use an iterative projection algorithm

Representation of Core

Definition

Let C be a convex set in $\mathbb{R}^n$. A point $x \in C$ is an extreme point (vertex) of C if it does not lie in any open line segment joining two points of C.

Theorem (Main theorem for polytopes)

The following are equivalent:

- *P is an intersection of finitely many halfspaces*
- *P is a convex hull of finitely many of its points*
- *P is a convex hull of its extreme points*

Representation of Core (ctnd.)

The core $\mathcal{C}(v)$ is the intersection of halfspaces

$$x(A) \geqslant v(A),\ A \subseteq N \quad \text{and} \quad x(N) \leqslant v(N).$$

Given $x \in \mathcal{C}(v)$, define $\mathcal{S}(x) = \{A \subseteq N \mid x(A) = v(A)\}$

Theorem

A payoff vector $x = (x_1, \ldots, x_n) \in \mathcal{C}(v)$ is an extreme point of $\mathcal{C}(v)$ if and only if the system of linear equations

$$x(A) = v(A), \quad \textit{for each } A \in \mathcal{S}(x)$$

has x as its unique solution.

Core of Convex Games

Theorem (Shapley; 1972)

Let v be a nonzero convex game. Then:

- *v is balanced*
- *there are at most $n!$ extreme points of $\mathcal{C}(v)$ and they coincide with the payoff vectors $x^\pi \in \mathbb{R}^n$ defined by*

$$x_i^\pi = v\big(\{j \in N \mid \pi(j) < \pi(i)\} \cup \{i\}\big) - v\big(\{j \in N \mid \pi(j) < \pi(i)\}\big)$$

for each $i \in N$ and each permutation π of N

Simple Games

Definition

- A game v is called simple if it is monotone, $v(A) \in \{0, 1\}$ for each $A \subseteq N$, and $v(N) = 1$.
- A player $i \in N$ is said to be a veto player if he belongs to each winning coalition.

Fact

Let v be a simple game. If there exists a veto player in v, then v is balanced.

Proof Let S be the set of all veto players. Take a payoff vector $x \in \mathbb{R}^n$ with $x_i = \frac{1}{|S|}, i \in S$, and $x_i = 0, i \notin S$. Then $x \in \mathcal{C}(v)$.

Assignment Games

(Shapley, Shubik; 1971) : "The assignment game is a model for a two-sided market in which a product that comes in large, indivisible units (e.g., houses, cars, etc.) is exchanged for money, and in which each participant either supplies or demands exactly one unit."

- $N = S \cup B$, $S, B \neq \emptyset$ and $S \cap B = \emptyset$
- each $i \in S$ is a seller who has a house of worth a_i, each $j \in B$ is a potential buyer whose reservation price for i's house is b_{ij}
- define the joint net profit of $\{i, j\}$ as $w\big(\{i, j\}\big) = \max\{b_{ij} - a_i, 0\}$
- an assignment for $A \subseteq N$ is a set $\mathcal{T} \subseteq 2^A$ such that for every $P, Q \in \mathcal{T}$ with $P \neq Q$ we have

$$P \cap Q = \emptyset \quad \text{and} \quad |P \cap S| = |P \cap B| = 1$$

Assignment Games (ctnd.)

Definition

The assignment game v with respect to $(a_i)_{i \in S}$ and $(b_{ij})_{i \in S, j \in B}$ is defined by

$$v(A) = \max \left\{ \sum_{P \in \mathcal{T}} w(P) \mid \mathcal{T} \text{ is an assignment for } A \right\}, \quad A \subseteq N.$$

Theorem

Every assignment game is balanced.

Example (Horse market)

$N = \{1, 2, 3\}, S = \{1\}, B = \{2, 3\}$ $\quad w(\{1, 2\}) = 90, w(\{1, 3\}) = 100$
Assignments for N are $\mathcal{T}_1 = \{\{1, 2\}\}$ and $\mathcal{T}_2 = \{\{1, 3\}\}$.

Minimum Cost Spanning Tree Games

- customers in N must be connected to a supplier of energy 0
- the complete undirected graph $\mathcal{G}$ with the vertex set $N \cup \{0\}$ captures all connections
- c_{ij} is the cost of connecting $i, j \in N \cup \{0\}, i \neq j$ by an edge e_{ij}

Definition

Let $A \subseteq N$. A minimum cost spanning tree for A is a tree with vertices $(A \cup \{0\})$ and a set of edges $\mathcal{E}_A$ that connects the members of A to 0 such that the total cost of all connections is minimal.

Definition (MCST game)

The cost function c is a game defined by $c(A) = \sum_{e_{ij} \in \mathcal{E}_A} c_{ij}$

for every $A \subseteq N$.

MCST Games (ctnd.)

The core of the cost game c is the set of cost allocations

$$\{x \in \mathbb{R} \mid x(N) = c(N),\ x(A) \leqslant c(A),\ \text{for every } A \subseteq N\}.$$

Fact

Every MCST game has the nonempty core.

Proof

- let $\mathcal{G}_N = (N \cup \{0\}, \mathcal{E}_N)$ be the m.c.s.t. for N
- for each $i \in N$, there is a unique path $(0, i_1, \ldots, i_r)$ in $\mathcal{G}_N$ with $i_r = i$
- the cost allocation $x \in \mathbb{R}^n$ such that $x_i = c_{i_{r-1} i_r}$ for all $i \in N$ belongs to the core

Core Issues

Example (Voting)

$N = \{1, 2, 3\}$
$v(A) = 0, \quad |A| < 2$
$v(A) = 1, \quad |A| \geqslant 2$
$\mathcal{C}(v) = \emptyset$

Example

$M = \{1, 2, 3, 4\}$
$u(A) = v(A), \quad A \subseteq N$
$u(A) = 0, \quad A \subset M$
$u(M) = \frac{3}{2}$
Then $\mathcal{C}(v) = \left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right)\right\}$

Core of Games with Coalition Structures

Definition

A coalition structure for the player set N is a partition $\mathcal{R}$ of N.

Definition

Let v be a game and $\mathcal{R}$ be a coalition structure. The core $\mathcal{C}(v, \mathcal{R})$ is a set

$$\left\{x \in \mathbb{R}^n \mid x(R) = v(R), R \in \mathcal{R},\;\; x(A) \geqslant v(A), A \subseteq N\right\}.$$

- observe that $\mathcal{C}(v) = \mathcal{C}(v, \{N\})$
- checking nonemptiness of the core with coalition structures amounts to verifying balancedness for some game with the coalition structure $\{N\}$

Motivation

- the need for a solution which is
  - **single-valued**
  - defined on the **whole space of games** Γ
- Shapley value (1953) $\varphi : \Gamma \to \mathbb{R}^n$
- it provides an **a priori evaluation** of every coalition game based on a set of axioms

Axioms

Let Γ be the set of all games with n players.

Definition

Let $v \in \Gamma$. A player $i \in N$ is a null player if $v(A) = v(A \cup \{i\})$ for every $A \subseteq N$. A single-valued solution $\varphi : \Gamma \to \mathbb{R}^n$ has the null player property if $\varphi_i(v) = 0$ for every $v \in \Gamma$ and each null player $i \in N$.

Theorem (Axiomatic characterization of Shapley value)

There is a unique single-valued solution (Shapley value) φ on Γ that satisfies the **equal treatment property**, *the* **null player property**, *and the* **additivity**.

See properties of solutions.

Formula

Theorem

The **Shapley value** φ *on* Γ *is given by*

$$\varphi_i(v) = \sum_{A \subseteq N \setminus \{i\}} \frac{|A|!(n-|A|-1)!}{n!}\big(v(A \cup \{i\}) - v(A)\big)$$

for every $v \in \Gamma$ *and for every* $i \in N$.

Fact (Main ingredient of the proof)

For each $\emptyset \neq B \subseteq N$, *let*

$$u_B(A) = \begin{cases} 1, & A \supseteq B \\ 0, & \textit{otherwise}, \end{cases} \qquad A \subseteq N.$$

The set of all games $\{u_B \mid \emptyset \neq B \subseteq N\}$ *is a* **linear basis** *of* Γ.

Example

Example (Small business)

Player 1 (the **owner**) runs the business. Each of the other players $2, \dots, 11$ (the **workers**) contributes an amount 10 000 to the total profit. No profits are generated without the owner.

$N = \{1, \dots, 11\}$

$$v(A) = \begin{cases} 0, & 1 \notin A, \\ 10\,000(|A| - 1), & 1 \in A, \end{cases} \quad A \subseteq N.$$

$$\varphi_1(v) = 50\,000, \quad \varphi_i(v) = 5\,000, \; i = 2, \dots, 11$$

Example (ctnd.)

The **workers** cannot object to the payoff distribution $\varphi(v)$ since $v(N \setminus \{1\}) = 0$. However, the **workers** may form a labor union to reduce the original game to the two-person symmetric game:

Example (Small business vs. labor unions)

$\widehat{N} = \{1, 2\}$

$$\hat{v}(A) = \begin{cases} 1, & A = \widehat{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad A \subseteq \widehat{N}.$$

$$\varphi_1(\hat{v}) = \varphi_2(\hat{v}) = \frac{1}{2}$$

Properties

Fact

The Shapley value φ is

- **Pareto optimal**
- **individually rational** *on the class of superadditive games*
- *mapping each convex game into its* **core**

The Shapley value $\varphi(v)$ can be viewed as a

- fair allocation of the profit generated by the grand coalition N
- voting power in simple games

Shapley Value for Simple Games

Fact

Let v be a ▸ simple game*. Then the Shapley value is*

$$\varphi_i(v) = \sum_{\substack{A \subseteq N \setminus \{i\}: \\ v(A \cup \{i\}) - v(A) = 1}} \frac{|A|!(n - |A| - 1)!}{n!}, \quad i \in N.$$

- the simple game v represents a vote on some issue
- the number $\varphi_i(v)$ is a probability that the player $i \in N$ determines an outcome of the vote

Examples

Example (Stockholders)

A company has 4 stockholders, each of them having $10, 25, 35$, and 40 shares of the company's stock. A decision is approved by a simple majority of all the shares.

$N = \{1, 2, 3, 4\}$
v is a **simple game** in which the only winning coalitions are:
$\{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N$

The Shapley value:

$$\varphi_1(v) = 0, \quad \varphi_i(v) = \frac{1}{3}, \; i = 2, 3, 4$$

Computational Issues

- computation of the Shapley value for real-world problems requires a prohibitive number of calculations
- for example, in a game v with $n = 100$ players the number of summands in $\varphi_i(v)$ is at most $2^{99} \approx 10^{29}$

Speeding up the computations

- multilinear extension
- statistical estimation of the Shapley value based on random sampling

Multilinear Extension

The set of all coalitions 2^N can be identified with the **vertices** of the unit n-cube $[0,1]^n$. Can a game $v : 2^N \to \mathbb{R}$ be extended to an n-**variable function**

$$\overline{v} : [0,1]^n \to \mathbb{R}$$

with "nice" properties?

Theorem (Owen (1972))

There exists a unique multilinear function $\overline{v} : [0,1]^n \to \mathbb{R}$ that coincides with v on 2^N. We have

$$\overline{v}(x_1, \ldots, x_n) = \sum_{A \subseteq N} \left(\prod_{i \in A} x_i \prod_{i \notin A} (1 - x_i) \right) v(A)$$

for every $(x_1, \ldots x_n) \in [0,1]^n$.

Example of Multilinear Extension

Example (Voting)

$N = \{1, 2, 3\}$
$v(\{i\}) = 0, \quad i = 1, 2, 3$
$v(S) = 1$, whenever $|S| \geqslant 2$

The multilinear extension of v is

$$\overline{v}(x) = x_1x_2(1 - x_3) + x_1x_3(1 - x_2) + x_2x_3(1 - x_1) + x_1x_2x_3$$

Stochastic Interpretation of Multilinear Extension

- 2^N can be identified with $\{0,1\}^N$
- every $x \in [0,1]^n$ defines a **product probability** on $\{0,1\}^N$ by

$$p_x(\chi_A) = \prod_{i \in A} x_i \prod_{i \notin A} (1 - x_i), \quad \text{for each } A \subseteq N$$

- $p_x(\chi_A)$ can be considered as the probability of the formation of a random coalition according to x and v as a real random variable on $\{0,1\}^N$
- hence $\overline{v}(x)$ is the **expected value** of worth of the random coalition:

$$\overline{v}(x) = \sum_{A \subseteq N} p_x(\chi_A) v(A) = E_{p_x}(v)$$

Multilinear Extension and Shapley Value

Theorem (Diagonal formula)

Let v be a game with the multilinear extension $\overline{v}$. Then, for every player $i \in N$,

$$\varphi_i(v) = \int_0^1 \frac{\partial \overline{v}}{\partial x_i}(t, \dots, t)\,\mathrm{d}t$$

- Shapley value is thus completely determined by behavior of the function $\overline{v}(x)$ in the neighborhood of the **diagonal** $\{(t, \dots, t) \in [0,1]^n \mid t \in [0,1]\}$

Example

Example (Voting)

$N = \{1, 2, 3\}$
$v(\{i\}) = 0, \quad i = 1, 2, 3$
$v(S) = 1$, whenever $|S| \geqslant 2$

$\overline{v}(x) = x_1x_2(1 - x_3) + x_1x_3(1 - x_2) + x_2x_3(1 - x_1) + x_1x_2x_3$
We have

$$\frac{\partial \overline{v}}{\partial x_1}(x) = x_2 + x_3 - 2x_2x_3$$

so that $\frac{\partial \overline{v}}{\partial x_1}(t, t, t) = 2t - 2t^2$ and thus

$$\varphi_1[v] = \int_0^1 2t - 2t^2 \, \mathrm{d}t = \left[t^2 - \frac{2t^3}{3}\right]_0^1 = \frac{1}{3}$$

Weighted Majority Games

Definition

A simple game v is a weighted majority game if there exist **weights** $\omega_1, \ldots, \omega_n \geqslant 0$ and a **quota** $q > 0$ such that for all $A \subseteq N$

$$v(A) = 1 \quad \text{iff} \quad \sum_{i \in A} \omega_i \geqslant q.$$

Example (Simple majority game)

$N = \{1, 2, 3\}$
$\omega_1 = \omega_2 = \omega_3 = 1, \quad q = 2$
A is winning iff $\omega(A) \geqslant 2$

The UNSC Voting (1)

Example

The United Nations Security Council consists of 5 permanent members and 10 other members. Every decision must be approved by 9 members including all the permanent members.

$N = \{1, \ldots, 15\}$

Representation as a weighted majority game with quota=39:

$$\omega_i = \begin{cases} 7, & i = 1, \ldots, 5 \\ 1, & i = 6, \ldots, 15 \end{cases}, \quad v(A) = \begin{cases} 1, & \sum_{i \in A} \omega_i \geqslant 39, \\ 0, & \text{otherwise,} \end{cases} \quad A \subseteq N.$$

Finding a weighted voting game representation for a simple game is equivalent to solving a system of linear inequalities! Every permanent member is a ▸ veto player.

The UNSC Voting (2)

Representation as a compound game: $N = N_1 \cup N_2$, where $N_1 = \{1, \dots, 5\}$, $N_2 = \{6, \dots, 15\}$
Define:

$$w_1(A) = \begin{cases} 1, & A = N_1, \\ 0, & A \neq N_1, \end{cases} \quad w_2(B) = \begin{cases} 1, & |B| \geqslant 4, \\ 0, & |B| \leqslant 3, \end{cases} \quad A \subseteq N_1, B \subseteq N_2$$

and

$$u(C) = \begin{cases} 1, & C = \{1, 2\} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$v(A) = u\big(\{i \in \{1, 2\} \mid w_i(A \cap N_i) = 1\}\big), \quad A \subseteq N$$

The UNSC Voting (3)

The multilinear extension $\overline{v} : [0,1]^{15} \to \mathbb{R}$ is composed in the same way as the game v. Defining $\overline{w} : \mathbb{R}^{15} \to \mathbb{R}^2$ as

$$\overline{w}(x_1, \dots, x_{15}) = \big(\overline{w_1}(x_1, \dots x_5), \overline{w}_2(x_6, \dots x_{15})\big),$$

we have $\overline{v} = \overline{u} \circ \overline{w}$

Due to **equal treatment property** and **Pareto optimality** axioms, it suffices to calculate the Shapley value of one player (say $i \in N_2$). We get

$$\varphi_i(v) = \int_0^1 \frac{\partial \overline{u}}{\partial y_2}(\overline{w}(\underbrace{t, \dots, t}_{15\times}))\frac{\partial \overline{w}_2}{\partial x_i}(\underbrace{t, \dots, t}_{10\times})\,\mathrm{d}t$$

The UNSC Voting (4)

- $\overline{w}_1(x_1, \ldots, x_5) = x_1 x_2 x_3 x_4 x_5$
- $\frac{\partial \overline{w}_2}{\partial x_i}(t, \ldots, t) = \sum_A t^{|S|}(1-t)^{9-|S|} = \binom{9}{3} t^3 (1-t)^6$, where the first sum runs over all $A \subseteq N_2 \setminus \{i\}$ such that A loses but $A \cup \{i\}$ wins
- $\overline{u}(y_1, y_2) = y_1 y_2$

Hence

$$\varphi_i(v) = \int_0^1 \overline{w}_1(t, \ldots, t) \frac{\partial \overline{w}_2}{\partial x_i}(t, \ldots, t)\, \mathrm{d}t = \int_0^1 t^5 \cdot 84 t^3 (1-t)^6\, \mathrm{d}t = \frac{4}{2145}$$

The UNSC Voting (5)

Since there are 10 players in N_2, each player from the set N_1 has the Shapley value

$$\frac{1}{5} \cdot \left(1 - 10 \cdot \frac{4}{2145}\right) = \frac{1}{5} \cdot \frac{2105}{2145} = \frac{421}{2145}$$

The UNSC game has the Shapley value

$$\varphi_j(v) = \begin{cases} 0.1963, & \text{if } j \text{ is a permanent member,} \\ 0.0019, & \text{otherwise.} \end{cases}$$

These values suggest that the permanent members of the UNSC have immense power in the voting!

U.S. Presidential Election Game

Two-stage procedure for electing a president can be modeled as a compound game:

1. **voters** from each state select Great Electors for Electoral College (51 simple majority games)
2. the **Great Electors** elect the president by a "weighted majority rule" (weighted majority game with 51 players)

A very thin majority for one candidate in a state with a large number of electoral votes can more than annul large majorities in several small states. Is this game in some sense "fair"?

Airport Game

(Littlechild, Owen; 1973)

- a runway needs to be built for m different types of aircrafts
- **costs** $c_1 \leqslant \cdots \leqslant c_m$ of building the runway to accomodate each aircraft
- let N_k be the set of aircraft landings of type k
- the player set is $N = \bigcup_k N_k$

Definition

An airport game is a cost-sharing game given by

$$c(A) = \max\{c_k \mid A \cap N_k \neq \emptyset\}, \quad c(\emptyset) = 0,$$

for each $A \subseteq N$.

Aumann-Drèze Value

Definition

Let v be a game and $\mathcal{R}$ be a coalition structure. The Aumann-Drèze value $\varphi^*(v, \mathcal{R})$ is given by

$$\varphi_i^*(v, \mathcal{R}) = \varphi_i(v_{\upharpoonright R}),$$

where R is the unique set $R \in \mathcal{R}$ with $i \in R$ and $\varphi(v_{\upharpoonright R})$ is the Shapley value of the **subgame** $v_{\upharpoonright R}$.

- if v is any game, then $\varphi(v) = \varphi^*(v, \{N\})$
- the Aumann-Drèze value can be characterized as a unique value satisfying certain axioms

Banzhaf Value

Definition

The (non-normalized) Banzhaf index β' on Γ is given by

$$\beta'_i(v) = \frac{1}{2^{n-1}} \sum_{A \subseteq N \setminus \{i\}} v(A \cup \{i\}) - v(A)$$

for every $v \in \Gamma$ and for every $i \in N$.

compare

The UNSC Voting (ctnd.)

Theorem

If v is a game, then $\beta_i'(v) = \frac{\partial \overline{v}}{\partial x_i}(\frac{1}{2}, \ldots, \frac{1}{2})$.

compare

Example

The UNSC game has the Banzhaf value

$$\beta_i'(v) = \begin{cases} 0.0518, & \text{if } i \text{ is a permanent member,} \\ 0.0051, & \text{otherwise.} \end{cases}$$

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