1. Show: $C \approx D(\varepsilon_F)\frac{\pi}{3} k_B^2 T$

pf: $U = \int_0^\infty d\varepsilon \; (\varepsilon)\,\varepsilon\, f(\varepsilon)$

let $g(\varepsilon) = \varepsilon D(\varepsilon) \rightarrow D'(\varepsilon) = g(\varepsilon) + \varepsilon g'(\varepsilon)$

$$U = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon) + \frac{\pi^2}{6}(k_B T)^2 \left(g(\varepsilon_F) + \varepsilon_F\, g'(\varepsilon_F)\right)$$

(for the lowest order)

$$U = \int_0^{\varepsilon_F} d\varepsilon \cdot \varepsilon D(\varepsilon) + \frac{\pi^2}{6}(k_B T)^2 \left(g(\varepsilon_F) + \varepsilon_F\, g'(\varepsilon_F)\right)$$

$$C = \left(\frac{\partial U}{\partial T}\right)_V =$$

$$= 0 + \frac{\pi^2}{6} \cdot 2 k_B T \cdot k_B \cdot \left(D'(\varepsilon_F)\right)$$

$$C \approx \frac{\pi^2}{3} k_B^2 T\, D(\varepsilon_F)$$

since $D'(\varepsilon_F) \approx D(\varepsilon_F)$ $\square$

2. (7.2) $\varepsilon >> mc^2$

$\varepsilon \simeq pc$

$V = L^3$

$p \sim \frac{\pi\hbar}{L}(n_x^2 + n_y^2 + n_z^2)^{1/2}$

$n = N/V$

Show: $U_0 = \frac{3}{4} N \varepsilon_F$

pf:

$n_F = \left(\frac{3N}{\pi}\right)^{1/3}$ (6)

$V = L^3 \longrightarrow n = N/L^3 \longrightarrow N = nL^3 \rightarrow n_F = L\left(\frac{3n}{\pi}\right)^{1/3}$

$\varepsilon_F \simeq p_F c$

$c \cdot p_F = \left(\frac{\pi\hbar}{L} \cdot n_F\right) c = c \cdot \frac{\pi\hbar}{L} \cdot L\left(\frac{3n}{\pi}\right)^{1/3} = \pi c\hbar\left(\frac{3n}{\pi}\right)^{1/3}$

$U_0 = 2\sum \varepsilon_n = \pi\int_0^{n_F} \varepsilon_n n^2\, dn$ (9)

$= \pi\int_0^{n_F}\left(\frac{\pi\hbar c}{L} n\right) n^2\, dn = \frac{\pi^2\hbar c}{L}\int_0^{n_F} n^3\, dn$

$= \frac{\pi^2\hbar c}{L}\left(\frac{1}{4}n^4\right)\Big|_0^{n_F} = \frac{\pi^2\hbar c}{4L}\left(n_F^4 - 0\right)$

$= \frac{\pi}{4}\left(\frac{\pi\hbar c}{L} n_F\right) n_F^3 = \frac{\pi}{4}\varepsilon_F n_F^3$

$= \frac{3}{4}\varepsilon_F\left(\frac{3N}{\pi}\right) = \frac{3}{4}\varepsilon_F N = U_0 \quad \square$

since $\pi \approx 3$

3. (7.11) show: $\langle(\Delta N)^2\rangle = \langle N\rangle(1-\langle N\rangle)$

$\Delta N \equiv N - \langle N\rangle$

$\langle N\rangle = 1$

pf: $\langle(\Delta N)^2\rangle = (N-\langle N\rangle)^2 = \langle N^2\rangle - \langle N\rangle^2$

for $N = 0, 1$ (fermions)

$0^2 = 0$, $1^2 = 1$

$\langle 0^2\rangle = \langle 0\rangle$, $\langle 1^2\rangle = \langle 1\rangle$

$\therefore \langle N^2\rangle = \langle N\rangle$, since $N^2 = N$

$\langle(\Delta N)^2\rangle = \langle N\rangle - \langle N\rangle^2$

$\langle(\Delta N)^2\rangle = \langle N\rangle(1-\langle N\rangle)$ □

4. (7.12) show: $\langle(\Delta N)^2\rangle = \langle N\rangle(1+\langle N\rangle)$

$\langle N\rangle \gg 1$

$$\frac{\langle(\Delta N^2)\rangle}{\langle N\rangle^2} \approx 1$$

pf: $\langle N^2\rangle \neq \langle N\rangle$ (bosons)

instead, $\langle N\rangle = f(\varepsilon)$

s.t. $\langle(\Delta N)^2\rangle = \tau \dfrac{\partial(\langle N\rangle)}{\partial\mu}$

$$= \tau \frac{\partial}{\partial\mu} \frac{1}{e^{(\varepsilon-\mu)/\tau} - 1}$$

$$= \frac{-\tau}{(e^{(\varepsilon-\mu)/\tau} - 1)^2} \cdot e^{(\varepsilon-\mu)/\tau} \cdot \left(-\frac{1}{\tau}\right)$$

$$= \frac{e^{(\varepsilon-\mu)/\tau}}{(e^{(\varepsilon-\mu)/\tau} - 1)^2}$$

$$\langle N\rangle(1+\langle N\rangle) = \frac{1}{e^{(\varepsilon-\mu)/\tau}-1}\left(1 + \frac{1}{e^{(\varepsilon-\mu)/\tau}-1}\right)$$

$$= \frac{1}{e^{(\varepsilon-\mu)/\tau}-1} + \frac{1}{(e^{(\varepsilon-\mu)/\tau}-1)^2} = \frac{e^{(\varepsilon-\mu)/\tau}-1+1}{(e^{(\varepsilon-\mu)/\tau}-1)^2}$$

$$= \frac{e^{(\varepsilon-\mu)/\tau}}{(e^{(\varepsilon-\mu)/\tau}-1)^2}$$

$\therefore \langle(\Delta N)^2\rangle = \langle N\rangle(1+\langle N\rangle)$ □

$N >> 1$

5. (7.14)

$$0: \quad f(0) = \frac{1}{e^{(0-\mu)/\tau} - 1}$$

$$\varepsilon: \quad f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau} - 1}$$

we want: $f(0) = 2 \cdot f(\varepsilon)$

let $f(0) = \frac{2N}{3} >> 1$

$f(\varepsilon) = \frac{N}{3} >> 1$

$$f(0) = \frac{2}{e^{(\varepsilon-\mu)/\tau} - 1} = \frac{2N}{3}$$

$$6 = 2N\left(e^{(\varepsilon-\mu)/\tau} - 1\right)$$

$$\ln 3 = \ln\left[N\left(e^{(\varepsilon-\mu)/\tau}\right)\right] - \ln N$$

$$\ln 3 = \ln N + \frac{(\varepsilon-\mu)}{\tau} - \ln N$$

$$\ln 3 = \frac{\varepsilon - \mu}{\tau}$$

$$\tau = \frac{\varepsilon - \mu}{\ln 3}$$