MATH 414/514 HOMEWORK 6

ELENA YANG

Exercise 1. a) Lebl 6.1.4: Suppose $\{f_n\}$ and $\{g_n\}$ defined on some set A converge to f and g respectively pointwise. Show that $\{f_n + g_n\}$ converges pointwise $f + g$.

b) Lebl 6.1.5: Suppose $\{f_n\}$ and $\{g_n\}$ defined on some set A converge to f and g respectively uniformly on A. Show that $\{f_n + g_n\}$ converges uniformly to $f + g$ on A.

Proof. a) By definition 1.1, for every $x \in S$ we have

$$f = \lim_{n\to\infty} f_n(x)$$

$$g = \lim_{n\to\infty} g_n(x)$$

Analyzing $\{f_n + g_n\}$ on A we have

$$(f + g)(x) = \lim_{n\to\infty} \{f_n(x) + g_n(x)\}$$

Date: March 24, 2021.

$$= \lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} g_n(x)$$

$$= f + g$$

b) Let $\epsilon > 0$ be given. Then by definition 1.2, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

$$|g_n(x) - g(x)| < \frac{\epsilon}{2}$$

for all $x \in \mathbb{A}$. Then, for all $n \geq N$ and $x \in A$, we have

$$|(f_n + g_n)(x) - (f + g)(x)| = |(f_n(x) + g_n(x)) - (f(x) + g(x))|$$

$$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus by definition $\{f_n + g_n\}$ converges uniformly to $f + g$ on A. □

Definition 1.1. For every $n \in \mathbb{N}$ let $f_n : S \to \mathbb{R}$ be a function. We say the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f : S \to \mathbb{R}$, if for every $x \in S$ we have

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Definition 1.2. Let $f_n : S \to \mathbb{R}$ and $f : S \to \mathbb{R}$ be functions. We say the sequence $\{f_n\}$ converges uniformly to f, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in S$.

Exercise 2. Lebl 6.1.6: Find an example of a sequence of functions $\{f_n\}$ and $\{g_n\}$ that converge uniformly to some f and g on some set A but such that $\{f_n g_n\}$ (the multiple) does not converge uniformly to fg on A. Hint: Let $A := \mathbb{R}$, let $f(x) := g(x) := x$. You can even pick $f_n = g_n$.

Answer. Let $A = \mathbb{R}$ and let $f_n = g_n = x$. Then let $f_n(x) = g_n(x) = x + \frac{1}{n}$.

First, we will show that $f_n(x) = g_n(x) = x + \frac{1}{n}$ converges uniformly. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ exist such that for every $n \geq N$ we have:

$$|f_n(x) - f(x)| = |x + \frac{1}{n} - x| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Now, we will show that $\{f_n g_n\}$ does not converge uniformly. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ exist such that for every $n \geq N$ we have:

$$|f_n g_n(x) - f(x)g(x)| = |(x^2 + \frac{2}{n}x + \frac{1}{n^2}) - x^2| = \frac{2}{n}|x| + \frac{1}{n^2} > \epsilon$$

for $|x| \geq \frac{n\epsilon}{2}$. Thus by definition 1.2, $\{f_n g_n\}$ does not converge uniformly. $\diamond$

Exercise 3. Lebl 6.1.10: Let $\{f_n\}$ be a sequence of functions defined on $[0,1]$.

Suppose there exists a sequence of distinct numbers $x_n \in [0,1]$ such that

$$f_n(x_n) = 1$$

Prove or disprove the following statements:

a) True or false: There exists $\{f_n\}$ as above that converges to 0 pointwise.

b) True or false: There exists $\{f_n\}$ as above that converges to 0 uniformly on $[0,1]$.

Answer. a) True. Let

$$f_n(x) = \begin{cases} 2nx & x \in [0, \frac{1}{2n}] \\ 2 - 2nx & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1] \end{cases}$$

This function converges to 0 pointwise but $f_n(\frac{1}{2n}) = 1$. $\diamond$

b) False.

Proof. Using definition 3.1, we see that $\{f_n\}$ converges to 0 uniformly if the following holds true:

$$\lim_{n\to\infty} ||f_n - 0||_u = 0$$

Rewriting the expression we see that:

$$\lim_{n\to\infty} ||f_n - 0||_u = \lim_{n\to\infty} \sup(f_n - 0) \geq \lim_{n\to\infty} 1 \neq 0$$

Therefore there does not exist $\{f_n\}$ that converges to 0 uniformly. $\square$

Definition 3.1. Proposition 6.1.10 states: A sequence of bounded functions $f_n : S \to \mathbb{R}$ converges uniformly to $f : S \to \mathbb{R}$ if and only if

$$\lim_{n\to\infty} ||f_n - f||_u = 0$$

Exercise 4. Lebl 6.2.7: For a continuously differentiable function $f : [a, b] \to \mathbb{R}$, define

$$||f||_{C^1} := ||f||_u + ||f'||_u$$

Suppose $\{f_n\}$ is a sequence of continuously differentiable functions such that for every $\epsilon > 0$, there exists an M such that for all $n, k \geq M$ we have

$$||f_n - f_k||_{C^1} < \epsilon$$

Show that $\{f_n\}$ converges uniformly to some continuously differentiable function $f : [a, b] \to \mathbb{R}$.

Proof. Rewriting $||f_n - f_k||_{C^1} < \epsilon$, we have

$$||f_n - f_k||_u + ||f'_n - f'_k||_u < \epsilon$$

Because $||$ denotes the supremum of an absolute value, both $||f_n - f_k||_u$ and $||f'_n - f'_k||_u$ are positive implying that they are also both less than ϵ. By definition 6.1.12, this means that the f_n is uniformly Cauchy and f_n converges pointwise to f. Fixing $c \in [a, b]$ we see that $\{f_n(c)\}(c)$. Thus, by theorem 6.2.10, f_n converges uniformly to a continuously differentiable function f on $[a, b]$. □