MATH 414/514 HOMEWORK 5T

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Exercise 1. Lebl 2.2.12:

(a) Suppose $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a sequence converging to 0. Show that $\{a_nb_n\}$ converges to 0.

(b) Find an example where $\{a_n\}$ is unbounded, $\{b_n\}$ converges to 0, and $\{a_nb_n\}$ is not convergent.

(c) Find an example where $\{a_n\}$ is bounded, $\{b_n\}$ converges to some $x \neq 0$, and $\{a_nb_n\}$ is not convergent.

Proof. a) We know that a_n is bounded which means there exists M such that $|a_n| \leq M$ for all n. We also know that since b_n converges to 0, there exists a $N \in \mathbb{N}$ such that $|b_n - 0| < \frac{\epsilon}{M}$ for all $n \geq N$. Thus, for all $n \geq N$, $|a_nb_n| = |a_n||b_n| \leq M|b_n| < M \cdot \frac{\epsilon}{M} = \epsilon$.

$\square$

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Answer. b) Let $\{a_n\} = n^2$, $b_n = \frac{1}{n}$. Then $\{a_n b_n\} = n$ which isn't convergent $\diamond$

Answer. c) Let $\{a_n\} = \sin(n)$, $b_n = (1 + \frac{1}{n}^n)$. $\diamond$

Exercise 2. Lebl 2.2.15: Prove $\lim_{n\to\infty}(n^2+1)^{\frac{1}{n}} = 1$.

Proof. Note that $(n)^{\frac{1}{n}} \leq (n^2+1)^{\frac{1}{n}} \leq (n^3)^{\frac{1}{n}}$ for all $n \geq 1$. By Lebl example 2.2.14, $\lim_{n\to\infty}(n)^{\frac{1}{n}} = 1$. We will now show that $\lim_{n\to\infty}(n^3)^{\frac{1}{n}} = 1$:

Let $\epsilon > 0$ be given. Consider the sequence $\{\frac{n^3}{(1+\epsilon)^n}\}$. Compute:

$$\frac{(n^3+1)/(1+\epsilon)^{n+1}}{(n^3)/(1+\epsilon)^n} = \frac{n^3+1}{n^3} = \frac{1}{1+\epsilon}$$

And so the limit of $\frac{n^3+1}{n^3} = 1 + \frac{1}{n^3}$ as $n \to \infty$ is 1. Thus:

$$\frac{(n^3+1)/(1+\epsilon)^{n+1}}{(n^3)/(1+\epsilon)^n} = \frac{1}{1+\epsilon} < 1$$

Therefore, $\{\frac{n^3}{(1+\epsilon)^n}\}$ converges to 0 by the ratio test for sequences. In particular, there exists an N such that for $n \geq N$, we have $\frac{n^3}{(1+\epsilon)^n} < 1$, or $n^3 < (1+\epsilon)^n$, or $n^{3^{\frac{1}{n}}} < 1+\epsilon$. As $n \geq 1$, then $n^{3^{\frac{1}{n}}} \geq 1$, and so $0 \leq n^{3^{\frac{1}{n}}} - 1 < \epsilon$. Consequently, $\lim n^{3^{\frac{1}{n}}} = 1$.

Thus by the Squeeze Theorem, $\lim_{n\to\infty}(n^2+1)^{\frac{1}{n}} = 1$. $\square$

Exercise 3. Lebl 2.5.14: Suppose $\sum x_n$ converges and $x_n \geq 0$ for all n. Prove that $\sum x_n^2$ converges. What if we drop the assumption $x_n \geq 0$?

Proof. Since $\sum x_n$ converges, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $0 < x_n < 1$. This is because if $x_n > 1$ for all n, the series would diverge. In this interval, $x_n^2 < x_n$ for all n. Using the comparison test (prop 2.5.16) on the "tail" of the summation, namely where $0 < x_n < 1$, we see that since x_n converges and $x_n^2 < x_n$, x_n^2 also converges. This is because x_n^2 converges on this "tail" and there are finitely many terms beforehand which have a finite sum. □

Answer. If we drop the assumption $x_n \geq 0$, then $\sum x_n^2$ does not have to converge. For example, we could have a series that displayed conditional convergence such as $\sum \frac{(-1)^n}{\sqrt{(n)}}$ which converges, but whose square is $\sum \frac{1}{n}$ which is the Harmonic Series which diverges. ◇

Exercise 4. TBB 3.12.8: Let $\{a_k\}$ be a monotonic sequence of real numbers such that $\sum_{k=1}^{\infty} a_k$ converges. Show that

$$\sum_{k=1}^{\infty} k(a_k - a_{k+1})$$

converges.

Proof. If we analyze the first few terms of the sequence, we get $1(a_1 - a_2) + 2(a_2 - a_3) + 3(a_3 - a_4) + ...$ which leaves us with a simplified expression of $a_1 + a_2 + a_3 + ...$ However, the last term which is unaccounted for is $-k \cdot a_{k+1}$. Since $a_1 + a_2 + a_3 + ... = a_k$ converges, we just need to show that $k \cdot a_{k+1}$ does as well. By the Monotone Convergence Theorem, since a_k converges and is monotone, is it bounded and in fact converges to that bound. Since $\sum a_k$ converges and is monotone, then $k \cdot a_{k+1}$ converges as well. $\square$