MATH 414/514 HOMEWORK 1

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Exercise 1. Read: `http://web.stonehill.edu/compsci/History_Math/math-read`

(a) What is one interesting or surprising idea you learned from this reading?

(b) What is one idea you will use when you read mathematics?

Answer. (a) I learned about mathematical maturity of books, and how books assumes the reader has a certain amount of knowledge before reading the book. I found this interesting as I often wondered why people on forums or professors choose a certain textbook or reading when there are so many options out there. It makes sense why certain editions or books would be more appropriate for certain audiences. (b) I will make sure to cross reference materials, talk to others, and compare sources. That way I can learn more from the material and understand it more deeply. ◇

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Exercise 2. 3.3.14: Suppose $f : [0, 1] \to (0, 1)$ is a bijection. Prove that f is not continuous.

Proof. For the sake of contradiction, assume f is continuous. Since f is closed continuous, then by the Minimum-Maximum Theorem we know that f achieves both an absolute maximum and minimum on $[0, 1]$. We know that there exists a point $c \in \mathbb{R}$ such that $f(c) \geq f(x)$ for all x in the interval. However, there still exists a range of values from $f(c)$ to 1. Let a be a point in $(f(c), 1)$. Since $f(c)$ is an absolute maximum so there can't exist an x such that $f(x) = a$. Since the rest of this range cannot be achieved through the bijection, f is not continuous. $\square$

Exercise 3. 3.4.5: Let A, B be intervals. Let f: $A \to \mathbb{R}$ and Let g: $B \to \mathbb{R}$ be uniformly continuous functions such that $f(x) = g(x)$ for $x \in A \cap B$.

Define the function $h : A \cup B \to \mathbb{R}$ by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B \setminus A$.

(a) Prove that if $A \cap B \neq \emptyset$, then h is uniformly continuous

(b) Find an example where $A \cap B = \emptyset$ and h is not even continuous

Proof. (a) By definition 3.4.1, uniform continuity exists in $S \subset \mathbb{R}$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x, c \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Without loss of generality, let $\sup(A) \leq \sup(B)$. Then, for any given $\epsilon > 0$ choose δ_1 such that $|f(x) - f(c)| < \frac{\epsilon}{2}$ whenever $|x - c| < \delta_1$ and $x, c \in A$ and δ_2 such that $|f(x) - f(c)| < \frac{\epsilon}{2}$ whenever $|x - c| < \delta_2$ and $x, c \in B$. Define δ to be the minimum of δ_1 and δ_2.

Now, we have three cases. The first two cases are when $x, c \in A$ or $x, c \in B$. These cases are true as f, g are defined to be uniformly continuous functions. The other case is if x, c are in different intervals. Without loss of generality, let $x \in A$ and $c \in B$. Since $A \cap B \neq \emptyset$, there exists $y \in A \cap B$ such that $y \in (x, c)$ and $x \leq y \leq c$. Now we check for uniform continuity in this case.

By definition, we have:

$$|h(x) - h(c)| = |h(x) - h(y) + h(y) - h(c)|$$

$$\leq |h(x) - h(y)| + |h(y) - h(c)|$$

Since $x, y \in A$ and $y, c \in B$, then we can write:

$$= |f(x) - f(y)| + |g(y) - g(c)|$$

The distance from x, c to y are both less than δ, thus:

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since $|x - y| < \delta \implies |h(x) - h(y)| < \epsilon$ for all x, y, then h is uniformly continuous if $A \cap B \neq \emptyset$. $\square$

Answer. (b) Let $f(x) = 0, g(x) = 2, A = [0, 1)$, and $B = [1, 2]$. At $x = 1$ the function is discontinuous. $\diamond$

Exercise 4. 4.1.11: Suppose $f : I \to \mathbb{R}$ is bounded and $g : I \to \mathbb{R}$ is differentiable at $c \in I$ and $g(c) = g'(c) = 0$. Show that $h(x) \coloneqq f(x)g(x)$ is differentiable at c. Hint: you cannot apply the product rule.

Proof. Recall from Lebl definition 4.1.1 that if I is an interval, $f : I \to \mathbb{R}$, and $c \in I$, then f is differentiable at c if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. In this case, if $\lim_{x\to c} \frac{h(x)-h(c)}{x-c}$ exists then we will have shown $h(x)$ is differentiable at c. Plugging in $h(x) \coloneqq f(x)g(x)$ we get:

$$h'(x) = \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x - c}$$

We know that $g(c) = 0$ resulting in:

$$h'(x) = \lim_{x\to c} \frac{f(x)g(x)}{x - c}$$

We are given that f is bounded. Thus there exists some $B \in \mathbb{R}$ such that $|f(x)| \leq B$ or $-B \leq f(x) \leq B$ for all $x \in \mathbb{R}$. We can combine $h'(x)$ and the second set of inequalities to get:

$$-B \cdot \lim_{x\to c} \frac{g(x)}{x - c} \leq \lim_{x\to c} \frac{f(x)g(x)}{x - c} \leq B \cdot \lim_{x\to c} \frac{g(x)}{x - c}$$

If we look at $\lim_{x\to c} \frac{g(x)}{x-c}$ we see that if we rewrite it as $\lim_{x\to c} \frac{g(x)-g(c)}{x-c}$ we don't change its value as $g(c) = 0$ and we get it in derivative form. This value is just $g'(c)$ which is given to be 0. Thus, we have:

$$-B \cdot 0 \leq \lim_{x\to c} \frac{f(x)g(x)}{x-c} \leq B \cdot 0$$

$$0 \leq \lim_{x\to c} \frac{f(x)g(x)}{x-c} \leq 0$$

The only possible value of $\lim_{x\to c} \frac{f(x)g(x)}{x-c}$ that makes this equation true is 0 which means the limit exists, and thus the derivative as well. $\square$

Exercise 5. 4.2.12: Suppose $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f'(x) = a$ for all x, and $f(0) = b$. Find f and prove that it is the unique differentiable function with this property.

Answer. The function is $f(x) = ax + b$. Then, $f'(x) = a$ and $f(0) = b$. This proves existence. Now we will prove that this is the only function with these properties. $\diamond$

Proof. Assume there exists a function $g(x) \neq f(x)$ such that $g'(x) = a$ and $g(0) = b$. The Mean Value Theorem states that if $f : [a, b] \to \mathbb{R}$ is a continuous function differentiable on (a, b) then there exists a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Let g be on the interval $[h, k] \in \mathbb{R}$ and $c \in (h, k)$. Plugging h, k into the Mean Value Theorem we get:

$$g(k) - g(h) = g'(c)(k - h)$$

Since we know $g'(k) = a$, we have:

$$g(k) - g(h) = a(k - h)$$

$$g(k) - g(h) = ak - ah$$

$$g(k) = ak - ah + g(h)$$

We now analyze point $x \in \mathbb{R}$ that is on $g(x)$. We know that $g(k) = ak - ah + g(h)$ holds for all points $h, k \in g(x)$. Fix any value of h and let $b = -ah + g(h)$. Then the equation reads $g(k) = ak + b$ which is identical to $f(x) = ax + b$. Thus $f(x) \neq g(x)$ a contradiction and $f(x) = ax + b$ is the unique function.

$\square$