MATH 414/514 HOMEWORK 3

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Exercise 1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b = 0$. Prove that $f(x) = 0$ for all x. Also, show that the hypothesis of continuity is necessary (give a counterexample to the version of the statement without the continuity hypothesis).

Proof. For the sake of contradiction, assume there exists $c \in [a, b]$ such that $f(c) \neq 0$. Definition 3.2.1 says that if f is continuous at c if for every every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \frac{f(c)}{2}$.

Then, there is an interval around $f(c)$ where the function is greater than 0. If $f \geq \frac{f(c)}{2}$ on $c - \delta, c + \delta$, then:

$$\int_a^b f(x)dx \geq= 2\delta \frac{f(c)}{2} = \delta f(c) > 0$$

Which is a contradiction since $\int_a^b = 0$ $\square$

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Answer. Let $f : [0, 10] \to \mathbb{R}$ and

$$f(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ \\ 1 & \text{if } x = \frac{1}{2} \end{cases}$$

Then the removable discontinuity makes it so that even though the integral is equal to 0, the point $x = \frac{1}{2}$ has $f(x) \neq 0$. $\diamond$

Exercise 2. Let $f : [a,b] \to \mathbb{R}$ be increasing. Show that f is Riemann integrable. Hint: use a uniform partition; each subinterval of same length.

Proof. Let $P_n = \{x_0, x_1, ..., x_n\}$ be a uniform partition of $[a,b]$, that is, $x_j := \frac{j}{n}$. Proposition 5.1.13 states that f is Riemann integrable if for every $\epsilon > 0$, there exists a partition P such that:

$$U(P,f) - L(P,f) < \epsilon$$

Let the partitions be fine enough such that $x_i - x_{i-1} < \frac{\epsilon}{f(b)-f(a)}$. Also note that $x_i - x_{i-1} = (a + i\frac{b-a}{n}) - (a + (i-1)\frac{b-a}{n}) = \frac{b-a}{n}$. Let $f(b) > f(a)$. We have:

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (\sup f(x) - \inf f(x))(x_i - x_{i-1})$$

Where $\sup f(x)$ and $\inf f(x)$ are each defined on their respective partitions. Since each partition is the same width, we can rewrite this as:

$$\sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_1 - x_0)$$

Note that the summation has telescoping values, so simplifying we have:

$$(f(b) - f(a))(x_1 - x_0)$$

Before we defined the difference between any two partitions, or $x_i - x_{i-1}$ to be $\frac{b-a}{n}$. Plugging this in we get:

$$(f(b) - f(a))\frac{b-a}{n}$$

Taking the limit as $n \to \infty$ we get 0, or $U(P, f) - L(P, f) < 0$. Since $0 < \epsilon$ this completes our proof. $\square$

Exercise 3. (a) Show that $f(x) := sin(\frac{1}{x})$ is integrable on any interval (you can define f(0) to be anything). (b) Compute $\int_{-1}^{1} sin(\frac{1}{x})dx$ (Mind the discontinuity)

Proof. (a) Define $f(x) = sin(\frac{1}{x})$ and $f(0) = 0$. Then, by theorem 5.2.9, since f is bounded and has finitely many discontinuities, f is Riemann integrable. Since $\sin(\frac{1}{x})$ has a finite number of discontinuities, namely one at $x = 0$, it is sufficient to say it meets the criterion and thus is Riemann integrable. We know that it is continuous everywhere other $x = 0$ since $\sin(\frac{1}{x})$ is the composition of $sin(x)$ and $\frac{1}{x}$ which is continuous everywhere but $x = 0$. □

Answer. (b) $\int_{-1}^{1} sin(\frac{1}{x})dx = \int_{-1}^{\epsilon} \sin(\frac{1}{x}) + \int_{\epsilon}^{1} \sin(\frac{1}{x})$. Using the fundamental theorem of calculus we see that if $F(x)$ is the anti-derivative of $sin(\frac{1}{x})$ then this value is $F(\epsilon) - F(-1) + F(1) - F(\epsilon) = F(1) - F(-1)$. where F is the antiderivative of $\sin(\frac{1}{x})$.

We can use theorem 5.3.5 which states that if $g : [a, b] \to \mathbb{R}$ is a continuously differentiable function, and $f : [c, d] \to \mathbb{R}$ is continuous, and $g([a, b]) \subset [c, d]$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(s)ds$$

Since $sin(\frac{1}{x})$ is an odd function so since it's symmetric around the origin, the integral is 0. $\diamond$

Exercise 4. TBB 8.3.10: If f and g are continuous on an interval [a,b], show that

$$(\int_a^b f(x)g(x)dx)^2 \leq (\int_a^b [f(x)]^2 dx)(\int_a^b [g(x)]^2 dx)$$

Proof. Define a function $f(t)$ to be $\int_a^b (tf(x)+g(x))^2$. Then $f(t) > 0$ is always non-negative. Expanding this out, we get:

$$0 \leq t^2 \int_a^b (f(x))^2 dx + t \int_a^b 2f(x)g(x)dx + \int_a^b (g(x))^2$$

Note that this is in the form of $At^2 + 2Bt + C$ for $A = \int_a^b (f(x))^2$, $B = \int_a^b f(x)g(x)$, and $C = \int_a^b (g(x))^2$.

We will now analyze the discriminant which is $(2B)^2 - 4AC$. We know that the discriminant is non-positive. This is because if it is positive, then both roots are real which implies that $At^2 + 2Bt + C$ is less than 0 for at least one t. But we defined it to be positive, so the discriminant can't be more than 0. So since it is non-positive, we have:

$$(2B)^2 - 4AC \leq 0$$

$$4B^2 - 4AC \leq 0$$

$$B^2 - AC \leq 0$$

$$B^2 \leq AC$$

This matches the form of our original inequality, thus it's true. $\square$