MATH 414/514 HOMEWORK 4

ELENA YANG

Exercise 1. Lebl 5.2.4: Prove the mean value theorem for integrals. That is, prove that if $f : [a,b] \to \mathbb{R}$ is continuous, then there exists a $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.

Proof. Consider $F(x) = \int_a^x f(t)dt$.

By definition 1.1, $F'(x) = f(x)$ for every $x \in [a,b]$ and $F(b) - F(a) = \int_a^b f(t)dt$.

By definition 1.2 on F, we have $F(b) - F(a) = F'(c)(b-a)$

Since $F'(x) = f(x)$, we have $F(b) - F(a) = f(c)(b-a) = \int_a^b f(t)dt$.

Since this c exists in the open interval by def 1.1 and def 1.2, it also exists in the closed interval $[a,b]$. $\square$

Date: February 17, 2021.

Definition 1.1. The first part of the Fundamental Theorem of Calculus states: let $F : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b), Let $f \in \mathbb{R}[a, b]$ and $f(x) = F'(x)$ for $x \in (a, b)$, then $\int_a^b f = F(b) - F(a)$.

Definition 1.2. The Mean Value Theorem states: if $f : [a, b] \to \mathbb{R}$ is a continuous function differentiable on (a, b), then there exists a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$

Exercise 2. TBB 6.7.4: Calculate $\omega_f(0)$ for each of the following functions:

a.

$$f(x) = \begin{cases} x & x \neq 0 \\ 4 & x = 0 \end{cases}$$

b.

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

c. Note that n is an integer.

$$f(x) = \begin{cases} n & x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

d.

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

e.

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 7 & x = 0 \end{cases}$$

f.

$$f(x) = \begin{cases} \frac{1}{x}\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Answer.

a. In any interval $(x-\delta, x+\delta)$, $\sup = \max(f(4, x+\delta))$ and $\inf = \min(f(x-\delta, 4)) = 0$. When $x = 0$, $\lim_{\delta\to 0}(x+\delta) = 0$. Thus, we have $\max(0,4) - \min(0,4) = 4 - 0 = \boxed{4}$

b. In any interval $(x-\delta, x+\delta)$, f has output values $[0] \cup [1]$. So $\sup f((x-\delta, x+\delta)) = 1$ and $\inf f((x-\delta, x+\delta)) = 0$.

$\omega_f((x-\delta, x+\delta)) = 1 - (0) = 1$

$\omega_f(0) = \lim_{\delta\to 0}\omega_f((x-\delta, x+\delta)) = lim_{\delta\to 0}1 = \boxed{1}$

c. In any interval $(x-\delta, x+\delta)$, f has output values $[0, \infty)$. So $\sup f((x-\delta, x+\delta)) = \infty$ and $\inf f((x-\delta, x+\delta)) = 0$.

$\omega_f((x-\delta, x+\delta)) = \infty - (0) = \infty$

$\omega_f(0) = \lim_{\delta\to 0}\omega_f((x-\delta, x+\delta)) = lim_{\delta\to 0}\infty = \boxed{\infty}$

d. In any interval $(x-\delta, x+\delta)$, f has output values $[-1, 1] \cup [7]$. So

$\sup f((x-\delta, x+\delta)) = 1$ and $\inf f((x-\delta, x+\delta)) = -1$.

$\omega_f((x-\delta, x+\delta)) = 1 - (-1) = 2$

$\omega_f(0) = \lim_{\delta \to 0} \omega_f((x-\delta, x+\delta)) = lim_{\delta \to 0} 2 = \boxed{2}$

e. In any interval $(x-\delta, x+\delta)$, f has output values $[-1, 1] \cup [7]$. So $\sup f((x-\delta, x+\delta)) = 7$ and $\inf f((x-\delta, x+\delta)) = -1$.

$\omega_f((x-\delta, x+\delta)) = 7 - (-1) = 8$

$\omega_f(0) = \lim_{\delta \to 0} \omega_f((x-\delta, x+\delta)) = lim_{\delta \to 0} 8 = \boxed{8}$

f. In any interval $(x-\delta, x+\delta)$, f has output values $[0, \infty)$. So $\sup f((x-\delta, x+\delta)) = \infty$ and $\inf f((x-\delta, x+\delta)) = 0$.

$\omega_f((x-\delta, x+\delta)) = \infty - (0) = \infty$

$\omega_f(0) = \lim_{\delta \to 0} \omega_f((x-\delta, x+\delta)) = lim_{\delta \to 0} \infty = \boxed{\infty}$ $\diamond$

Exercise 3. TBB 8.6.2: Show that the product of two Riemann integrable functions is itself Riemann integrable.

Proof. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be Riemann integrable. By definition 3.1, f, g are both bounded and D_f, D_g (set of discontinuites) have measure zero.

First, we'll show that $f \cdot g$ is bounded. By definition 3.2, let f be bounded by M_f, or $|f| \leq M_f$ and let g be bounded by M_g, or $|g| \leq M_g$. Thus, $|f \cdot g| \leq M_f M_g$ which means $f \cdot g$ is also bounded.

Now we'll show that $D_{f \cdot g}$ has measure zero. Suppose there exists a point $x \notin D_f \cup D_g$. Then, $x \notin D_f$ and $x \notin D_g$. This means that f, g are both continuous at x, so $f \cdot g$ is also continuous at x by Lebl Proposition 3.2.5 (iii). Thus, $x \notin D_{f \cdot g}$.

We want to show that $D_{f \cdot g} \subseteq D_f \cup D_g$. We will prove this using the contrapositive - if a point $k \notin D_f \cup D_g$ then $k \notin D_{f \cdot g}$. Since k is in neither of the sets of discontinuities, then f and g are both continuous at k. By proposition 3.2.5 (iii), $f \cdot g$ is also continuous at k, which means that $k \notin D_{f \cdot g}$.

Since $D_{f \cdot g}$ is subset of a union of two measure zero sets, it also has measure zero. Thus, by Riemann-Lebesgue, $f \cdot g$ is integrable. $\square$

Definition 3.1. Riemann-Lebesgue Theorem: Let $f : [a, b] \to \mathbb{R}$. Then $f \in \mathbb{R}[a, b]$ if and only if (1) f is bounded, and (2) the set of points of discontinuity of f has measure zero.

Definition 3.2. Suppose $f : D \to \mathbb{R}$ is a function. We say f is bounded if there exists a number M such that $|f(x)| \leq M$ for all $x \in D$.

Exercise 4. TBB 6.8.12: Show that the set of real numbers in the interval [0,1] that do not have a 7 in their infinite decimal expansion is of measure zero.

Proof. We will construct a union of numbers that do not have a 7 in their infinite decimal expansion by restricting first the tens, then the hundreds, and so on decimal place.

Our first set will be S_1 where we construct an interval of numbers without a 7 in the tens place. This is:

$$S_1 = [0, 0.7) \cup [0.8, 1]$$

Here, we deleted $[0.7, 0.8)$. Note that this is $0.9 = \frac{9}{10^1}$ of the original interval of $(0, 1)$.

We'll no construct a second set which is the union of these intervals but with the added constraint that the second digit cannot be 7. This is:

$$S_2 = ([0, 0.07) \cup [0.08, 0.1]) \cup ([0.1, 0.17)] \cup [0.18, 0.2]) \cup ([0.2, 0.27)] \cup [0.28, 0.3])$$

$$\cup([0.3, 0.37)] \cup [0.38, 0.4]) \cup ([0.4, 0.47)] \cup [0.48, 0.5]) \cup ([0.5, 0.57)] \cup [0.58, 0.6])$$

$$\cup([0.6, 0.67)] \cup [0.68, 0.7)) \cup ([0.8, 0.87)] \cup [0.88, 0.9]) \cup ([0.9, 0.97)] \cup [0.98, 0.1])$$

The total length of this interval is $9 \cdot (\frac{9}{10^2})$. Note that the total length of set $S_n = (\frac{9}{10})^n$ because each time we're removing one digit (7) from a total of 10 digits in each place value holder. This value approaches 0 as $n \to \infty$, it has measure 0. $\square$