Soshnikov gluing procedure and the counterexample

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Abstract

Here is a short overview of Soshnikov's gluing procedure that made it possible to improve the bound on the spectral norm of a random symmetric matrix with independent bounded entries, not necesserally symmetrically or identically (but they have to have the same third moment) distributed. Additionally, in the second part there is a counterexample to one of the key assertions used by Soshnikov.

1 Soshnikov result

The best known result for the spectral norm of symmetric random matrices is attributed to Soshnikov and Peche. In their paper ([1], 2007) they claim the following:

Theorem 1:

$$\lambda(A) \leq 2\sigma\sqrt{n} + o(n^{-1/22+\varepsilon}),$$

where ε is an arbitrary small positive number.

Consider a symmetric random $n \times n$ matrix A with independent random entries $A[i,j]$. Additionally, they satisfy sub-Gaussian assumption:

$$\mathbb{E}(A[i,j])^{2\kappa} \leq (const\ \kappa)^{\kappa} \tag{1}$$

The key part of the proof is the connection between symmetric random matrices with independent entries and standard Wigner matrices with symmetrically distributed entries:

Theorem 2: Assume that $s_n = O(n^{1/2+\eta})$ where $\eta < 1/22$. Then

$$\mathbb{E}[trace\ A_n^{2s_n}] = \mathbb{E}[trace\ W_n^{2s_n}](1 + o(1)),$$

where W_n is a standard Wigner matrix with symmetrically distributed sub-Gaussian entries of variance σ^2. And

$$\mathbb{E}[trace\ W_n^{2s_n}] = \frac{n^{s_n+1}}{\pi^{1/2} s_n^{3/2}}(2\sigma)^{2s_n}(1 + o(1))$$

as long as $s_n = o(n^{2/3})$. Or equivalently:

$$\mathbb{E}[trace\ A_n^{2s_n}] = Z_e(1 + o(1)) = (1 + o(1))nC_{s_n}\sigma^{2s_n},$$

where Z_e is the contribution from the paths where each edge is used exactly twice (for paths see: https://elidechse.github.io/VanVu.pdf) and C_{s_n} is a Catalan number.

Similarly to Vu's approach they bound the number of closed walks of length $k = 2s_n$ on a graph. Consider a walk

$$\mathcal{W} = [w_0, w_1, ..., w_{2s_n-1}, w_0].$$

Then the l^{th} step $w_{l-1} \to w_l$ is called *marked* if during the first l steps the edge $[w_{l-1}, w_l]$ appeared odd number of times. (That is the first step is always marked.) If an edge $[w_i, w_j]$

appears odd number of times in the walk we call its last appearance a *non-returned* edge. If there is at least one odd edge in a walk then the walk is called *odd*, otherwise we call it *even*.

Lemma 1: The non-returned edges can be concatenated into a set of closed walk themselves.

Proof: We will illustrate our proof with an example. Consider a walk:

$$\mathcal{W} = [1, 6, 5, 1, 2, 4, 3, 2, 1, 5, 4, 5, 1]$$

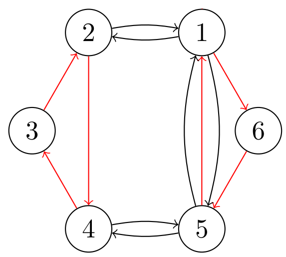


Figure 1: *The non-returned edges are colored red.*

First, for any closed walk the degree of each node must be even. To see that note that for any edge $[w_{j-1}, w_j]$ arriving at a node j, where $j \neq 0$, there is a corresponding edge $[w_j, w_{j+1}]$ leaving j. For $j = 0$ we say that the edge $[w_{k-1}, w_0]$ corresponds to the edge $[w_0, w_1]$. Thus for each node its 'in' and 'out' degrees are the same resulting in an even total degree.

Second, note that if we erase all the edges except the non-returned ones then the degree of each node stays even (possibly zero) by definition of the non-returned edges (for any edge it's either used even number of times in the walk or its last appearance is a non-returned edge).

And finally, construct the closed walks that consist only of the non-returned edges of the original walk in the following way: start at any node which degree is greater than zero (that is greater or equal to 2) and follow the edges at random erasing them at each step until it arrives at the original node. The fact that all nodes have even degrees guarantees that for each time we arrive at a node there will be a "way out" except for the first node. So, we constructed a first closed walk of the set. Proceed until there are no edges left. $\square$

Next they introduce a nice gluing procedure and its reverse insertion procedure to calculate the number of the walks of interest.

Gluing procedure:

1. Define $[w_{r_i}, w_{r_i+1}]$, $1 \leq i \leq 2l$, to be the non-returned edges of the walk. Choose sub-walks of the $\mathcal{W}$ such that

$$\mathcal{W}_i = \begin{cases} [w_0, w_1, ..., w_{r_i-1}] & \text{if } i = 0 \\ [w_{r_{2l}+1}, ..., w_{2s_n-1}, w_0] & \text{if } i = 2l \\ [w_{r_i+1}, ..., w_{r_{i+1}-1}] & \text{otherwise.} \end{cases}$$

   Similarly to the procedure used in the proof of lemma 2 it is possible to concatenate (*glue*) the sub-walks $\mathcal{W}_i$ into a set of closed walks. Denote this set $\mathcal{W}'$.

2. Define $i_0, i_1, ..., i_I$ to be the origins of the closed walks in $\mathcal{W}'$. W.l.o.g. we can assume that all i_j are different: if not we can glue two walks with the same origin into one. There are three possible outcomes of the above gluing:

   (a) $I = 1$, that is essentially $\mathcal{W}'$ is a one even walk.

(b) $\mathcal{W}'$ is a set of even walks.

(c) $\mathcal{W}'$ is a set of walks some of which are odd.

3. In case when $\mathcal{W}'$ is a set of walks some of which are odd there is additional gluing procedure needed: let $\mathcal{W}_1 := [w_0^1, ..., w_{s_1}^1]$ and $\mathcal{W}_2 := [w_0^2, ..., w_{s_2}^2]$ be two walks with the same edge $[w_{i-1}, w_i]$ that they both use odd number of times (note that for any odd edge there must exist two such walks because $\mathcal{W}'$ uses all edges even number of times by definition). Let t_1, t_2 be the first occurrences of this edge in $\mathcal{W}_1$ and $\mathcal{W}_2$ respectively, i.e $[w_{t_1-1}^1, w_{t_1}^1]$ and $[w_{t_2-1}^2, w_{t_2}^2]$ are the same edges as $[w_{i-1}, w_i]$. Then we glue $\mathcal{W}_1$ and $\mathcal{W}_2$ in the following way:

   - We start at the origin of $\mathcal{W}_1$ and follow its first t_1 edges.
   - If the orientation of $[w_{t_1-1}^1, w_{t_1}^1]$ is the same as of $[w_{t_2-1}^2, w_{t_2}^2]$ (i.e. $w_{t_1-1}^1 = w_{t_2-1}^2$ and $w_{t_1}^1 = w_{t_2}^2$) we continue with edge $[w_{t_2+1}^2, w_{t_2+2}^2]$ and follow the walk $\mathcal{W}_2$ to the end. Then we continue by following the first t_2 edges of the $\mathcal{W}_2$ and finally, we follow the edges of $\mathcal{W}_1$ from $[w_{t_1+1}^1, w_{t_1+2}^1]$ till the end, thus creating a closed walk that uses edge $[w_{i-1}, w_i]$ even number of times.
   - Similarly, if the orientation of $[w_{t_1-1}^1, w_{t_1}^1]$ is not the same as of $[w_{t_2-1}^2, w_{t_2}^2]$ we continue in the opposite direction of $\mathcal{W}_2$, i.e we continue with edge $[w_{t_2-1}^2, w_{t_2-2}^2]$ and follow the walk $\mathcal{W}_2$ to the beginning. Then we continue by following $\mathcal{W}_2$ from the end to the edge $[w_{t_2}^2, w_{t_2-1}^2]$ and again we finish the walk by following the edges of $\mathcal{W}_1$ from $[w_{t_1+1}^1, w_{t_1+2}^1]$ till the end.

   Let us illustrate the procedure with an example:

   Example: Let

$$\mathcal{W}_1 = [1, 2, 3, 1, 2, 3, 2, 1] \text{ and } \mathcal{W}_2 = [1, 3, 4, 5, 1, 3, 1]$$

   We colored red the odd edge they share. Then the result of the gluing will be:

$$[1, 2, 3, 1, 3, 1, 5, 4, 3, 1, 2, 3, 2, 1],$$

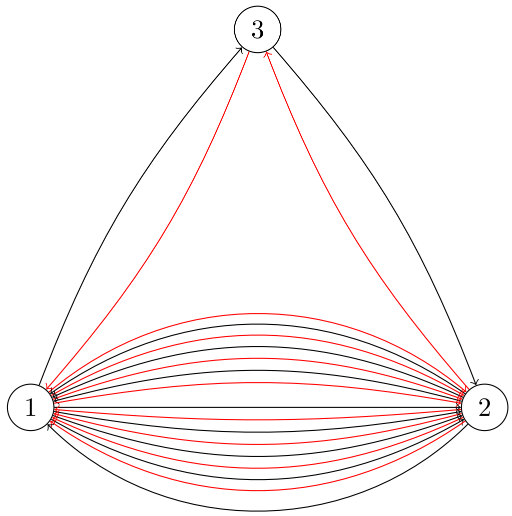
   that is the edge $[3, 1]$ is used an even number of times.

Then they prove that the contribution from the sets $\mathcal{W}'$ that are not real even walk, i.e $I \neq 1$, is negligible compared to the contribution of the sets $\mathcal{W}'$ with $I = 1$ using techniques they created in three previous papers: ([2], [3], [4]) starting from 1998. And while reading the first one ([2]) we found a false statement that might or might not affect the whole proof:

Claim (false): We say that a vertex u belongs to $\mathcal{N}_k$ if the number of times we arrived at u by marked steps equals to k. Denote $n_k := |\mathcal{N}_k|$. Then if the matrix entries satisfy the sub-Gaussian condition (1), then

$$\left|\mathbb{E}A[i_0, i_1]A[i_1, i_2]...A[i_{p-1}i_0]\right| \leq \prod_{k=1}^{s} (const \cdot k)^{k \cdot n_k}.$$

Counterexample: Consider a walk $\mathcal{W} = [1212121231212121213 21]$. (Marked steps are colored red.)



In this case:

$$LHS = \left|\mathbb{E}A[1,2]^{16}A[1,3]^2A[2,3]^2\right| \leq (const \cdot 8)^8 (const \cdot 1)^1 (const \cdot 1)^1,$$

using the subgaussian condition (1). And

$$RHS = (const \cdot 4)^{4 \cdot 1} (const \cdot 5)^{5 \cdot 1} (const \cdot 1)^{1 \cdot 1},$$

but $8^8 > 4^4 \cdot 5^5$, so adding enough edges $[1,2]$ and $[2,1]$ at the beginning and the end of the walk we can get rid of the effect of the constants. Thus the subgaussian condition is not sufficient.

References

[1] Sandrine Péché and Alexander Soshnikov. Wigner random matrices with non-symmetrically distributed entries. *Journal of Statistical Physics*, 129(5):857–884, 2007.

[2] Ya Sinai and Alexander Soshnikov. Central limit theorem for traces of large random symmetric matrices with independent matrix elements. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, 29(1):1–24, 1998.

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[4] Alexander Soshnikov. Universality at the edge of the spectrum in wigner random matrices. *Communications in mathematical physics*, 207(3):697–733, 1999.