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COURSE OF MATHEMATICS

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Abstract. This paper is devoted to problems of modern mathematical education. Fundamental problems of mathematics are considered in their unity.

The goal of this article is to present an outline of the theoretical base of Russian mathematical education at school, Higher College (in the Russian tradition, higher schools of this kind are called *institutes*), and University levels. Thus, the material is presented on three levels: (A) special mathematical school, (B) technical or economic institutes, (C) universities where natural science is studied.

The question of reasonable methods for mathematical education is discussed all over the world. In this article, the author presents his own point of view on the style and methodology of mathematical education.

Traditionally, the teaching of mathematics in Russia is divided into different subjects, or "courses": algebra, mathematical analysis, differential equations, etc. In this article, mathematics is considered as an organic whole in its unity with its applications.

The article is designed for a discussion of problems of mathematical education with a school teacher, a lecturer of an institute, and my university colleagues. I want to show to my readers that the mathematics that we are teaching at the corresponding levels is simple and essentially united. It is possible to formulate the main notions of mathematics in a universal manner, and the distance between school mathematics and the highest level of university education is amazingly small.

I. Notions

1. Real Numbers and Vector Spaces.

(A) *The real line* $\mathbb{R}$ *and Cartesian plane* $\mathbb{R}^2$. The real line is one of the most important mathematical notions. It is denoted as $\mathbb{R}$. It contains natural numbers $\mathbb{N}$, integer numbers $\mathbb{Z}$, and rational fractions $\mathbb{Q}$. The real line has a double description, namely an axiomatic one and by means of an arithmetical model. From an axiomatic point of view, $\mathbb{R}$ is a *complete ordered field*. This means that there are two operations in $\mathbb{R}$, which are called addition and multiplication. These operations satisfy commutative, associative, and distributive laws. The ordering of numbers also follows some natural laws. This means that $\mathbb{R}$ is an ordered field. The set of all fractions is also an ordered field. But there are some values that are impossible to describe by means of rational numbers. For example, the diagonal of the square with a side of length a is not a rational number. A real line is a completion of the set of rational numbers. There exist many equivalent axioms of completeness, which are due to Bolzano, Weierstrass, Dedekind, Cantor, and Cauchy. Now we will introduce only Cantor's axiom on nested segments, according to which *a sequence of nested segments whose lengths tend to zero has a unique common point.* Thus, it is possible to represent every real number uniquely as a decimal fraction $n.q_1q_2\ldots$, where n is an integer number, $q_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and all sequences in which a final part consist only of nines are excluded. This is *the arithmetical model of* $\mathbb{R}$.

The Cartesian plain $\mathbb{R}^2$ consists of column vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

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Real numbers x_1 and x_2 are called *coordinates of the vector* x. It is possible to add vectors and multiply vectors by real numbers coordinate-wise.

(B) The *finite-dimensional space* $\mathbb{R}^n$ is the nth degree of $\mathbb{R}$. It consists of column vectors

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

(sometimes we write $(x_1, \dots, x_n)^{\mathrm{T}}$, where T denotes transposition). The numbers $x_1, \dots, x_n$ are called *coordinates of the vector* x. Vectors can be added up and multiplied by real numbers coordinate-wise.

A set of elements $X = \{x\}$ is called a *vector space* if the operations of summation of vectors $x, x' \in X \to x + x'$ and multiplication of vectors by real numbers ($x \in X$, $\alpha \in \mathbb{R} \to \alpha x$) are defined. Thus, the natural rules similar to rules for numbers are satisfied.

(C) *A normed space* $(X, \|\cdot\|_X)$. A normed space is a vector space equipped with a *norm*, i.e., a function $\|\cdot\|_X \colon X \to \mathbb{R}$, with the following properties:

(a) $\|x\|_X \geq 0\ \forall\, x \in X$ and $\|x\|_X = 0 \implies x = 0$,
(b) $\|\alpha x\|_X = |\alpha|\, \|x\|_X\ \forall\, x \in X$, $\alpha \in \mathbb{R}$,
(c) $\|x + x'\|_X \leq \|x\|_X + \|x'\|_X\ \forall\, x, x' \in X$ (the triangle inequality).

In spaces $\mathbb{R}$, $\mathbb{R}^n$, $(X, \|\cdot\|_X)$, the distance $d_X(x, x')$ between elements x and x' can be introduced:

$$d_{\mathbb{R}}(x, x') = |x - x'|, \quad d_{\mathbb{R}^n}(x, x') = |x - x'| = \left(\sum_{i=1}^{n} (x_i - x_i') \right)^{1/2}, \quad d_X(x, x') = \|x - x'\|_X.$$

2. Order and General Topology. In these branches of mathematics, properties of ordering, continuity, and compactness are investigated.

First, we will use the following ordering definition.

A number a, which is denoted as $\sup A$ ($\inf A$), is called the *least upper* (*greatest lower*) *bound of a set* A if for all $\varepsilon > 0$ the intersection $A \cap [a - \varepsilon, a]$ ($A \cup [a, a + \varepsilon]$) is a nonempty set. The following proposition can be easily deduced from Cantor's axiom.

Weierstrass lemma on upper and lower bounds. *A set of numbers bounded from above* (*from below*) *has an upper* (*lower*) *bound.*

Let X be $\mathbb{R}$ in (A), be $\mathbb{R}^n$ in (B), or a normed space $(X, \|\cdot\|_X)$ in (C).

Here are the main definitions of general topology (which are formulated identically for (A), (B), and (C)).

Definition 1 (limit)**.** An element $\hat{x}$ is called the *limit of the sequence* $\{x_k\}_{k \in \mathbb{N}}$ ($x_k \in X$) if for all $\varepsilon > 0$ there exists a number N such that $d_X(x_k, \hat{x}) < \varepsilon$ for all $k > N$. Herewith we write $x_n \to \hat{x}$. A sequence that has a limit is called *convergent*.

Definition 2 (completeness)**.** A sequence $\{x_k\}_{k \in \mathbb{N}}$ is called *fundamental* in the space X if for all $\varepsilon > 0$ there exists a number N such that if $k_i > N$, $i = 1, 2$, then $d_X(x_{k_1}, x_{k_2}) < \varepsilon$. A space in which every fundamental sequence is convergent is called a *complete space*. Complete normed spaces are called *Banach spaces*.

The completeness of $\mathbb{R}$ can be deduced from Cantor's axiom, whence the spaces $\mathbb{R}^n$ are Banach spaces. Banach spaces are the main spaces in this article.

The set $U_X(\hat{x}, \varepsilon) = \{x \in X \mid d_X(x, \hat{x}) < \varepsilon\}$ ($B_X(\hat{x}, \varepsilon) = \{x \in X \mid d_X(x, \hat{x}) \leq \varepsilon\}$) is called the *open* (*closed*) *ball with the center* $\hat{x}$ *and radius* ε. A subset $V \subset X$ is called an *open set* if for any point $x \in V$ there exists a number $\varepsilon = \varepsilon(x)$ such that $U_X(x, \varepsilon) \subset V$. If an open set V contains a point $\hat{x}$, then this set is called a *neighborhood of* $\hat{x}$. The set of all neighborhoods of $\hat{x}$ will be denoted $\mathcal{O}(\hat{x}, X)$. A set is called *closed* if its complement is an open set.

Definition 3 (continuity)**.** Let X be $\mathbb{R}$ in (A), $\mathbb{R}^n$ in (B), or a normed space $(X, \|\cdot\|_X)$ in (C), let Y be $\mathbb{R}$ in (A), $\mathbb{R}^m$ in (B), or a normed space $(Y, \|\cdot\|_Y)$ in (C), let $V \in \mathcal{O}(\hat{x}, X)$, and let $F\colon V \to Y$ be a mapping from V to Y. The mapping F is called *continuous at* $\hat{x} \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(F(x), F(\hat{x})) < \varepsilon$ for all x such that $d_X(x, \hat{x}) < \delta$. This definition is equivalent to the definition of *sequential continuity or Heine continuity*, according to which a mapping F is *continuous at a point* $\hat{x}$ if for each sequence $\{x_k\}_{k\in\mathbb{N}}$ converging to $\hat{x}$ the sequence $\{F(x_k)\}_{k\in\mathbb{N}}$ converges to $F(\hat{x})$. If F is continuous at all points of a subset $T \subset X$, then it is called *continuous at* T and it is denoted as $F \in C(T, Y)$. A set $C \subset X$ is called *everywhere dense in* X if for all $x \in X$ and any $\varepsilon > 0$ the intersection $U_X(x, \varepsilon) \cap A$ is nonempty. A space (X, d_X) is called a *separable set* if it contains a countable everywhere dense subset.

Let X be $\mathbb{R}$ in (A), $\mathbb{R}^n$ in (B), or a normed space $(X, \|\cdot\|_X)$ in (C).

Definition 4 (compactness)**.** A set $C \subset X$ is called *compact* if from any sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in C$, it is possible to extract a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, $x_{n_k} \to \hat{x} \in C$.

3. Linearity. The class of linear mappings from one vector space to another plays a fundamental role in mathematics.

(A) Let $a \in \mathbb{R}$. The function $x \mapsto ax$, or $y = ax$, is called a *one-dimensional linear function.*

Let a_1 and a_2 be real numbers and

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (x \in \mathbb{R}^2).$$

The function $x \mapsto a_1x_1 + a_2x_2$, or $y = a_1x_1 + a_2x_2$, is called a *two-dimensional linear function.*

Let a_{11}, a_{12}, a_{21}, and a_{22} be four real numbers. The mapping

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} \in \mathbb{R}^2$$

is called a *linear mapping from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$.

This mapping can be written as $x \mapsto Ax$, where A is the table

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

(which is called a 2×2 *matrix*) and $x, y \in \mathbb{R}^2$.

(B) Let $a_1, \dots, a_n$ be n natural numbers and

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

be an n-dimensional vector. The function $x \mapsto a_1x_1 + \dots + a_nx_n$, or $y = a_1x_1 + \dots + a_nx_n$, is called a *linear function of* n *variables.* Let us assign to this function the n-tuple vector $a = (a_1, \dots, a_n)$. This vector defines a linear functional

$$x \mapsto \sum_{i=1}^{n} a_ix_i.$$

The set of all linear functionals on $\mathbb{R}^n$ is denoted $(\mathbb{R}^n)^*$. The expression $a_1x_1 + \dots + a_nx_n$ is written as $\langle a, x\rangle$.

Let a_{ij}, $1 \le i \le m$, $1 \le j \le n$, be mn real numbers. The mapping

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} a_{11}x_1 + \ldots + a_{1n}x_2 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{pmatrix} \in \mathbb{R}^m$$

is called a *linear mapping from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$. This mapping can be written as $x \mapsto Ax$, where A is the $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix},$$

which consists of m rows and n columns. It defines the linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(C) Let X and Y be two vector spaces.

Definition 5 (linear mapping)**.** A mapping $A\colon X \to Y$ is called a *linear mapping* if for x, x' from X and α, α' from $\mathbb{R}$ the following equality holds: $A(\alpha x + \alpha' x') = \alpha Ax + \alpha' Ax'$. Such linear mappings are also called *linear operators*. The set of linear continuous mappings from a Banach space $(X, \|\cdot\|_X)$ into $(Y, \|\cdot\|_Y)$ is denoted $\mathcal{L}(X, Y)$. The set of all *continuous linear functionals* (i.e., $\mathcal{L}(X, \mathbb{R})$) is called the *conjugate space* of X and is denoted X^*.

4. Mathematical Analysis. Differentiability. Mathematical analysis is a fundamental branch of mathematics. It has become a language of natural science. It consists of two chapters, which are called *differential calculus* and *integral calculus*.

The derivative and integral are the main notions of mathematical analysis. They were introduced by Newton and Leibniz. Let us quote the words of Newton in which he explains the crux of mathematical analysis: "Now it remains that, for an Illustration of the Analytic Art, I should give some Specimens of Problems ..." Here are these examples. "(1) Assume that the path length is known. Find the velocity at a given time, (2) Assume that the velocity is known. Find the length of the path described."

(A) Let us comment on Newton's words at the school level. Assume that by time t a carriage moving on a straight path covered a distance of length $s(t)$. Assume that this function, which associates distance covered $s(t)$ with any time t, is known. Analyzing the first question about the velocity, we naturally come to the idea that the velocity $v(\tau)$ of the carriage at some time instant τ (the *instantaneous velocity*) is approximately the *average velocity*

$$\frac{s(\tau + \Delta t) - s(\tau)}{\Delta t}$$

on a small time interval from τ to $\tau + \Delta t$. So velocity itself is the *limit of this ratio as* $\Delta t \to 0$. Here, the word "limit" means that the smaller the time increment Δt, the smaller the difference between the average velocity

$$\frac{s(\tau + \Delta t) - s(\tau)}{\Delta t}$$

and $v(\tau)$. In mathematical language, this is written as

$$v(\tau) = \lim_{\Delta t \to 0} \frac{s(\tau + \Delta t - s(\tau)}{\Delta t},$$

meaning that the velocity is the *derivative of the path with respect to time.* Newton used the overdot to denote this limit: $v(\tau) = \dot{s}(\tau)$. Thus, *the derivative in the sense of Newton is velocity.*

Whereas Newton's analysis proceeded from natural science, Leibniz took geometry as a basis to a large extent. When explaining Leibniz's concept, it seems natural to use a more standard notation for an argument and a function, where the argument is denoted by the letter x, and a function is denoted by f. In this notation, the derivative of a function f at a point $\hat{x}$ (written $f'(\hat{x})$) is the limit

$$\lim_{x \to \hat{x}} \frac{f(x) - f(\hat{x})}{x - \hat{x}}.$$

This definition is due to Cauchy. From the geometrical point of view, the derivative is the trigonometric tangent of the angle of inclination of the tangent line to the graph of the function f at a point $\hat{x}$. Leibniz understood the most important property of the derivative: it delivers *the best local linear approximation to a nonlinear function.* This property of the derivative admits both finite- and infinite-dimensional interpretations and is the basis for the differential calculus at both college and university levels. We give the definition of the derivative at these levels.

Let again X be $\mathbb{R}$ in (A), $\mathbb{R}^n$ in (B), and a normed space $(X, \|\cdot\|_X)$ in (C), let Y be $\mathbb{R}$ in (A), $\mathbb{R}^m$ in (B), and a normed space $(Y, \|\cdot\|_Y)$ in (C), and let V be a neighborhood of a point $\hat{x}$ in X.

Definition 6 (differentiability)**.** A mapping $F\colon V \to Y$ is called *differentiable at point* $\hat{x}$ if there exists a linear (continuous in the infinite-dimensional case) operator $A\colon X \to Y$ such that for all numbers $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\|x-\hat{x}\|_X < \delta$ implies $\|F(x)-F(\hat{x})-A(x-\hat{x})\|_Y < \varepsilon\|x-\hat{x}\|_X$.

It can be easily shown that the operator A is defined uniquely. It is called the *derivative of* F *at* $\hat{x}$ and is denoted $F'(\hat{x})$. If F is differentiable at $\hat{x}$, then we write $F \in D^1(\hat{x})$.

The linear mapping $x \mapsto F'(\hat{x})x$ is called the *differential of* F *at* $\hat{x}$.

(C) At the university level, we will give a definition of strict differentiability. This notion is frequently useful. A mapping F is called *strictly differentiable at a point* $\hat{x}$ if it is differentiable at $\hat{x}$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|F(x_1) - F(x_2) - F'(\hat{x})(x_1 - x_2)\|_Y < \varepsilon\|x_1 - x_2\|_X$ whenever $\|x_i - \hat{x}\|_X < \delta$, $i = 1, 2$. In this case, we write $F \in SD^1(\hat{x}, X)$.

5. Separability. Convex analysis is a branch of mathematics that studies convex sets, convex functions, and convex extremal problems. Convex analysis was born in the 20th century. The main notion of convex analysis is separability.

Let X be a vector space and X^* be the (nontrivial) conjugate space (nontriviality means that for all $x \in X$ there exists $x^* \in X^*$ such that $\langle x^*, x\rangle \neq 0$ and inversely for all $x^* \in X^*$ there exists $x \in X$ with analogous property). Of course, $(\mathbb{R}^n)^*$ is a dual space to $\mathbb{R}^n$, and every Banach space has a nontrivial conjugate. There is a wide class of vector spaces with nontrivial conjugates, which is called locally convex spaces. It is the real “areal” of convex analysis, but in considering convex problems we will confine ourselves to Banach spaces.

Definition 7 (separability)**.** Let X be a vector space, X^* be its conjugate, and A and B be subsets of X. It is said that the sets A and B are *separated* if there exists an element $x^* \in X^*$, $x^* \neq 0$, such that $\inf\limits_{x\in A}\langle x^*, x\rangle \geq \sup\limits_{\xi\in B}\langle x^*, \xi\rangle$. If $\inf\limits_{x\in A}\langle x^*, x\rangle > \sup\limits_{\xi\in B}\langle x^*, \xi\rangle$, then the sets A and B are called *strictly separated.*

6. Two Examples of Banach Spaces.

(1) The space $C([a,b])$ consists of functions continuous on $[a,b]$. It is a vector space with the norm $\|x(\cdot)\|_{C([a,b])} = \max\limits_{t\in[a,b]} |x(t)|$. It is a Banach space.

The space $C([a,b])$ is a *universal space* for all separable Banach spaces, because any separable Banach space can be isometrically included into $C([a,b])$.

(2) The space l_2 consists of all sequences $x = (x_1, x_2, \ldots)$ for which

$$\sum_{k\in\mathbb{N}} x_k^2 < \infty,$$

with the scalar product

$$\langle x, y\rangle = \sum_{k\in\mathbb{N}} x_k y_k.$$

The finite-dimensional space consisting of vectors $x = (x_1, \ldots, x_n)$ with the scalar product

$$\langle x, y\rangle = \sum_{k\in\mathbb{N}}^{n} x_k y_k$$

is denoted as l_2^n or $\mathbb{E}^n$. The space l_2 is a unique space, because any infinite-dimensional closed subspace of l_2 is isometrically isomorphic to l_2.

A vector space $(X, \langle\cdot,\cdot\rangle)$ in which the scalar product $\langle\cdot,\cdot\rangle\colon X\times X\to\mathbb{R}$ with the properties

$$\langle x,x\rangle \ge 0 \quad\text{and}\quad \langle x,x\rangle = 0 \iff x=0,$$
$$\langle x,y\rangle = \langle y,x\rangle, \quad \langle \alpha x+\beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$$

is defined is called *Euclidean space* in the finite-dimensional case, and it is called *Hilbert space* in the infinite-dimensional case (if it is a complete space). A linear operator from l_2 to itself is called a *compact operator* if it transforms closed balls into compact sets.

7. Integrability. Consider the second question of Newton. Let a carriage move on a straight path and assume that we know the function that associates with any time t the velocity $v(t)$ of the carriage. Now we search for the length $s(t)$ of the path that the carriage has followed during the time t. It is natural to assume that on a short segment $\Delta = [\tau, \tau+\Delta\tau]$ the velocity undergoes a small change, so the way is approximately equal to $|v(\theta)\Delta\tau|$, where θ is a point of the segment $[\tau,\tau+\Delta\tau]$.

Let f be a bounded function on a finite interval $[a,b]$ and let D ("D" comes from the "division") be a division of $[a,b]$ by points x_j: $a = x_0 \le x_1 \le \dots \le x_N = b$. Next,

$$M(D) = \max\{x_j - x_{j-1},\ 1 \le j \le N\}$$

is the "diameter" of the division D, and S ("S" comes from the "selection") is a set of points ξ_j, $\xi_j \in \Delta_j = [x_j, x_{j-1}]$, $1\le j\le N$. The expression

$$R(f,D,S) = \sum_{j=1}^{N} f(\xi_j)(x_j - x_{j-1})$$

is called the *Riemann sum of the function f with respect to the division D and the selection set S.*

A number I (if it exists) is called the *definite Riemann integral of f on* $[a,b]$, if for all $\delta>0$ there exists a number $\varepsilon>0$, such that for all D with $M(D)<\varepsilon$ for all S the following inequality holds:

$$|R(f,D,S) - I| < \varepsilon.$$

This integral is denoted

$$\int_a^b f(x)\,dx.$$

The sum

$$\overline{R}(f,D) = \sum_{j=1}^{n} \sup_{x\in\Delta_j} f(x)|\Delta_j|$$

is called the *upper Darboux sum*, and the sum

$$\underline{R}(f,D) = \sum_{j=1}^{n} \inf_{x\in\Delta_j} f(x)|\Delta_j|$$

is called the *lower Darboux sum of f with respect to division D* (*when S is given*). The inequalities

$$\underline{R}(f,D) \le R(f,D,S) \le \overline{R}(f,D) \quad \text{for all} \quad S$$

are evident.

Lemma I.1. *Let*

$$D = \bigcup_{i=1}^{N} \Delta_i \quad \textit{and} \quad D' = \bigcup_{j=1}^{N'} \Delta'_j$$

be two divisions of $[a,b]$. *Then* $\underline{R}(f,D) \le \overline{R}(f,D')$.

Proof. Denote by Δ_{ij} the intersection of segments Δ_i and Δ'_j,

$$m_{ij} = \min_{x\in\Delta_{ij}} f(x), \quad M_{ij} = \max_{x\in\Delta_{ij}} f(x).$$

Let m'_{ij} and M'_{ij} be analogous values for the division D'. We have

$$\underline{R}(f,D) \overset{\text{def}}{=} \sum_{i=1}^{N} m_i|\Delta_i| \overset{\text{def}}{=} \sum_{i=1}^{N}\sum_{j=1}^{N'} m_i|\Delta_{ij}|$$
$$\overset{m_i\le m_{ij}}{\le} \sum_{i=1}^{N}\sum_{j=1}^{N'} m_{ij}|\Delta_{ij}| \overset{m_{ij}\le M_{ij}}{\le} \sum_{i=1}^{N}\sum_{j=1}^{N'} M_{ij}|\Delta_{ij}| \overset{M_{ij}\le M'_i}{\le} \sum_{j=1}^{N'} M'_j|\Delta'_j| \overset{\text{def}}{=} \overline{R}(f,D'). \quad \square$$

Existence of definite integral. *There exists a definite integral of a continuous function on a segment.*

Proof. In this proof, we will use the Weierstrass and Cantor theorems. Their proofs can be found at the beginning of Sec. II. It follows from the Weierstrass theorem that upper and lower Darboux sums are bounded:

$$\overline{R}(f,D) \ge \Big(\min_{x\in[a,b]} f(x)\Big)(b-a), \quad \underline{R}(f,D) \le \Big(\max_{x\in[a,b]} f(x)\Big)(b-a).$$

Thus, they have lower $\underline{I} = \inf_D \underline{R}(f,D)$ and upper $\overline{I} = \sup_D \bar{R}(f,d)$ bounds. The lemma implies that $\underline{I} \le \overline{I}$, and from the Cantor theorem it follows that $I = \underline{I} = \overline{I}$. This number is the definite integral we are looking for. $\square$

So the geometric sense of definite integral $\int\limits_a^b f(x)\,dx$ of a positive function f is the area under the graph of f.

II. Theory

Preliminaries

Proposition II.1. *Any sequence $\{x_k\}_{k\in\mathbb{N}}$ of points of a closed interval (in (A)) or of points of a closed bounded set (in (B)) contains a convergent subsequence. Thus, at the college level this means that an interval on the real line is a compact set, and at the institute level this means that bounded and closed subsets of $\mathbb{R}^n$ are compacts.*

Proof.

(A) The proof is based on the *bisection method.* Taking any element of the sequence as the initial one, we split the interval into two equal parts and take the one that contains an infinite number of elements of the sequence. In turn, in this part we choose as a second one an element with number exceeding the number of the first one, and then repeat the entire process. The required result now follows from Cantor's axiom.

(B) A multivariate analogue of this proposition is based on the fact that a bounded closed subset of $\mathbb{R}^n$ is compact. To prove this fact, we place a set in an n-dimensional cube and then partition this cube "into 2^n equal cubes." $\square$

Weierstrass theorem on the maximum and minimum of a continuous function. *A continuous function on a closed interval (in (A)), on a closed bounded subset of $\mathbb{R}^n$ (in (B)), and on a compact set (in (C)) attains its maximum and minimum values.*

The proof of the Weierstrass theorem may be carried out simultaneously at the school, institute, and university level. Let f be a real continuous function on a closed interval (on a closed bounded subset of $\mathbb{R}^n$ or on a compact set) C. If we assume that $\sup_{x\in C} f(x) = +\infty$, then, by definition, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ of elements from X such that $f(x_k) \to +\infty$. Because of Proposition II.1, there exists a point $\hat{x}$ that is the limit of some subsequence of the given sequence. But this contradicts the definition of continuity of the function f at the point $\hat{x}$ (because f must be bounded in some neighborhood of $\hat{x}$). The case $\inf_{x\in C} f(x) = -\infty$ is treated similarly.

Let us prove that the maximum value is attained. We set $M = \sup_{x\in C} f(x)$ (this follows from the Weierstrass lemma). If we assume that $f(x) \neq M$ for all x, then by the definition of the strict upper bound there exists a sequence $\{\xi_k \in X\}_{k\in\mathbb{N}}$ such that $f(\xi_k) \to M$. By Proposition II.1 there exists a point $\hat{\xi}$ that is the limit of some subsequence of the given sequence. By the definition of continuity of f at $\hat{\xi}$ we have $f(\hat{\xi}) = M$. The case of the minimum is treated similarly. □

Theorem on uniform continuity. *A continuous function on a closed interval* (*in* (A)), *on a closed bounded subset of* $\mathbb{R}^n$ (*in* (B)), *and on a compact set in a normed space* (*in* (C)) *is uniformly continuous.*

In our literature, this result is called the Cantor theorem; Bourbaki calls this result the Heine theorem.

Proof. Assume that for some $\varepsilon > 0$ there exist two sequences $\{x_j\}_{j\in\mathbb{N}}$ and $\{\xi_j\}_{j\in\mathbb{N}}$ of points of the compact set such that $|x_j - \xi_j| \to 0$ as $j \to \infty$ in the case of $\mathbb{R}$ (or $\mathbb{R}^n$) or $\|x_j - \xi_j\|_X \to 0$ as $j \to \infty$ in the normed case, whereas

$$\text{the distance between the points } f(x_j) \text{ and } f(\xi_j) \text{ is larger than } \varepsilon. \tag{$*$}$$

Since X is a closed interval (respectively, a closed bounded subset of $\mathbb{R}^n$ or a compact set) it follows that there exists a point $\hat{x} \in X$ such that $x_{k_j} \to \hat{x}$ as $j \to \infty$. Consequently, $\xi_{k_j} \to \hat{x}$ as $j \to \infty$ and now $(*)$ contradicts the definition of continuity of the function f at $\hat{x}$. □

The first separation theorem. *Let a space X be* $\mathbb{R}^2$ (*in* (A)), $\mathbb{R}^n$ (*in* (B)), *a Banach space* $(X, \|\cdot\|_X)$ (*or a locally convex space*) (*in* (C)), *and A and B be convex subsets of X such that the interior of A is nonempty and does not intersect B. Then A and B are separated.*

Proof. Consider the difference $C = A - B = \{x = a - b \mid a \in A,\ b \in B\}$. This consideration reduces the theorem to the case of separation of the zero point and a set C with nonempty interior. We give a sketch of proof of the reduced variant of the theorem. Further we suppose that zero is contained in the closure of C. In this case, a hyperplane that separates C and zero is called a *support hyperplane*.

Let in the case (A) the set C be a subset of $\mathbb{R}^2$ such that $\operatorname{int} C \cap \{0\} = \varnothing$. Then the closure of C possesses the same property. So it is possible to presuppose that C is a closed set. Let c be an inner point of C. Consequently, C contains a circle $U_{\mathbb{R}^2}(c, \varepsilon) = \{x \in c \mid |x - c| < \varepsilon\}$. Let us draw a straight line l through zero and c. This line is divided into three parts: the open ray l_+ that contains c, the opposite open ray l_-, which contains no points of C (this assertion is evident), and the zero point. Any point $\xi \in l_-$ can be separated from C. In fact, from the Weierstrass theorem it follows that in a closed set C there exists a point η nearest to ξ. Then the perpendicular p_η to the segment $[\xi, \eta]$ that contains η separates C from ξ. Thus, we can take a sequence $\{\eta_n \in l_-\}$, $\eta_n \to 0$ and construct the corresponding perpendiculars p_{η_n}. The limit line separates zero and C.

In the case (B), one uses induction. Let the theorem be proved for m-dimensional convex sets. We must prove it for $(m+1)$-dimensional sets. To do this, we draw the straight line l through zero and $c \in \operatorname{int} C$ and then include this line in the m-dimensional subspace L_m.

According to the inductive hypothesis, there exists an $(m-1)$-dimensional plane $\hat{L}_{m-1}$ that is a support to the section $C \cap L_m$. The factor space $\mathbb{R}^{m+1}/\hat{L}_{m-1}$ is a plane in which the image of C is a convex set with nonempty interior. The existence of a support line $\hat{l}$ for this plane set follows from point (A). Then the sum $\hat{L}_{m-1} + \hat{l}$ is the support plane we looked for.

In the case (C), it is necessary to use Zorn's lemma. Let there be given a family of subspaces that do not intersect the interior of a convex set and are included in each other. The union of these spaces is a majorant of this family. According to Zorn's lemma there is a maximal subspace with this property. The closure of this majorant does not intersect the interior of the set. If the span of this space and the internal point c is not the whole space X, then by means of standard arguments we come to a contradiction with the maximality of the majorant. □

Corollary (the second separability theorem)**.** *A nonempty convex closed set A and a point $a \notin A$ are strictly separated.*

Proof. It suffices to take a convex neighborhood V of the point a such that $V \cap A = \varnothing$ and to separate it from A. □

The Main Part (Theorems)

At all times, solving linear equations was a subject of primary mathematical education. The theory of linear equations is taught at introductory courses in mathematics at colleges and universities. Solving systems of linear equations underlies a huge number of applications of modern mathematics, and moreover, the differential calculus is based on linear approximations. This motivates this topic in mathematical education. The presentation now follows the path from the origins to the beginning of the 20th century.

1. The Theory of Linear Equations.

Theorem II.1 (Fredholm)**.** *Let X be $\mathbb{R}^2$ (in (A)), X be $\mathbb{R}^n$ (in (B)), $(X, \|\cdot\|_X)$ be a Banach space (in (C)), and $A\colon X \to X$ be a linear operator (in the Banach infinite-dimensional case it is a linear continuous operator that is the sum of unity and compact operators).*[1] *Then the system $Ax = y$ is either solvable for any right-hand side or the homogeneous system has a nonzero solution (the Fredholm alternative).*

In the infinite-dimensional case, we prove the theorem for $X = l_2$ and $A = I + B$, where I is the identity operator and the operator B is represented by a matrix with the following property: $\sum\limits_{i,j\in\mathbb{N}} b_{ij}^2 < \infty$.

The proof consists of two parts.

(1) Let us prove that if $\operatorname{Im} A = X$, then $\operatorname{Ker} A = 0$.

We begin with (A). Indeed, assume that $\operatorname{Im} A = \mathbb{R}^2$, but there exists a vector $y^1 \neq 0$ such that $Ay^1 = 0$. Then we find a vector y^2 such that $Ay^2 = y^1$ and a vector y^3 such that $Ay^3 = y^2$. Any three vectors in $\mathbb{R}^2$ are linearly dependent, and hence there exist numbers a_1, a_2, and a_3, not all zero, such that $a_1y^1 + a_2y^2 + a_3y^3 = 0$ (this fact is easily reduced to the result that any two homogeneous linear equations with three unknowns have a nonzero solution). Acting by the operator A^2 on this equality and using the fact that this operator is linear, we obtain that $a_3y^1 = 0$; i.e., $a_3 = 0$ (because $y^1 \neq 0$). In a similar way we show that $a_2 = a_1 = 0$, which contradicts the fact that not all a_i are zero.

(B) The proof in the n-dimensional case is in complete analogy with the case (A). We construct $y^1 \neq 0$ such that $Ay^1 = 0$ and then $n+1$ vectors $Ay^{i+1} = y^i$, $1, \dots, n$. They are linearly dependent, whence there exist numbers $\{\alpha_k\}_{k=1}^{n+1}$, not all zero, such that $\sum\limits_{k=1}^{n+1} \alpha_k y^k = 0$. Acting on this equality by the operator A^n, we see that $\alpha_{n+1}y^1 = 0$, where $\alpha_{n+1} = 0$. One similarly shows that $\alpha_n = \alpha_{n-1} = \dots = \alpha_1 = 0$. We have a contradiction.

(C) The proof of the theorem at the university level is based on excursions into infinite-dimensional space.

Lemma II.1 (on a sequence of unit vectors)**.** *There is no sequence of unit vectors $\{f^i\}_{i\in\mathbb{N}}$ such that $\|Bf^i - Bf^j\| > \alpha > 0$, $i \neq j$.*

This is our first excursion into infinite-dimensional space. The proof of this lemma can be found in the Appendix.

Using Lemma II.1, we proceed to prove the first assertion of the theorem. Assume that $\operatorname{Im} A = l_2$, but there exists a vector $e^1 \neq 0$ such that $Ae^1 = 0$ (we may assume that $\|e^1\|_{l_2} = 1$). We set $L_k = \operatorname{Ker} A^k$. We already know that L_1 is nontrivial. Solving the equation $Ae^2 = e^1$ and, next, the equation

[1] The reason for this condition will be revealed later.

$Ae^{k+1} = e^k$ shows that all L_k are nontrivial. Moreover, it follows from Lemma II.1 that all L_k are finite-dimensional (otherwise we will construct in L_k an orthonormal system of any number of vectors g^k such that $\|Bg^i - Bg^j\|_X = \|g^i - g^j\|_X = \sqrt{2}$). Consider the sequence Be^n. Let $m > n$. It is easily checked that $z_n = e^n - Ae^n + Ae^m \in \operatorname{Ker} A^{m-1}$. Hence, $\|Be^m - Be^n\|_X = \|z_n - e_m\|_X \geq 1/2$. But this is a contradiction with the fact that the operator B is compact. Hence, $\operatorname{Ker} A = 0$.

(2) Let us prove that if $AX \neq X$, then $\operatorname{Ker} A \neq 0$ (see Fig. 1).

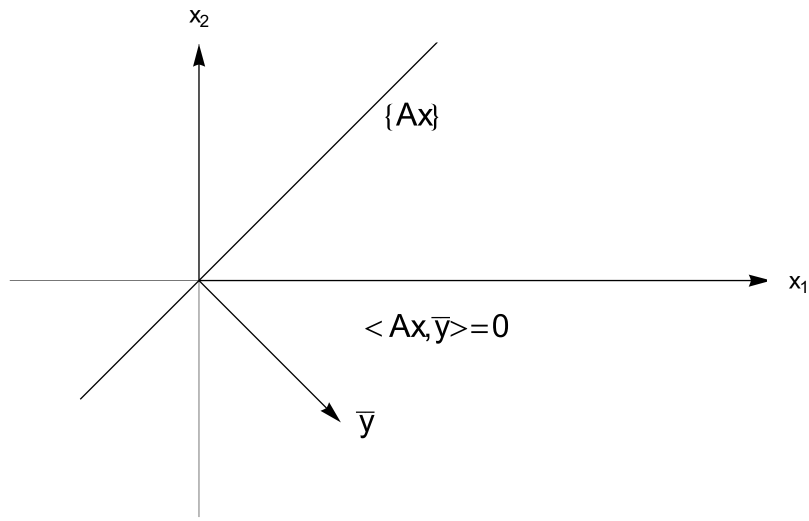

Fig. 1

All over again, we will begin from (A). If $\operatorname{Im} A \neq \mathbb{R}^2$, then $\operatorname{Im} A$ is either the origin or a line passing through the origin. In the first case, the required assertion holds, because any vector lies in the kernel, and in the second case, one may find an orthogonal vector to the line $\operatorname{Im} A$; i.e., $0 = \langle \bar{y}, Ax \rangle = \langle A^{\mathrm{T}}\bar{y}, x \rangle$. Hence, $\langle A^{\mathrm{T}}\bar{y}, x \rangle = 0$ for all $x \in \mathbb{R}^2$, and so, $A^{\mathrm{T}}\bar{y} = 0$; i.e., $\bar{y}$ is a solution of the homogeneous equation for the transposed matrix. Thus, the operator A^{T} is not surjective. Applying to A^{T} the result just proved, it follows that the homogeneous equation $A^{\mathrm{TT}} = A$ has a nonzero solution.

In the case (B), the proof is analogous. If $\operatorname{Im} A \neq \mathbb{R}^n$, then this image is a proper subspace of $\mathbb{R}^n$. A proper subspace is given by a system of linear homogeneous equations. Let $\sum\limits_{k=1}^{n} \bar{y}_k x_k = 0$ be one of these equations, i.e., $0 = \langle \bar{y}, Ax \rangle = \langle A^{\mathrm{T}}\bar{y}, x \rangle$. We have $\langle A^{\mathrm{T}}\bar{y}, x \rangle = 0$ for all $x \in \mathbb{R}^n$, and hence, $A^{\mathrm{T}}\bar{y} = 0$; i.e., $\bar{y}$ is a solution of the homogeneous equation with transposed matrix. By the first assertion of the Fredholm alternative, the operator A^{T} is not surjective. Applying the argument just used to this operator, it follows that the homogeneous equation $A^{\mathrm{TT}} = A$ has a nonzero solution. This proves the Fredholm alternative.

(C) In the infinite-dimensional case, the proof is also analogous, but we need two excursions into infinite-dimensional space.

Lemma II.2 (on the closeness)**.** *The image of the sum of the identity and a compact operator is closed.*

Lemma II.3 (on the perpendicular)**.** *Let X be a Hilbert space and let L be a proper closed subspace of X. Then there exists an element $y \neq 0$ perpendicular to L.*

These two technical lemmas of general nature will be proved in the Appendix.

Now to prove the second assertion of the theorem we must repeat the same arguments as in the finite-dimensional setting. □

2. Solvability of Nonlinear Equations. Nonlinear equations appear in almost all mathematical models describing various phenomena and processes: in natural and engineering sciences, in control theory, economics, etc. Consequently, finding approximate solutions to such equations is one of the fundamental problems in mathematics and its applications. We start from the solvability of linear equations.

Lemma on right-solvability in linear case. *Let X and Y be either $\mathbb{R}$ and $\mathbb{R}$ (in (A)), or $\mathbb{R}^n$ and $\mathbb{R}^m$ (in (B)), or Banach spaces (in (C)) and A be a linear (continuous in the Banach case) surjective operator from X to Y. Then there exist a right-inverse mapping $R\colon Y \to X$ and a number $\gamma > 0$ such that $AR(y) = y$ and $\|Ry\|_X \leq \gamma \|y\|_X$ for all $y \in Y$.*

Proof. In the case (A), "operator A" is nothing else but a real number $A \neq 0$ and $R = A^{-1}$. In the case (B), consider the system of basis vectors $\{e_j\}_{j=1}^m$ in $\mathbb{R}^m$. By the hypothesis, there exist vectors $\{f_k\}_{k=1}^m$, $f_k \in \mathbb{R}^n$, $1 \leq k \leq m$, such that $Af_k = e_k$. Given any vector

$$y = \sum_{k=1}^m y_k e_k \in \mathbb{R}^m,$$

we set

$$R(y) = \sum_{k=1}^m y_k f_k.$$

Now the equality $AR(y) = y$ follows from the linearity of A, and the second inequality follows from the following obvious inequalities (and from the fact that the length of a vector is not smaller than any of its coordinates):

$$|Ry| \leq \sum_{i=1}^m |y_k|\,|f_k| \leq \max_{1\leq k\leq m} |y_k| \sum_{k=1}^m |f_k| \leq \gamma |y|,$$

where $\gamma = \sum\limits_{k=1}^m |f_k|$.

In the case (C), using the Banach inverse mapping theorem (see, for example, the textbook of Kolmogorov and Fomin; in the Appendix this lemma will be proved directly), one finds a number $\delta > 0$ such that $\Lambda U_X(0,1) \supset U_Y(0,\delta)$. This means that for all $y \in U_Y(0,\delta)$ there exists $x(y) \in U_X(0,1)$ such that $\Lambda x(y) = y$ and we must put

$$R(y) = \frac{2\|y\|_Y}{\delta} x\left(\frac{\delta y}{2\|y\|_Y}\right)$$

and check fulfillment of conditions of the theorem. □

The proof of the next main theorem of this section is the same in the one-dimensional, finite-dimensional, and infinite-dimensional cases. So we formulate this result analogously to the lemma at once for all cases.

Theorem II.2 (on right-inverse mapping). *Let either X be $\mathbb{R}$ with $\|x\|_X = |x|$ and Y be $\mathbb{R}$ with $\|y\|_Y = |y|$ (in (A)), or X be $\mathbb{R}^n$ with*

$$\|x\|_X = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$$

and Y be $\mathbb{R}^m$ with

$$\|y\|_Y = \left(\sum_{i=1}^m y_i^2\right)^{1/2}$$

(in (B)), or $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces (in (C)), V be a neighborhood of a point $\hat{x} \in X$, and $F\colon V \to Y$ be a (nonlinear) mapping. To solve the equation $F(x) = y$, $y \in Y$, we apply the modified Newton method:

$$x_k = x_{k-1} + R\big(y - F(x_{k-1})\big), \quad k \in \mathbb{N}, \tag{II.1}$$

where R is the right inverse of the linear surjective operator $A\colon X \to Y$ which is such that, for $0 < \vartheta < 1$ and $\delta > 0$,

$$\|F(x) - F(x') - A(x - x')\|_Y \le \frac{\vartheta}{\gamma}\|x - x'\|_X, \tag{II.2}$$

for any x and x' such that $\|x - \hat{x}\|_X < \delta$, $\|x' - \hat{x}\|_X < \delta$. Then, for any vector $y \in Y$ lying at the distance at most $\delta(1 - \vartheta)/|A|$ from $F(\hat{x})$, the sequence (II.1) converges to the vector $\varphi(y)$ lying at a distance at most δ from $\hat{x}$. Moreover, $F\bigl(\varphi(y)\bigr) = y$ and $\|\varphi(y) - \hat{x}\|_X \le K\|y - F(\hat{x})\|_Y$, where $K = \gamma/(1 - \vartheta)$.

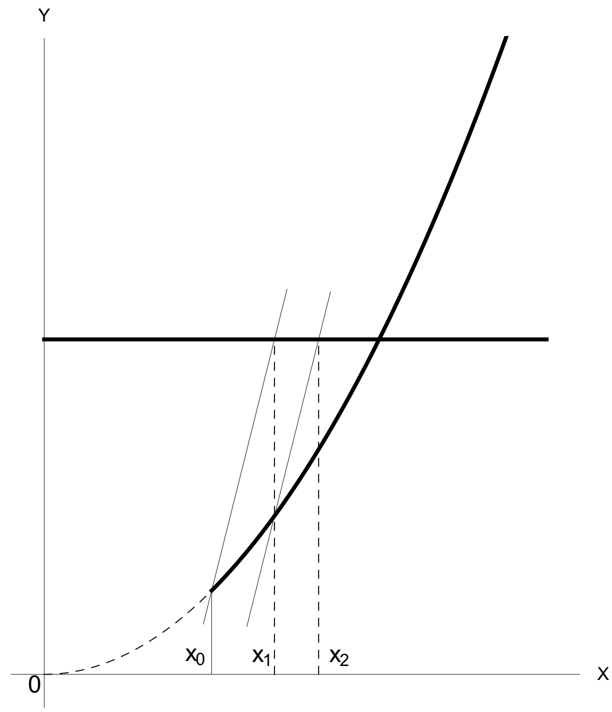

Fig. 2

The proof is illustrated by Fig. 2. We claim that

(a) all x_k for $k \ge 0$ lie in $U_X(x_0, \delta)$;

(b) the sequence $\{x_k\}_{k \ge 0}$ is fundamental.

We prove (a) by induction. By definition, x_0 lies in $U_X(x_0, \delta)$. Assuming that $x_s \in U_X(x_0, \delta)$ for $1 \le s \le k$, we claim that $x_{k+1} \in U_X(x_0, \delta)$. Using, in sequence, (II.2) and the equality

$$A(x_s - x_{s-1}) - y + F(x_{s-1}) = 0,$$

which follows from (II.2), applying (II.1), and then iterating, we have

$$\begin{aligned}\|x_{k+1} - x_k\|_X &= \|R(y - F(x_{k-1}))\|_X \le \gamma\|y - F(x_k)\|_Y \\ &= \gamma\|y - F(x_k) - y + F(x_{k-1}) + A(x_k - x_{k-1})\|_Y \le \theta\|x_k - x_{k-1}\|_X \le \dots \le \theta^n\|x_1 - x_0\|_X.\end{aligned} \tag{II.3}$$

Now, applying the inequality for absolute values and the formula for geometric progression, and (II.3) with $k = 1$ and choosing y, we have

$$\|x_{k+1} - \hat{x}\|_X \le \|x_{k+1} - x_k\|_X + \dots + \|x_1 - \hat{x}\|_X \le (\theta^k + \theta^{k-1} + \dots + 1)\|x_1 - \hat{x}\|_X < \frac{1}{(1 - \theta)}\|y - F(\hat{x})\|_Y \le \delta;$$

i.e., the elements x_k are defined for all k.

Let us prove (b). For any $k, l \in \mathbb{N}$,

$$\|x_{k+l} - x_k\|_X \le \|x_{k+l} - x_{k+l-1}| + \cdots + |x_{k+1} - x_k\|_X$$
$$\le \left(\sum_{i=k}^{k+l} \theta^i\right) \|x_1 - \hat{x}\|_X < \frac{\theta^k}{1-\theta} \|y - F(\hat{x})\|_Y \le \delta\theta^k, \quad \text{(II.4)}$$

whence $\{x_k\}_{n\in\mathbb{N}}$ is a fundamental sequence. Hence, $\{x_k\}_{n\in\mathbb{N}}$ is convergent. Let $\varphi(y)$ be its limit. Letting $k \to \infty$ in (II.2), we obtain that $F\big(\varphi(y)\big) = y$, and letting $k \to \infty$ in (II.4) and taking into account that $\|x_1 - \hat{x}\|_X \le \gamma \|y - F(\hat{x})\|_Y$, we establish the inequality $\|\varphi(y) - \hat{x}\|_X \le K\|y - F(\hat{x})\|_Y$ with $K = \gamma/(1-\theta)$. □

3. Elements of the Theory of Extremum Problems. The introduction of the theory of extremum in education is motivated by the fact that the calculus of variations provides a basis for mathematical natural science, the optimal control theory underlies the theory of control in dynamical systems, and convex problems are fundamental for mathematical economics.

Theorem II.3 (Fermat theorem and Lagrange multiplier rule)**.** *Let X and Y be the same as in Theorem* II.2*, V be a neighborhood of a point $\hat{x} \in X$, $f_0\colon V \to \mathbb{R}$, $F\colon V \to Y$, f_0 and F be strictly differentiable at $\hat{x}$,* $\operatorname{Im} F'(\hat{x})$ *be a closed subspace, and $\hat{x}$ be a local extremum of the problem*

$$f_0(x) \to \min, \quad F(x) = 0. \tag{P}$$

Then

(a) *if $F = 0$ (i.e., in the problem without constraint) the stationary condition $f_0'(\hat{x}) = 0$ holds;*
(b) *in the general case, there exists a nonzero vector of Lagrange multipliers $\bar{\lambda} = (1, \lambda) \in \mathbb{R} \times Y^*$ such that the point $\hat{x}$ is a stationary point of the Lagrange function $\mathcal{L}(x, \bar{\lambda}) = f_0(x) + \langle \lambda, F(x)\rangle$ (i.e., $\mathcal{L}_x(\hat{x}, \bar{\lambda}) = 0$) (Lyusternik's theorem on Lagrange multipliers, 1934).*

Proof. The statement (a) follows from the definition of differentiability.

The proof of the statement (b).

Lemma on nontriviality of annihilator. *Let X be $\mathbb{R}^2$ (in* (A)*), X be $\mathbb{R}^n$ (in* (B)*), X be a normed space (in* (C)*) and L be a proper closed subspace. Then the annihilator $L^\perp = \{x^* \in X^* \mid \langle x^*, x\rangle = 0$ for all $x \in L\}$ contains a nonzero element.*

Proof. One takes an element out of L and strictly separates it from L applying the second separation theorem. □

Lemma on the annihilator of the kernel of a surjective operator. *Let X and Y be Banach spaces, $\Lambda\colon X \to Y$ be a linear continuous surjective operator, and Λ^* be its conjugate. Then* $(\operatorname{Ker}\Lambda)^\perp = \operatorname{Im}\Lambda^*$.

Proof. Let $x^* \in (\operatorname{Ker}\Lambda)^\perp$. At the beginning, we show that the space MX, where $Mx = (\langle x^*, x\rangle, Ax)$, is a closed subspace in $\mathbb{R} \times Y$. Indeed, if $(\alpha, y) \in \operatorname{cl} MX$, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\langle x^*, x_n\rangle \to \alpha$, and $Ax_n \to y$. Let us put $h_n = R(Ax_n - y)$, where R is a right-inverse operator for Λ. From the properties of the operator R it follows that $h_n \to 0$; consequently, $\langle x^*, h_n\rangle \to 0$. If we put $\xi_n = x_n - h_n$, one can see that $M(\xi_n) \to (\alpha, y)$, i.e., $(\alpha, y) \in \operatorname{cl} MX$. After application of the second separation theorem that separates the point $(1, 0) \notin MX$ and the closed set MX, we obtain the result of the lemma. □

Now it is possible to prove the Lagrange multiplier rule. If $F'(\hat{x})X$ is a proper subspace, then one can apply the lemma on nontriviality of the annihilator. If $F'(\hat{x})X = Y$, then we consider the mapping $\mathcal{F} = (f_0, F)\colon V \to \mathbb{R} \times Y$. If $\mathcal{F}'(\hat{x})X = \mathbb{R} \times Y$, then we should apply the right-inverse mapping theorem. It shows that $\hat{x}$ is not a local minimum. Thus, $\operatorname{Im}\mathcal{F}'$ is a proper subspace of $U_X(x_0, \delta)$, so application of the lemma on the annihilator leads to the theorem. □

4. Quadrics. We study the solution of quadratic equations at school. The reduction of quadratic forms to canonical form is an essential part of college and university educational programs. The Hilbert–Schmidt theorem is the main result of this section. This theorem plays a fundamental role not only in mathematics, but also in mathematical physics and quantum mechanics.

A quadric is a level set of a quadratic form $Q(x) = \langle Ax, x\rangle$, where A is a symmetric operator in Euclidean or Hilbert space.

Theorem II.4 (Hilbert–Schmidt theorem on diagonalization of symmetric operators)**.** *Let X be the Euclidean plane* (*in* (A))*, X be n-dimensional Euclidean space* (*in* (B))*, $(X, \langle\cdot,\cdot\rangle)$ be Hilbert space* (*in* (C))*, and $A\colon X \to X$ be a linear symmetric operator* (*in Hilbert space, a linear symmetric and compact operator*)*. Then there exist $m \in \mathbb{N} \cup \{+\infty\}$, mutually orthogonal unit vectors $\{f^i\}_{1\le i\le m}$, and m nonzero numbers λ_i for which the quadratic form $Q(x) = \langle Ax, x\rangle$ assumes the form*

$$Q(x) = \sum_{i=1}^{m} \lambda_i \langle f^i, x\rangle^2.$$

If $M = \infty$, then $|\lambda_i| \downarrow 0$.

Proof. In the case (A) assume that the form $Q(x) = \langle Ax, x\rangle$, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

adopts positive values (if not, then we consider the minimization problem).

Consider the maximization problem with equality constraints:

$$f_0(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \to \max, \quad f_1(x) = x_1^2 + x_2^2 = 1,$$

or, in abbreviated notation,

$$\langle Ax, x\rangle \to \max, \quad \langle x, x\rangle = 1.$$

The unit circle $\mathbb{S}^1 = \{x \mid x_1^2 + x_2^2 = 1\}$ is a compact subset of $\mathbb{E}^2$, the function f_0 is continuous on $\mathbb{E}^2$, and hence, by the Weierstrass theorem on continuous functions on a compact set, this problem has a solution (which we denote f^1). From the Lagrange multiplier rule it follows that there exists a number λ_1 such that $Af^1 = \lambda_1 f^1$. (Such vectors are called *eigenvectors*.) Let f^2 be a unit vector which is orthogonal to f^1. We have $\langle f^1, Af^2\rangle = \langle Af^1, f^2\rangle = \lambda_1\langle f^1, f^2\rangle = 0$. So, the vector Af^2 is orthogonal to both f^1 and f^2, and hence, they are proportional: $Af^2 = \lambda_2 f^2$. We expand the vector x into the mutually orthogonal unit vectors f^1 and f^2: $x = \langle f^1, x\rangle f^1 + \langle f^2, x\rangle f^2$. As a result, $\langle Ax, x\rangle = \lambda_1\langle f^1, x\rangle^2 + \lambda_2\langle f^2, x\rangle^2$.

Cases (B) and (C) are considered analogously with a small difference. In the finite-dimensional case at the beginning one considers the problem $\langle Ax, x\rangle \to \max$, $\langle x, x\rangle = 1$. In the infinite-dimensional case it is necessary to consider the inequality restriction: $\langle x, x\rangle \le 1$. The fact is that in $\mathbb{R}^n$ the sphere $\mathbb{S}^{n-1}$ is a compact, but in l_2 the sphere $\{x \mid \langle x, x\rangle = 1\}$ is not a compact set. On the other hand, the ball $\langle x, x\rangle \le 1$ is a compact in some weak sense. So the Weierstrass theorem leads to the existence of the first eigenpair f_1, λ_1 in both cases. If we put then the new equality restriction $\langle f^1, x\rangle = 0$, and apply the Weierstrass theorem and then the Lagrange multiplier rule, we obtain the second eigenpair. Repeating this procedure in the finite-dimensional case we will construct the desired basis after a finite number of steps. In the infinite-dimensional case, this procedure may consist of an infinite number of steps. From compactness it follows that $|\lambda_k| \downarrow 0$. $\square$

5. Differentials and Integrals. The operations of differentiation and integration are inverse to each other—this is one of the most fundamental results both in mathematics and in mathematical natural science. To formulate a general result on this subject, we need to become more familiar with the concept of the differential form.

(A) *Differential forms of one variable.* There are two of these: $\omega_{10} = f(x)$ and $\omega_{11} = f_1(x)\,dx$ (functions f and f_1 are assumed to be smooth). It will be required to differentiate and integrate the forms ω_{1j}, $j = 0, 1$. Here, the integration domain is a closed interval $\Omega^1 = [a, b]$, whose boundary consists of two points $\{-a, +b\}$. For differentials, we have $d\omega_{10} = f'(x)\,dx$, $d\omega_{11} = 0$, and the integration formulas are as follows:

$$\int\limits_{\partial\Omega^1} \omega_{10} = f(b) - f(a), \qquad \int\limits_{\Omega^1} \omega_{11} = \int\limits_a^b f_1(x)\,dx.$$

We have proved above the fundamental theorem of integral calculus (the Newton–Leibniz formula) on the interval $[-1, 1]$. On the basis of this formula, one may prove the principal one-dimensional theorem

$$\int\limits_{\Omega^1} d\omega_{10} \overset{\text{def}}{=} \int\limits_{\Omega^1} f'(x)\,dx \overset{\text{CV}}{=} \int\limits_{-1}^{1} f'(y)\,dy \overset{\text{NL}}{=} f(b) - f(a) \overset{\text{def}}{=} \int\limits_{\partial\Omega^1} f(x) \iff \int\limits_{\Omega^1} d\omega_{10} = \int\limits_{\partial\Omega^1} \omega_{10}$$

("CV" means the change of variables $x = y(t)$, $y(-1) = a$, $y(1) = b$).

With this proviso, now it is possible to give a general definition of a differential form and prove two college-level formulas.

Differential forms ω_{nm} of n variables of order m are sums of terms of the form

$$f_{i_1\dots i_m}(x)\,dx_{i_1} \wedge \cdots \wedge dx_{i_m},$$

where $x \in \mathbb{R}^n$, $m \le n$, $dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ is the determinant of the matrix $\{dx_{i_1}e_{i_1}, \dots, dx_{i_m}e_{i_m}\}$, dx_i are numbers, and $\{e_i\}_{i=1}^n$ is the canonical basis for $\mathbb{R}^n$. The differential of a single-term form is defined as follows:

$$df_{i_1\dots i_m}(x)\,dx_{i_1} \wedge \cdots \wedge dx_{i_m} = \sum_{j=1}^{m} \frac{\partial f_{i_1\dots i_m}(x)}{\partial x_j}\,dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

(if for some term the differentials are equal, then this term is neglected).

Now let us prove some extensions of the fundamental theorem of integral calculus (the Newton–Leibniz formula).

(B) *Differential forms of two and three variables*, when $n = 2$ or $n = 3$, and $m = n - 1$. For $n = 2$ and $m = 1$, the differential forms ω_{2i}, $i = 1, 2$, are given by

$$\omega_{21} = f_1(x_1, x_2)\,dx_1 + f_2(x_1, x_2)\,dx_2, \quad \omega_{22} = f(x_1, x_2)\,dx_1 \wedge dx_2.$$

Here, the integration domain Ω^2 is a smooth one-to-one image of the square $B^2 = [-1, 1]^2$; the boundary of Ω^2 is the image of the square's boundary (i.e., it is composed of its sides, which are traversed, say, in the counterclockwise direction). The formulas for the differentials are as follows:

$$d\omega_{21} = \left(\frac{\partial f_2(x_1, x_2)}{\partial x_1} - \frac{\partial f_1(x_1, x_2)}{\partial x_2}\right) dx_1 \wedge dx_2, \quad d\omega_{22} = 0,$$

and the integration formulas are standard formulas from textbooks on integral calculus:

$$\int\limits_{\partial\Omega^2} f_1(x)\,dx_1 + f_2(x)\,dx_2$$

is the integral over the boundary of the domain Ω^2, and

$$\int\limits_{\Omega^2} f(x_1, x_2)\,dx_1 \wedge dx_2$$

is the usual integral

$$\int\limits_{\Omega^2} f(x_1, x_2)\,dx_1\,dx_2$$

over the domain Ω^2. The Newton–Leibniz formula extends almost verbatim to the two-dimensional setting as in the proof of the one-dimensional Newton–Leibniz formula:

$$\int\limits_{\Omega^2} d\omega_{21} \overset{\text{def}}{=} \int\limits_{\Omega^2} d(f_1(x)\,dx_1 + f_2(x)\,dx_2) \overset{\text{CV}}{=} \int\limits_{B^2} d(f_1(y)dy_1 + f_2(y)dy_2)$$
$$\overset{\text{def}}{=} \int\limits_{B^2} \Big(\frac{\partial f_2(y)}{\partial y_1} - \frac{\partial f_1(y)}{\partial y_2}\big)\,dy_1\,dy_2\Big) \overset{\text{(NL)}}{=} \int\limits_{\partial B^2} f_1(y)\,dy_1 + f_2(y)\,dy_2 \overset{\text{CV}}{=} \int\limits_{\partial\Omega^2} \omega_{21}$$

("NL" means that the implication is proved using the fundamental theorem of integral calculus (the Newton–Leibniz formula) after a change in the order of integration; "CV" is the change of variables $x = y(t)$, under which Ω^2 is mapped onto B^2). This formula is known as *Green's formula.*

For $n = 3$ and $m = 2$, the differential forms ω_{3i}, $i = 2, 3$, are as follows:

$$\omega_{32} = f_1(x_1, x_2, x_3)\,dx_2 \wedge dx_3 + f_2(x_1, x_2, x_3)\,dx_3 \wedge dx_1 + f_3(x_1, x_2, x_3)\,dx_1 \wedge dx_2,$$
$$\omega_{33} = f(x_1, x_2, x_3)\,dx_1 \wedge dx_2 \wedge dx_3.$$

In this case, the integration domain Ω^3 is a smooth one-to-one image of the cube $B^3 = [-1, 1]^3$, and the boundary of Ω^3 is the oriented boundary of this cube. For differentials, we have

$$d\omega_{32} = \left(\frac{\partial f_1(x)}{\partial x_1} + \frac{\partial f_2(x)}{\partial x_2} + \frac{\partial f_3(x)}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3, \quad d\omega_{33} = 0,$$

and the integration formulas are the standard ones that can be found in textbooks on integral calculus:

$$\int\limits_{\partial\Omega^3} f_1(x)\,dx_2 \wedge dx_3 + f_2(x)\,dx_3 \wedge dx_1 + f_3(x)\,dx_1 \wedge dx_2$$

is the integral over the boundary of the domain Ω^3, and

$$\int\limits_{\partial\Omega^3} f(x_1, x_2, x_3)\,dx_1 \wedge dx_2 \wedge dx_3$$

is the conventional integral

$$\int\limits_{\Omega} f(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3$$

over the domain Ω^3. The Newton–Leibniz formula extends almost verbatim to the three-dimensional case as in the proof of the fundamental theorem of integral calculus (the Newton–Leibniz formula) or Green's formula ("CV" means the change of variables $x = y(t)$, which maps Ω^3 onto B^3):

$$\int\limits_{\Omega^3} d\omega_{32} \overset{\text{def}}{=} \int\limits_{\Omega^3} d(f_1(x)\,dx_2 \wedge dx_3 + f_2(x)\,dx_3 \wedge dx_1 + f_3(x)\,dx_1 \wedge dx_2)$$
$$\overset{\text{CV}}{=} \int\limits_{B^3} d(f_1(y)\,dy_2 \wedge dx_3 + f_2(y)\,dy_3 \wedge dx_2) + f_3(y)\,dy_1 \wedge dx_2$$
$$\overset{\text{def}}{=} \int\limits_{B^3} \left(\frac{\partial f_1(y)}{\partial y_1} + \frac{\partial f_2(y)}{\partial y_2}\right) dy_1\,dy_2 + \frac{\partial f_3(y)}{\partial y_3}\big)\,dy_1 \wedge dy_2\,dy_3$$
$$\overset{\text{(NL)}}{=} \int\limits_{\partial B^3} f_1(y)\,dy_2 \wedge dy_3 + f_2(y)\,dy_3 \wedge dy_1 + f_3(y)\,dy_1 \wedge dy_2 = \int\limits_{\partial\Omega^3} \omega_{32}$$

("NL" means that this implication is proved using the fundamental theorem of integral calculus (the Newton–Leibniz formula) after changing the order of integration). This formula is known as the *Gauss–Ostrogradskii formula* (the divergence theorem).

(C) *Differential forms, where n and m are arbitrary.*

Using the Newton–Leibniz formula, Green's formula, and the Gauss–Ostrogradskii formula proved above, one may propose the following general assertion on invertibility of the differentiation and integration operators of differential forms: the integral of the differential of a differential form over a domain equals the integral of the form over the domain's boundary. Namely, the following formula holds:

$$\int\limits_{\Omega^n} d\omega_{nm} = \int\limits_{\partial\Omega^m} \omega_{nm}.$$

This formula is sometimes referred to as the *Stokes formula* (Stokes has published this formulas with $n = 2$, $m = 3$) or the *Stokes–Poincaré formula* or the *Newton–Leibniz–Green–Gauss–Ostrogradskii–Stokes–Poincaré formula*. The proof of this formula by the above schemes is left to the reader.

III. Applications

Survey of Mechmath Mathematical Courses

Here we touch upon Mechmath courses that are reflected in the program of the final exam of the department (the so-called *State Examination*).

1. Analytic Geometry. The general theoretical content of this course consists of a geometric interpretation of the theory of linear equations and the theory of quadrics in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Only one question is included in the final exam:

> *Affine and metric classification of curves and surfaces of the second order. Projective classification of curves.*

The theory underlying this question is based on two-and three-dimensional versions of Theorem II.4 on diagonalization of symmetric operators. After application of this result, the classification of curves and surfaces is rather simple. We will demonstrate it on the example of affine classification of curves.

Theorem III.1 (affine classification of curves of the second order)**.** *A plane curve can be transformed by an affine mapping to one of the following types: a circle, a hyperbola, a parabola, a pair of lines parallel or intersecting, a point, or the empty set.*

Proof. A plane curve of the second order is also called a quadric plane curve or "a conic" (the last name is due to Apollonius). It is a level curve of a quadratic function, i.e., the set

$$\mathcal{K} = \{(x_1, x_2) \mid a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2b_1x_1 + 2b_2x_2 + c = 0\}.$$

The goal here is to give the affine classification of different types of second order plane curves.

The application of Theorem II.4 with dilatation leads to the following equations of curves: $y_1^2 + \varepsilon y_2^2$ (where $\varepsilon = \{1, 0, -1\}$) plus affine members equals to zero. If $\varepsilon = 1$ or $\varepsilon = -1$, then the shift $y_i \mapsto y_i + \alpha_i$, $i = 1, 2$, transforms the equation of quadrics to the following equations $z_1^2 + z_2^2 = \{-1, 0, 1\}$ or $z_1^2 - z_2^2 = \{1, 0\}$, and in the case $\varepsilon = 0$ to $z_1^2 = \{1, 0, -1\}$ or $z_2 = z_1^2$. This leads to the answer: for example, if $z_1^2 + z_2^2$ is equal to 1, then the curve is the unit circle, if it is equal to zero, then it is the zero point, if it is -1, then it is the empty set. □

2. Higher Algebra. The Kronecker–Capelli theorem and the fundamental theorem of algebra are the two most important results of the course of "Higher Algebra" on the first semester of Mechmath.

The maximal number of linearly independent column-vectors of a matrix is called the *rank of the matrix*. We call an $m \times m$ matrix A *solvable* if the equation $Ax = y$ is solvable for all column-vectors $y \in \mathbb{R}^m$.

Lemma on the rank. *The rank of a matrix is equal to the highest order of a solvable submatrix of the given matrix.*

Proof. Let us denote by r the rank of a matrix A and by r' the highest order of a solvable submatrix A' of A. Without loss of generality, one can assume that the column vectors of the matrix A' have numbers $1 \le i \le r'$. From the Fredholm theorem it follows that the column vectors of the solvable matrix A' are linearly independent, whence $r \ge r'$. Let us suppose that $r > r'$ and the $(r'+1)$th column vector of the matrix A is linearly independent of the first r' column vectors of this matrix. But again from the Fredholm theorem there exists a linear combination of column vectors of matrix A' such that the first r' coordinates of this combination coincide with the first r' coordinates of the $(r'+1)$th column vector of matrix A. Then from the definition of the $(r'+1)$th column vector it follows that there exists an element $a_{i(r'+1)}$, $i > r'+1$, that does not coincide with the element with the same number of linear combinations. This means that the quadratic $(r'+1) \times (r'+1)$ matrix "befringing" the matrix A' by the column vector $(a_{1(r'+1)}, \dots, a_{r'(r'+1)}, a_{i(r'+1)})^{\mathrm{T}}$ and the row vector $(a_{i1}, \dots, a_{ir'}, a_{i(r'+1)})$ is solvable (again from the Fredholm theorem). But this is contradicts the definition of the matrix A'. □

Theorem III.2 (Kronecker–Kapelli theorem)**.** *A system of linear equations in $\mathbb{R}^n$ is solvable if and only if the rank of the matrix is equal to the rank of the extended matrix* (*which consists of the matrix of the system being complemented by the column of right sides of the equations*)*.*

Proof. If a system $Ax = y$ in which the matrix A has m row vectors and n column vectors is solvable, then the vector y is expressed in terms of column vectors of matrix A. In view of the lemma on the rank, it is expressed in terms of r column vectors of A, where r is the rank of A. So the rank of the extended matrix (according to the definition of rank) is equal to r. But if the system is not solvable, this means that the vector y cannot be expressed in terms of column vectors of matrix A, i.e., the rank of the extended matrix is larger than the rank of A. □

Theorem III.3 (d'Alembert–Gauss theorem, fundamental theorem of algebra)**.** *Every nonconstant single-variable polynomial with complex coefficients has at least one complex root.*

Proof. Let $p(\cdot)$ be a polynomial of degree n. Consider a smooth problem without constraints $|p(z)|^2 \to \min$. The modulus of the polynomial tends to infinity when the argument tends to infinity:

$$|p(z)| = \left| a_n z^n \left(1 + \sum_{k=0}^{n-1} \frac{a_k}{a_n} z^{k-n}\right)\right| = |a_n|\,|z|^n\big(1 + o(1)\big) \to \infty,$$

when $|z| \to \infty$. From the Weierstrass theorem we obtain that the problem attains its minimum. Without loss of generality, the minimum is attained at zero. Then $p(z) = a_0 + a_k z^k + \dots$, where $a_1 = \dots = a_{k-1} = 0$, $a_k \ne 0$. We have

$$f(t) = |p(t^{1/k} e^{i\theta})|^2 = |a_0|^2 + 2|a_0|\,|a_k| t \cos(k\theta + \beta) + o(t), \quad t \ge 0,$$

consequently, $|a_k|\,|a_0| \cos(k\theta + \beta) \ge 0$ for all θ. But this is possible if and only if $a_0 = 0$. □

3. Linear Algebra and Geometry. Theorems on diagonalization of symmetric matrices are no doubt among the most important results of the course "Linear algebra and geometry."

The following three theorems are included in the final exam.

Theorem III.4 (on diagonal reduction of symmetric matrices)**.** *For any symmetric matrix, there exists an orthogonal diagonalizing transformation.*

This is nothing else but finite-dimensional Theorem II.4 on diagonalization of symmetric operators.

Theorem III.5 (on reduction to normal form)**.** *A linear change of variables $x = Cy$, where C is an invertible matrix, leads to the normal form*: $Q(x) = \sum_{j=1}^{n} \varepsilon_j y_j$, *where $\varepsilon_j = \{-1, 0, 1\}$.*

Proof. From the theorem on diagonalization, the quadratic form in some orthonormal basis $\{f_j\}$ is written in the form $\sum_{j=1}^{m} \lambda_j z_j^2$, where $\lambda_j \neq 0$, $m \leq n$. Then we come to the final result after the following changes of variables: $y_j = \sqrt{\lambda_j} z_j$ if $\lambda_j > 0$ and $y_j = -\sqrt{-\lambda_j} z_j$ if $\lambda_j < 0$. □

Theorem III.6 (the inertia law)**.** *The number of positive and negative squares in a normal form of a quadratic form does not depend on the selected linear transformation.*

Proof. Suppose that two linear transformations $x = By$ and $x = Cz$ with nondegenerate matrices B and C diagonalize a form $x \mapsto Q(x)$ with two different normal forms:

$$Q = y_1^2 + \cdots + y_k^2 - y_{k+1}^2 - \cdots - y_r^2 = z_1^2 + \cdots + z_l^2 - z_{l+1}^2 - \cdots - z_r^2, \tag{$**$}$$

where $k < l$. Thus, it is possible to write out the system of $n + k - l < n$ homogeneous equations over $x \in \mathbb{R}^n$:

$$y_1(x) = \cdots = y_k(x) = z_{l+1}(x) = \cdots = z_n(x) = 0.$$

From the Kronecker–Capelli theorem, it follows that there exists a nonzero solution $\hat{x} \neq 0$ of this system. The substitution of this solution into $(**)$ leads to the equation

$$-y_{k+1}^2(\hat{x}) - \cdots - y_r^2(\hat{x}) = z_1^2(\hat{x}) + \cdots + z_l^2(\hat{x}).$$

Consequently, $z_j(\hat{x}) = 0$, $1 \leq j \leq l$, i.e., the system $z = C^{-1}x$ has a nonzero solution, a contradiction. □

4. Mathematical Analysis. This is a rich two-year course. Let us look at it from the State examination viewpoint. In the State examination, there are 11 questions on "Mathematical analysis":

(1) *Continuity of functions of one variable. Properties of continuous functions.*
(2) *Multivariate functions, the total differential, and its geometrical meaning. Sufficient conditions for differentiability. Gradient.*
(3) *Definite integral. Integrability of continuous function. The primitive of a continuous function.*
(4) *Implicit functions. Existence, continuity, and differentiability of implicit functions.*
(5) *Numerical series. Convergence of series. Cauchy's convergence test. Sufficient conditions for convergence.*
(6) *Absolute and conditional convergence of a series. Properties of absolutely convergent series. Multiplication of series.*
(7) *Series of functions. Uniform convergence. Weierstrass test for uniform convergence. Properties of uniformly convergent series* (*the continuity of the sum, termwise differentiability and integration*)*.*
(8) *Power series in a real and complex domain. The radius of convergence, properties of power series* (*termwise differentiability and integration*)*. Expansion of elementary functions.*
(9) *Improper integrals and their convergence. Uniform convergence of integrals depending on a parameter. Properties of uniformly convergent integrals.*
(10) *Fourier series. Sufficient conditions for representation of a function by Fourier series.*
(11) *Ostrogradskii and Stokes theorems.*

Many topics in this list (continuous functions, differentiability, implicit functions) were outlined at a more sophisticated level. Questions (5)–(7) and (9) pertain to calculations where the role of the theory is fairly insignificant; below we shall dwell further on power series, expansions of elementary functions, and on Fourier series. Question (11) has already been elucidated. Here, we shall be concerned only with the Taylor formula and Fourier series.

Taylor's formula with integral remainder. Assume (for definiteness) that a function $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Any function from the family

$$x \mapsto F(x) = \int\limits_a^x f(t)\,dt + C$$

is called a *primitive of the function* f, and the family itself is called the *indefinite integral of* f, denoted by

$$\int f(x)\,dx.$$

From the definition of the derivative it easily follows that $F'(x) = f(x)$. Sometimes this equality is taken as the definition of the primitive of a function (for the equivalence of the definitions one should be able to prove that the equation $F'(x) = 0$ is satisfied only by the constant functions). If f is continuously differentiable, then from the above we have

$$f(x) = f(a) + \int\limits_a^x f'(t)\,dt.$$

If one assumes that f is twice continuously differentiable and if the first derivative is given by

$$f'(t) = f'(a) + \int\limits_a^t f''(\tau)\,d\tau,$$

then after substituting the second formula into the first one and permuting the order of integration, we arrive at the formula

$$f(x) = f(a) + f'(a)(x-a) + \int\limits_a^x (t-a)f''(t)\,dt.$$

Repeating this operation $n-1$ times, this gives Taylor's formula with integral remainder:

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(a)(x-a)^{n-1} + \frac{1}{(n-1)!}\int\limits_a^x (x-t)^{n-1}f^{(n)}(t)\,dt.$$

Elementary functions and their Taylor series expansions. Polynomials, exponents, trigonometric functions, and their inverse functions are attributed to elementary functions. Polynomials are Taylor sums thereof. The most important exponent is the function

$$x \mapsto e^x.$$

It is the solution of the Cauchy problem

$$\dot{x} = x, \quad x(0) = 1$$

or in integral form

$$x(t) = 1 + \int\limits_0^t x(\tau)\,d\tau.$$

The iterative sequence

$$x_{n+1}(t) = 1 + \int\limits_0^t x_n(\tau)\,d\tau, \quad x_0(t) = 1$$

leads to the series

$$e^t = \sum_{k\geq 0} \frac{t^k}{k!}.$$

The convergence of this series can be easily proved.

Similarly, the function

$$x \mapsto \sin t$$

is the solution of the Cauchy problem

$$\ddot{x} + x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1$$

or in integral form

$$x(t) = t + \int_0^t (t-\tau)x(\tau)\,d\tau.$$

The iterated sequence

$$x_{n+1}(t) = t + \int_0^t (t-\tau)x_n(\tau)\,d\tau, \quad x_0(t) = t$$

leads to the series

$$\sin t = \sum_{k\in\mathbb{N}} (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!}.$$

The series for logarithm and arctangent are obtained by integration of series:

$$\ln'(1+x) = \frac{1}{1+x},$$

hence

$$\ln(1+x) = \sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{x^k}{k}.$$

Fourier series and special functions. A circle is the simplest compact homogeneous space (a compact manifold on which the group of distance-preserving translations acts transitively). We realize this manifold as the interval $[-\pi, \pi]$ with the extremities identified.

On compact homogeneous spaces, there exist second-order translation-invariant differential operators. On the circle (in such realization) one may take as such an operator the operator

$$\Delta = \frac{d^2}{dt^2}$$

with periodic boundary conditions $x(-\pi) = x(\pi)$, $\dot{x}(-\pi) = \dot{x}(\pi)$. This operator is not bounded, but its inverse (the integral operator) is a compact symmetric operator. In order to write it down, one needs to solve the equation

$$\frac{d^2}{dt^2}x(t) = \delta(t),$$

where $\delta(\cdot)$ is the Dirac δ-function, with periodic conditions at the extremities and which vanishes on solution of the equation. Integrating "heuristically" the δ-function, we get the function

$$B_1(t) = \begin{cases} -\dfrac{t+\pi}{2\pi} & \text{for } -\pi \leq t < 0; \\ -\dfrac{t-\pi}{2\pi} & \text{for } 0 < t \leq \pi. \end{cases}$$

Repeated integration gives the formula

$$B_2(t) = c + \begin{cases} -\dfrac{(t+\pi)^2}{4\pi} & \text{for } -\pi \le t < 0; \\ -\dfrac{(t-\pi)^2}{4\pi} & \text{for } 0 < t \le \pi. \end{cases}$$

Here, the constant is chosen so that the integral over the period of the function $B_2(\cdot)$ is zero. As a result, we arrive at the inversion formula

$$\xi(t) = \int\limits_{\mathbb{T}} B_2(t+\tau)\ddot{\xi}(\tau)\, d\tau,$$

which is verified by direct calculation for functions from (say) $C^2(\mathbb{T})$ (for which the integral over the period is zero). Hence, the inversion formula also holds in the completion of this space in the root mean square norm. Such a completion, which is denoted by $L_2(\mathbb{T})$, is known to be isometrically isomorphic to l_2. The operator

$$\mathcal{B}x(t) = \int\limits_{\mathbb{T}} B_2(t+\tau)x(\tau)\, d\tau$$

is such that the Hilbert–Schmidt theorem can be applied to the operator $x(\cdot) \mapsto x(\cdot) + \mathcal{B}x(\cdot)$. An application of this theorem shows that a function from $L_2(\mathbb{T})$ can be expanded into a series in eigenfunctions of the operator $\mathcal{B}$ or, what is the same, into eigenfunctions of the operator

$$\frac{d^2}{dt^2}$$

(with periodic boundary conditions). Thus, we obtain the following basis for $L_2(\mathbb{T})$:

$$e_0(t) = \frac{1}{2\pi}, \quad e_{2k-1}(t) = \frac{1}{\pi}\sin kt, \quad e_{2k}(t) = \frac{1}{\pi}\cos kt, \qquad k \in \mathbb{N}.$$

The expansion described in the Hilbert–Schmidt theorem is nothing else but the expansion of a function into a standard series.

In this way, one may construct a very large number of special functions, starting from the spherical functions on $\mathbb{S}^2$, which are expressed in terms of the main and associated Legendre polynomials.

5. Ordinary Differential Equations.

Theorem III.7 (local theorem on existence and uniqueness of the solution to the Cauchy problem)**.** *Let V be a neighborhood of a point $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and a function $f\colon V \to \mathbb{R}^n$ be defined and continuous on V and satisfy there the Lipschitz condition in x. Then there is a unique solution to the Cauchy problem*

$$\dot{x} = f(t, x), \quad x(\tau) = \xi, \tag{III.1}$$

defined on an interval $\Delta = [\tau - \alpha, \tau + \alpha]$.

Proof. Let V contain the set $D = [\tau - a, \tau + a] \times B_{\mathbb{R}^n}(\xi, b)$ (for some positive a and b) and let

$$|f(t, x_2) - f(t, x_1)| \le L|x_2 - x_1|.$$

Apply the corollary to the theorem on right inverse (the Graves theorem) to the case where $X = Y = C([\tau - a, \tau + a], \mathbb{R}^n)$,

$$F\big(x(\cdot)\big)(t) = x(t) - \xi - \int\limits_{\tau}^{t} f(x, s)\, ds,$$

$\Lambda = \mathrm{Id}$ (the identity operator), and

$$\alpha = \min\left\{a, \frac{1}{2L}, \frac{b}{2M}\right\},$$

where $M = \max\limits_{t\in[\tau-a,\tau+a]} f(t,\xi)$. The reader can easily verify that in this case condition $(**)$ of the theorem on right inverse is fulfilled and, therefore, application of the Graves theorem yields the result. $\square$

Theorem III.8 (global theorem on existence and uniqueness of the solution to the Cauchy problem for linear systems)**.** *Let in the differential equation* (III.1) $f(t,x)$ *be* $A(t)x+b(t)$*, where* $A(\cdot)$ *is a matrix-function continuous on* $[t_0,t_1]$ *and* $b(\cdot)$ *is a continuous vector-function. Then for every* $\tau\in[t_0,t_1]$ *and* $\xi\in\mathbb{R}^n$*, there exists a unique solution of the equation* (III.1) *defined on the whole segment* $[t_0,t_1]$*.*

Proof. Application of the local theorem to our linear case gives a solution $\hat{x}(\cdot)$ on the segment $[\tau-\alpha, \tau+\alpha]$, where α does not depend on the position τ of the initial point. Then we can continue our solution on the segment $[\tau+\alpha,\tau+1\alpha]$, solving the Cauchy problem $x(\tau+\alpha)=\hat{x}(\tau+\alpha)$. Thus, after a finite number of steps we construct a solution on the whole segment $[t_0,t_1]$. $\square$

6. Complex Analysis.

Definition of derivative of complex functions. Let a function $f\colon V\to\mathbb{C}$ be defined in a neighborhood V of a point $\hat{z}$. One says that f is *differentiable at* $\hat{z}$ if there exists a number $a=\alpha+i\beta\in\mathbb{C}$ such that $f(\hat{z}+z)=f(\hat{z})+az+r(z)$, where $r(z)=o(|z|)$, i.e., $\lim\limits_{z\to 0}(r(z)/|z|)=0$. This number a is defined uniquely. It is called the *derivative of* f *at the point* $\hat{z}$. It is denoted $f'(\hat{z})$. If f is differentiated in a neighborhood W of the point $\hat{z}$ and the function $z\mapsto f'(z)$ is continuous in W, then one says that f is an analytic (or holomorphic) function in W.

Lemma III.1 (Cauchy–Riemann conditions)**.** *Let* f *be an analytic function in a neighborhood of a point* $\hat{z}=(\hat{x},\hat{y})$*. Then for* $u(\hat{x},\hat{y})=\operatorname{Re} f(\hat{z})$ *and* $v(\hat{x},\hat{y})=\operatorname{Im} f(\hat{z})$ *the formulas*

$$\frac{\partial u(\hat{x},\hat{y})}{\partial x}=\frac{\partial v(\hat{x},\hat{y})}{\partial y},\quad \frac{\partial v(\hat{x},\hat{y})}{\partial x}=-\frac{\partial u(\hat{x},\hat{y})}{\partial y}$$

hold. They are called the Cauchy–Riemann conditions.

Proof. We have

$$f(\hat{z}+z)=f(\hat{z})+f'(\hat{z})z+o(|z|)=u(\hat{x}+x,\hat{y}+y)+iv(\hat{x}+x,\hat{y}+y)+o(|z|)$$
$$\overset{\mathrm{Id}}{=} f(\hat{z})+\frac{\partial u}{\partial x}x+\frac{\partial v}{\partial y}y+i\left(\frac{\partial v}{\partial x}x+\frac{\partial v}{\partial y}y\right)+o(|z|).$$

Putting $f'(\hat{z})=\xi+i\eta$ and setting equal the real and imaginary parts of the left and right components of the equality, we obtain

$$\xi x-\eta y+i(\xi y+\eta x)=\frac{\partial u}{\partial x}x+\frac{\partial v}{\partial x}y+i\left(\frac{\partial v}{\partial x}x+\frac{\partial v}{\partial y}y\right).$$

From this we have

$$\frac{\partial u}{\partial x}=\xi,\quad \frac{\partial u}{\partial y}=-\eta,\quad \frac{\partial v}{\partial x}=\eta,\quad \frac{\partial v}{\partial y}=\xi;$$

consequently,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

$\square$

Theorem III.9 (Cauchy theorem on circulation integral)**.** *Let* U *be an open subset of* $\mathbb{C}$ *that is simply connected, let* $f\colon U\to\mathbb{C}$ *be a holomorphic function, and let* γ *be a rectifiable path in* U *whose start point is equal to its end point. Then*

$$\int\limits_{\gamma} f(z)\,dz=0.$$

Proof. Indeed,

$$\int\limits_{\partial\Omega} f(z)\,dz \overset{\text{def}}{=} \int\limits_{\partial\Omega} (u+iv)(dx+i\,dy) \overset{\text{Id}}{=} \int\limits_{\partial\Omega} (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

$$\overset{\text{Green}}{=} \int\limits_{\Omega} \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dx\,dy + i \int\limits_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \overset{\text{Cauchy–Riemann}}{=} 0. \quad \square$$

Theorem III.10 (Cauchy formula)**.** *Let the function f be holomorphic on and inside a positively oriented simple closed contour γ and z be any point inside γ. Then*

$$f(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof. Consider the circle

$$\gamma_\varepsilon = \{\zeta \in \mathbb{C} \mid |\zeta - z| = \varepsilon\}.$$

If ε is small enough, then the disc

$$D_\varepsilon = \{\zeta \in \mathbb{C} \mid |\zeta - z| \le \varepsilon\}$$

lies in the domain Ω bounded by the contour γ. From the Cauchy theorem it follows that

$$\int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int\limits_{|\zeta - z| = \varepsilon} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 0.$$

Hence

$$\int\limits_{\gamma} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

is equal to

$$J_\varepsilon = \int\limits_{|\zeta - z| = \varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

If we put $\zeta = z + \varepsilon e^{it}$, $0 \le t \le 2\pi$, then we obtain that

$$J_\varepsilon = \int\limits_0^{2\pi} \frac{f(z + \varepsilon e^{it})\varepsilon i e^{it}\,dt}{\varepsilon e^{it}} = i \int\limits_0^{2\pi} f(z + \varepsilon e^{it})\,dt = 2\pi i\big(f(z) + o(\varepsilon)\big).$$

Thus, $J = \lim\limits_{\varepsilon \to 0} J_\varepsilon = 2\pi i f(z)$. $\square$

7. Differential Geometry. Only two results of differential geometry are included in the final exam: the Meusnier theorem and Euler's formula. We will talk about them, but first of all it is necessary to define notions indispensable for the formulation of these results.

A smooth plane curve locally is a smooth image of a unit interval from $\mathbb{R}$ into the Euclidean plane. Let γ be a smooth plane curve, $\xi \in \gamma$, $T_\xi\gamma$ and $N_\xi\gamma$ be the tangent line and normal to γ at ξ, respectively. Let $T_\xi\gamma$ and $N_\xi\gamma$ be coordinate axes in the Euclidean plane. The curve γ in these axes can be described by the following equation:

$$x_2 = f_\gamma(x_1) = \frac{kx_1^2}{2} + o(x_1^2), \quad r \ge 0.$$

If $k > 0$, this means that in a neighborhood of the point ξ the curve γ to an accuracy of the second order can be approximate by a parabola. In this case, there exists a unique point $(0, r)$ such that the circle with center at $(0, r)$ and radius r approximates γ at $\xi \leftrightarrow (0, 0)$ to an accuracy greater than second order. It can easily be calculated that $r = k^{-1}$. The number k is called the *curvature of γ at ξ* and the point $(0, k^{-1})$ is called the *center of curvature of γ at ξ*.

Now consider the curve ℓ in three-dimensional Euclidean space. Let p be a point of ℓ and $T_p\ell$ be a tangent line to ℓ at p. Among the planes that contain $T_p\ell$, there exists a plane of the locally best approximation of ℓ near p. Either this plane is unique (this is the nondegenerate case) or any plane that contains $T_p\ell$ has the property of the best approximation (the degenerate case). In the first case, the plane of the best approximation is called the *osculating plane*, which we denote $\mathcal{P}$. Let ℓ' be the image of projection of ℓ into $\mathcal{P}$. The center of curvature of ℓ' at p is called the *center of curvature of the curve ℓ at p*.

A smooth surface in three-dimensional Euclidean space locally is a smooth image of a unit disk from $\mathbb{E}^2$ into this space. Let S be a smooth surface, P be a point in S, and T_PS and N_PS be the tangent plane of S and the normal to S at P, respectively. Let e_3 be a unit vector lying in N_PS, and f_1, f_2 be a basis in T_PS. The surface S in the basis f_1, f_2, e_3 is described in the neighborhood of P by the following equation:

$$x_3 = Q(x_1, x_2) + o(x_1^2 + x_2^2),$$

where Q is a quadratic form. If in T_PS we take the orthonormal basis e_1, e_2 in correspondence with Theorem II.4 on diagonalization, then the equation of the space S will have the form

$$x_3 = \frac{k_1x_1^2 + k_2x_2^2}{2} + o(x_1^2 + x_2^2).$$

The directions along e_1 and e_2 are called the *principal directions* on S, and the numbers k_1 and k_2 are called the principal curvatures. The following results hold:

Theorem III.11 (Meusnier theorem)**.** *The center of curvature of a smooth curve on the surface S at ℓ at P is the projection of the center of curvature of the normal section of S with the same tangent line.*

Theorem III.12 (Euler formula)**.** *The curvature of the surface S at the point P satisfies the formula $k_\alpha = k_1 \cos\alpha + k_2 \sin\alpha$, where k_α is the normal curvature in a direction making an angle α with the first principal direction and k_1 and k_2 are the principal curvatures.*

In fact, both theorems follow from the definitions.

Finally, we say a few words about courses that are not reflected in the final exam.

In *Functional Analysis*, three theorems of linear analysis play a very important role. They are the Banach open mapping theorem, the Hahn–Banach theorem, and the uniform boundedness principle or the Banach–Steinhaus theorem. All the theorems cited above were proved in this article somewhere in some different form. Indeed, the lemma on right inverse mapping is equivalent to the Banach open mapping theorem, the Hahn–Banach theorem properly speaking is equivalent to the separation theorem, and the Banach–Steinhaus theorem can be proved analogously to Lemma II.2.

Main concrete problems of mathematical physics that are studied in the mechmath course *Partial Differential Equations* are solved by the Fourier method, which reduces problems of mathematical physics to the Hilbert–Schmidt theorem (Theorem II.4).

We will illustrate it on an example of string vibration. Let $u(t, x)$ be the position of a string fastened at the ends of the segment $[0, 1]$. Let the initial position of the string at the point x at the initial moment be equal to $\varphi(x)$, and the velocity at x at the same moment be $\psi(x)$.

Let the function $u(x, t)$ describe the vertical movement of the string. The equation of the vibrating string is the following:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{$*$}$$

Let $u(t, 0) = u(t, 1) = 0$ be the boundary conditions and $u(0, x) = \varphi(x)$ and $u_t(0, x) = \psi(x)$ be the initial conditions of the string. Equation $(*)$ is qualified as a *hyperbolic equation.*

Solution. We will solve the problem $(*)$ by the method of separation of variables. Let us try to present the solution as a product $T(t)X(x)$. Then the equation $(*)$ leads to the equality

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Hence

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{const} = \lambda.$$

The boundary conditions lead to the equalities $X(0) = X(1) = 0$. Thus, the function $X(\cdot)$ is the solution of the Cauchy problem

$$X'' = \lambda X, \quad X(0) = X(1) = 0.$$

This implies that $\lambda_k = -k^2$, and the solutions of the Cauchy problem are the following functions: $x \mapsto \sin k\pi x$, $k \in \mathbb{N}$. Now it is necessary to apply the Hilbert–Schmidt theorem on diagonalization. Assume that $G(x,\xi) = (1-\xi)x$ for $0 \le \xi \le x$ and $G(x,\xi) = (1-x)\xi$ for $x \le \xi \le 1$ and

$$z(x) = \int\limits_0^1 G(x,\xi)y(\xi)\,d\xi.$$

Then $z''(x) = y(x)$ and, moreover, $z(0) = z(1) = 0$. The system of functions $\{2^{-1/2}\sin k\pi x\}_{k\in\mathbb{N}}$ is a complete orthonormal system in $L_2([0,1])$. Now it is necessary to solve the equations $T''(t) + n^2T(t) = 0$ and then the general solution of the vibrating string is the following:

$$u(t,x) = \sum_{k\in\mathbb{N}} (c_k \sin nt + d_k \cos nt) \sin k\pi x.$$

The values c_k and d_k are defined uniquely from the initial data. □

The Schrödinger equation, which describes the time evolution of the system's wave function (also called a "state function"), can be solved analogously in many cases.

Application to Natural Science

At the beginning of this paper, we pointed out the unity of mathematics and mathematical natural sciences. Galilei (1564–1642) wrote: "*The Book [of Nature] is written in mathematical language.*" Since then, humankind has learned a lot and now is ready to really begin reading this book.

Euler (1707–1783) formulated a characteristic feature of the laws written in the Book of Nature: "*Nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.*" This idea is sometimes formulated as follows: "*The laws of nature are largely described by extremal principles.*"

Let us get back to discussing ideas on three levels.

(A) Knowledge of the extremum theory presented above suffices even at the school level to derive the law of refraction of light when passing through the boundary between two isotropic media with different velocities of light in them. (In experiments, this law was established by the Dutch astronomer W. Snellius (1580–1626).)

To this end, we must apply the condition for minimum in the simple problem

$$f(x) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(1-x)^2}}{v_2} \to \min,$$

which formalizes the first extremal principle in natural sciences introduced by Fermat (1601–1665) in 1662. (Fermat's principle in optics or the principle of least time is the principle that the path taken between two points by a ray of light is the path that can be traversed in the least time.) In the problem at hand, the light travels from the point $(0,a)$ in the upper half-plane, where its velocity is v_1, to the point $(l,-b)$ in the lower half-plane, where the velocity is v_2.

In 1687 there appeared Newton's (1643–1727) book "Mathematical Principles of Natural Philosophy," the greatest treatise in the history of science. It contained the whole corpus of laws of nature based on a common conception. When thinking about the problems of school mathematical education, A. N. Kolmogorov wrote: "It hardly needs to be proved that from the general educational point of view it is highly desirable to achieve quite concrete comprehending by all schoolchildren of at least Newton's conception of mathematical natural science."

As applied to the laws of dynamics, Newton's conception of mathematical education can be briefly stated as: "The laws of dynamics are described by differential equations." This motivates the study of the theory of ordinary differential equations and their calculus.

In particular, the motion $t \mapsto x(t)$ of a particle with mass m along a straight line under the force $F(x)$ (depending on the coordinate of the particle) satisfies the differential equation $m\ddot{x} = F(x)$. For $F(x) = 0$ and $F(x) = \text{const}$ this equation describes the Galileo laws, which are the first laws of dynamics, viz., the inertia law and the law of falling bodies, while in the case of Hooke's law, $F(x) = -\alpha x$, the law of harmonic oscillations. If the acting force of the center decreases proportionally to the distance from the center, the position of the particle decreases exponentially with time. The investigation of the simplest one-dimensional motions already motivates the study of elementary functions (polynomials, trigonometric and exponential functions, and, to solve equations involving them, the radicals, inverse trigonometric functions, and logarithms) in high school, institute, and university courses.

(B) In general, the majority of the laws of dynamics are derived from *Hamilton's principle*, which says that any dynamical system is characterized by the *Lagrangian* $L = L(t, x, \dot{x})$, where $L\colon \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$. The functional

$$S(x(\cdot)) = \int\limits_{t_0}^{t_1} L\bigl(t, x(t), \dot{x}(t)\bigr)\, dt$$

is called the *action*. Similarly to Fermat' principle concerning the trajectory of light, a mechanical system moving from a point (t_0, x_0) to (t_1, x_1) "chooses" the trajectory that requires (locally, for sufficiently close initial and final points) the least action to overcome the way. This means, in particular, that *Euler's equation*

$$-\frac{d}{dt}L_{\dot{x}} + L_x = 0,$$

which we discussed above, is fulfilled. Application of Euler's equation to the motion of a satellite around the central body when the force of their interaction is inversely proportional to the squared distance between them yields integrable differential equations (which refers to the higher education level). Their integration resulted in Kepler's three laws, which was a great triumph for science.

(C) Let a moving body of mass m be drawn to a planet of mass M located at the zero point. Orthogonal coordinates of the body are $\bigl(x(t), y(t)\bigr)$, and polar coordinates of this body are $\bigl(r(t), \varphi(t)\bigr)$. The force F between the two bodies according to the gravity law is equal to

$$F(x,y) = -kr^{-2} = -\frac{k}{\sqrt{x^2+y^2}}, \quad k = cMm.$$

The kinetic energy T of the body is equal to

$$\frac{m}{2}\bigl(\dot{x}^2(t) + \dot{y}^2(t)\bigr) = \frac{m}{2}\left(\frac{d}{dt}\bigl(r(t)\cos\varphi(t)\bigr)^2 + \frac{d}{dt}\bigl(r(t)\sin\varphi(t)\bigr)^2\right) = \frac{m}{2}\bigl(\dot{r}^2(t) + r^2(t)\dot{\varphi}^2(t)\bigr),$$

and the potential energy U is equal to

$$\frac{k}{r(t)}.$$

The Lagrangian $T - U$ does not depend on time, so the Euler equation has an integral of energy (which expresses the *energy conservation law*)

$$\frac{m}{2}\left(\dot{r}^2(t) + r^2(t)\dot{\varphi}^2(t)\right) + \frac{k}{r(t)} = H.$$

The Lagrangian does not depend on the phase coordinate φ, so it has the integral of impulse

$$mr^2(t)\dot{\varphi}(t) = mL.$$

It can be rewritten in the form of Kepler's second *law of squares*:

$$\frac{d\varphi}{dt} = \frac{L}{r^2}.$$

The energy conservation law in combination with Kepler's law of squares can be rewritten in the form

$$\frac{m}{2}\left(\dot{r}^2(t) + \frac{L}{mr^2(t)}\right) + \frac{k}{r(t)} = H.$$

This is equivalent to the following differential equation:

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(H - \frac{mk}{r}\right) - \frac{L^2}{m^2r^2}}.$$

Dividing $d\varphi/dt$ by dr/dt, we come to the differential equation

$$\frac{d\varphi}{dr} = \frac{L}{r^2\sqrt{\frac{2}{m}\left(H - \frac{mk}{r}\right) - \frac{L^2}{m^2r^2}}},$$

which can be integrated by means Euler's substitution. After integrating, we have

$$r = \frac{p}{1 + e\cos\varphi}, \quad p = \frac{L^2}{|k|}, \quad e = \sqrt{\frac{2HL^2}{mk^2} + 1}, \quad L = \sqrt{|k|p}.$$

Thus, the orbit of a body is a curve of the second order. If $e < 1$, then the orbit of the body (a planet) is an ellipse with the main body (the Sun) located at one of the two foci (this is Kepler's first law). A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time (the second law). The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit (the third law).

Appendix

Proof of the lemma on the right inverse. In this proof, the Baire category theorem and the modified Newton method are used.

Let $k \in \mathbb{N}$ and $U_k = U_X(0,k) = \{x \in X \mid \|x\|_X < n\}$ be the open ball in X with center at the origin and radius n. The operator A is surjective, $\bigcup\limits_{k\in\mathbb{N}} AU_k = Y$, and so from the Baire category theorem there exist $N \in \mathbb{N}$, $y_0 \in Y$, and $r > 0$ such that $y_0 + U_Y(0,r) \subset \operatorname{cl}\Lambda U_N$, where cl denotes the closure. The set AU_N is convex and symmetric, and so is its closure. Hence, $U_Y(0,r) \subset \operatorname{cl} AU_N$.

Let $\eta \in U_Y(0,r)$. We find an element $x(\eta)$ such that

$$\|\eta - Ax(\eta)\|_Y \le \frac{r}{4}. \tag{i}$$

Now, given arbitrary $y \in Y \setminus \{0\}$, we set

$$\bar{R}(y) = 2r^{-1}\|y\|_Y x(r2^{-1}\|y\|_Y^{-1}y), \quad \bar{R}(0) = 0.$$

Hence, by (i) and the definition of $\bar{R}$,

$$\left\|\frac{ry}{2\|y\|_Y} - x\left(\frac{ry}{2\|y\|_Y}\right)\right\|_Y \le \frac{r}{4},$$

whence

$$\|y - A\bar{R}(y)\|_Y \leq \frac{\|y\|_Y}{2}, \quad \|\bar{R}(y)\|_X \leq C_1\|y\|_Y, \tag{ii}$$

where $C_1 = 2N/r$. The mapping $\bar{R}$ is the "almost" right inverse of A. To construct the right inverse, we employ the modified Newton method:

$$x_k = x_{k-1} + \bar{R}(y - Ax_{k-1}), \quad k \in \mathbb{N}, \quad x_0 = 0. \tag{iii}$$

Consequently,

$$\|y - Ax_k\|_Y \overset{\text{(iii)}}{=} \|y - Ax_{k-1} - \Lambda\bar{R}(y - \Lambda x_{k-1})\|_Y \overset{\text{(i)}}{\leq} \frac{\|y - Ax_{k-1}\|_Y}{2} \leq \cdots \leq \frac{\|y\|_Y}{2^k}, \tag{iv}$$

and hence,

$$\|x_{k+1} - x_k\|_X \overset{\text{(iii)}}{\leq} \|\bar{R}(y - Ax_k)\|_Y \overset{\text{(ii)}}{\leq} C_1\|y - Ax_k\|_Y \overset{\text{(iv)}}{\leq} \frac{C_1\|y\|_Y}{2^k}$$

and

$$\|x_{k+m} - x_k\|_X \leq \frac{C_1}{2^{k-1}}\|y\|_Y, \tag{v}$$

which gives that $\{x_k\}_{k\in\mathbb{Z}_+}$ is fundamental. Now, putting $R(y) = \lim\limits_{k\to\infty} x_k$ and making $k \to \infty$ in (iv), we see that $\Lambda R(y) = y$. Further, letting $k \to \infty$ in the inequality

$$\|x_k\|_X \leq \|x_k - x_{k-1}\|_X + \cdots + \|x_1 - x_0\|_X \overset{\text{(v)}}{\leq} 2C_1\|y\|_Y,$$

this establishes $\|R(y)\|_Y \leq \gamma\|y\|_Y$, where $\gamma = 2C_1$, which is the required result. □

Proof of Lemma II.1. Consider $\|Bf^i - Bf^j\|_X \geq \alpha$, $i \neq j$. Let us choose N so large that

$$\|B - B_N\|_{\mathcal{L}(X,X)} \leq \frac{\alpha}{4}, \tag{vi}$$

where $B_N = (b_{ij})_{1\leq i,j\leq N}$. Then

$$\begin{aligned}\|B_Nf^i - B_Nf^j\|_X &\overset{\text{Id}}{=} \|Bf^i - Bf^j + B_Nf^i - Bf^i - B_Nf^j + Bf^j\|_X \\ &\overset{\text{TI}}{\geq} \|Bf^i - Bf^j\|_X - \|B_Nf^i - Bf^i\|_X - \|B_Nf^j - Bf^j\|_X \end{aligned} \tag{vii}$$

(here we use "TI" for "the triangle inequality"). But from the definition of norm it follows that

$$\|B_Nf^i - Bf^i\|_X \leq \|B_N - B\|_{\mathcal{L}(X,X)}\|f^i\|_X \overset{\text{(vi)}}{\leq} \frac{\alpha}{4},$$

and analogously

$$\|B_Nf^j - Bf^j\|_X \leq \frac{\alpha}{4}.$$

From this and (vii) we obtain that

$$\|B_Nf^i - B_Nf^j\|_X \geq \frac{\alpha}{2}.$$

So we have reduced the infinite-dimensional situation to the finite-dimensional case. The ball $B_{\mathbb{R}^N}(0,R)$ is a compact set in $\mathbb{R}^N$; therefore it is impossible to put in it infinitely many points x^i such that

$$\|x_i - xj\| > \frac{\alpha}{2}, \quad i \neq j.$$ □

Proof of Lemma II.2. Let X be l_2, $B\colon X \to X$ be a precompact linear operator, $y \in \operatorname{cl}\operatorname{Im}A$, $A = \mathrm{Id} - B$ (Id is the identity operator), and $Ax_n \to y$. It is assumed that x_n are orthogonal to $\operatorname{Ker} A$ (because it is possible to take x_n minus the projection of x_n into $\operatorname{Ker} A$). But then the norms $\|x_n\|_X$ are bounded. Indeed, if $\|x_n\|_X \to \infty$, then it is possible to select from the sequence

$$\left\{B\frac{x_n}{\|x_n\|_X}\right\}_{n\in\mathbb{N}}$$

a convergent subsequence

$$\left\{B\frac{x_{n_k}}{\|x_{n_k}\|_X}\right\}_{k\in\mathbb{N}}.$$

Then the sequence

$$\left\{\frac{x_{n_k}}{\|x_{n_k}\|_X}\right\}_{k\in\mathbb{N}}$$

itself is convergent, because

$$\frac{x_{n_k}}{\|x_{n_k}\|_X}=A\frac{x_{n_k}}{\|x_{n_k}\|_X}+B\frac{x_{n_k}}{\|x_{n_k}\|_X},$$

and the first item tends to zero. If the limit of the sequence

$$\left\{\frac{x_{n_k}}{\|x_{n_k}\|_X}\right\}_{k\in\mathbb{N}}$$

is denoted by z, then $\|z\|_X=1$ and $Az=0$. We have come to a contradiction and, therefore, we have proved that the norms $\|x_n\|_X$ are bounded. Then it is possible to take from the sequence $\{Bx_n\}_{n\in\mathbb{N}}$ a convergent subsequence $\{Bx_{n_l}\}_{l\in\mathbb{N}}$ (because B is a compact operator). Thus, we obtain that the subsequence $\{x_{n_l}\}_{l\in\mathbb{N}}=\{y+Bx_{n_l}\}_{l\in\mathbb{N}}$ itself is convergent. Denoting the limit by ζ, we obtain that $y=\zeta-B\zeta$. □

Proof of Lemma II.3. Let L be a closed proper subspace of X, $\bar{x}\notin L$, and $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of vectors from L such that

$$\|\bar{x}-y_n\|_X\to\inf_{y\in Y}\|\bar{x}-y\|_X=:d.$$

From the parallelogram equality

$$\|y_n-y_m\|_X^2=2\|\bar{x}-y_m\|_X^2+2\|\bar{x}-y_n\|_X^2-4\left\|\bar{x}-\frac{y_1+y_2}{2}\right\|_X^2$$

it follows that if $\|\bar{x}-y_n\|_X$ and $\|\bar{x}-y_m\|_X$ are small enough, then the norm $\|y_n-y_m\|_X$ is small, i.e., the sequence $\{y_n\}_{n\in\mathbb{N}}$ is fundamental. From the completeness of X and closeness of L it follows that the sequence y_n converges to an element $\bar{y}\in L$. Then $d=\|\bar{x}-\bar{y}\|_X$. So the function $f(t)=\langle\bar{x}-\bar{y}-ty,\bar{x}-\bar{y}-ty\rangle$ attains its minimum at $t=0$ for all $y\in L$. Thus, $0=f'(0)=\langle\bar{x}-\bar{y},y\rangle$, i.e., $\bar{x}-\bar{y}\in L^{\perp}$. □

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