Bell Inequalities for Entanglement Verification: Are Higher Dimensions More Resistant to Noise?

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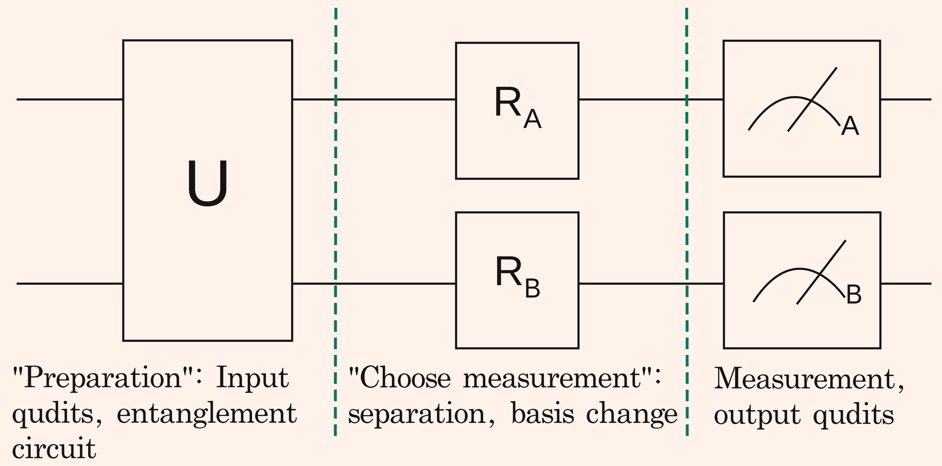
Motivation and Overview

We investigate the claim by Collins et. al that "Bell inequalities for bipartite quantum systems of arbitrarily high dimensionality…are strongly resistant to noise"[1] by testing two known Bell inequalities for higher-dimensional robustness against non-, log-, and linear-scaling models of depolarizing, dephasing, and amplitude damping quantum noise.

Bell Inequalities

A Bell inequality is an algebraic expression (in which the terms are calculable properties of a specific system) that assumes one bound under local realistic theories and another under the laws of quantum mechanics. We focus on the CGLM[1] and ZG[2] inequalities for bipartite qudit systems, which characterize expected outcomes from a general Bell test procedure, circuit representation below:



Once entangled according to a procedure U, the qudits are separated and delivered to two parties, Alice and Bob. Alice may choose one of two possible measurements, A1 or A2, to perform on her qudit; similarly, Bob may check his qudit for either B1 or B2. Each measurement may return one of d possible outcomes labeled 0, 1, … d-1.

Denote with $j \in \{0…d\text{-}1\}$ the outcome of Aa, $a \in \{1,2\}$. We can then introduce offset probabilities describing the probability that measurements Aa and Bb have outcomes differing by c (mod d):

$$\mathrm{P}(A_a = B_b + c) = \sum_{j=0}^{d-1} \mathrm{P}(A_a, B_b = j, j + c \bmod d)$$

Allowing expression of the CGLM inequality:

$$I_d = \sum_{c=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1}\right) [\mathrm{P}(A_1 = B_1 + c) + \mathrm{P}(B_1 = A_1 + c + 1) \\ + \mathrm{P}(A_2 = B_2 + c) + \mathrm{P}(B_2 = A_1 + c) \\ - \mathrm{P}(A_1 = B_1 - c - 1) - \mathrm{P}(B_1 = A_2 - c) \\ - \mathrm{P}(A_2 = B_2 - c - 1) - \mathrm{P}(B_2 = A_1 - c - 1)]$$

For local realistic methods of calculating outcome probabilities, I_d is less than or equal to two for all possible input states and entanglement circuits U.

Another possible inequality is the ZG inequality, below:

$$Z_d = \mathrm{P}(A_2 < B_2) + \mathrm{P}(B_2 < A_2) + \mathrm{P}(A_1 < B_1) + \mathrm{P}(B_1 \leq A_2)$$

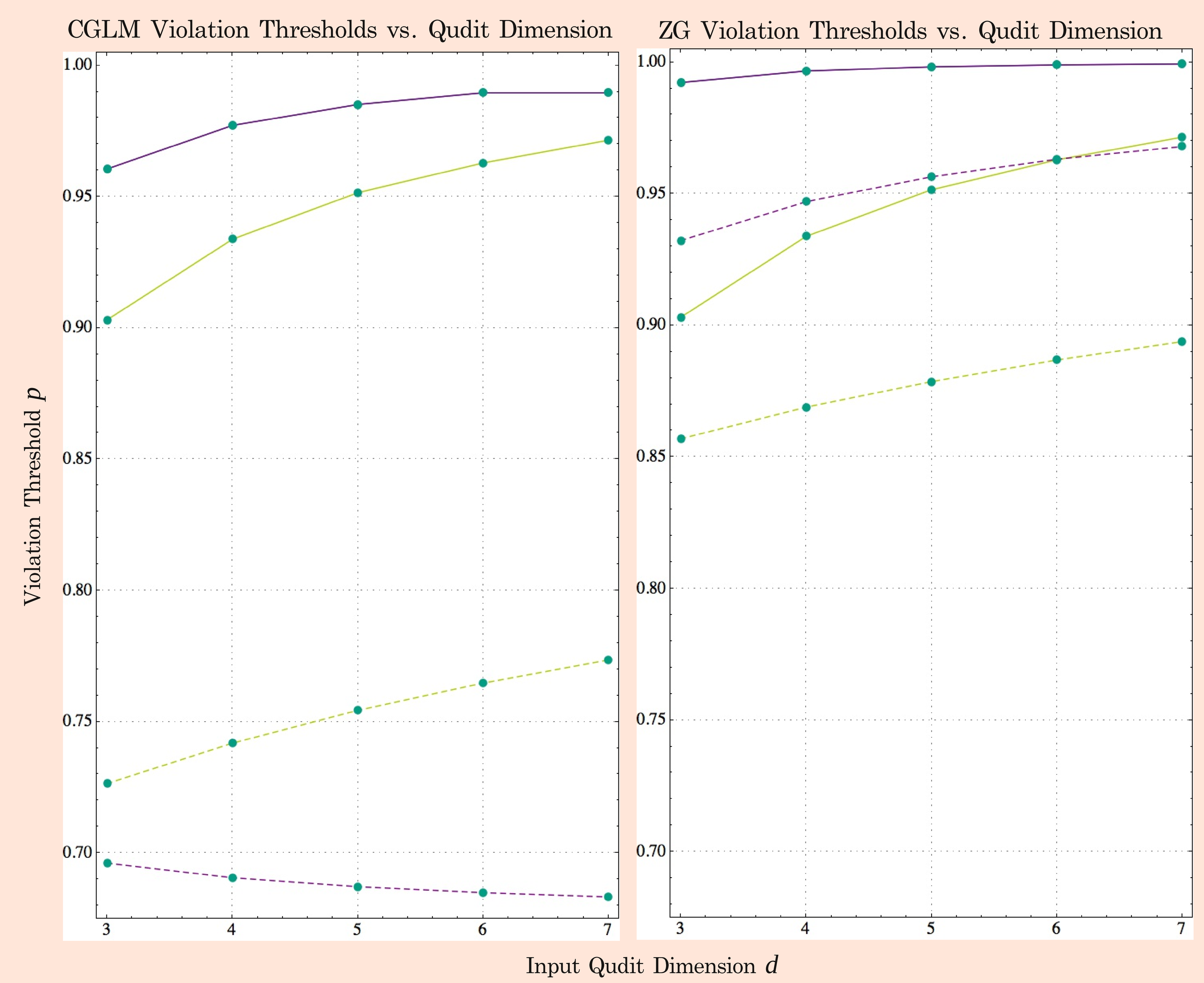
For local realistic methods of calculating outcome probabilities, Z_d is greater than or equal to one.

Violation

$$|\psi\rangle = \frac{1}{\sqrt{d}}(|00\rangle + |11\rangle + ... + |(d-1)(d-1)\rangle)$$

Quantum mechanics allows for the existence of states which are both nonlocal and nonrealistic, such as the arbitrary-dimensional singlet state above. States such as this, when passed into a Bell inequality, will return a value in conflict with the local realistic bound. For example, $I_3 = 2.87923 > 2$ and $Z_2 = 0.79289 < 1$. For both inequalities, the magnitude of violation increases with d.

Results



At left: plots of critical p value vs. entangled qudit dimension for the maximally entangled singlet state. At the threshold p value, the inequality takes on the value of the classical limit. Above the p value, it violates; below, it does not.

To calculate these values, the appropriate noise models were applied to input density matrices prior to calculation of inequality terms. Phase shifts from [1] were used.

With the single exception of the non-scaling depolarizing noise case, known higher-dimension Bell inequalities consistently require higher-fidelity entangled qudits as input in order to register entanglement.

In every case, dephasing proved eqivalent to depolarizing noise.

The CGLMP inequality is more resistant than the ZG inequality to every noise type except linear amplitude damping, in which case the two inequalities perform approximately equally.

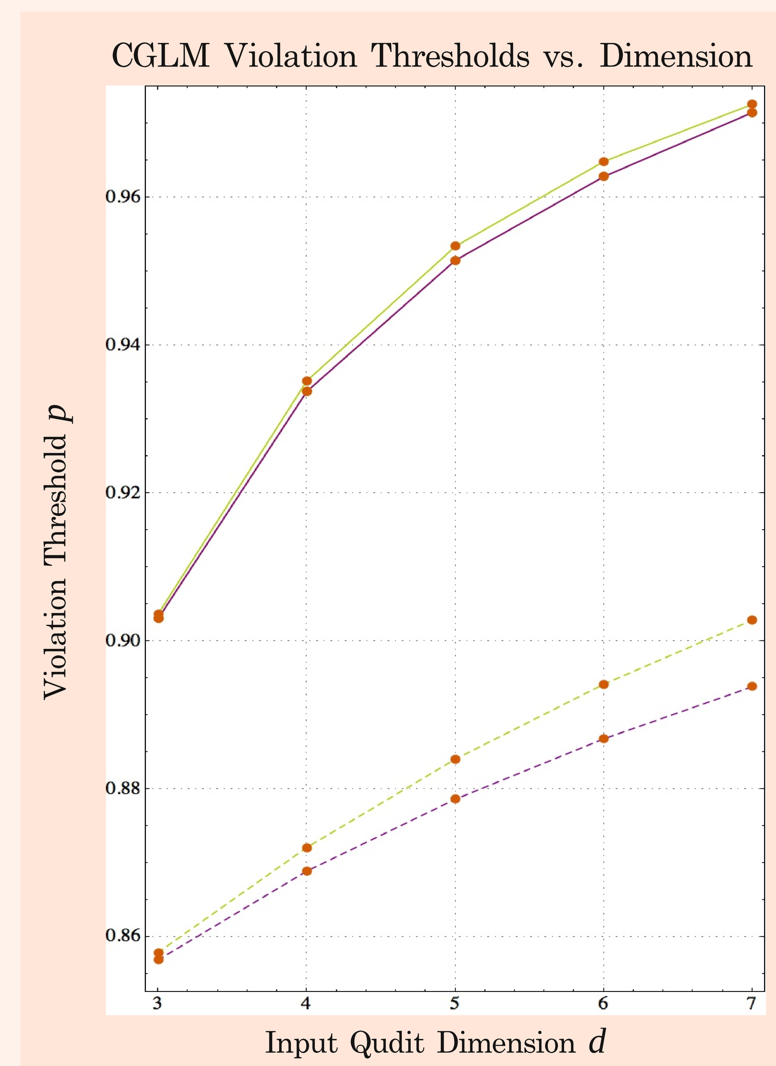
Logarithmic scaling consistently acted as an intermediate between linear and nonscaling noise.

Effect of Initial States

Zohren et al. report the existence of a two-qdit entangled state that in pure form violates the CGLM inequality to a greater degree than the singlet state. We compared violation thresholds of this maximally violating state[2] to those for the singlet state and found that the state's ability to attain higher absolute values of violation did not translate into meaningful gains in noise resistance. (The maximally violating state slightly underperformed the singlet state.)

A similar pattern was observed for the Z inequality.

The pattern additionally remained consistent for depolarizing and dephasing noise.



Models Used

Types of Scaling

Preparation of an initial state scales ~linearly with d, the measurement selection/basis change step employs the discrete Fourier transform and thus scales with $d\log d$, and several known qudit measurement circuits scale with the square of $\log d$.[3] We accordingly chose to investigate non-, log-, and linear-scaling noise models.

Density Matrix Model of Decoherent Qudits

Using the trace form of the standard quantum-mechanical joint probability, below, allows representation of noise processes as operators on density matrices of pure initial states.

$$\mathrm{P}(A_a, B_b = j, k) = Tr[(\Pi_j \otimes \Pi_k)(\hat{U}_{\phi_a} \otimes \hat{U}_{\phi_b})(\rho)(\hat{U}^\dagger_{\phi_a} \otimes \hat{U}^\dagger_{\phi_b})]$$

where ρ is the density matrix for input qudits; Π_j and Π_k are the projectors onto states $|j\rangle$ and $|k\rangle$, respectively; ϕ_a and ϕ_b are the d-dimensional phase shifts corresponding to settings A_a and B_b; and $U\phi_a$ is the matrix with entries $U_{nm} = \alpha_{nm} * \exp(i * \phi_m)$, where α_{nm} is the entry in the nth row and mth column of the d-dimensional discrete Fourier transform.

Types of Noise

Each type of noise we study transforms the initial density matrix according to some intensity parameter p, defined as the probability of no error per unit time interval.

$$\rho_o = |\psi\rangle\langle\psi| \xrightarrow{\text{Noise(p)}} \rho_n = \mathrm{N}|\psi\rangle\langle\psi|\mathrm{N}^\dagger$$

Depolarizing noise simulates environmental interference. With each iteration, each qudit has probability 1-p of becoming depolarized, i.e. replaced by the completely mixed state $\mathbb{I}/d$.

We model nonscaling and linear depolarizing with the respective equations:

$$\rho_n = \mathrm{p}\rho_o + (1-\mathrm{p})\frac{1}{d^2}\mathbb{I} \qquad \rho_n = \mathrm{p}^d\rho_o + (1-\mathrm{p}^d)\frac{1}{d^2}\mathbb{I}$$

Amplitude damping describes the effects of energy dissipation on the quantum system. In our simplified model, $1\text{-}p^n$, $n \in \{0…d\text{-}1\}$ represents the probability of a qdit in state $|n\rangle$ decaying into state $|n\text{-}1\rangle$. One iteration of amplitude damping performs the following transformation:

$$\rho_n = (\mathrm{E}_0 \otimes \mathrm{E}_0)\rho(\mathrm{E}_0 \otimes \mathrm{E}_0)^\dagger + (\mathrm{E}_1 \otimes \mathrm{E}_0)\rho(\mathrm{E}_1 \otimes \mathrm{E}_0)^\dagger \\ + (\mathrm{E}_0 \otimes \mathrm{E}_1)\rho(\mathrm{E}_0 \otimes \mathrm{E}_1)^\dagger + (\mathrm{E}_1 \otimes \mathrm{E}_1)\rho(\mathrm{E}_1 \otimes \mathrm{E}_1)^\dagger$$

$$\mathrm{E}_0 = \begin{pmatrix} 1 & 0 & & & \\ 0 & \sqrt{\mathrm{p}} & \cdot & & \\ & \cdot & \sqrt{\mathrm{p}^2} & \cdot & \\ & & \cdot & \cdot & 0 \\ & & & 0 & \sqrt{\mathrm{p}^{d-1}} \end{pmatrix} \qquad \mathrm{E}_1 = \begin{pmatrix} 0 & \sqrt{1-\mathrm{p}} & & & \\ & \cdot & \sqrt{1-\mathrm{p}^2} & & \\ & & \cdot & \cdot & \\ & & & \cdot & \sqrt{1-\mathrm{p}^{d-1}} \\ & & & & 0 \end{pmatrix}$$

Dephasing describes loss of quantum information without loss of energy: rather than changing as a function of time, energy eigenstates of a system accrue phases proportional to their eigenvalues. Below, ρ_n for nonscaling and scaling dephasing.

$$\begin{pmatrix} \rho_{11} & \mathrm{p}\rho_{12} & \mathrm{p}\rho_{13} \\ \mathrm{p}\rho_{21} & \rho_{22} & \mathrm{p}\rho_{23} \\ \mathrm{p}\rho_{31} & \mathrm{p}\rho_{32} & \rho_{33} \end{pmatrix} \qquad \begin{pmatrix} \rho_{11} & \mathrm{p}^d\rho_{12} & \mathrm{p}^d\rho_{13} \\ \mathrm{p}^d\rho_{21} & \rho_{22} & \mathrm{p}^d\rho_{23} \\ \mathrm{p}^d\rho_{31} & \mathrm{p}^d\rho_{32} & \rho_{33} \end{pmatrix}$$

Acknowledgements and References

[1] Collins, D., Gisin, N., Linden, N., Massar, S. & Popescu, S. Bell inequalities for arbitrarily high-dimensional systems. Phys. Rev. Lett. 88, 040404 (2002).

[2] S. Zohren and R.D. Gill, Phys. Rev. Lett. 100, 120406 (2008).

[3] F. W. Strauch, D. Onyango, K. Jacobs, and R. W. Simmonds, arXiv:1110.5801.

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