0 Formal Laurent series

Let $V[[x, x^{-1}]]$ be the space of doubly infinite formal Laurent series in x with coefficients in a vector space V:

$$V[[x, x^{-1}]] = \left\{ v(x) = \sum_{n \in \mathbb{Z}} v_n x^n \mid v_n \in V \right\}.$$

We could specialize to $\mathbb{C}$, but for the sake of generality, all we assume is a field of characteristic 0.

A *formal* sum/product exists if and only if every monomial coefficient acts as a finite sum when applied to a fixed but arbitrary vector. We should interpret the coefficients as scalar operators in the endomorphism ring of V. This purely algebraic approach allows us to restrict attention to the formal calculus without having to check convergence.

0.1 How not to multiply

We say the formal Dirac delta function

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n$$

is an "expansion of zero" in the sense that it is the Laurent expansion of the same function in two different domains (around 0 and ∞):

$$\begin{aligned} \sum_{n \in \mathbb{Z}} x^n &= \sum_{n \geq 0} x^n + \sum_{n < 0} x^n \\ &= (1 - x)^{-1} + (x - 1)^{-1}. \end{aligned}$$

If we did not impose this restriction, then we would have fake equations such as

$$\begin{aligned} \delta(x) &= \left(\left(\sum_{n \geq 0} x^n \right) (1 - x) \right) \delta(x) \\ &= \left(\sum_{n \geq 0} x^n \right) ((1 - x)\delta(x)) \\ &= \left(\sum_{n \geq 0} x^n \right) 0 \\ &= 0. \end{aligned}$$

In some sense, $\delta(x)$ captures what is conceptually different about working with these doubly infinite sequences versus (singly infinite) formal power series.

0.2 N.B.: formal $\neq$ analytic

This *formal* definition of series multiplication rules out certain cases that would make sense analytically. For example,

$$\left(\sum_{n\geq 0} x^n\right)\left(\sum_{n\leq 0} \frac{x^n}{2^n}\right)$$

is not a well-defined product even though the coefficients as formal sums are convergent in $\mathbb{C}$, just as we cannot argue that $(1-x)^{-1} = -(x-1)^{-1}$ to get $\delta(x) = 0$. (In fact, with this restriction, there exist products with non-existent subproducts, e.g. the non-existent product above multiplied by 0. Moreover, $f(x)g(x)$ can exist *even if* both f and g have infinitely many nonzero coefficients; finite support is not strictly necessary.)

1 Formal multiplication and convolution

If $f(x)g(x) \in V[[x, x^{-1}]]$ exists, then

$$f(x)g(x) = \sum_{n\in\mathbb{Z}}\left(\sum_{n_1+n_2=n} f_{n_1}g_{n_2}\right)x^n.$$

So we can define convolution as usual:

$$(f*g)[n] = \sum_{n_1+n_2=n} f_{n_1}g_{n_2}.$$

Our finiteness condition tells us that a formal product exists if and only if formal convolution exists (= is finitely summable) for every $n \in \mathbb{Z}$. Note that this readily extends to products of arbitrary (finite) length:

$$F_1(x)\cdots F_r(x) = \sum_{n\in\mathbb{Z}}\left(\sum_{n_1+\cdots+n_r=n} f_1(n_1)\cdots f_r(n_r)\right)x^n$$

where $f_k(n_r)$ is the coefficient of x^{n_r} in the series expansion of $F_k(x)$. Then we can analogously define

$$(F_1 * \cdots * F_r)[n] = \sum_{n_1+\cdots+n_r=n} f_1(n_1)\cdots f_r(n_r).$$

Thus for convolution to make sense, the image of the hyperplane $n_1+\cdots+n_r = n$ under the map $(n_1,\ldots,n_r) \mapsto f_1(n_1)\ldots f_r(n_r)$ must have finite support.

Since the coefficients live in a field, if formal multiplication exists at all, then it is both associative and commutative.

1.1 Two examples

We define the following series:

$$\begin{aligned} a(x) &= 2x(1-x^2)^{-1} + x(x^2-1)^{-1}, \\ b(x) &= -x + x^{-1}, \\ c(x) &= x(1-x^2)^{-1} + x(x^2-1)^{-1}. \end{aligned}$$

Question: does $(a * b)[n]$ exist? Yes: the image of the hyperplane vanishes everywhere for n odd, and everywhere but 2 points for n even. In particular, $a(x)b(x) = 1$, hence $(a * b)[n] = \delta_0$.

What about $(b * c)[n]$? Yes, by the same argument, we get $b(x)c(x) = 0$, so the convolution must always vanish, too.

1.2 A non-example

What about $(a * b * c)[n]$? If it exists, then

$$(a * b * c)[n] = \sum_{n_1+n_2+n_3=n} a(n_1)b(n_2)c(n_3).$$

Fix $n = n_2 = 1$. Then for any odd k, the summand $-a_k c_{-k} \neq 0$, which violates the finiteness condition.

Alternatively, if multiplication/convolution exists, then it is associative. But

$$(a(x)b(x))c(x) = (1)c(x) = c(x) \neq 0 = a(x)(0) = a(x)(b(x)c(x)).$$

So it is not associative, hence it does not exist!

2 Multiplicative inverses

Given $a(x)$, we might want to find a multiplicative inverse $a^{-1}(x)$ that satisfies

$$\begin{aligned} a(x)a^{-1}(x) &= 1, \\ (a * a^{-1})[n] &= \delta_0. \end{aligned}$$

We have seen an example above already: if $a(x) = 2x(1-x^2)^{-1} + x(x^2-1)^{-1}$, then $b(x) = -x + x^{-1}$ is an inverse.

There is no reason to expect existence or uniqueness of inverses; indeed, $V[[x, x^{-1}]]$ is not even a ring, since (unlike, say, in the case of $V[x, x^{-1}]$, alias the ring of formal Laurent polynomials) multiplication is only partially defined.

Example: for any $k \in \mathbb{C}$, we may define

$$\beta(x) = kx(1-x^2)^{-1} + (k-1)x(x^2-1)^{-1}.$$

Then $\beta(x)b(x) = 1$, so $b(x)$ has uncountably many inverses having infinitely many nonzero coefficients.