How do we get inverses to behave in the doubly infinite world?

Define the innocent-looking sequence $(f_n)_{n\in\mathbb{Z}}$ such that $f_n = \delta_{\pm 1}$. Equivalently, we have its generating function $f(x) = x^{-1} + x$. Say we pretend x is a complex rather than a formal variable. Then we have the inverse

$$\left(x^{-1} + x\right)^{-1} = \frac{1}{\frac{1}{x} + x} = \frac{x}{1 + x^2}.$$

Now we smuggle this answer back with the series expansion about $x = 0$:

$$(x^{-1} + x)(x - x^3 + x^5 - x^7 + \ldots) = 1.$$

But we could *also* expand about $x = \infty$:

$$(x^{-1} + x)(x^{-1} - x^{-3} + x^{-5} - x^{-7} + \ldots) = 1.$$

What is happening here? In the complex world, *Laurent expansions are unique*, but when x is formal, we have expansions of zero such as the Dirac delta function. In particular, we might subtract one inverse from another:

$$(x - x^3 + x^5 - \ldots) - (x^{-1} - x^{-3} + x^{-5} - \ldots) = \quad \ldots + x^{-3} - x^{-1} + x - x^3 + \ldots$$

This yields another expansion of zero, which we can rewrite as (for example) $x\delta(x^2) - 2x^3\delta(x^4)$. So to get one inverse from another, we are adding an expansion of zero. Indeed, we could add any annihilator of $f(x)$ (which many expansions of zero will be) to any inverse to get another inverse. In other words, it is not the sequence's fault; it is because expansions of zeros live here.

If we want unique inverses, then we could try taking the space of doubly infinite formal Laurent series modulo the equivalence relation that sends all expansions of zero to 0. So, for example,

$$\delta(x) = (1 - x)^{-1} + (x - 1)^{-1} \sim 0.$$

But this just means $-(x - 1)^{-1} \sim (1 - x)^{-1}$, that is,

$$\ldots - x^{-3} - x^{-2} - x^{-1} \sim 1 + x + x^2 + x^3 + \ldots$$

This illustrates how the equivalence relation also establishes an equivalence between left- and right-infinite series.

Hence if we want inverses to work, but we also want to keep expansions of zero around, we could say inverses exist *up to equivalence*. Then we would just pick a canonical representative, (e.g. the inverse in the subspace of formal Laurent series that are truncated below.) For example, the inverse of $x^{-1} + x$ up to addition of expansions of zero is, as expected, $x(1 + x^2)^{-1}$.

(Note that working with series that are truncated below also solves the multiplication problem, since we get closure. In the broader setting, multiplication fails to be well-defined because squaring the formal Dirac delta fails, just like in classical distribution theory.)