Task 1: The fast algorithm

$$\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = |b\rangle \qquad \begin{pmatrix} x(\tau+t) \\ y(\tau+t) \end{pmatrix} = HHL \begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = HHL|b\rangle$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} 1 & \omega t \\ -\omega t & 1 \end{pmatrix} \begin{pmatrix} x(\tau+t) \\ y(\tau+t) \end{pmatrix} = \begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} \qquad \hat{Z} \cdot \begin{pmatrix} 1 & \omega t \\ -\omega t & 1 \end{pmatrix} \begin{pmatrix} x(\tau+t) \\ y(\tau+t) \end{pmatrix} = \hat{Z} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Hermitian

$$\begin{pmatrix} 1 & \omega t \\ \omega t & -1 \end{pmatrix} \begin{pmatrix} x(\tau+t) \\ y(\tau+t) \end{pmatrix} = \begin{pmatrix} x(\tau) \\ -y(\tau) \end{pmatrix}$$

Eigenvalues

$$\lambda = \pm\sqrt{1+\omega^2 t^2} = \pm v$$

Eigenvectors

$$|u\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{v+1}{2v}} \\ \pm\sqrt{\frac{v-1}{2v}} \end{pmatrix}$$

$$H = \begin{pmatrix} \sqrt{\frac{v+1}{2v}} & \sqrt{\frac{v-1}{2v}} \\ \sqrt{\frac{v-1}{2v}} & \sqrt{\frac{v+1}{2v}} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} \begin{pmatrix} \sqrt{\frac{v+1}{2v}} & \sqrt{\frac{v-1}{2v}} \\ \sqrt{\frac{v-1}{2v}} & -\sqrt{\frac{v+1}{2v}} \end{pmatrix}$$

$$U^n = e^{iHTn} = \begin{pmatrix} \sqrt{\frac{v+1}{2v}} & \sqrt{\frac{v-1}{2v}} \\ \sqrt{\frac{v-1}{2v}} & -\sqrt{\frac{v+1}{2v}} \end{pmatrix} \begin{pmatrix} e^{ivTn} & 0 \\ 0 & e^{-ivTn} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{v+1}{2v}} & \sqrt{\frac{v-1}{2v}} \\ \sqrt{\frac{v-1}{2v}} & -\sqrt{\frac{v+1}{2v}} \end{pmatrix} = \begin{pmatrix} \cos(vTn) + i\sin(vTn) & \frac{i\omega t}{v}\sin(vTn) \\ \frac{i\omega T}{v}\sin(vTn) & \cos(vTn) - i\sin(vTn) \end{pmatrix}$$

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Orbital period $T_{op} = 6$ months $= 24$ weeks

Rotational velocity $\omega = \frac{2\pi}{T_{op}} = \frac{\pi}{12}$ radian per week

Time step $t = 1$ week

Angle step $\omega t = \frac{\pi}{12}$ radian

Eigenvalues $\lambda = \pm\, 1{,}0337\ldots_{10} = \pm\, 1{,}00001001\ldots_2$

Time evolution step $T = \pi$

QPE eigenvalue $\theta = \frac{\lambda T}{2\pi} = \pm\, 0{,}\boxed{100001}001\ldots_2$

6-qubit register $|\ell\rangle$ is required

QPE returns valid $|100001\rangle$ for positive θ

Otherwise QPE returns not valid $|011111\rangle$ corresponds to $1{,}000000_2 - 0{,}100001_2 = 0{,}011111_2$

Trick: $|\theta| > \frac{1}{2}$ since $T = \pi$ and $v > 1$ anyway. That's why MSB of $|\ell\rangle$ ($\ell[0]$ in the circuit) is a kind of flag ($|1\rangle$ for $\theta > 0$ and $|0\rangle$ for $\theta < 0$) and it acts like an extra control qubits

The trick works if $|\theta|$ is strictly greater than $\frac{1}{2}$ and it is odd. So we need non-zero MSB and LSB. It defines a length of $|\ell\rangle$. Time step 1 day (angle step $\frac{\pi}{90}$ radians) requires 12 qubits in $|\ell\rangle$