Handout 3 **January 22nd , 2020**

CSci 242. Algorithms and Data Structure **Dr. M. E. Kim**

Asymptotic Upper/Lower/Tight Bound of Running Time

- Proof by Definition

Example 1 (Ex. 1.3.) : Prove $\boldsymbol{20n^3 + 10n \log n + 5 = O(n^3)}$.

Proof) Assume that the base of log = 2.

Show that $20n^3 + 10n \log n + 5 \le c \cdot n^3$ for a positive constant c and for any $n \ge n_0$.

1) Let's choose a reasonable value of positive constant c.

Since $20n^3 + 10n \log n + 5$ is an increasing function, $20 \cdot 1^3 + 10 \cdot \log 1 + 5 = 25 \le c \cdot 1^3$ for $n = 1$.

For n=2, $20 \cdot 2^3 + 10 \cdot 2 \log 2 + 5 \le c \cdot 2^3 \Leftrightarrow 25 \le (c - 20) \cdot 2^3 \Leftrightarrow 25/8 \le (c - 20) \Leftrightarrow 23.125 \le c$

As n increases, the lower bound value of c decreases.

So, it's reasonable to choose c = 25.

2) Now, let's show that $20n^3 + 10n \log n + 5 \le 25 \cdot n^3$ for $n \ge n_0$

$20n^3 + 10n \log n + 5 \le 25 \cdot n^3 \Leftrightarrow 10n \log n + 5 \le 5 n^3$

For n=1, $10 \cdot 2 \log 1 + 5 \le 5 \cdot 1^3 \Leftrightarrow 5 \le 5$.

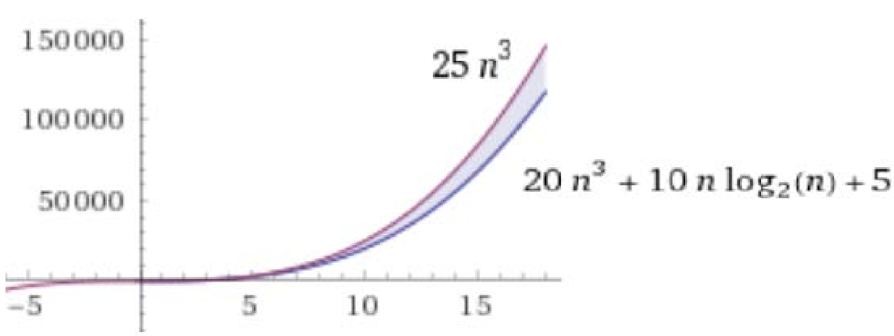
For n=2, $10 \cdot 2 \log 2 + 5 \le 5 \cdot 2^3 \Leftrightarrow 25 \le 5 \cdot 2^3 \Leftrightarrow 25 \le 40$.

Similarly, for any $n \ge$ 3, 2, 1, $20n^3 + 10n \log n + 5 \le 25 \cdot n^3$

Thus, there exists a positive constant c = 25 and $n_0 = 1$ such that

$20n^3 + 10n \log n + 5 \le c \cdot n^3$ for any $n \ge n_0$.

Therefore, $\boldsymbol{20n^3 + 10n \log n + 5 = O(n^3)}$.



Alternative Proof)

$$\lim_{n\to\infty}\frac{20n^3+10n\log n+5}{n^3} = \lim_{n\to\infty}\frac{60n^2+10n\cdot\frac{1}{n}+10\log n}{3n^2} = \lim_{n\to\infty}\frac{120n+\frac{10}{n}}{6n} = \lim_{n\to\infty}\frac{120}{6} = 20$$ is a constant.

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by L'Hôspital's rule.

Therefore, $\boldsymbol{20n^3 + 10n\log n + 5 = O(n^3)}$.

Example 2: Prove $3n^2 - 3n + 1 = O(n^3)$

Proof)

Show that $3n^2 - 3n + 1 \le c \cdot n^3$ *for any* $n \ge n_0$. for a positive constant c and for any $n \ge n_0$.

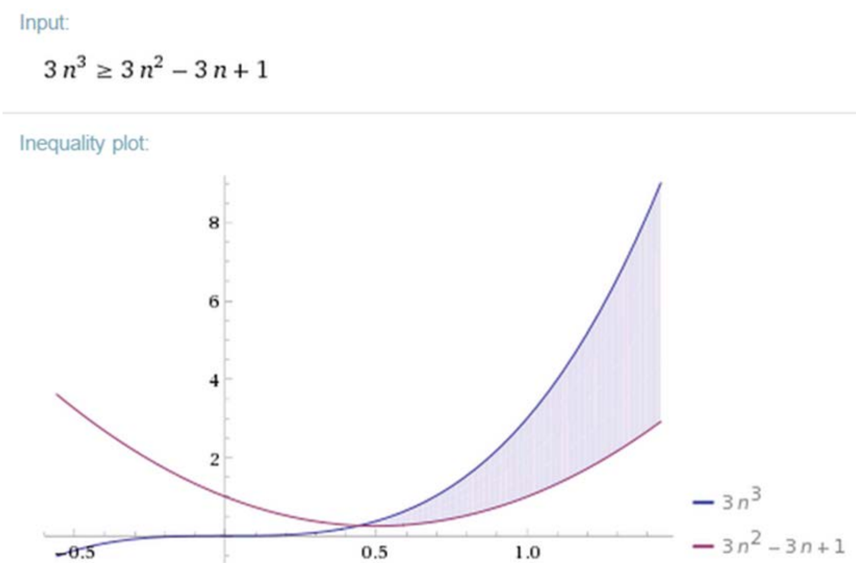
Let us find the positive constants c and n_0 such that $f(n) \le cn^3$ for $n \ge n_0$.

1) Since $-3n + 1$ is a decreasing function, let's choose $c = 3$.
   Then, $3n^2 - 3n + 1 \le 3n^3 \Leftrightarrow 1 \le 3n \Leftrightarrow 1/3 \le n$.
2) From 1), $3n^2 - 3n + 1 \le 3n^3$ for any $n \ge n_0 = 1$. As the graph show below, $3n^2 - 3n + 1 \le 3n^3$ for *any* $n \ge 1$.

Thus, there exists a positive constant $c = 3$ and $n_0 = 1$ such that

$3n^2 - 3n + 1 \le c \cdot n^3$ for any $n \ge n_0$.

Therefore, $\boldsymbol{3n^2 - 3n + 1 = O(n^3)}$.



Alternative Proof)

$$\lim_{n\to\infty}\frac{3n^2-3n+1}{n^3} = \lim_{n\to\infty}\frac{6n-3}{3n^2} = \lim_{n\to\infty}\frac{6}{6n} = 0$$ is a constant.

Therefore, $\boldsymbol{3n^2 - 3n + 1 = O(n^3)}$. Note that $\boldsymbol{3n^2 - 3n + 1 = o(n^3)}$.

Example 3 (Ex. 1.4): Prove ***$3 \log n + \log\log n = O(\log n)$***

Proof) Show that $3 \log n + \log\log n \le c \cdot \log n$ *for any* $n \ge n_0$. for $c > 0$ and for any $n \ge n_0$.
Let us find the positive constants c and n_0 such that f(n) $3 \log n + \log\log n \le c \log n$ for $n \ge n_0$.

1) Since log $n < n$ for any $n \ge 1$, $\log\log n < \log n$ for any $n \ge 1$ after applying log function to both side. Then, $3 \log n + \log\log n < 3 \log n + \log n = 4 \log n$.
   So, let's choose $c = 4$
2) From 1), $3 \log n + \log\log n \le 4 \log n$ for any $n \ge n_0 = 1$. .

Thus, there exists a positive constant c = 3 and n_0 = 1 such that

$3 \log n + \log\log n \le c \cdot \log n$ *for any* $n \ge n_0$. for $c > 0$ and for any $n \ge n_0$.

Therefore, ***$3 \log n + \log\log n = O(\log n)$***

Alternative Proof)

$$\lim_{n\to\infty} \frac{3\log n + \log\log n}{\log n} = 3$$ is a constant.

Therefore, ***$5n \log n + 2n = \Omega(n \log n)$***

Example 4 (Ex. 1.5): Prove ***$2^{100} = O(1)$***

Proof) Show that $2^{100} \le c \cdot 1$ *for any* $n \ge n_0$. for $c > 0$ and for any $n \ge n_0$.
Let us find the positive constants c and n_0 such that $2^{100} \le c \cdot 1$ for $n \ge n_0$.

1) Since 2^{100} is a fixed constant value, let's choose $c = 2^{101}$. *Then,* $2^{100} \le 2^{101} \cdot 1$.
2) From 1), $2^{100} \le 2^{101} \cdot 1$ regardless n or for any $n \ge n_0 = 1$. .

Thus, there exists a positive constant c = 2^{101} and n_0 = 1 such that

$2^{100} \le c \cdot 1$ for $c > 0$ and for any $n \ge n_0$.

Therefore, ***$2^{100} = O(1)$***

Alternative Proof) $$\lim_{n\to\infty} \frac{2^{100}}{1} = 2^{100}$$ is a constant.

So, ***$2^{100} = O(1)$***

Example 5 (Ex. 1.8): Prove ***$2n^3 + 4n^2 \log n = O(n^3)$***

Proof) Assume that the base of log is 2.
Let's prove it using the properties of big-Oh in Theorem 1.7.

log n = O(n) since n > log n for any n > 0.
Since $4n^2 \log n = 4n^2 \times \log n$, $4n^2 \log n = O(n^2)O(n) = O(n^3)$ by Rule 3 in Thm. 1.7
Since $2n^3 = O(n^3)$, $2n^3 + 4n^2 \log n = O(n^3) + O(n^3) = \mathbf{max}(O(n^3), O(n^3)) = O(n^3)$.
Therefore, $2n^3 + 4n^2 \log n = O(n^3)$.

Example 6 (Ex. 1.6'): Prove ***$5n \log n + 2n = \Omega(n \log n)$***

Proof) Assume that the base of log is 2.

Show that $5n \log n + 2n \geq c \cdot n \log n$ *for any* $n \geq n_0$. for $c > 0$ and for any $n \geq n_0$.

Let us find the positive constants c and n_0 such that $5n \log n + 2n \geq c \cdot n \log n$ for $n \geq n_0$.

1) Since $5n \log n + 2n$ is an increasing function and $5n \log n \geq n \log n$ for any n it's reasonable to choose $c = 1$. *Then,* $5n \log n + 2n \geq n \log n$ for any n.
2) From 1), $5n \log n + 2n \geq n \log n$ for any $n \geq n_0 = 1$. .

Thus, there exists a positive constant $c = 1$ and $n_0 = 1$ such that

$5n \log n + 2n \geq n \log n$ for $c > 0$ and for any $n \geq n_0$.

Therefore, ***$5n \log n + 2n = \Omega(n \log n)$***

Alternative Proof)

$$\lim_{n\to\infty} \frac{n \log n}{5n \log n + 2n} = \lim_{n\to\infty} \frac{n \frac{1}{n} + \log n}{5n \cdot \frac{1}{n} + 5 \log n + 2} = \lim_{n\to\infty} \frac{1 + \log n}{7 + 5 \log n} = \frac{1/n}{5/n} = \frac{1}{5}$$ is a constant.

Therefore, ***$5n \log n + 2n = \Omega(n \log n)$***

Example 7 (Ex. 1.6''): Prove that ***$5n \log n + 2n = \Theta(n \log n)$.***

Proof) Assume that the base of log is 2.

Prove it both for ***$5n \log n + 2n = O(n \log n)$*** and for ***$5n \log n + 2n = \Omega(n \log n)$.***

Since $5n \log n + 2n = \Omega(n \log n)$ was proven above, let's prove $5n \log n + 2n = O(n \log n)$.

Show that $5n \log n + 2n \leq c \cdot n \log n$ *for any* $n \geq n_0$. for $c > 0$ and for any $n \geq n_0$.

Let us find the positive constants c and n_0 such that $5n \log n + 2n \leq c \cdot n \log n$ for $n \geq n_0$.

1) To decide c:

   When $n=2$, $5n \log n + 2n \leq c \cdot n \log n \Leftrightarrow 5 \cdot 2 \log 2 + 2 \cdot 2 \leq c\, 2 \cdot \log 2 \Leftrightarrow 14 \leq 2c \Leftrightarrow 7 \leq c$.

   When $n=4$, $5n \log n + 2n \leq c \cdot n \log n \Leftrightarrow 5 \cdot 4 \log 4 + 2 \cdot 4 \leq c\, 4 \cdot \log 4 \Leftrightarrow 48 \leq 8c \Leftrightarrow 6 \leq c$.

   etc.

   So, it's reasonable to choose $c = 7$. *Then,* $5n \log n + 2n \leq 7 n \log n$ for any n.
2) From 1), $5n \log n + 2n \leq 7n \log n \Leftrightarrow 2n \leq 2n \log n \Leftrightarrow 1 \leq \log n$ for any $n \geq n_0 = 2$.

Thus, there exists a positive constant $c = 7$ and $n_0 = 2$ such that

$5n \log n + 2n \leq 7 n \log n$ for $c > 0$ and for any $n \geq n_0$.

Hence, ***$5n \log n + 2n = O(n \log n)$.***

Since both $5n \log n + 2n = O(n \log n)$ and $5n \log n + 2n = \Omega(n \log n)$,

$n \log n \leq 5n \log n + 2n \leq 7 n \log n$ for $c_1 = 1$ and $c_2 = 7$ for any $n \geq n_0 = 2$.

$5n \log n + 2n = \Theta(n \log n)$. *Q.E.D.*