Handout 1: `CSci 242. Algorithms and Data Structures` **Spring, 2020**
Date: 1 - 15 - 2020 Instructor: Dr. M. E. Kim

Mathematical Facts

Refer to Appendix A of the textbook (pg.761-764).

1. **Summations:**

(a) $\sum_{i=1}^{n} c = cn \quad \text{vs.} \quad \sum_{i=0}^{n} c = c(n+1)$

(b) $\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(c) $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(d) $\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

(e) $\sum_{i=0}^{n} ar^i = a\sum_{i=0}^{n} r^i = \begin{cases} a\frac{(1-r^{n+1})}{(1-r)} & r < 1 \\ a\frac{(r^{n+1}-1)}{(r-1)} & r > 1 \end{cases}$

vs.

$$\sum_{i=1}^{n} ar^i = a\sum_{i=1}^{n} r^i = a(\sum_{i=0}^{n} r^i - r^0) = a(\sum_{i=0}^{n} r^i - 1) = \begin{cases} a\frac{r(1-r^n)}{1-r} & r < 1 \\ a\frac{r(r^n-1)}{r-1} & r > 1 \end{cases}$$

(f) $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}.$

(g) $\sum_{i=1}^{n} \frac{1}{i} = \ln n + c$ where c is a constant.

Proof:

(a) $\sum_{i=1}^{n} c = c\sum_{i=1}^{n} 1 = c(1+1+\cdots+1) = cn.$ Q.E.D.

(b) Let $\sum_{i=1}^{n} i$ be S.

$$\begin{array}{lll} & S & = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n \\ & S & = n + (n-1) + (n-2) + \cdots + 1 \\ + & \text{-----------------------------------} & \\ & 2S & = (n+1) + (n+1) + (n+1) + \cdots + (n+1) \\ & & = n \cdot (n+1) \end{array}$$

Therefore, $S = \frac{n(n+1)}{2}$. Q.E.D.

(e) Let $\sum_{i=0}^{n} ar^i$ be S.

$$S = \sum_{i=0}^{n} ar^i = a + ar^1 + ar^2 + \cdots + ar^{n-1} + r^n = a(1 + r^1 + r^2 + \cdots + r^{n-1} + r^n)$$

$$rS = a(r^1 + r^2 + r^3 + \cdots + r^n + r^{n+1})$$

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$$(1-r)S = a(1 - r^{n+1})$$

Therefore, $S = a \cdot \frac{(1-r^{n+1})}{1-r}$. Q.E.D.

2. **Exponents:**

The following identities hold for exponents with all real $a > 0,\ b > 0,\ c > 0$ and $n \in \mathcal{N}$:

(a) $a^0 = 1$
(b) $a^1 = a$
(c) $a^{-1} = 1/a$
(d) $(b^a)^c = (b^c)^a = b^{ac}$
(e) $b^a\ b^c = b^{a+c}$
(f) $b^a / b^c = b^{a-c}$

3. **Logarithms:**

The logarithm function is defined as

$$\log_b a = c \quad \text{if} \quad a = b^c.$$

The following identities hold for logarithms with all real $a > 0,\ b > 0,\ c > 0$ and $n \in \mathcal{N}$:

(a) $b^c = a \iff c = log_b\ a$
(b) $log_b\ 1 = 0$

(c) $log_b\ (ac) = log_b\ a + log_b\ c$
(d) $log_b\ (a/c) = log_b\ a - log_b\ c$

(e) $log_b^c\ a = (log_b\ a)^c$
(f) $log_b\ a^c = c\ log_b\ a$

(g) $log_b\ a = \frac{log_c\ a}{log_c\ b}$
(h) $log_b\ (1/a) = -\ log_b\ a$

(i) $log_b\ a = \frac{1}{log_a\ b}$
(j) $a^{log_b\ c} = c^{log_b\ a}$

4. **Integer Functions and Relations**

   (a) Floor and Ceiling:

      i. $\lfloor x \rfloor$ = the largest integer $\leq x$

      ii. $\lceil x \rceil$ = the smallest integer $\geq x$

   (b) Modulo function: The Remainder after Division

   $a \bmod b = a - \lfloor \frac{a}{b} \rfloor \cdot b.$

   (c) Factorial function:

   $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n.$

   (d) Binomial Coefficient:

   $\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$

5. **Derivatives:**

   (a) $\frac{d}{dx} e^{f(x)} = \frac{df(x)}{dx} e^{f(x)} = f'(x) \cdot e^{f(x)}$

   (b) $\frac{d}{dx} x^n = n \cdot x^{n-1}$

   (c) $\frac{d}{dx} \ln f(x) = \frac{\frac{d}{dx} f(x)}{f(x)} = \frac{f'(x)}{f(x)}$; so, $\frac{d}{dx} \ln x = \frac{1}{x}$

6. **L'Hôspital's Rule:**

   Let f and g be differentiable functions. Suppose that

   (a) as $x \to a$,

   either (i) $f(x) \to 0$ and $g(x) \to 0$

   or (ii) $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$;

   AND

   (b) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

   Then, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

   Example:

$$\lim_{x \to 1} \frac{x^3 + x^2 - 5x + 3}{x^3 - 7x^2 + 11x - 5} = \lim_{x \to 1} \frac{3x^2 + 2x^1 - 5}{3x^2 - 14x^1 + 11} = \lim_{x \to 1} \frac{6x^1 + 2}{6x^1 - 14} = \frac{8}{-8} = -1$$

7. **The Principle of (Mathematical) Induction**

In order to prove a property $P(n)$ for all $n \in \mathcal{N}$, it is sufficient to prove

(a) *Base Case:* $P(0)$ (or $P(1)$) is true; and

(b) *Induction Step:* If $P(k)$ holds for all $k \leq n$, then $P(n+1)$ is true.

Example:

Prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ using Mathematical Induction.

Proof:

Let a property $P(n)$ be $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Base Case:

For $n = 0$, $\sum_{i=0}^{0} i^2 = 0^2 = 0$ – LHD.

$$\sum_{i=0}^{0} i^2 = \frac{0(0+1)(2*0+1)}{6} = \frac{0}{6} = 0 \quad \text{– RHD.} \qquad \text{So, LHD = RHD}$$

For $n = 1$, $\sum_{i=0}^{1} i^2 = 0^2 + 1^2 = 1$ – LHD.

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2*1+1)}{6} = \frac{6}{6} = 1 \quad \text{– RHD.} \qquad \text{So, LHD = RHD}$$

Therefore, both $P(0)$ and $P(1)$ are true.

Inductive Step:

Show that if the above $P(k)$ holds for all $k \leq n$, then $P(n+1)$ is true.

Inductive Hypothesis:

Assume that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ is true for $k \leq n$.

Then, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, for $k = n$.

Claim:

Show that $P(k)$ is true for $k = n+1$, i.e.

$$P(n+1) = \sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} \text{ is true,}$$

using the above *Inductive Hypothesis* (and the *Base Case*).

$$\begin{aligned} P(n+1) &= \sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \text{from the \textit{Inductive Hypothesis}} \\ &= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} \\ &= \frac{(n+1)((2n^2+n)+(6n+6))}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \end{aligned}$$

Therefore, $P(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ holds for all $n \in \mathcal{N}$. Q.E.D.