

COM 3240

# REINFORCEMENT LEARNING



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# OPTIMISATION

## GRADIENT DESCENT

**Desirable:** A system that performs a specific task

The system has parameters that we need to select appropriately (optimise) for that task

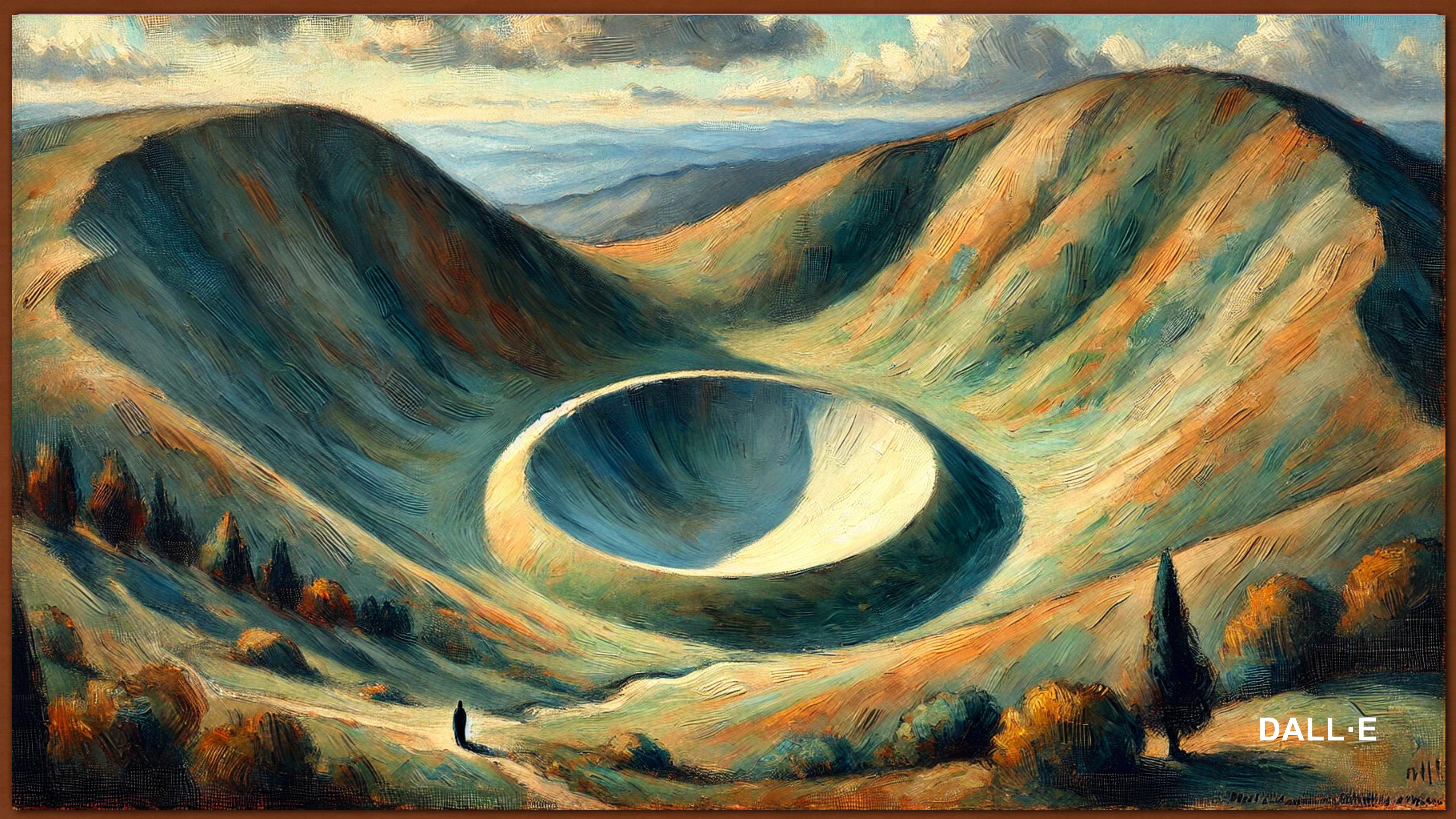
$f(x_1, x_2, \dots, x_n)$   
parameters

$f$  System's performance: maximise  
System's error : minimise

# OPTIMISATION

## ARTIFICIAL NEURAL NETWORKS

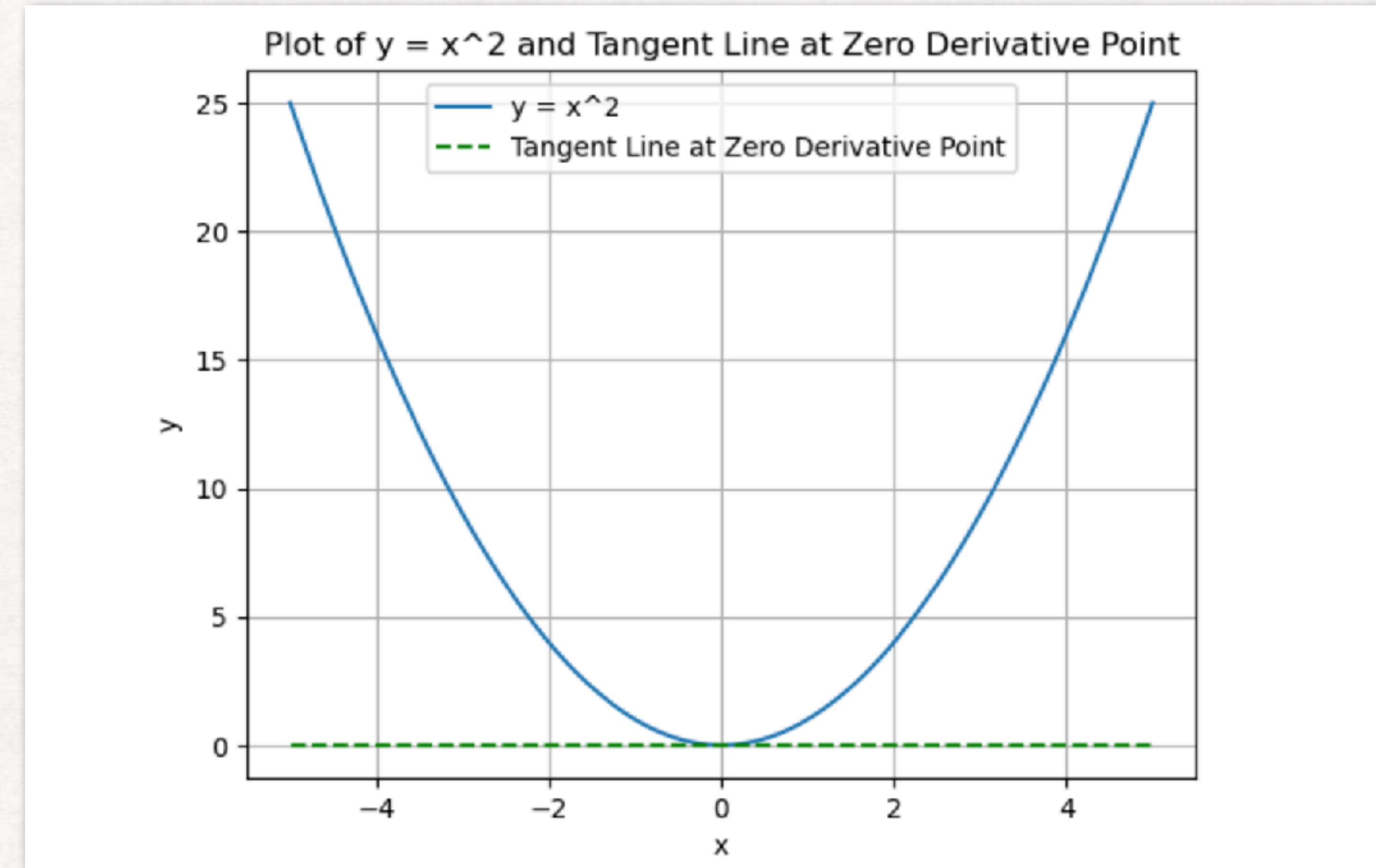




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# DERIVATIVES

## OPTIMISATION - GRADIENT METHODS

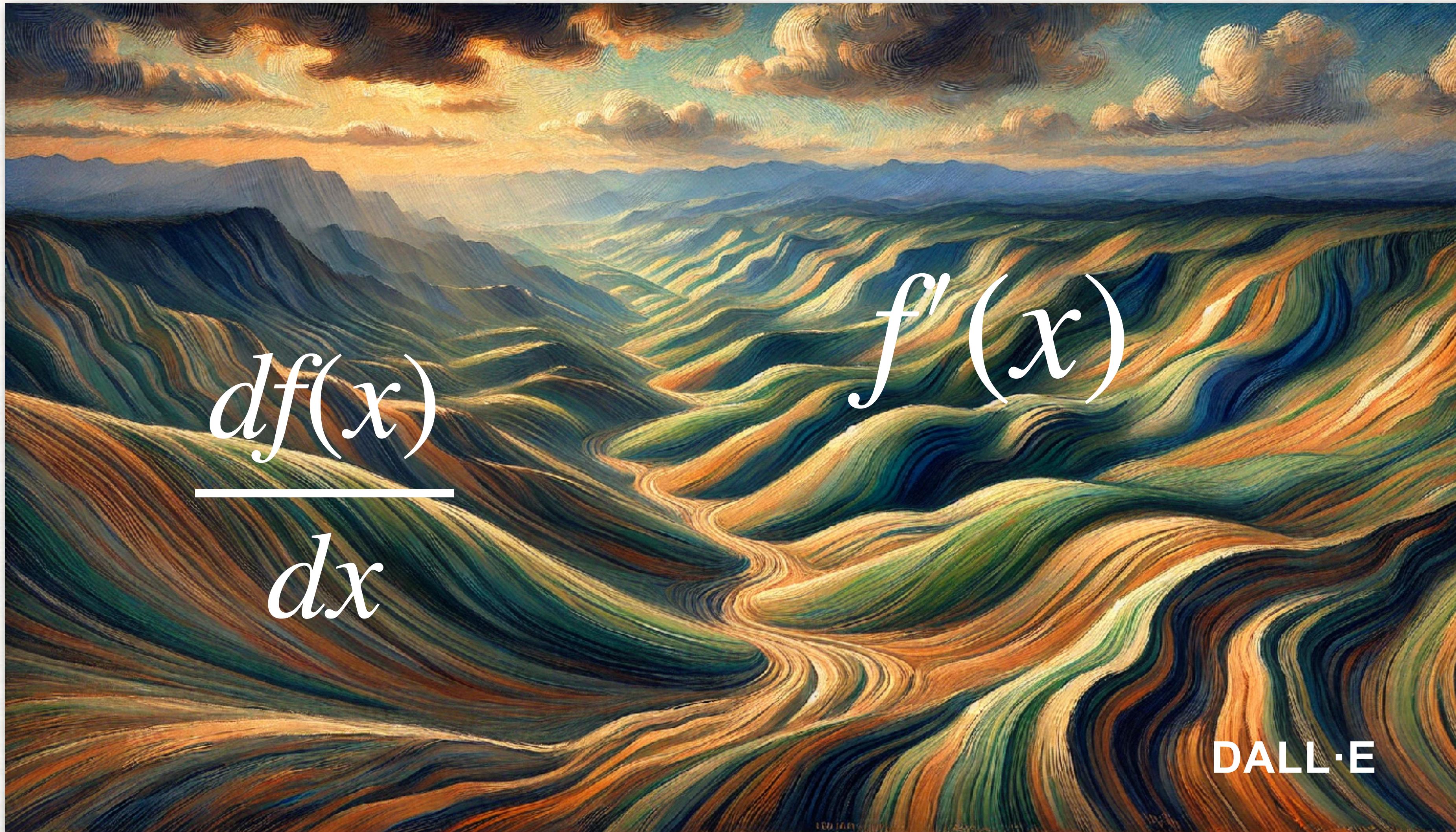


Slope

Error (Loss) function - Machine Learning

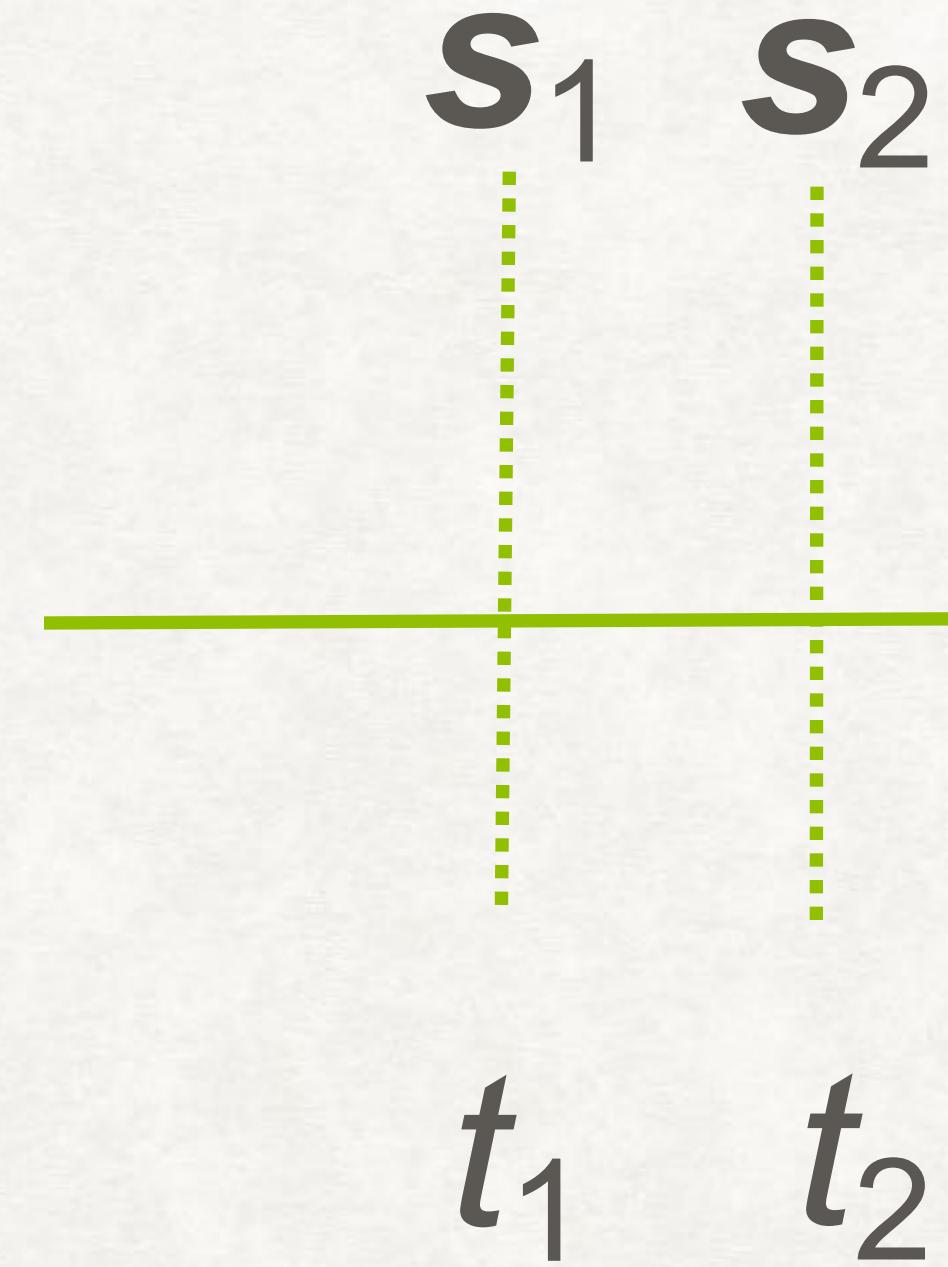
# DERIVATIVES

## OPTIMISATION - GRADIENT METHODS



A painting of a landscape with a long line of dominoes.

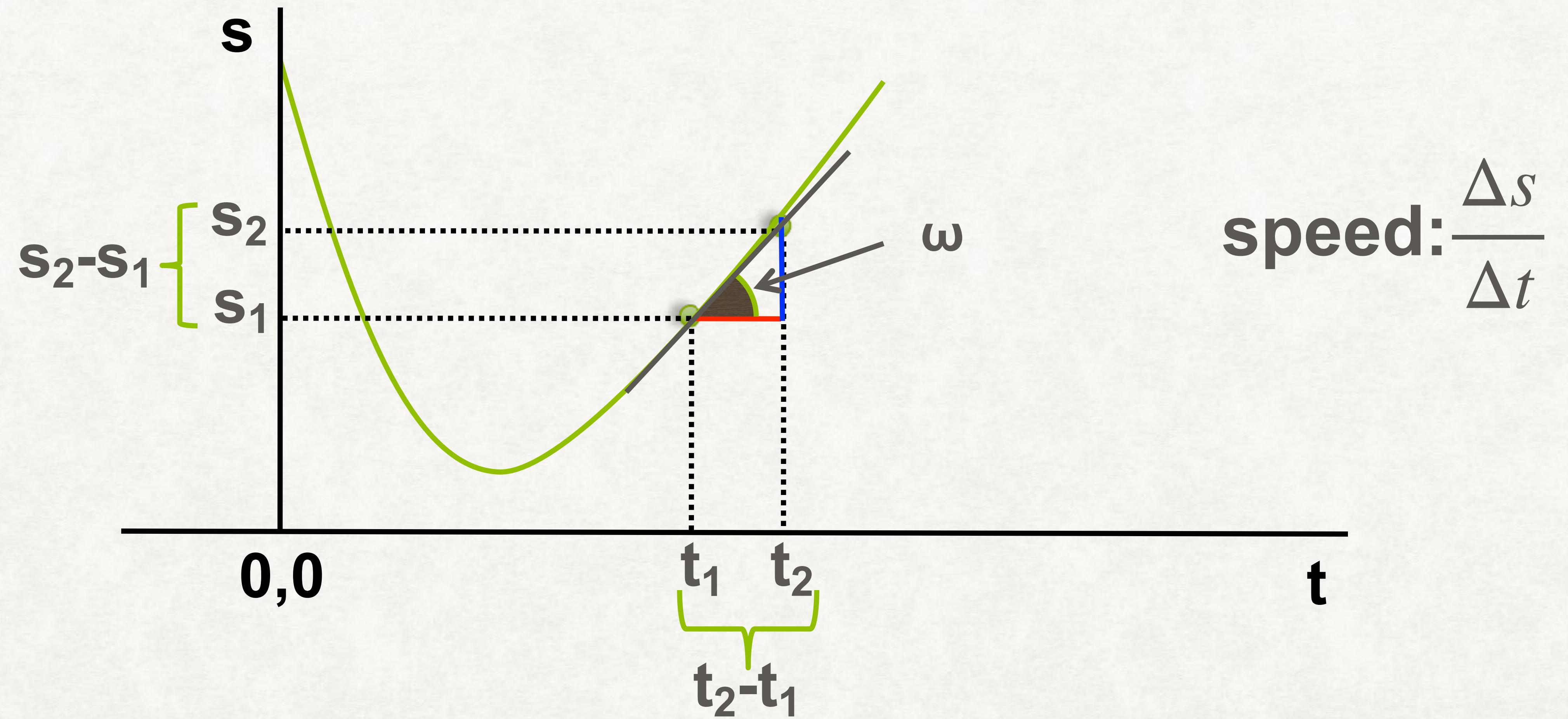
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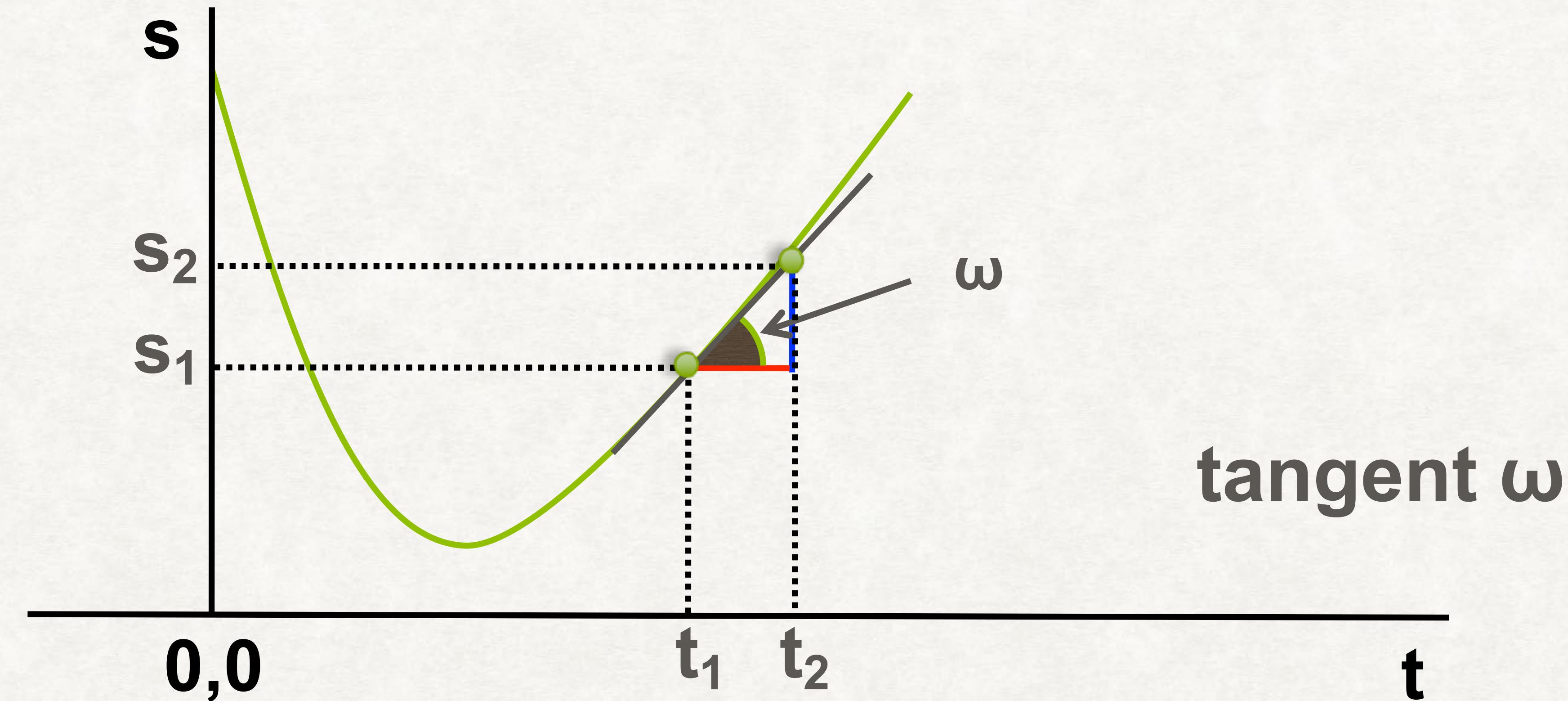
**speed:**  $\frac{\Delta s}{\Delta t}$

$$\Delta s = s_2 - s_1$$

$$\Delta t = t_2 - t_1$$



$$\frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$



# DERIVATIVES

USE THE DEFINITION TO CALCULATE DERIVATIVES OF FUNCTIONS

More generally:

$$\frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

What is the effect of incrementing  $x$  by a small amount,  $\Delta x$ , on the value of  $f(x)$ ?

$$f(x) = 7$$

$$f(x) = 7x$$

$$f(x) = 7x^2$$

# DERIVATIVES IN PRACTICE

## MEMORISE BASIC FUNCTIONS AND USE RULES

**Power function:**  $f(x) = x^n$        $f'(x) = nx^{n-1}$

**Exponential function:**  $f(x) = e^x$        $f'(x) = e^x$

**Natural logarithm:**  $f(x) = \ln(x)$        $f'(x) = \frac{1}{x}$

# DERIVATIVES - RULES

## ADDITION

$$\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$h(x) = x + x^2$$

# DERIVATIVES - RULES

## MULTIPLICATION

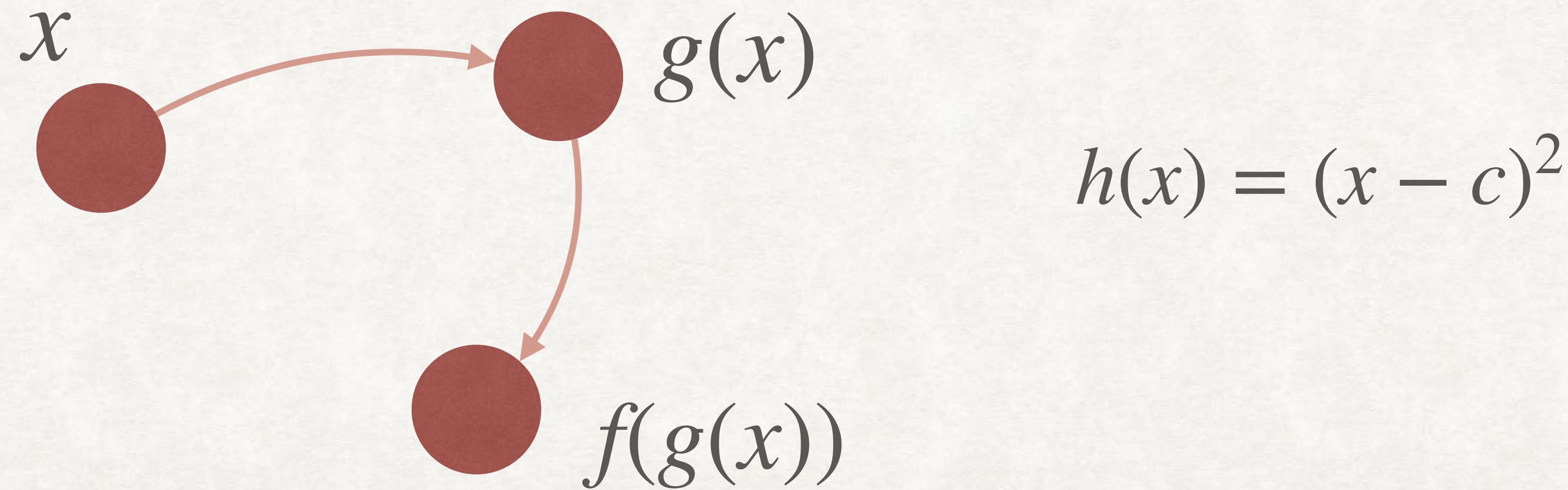
$$\frac{d}{dx} (f(x) \cdot g(x)) = \frac{df(x)}{dx} \cdot g(x) + f(x) \cdot \frac{dg(x)}{dx}$$

$$h(x) = x \cdot \ln(x)$$

# DERIVATIVES - RULES

## THE CHAIN RULE

$$\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$$



# MOTIVATION FOR PARTIAL DERIVATIVES

## LIGHTS AND BUTTONS PUZZLE



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$$f(x, y) = x^2y + y^2$$

A more general way to  
write the same:

$$f(x_1, x_2) = x_1^2x_2 + x_2^2$$

How do we estimate the gradient?

# PARTIAL DERIVATIVES AND THE NABLA NOTATION

$f(x_1, x_2, \dots)$

$$\frac{\partial f}{\partial x_1} = \lim_{\Delta x_1 \rightarrow 0} \frac{f(x_1 + \Delta x_1, x_2, \dots) - f(x_1, x_2, \dots)}{\Delta x_1}$$

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right)$$

$$\frac{\partial f}{\partial x_2} = \lim_{\Delta x_2 \rightarrow 0} \frac{f(x_1, x_2 + \Delta x_2, \dots) - f(x_1, x_2, \dots)}{\Delta x_2}$$

# PARTIAL DERIVATIVES AND THE NABLA NOTATION

**One variable at the time, treat others as constants!**

# PARTIAL DERIVATIVES AND THE NABLA NOTATION

$$f(x_1, x_2) = x_1^2 x_2 + x_2^2$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^2 x_2 + x_2^2) = 2x_1 x_2$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2}(x_1^2 x_2 + x_2^2) = x_1^2 + 2x_2$$

# INTRODUCE (PARTIAL) DIFFERENTIAL EQUATIONS

## EVERYTHING CHANGES

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) = 0$$

**Unknown:**  $y(x)$

$$F\left(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right) = 0$$

$u(x_1, x_2, \dots)$

# DIFFERENTIAL EQUATIONS

## RELAXATION

$$\tau \frac{dx(t)}{dt} = -x(t)$$

**$x(t)$  : here it is a function of time  $f(t) = x(t)$ ; not a variable!**

**$\tau$  : time constant, if a large value you relax slow (you have a long memory:)**

# DIFFERENTIAL EQUATIONS

## LEAKY INTEGRATOR

$$\tau \frac{dx(t)}{dt} = -x(t) + s(t)$$

$x(t)$  : a function

$s(t)$  : sample, input

$t$  : variable



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# DEFINITION OF DERIVATIVES AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$\frac{dx(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Approximate the derivative as difference.

$$\tau \frac{dx(t)}{dt} = -x(t) + s(t)$$

You need one initial point and a time step.

e.g.,  $x(0), \Delta t = 0.1$  (seconds)

Its discrete form is known as “exponential moving average”

# DEFINITION OF DERIVATIVES AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$\frac{dx(t)}{dt} \approx \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Approximate the derivative as difference.

$$\tau \frac{x(t + \Delta t) - x(t)}{\Delta t} = -x(t) + s(t)$$

You need one initial point and a time step.

e.g.,  $x(0), \Delta t = 0.1$  (seconds)

Its discrete form is known as “exponential moving average”

# DEFINITION OF DERIVATIVES AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$x(t + \Delta t) - x(t) = \frac{\Delta t}{\tau} (-x(t) + s(t))$$

Only unknown!

e.g.  $x(t = 0)$ ,  $\Delta t = 0.1$  (seconds)

Euler's method: I can start from, e.g. time 0 (sec)  
and iteratively calculate the value of x at any time

# DEFINITION OF DERIVATIVES AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$x(t + \Delta t) = x(t) \left( 1 - \frac{\Delta t}{\tau} \right) + \frac{\Delta t}{\tau} s(t)$$

e.g.  $x(t = 0), \Delta t = 0.1$  (seconds)

Useful equation for “forgetful” averaging, filtering high frequencies (noise)

# DEFINITION OF DERIVATIVES AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS



$$\tau \frac{d\mathbf{w}(t)}{dt} = - \frac{\partial \mathcal{L}(t)}{\partial \mathbf{w}(t)}$$

$$\Delta \mathbf{w}(t) = - \eta \frac{\partial \mathcal{L}(t)}{\partial \mathbf{w}(t)}$$

$\eta$ : learning rate

# SUMMARY

## PREREQUISITES

- Concept of Optimisation, Gradient Learning
- Derivatives & partial derivatives
- Differential equations, Euler's method & leaky integrators

# **WARNING!**

## **GENERATIVE AI AND THEORY**

**Unless specialised in maths and reasoning, answers to questions regarding theory might be wrong!**

**Safe to use in programming because it is possible to test the answer.**

**THANK YOU!**