COM 3240

# REINFORCEMENT LEARNING


DALL·E

# OPTIMISATION

## GRADIENT DESCENT

**Desirable: A system that performs a specific task**

**The system has parameters that we need to select appropriately (optimise) for that task**

$$f(x_1, x_2, \ldots, x_n)$$

**parameters**

$$f$$

**System's performance: maximise**

**System's error: minimise**

# OPTIMISATION

## ARTIFICIAL NEURAL NETWORKS


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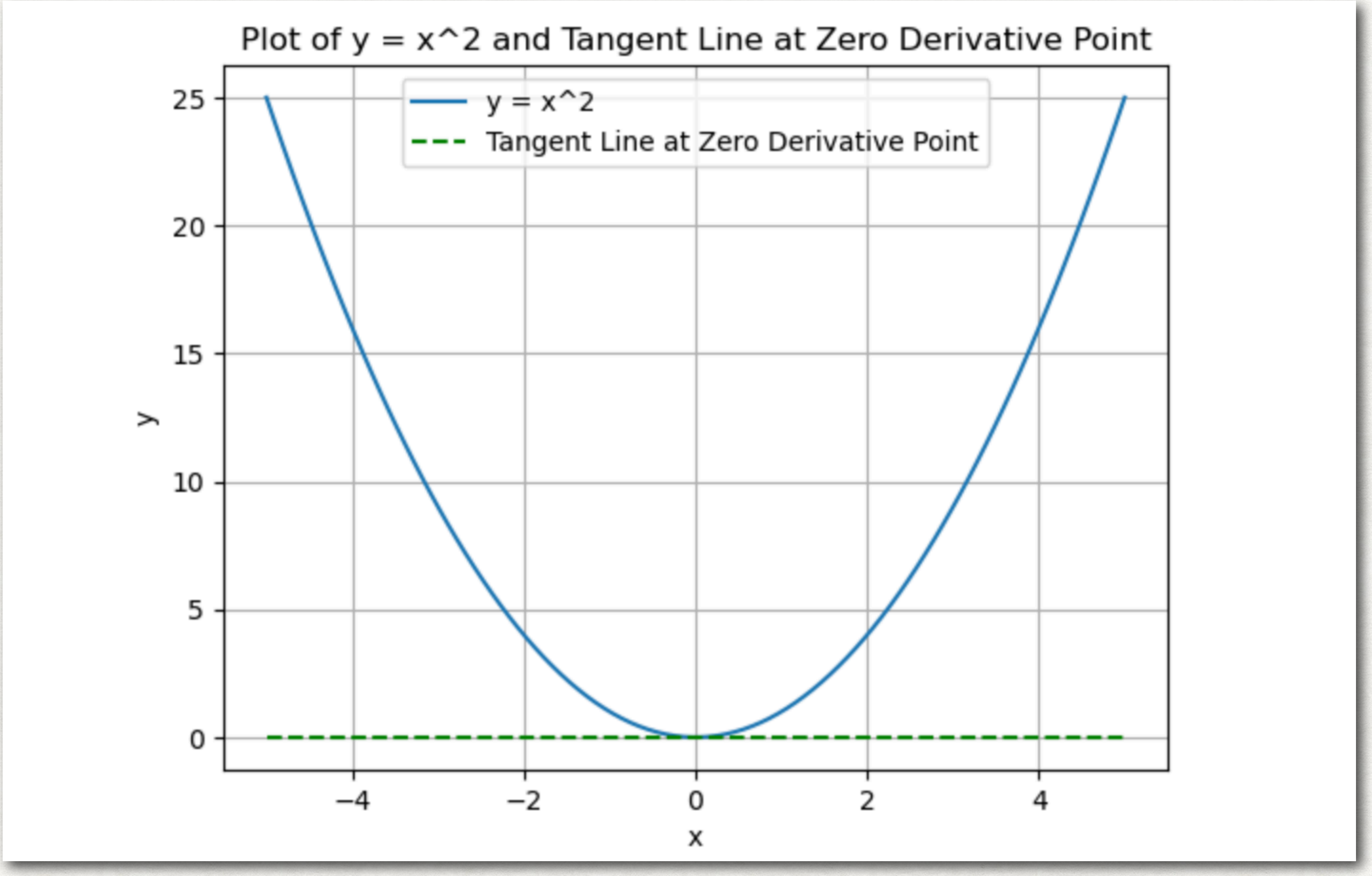
# DERIVATIVES

## OPTIMISATION - GRADIENT METHODS

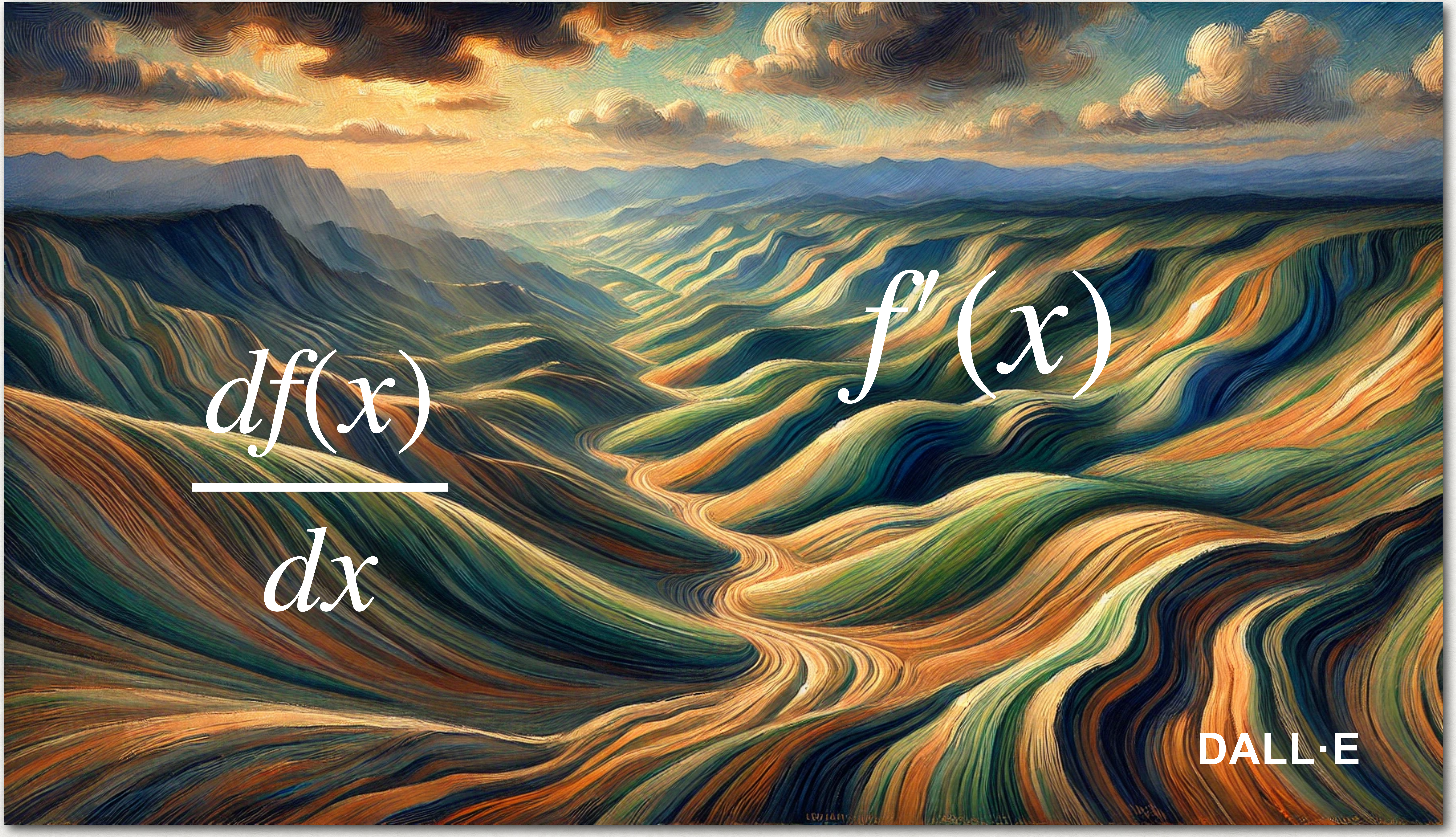



**Slope**

**Error (Loss) function - Machine Learning**

# DERIVATIVES

## OPTIMISATION - GRADIENT METHODS

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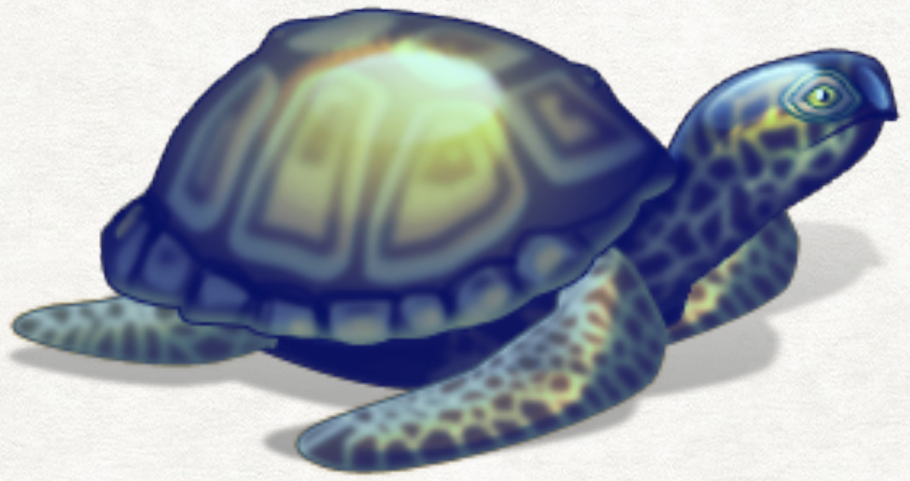


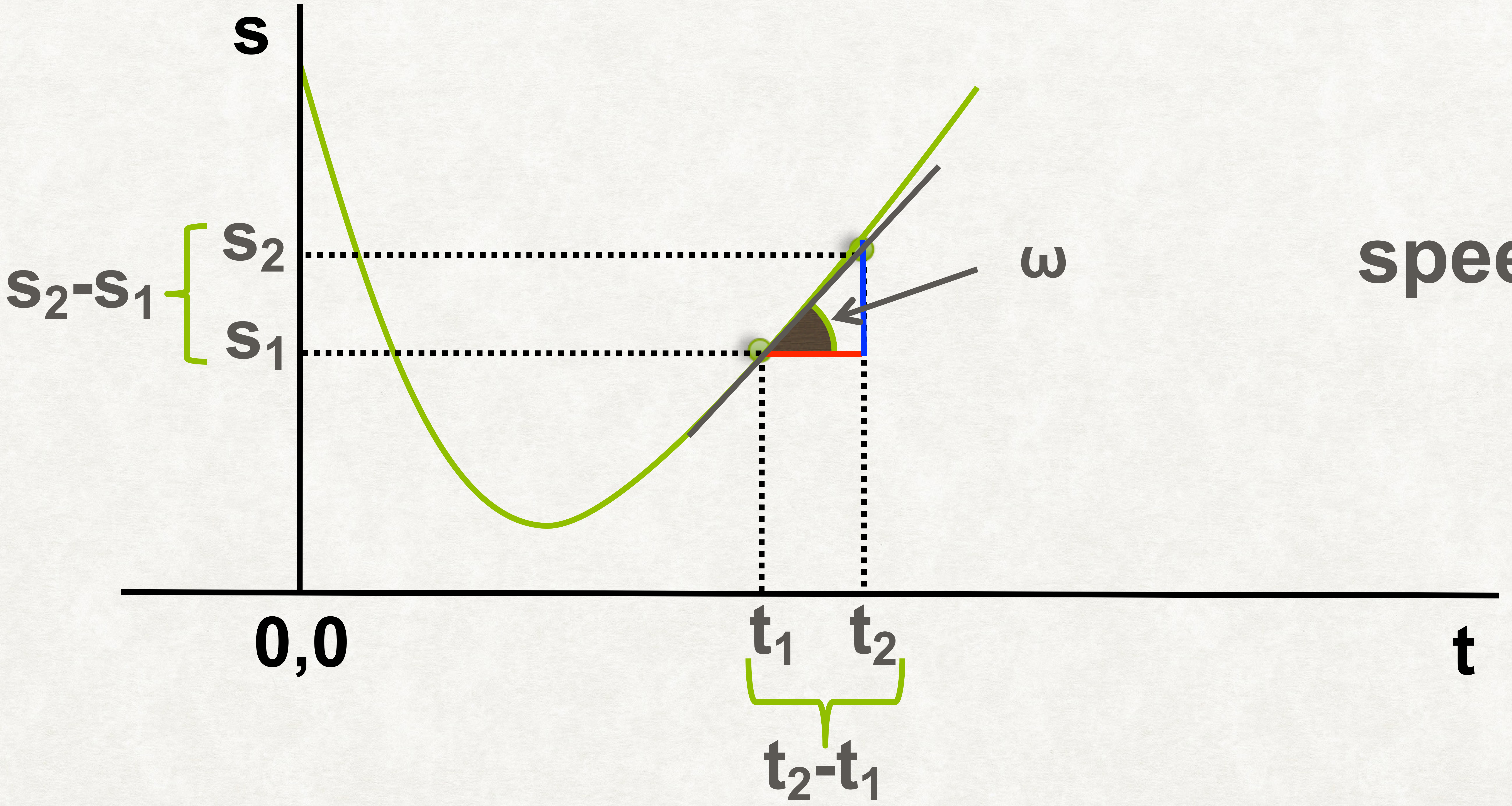
speed: $\dfrac{\Delta s}{\Delta t}$

$s_1$ $s_2$

$t_1$ $t_2$

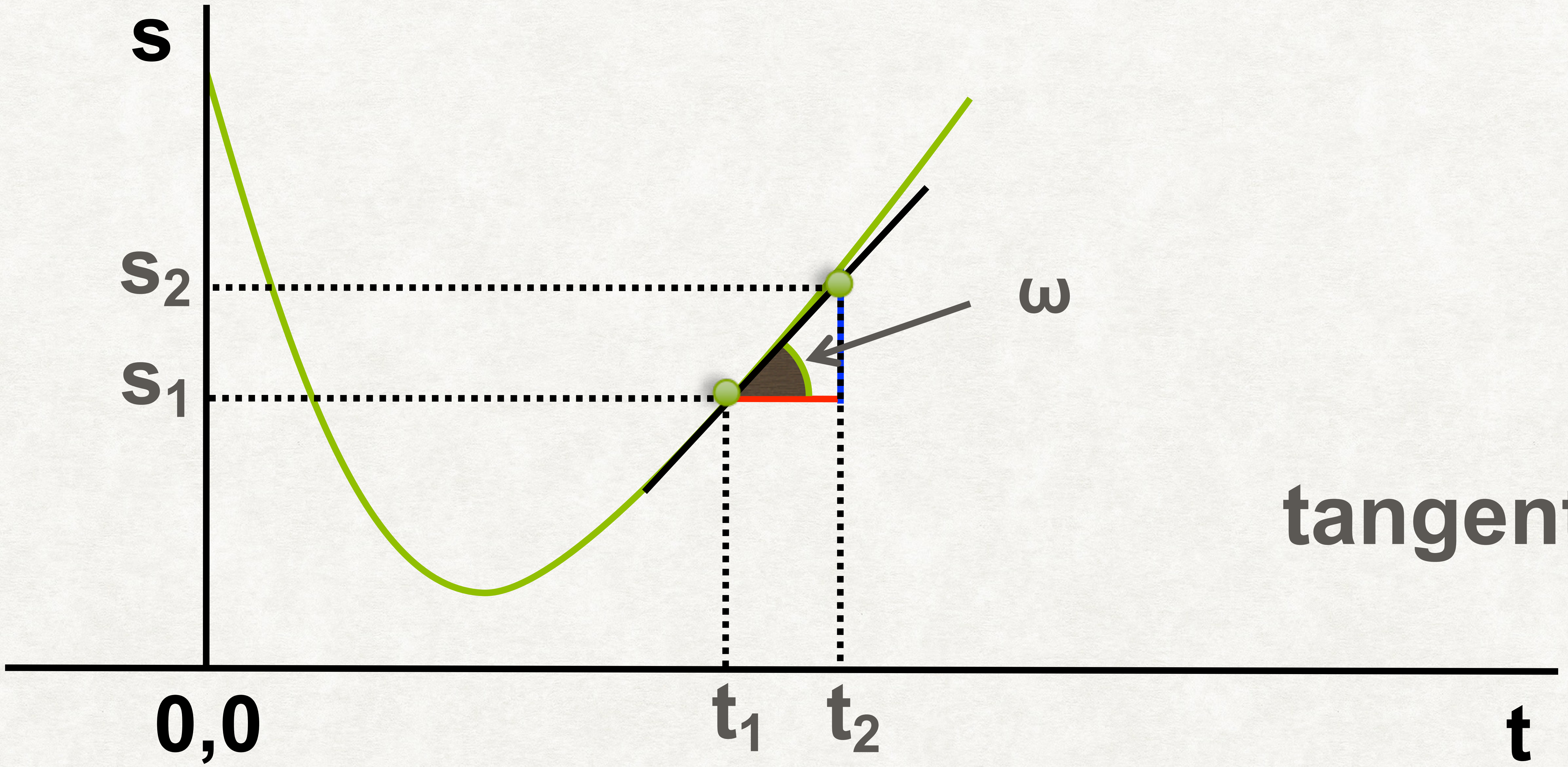
$$\Delta s = s_2 - s_1$$

$$\Delta t = t_2 - t_1$$




speed: $\dfrac{\Delta s}{\Delta t}$

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$




**tangent ω**

# DERIVATIVES

## USE THE DEFINITION TO CALCULATE DERIVATIVES OF FUNCTIONS

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

**More generally:**

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

**What is the effect of incrementing x by a small amount, Δx, on the value of f(x)?**

$$f(x) = 7$$

$$f(x) = 7x$$

$$f(x) = 7x^2$$

# DERIVATIVES IN PRACTICE

## MEMORISE BASIC FUNCTIONS AND USE RULES

**Power function:** $f(x) = x^n$ $\qquad f'(x) = nx^{n-1}$

**Exponential function:** $f(x) = e^x$ $\qquad f'(x) = e^x$

**Natural logarithm:** $f(x) = \ln(x)$ $\qquad f'(x) = \dfrac{1}{x}$

# DERIVATIVES - RULES

## ADDITION

$$\frac{d}{dx}\Big(f(x) + g(x)\Big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$h(x) = x + x^2$$

# DERIVATIVES - RULES

## MULTIPLICATION

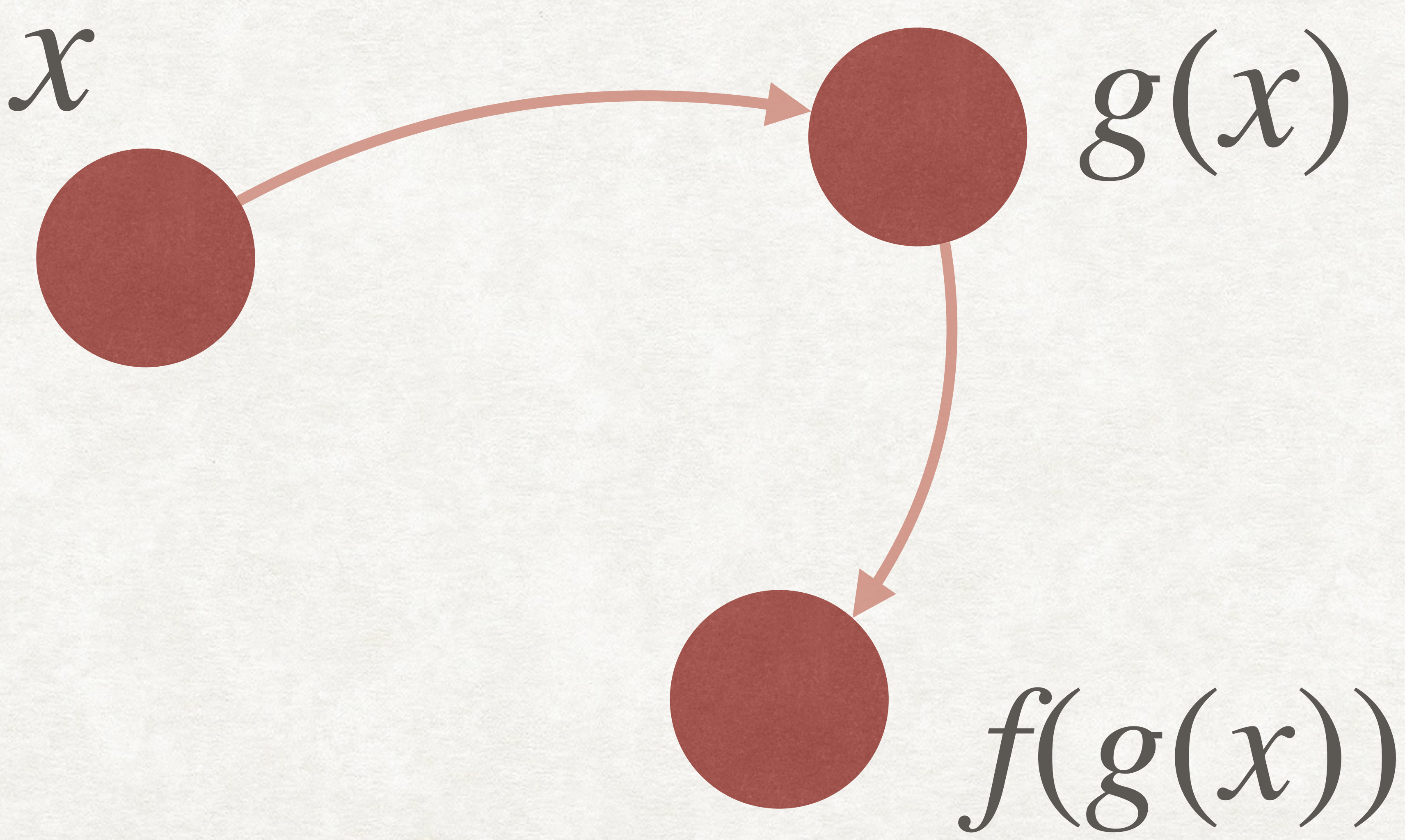
$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{df(x)}{dx} \cdot g(x) + f(x) \cdot \frac{dg(x)}{dx}$$

$$h(x) = x \cdot ln(x)$$

# DERIVATIVES - RULES

## THE CHAIN RULE

$$\frac{df\left(g(x)\right)}{dx} = \frac{df\left(g(x)\right)}{dg(x)} \cdot \frac{dg(x)}{dx}$$




$$h(x) = (x - c)^2$$

# MOTIVATION FOR PARTIAL DERIVATIVES

## LIGHTS AND BUTTONS PUZZLE

$$f(x, y) = x^2 y + y^2$$

**A more general way to write the same:**

$$f(x_1, x_2) = x_1^2 x_2 + x_2^2$$

**How do we estimate the gradient?**

# PARTIAL DERIVATIVES

## AND THE NABLA NOTATION

$$f(x_1, x_2, \ldots)$$

$$\frac{\partial f}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2, \ldots) - f(x_1, x_2, \ldots)}{\Delta x_1}$$

$$\frac{\partial f}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2, \ldots) - f(x_1, x_2, \ldots)}{\Delta x_2}$$

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right)$$

# PARTIAL DERIVATIVES

## AND THE NABLA NOTATION

**One variable at a time, treat others as constants!**

# PARTIAL DERIVATIVES

## AND THE NABLA NOTATION

$$f(x_1, x_2) = x_1^2 x_2 + x_2^2$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^2 x_2 + x_2^2) = 2x_1 x_2$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2}(x_1^2 x_2 + x_2^2) = x_1^2 + 2x_2$$

# INTRODUCE (PARTIAL) DIFFERENTIAL EQUATIONS

## EVERYTHING CHANGES

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right) = 0$$

**Unknown:** $y(x)$

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0$$

$u(x_1, x_2, \ldots)$

# DIFFERENTIAL EQUATIONS

## RELAXATION

$$\tau \frac{dx(t)}{dt} = -\,x(t)$$

$x(t)$ : **here it is a function of time** $f(t) = x(t)$**; not a variable!**

$\tau$ : **time constant, if it is large, you relax slowly (you have a long memory:)**

# DIFFERENTIAL EQUATIONS

## LEAKY INTEGRATOR

$$\tau \frac{dx(t)}{dt} = -x(t) + s(t)$$

$x(t)$ : **a function**

$s(t)$ : **sample, input**

$t$ : **variable**

# DEFINITION OF DERIVATIVES

## AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$\frac{dx(t)}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

**Approximate the derivative as difference.**

$$\tau \frac{dx(t)}{dt} = -x(t) + s(t)$$

**You need one initial point and a time step.**

**e.g.,** $x(0), \Delta t = 0.1$ **(seconds)**

**Its discrete form is known as "exponential moving average"**

# DEFINITION OF DERIVATIVES

## AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$\frac{dx(t)}{dt} \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

**Approximate the derivative as difference.**

$$\tau \frac{x(t+\Delta t) - x(t)}{\Delta t} = -x(t) + s(t)$$

**You need one initial point and a time step.**

**e.g.,** $x(0), \Delta t = 0.1$ **(seconds)**

**Its discrete form is known as "exponential moving average"**

# DEFINITION OF DERIVATIVES

## AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$x(t + \Delta t) = x(t) + \frac{\Delta t}{\tau}\left(-x(t) + s(t)\right)$$

**Only unknown!**

**Euler's method: I can start from, e.g. time 0 (sec) and iteratively calculate the value of x at any time**

# DEFINITION OF DERIVATIVES

## AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$x(t + \Delta t) = x(t)\left(1 - \frac{\Delta t}{\tau}\right) + \frac{\Delta t}{\tau} s(t)$$

**Useful equation for "forgetful" averaging, filtering high frequencies (noise)**

# DEFINITION OF DERIVATIVES

## AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

$$\tau \frac{d\mathbf{w}(\mathbf{t})}{dt} = - \frac{\partial \mathscr{L}(\mathbf{t})}{\partial \mathbf{w}(\mathbf{t})}$$

$$\Delta \mathbf{w}(\mathbf{t}) = - \eta \frac{\partial \mathscr{L}(\mathbf{t})}{\partial \mathbf{w}(\mathbf{t})}$$

$\eta \approx \Delta t / \tau$: **learning rate**

# SUMMARY

## PREREQUISITES

- Concept of Optimisation, Gradient Learning
- Derivatives & partial derivatives
- Differential equations, Euler's method & leaky integrators

# THANK YOU!