

Different priors

1 Dirichlet Process

1.1 Class Model (Stochastic Block Model, SBM)

Likelihood

$$\begin{aligned} y_{i,i'} \mid \xi_i, \xi_{i'}, \{\eta_{k,\ell}\} &\stackrel{\text{ind}}{\sim} \text{Ber}(\text{expit } \eta_{\phi(\xi_i, \xi_{i'})}) \\ p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) &= \prod_{i=1}^{I-1} \prod_{i'=i+1}^I (\text{expit } \eta_{\phi(\xi_i, \xi_{i'})})^{y_{i,i'}} (1 - \text{expit } \eta_{\phi(\xi_i, \xi_{i'})})^{1-y_{i,i'}} \\ &= \prod_{k=1}^K \prod_{\ell=k}^K (\text{expit } \eta_{k,\ell})^{s_{k,\ell}} (1 - \text{expit } \eta_{k,\ell})^{n_{k,\ell} - s_{k,\ell}} \end{aligned}$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$, $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i, i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k, \ell)\}$, and $\phi(x, y) = (\min\{x, y\}, \max\{x, y\})$ is a function to take into account that $\mathbf{Y} = [y_{i,i'}]$ is a symmetric matrix.

Prior

$$\begin{aligned} \eta_{k,\ell} \mid \zeta, \tau^2 &\stackrel{\text{iid}}{\sim} \text{N}(\zeta, \tau^2) \\ \zeta &\sim \text{N}(\mu_\zeta, \sigma_\zeta^2) \\ \tau^2 &\sim \text{IGam}(a_\tau, b_\tau) \\ \xi_i \mid \alpha &\sim \text{CRP}(\alpha) \\ \alpha &\sim \text{might add this later} \end{aligned}$$

Parameters

$$\Upsilon = (\eta_{1,1}, \eta_{1,2}, \dots, \eta_{K,K}, \xi_1, \dots, \xi_K, \zeta, \tau^2, \alpha),$$

where $\xi_i \in \{1, \dots, K\}$, $i = 1, \dots, I$, are the cluster assignments ($\xi_i = k$ means that actor i belongs to cluster k).

Hyper-parameters

$$(\mu_\zeta, \sigma_\zeta^2, a_\tau, b_\tau).$$

Posterior

$$p(\Upsilon \mid \mathbf{Y}) = p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) p(\{\eta_{k,\ell}\} \mid \zeta, \tau^2) p(\zeta) p(\tau^2) p(\{\xi_i\} \mid \alpha) p(\alpha)$$

$$\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^I \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\frac{1}{2\sigma_\zeta^2}(\zeta - \mu_\zeta)^2\right\} \times (\tau^2)^{-(a_\tau-1)} \exp\left\{-\frac{b_\tau}{\tau^2}\right\} \\ \times \prod_{k=1}^K \prod_{\ell=k}^K (\tau^2)^{-1/2} \exp\left\{-\frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2\right\} \times \begin{cases} n_k, & \text{for } k = 1, \dots, K \\ \alpha, & \text{for } k = K + 1 \end{cases}$$

where $\theta_{i,i'} = \text{expit}(\eta_{\phi(\xi_i, \xi_{i'})})$.

MCMC

The algorithm proceeds by generating a new state $\Upsilon^{(b+1)}$ from a current state $\Upsilon^{(b)}$, $b = 1, \dots, B$, as follows:

1. Sample $\eta_{k,\ell}^{(b+1)}$, $\ell = k, \dots, K$ and $k = 1, \dots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\log p(\eta_{k,\ell} \mid \text{rest}) \propto s_{k,\ell} \log(\text{expit } \eta_{k,\ell}) + (n_{k,\ell} - s_{k,\ell}) \log(1 - \text{expit } \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2 \\ = s_{k,\ell} \eta_{k,\ell} - n_{k,\ell} \log(1 + \exp \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2,$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$ and $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i, i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k, \ell)\}$.

2. Sample $\xi_i^{(b+1)}$, $i = 1, \dots, I$, from a categorical distribution prop to $\begin{cases} n_k, & \text{for } k = 1, \dots, K \\ \alpha, & \text{for } k = K + 1 \end{cases}$
3. Sample $\zeta^{(b+1)}$ from $\mathbf{N}(m, v^2)$, where

$$v^2 = \left(\frac{1}{\sigma_\zeta^2} + \frac{H(H+1)/2}{\tau^2} \right)^{-1} \quad \text{and} \quad m = v^2 \left(\frac{\mu_\zeta}{\sigma_\zeta^2} + \frac{1}{\tau^2} \sum_{k=1}^K \sum_{\ell=k}^K \eta_{k,\ell} \right).$$

4. Sample $(\tau^2)^{(b+1)}$ from $p(\tau^2 \mid \text{rest}) = \text{IGam}\left(\tau^2 \mid a_\tau + \frac{H(H+1)}{4}, b_\tau + \frac{1}{2} \sum_{k=1}^K \sum_{\ell=k}^K (\eta_{k,\ell} - \zeta)^2\right)$.

1.2 Class - Distance Model

Likelihood

$$y_{i,i'} \mid \zeta, \{\mathbf{u}_k\}, \xi_i, \xi_{i'} \stackrel{\text{ind}}{\sim} \text{Ber}(\text{expit } \eta_{\phi(\xi_i, \xi_{i'})})$$

where

$$\eta_{k,\ell} = \zeta - \|\mathbf{u}_k - \mathbf{u}_\ell\|$$

Prior

$$\begin{aligned}
\mathbf{u}_k &| \sigma^2 \stackrel{\text{iid}}{\sim} \mathbf{N}(\mathbf{0}, \sigma^2 \mathbf{I}) \\
\sigma^2 &\sim \text{IGam}(a_\sigma, b_\sigma) \\
\zeta &| \omega^2 \sim \mathbf{N}(0, \omega^2) \\
\omega^2 &\sim \text{IGam}(a_\omega, b_\omega) \\
\xi_i &| \alpha \sim \text{CRP}(\alpha) \\
\alpha &\sim \text{might add this later}
\end{aligned}$$

Parameters

$$\mathbf{\Upsilon} = (\mathbf{u}_1, \dots, \mathbf{u}_K, \sigma^2, \zeta, \omega^2, \xi_1, \dots, \xi_K, \alpha),$$

where $\zeta \in \mathbb{R}$ and $\mathbf{u}_k = (u_{k,1}, \dots, u_{k,Q}) \in \mathbb{R}^Q$.

Hyper-parameters

$$(a_\sigma, b_\sigma, a_\omega, b_\omega).$$

Posterior

$$\begin{aligned}
p(\mathbf{\Upsilon} | \mathbf{Y}) &= p(\mathbf{Y} | \zeta, \{\mathbf{u}_k\}, \{\xi_i\}, \{\xi_{i'}\}) p(\{\mathbf{u}_k\} | \sigma^2) p(\sigma^2) p(\zeta | \omega^2) p(\omega^2) p(\{\xi_i\} | \alpha) p(\alpha) \\
&\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^I \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \prod_{k=1}^K (\sigma^2)^{-H/2} \exp\left\{-\frac{1}{2\sigma^2} \|\mathbf{u}_k\|^2\right\} \times (\sigma^2)^{-(a_\sigma+1)} \exp\left\{-\frac{b_\sigma}{\sigma^2}\right\} \\
&\quad \times (\omega^2)^{-1/2} \exp\left\{-\frac{1}{2\omega^2} \zeta^2\right\} \times (\omega^2)^{-(a_\omega+1)} \exp\left\{-\frac{b_\omega}{\omega^2}\right\} \times \begin{cases} n_k, & \text{for } k = 1, \dots, K \\ \alpha, & \text{for } k = K + 1 \end{cases}
\end{aligned}$$

where $\theta_{i,i'} = \text{expit } \eta_{\phi(\xi_i, \xi_{i'})}$.

MCMC

The algorithm proceeds by generating a new state $\mathbf{\Upsilon}^{(b+1)}$ from a current state $\mathbf{\Upsilon}^{(b)}$, $b = 1, \dots, B$, as follows:

1. Sample $\mathbf{u}_k^{(b+1)}$, $k = 1, \dots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\mathbf{u}_k | \text{rest}) \propto \prod_{\substack{i,i': i < i' \\ \xi_i = k \text{ or } \xi_{i'} = k}} \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\frac{1}{2\sigma^2} \|\mathbf{u}_k\|^2\right\}.$$

2. Sample $\zeta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\zeta \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^I \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\frac{1}{\omega^2} \zeta^2\right\}.$$

3. Sample $\xi_i^{(b+1)}$, $i = 1, \dots, I$, from a categorical distribution proportional to $\begin{cases} n_k, & \text{for } k = 1, \dots, K \\ \alpha, & \text{for } k = K + 1 \end{cases}$

4. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \text{IGam}\left(\sigma^2 \mid a_\sigma + \frac{HQ}{2}, b_\sigma + \frac{1}{2} \sum_{k=1}^K \|\mathbf{u}_k\|^2\right)$.

5. Sample $(\omega^2)^{(b+1)}$ from $p(\omega^2 \mid \text{rest}) = \text{IGam}\left(\omega^2 \mid a_\omega + \frac{1}{2}, b_\omega + \frac{1}{2} \zeta^2\right)$.

6. Sample $\alpha^{(b+1)}$ according to ...

2 Pitman-Yor prior

2.1 Class Model (Stochastic Block Model, SBM)

Likelihood

$$y_{i,i'} \mid \xi_i, \xi_{i'}, \{\eta_{k,\ell}\} \stackrel{\text{ind}}{\sim} \text{Ber}(\text{expit } \eta_{\phi(\xi_i, \xi_{i'})})$$

$$\begin{aligned} p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) &= \prod_{i=1}^{I-1} \prod_{i'=i+1}^I (\text{expit } \eta_{\phi(\xi_i, \xi_{i'})})^{y_{i,i'}} (1 - \text{expit } \eta_{\phi(\xi_i, \xi_{i'})})^{1-y_{i,i'}} \\ &= \prod_{k=1}^K \prod_{\ell=k}^K (\text{expit } \eta_{k,\ell})^{s_{k,\ell}} (1 - \text{expit } \eta_{k,\ell})^{n_{k,\ell} - s_{k,\ell}} \end{aligned}$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$, $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i, i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k, \ell)\}$, and $\phi(x, y) = (\min\{x, y\}, \max\{x, y\})$ is a function to take into account that $\mathbf{Y} = [y_{i,i'}]$ is a symmetric matrix.

Prior

$$\eta_{k,\ell} \mid \zeta, \tau^2 \stackrel{\text{iid}}{\sim} \text{N}(\zeta, \tau^2)$$

$$\zeta \sim \text{N}(\mu_\zeta, \sigma_\zeta^2)$$

$$\tau^2 \sim \text{IGam}(a_\tau, b_\tau)$$

$$\xi_i \mid \alpha, \sigma \sim \text{CRP}(\alpha, \sigma)$$

$$\alpha, \sigma \sim \text{might add this later}$$

Parameters

$$\Upsilon = (\eta_{1,1}, \eta_{1,2}, \dots, \eta_{K,K}, \xi_1, \dots, \xi_K, \zeta, \tau^2, \alpha, \sigma),$$

where $\xi_i \in \{1, \dots, K\}$, $i = 1, \dots, I$, are the cluster assignments ($\xi_i = k$ means that actor i belongs to cluster k).

Hyper-parameters

$$(\mu_\zeta, \sigma_\zeta^2, a_\tau, b_\tau).$$

Posterior

$$\begin{aligned} p(\Upsilon \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) p(\{\eta_{k,\ell}\} \mid \zeta, \tau^2) p(\zeta) p(\tau^2) p(\{\xi_i\} \mid \alpha, \sigma) p(\alpha) p(\sigma) \\ &\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^I \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\frac{1}{2\sigma_\zeta^2}(\zeta - \mu_\zeta)^2\right\} \times (\tau^2)^{-(a_\tau-1)} \exp\left\{-\frac{b_\tau}{\tau^2}\right\} \\ &\quad \times \prod_{k=1}^K \prod_{\ell=k}^K (\tau^2)^{-1/2} \exp\left\{-\frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2\right\} \times \begin{cases} n_k - \sigma, & \text{for } k = 1, \dots, K \\ \alpha + H\sigma, & \text{for } k = K + 1 \end{cases} \end{aligned}$$

where $\theta_{i,i'} = \text{expit}(\eta_{\phi(\xi_i, \xi_{i'})})$.

MCMC

The algorithm proceeds by generating a new state $\Upsilon^{(b+1)}$ from a current state $\Upsilon^{(b)}$, $b = 1, \dots, B$, as follows:

1. Sample $\eta_{k,\ell}^{(b+1)}$, $\ell = k, \dots, K$ and $k = 1, \dots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\begin{aligned} \log p(\eta_{k,\ell} \mid \text{rest}) &\propto s_{k,\ell} \log(\text{expit } \eta_{k,\ell}) + (n_{k,\ell} - s_{k,\ell}) \log(1 - \text{expit } \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2 \\ &= s_{k,\ell} \eta_{k,\ell} - n_{k,\ell} \log(1 + \exp \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2, \end{aligned}$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$ and $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i, i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k, \ell)\}$.

2. Sample $\xi_i^{(b+1)}$, $i = 1, \dots, I$, from a categorical distribution prop to $\begin{cases} n_k - \sigma, & \text{for } k = 1, \dots, K \\ \alpha + H\sigma, & \text{for } k = K + 1 \end{cases}$
3. Sample $\zeta^{(b+1)}$ from $\mathcal{N}(m, v^2)$, where

$$v^2 = \left(\frac{1}{\sigma_\zeta^2} + \frac{H(H+1)/2}{\tau^2} \right)^{-1} \quad \text{and} \quad m = v^2 \left(\frac{\mu_\zeta}{\sigma_\zeta^2} + \frac{1}{\tau^2} \sum_{k=1}^K \sum_{\ell=k}^K \eta_{k,\ell} \right).$$

4. Sample $(\tau^2)^{(b+1)}$ from $p(\tau^2 \mid \text{rest}) = \text{IGam}\left(\tau^2 \mid a_\tau + \frac{H(H+1)}{4}, b_\tau + \frac{1}{2} \sum_{k=1}^K \sum_{\ell=k}^K (\eta_{k,\ell} - \zeta)^2\right)$.
5. Sample $\alpha^{(b+1)}$ according to ...
6. Sample $\sigma^{(b+1)}$ according to ...

2.2 Class - Distance Model

Likelihood

$$y_{i,i'} \mid \zeta, \{\mathbf{u}_k\}, \xi_i, \xi_{i'} \stackrel{\text{ind}}{\sim} \text{Ber}(\text{expit } \eta_{\phi(\xi_i, \xi_{i'})})$$

where

$$\eta_{k,\ell} = \zeta - \|\mathbf{u}_k - \mathbf{u}_\ell\|$$

Prior

$$\mathbf{u}_k \mid \sigma^2 \stackrel{\text{iid}}{\sim} \text{N}(\mathbf{0}, \sigma^2 \mathbf{I})$$

$$\sigma^2 \sim \text{IGam}(a_\sigma, b_\sigma)$$

$$\zeta \mid \omega^2 \sim \text{N}(0, \omega^2)$$

$$\omega^2 \sim \text{IGam}(a_\omega, b_\omega)$$

$$\xi_i \mid \alpha, \sigma \sim \text{CRP}(\alpha, \sigma)$$

$$\alpha, \sigma \sim \text{might add this later}$$

Parameters

$$\Upsilon = (\mathbf{u}_1, \dots, \mathbf{u}_K, \sigma^2, \zeta, \omega^2, \xi_1, \dots, \xi_K, \alpha, \sigma),$$

where $\zeta \in \mathbb{R}$ and $\mathbf{u}_k = (u_{k,1}, \dots, u_{k,Q}) \in \mathbb{R}^Q$.

Hyper-parameters

$$(a_\sigma, b_\sigma, a_\omega, b_\omega).$$

Posterior

$$\begin{aligned} p(\Upsilon \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \zeta, \{\mathbf{u}_k\}, \{\xi_i\}, \{\xi_{i'}\}) p(\{\mathbf{u}_k\} \mid \sigma^2) p(\sigma^2) p(\zeta \mid \omega^2) p(\omega^2) p(\{\xi_i\} \mid \alpha, \sigma) p(\alpha) p(\sigma) \\ &\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^I \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \prod_{k=1}^K (\sigma^2)^{-Q/2} \exp\left\{-\frac{1}{2\sigma^2} \|\mathbf{u}_k\|^2\right\} \times (\sigma^2)^{-(a_\sigma+1)} \exp\left\{-\frac{b_\sigma}{\sigma^2}\right\} \\ &\quad \times (\omega^2)^{-1/2} \exp\left\{-\frac{1}{2\omega^2} \zeta^2\right\} \times (\omega^2)^{-(a_\omega+1)} \exp\left\{-\frac{b_\omega}{\omega^2}\right\} \times \begin{cases} n_k - \sigma, & \text{for } k = 1, \dots, K \\ \alpha + H\sigma, & \text{for } k = K + 1 \end{cases} \end{aligned}$$

where $\theta_{i,i'} = \text{expit } \eta_{\phi(\xi_i, \xi_{i'})}$.

MCMC

The algorithm proceeds by generating a new state $\Upsilon^{(b+1)}$ from a current state $\Upsilon^{(b)}$, $b = 1, \dots, B$, as follows:

1. Sample $\mathbf{u}_k^{(b+1)}$, $k = 1, \dots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\mathbf{u}_k \mid \text{rest}) \propto \prod_{\substack{i,i': i < i' \\ \xi_i = k \text{ or } \xi_{i'} = k}} \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\frac{1}{2\sigma^2} \|\mathbf{u}_k\|^2\right\}.$$

2. Sample $\zeta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\zeta \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^I \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\frac{1}{\omega^2} \zeta^2\right\}.$$

3. Sample $\xi_i^{(b+1)}$, $i = 1, \dots, I$, from a categorical distribution prop to $\begin{cases} n_k - \sigma, & \text{for } k = 1, \dots, K \\ \alpha + H\sigma, & \text{for } k = K + 1 \end{cases}$
4. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \text{IGam}\left(\sigma^2 \mid a_\sigma + \frac{HQ}{2}, b_\sigma + \frac{1}{2} \sum_{k=1}^K \|\mathbf{u}_k\|^2\right)$.
5. Sample $(\omega^2)^{(b+1)}$ from $p(\omega^2 \mid \text{rest}) = \text{IGam}\left(\omega^2 \mid a_\sigma + \frac{1}{2}, b_\sigma + \frac{1}{2} \zeta^2\right)$.
6. Sample $\alpha^{(b+1)}$ according to ...
7. Sample $\sigma^{(b+1)}$ according to ...