Models

1 Data

Adjacency matrix $\mathbf{Y} = [y_{i,i'}]$ corresponding to an undirected, binary network.

2 Erdos–Renyi Model

Likelihood

$$y_{i,i'} \mid \theta \overset{\text{ind}}{\sim} \mathsf{Ber}\,(\theta)$$

Prior

$$\theta \sim \mathsf{Beta}(a_\theta, b_\theta)$$

Parameters

$$\boldsymbol{\Upsilon} = (\theta)\,.$$

Hyper-parameters

$$(a_\theta, b_\theta)\,.$$

Posterior

$$p(\theta \mid \mathbf{Y}) = p(\mathbf{Y} \mid \theta)\, p(\theta) = \theta^{a_\theta + s_y - 1}(1-\theta)^{b_\theta + N - s_y - 1} = \mathsf{Beta}(\theta \mid a_\theta + s_y, b_\theta + N - s_y)\,,$$

where $s_y = \sum_{i=1}^{I-1} \sum_{i'=i+1}^{I} y_{i,i'}$ and $N = I(I-1)/2$.

Prior Elicitation

$$a_\theta = 1\,, \qquad b_\theta = 1\,.$$

3 Distance Model

Likelihood

$$y_{i,i'} \mid \zeta, \boldsymbol{u}_i, \boldsymbol{u}_{i'} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\text{expit}(\zeta - \|\boldsymbol{u}_i - \boldsymbol{u}_{i'}\|)\right)$$

Prior

$$\begin{aligned}
\boldsymbol{u}_i \mid \sigma^2 &\overset{\text{iid}}{\sim} \mathsf{N}(\mathbf{0}, \sigma^2 \mathbf{I}) \\
\sigma^2 &\sim \mathsf{IGam}(a_\sigma, b_\sigma) \\
\zeta \mid \omega^2 &\sim \mathsf{N}(0, \omega^2) \\
\omega^2 &\sim \mathsf{IGam}(a_\omega, b_\omega)
\end{aligned}$$

Parameters

$$\boldsymbol{\Upsilon} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_I, \zeta, \sigma^2, \omega^2)\,,$$

where $\zeta \in \mathbb{R}$ and $\boldsymbol{u}_i = (u_{i,1}, \ldots, u_{i,K}) \in \mathbb{R}^K$.

Hyper-parameters

$$(a_\sigma, b_\sigma, a_\omega, b_\omega)\,.$$

Posterior

$$\begin{aligned}
p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \zeta, \{\boldsymbol{u}_i\})\, p(\{\boldsymbol{u}_i\} \mid \sigma^2)\, p(\sigma^2)\, p(\zeta \mid \omega^2)\, p(\omega^2) \\
&\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1 - y_{i,i'}} \times \prod_{i=1}^{I} (\sigma^2)^{-K/2} \exp\left\{-\tfrac{1}{2\sigma^2} \|\boldsymbol{u}_i\|^2\right\} \times (\sigma^2)^{-(a_\sigma+1)} \exp\left\{-\frac{b_\sigma}{\sigma^2}\right\} \\
&\quad \times (\omega^2)^{-1/2} \exp\left\{-\tfrac{1}{2\omega^2}\, \zeta^2\right\} \times (\omega^2)^{-(a_\omega+1)} \exp\left\{-\frac{b_\omega}{\omega^2}\right\},
\end{aligned}$$

where $\theta_{i,i'} = \text{expit}(\zeta - \|\boldsymbol{u}_i - \boldsymbol{u}_{i'}\|)$.

MCMC Algorithm

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \ldots, B$, as follows:

1. Sample $\boldsymbol{u}_i^{(b+1)}$, $i = 1, \ldots, I$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\boldsymbol{u}_i \mid \text{rest}) \propto \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1 - y_{i,i'}} \times \prod_{i'=1}^{i-1} \theta_{i',i}^{y_{i',i}} (1 - \theta_{i',i})^{1 - y_{i',i}} \times \exp\left\{-\tfrac{1}{2\sigma^2} \|\boldsymbol{u}_i\|^2\right\}.$$

2. Sample $\zeta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\zeta \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{2\omega^2}\,\zeta^2\right\}.$$

3. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \mathsf{IGam}\left(\sigma^2 \mid a_\sigma + \frac{I\,K}{2}, b_\sigma + \frac{1}{2}\sum_{i=1}^{I} \|\boldsymbol{u}_i\|^2\right)$.

4. Sample $(\omega^2)^{(b+1)}$ from $p(\omega^2 \mid \text{rest}) = \mathsf{IGam}\left(\omega^2 \mid a_\omega + \frac{1}{2}, b_\omega + \frac{1}{2}\zeta^2\right)$.

Prior Elicitation

Setting a priori $\mathbb{E}\left[\sigma^2\right] = V_2\left(I^{1/K}\right)$ and $\mathbb{E}\left[\omega^2\right] = 100$, with $\mathbb{CV}\left[\sigma^2\right] = \mathbb{CV}\left[\omega^2\right] = \infty$, it follows that

$$a_\sigma = 2\,, \qquad b_\sigma = V_2\left(I^{1/K}\right)\,, \qquad a_\omega = 2\,, \qquad b_\omega = 100\,,$$

where $V_2(I^{1/K})$ is the volume of a 2-dimensional Euclidean ball of radius $I^{1/K}$.

4 Class Model (Stochastic Block Model, SBM)

Likelihood

$$y_{i,i'} \mid \xi_i, \xi_{i'}, \{\eta_{k,\ell}\} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})}\right)$$

$$\begin{aligned} p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) &= \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} (\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{y_{i,i'}} (1-\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{1-y_{i,i'}} \\ &= \prod_{k=1}^{K} \prod_{\ell=k}^{K} (\operatorname{expit} \eta_{k,\ell})^{s_{k,\ell}} (1-\operatorname{expit} \eta_{k,\ell})^{n_{k,\ell}-s_{k,\ell}} \end{aligned}$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$, $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i < i' \text{ and } \phi(\xi_i,\xi_{i'}) = (k,\ell)\}$, and $\phi(x,y) = (\min\{x,y\}, \max\{x,y\})$ is a function to take into account that $\mathbf{Y} = [y_{i,i'}]$ is a symmetric matrix.

Prior

$$\begin{aligned} \eta_{k,\ell} \mid \zeta, \tau^2 &\overset{\text{iid}}{\sim} \mathsf{N}(\zeta, \tau^2) \\ \zeta &\sim \mathsf{N}(\mu_\zeta, \sigma_\zeta^2) \\ \tau^2 &\sim \mathsf{IGam}(a_\tau, b_\tau) \\ \xi_i \mid \boldsymbol{\omega} &\overset{\text{iid}}{\sim} \mathsf{Cat}(\boldsymbol{\omega}) \end{aligned}$$

$$\boldsymbol{\omega} \mid \alpha \sim \mathsf{Dir}\left(\tfrac{\alpha}{K}, \ldots, \tfrac{\alpha}{K}\right)$$
$$\alpha \sim \mathsf{Gam}(a_\alpha, b_\alpha)$$

Parameters

$$\boldsymbol{\Upsilon} = (\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{K,K}, \xi_1, \ldots, \xi_I, \omega_1, \ldots, \omega_K, \zeta, \tau^2, \alpha)\,,$$

where $\xi_i \in \{1, \ldots, K\}$, $i = 1, \ldots, I$, are the cluster assignments ($\xi_i = k$ means that actor i belongs to cluster k), and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_K)$ is a probability vector such that $\Pr\left[\xi_i = k \mid w_k\right] = w_k$, $k = 1, \ldots, K$.

Hyper-parameters

$$(\mu_\zeta, \sigma^2_\zeta, a_\tau, b_\tau, a_\alpha, b_\alpha)\,.$$

Posterior

$$\begin{aligned}
p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\})\, p(\{\eta_{k,\ell}\} \mid \zeta, \tau^2)\, p(\zeta)\, p(\tau^2)\, p(\{\xi_i\} \mid \boldsymbol{\omega})\, p(\boldsymbol{\omega} \mid \alpha)\, p(\alpha) \\
&\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{\xi_i,\xi_{i'}}^{y_{i,i'}} (1 - \theta_{\xi_i,\xi_{i'}})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{2\sigma^2_\zeta}(\zeta - \mu_\zeta)^2\right\} \times (\tau^2)^{-(a_\tau - 1)} \exp\left\{-\frac{b_\tau}{\tau^2}\right\} \\
&\quad \times \prod_{k=1}^{K} \prod_{\ell=k}^{K} (\tau^2)^{-1/2} \exp\left\{-\tfrac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2\right\} \times \prod_{i=1}^{I} \prod_{k=1}^{K} \omega_k^{[\xi_i = k]} \times \frac{\Gamma\left(\frac{\alpha}{K}\right)^K}{\Gamma(\alpha)} \prod_{k=1}^{K} \omega_k^{\frac{\alpha}{K} - 1} \\
&\quad \times \alpha^{a_\alpha - 1} \exp\{-b_\alpha\, \alpha\}\,,
\end{aligned}$$

where $\theta_{i,i'} = \mathrm{expit}(\eta_{\phi(\xi_i,\xi_{i'})})$ and $[\cdot]$ is the Iverson bracket.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \ldots, B$, as follows:

1. Sample $\eta_{k,\ell}^{(b+1)}$, $\ell = k, \ldots, K$ and $k = 1, \ldots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\begin{aligned}
\log p(\eta_{k,\ell} \mid \text{rest}) &\propto s_{k,\ell} \log(\mathrm{expit}\, \eta_{k,\ell}) + (n_{k,\ell} - s_{k,\ell}) \log(1 - \mathrm{expit}\, \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2 \\
&= s_{k,\ell}\, \eta_{k,\ell} - n_{k,\ell} \log(1 + \exp \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2\,,
\end{aligned}$$

   where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$ and $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k,\ell)\}$.

2. Sample $\xi_i^{(b+1)}$, $i = 1, \ldots, I$, from a categorical distribution on $\{1, \ldots, K\}$, such that:

$$\mathbb{P}\mathrm{r}\left[\xi_i = k \mid \text{rest}\right] \propto \omega_k \times \prod_{i'=i+1}^{I} \theta_{k,\xi_{i'}}^{y_{i,i'}} (1 - \theta_{k,\xi_{i'}})^{1-y_{i,i'}} \times \prod_{i'=1}^{i-1} \theta_{\xi_{i'},k}^{y_{i',i}} (1 - \theta_{\xi_{i'},k})^{1-y_{i',i}} \,.$$

Note that the term above actually comes from the following:

$$\mathbb{P}\mathrm{r}\left[\xi_i = k \mid \boldsymbol{\xi}_{-i}, \text{rest}\right] \propto \pi(\boldsymbol{\xi}) \frac{\mathbb{P}\mathrm{r}\left[\xi_i = k, \boldsymbol{\xi}_{-i} \mid \text{rest}\right]}{\mathbb{P}\mathrm{r}\left[\boldsymbol{\xi}_{-i} \mid \text{rest}\right]},$$

where $\boldsymbol{\xi}_{-i}$ is the vector of cluster assignments excluding element i and $\pi(\boldsymbol{\xi})$ is the prior distribution on cluster assignments (i.e. it could be DM, DP, PY or Gnedin). Note that the likelihood term is:

$$\frac{\mathbb{P}\mathrm{r}\left[\xi_i = k, \boldsymbol{\xi}_{-i} \mid \text{rest}\right]}{\mathbb{P}\mathrm{r}\left[\boldsymbol{\xi}_{-i} \mid \text{rest}\right]} = \frac{\prod_{i'=1}^{I-1} \prod_{j=i'+1}^{I} \theta_{\xi_{i'},\xi_j}^{y_{i',j}} (1 - \theta_{\xi_{i'},\xi_j})^{1-y_{i',j}}}{\prod_{i'=1, i' \neq i}^{I-1} \prod_{j=i'+1}^{I} \theta_{\xi_{i'},\xi_j}^{y_{i',j}} (1 - \theta_{\xi_{i'},\xi_j})^{1-y_{i',j}}}$$

Note that all the terms in the numerator will cancel out with exception of the one for $i' = i$ where $\xi_i = k$.

3. Sample $\boldsymbol{\omega}^{(b+1)}$ from $p(\boldsymbol{\omega} \mid \text{rest}) = \mathsf{Dir}\left(\boldsymbol{\omega} \mid \frac{\alpha}{K} + n_1, \ldots, \frac{\alpha}{K} + n_K\right)$, where n_k is the number of actors in cluster $k \in \{1, \ldots, K\}$.

4. Sample $\zeta^{(b+1)}$ from $\mathsf{N}(m, v^2)$, where

$$v^2 = \left(\frac{1}{\sigma_\zeta^2} + \frac{K(K+1)/2}{\tau^2}\right)^{-1} \qquad \text{and} \qquad m = v^2 \left(\frac{\mu_\zeta}{\sigma_\zeta^2} + \frac{1}{\tau^2} \sum_{k=1}^{K} \sum_{\ell=k}^{K} \eta_{k,\ell}\right) .$$

5. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \mathsf{IGam}\left(\sigma^2 \mid a_\tau + \frac{K(K+1)}{4}, b_\tau + \frac{1}{2} \sum_{k=1}^{K} \sum_{\ell=k}^{K} (\eta_{k,\ell} - \zeta)^2\right)$.

6. Sample $\alpha^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\log p(\alpha \mid \text{rest}) \propto \log \Gamma(\alpha) - K \log \Gamma(\alpha/K) + \frac{\alpha}{K} \sum_{k=1}^{K} \log \omega_k - (a_\beta - 1) \log \alpha - b_\alpha \, \alpha \,.$$

Prior Elicitation

$$\mu_\zeta = 0 \,, \qquad \sigma_\zeta^2 = 3 \,, \qquad a_\tau = 2 \,, \qquad b_\tau = 3 \,, \qquad a_\alpha = 1 \,, \qquad b_\alpha = 1 \,.$$

5 Eigen Model

Likelihood

$$y_{i,i'} \mid \zeta, \boldsymbol{u}_i, \boldsymbol{u}_{i'}, \boldsymbol{\Lambda} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\operatorname{expit}(\zeta + \boldsymbol{u}_i^T \boldsymbol{\Lambda} \boldsymbol{u}_{i'})\right)$$

Prior

$$\begin{aligned}
\boldsymbol{u}_i \mid \sigma^2 &\overset{\text{iid}}{\sim} \mathsf{N}(\mathbf{0}, \sigma^2 \mathbf{I}) \\
\sigma^2 &\sim \mathsf{IGam}(a_\sigma, b_\sigma) \\
\lambda_k \mid \kappa^2 &\overset{\text{iid}}{\sim} \mathsf{N}(0, \kappa^2) \\
\kappa^2 &\sim \mathsf{IGam}(a_\kappa, b_\kappa) \\
\zeta \mid \omega^2 &\sim \mathsf{N}(0, \omega^2) \\
\omega^2 &\sim \mathsf{IGam}(a_\omega, b_\omega)
\end{aligned}$$

Parameters

$$\boldsymbol{\Upsilon} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_I, \lambda_1, \ldots, \lambda_K, \zeta, \sigma^2, \kappa^2, \omega^2)\,,$$

where $\zeta \in \mathbb{R}$ and $\boldsymbol{u}_i = (u_{i,1}, \ldots, u_{i,K}) \in \mathbb{R}^K$, and $\boldsymbol{\Lambda} = \operatorname{diag}[\lambda_1, \ldots, \lambda_K]$.

Hyper-parameters

$$(a_\sigma, b_\sigma, a_\kappa, b_\kappa, a_\omega, b_\omega)\,.$$

Posterior

$$\begin{aligned}
p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \zeta, \{\boldsymbol{u}_i\}, \{\lambda_k\})\, p(\{\boldsymbol{u}_i\} \mid \sigma^2)\, p(\sigma^2)\, p(\{\lambda_k\} \mid \kappa^2)\, p(\kappa^2)\, p(\zeta \mid \omega^2)\, p(\omega^2) \\
&\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \prod_{i=1}^{I} (\sigma^2)^{-K/2} \exp\left\{-\tfrac{1}{2\sigma^2} \|\boldsymbol{u}_i\|^2\right\} \times (\sigma^2)^{-(a_\sigma+1)} \exp\left\{-\frac{b_\sigma}{\sigma^2}\right\} \\
&\quad \times \prod_{k=1}^{K} (\kappa^2)^{-1/2} \exp\left\{-\tfrac{1}{2\kappa^2}\, \lambda_k^2\right\} \times (\kappa^2)^{-(a_\kappa+1)} \exp\left\{-\frac{b_\kappa}{\kappa^2}\right\} \times (\omega^2)^{-1/2} \exp\left\{-\tfrac{1}{2\omega^2}\, \zeta^2\right\} \\
&\quad \times (\omega^2)^{-(a_\omega+1)} \exp\left\{-\frac{b_\omega}{\omega^2}\right\},
\end{aligned}$$

where $\theta_{i,i'} = \operatorname{expit}(\zeta + \boldsymbol{u}_i \boldsymbol{\Lambda} \boldsymbol{u}_{i'})$.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \ldots, B$, as follows:

1. Sample $\boldsymbol{u}_i^{(b+1)}$, $i = 1, \dots, I$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\boldsymbol{u}_i \mid \text{rest}) \propto \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \prod_{i'=1}^{i-1} \theta_{i',i}^{y_{i',i}} (1-\theta_{i',i})^{1-y_{i',i}} \times \exp\left\{-\tfrac{1}{2\sigma^2} \|\boldsymbol{u}_i\|^2\right\}.$$

2. Sample $\lambda_k^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\lambda_k \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{2\kappa^2} \lambda_k^2\right\}.$$

3. Sample $\zeta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\zeta \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{\omega^2} \zeta^2\right\}.$$

4. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \mathsf{IGam}\left(\sigma^2 \mid a_\sigma + \frac{IK}{2}, b_\sigma + \frac{1}{2}\sum_{i=1}^{I} \|\boldsymbol{u}_i\|^2\right)$.
5. Sample $(\kappa^2)^{(b+1)}$ from $p(\kappa^2 \mid \text{rest}) = \mathsf{IGam}\left(\kappa^2 \mid a_\kappa + \frac{Q}{2}, b_\kappa + \frac{1}{2}\sum_{k=1}^{K} \lambda_k^2\right)$.
6. Sample $(\omega^2)^{(b+1)}$ from $p(\omega^2 \mid \text{rest}) = \mathsf{IGam}\left(\omega^2 \mid a_\sigma + \frac{1}{2}, b_\sigma + \frac{1}{2}\zeta^2\right)$.

Prior Elicitation

Setting a priori $\mathbb{E}[\sigma^2] = \mathbb{E}[\kappa^2] = \mathbb{E}[\omega^2] = 3$, with $\mathbb{CV}[\sigma^2] = \mathbb{CV}[\kappa^2] = \mathbb{CV}[\omega^2] = \infty$, it follows that

$$a_\sigma = 2\,, \qquad b_\sigma = 3\,, \qquad a_\kappa = 2\,, \qquad b_\kappa = 3\,, \qquad a_\omega = 2\,, \qquad b_\omega = 3\,.$$

6 Class–Distance Model

Likelihood

$$y_{i,i'} \mid \zeta, \{\boldsymbol{u}_k\}, \xi_i, \xi_{i'} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})}\right)$$

where

$$\eta_{k,\ell} = \zeta - \|\boldsymbol{u}_k - \boldsymbol{u}_\ell\|$$

Prior

$$\boldsymbol{u}_k \mid \sigma^2 \overset{\text{iid}}{\sim} \mathsf{N}(\boldsymbol{0}, \sigma^2 \mathbf{I})$$

$$
\begin{aligned}
\sigma^2 &\sim \mathsf{IGam}(a_\sigma, b_\sigma)\\
\zeta \mid \omega^2 &\sim \mathsf{N}(0, \omega^2)\\
\omega^2 &\sim \mathsf{IGam}(a_\omega, b_\omega)\\
\xi_i \mid \boldsymbol{\omega} &\overset{\text{iid}}{\sim} \mathsf{Cat}(\boldsymbol{\omega})\\
\boldsymbol{\omega} \mid \alpha &\sim \mathsf{Dir}\left(\tfrac{\alpha}{K}, \ldots, \tfrac{\alpha}{K}\right)\\
\alpha &\sim \mathsf{Gam}(a_\alpha, b_\alpha)
\end{aligned}
$$

Parameters

$$
\boldsymbol{\Upsilon} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K, \xi_1, \ldots, \xi_I, \omega_1, \ldots, \omega_K, \sigma^2, \omega^2, \alpha)\,,
$$

where $\zeta \in \mathbb{R}$ and $\boldsymbol{u}_k = (u_{k,1}, \ldots, u_{k,Q}) \in \mathbb{R}^Q$.

Hyper-parameters

$$
(a_\sigma, b_\sigma, a_\omega, b_\omega, a_\alpha, b_\alpha)\,.
$$

Posterior

$$
\begin{aligned}
p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \zeta, \{\boldsymbol{u}_k\}, \{\xi_i\}, \{\xi_{i'}\})\, p(\{\boldsymbol{u}_k\} \mid \sigma^2)\, p(\sigma^2)\, p(\zeta \mid \omega^2)\, p(\omega^2)\, p(\{\xi_i\} \mid \boldsymbol{\omega})\, p(\boldsymbol{\omega} \mid \alpha)\, p(\alpha)\\
&\propto \prod_{i=1}^{I-1}\prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}}(1-\theta_{i,i'})^{1-y_{i,i'}} \times \prod_{k=1}^{K} (\sigma^2)^{-K/2} \exp\left\{-\tfrac{1}{2\sigma^2}\,\|\boldsymbol{u}_k\|^2\right\} \times (\sigma^2)^{-(a_\sigma+1)} \exp\left\{-\frac{b_\sigma}{\sigma^2}\right\}\\
&\quad \times (\omega^2)^{-1/2} \exp\left\{-\tfrac{1}{2\omega^2}\,\zeta^2\right\} \times (\omega^2)^{-(a_\omega+1)} \exp\left\{-\frac{b_\omega}{\omega^2}\right\} \times \prod_{i=1}^{I}\prod_{k=1}^{K} \omega_k^{[\xi_i=k]} \times \frac{\Gamma\left(\frac{\alpha}{K}\right)^K}{\Gamma(\alpha)} \prod_{k=1}^{K} \omega_k^{\frac{\alpha}{K}-1}\\
&\quad \times \alpha^{a_\alpha-1} \exp\{-b_\alpha\,\alpha\}
\end{aligned}
$$

where $\theta_{i,i'} = \operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})}$ and $[\cdot]$ is the Iverson bracket.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \ldots, B$, as follows:

1. Sample $\boldsymbol{u}_k^{(b+1)}$, $k = 1, \ldots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$
p(\boldsymbol{u}_k \mid \text{rest}) \propto \prod_{\substack{i,i' : i<i' \\ \xi_i = k \text{ or } \xi_{i'} = k}} \theta_{i,i'}^{y_{i,i'}}(1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{2\sigma^2}\,\|\boldsymbol{u}_k\|^2\right\}.
$$

2. Sample $\zeta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\zeta \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\{-\tfrac{1}{\omega^2}\,\zeta^2\}.$$

3. Sample $\xi_i^{(b+1)}$, $i = 1, \dots, I$, from a categorical distribution on $\{1, \dots, K\}$, such that:

$$\Pr[\xi_i = k \mid \text{rest}] \propto \omega_k \times \prod_{i'=i+1}^{I} \eta_{\phi(k,\xi_{i'})}^{y_{i,i'}} (1-\eta_{\phi(k,\xi_{i'})})^{1-y_{i,i'}} \times \prod_{i'=1}^{i-1} \eta_{\phi(\xi_{i'},k)}^{y_{i',i}} (1-\eta_{\phi(\xi_{i'},k)})^{1-y_{i',i}}\,.$$

4. Sample $\boldsymbol{\omega}^{(b+1)}$ from $p(\boldsymbol{\omega} \mid \text{rest}) = \mathsf{Dir}(\boldsymbol{\omega} \mid \frac{\alpha}{K} + n_1, \dots, \frac{\alpha}{K} + n_K)$, where n_k is the number of actors in cluster $k \in \{1, \dots, K\}$.

5. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \mathsf{IGam}\left(\sigma^2 \mid a_\sigma + \frac{K\,Q}{2}, b_\sigma + \frac{1}{2}\sum_{k=1}^{K} \|\boldsymbol{u}_k\|^2\right)$.

6. Sample $(\omega^2)^{(b+1)}$ from $p(\omega^2 \mid \text{rest}) = \mathsf{IGam}\left(\omega^2 \mid a_\sigma + \frac{1}{2}, b_\sigma + \frac{1}{2}\zeta^2\right)$.

7. Sample $\alpha^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\log p(\alpha \mid \text{rest}) \propto \log\Gamma(\alpha) - K\log\Gamma(\alpha/K) + \frac{\alpha}{K}\sum_{k=1}^{K}\log\omega_k - (a_\alpha - 1)\log\alpha - b_\alpha\,\alpha\,.$$

Prior Elicitation Setting a priori $\mathbb{E}[\sigma^2] = V_2\left(K^{1/Q}\right)$, $\mathbb{E}[\omega^2] = 3$ and $\mathbb{E}[\alpha] = 1$, with $\mathbb{CV}[\sigma^2] = \mathbb{CV}[\omega^2] = \infty$ and $\mathbb{CV}[\alpha] = 1$, it follows that

$$a_\sigma = 2\,, \qquad b_\sigma = V_2\left(K^{1/Q}\right)\,, \qquad a_\omega = 2\,, \qquad b_\omega = 3\,, \qquad a_\alpha = 1\,, \qquad b_\alpha = 1\,,$$

where $V_2(K^{1/Q})$ is the volume of a 2-dimensional Euclidean ball of radius $I^{1/Q}$.

7 Class–Eigen Model

Likelihood

$$y_{i,i'} \mid \zeta, \{\boldsymbol{u}_k\}, \boldsymbol{\Lambda}, \xi_i, \xi_{i'} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\text{expit}\,\eta_{\phi(\xi_i,\xi_{i'})}\right)$$

where

$$\eta_{k,\ell} = \zeta + \boldsymbol{u}_k^T \boldsymbol{\Lambda} \boldsymbol{u}_\ell$$

Prior

$$\boldsymbol{u}_k \mid \sigma^2 \overset{\text{iid}}{\sim} \mathsf{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

$$
\begin{aligned}
\sigma^2 &\sim \mathsf{IGam}(a_\sigma, b_\sigma) \\
\lambda_q \mid \kappa^2 &\overset{\text{iid}}{\sim} \mathsf{N}(0, \kappa^2) \\
\kappa^2 &\sim \mathsf{IGam}(a_\kappa, b_\kappa) \\
\zeta \mid \omega^2 &\sim \mathsf{N}(0, \omega^2) \\
\omega^2 &\sim \mathsf{IGam}(a_\omega, b_\omega) \\
\xi_i \mid \boldsymbol{\omega} &\overset{\text{iid}}{\sim} \mathsf{Cat}(\boldsymbol{\omega}) \\
\boldsymbol{\omega} \mid \alpha &\sim \mathsf{Dir}\left(\tfrac{\alpha}{K}, \dots, \tfrac{\alpha}{K}\right) \\
\alpha &\sim \mathsf{Gam}(a_\alpha, b_\alpha)
\end{aligned}
$$

Parameters

$$
\boldsymbol{\Upsilon} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_K, \lambda_1, \dots, \lambda_Q, \xi_1, \dots, \xi_I, \omega_1, \dots, \omega_K, \sigma^2, \kappa^2, \omega^2, \alpha)\,,
$$

where $\zeta \in \mathbb{R}$ and $\boldsymbol{u}_k = (u_{k,1}, \dots, u_{k,Q}) \in \mathbb{R}^Q$, and $\boldsymbol{\Lambda} = \operatorname{diag}[\lambda_1, \dots, \lambda_Q]$.

Hyper-parameters

$$
(a_\sigma, b_\sigma, a_\kappa, b_\kappa, a_\omega, b_\omega, a_\alpha, b_\alpha)\,.
$$

Posterior

$$
\begin{aligned}
p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) = {} & p(\mathbf{Y} \mid \zeta, \{\boldsymbol{u}_k\}, \{\lambda_q\}, \{\xi_i\})\, p(\{\boldsymbol{u}_k\} \mid \sigma^2)\, p(\sigma^2)\, p(\{\lambda_q\} \mid \kappa^2)\, p(\kappa^2)\; p(\zeta \mid \omega^2)\, p(\omega^2) \\
& p(\{\xi_i\} \mid \boldsymbol{\omega})\, p(\boldsymbol{\omega} \mid \alpha)\, p(\alpha) \\
\propto {} & \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \prod_{k=1}^{K} (\sigma^2)^{-K/2} \exp\left\{-\tfrac{1}{2\sigma^2}\, \|\boldsymbol{u}_k\|^2\right\} \times (\sigma^2)^{-(a_\sigma+1)} \exp\left\{-\frac{b_\sigma}{\sigma^2}\right\} \\
& \times \prod_{q=1}^{Q} (\kappa^2)^{-1/2} \exp\left\{-\tfrac{1}{2\kappa^2}\, \lambda_q^2\right\} \times (\kappa^2)^{-(a_\kappa+1)} \exp\left\{-\frac{b_\kappa}{\kappa^2}\right\} \times (\omega^2)^{-1/2} \exp\left\{-\tfrac{1}{2\omega^2}\, \zeta^2\right\} \\
& \times (\omega^2)^{-(a_\omega+1)} \exp\left\{-\frac{b_\omega}{\omega^2}\right\} \times \prod_{i=1}^{I} \prod_{k=1}^{K} \omega_k^{[\xi_i = k]} \times \frac{\Gamma\left(\frac{\alpha}{K}\right)^K}{\Gamma(\alpha)} \prod_{k=1}^{K} \omega_k^{\frac{\alpha}{K}-1} \times \alpha^{a_\alpha - 1} \exp\{-b_\alpha\, \alpha\}\,,
\end{aligned}
$$

where $\theta_{i,i'} = \operatorname{expit} \eta_{\phi(\xi_i, \xi_{i'})}$ and $[\cdot]$ is the Iverson bracket.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \dots, B$, as follows:

1. Sample $\boldsymbol{u}_k^{(b+1)}$, $k = 1, \dots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\boldsymbol{u}_k \mid \text{rest}) \propto \prod_{\substack{i<i' \\ \xi_i = k \text{ or } \xi_{i'} = k}} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{2\sigma^2} \|\boldsymbol{u}_k\|^2\right\}.$$

2. Sample $\lambda_q^{(b+1)}$, $q = 1, \dots, Q$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\lambda_q \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{2\kappa^2} \lambda_q^2\right\}.$$

3. Sample $\zeta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\zeta \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{\omega^2} \zeta^2\right\}.$$

4. Sample $\xi_i^{(s+1)}$, $i = 1, \dots, I$, from a categorical distribution on $\{1, \dots, K\}$, such that:

$$\Pr[\xi_i = k \mid \text{rest}] \propto \omega_k \times \prod_{i'=i+1}^{I} \eta_{\phi(k,\xi_{i'})}^{y_{i,i'}} (1-\eta_{\phi(k,\xi_{i'})})^{1-y_{i,i'}} \times \prod_{i'=1}^{i-1} \eta_{\phi(\xi_{i'},k)}^{y_{i',i}} (1-\eta_{\phi(\xi_{i'},k)})^{1-y_{i',i}}.$$

5. Sample $\boldsymbol{\omega}^{(b+1)}$ from $p(\boldsymbol{\omega} \mid \text{rest}) = \mathsf{Dir}(\boldsymbol{\omega} \mid \frac{\alpha}{K} + n_1, \dots, \frac{\alpha}{K} + n_K)$, where n_k is the number of actors in cluster $k \in \{1, \dots, K\}$.

6. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \mathsf{IGam}\left(\sigma^2 \mid a_\sigma + \frac{KQ}{2}, b_\sigma + \frac{1}{2}\sum_{k=1}^{K} \|\boldsymbol{u}_k\|^2\right)$.

7. Sample $(\kappa^2)^{(b+1)}$ from $p(\kappa^2 \mid \text{rest}) = \mathsf{IGam}\left(\kappa^2 \mid a_\kappa + \frac{Q}{2}, b_\kappa + \frac{1}{2}\sum_{q=1}^{Q} \lambda_q^2\right)$.

8. Sample $(\omega^2)^{(b+1)}$ from $p(\omega^2 \mid \text{rest}) = \mathsf{IGam}\left(\omega^2 \mid a_\sigma + \frac{1}{2}, b_\sigma + \frac{1}{2}\zeta^2\right)$.

9. Sample $\alpha^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\log p(\alpha \mid \text{rest}) \propto \log \Gamma(\alpha) - K \log \Gamma(\alpha/K) + \frac{\alpha}{K} \sum_{k=1}^{K} \log \omega_k - (a_\alpha - 1) \log \alpha - b_\alpha \, \alpha.$$

Prior Elicitation

Setting a priori $\mathbb{E}[\sigma^2] = \mathbb{E}[\kappa^2] = \mathbb{E}[\omega^2] = 3$, $\mathbb{E}[\alpha^2] = 1$, with $\mathbb{CV}[\sigma^2] = \mathbb{CV}[\kappa^2] = \mathbb{CV}[\omega^2] = \infty$ and $\mathbb{CV}[\alpha] = 1$, it follows that

$$a_\sigma = 2, \quad b_\sigma = 3, \quad a_\kappa = 2, \quad b_\kappa = 3, \quad a_\omega = 2, \quad b_\omega = 3, \quad a_\alpha = 1, \quad b_\alpha = 1.$$

8 Multilevel–Class Model

Likelihood

$$y_{i,i'} \mid \xi_i, \xi_{i'}, \{\eta_{k,\ell}\} \overset{\text{ind}}{\sim} \textsf{Ber}\left(\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})}\right)$$

$$\begin{aligned} p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) &= \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} (\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{y_{i,i'}} (1 - \operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{1-y_{i,i'}} \\ &= \prod_{k=1}^{K} \prod_{\ell=k}^{K} (\operatorname{expit} \eta_{k,\ell})^{s_{k,\ell}} (1 - \operatorname{expit} \eta_{k,\ell})^{n_{k,\ell} - s_{k,\ell}} \end{aligned}$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$ and $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k,\ell)\}$.

Prior

$$\begin{aligned} \eta_{k,\ell} \mid \gamma_k, \gamma_\ell, \{\mu_{q,r}\}, \sigma^2 &\sim \textsf{N}(\mu_{\phi(\gamma_k,\gamma_\ell)}, \sigma^2) \\ \mu_{q,r} \mid \zeta, \tau^2 &\overset{\text{ind}}{\sim} \textsf{N}(\zeta, \tau^2) \\ \zeta &\sim \textsf{N}(\mu_\zeta, \sigma^2_\zeta) \\ \tau^2 &\sim \textsf{IGam}(a_\tau, b_\tau) \\ \sigma^2 &\sim \textsf{IGam}(a_\sigma, b_\sigma) \\ \gamma_k \mid \boldsymbol{\vartheta} &\overset{\text{iid}}{\sim} \textsf{Cat}(\boldsymbol{\vartheta}) \\ \boldsymbol{\vartheta} \mid \beta &\sim \textsf{Dir}\left(\tfrac{\beta}{Q}, \ldots, \tfrac{\beta}{Q}\right) \\ \beta &\sim \textsf{Gam}(a_\beta, b_\beta) \\ \xi_i \mid \boldsymbol{\omega} &\overset{\text{iid}}{\sim} \textsf{Cat}(\boldsymbol{\omega}) \\ \boldsymbol{\omega} \mid \alpha &\sim \textsf{Dir}\left(\tfrac{\alpha}{K}, \ldots, \tfrac{\alpha}{K}\right) \\ \alpha &\sim \textsf{Gam}(a_\alpha, b_\alpha) \end{aligned}$$

Parameters

$$\boldsymbol{\Upsilon} = \left(\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{K,K}, \mu_{1,1}, \mu_{1,2}, \ldots, \mu_{Q,Q}, \zeta, \tau^2, \sigma^2, \gamma_1, \ldots, \gamma_K, \vartheta_1, \ldots, \vartheta_Q, \beta, \xi_1, \ldots, \xi_I, \omega_1, \ldots, \omega_K, \alpha\right),$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_I)$, $\xi_i \in \{1, \ldots, K\}$, are the actors-cluster assignments ($\xi_i = k$ means that actor i belongs to cluster k), $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_K)$, $\gamma_k \in \{1, \ldots, Q\}$, are the super-cluster assignments ($\gamma_k = q$ means that cluster k belongs to super-cluster q), $\phi(x,y) = (\min\{x,y\}, \max\{x,y\})$ is a function to take into account that both $\mathbf{Y} = [y_{i,i'}]$ and $\boldsymbol{\eta} = [\eta_{k,\ell}]$ are symmetric matrices, and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_K)$ and $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_Q)$ are probability vectors such that $\mathbb{P}\mathrm{r}\left[\xi_i = k \mid \omega_k\right] = \omega_k$ and $\mathbb{P}\mathrm{r}\left[\gamma_k = q \mid \vartheta_q\right] = \vartheta_q$, respectively.

Hyper-parameters

$$(\mu_\zeta, \sigma^2_\zeta, a_\tau, b_\tau, a_\sigma, b_\sigma, a_\beta, b_\beta, a_\alpha, b_\alpha)\,.$$

Posterior

$$\begin{aligned}
p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) \propto & \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \prod_{k=1}^{K}\prod_{\ell=k}^{K} (\sigma^2)^{-1/2} \exp\Big\{-\tfrac{1}{2\sigma^2}\left(\eta_{k,\ell} - \mu_{\phi(\gamma_k,\gamma_\ell)}\right)^2\Big\} \\
& \times \prod_{q=1}^{Q}\prod_{r=q}^{Q} (\tau^2)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2\tau^2}(\mu_{q,r}-\zeta)^2\} \times \exp\Big\{-\tfrac{1}{2\sigma^2_\zeta}(\zeta-\mu_\zeta)^2\Big\} \times (\tau^2)^{-(a_\tau-1)} \exp\Big\{-\frac{b_\tau}{\tau^2}\Big\} \\
& \times (\sigma^2)^{-(a_\sigma-1)} \exp\Big\{-\frac{b_\sigma}{\sigma^2}\Big\} \times \prod_{k=1}^{K}\prod_{q=1}^{Q} \vartheta_q^{[\gamma_k=q]} \times \frac{\Gamma\left(\frac{\beta}{Q}\right)^Q}{\Gamma(\beta)} \prod_{q=1}^{Q} \vartheta_q^{\frac{\beta}{Q}-1} \times \beta^{a_\beta-1}\exp\{-b_\beta\,\beta\} \\
& \times \prod_{i=1}^{I}\prod_{k=1}^{K} \omega_k^{[\xi_i=k]} \times \frac{\Gamma\left(\frac{\alpha}{K}\right)^K}{\Gamma(\alpha)} \prod_{k=1}^{K} \omega_k^{\frac{\alpha}{K}-1} \times \alpha^{a_\alpha-1}\exp\{-b_\alpha\,\alpha\}\,,
\end{aligned}$$

where $\theta_{i,i'} = \operatorname{expit}\eta_{\phi(\xi_i,\xi_{i'})}$ and $[\cdot]$ is the Iverson bracket.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1,\ldots,B$, as follows:

1. Sample $\eta_{k,\ell}^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\begin{aligned}
\log p(\eta_{k,\ell} \mid \text{rest}) &\propto s_{k,\ell}\log(\operatorname{expit}\eta_{k,\ell}) + (n_{k,\ell}-s_{k,\ell})\log(1-\operatorname{expit}\eta_{k,\ell}) - \frac{1}{2\sigma^2}(\eta_{k,\ell}-\mu_{\phi(\gamma_k,\gamma_\ell)})^2 \\
&= s_{k,\ell}\,\eta_{k,\ell} - n_{k,\ell}\log(1+\exp\eta_{k,\ell}) - \frac{1}{2\sigma^2}(\eta_{k,\ell}-\mu_{\phi(\gamma_k,\gamma_\ell)})^2\,,
\end{aligned}$$

   where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$ and $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i < i' \text{ and } \phi(\xi_i,\xi_{i'}) = (k,\ell)\}$.

2. Sample $\mu_{q,r}^{(b+1)}$ from $\mathsf{N}(m, v^2)$, where

$$v^2 = \left(\frac{1}{\tau^2} + \frac{n_{q,r}}{\sigma^2}\right)^{-1} \qquad \text{and} \qquad m = v^2\left(\frac{\zeta}{\tau^2} + \frac{s_{q,r}}{\sigma^2}\right)\,,$$

   where $s_{q,r} = \sum_{\mathcal{S}_{q,r}} \eta_{k,\ell}$ and $n_{q,r} = \sum_{\mathcal{S}_{q,r}} 1$, with $\mathcal{S}_{q,r} = \{(k,\ell) : k \le \ell \text{ and } \phi(\gamma_k,\gamma_\ell) = (q,r)\}$.

3. Sample $\zeta^{(b+1)}$ from $\mathsf{N}(m, v^2)$, where

$$v^2 = \left(\frac{1}{\sigma^2_\zeta} + \frac{n_Q}{\tau^2}\right)^{-1} \qquad \text{and} \qquad m = v^2\left(\frac{\mu_\zeta}{\sigma^2_\zeta} + \frac{\mu_{\cdot\cdot}}{\tau^2}\right)\,,$$

   where $n_Q = Q(Q+1)/2$ and $\mu_{\cdot\cdot} = \sum_{q=1}^{Q}\sum_{r=q}^{Q}\mu_{q,r}$.

4. Sample $(\tau^2)^{(b+1)}$ from $\mathsf{IGam}(a, b)$, where

$$a = a_\tau + n_Q/2 \qquad \text{and} \qquad b = b_\tau + \tfrac{1}{2}\sum_{q=1}^{Q}\sum_{r=q}^{Q}(\mu_{q,r} - \zeta)^2\,,$$

   where $n_Q = Q(Q+1)/2$.

5. Sample $(\sigma^2)^{(b+1)}$ from $\mathsf{IGam}(a, b)$, where

$$a = a_\sigma + n_K/2 \qquad \text{and} \qquad b = b_\sigma + \tfrac{1}{2}\sum_{k=1}^{K}\sum_{\ell=k}^{K}(\eta_{k,\ell} - \mu_{\phi(\gamma_k,\gamma_\ell)})^2\,,$$

   where $n_K = K(K+1)/2$.

6. Sample $\gamma_k^{(b+1)}$ from a categorical distribution on $\{1, \ldots, Q\}$, such that:

$$\mathbb{Pr}\left[\gamma_k = q \mid \text{rest}\right] \propto \vartheta_q \times \prod_{\ell=k}^{K}\mathsf{N}(\eta_{k,\ell} \mid \mu_{\phi(q,\gamma_\ell)}, \sigma^2) \times \prod_{\ell=1}^{k-1}\mathsf{N}(\eta_{\ell,k} \mid \mu_{\phi(q,\gamma_\ell)}, \sigma^2)\,,$$

   for $q \in \{1, \ldots, Q\}$.

7. Sample $\boldsymbol{\vartheta}^{(b+1)}$ from $p(\boldsymbol{\vartheta} \mid \text{rest}) = \mathsf{Dir}\left(\boldsymbol{\vartheta} \mid \frac{\beta}{Q} + n_1, \ldots, \frac{\beta}{Q} + n_Q\right)$, where n_q is the number of clusters in super-cluster $q \in \{1, \ldots, Q\}$.

8. Sample $\beta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\beta \mid \text{rest}) \propto \log\Gamma(\beta) - K\log\Gamma(\beta/Q) + \frac{\beta}{Q}\sum_{q=1}^{Q}\log\vartheta_q - (a_\beta - 1)\log\beta - b_\beta\,\beta\,.$$

9. Sample $\xi_i^{(b+1)}$ from a categorical distribution on $\{1, \ldots, K\}$, such that:

$$\mathbb{Pr}\left[\xi_i = k \mid \text{rest}\right] \propto \omega_k \times \prod_{i'=i+1}^{I}\eta_{\phi(k,\xi_{i'})}^{y_{i,i'}}(1 - \eta_{\phi(k,\xi_{i'})})^{1-y_{i,i'}} \times \prod_{i'=1}^{i-1}\eta_{\phi(k,\xi_{i'})}^{y_{i',i}}(1 - \eta_{\phi(k,\xi_{i'})})^{1-y_{i',i}}\,.$$

   for $k \in \{1, \ldots, K\}$.

10. Sample $\boldsymbol{\omega}^{(b+1)}$ from $p(\boldsymbol{\omega} \mid \text{rest}) = \mathsf{Dir}\left(\boldsymbol{\omega} \mid \frac{\alpha}{K} + n_1, \ldots, \frac{\alpha}{K} + n_K\right)$, where n_k is the number of actors in cluster $k \in \{1, \ldots, K\}$.

11. Sample $\alpha^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\alpha \mid \text{rest}) \propto \log\Gamma(\alpha) - K\log\Gamma(\alpha/K) + \frac{\alpha}{K}\sum_{k=1}^{K}\log\omega_k - (a_\alpha - 1)\log\alpha - b_\alpha\,\alpha\,.$$

Prior Elicitation

$$\mu_\zeta = 0, \quad \sigma_\zeta^2 = 3, \quad a_\tau = 2, \quad b_\tau = 3, \quad a_\sigma = 2, \quad b_\sigma = 3, \quad a_\beta = 1, \quad b_\beta = 3, \quad a_\alpha = 1, \quad b_\alpha = 1\,.$$

9 WAIC

$$\mathrm{lppd} = \sum_{i=1}^{n} \log\left(\frac{1}{B}\sum_{b=1}^{B} p(y_i \mid \theta^b)\right)$$

$$p_{\mathrm{WAIC}_1} = 2\sum_{i=1}^{n}\left[\log\left(\frac{1}{B}\sum_{b=1}^{B} p(y_i \mid \theta^b)\right) - \frac{1}{B}\sum_{b=1}^{B}\log p(y_i \mid \theta^b)\right]$$

$$p_{\mathrm{WAIC}_2} = \frac{1}{B-1}\sum_{i=1}^{n}\sum_{b=1}^{B}(a_{i,b} - \bar{a}_i)^2\,, \qquad \bar{a}_i = \frac{1}{B}\sum_{b=1}^{B} a_{i,b}\,, \qquad a_{i,b} = \log p(y_i \mid \theta^b)$$

$$\mathrm{WAIC} = -2\,\mathrm{lppd} + 2\,p_{\mathrm{WAIC}}$$