Different priors

1 Class Model (Stochastic Block Model, SBM)

Likelihood

$$y_{i,i'} \mid \xi_i, \xi_{i'}, \{\eta_{k,\ell}\} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})}\right)$$

$$\begin{aligned} p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) &= \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} (\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{y_{i,i'}} (1 - \operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{1-y_{i,i'}} \\ &= \prod_{k=1}^{K} \prod_{\ell=k}^{K} (\operatorname{expit} \eta_{k,\ell})^{s_{k,\ell}} (1 - \operatorname{expit} \eta_{k,\ell})^{n_{k,\ell}-s_{k,\ell}} \end{aligned}$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$, $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k,\ell)\}$, and $\phi(x,y) = (\min\{x,y\}, \max\{x,y\})$ is a function to take into account that $\mathbf{Y} = [y_{i,i'}]$ is a symmetric matrix.

Prior

$$\begin{aligned} \eta_{k,\ell} \mid \zeta, \tau^2 &\overset{\text{iid}}{\sim} \mathsf{N}(\zeta, \tau^2) \\ \zeta &\sim \mathsf{N}(\mu_\zeta, \sigma_\zeta^2) \\ \tau^2 &\sim \mathsf{IGam}(a_\tau, b_\tau) \\ p(K, n) &\sim DP(\alpha) \\ \alpha &\sim \text{might add this later} \end{aligned}$$

Parameters

$$\boldsymbol{\Upsilon} = (\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{K,K}, K, n_1, \ldots, n_K, \alpha)\,,$$

where $\xi_i \in \{1, \ldots, K\}$, $i = 1, \ldots, I$, are the cluster assignments ($\xi_i = k$ means that actor i belongs to cluster k).

Hyper-parameters

$$(\mu_\zeta, \sigma_\zeta^2, a_\tau, b_\tau)\,.$$

Posterior

$$p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) = p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\})\, p(\{\eta_{k,\ell}\} \mid \zeta, \tau^2)\, p(\zeta)\, p(\tau^2)\, p(K, n\} \mid \alpha)\, p(\alpha)$$

$$\propto \prod_{i=1}^{I-1}\prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}}(1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\Big\{-\tfrac{1}{2\sigma_\zeta^2}(\zeta-\mu_\zeta)^2\Big\} \times (\tau^2)^{-(a_\tau-1)}\exp\Big\{-\frac{b_\tau}{\tau^2}\Big\}$$
$$\times \prod_{k=1}^{K}\prod_{\ell=k}^{K}(\tau^2)^{-1/2}\exp\{-\tfrac{1}{2\tau^2}(\eta_{k,\ell}-\zeta)^2\} \times \begin{cases} n_h, & \text{for } h=1,...,H \\ \alpha, & \text{for } h=H+1 \end{cases}$$

where $\theta_{i,i'} = \operatorname{expit}(\eta_{\phi(\xi_i,\xi_{i'})})$ and $[\cdot]$ is the Iverson bracket.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b=1,\ldots,B$, as follows:

1. Sample $\eta_{k,\ell}^{(b+1)}$, $\ell=k,\ldots,K$ and $k=1,\ldots,K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\log p(\eta_{k,\ell} \mid \text{rest}) \propto s_{k,\ell}\log(\operatorname{expit}\eta_{k,\ell}) + (n_{k,\ell}-s_{k,\ell})\log(1-\operatorname{expit}\eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell}-\zeta)^2$$
$$= s_{k,\ell}\,\eta_{k,\ell} - n_{k,\ell}\log(1+\exp\eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell}-\zeta)^2\,,$$

   where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$ and $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i<i' \text{ and } \phi(\xi_i,\xi_{i'}) = (k,\ell)\}$.

2. Sample $\xi_i^{(b+1)}$, $i=1,\ldots,I$, from a categorical distribution on $\{1,\ldots,K\}$, such that:

$$\Pr[\xi_i = k \mid \text{rest}] \propto \prod_{i'=i+1}^{I}\eta_{\phi(k,\xi_{i'})}^{y_{i,i'}}(1-\eta_{\phi(k,\xi_{i'})})^{1-y_{i,i'}} \times \prod_{i'=1}^{i-1}\eta_{\phi(\xi_{i'},k)}^{y_{i',i}}(1-\eta_{\phi(\xi_{i'},k)})^{1-y_{i',i}} \times \begin{cases} n_h, & h=1,...,H \\ \alpha, & h=H+1 \end{cases}$$

3. Sample $\zeta^{(b+1)}$ from $\mathsf{N}(m,v^2)$, where

$$v^2 = \left(\frac{1}{\sigma_\zeta^2} + \frac{K(K+1)/2}{\tau^2}\right)^{-1} \qquad \text{and} \qquad m = v^2\left(\frac{\mu_\zeta}{\sigma_\zeta^2} + \frac{1}{\tau^2}\sum_{k=1}^{K}\sum_{\ell=k}^{K}\eta_{k,\ell}\right).$$

4. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \mathsf{IGam}\left(\sigma^2 \mid a_\tau + \frac{K(K+1)}{4}, b_\tau + \frac{1}{2}\sum_{k=1}^{K}\sum_{\ell=k}^{K}(\eta_{k,\ell}-\zeta)^2\right)$.