Different priors

1 Dirichlet Process

1.1 Class Model (Stochastic Block Model, SBM)

Likelihood

$$y_{i,i'} \mid \xi_i, \xi_{i'}, \{\eta_{k,\ell}\} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})}\right)$$

$$\begin{aligned} p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) &= \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} (\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{y_{i,i'}} (1 - \operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{1-y_{i,i'}} \\ &= \prod_{k=1}^{K} \prod_{\ell=k}^{K} (\operatorname{expit} \eta_{k,\ell})^{s_{k,\ell}} (1 - \operatorname{expit} \eta_{k,\ell})^{n_{k,\ell}-s_{k,\ell}} \end{aligned}$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$, $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k,\ell)\}$, and $\phi(x,y) = (\min\{x,y\}, \max\{x,y\})$ is a function to take into account that $\mathbf{Y} = [y_{i,i'}]$ is a symmetric matrix.

Prior

$$\begin{aligned} \eta_{k,\ell} \mid \zeta, \tau^2 &\overset{\text{iid}}{\sim} \mathsf{N}(\zeta, \tau^2) \\ \zeta &\sim \mathsf{N}(\mu_\zeta, \sigma_\zeta^2) \\ \tau^2 &\sim \mathsf{IGam}(a_\tau, b_\tau) \\ \xi_i \mid \alpha &\sim CRP(\alpha) \\ \alpha &\sim \text{might add this later} \end{aligned}$$

Parameters

$$\boldsymbol{\Upsilon} = (\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{K,K}, \xi_1, \ldots, \xi_K, \zeta, \tau^2, \alpha)\,,$$

where $\xi_i \in \{1, \ldots, K\}$, $i = 1, \ldots, I$, are the cluster assignments ($\xi_i = k$ means that actor i belongs to cluster k).

Hyper-parameters

$$(\mu_\zeta, \sigma_\zeta^2, a_\tau, b_\tau)\,.$$

Posterior

$$p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) = p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\})\, p(\{\eta_{k,\ell}\} \mid \zeta, \tau^2)\, p(\zeta)\, p(\tau^2)\, p(\{\xi_i\} \mid \alpha)\, p(\alpha)$$

$$
\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\Big\{-\tfrac{1}{2\sigma_\zeta^2}(\zeta-\mu_\zeta)^2\Big\} \times (\tau^2)^{-(a_\tau-1)} \exp\Big\{-\frac{b_\tau}{\tau^2}\Big\}
$$
$$
\times \prod_{k=1}^{K} \prod_{\ell=k}^{K} (\tau^2)^{-1/2} \exp\{-\tfrac{1}{2\tau^2}(\eta_{k,\ell}-\zeta)^2\} \times \begin{cases} n_k, & \text{for } k=1,...,K \\ \alpha, & \text{for } k=K+1 \end{cases}
$$

where $\theta_{i,i'} = \mathrm{expit}(\eta_{\phi(\xi_i,\xi_{i'})})$.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \ldots, B$, as follows:

1. Sample $\eta_{k,\ell}^{(b+1)}$, $\ell = k, \ldots, K$ and $k = 1, \ldots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:
$$
\begin{aligned}
\log p(\eta_{k,\ell} \mid \text{rest}) &\propto s_{k,\ell} \log(\mathrm{expit}\, \eta_{k,\ell}) + (n_{k,\ell} - s_{k,\ell}) \log(1 - \mathrm{expit}\, \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell}-\zeta)^2 \\
&= s_{k,\ell}\, \eta_{k,\ell} - n_{k,\ell} \log(1 + \exp \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell}-\zeta)^2 \,,
\end{aligned}
$$
where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$ and $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k,\ell)\}$.

2. Sample $\xi_i^{(b+1)}$, $i = 1, \ldots, I$, from a categorical distribution prop to $\begin{cases} n_k, & \text{for } k=1,...,K \\ \alpha, & \text{for } k=K+1 \end{cases}$

3. Sample $\zeta^{(b+1)}$ from $\mathsf{N}(m, v^2)$, where
$$
v^2 = \left(\frac{1}{\sigma_\zeta^2} + \frac{K(K+1)/2}{\tau^2}\right)^{-1} \qquad \text{and} \qquad m = v^2\left(\frac{\mu_\zeta}{\sigma_\zeta^2} + \frac{1}{\tau^2}\sum_{k=1}^{K}\sum_{\ell=k}^{K} \eta_{k,\ell}\right) .
$$

4. Sample $(\tau^2)^{(b+1)}$ from $p(\tau^2 \mid \text{rest}) = \mathsf{IGam}\left(\tau^2 \mid a_\tau + \frac{K(K+1)}{4}, b_\tau + \frac{1}{2}\sum_{k=1}^{K}\sum_{\ell=k}^{K}(\eta_{k,\ell}-\zeta)^2\right)$.

1.2 Class - Distance Model

Likelihood

$$
y_{i,i'} \mid \zeta, \{\boldsymbol{u}_k\}, \xi_i, \xi_{i'} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\mathrm{expit}\, \eta_{\phi(\xi_i,\xi_{i'})}\right)
$$

where

$$
\eta_{k,\ell} = \zeta - \|\boldsymbol{u}_k - \boldsymbol{u}_\ell\|
$$

Prior

$$
\begin{aligned}
\boldsymbol{u}_k \mid \sigma^2 &\overset{\mathsf{iid}}{\sim} \mathsf{N}(\mathbf{0}, \sigma^2 \mathbf{I}) \\
\sigma^2 &\sim \mathsf{IGam}(a_\sigma, b_\sigma) \\
\zeta \mid \omega^2 &\sim \mathsf{N}(0, \omega^2) \\
\omega^2 &\sim \mathsf{IGam}(a_\omega, b_\omega) \\
\xi_i \mid \alpha &\sim CRP(\alpha) \\
\alpha &\sim \text{might add this later}
\end{aligned}
$$

Parameters

$$
\boldsymbol{\Upsilon} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K, \sigma^2, \zeta, \omega^2, \xi_1, \ldots, \xi_K, \alpha)\,,
$$

where $\zeta \in \mathbb{R}$ and $\boldsymbol{u}_k = (u_{k,1}, \ldots, u_{k,Q}) \in \mathbb{R}^Q$.

Hyper-parameters

$$
(a_\sigma, b_\sigma, a_\omega, b_\omega)\,.
$$

Posterior

$$
\begin{aligned}
p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \zeta, \{\boldsymbol{u}_k\}, \{\xi_i\}, \{\xi_{i'}\})\, p(\{\boldsymbol{u}_k\} \mid \sigma^2)\, p(\sigma^2)\, p(\zeta \mid \omega^2)\, p(\omega^2)\, p(\{\xi_i\} \mid \alpha)\, p(\alpha) \\
&\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \prod_{k=1}^{K} (\sigma^2)^{-K/2} \exp\left\{-\tfrac{1}{2\sigma^2}\, \|\boldsymbol{u}_k\|^2\right\} \times (\sigma^2)^{-(a_\sigma+1)} \exp\left\{-\frac{b_\sigma}{\sigma^2}\right\} \\
&\quad \times (\omega^2)^{-1/2} \exp\left\{-\tfrac{1}{2\omega^2}\, \zeta^2\right\} \times (\omega^2)^{-(a_\omega+1)} \exp\left\{-\frac{b_\omega}{\omega^2}\right\} \times \begin{cases} n_k, & \text{for } k = 1, ..., K \\ \alpha, & \text{for } k = K+1 \end{cases}
\end{aligned}
$$

where $\theta_{i,i'} = \operatorname{expit} \eta_{\phi(\xi_i, \xi_{i'})}$.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \ldots, B$, as follows:

1. Sample $\boldsymbol{u}_k^{(b+1)}$, $k = 1, \ldots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$
p(\boldsymbol{u}_k \mid \text{rest}) \propto \prod_{\substack{i,i':i<i' \\ \xi_i = k \text{ or } \xi_{i'} = k}} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{2\sigma^2}\, \|\boldsymbol{u}_k\|^2\right\}.
$$

2. Sample $\zeta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:

$$p(\zeta \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\{-\tfrac{1}{\omega^2}\,\zeta^2\}.$$

3. Sample $\xi_i^{(b+1)}$, $i = 1, \ldots, I$, from a categorical distribution proportional to $\begin{cases} n_k, & \text{for } k = 1, ..., K \\ \alpha, & \text{for } k = K+1 \end{cases}$

4. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \mathsf{IGam}\left(\sigma^2 \mid a_\sigma + \frac{K\,Q}{2}, b_\sigma + \frac{1}{2}\sum_{k=1}^{K} \|\boldsymbol{u}_k\|^2\right)$.

5. Sample $(\omega^2)^{(b+1)}$ from $p(\omega^2 \mid \text{rest}) = \mathsf{IGam}\left(\omega^2 \mid a_\sigma + \frac{1}{2}, b_\sigma + \frac{1}{2}\zeta^2\right)$.

6. Sample $\alpha^{(b+1)}$ according to ...

2 Pitman-Yor prior

2.1 Class Model (Stochastic Block Model, SBM)

Likelihood

$$y_{i,i'} \mid \xi_i, \xi_{i'}, \{\eta_{k,\ell}\} \overset{\text{ind}}{\sim} \mathsf{Ber}\left(\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})}\right)$$

$$\begin{aligned} p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\}) &= \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} (\operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{y_{i,i'}} (1 - \operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})})^{1-y_{i,i'}} \\ &= \prod_{k=1}^{K} \prod_{\ell=k}^{K} (\operatorname{expit} \eta_{k,\ell})^{s_{k,\ell}} (1 - \operatorname{expit} \eta_{k,\ell})^{n_{k,\ell}-s_{k,\ell}} \end{aligned}$$

where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$, $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i,i') : i < i' \text{ and } \phi(\xi_i,\xi_{i'}) = (k,\ell)\}$, and $\phi(x,y) = (\min\{x,y\}, \max\{x,y\})$ is a function to take into account that $\mathbf{Y} = [y_{i,i'}]$ is a symmetric matrix.

Prior

$$\begin{aligned} \eta_{k,\ell} \mid \zeta, \tau^2 &\overset{\text{iid}}{\sim} \mathsf{N}(\zeta, \tau^2) \\ \zeta &\sim \mathsf{N}(\mu_\zeta, \sigma_\zeta^2) \\ \tau^2 &\sim \mathsf{IGam}(a_\tau, b_\tau) \\ \xi_i \mid \alpha, \sigma &\sim CRP(\alpha, \sigma) \\ \alpha, \sigma &\sim \text{might add this later} \end{aligned}$$

Parameters

$$\boldsymbol{\Upsilon} = (\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{K,K}, \xi_1, \ldots, \xi_K, \zeta, \tau^2, \alpha, \sigma)\,,$$

where $\xi_i \in \{1, \ldots, K\}$, $i = 1, \ldots, I$, are the cluster assignments ($\xi_i = k$ means that actor i belongs to cluster k).

Hyper-parameters

$$(\mu_\zeta, \sigma^2_\zeta, a_\tau, b_\tau)\,.$$

Posterior

$$\begin{aligned} p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \{\xi_i\}, \{\eta_{k,\ell}\})\, p(\{\eta_{k,\ell}\} \mid \zeta, \tau^2)\, p(\zeta)\, p(\tau^2)\, p(\{\xi_i\} \mid \alpha, \sigma)\, p(\alpha)\, p(\sigma) \\ &\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \exp\Big\{-\tfrac{1}{2\sigma^2_\zeta}(\zeta - \mu_\zeta)^2\Big\} \times (\tau^2)^{-(a_\tau - 1)} \exp\Big\{-\frac{b_\tau}{\tau^2}\Big\} \\ &\quad \times \prod_{k=1}^{K} \prod_{\ell=k}^{K} (\tau^2)^{-1/2} \exp\{-\tfrac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2\} \times \begin{cases} n_k - \sigma, & \text{for } k = 1, ..., K \\ \alpha + K\sigma, & \text{for } k = K+1 \end{cases} \end{aligned}$$

where $\theta_{i,i'} = \operatorname{expit}(\eta_{\phi(\xi_i, \xi_{i'})})$.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \ldots, B$, as follows:

1. Sample $\eta_{k,\ell}^{(b+1)}$, $\ell = k, \ldots, K$ and $k = 1, \ldots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:

$$\begin{aligned} \log p(\eta_{k,\ell} \mid \text{rest}) &\propto s_{k,\ell} \log(\operatorname{expit} \eta_{k,\ell}) + (n_{k,\ell} - s_{k,\ell}) \log(1 - \operatorname{expit} \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2 \\ &= s_{k,\ell}\, \eta_{k,\ell} - n_{k,\ell} \log(1 + \exp \eta_{k,\ell}) - \frac{1}{2\tau^2}(\eta_{k,\ell} - \zeta)^2\,, \end{aligned}$$

   where $s_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} y_{i,i'}$ and $n_{k,\ell} = \sum_{\mathcal{S}_{k,\ell}} 1$, with $\mathcal{S}_{k,\ell} = \{(i, i') : i < i' \text{ and } \phi(\xi_i, \xi_{i'}) = (k, \ell)\}$.

2. Sample $\xi_i^{(b+1)}$, $i = 1, \ldots, I$, from a categorical distribution prop to $\begin{cases} n_k - \sigma, & \text{for } k = 1, ..., K \\ \alpha + K\sigma, & \text{for } k = K+1 \end{cases}$

3. Sample $\zeta^{(b+1)}$ from $\mathsf{N}(m, v^2)$, where

$$v^2 = \left(\frac{1}{\sigma^2_\zeta} + \frac{K(K+1)/2}{\tau^2}\right)^{-1} \qquad \text{and} \qquad m = v^2 \left(\frac{\mu_\zeta}{\sigma^2_\zeta} + \frac{1}{\tau^2} \sum_{k=1}^{K} \sum_{\ell=k}^{K} \eta_{k,\ell}\right).$$

4. Sample $(\tau^2)^{(b+1)}$ from $p(\tau^2 \mid \text{rest}) = \mathsf{IGam}\left(\tau^2 \mid a_\tau + \frac{K(K+1)}{4}, b_\tau + \frac{1}{2}\sum_{k=1}^{K}\sum_{\ell=k}^{K}(\eta_{k,\ell} - \zeta)^2\right)$.

5. Sample $\alpha^{(b+1)}$ according to ...

6. Sample $\sigma^{(b+1)}$ according to ...

2.2 Class - Distance Model

Likelihood

$$y_{i,i'} \mid \zeta, \{\boldsymbol{u}_k\}, \xi_i, \xi_{i'} \stackrel{\text{ind}}{\sim} \mathsf{Ber}\left(\operatorname{expit} \eta_{\phi(\xi_i, \xi_{i'})}\right)$$

where

$$\eta_{k,\ell} = \zeta - \|\boldsymbol{u}_k - \boldsymbol{u}_\ell\|$$

Prior

$$\begin{aligned}
\boldsymbol{u}_k \mid \sigma^2 &\stackrel{\text{iid}}{\sim} \mathsf{N}(\mathbf{0}, \sigma^2\mathbf{I}) \\
\sigma^2 &\sim \mathsf{IGam}(a_\sigma, b_\sigma) \\
\zeta \mid \omega^2 &\sim \mathsf{N}(0, \omega^2) \\
\omega^2 &\sim \mathsf{IGam}(a_\omega, b_\omega) \\
\xi_i \mid \alpha, \sigma &\sim CRP(\alpha, \sigma) \\
\alpha, \sigma &\sim \text{might add this later}
\end{aligned}$$

Parameters

$$\boldsymbol{\Upsilon} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K, \sigma^2, \zeta, \omega^2, \xi_1, \ldots, \xi_K, \alpha, \sigma)\,,$$

where $\zeta \in \mathbb{R}$ and $\boldsymbol{u}_k = (u_{k,1}, \ldots, u_{k,Q}) \in \mathbb{R}^Q$.

Hyper-parameters

$$(a_\sigma, b_\sigma, a_\omega, b_\omega)\,.$$

Posterior

$$\begin{aligned}
p(\boldsymbol{\Upsilon} \mid \mathbf{Y}) &= p(\mathbf{Y} \mid \zeta, \{\boldsymbol{u}_k\}, \{\xi_i\}, \{\xi_{i'}\})\, p(\{\boldsymbol{u}_k\} \mid \sigma^2)\, p(\sigma^2)\, p(\zeta \mid \omega^2)\, p(\omega^2)\, p(\{\xi_i\} \mid \alpha, \sigma)\, p(\alpha)\, p(\sigma) \\
&\propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1 - \theta_{i,i'})^{1-y_{i,i'}} \times \prod_{k=1}^{K} (\sigma^2)^{-K/2} \exp\left\{-\tfrac{1}{2\sigma^2}\|\boldsymbol{u}_k\|^2\right\} \times (\sigma^2)^{-(a_\sigma+1)} \exp\left\{-\frac{b_\sigma}{\sigma^2}\right\} \\
&\quad \times (\omega^2)^{-1/2} \exp\left\{-\tfrac{1}{2\omega^2}\zeta^2\right\} \times (\omega^2)^{-(a_\omega+1)} \exp\left\{-\frac{b_\omega}{\omega^2}\right\} \times \begin{cases} n_k - \sigma, & \text{for } k = 1, ..., K \\ \alpha + K\sigma, & \text{for } k = K+1 \end{cases}
\end{aligned}$$

where $\theta_{i,i'} = \operatorname{expit} \eta_{\phi(\xi_i,\xi_{i'})}$.

MCMC

The algorithm proceeds by generating a new state $\boldsymbol{\Upsilon}^{(b+1)}$ from a current state $\boldsymbol{\Upsilon}^{(b)}$, $b = 1, \ldots, B$, as follows:

1. Sample $\boldsymbol{u}_k^{(b+1)}$, $k = 1, \ldots, K$, according to a Metropolis–Hastings Algorithm, considering the fcd:
$$p(\boldsymbol{u}_k \mid \text{rest}) \propto \prod_{\substack{i,i':i<i' \\ \xi_i=k \text{ or } \xi_{i'}=k}} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{2\sigma^2} \, \|\boldsymbol{u}_k\|^2\right\}.$$

2. Sample $\zeta^{(b+1)}$ according to a Metropolis–Hastings Algorithm, considering the fcd:
$$p(\zeta \mid \text{rest}) \propto \prod_{i=1}^{I-1} \prod_{i'=i+1}^{I} \theta_{i,i'}^{y_{i,i'}} (1-\theta_{i,i'})^{1-y_{i,i'}} \times \exp\left\{-\tfrac{1}{\omega^2} \, \zeta^2\right\}.$$

3. Sample $\xi_i^{(b+1)}$, $i = 1, \ldots, I$, from a categorical distribution prop to $\begin{cases} n_k - \sigma, & \text{for } k = 1, ..., K \\ \alpha + K\sigma, & \text{for } k = K+1 \end{cases}$

4. Sample $(\sigma^2)^{(b+1)}$ from $p(\sigma^2 \mid \text{rest}) = \mathsf{IGam}\left(\sigma^2 \mid a_\sigma + \frac{K\,Q}{2}, b_\sigma + \frac{1}{2}\sum_{k=1}^{K} \|\boldsymbol{u}_k\|^2\right)$.

5. Sample $(\omega^2)^{(b+1)}$ from $p(\omega^2 \mid \text{rest}) = \mathsf{IGam}\left(\omega^2 \mid a_\sigma + \frac{1}{2}, b_\sigma + \frac{1}{2}\zeta^2\right)$.

6. Sample $\alpha^{(b+1)}$ according to ...

7. Sample $\sigma^{(b+1)}$ according to ...