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Part I

Week 1: Markets and Uncertainty

1 The model Walrasian Equi

We work with N agents in a exchange economy. We will think about consumption as consumption bundles with l goods $X_i \in R^l_+$. Agents are endowed by $e_i \in R^l_+$. We will be thinking in terms of cardinal of preferences $u_i : R^+ \to R$. We are assuming static model and also that people are able to evaluate their bundle.

So this is going to be a **Walrasian Equilibrium**. The primitive is everything except their consumption. Indeed, the equilibirum provides the "efficient" allocation of goods among agents, and consumption bundles are vector for each agent of size l as there are l goods in the economy. Hence the results of the economy is going to be $(x_1, ..x_n) \in R^{n\times l}_+$ under the assumption that $\sum^n x_i \leq \sum^n e_i$, and positive prices such that $p \in R^l_+ \implies p \times x_i \leq p \times e_i$. Hence, we have that $x_i \in \arg\max u_i$ such that $x_i' : p \times x_i \leq p \times e_i$.

Another way to write this is $u_i(x_i') > u_i(x_i) \implies px_i' > pe_i$ (in words: if there exist another allocation x' that increases utility of agent i, then it is not chosen because it is not affordable).

This model under some conditions (1) provide existence of a non-empty set of consumption and (2) achieve Pareto Efficient. The latter is interesting but it take no stand on redistribution or initial allocation of endowments.

1.1 Pareto Efficient

Definitions:

- $(x_1, ..x_n) \in R^{n\times l}_+$ such that it is feasible $\sum x_i \leq \sum e_i$
- and there does not exist $(x_1', ..x_n')$ such that $\sum x_i' \leq \sum e_i$ and $u_i(x_i') \geq u_i(x_i)$ for all i and > for some.
- *Weak PE* is same definition but with strict inequality for all: $u_i(x_i') > u_i(x_i)$ for all i

This mainly rely on the *Local non-satiation* assumption:

$$\text{If } \forall x_i \in R^l_+ \text{ and all } \epsilon > 0$$
$$\text{then there exists } x'_i \text{ s.t. } u_i(x'_i) > u_i(x_i) \text{ and } |x_i - x'_i| < \epsilon$$

1.2 1st Welfare Theorem

If $(x_1, ..x_n)$ is a Walrasian Equi for an economy, then $(x_1, ..x_n)$ is weakly Pareto Efficient. And there is no additional condition that applies.

If we add Local Non-Satiation, then for all i, we have Pareto Efficiency.

1.2.1 Proof of the First WE

W.Equi says that:

- x_i demanded by i at some price p implies that $\implies u_i(x'_i) > u_i(x_i) \implies px'_i > px_i$
- if you add LNS, then in additional $u_i(x'_i) \geq u_i(x_i) \implies px'_i \geq px_i$

Now we suppose that there exist $(x'_1, ..x'_n)$ such that $u_i(x'_i) > u_i(x_i), \forall i$ and $\sum x'_i \leq \sum e_i$ and $px'_i > px_i$. Then we have $p \sum x'_i > p \sum e_i$ which is a CONTRADICTION.

1.2.2 Existence

To ensure existence we need some condition on u_i:

- continuous
- local non-satiation / non-decreasing
- quasi concavity

Then, we consider $e_i \in R^l_{++}$ (++ = out of the boundaries), then there exist a W. Equi.

1.3 Unpacking other implicit conditions

Previously, we assumed implicitly:

- UTILITY MAXIMIZER which means the problem is static and has full scope.
- PRIVATE PREFERENCE $u_i(x_i)$ means there is no externality (no altruism for example)
- PRICES, we assume agents are prices taker and the prices are exogenous. And they can lead to different equilibria.

- MARKET ARE COMPLETE
- PRIVATE GOODS no national defence or clean air. we only consider goods that a single person can buy excluding other buyers.

Allowing for incomplete information This will lead to two problems, the discontinuity of the info set will question the existence of equilibrium and it will also create inefficiency.

2 Sunspot example

Note that there is no trading in this example.

- $n = 2$ and $l = 2$ and $e_1 = (1, 0)$ and $e_2 = (0, 1)$
- $p_1 = 1$ and $p_2 = p$
- $u_i(x_{i,1}, x_{i,2}) = (\min\{x_{i,1}, x_{i,2}\})^\alpha$ this means complement goods type of IC.
- Pareto set is the 45 degree line
- as such the demand are
  - $x_1 = \left(\frac{1}{1+p}, \frac{1}{1+p}\right)$
  - $x_2 = \left(\frac{p}{1+p}, \frac{p}{1+p}\right)$

We then draw and edgeworth box.

The sunspots (Cass and Shell) - 1983 We then add states. Two states s and s' and no insurance against the sunspot.

- Equilibrium 1: $p = 1$ and we ignore the states
- Equilibrium 2: $p = 1/3$ in s and $p = 3$ in s'.
  - in state s: $x_1 = \left(\frac{3}{4}, \frac{3}{4}\right)$ and $x_2 = \left(\frac{1}{4}, \frac{1}{4}\right)$
  - in state s': $x_1 = \left(\frac{1}{4}, \frac{1}{4}\right)$ and $x_2 = \left(\frac{3}{4}, \frac{3}{4}\right)$

We now assume that there are equal chances to be in either of the states. Then, the utility are:

- Equilibrium 1: each agent gets $(1/2)^\alpha$
- Equilibrium 2: $u_i = 0.5(1/4)^\alpha + 0.5(3/4)^\alpha$

The ranking of those utility now relies on α, and their is a Pareto order and the FWT fails to apply.

3 Arrow Debreu contingent state

To resolve the FWT, AD proposes a set of contingent claims. However, this creates two problems. First, this would create a huge state space and thus not all those contingent claims might not be tradable ex post. Also, this creates asymmetry of information.

Hence, REE (= Rational Expectation Equi) will try to deal with both problems, but down this road we will lose both existence property and Pareto Efficiency.

3.1 Model

Primitive:

- S states (finite state space)
- I^i information over states, which is a partition of S. We think of a partition such that for state that you do not observe you can't distinguish them from one another. This is coming from Baysian rule. Indeed if I see 3 states and you see only 2 but not the third then this would create contradiction. Indeed, I would be able to make inference about what I am not observing which I should not.
  - The union of all partitions is equal to the whole set S
  - The intersection between two info sets is empty

Definition: Let us consider 2 info set I and I'. We say that I is coarser than I' if $\forall \pi' \in I'$ there exist $\pi \in I$ s.t. $\pi' \subset \pi$. This is a partial order. And we say that I' is finer than I, because I can better distinguish between states in I'.

Definition: To talk about common knowledge, we want to distinguish between "meet" (= common knowledge, they both know it) and "joint" (=pooling info, what could they know if they were sharing info). The "meet" is the finest partition coarser than I and I'. The "join" is coarsest partition that is finer than both I and I'.

Example of Meet/Join:

- Let us define two info sets $I = ((s_1),(s_2),(s_3,s_4))$ and $I' = ((s_1),(s_2,s_3),(s_4)$.
- Here the common knowledge is that s_1 is a singleton and happen along and the rest need to be coarser. So the meet is $I \wedge I' = ((s_1),(s_2,s_3,s_4))$
- Then, the joint must be finer than both info set, so the join in $I \vee I' = ((s_1),(s_2),(s_3),(s_4))$. This means that if players were talking they would be able to identify all state as singleton.

We know want to define a function $z_i : S \to R^l_+$ that maps states S, to some outcome, consumption bundles for example. The question is now to know if this is measurable. This means that when taking different actions, the agent knows that the states is different. Measurable can be defined as $z_i(s) \neq z_i(s') \implies I^i(s) \neq I^i(s')$. The function is thus state dependent and the function z is constant over all the elements of the partition of I.

Definition: We now want to generate more information based on prices. Knowing the algorithm used to announce the prices, agents should be able to infer something about the states. We define I_z as the finest partition to measure wrt z. Then we have $I_z(s) \neq I_z(s') \Leftrightarrow z(s) \neq z(s')$, which means that everyone obtains the same information from prices.

4 Rational Expectations Equilibria

Exchange Economy:

- N agents / l goods / R^l_+ consumption space (as before)
- S state space
- each person i have I^i partition on S
- Endowment is now a mapping $e_i : S \to R^l_+$ and we want this to be measurable wrt I^i.
- utility function $u_i : R^{l \times S}_+ \to R$

Then the REE is a pair $(p, \{x_i\}_i)$ such that:

- price function $p : S \to \triangle^l_+ = \{p \in R^l_+ \text{ s.t. } \sum p_i = 1\}$. Here, normalisation of the price matters, the sum to 1 makes it s.t. we call out in relative prices.
- consumption bundle or state contingent plan $x_i : S \to R^l_+$ such that:
  - x_i is measurable wrt to I^i and I_p (this is where the REE kicks in), so wrt $I^i \vee I_p$.
  - market clearing conditions $\sum x_i \leq \sum e_i$
  - (ex ante) $x_i \in \arg\max u_i(x_i)$ under the condition described above.

Sunspot Example We have agent's info set defined as $I^i = ((s),(s'))$ and $p(s) = [1,1/3]$ and $p(s') = [1,3]$. But the interesting part is that the Walrasian auctioneer announcing the price know which state is realized. As such, even if ex-ante agents do not know in which state we are in, once the prices are announced then $I^i \vee I_p = ((s),(s'))$ (each state is a singleton)

Hence, we see that the FWT does not hold and the REE fails to accomplish the Pareto Efficiency.

4.1 Existence Fails - Kreps 1977

We know that REE does not imply efficiency. Moreover, existence of REE is not ensure. Let's consider an example.

- $n = 2$ / $l = 2$ / $S = ((s),(s'))$
- $e_i(s) = e_i(s') = (1,10)$ agents have the same endowment in both states
- However, agent 1 knows the states, $I^1 = ((s),(s'))$, and agent 2 don't, $I^2 = ((s,s'))$.
- $u_i(x_i(s),s) = a_i(s)\ln(x_{i1}(s)) + x_{i2}(s)$ and $a_1(s) = 1$ and $a_1(s') = 2$ and $a_2(s) = 2$ and $a_2(s') = 1$.

Claim There does not exist an equilibrium in this economy.

- Case 1 *REE non revealing*: $p(s) = p(s')$. Agent 1 wants a different consumption in each state, but agent 2 does not know the states and prices are not telling him anything so he wants to ask for the same consumption in both states. But then, market don't clear.
- Case 2 *REE non revealing* : $p(s) \neq p(s')$. As such, when the state is realized agent 2 can also infer in which states we are and adjust his demand. Then, market clears and each agent demand differs in each states according to their preferences. BUT, when we do the computation, it is necessary to have prices equal which is a contradiction. So again, an equilibrium does not exist.

Demands We compute the demand for each goods

- $x_{i,1}(s) = \dfrac{E_i[a_i(s)|I_i \vee I_P(s)]}{p(s)}$ for $I_1 = \{(s),(s')\} = I_2$
  - $x_{11}(s) = 1/p(s)$ and $x_{11}(s') = 2/p(s')$
  - $x_{21}(s) = 2/p(s)$ and $x_{21}(s') = 1/p(s')$

- Putting those together, when we compute market clearing condition we have that $x_{11}(s)+x_{21}(s) = 2 = x_{11}(s')+x_{21}(s') \implies$ we have prices equal whereas we started with different prices

4.2 Allen - 1981

- N individuals / l goods / S is finite
- $e_i(s) \in R_+^{l,|S|}$
- $u_i(x_i(s),s) : R_+^{l,|S|} \to R$.

As long as there is a dense set of utility function which do not overlap to much, then existence can be restored. Allen defined a generic specification economy where there exist REE which is fully revealing.

$\implies$ There exist a WE for all s, if utility all differ, then prices are revealing $p(s) \neq p(s'), \forall s, s'$ and I_p are singleton for each states.

She shows that there exist a fix point

Grossam Stiglits - Paradox We defined two state space. In one, we can't distinguish any state and there other is only populated with singleton for each state. We now model a cost to go from the "don't know" to the "know". However, is prices are fully revealing then, no one want's to pay the cost. But then, if no one pay the cost and stays with the "don't know" info set, then how can prices be revealing as price taker's agents.

Where to go from there you need to put restrictions on prices + prices should be measurable wrt $(x_i, ..., ...)$ the joint.

5 Incomplete information and Common Knowledge

The incomplete information will make us loose Pareto Efficient, Existence and challenge the coherent definition of economic equilibrium and predictions. Finally, Aumann's paper has implication of common knowledge and then agents is rational cannot agree to disagree.

Model:

- $n = 2$ / S is finite
- I_i is the info set of i about S
- $P(s)$ prior prob of $s \in S$, common to both agents.

Common Knowledge

Let us define some subset of S, for E events s.t. $E \subset S$. Then, we define E is a common knowledge at s, if $I^m = I_1 \wedge I_2 \subset E$.

Examples

- $I_1 = \{(1), (2,3), (4,5,6)\}$
- $I_2 = \{(1), (2), (3,4), (5,6)\}$
- The meet: $I^m = \{(1), (2,3,4,5,6)\}$

From this example, $E = \{1,2\}$ is c.k. at $s = 1$ but not at $s = 2$

Conditional probability

Fro a given event A, the conditional probability is:

$$q_i(A|s) = \frac{p(A \cap I^i(s))}{p(I^i(s))}$$

For example: for $I^i = \{\{1\}, \{2,3,4\}$ where all outcome are equally likely, we have that for $A = \{1,2\}$:

$$\begin{aligned}
q_i(A|1) &= 1 \\
q_i(A|2) &= \frac{p(2)}{p(\{2,3,4\})} = 1/3 \\
q_i(A|3) &= \frac{p(2)}{p(\{2,3,4\})} = 1/3 \\
q_i(A|4) &= \frac{p(2)}{p(\{2,3,4\})} = 1/3
\end{aligned}$$

Theorem - Aumann

If $q_1(A|S) = q_1$ and $q_2(A|S) = q_2$ is c.k. at s, then $q_1 = q_2$.
More formally, if $E = \{s|q_1(A|S) = q_1, q_2(A|S) = q_2\}$ s.t. $I^M(s) \subset E$, then $q_1 = q_2$
Converse is not true.

Proof

- $E = \{s|q_1(A|S) = q_1, q_2(A|S) = q_2\}$
- The meet: $I^m(s) \subset E$

Then we can re-write $I^m(s) = \cup B^k$ for $B^k \in I^1 \forall k$. Then we have that:

$$\begin{aligned} q_1(A|S) &= \frac{P(A \cup B^k)}{P(B^k)} \text{ for } s \in B^k \\ &\implies q_1 \cdot P(B^k) = P(A \cup B^k) \\ &\implies q_1 \sum_k P(B^k) = \sum_k P(A \cup B^k) \\ &\implies q_1 P(I^m(s)) = P(A \cup I^m(s)) \\ &\implies q_1 = \frac{P(A \cup I^m(s))}{P(I^m(s))} = q_2 \end{aligned}$$

Theorem - No "Trade" = No betting

Let us take risk-neutral agents, an considered an event $A \subset S$, the event both agent are betting on.

- Agent 1 pays x to 2 is A occurs
- Agent 2 pays y to 1 if not A.

Then, what are the condition for trade to occur? To answer, we considere to information set $I^i(s)$ and there posterior are compute as:

$$\begin{aligned} -x \cdot q_1(A|s) + y \cdot (1 - q_1(A|S)) > 0 \text{ that is preferred to bet} \\ x \cdot q_1(A|S) - y \cdot (1 - q_2(A|S)) > 0 \text{ that is preferred to bet} \end{aligned}$$

For trade to occur the two inequality must hold such that $q_1(A|S) < \frac{y}{x+y} < q_2(A|S)$

The theorem then says that if both agent agree to this bet, and thus prefer is and make is public information and thus c.k. at s comes in. And we can define a new $I^m(s) \subset E = \{s | q_1(A|S) < \frac{y}{x+y} < q_2(A|S)\}$ which is not possible. Actually the middle term is $\frac{P(A|B)}{P(B)}$ with the same prior, it is impossible that q_1 is less and q_2 is more systematically.

Rubinstein - Email game

Each agent have a decision to invert or not. If an agent invest the cost is 30 for each agent, but is they do not they get 0. If they both invest, then the return is 80 for both with 50% change or 40 with 50%. If only one invest, the return is 0.

Information Agent 1 know the state and need to communicate. But, the email can get lost. In the Bad state, Agent 1 does not get the email, if it is the Good state, Agent 1 send an email but it can get lost with probability ϵ.

Good state:

	not	inv
not	(0,0)	(0,-30)
inv	(-30,0)	(10,10)

Bad state:

	not	inv
not	(0,0)	(0,-30)
inv	(-30,0)	(-10,-10)

We only need to know how many emails were sent by Agent 1 and how many emails were sent by Agent 2.

Hence, $I^1 = \{(0,0),[(1,0),(1,1)],\}...$, that is for a fix nb of email sent by 1, 1 can never distinguish between the two state where 2 sent a confirmation or not. For agent 2, the info set is the inverse.

In this game, there is a unique equilibrium which is not to invest.

From the point of you of agent 2, he can't distinguish between $(0,0)$ and $(0,1)$, but then the first info set is 1/2 likely, but $(1/2)\epsilon$ which is smaller, in subsequent rounds we have $\epsilon > (1-\epsilon)\epsilon$. So, then from agent 2 point of view, being in the bad state is more likely. And this is infinite.

Now, we can break the game to make it finite. On the last round, we say that one agent is no longer expecting a confirmation. Hence, on the last round, the last receiver "Invest" and the last sender now had $\epsilon < (1-\epsilon)$, so the other is also investing.

Hence, this game has now two equilibrium.

Part II

Week 3/4: Collusion and Cooperation

6 Manipulation and Coalition of WE

The subject of collusion started with Egdeworth. There was the idea that Walrasian equilibrium is robust against collusion and manipulations. Suppose that we have a set of agents $N = \{1, 2, ..., N\}$ and l goods, and we assume an exchange economy. Thus we have:

- $C \subset N = \{1, .., N\}$ / $\{e_1, ..., e_n\} \in R^l_+$ / $u_i : R^L_+ \to R$

- Secession of a coalition: $X(C) \subset R_+^{l|C|}$ (no production function) and $X(C) = \{(X_i)_{i\in C} : \sum_{i\in C} x_i \leq \sum_{i\in C} e_i, x_i \in R_+^l\}$

Let us define the **"Core"** (Gilles 1959): for an allocation $y = (x_1, ..x_n) \in R_+^{nl}$, we say that $x \in X(C)$ is blocked by $C \subset N$:

1. if $u_i(y_i) > u_i(x_i), \forall i \in C$. Strong Blocking
2. if $u_i(y_i) \geq u_i(x_i), \forall i \in C$ and $>$ for some i. (this is easier to block). Blocking*
3. if we have the second definition of Blocking + u_i continuous, locally non-satiated, and strictly increasing, then we get the strong blocking definition.

And the Core Economy is define as all $x \in X(N)$ that are not blocked.

Then for $x \in Core(E)$ where E includes $(N/e_i^l/u_i)$:

- $C = N \implies$ Pareto Efficiency
- $C = \{i\} \implies$ that individuals must be individually rational based on their endowments.
- For $N = 2$, Core = IR $\cap$ PE $\implies$ WE $\subset$ Core.

Theorem 1. For $n \geq 2$ the WE is a subset of the Core(E).

Proof: We want to prove that $x \in WE(E) \implies x \in Core(E)$.

Suppose the contrary, that $x \notin Core$ meaning that x is blocked by C, which translate into having $u_i(y_i) > u_i(x_i), \forall i \in C$ (this reads as x was demanded, but y makes everybody better off, so it must be that y is not affordable, otherwise it would have been demanded). Formally,
$x \in WE \implies py_i > pe_i, \forall i \in C \implies p\sum_{i\in C} y_i > p\sum_{i\in C} e_i \implies \sum_{i\in C} y_i > \sum_{i\in C} e_i$,
thus unfeasible which contradict that $y \in X(C)$.

The **Corollary** is that if WE exists, the core is nonempty in exchange economy. Also this strengthens the First Welfare Theorem.

Theorem 2. Debreu and Scarf (1963) The Core Convergence says that in " large economy", Walrasian Equi allocation are the only allocation that survive in the core.

The idea is that from a small economy, we will replicated the economy until it is large enough and have nice properties, especially by making sure there is enough competition in between agent, so that no one is too powerful. The argument of convergence relies on how the economy grows.

Formally, let us define a replicated economy as:

- $E = \{(1, ..., n), (u_1, ..., u_n), (e_1, ..., e_n)\}$
- $E^2 = \{(n+1, ..., 2n), (u_{n+1}, ..., u_{2n}), (e_{n+1}, ..., e_{2n})\}$
- $E^k = \{((k-1)n+1, ..., kn), (u_{(k-1)n+1}, ..., u_{kn}), (e_{(k-1)n+1}, ..., e_{kn})\}$

Theorem 3 Let us consider E s.t. u_i continuous + strictly quasi-concave + increasing (for $x_i \geq x_i'$ and $x_i' \neq x_i \implies u_i(x_i') > u_i(x_i)$) and $E_i \in R^l_{++}, e_{ik} > 0, \forall k$.

Then, take any $x \in X(N)$ and replicated k times, such that $(x_1, ..., x_k) \in Core(E^k) \forall k$, then it must be that x is a WE and Core is not empty.

In words, this says that if we start from something that is not a WE, then there exists a k for which the replication of this allocation will not belong to the core.

Example 1. Let us suppose that endowment are $e_1 = (1,0)$ and $e_2 = (0,x)$ and utility is defined as $u_i = x_{i1}x_{i2}$. The unique WE is $x_i = (1/2, 1/2)$ for all i and prices are $p = (1,1)$. In this problem the core is given by $Core(E) = \{(x,x),(1-x,1-x) : x \in [0,1]\}$ (the 45 degree line in the Edgeworth box). *The idea of the theorem is to prove that as we replicate the economy all the points different than the WE will disappear from the core.*

Replication of the original economy allocation are agent 1 and 3 (good 1, good 2) and agent 2 and 4: $\left(\frac{7}{8}, \frac{7}{8}\right); \left(\frac{1}{8}, \frac{1}{8}\right)$

We now consider 3 people out of the 4, which are agent 1, 2 and 4, blocking out agent 3, we can use his endowment to share among the coalition which in this case would be agent 1,2 and 4.

- Agent 1 is given a whole unit of 2, so he is better off: $\left(\frac{7}{8}, 1\right)$
- Agent 2 and 3 share the remaining of good 2 and good 1 given agent 3 is not given anything: $\left(\frac{1}{16}, \frac{1}{2}\right)$ for each agent 2 and 4.

Formally, suppose that we stand at $x \in Core(E)$ that we want to rule out. Consider the vector $z_i = x_i - e_i$ (demand surplus), by concavity we know that $u_i(e_i + t \cdot z_i) > u_i(e_i + z_i)$ and we know that t exists because the rationals are dense. Suppose that $t = a/b$, where b is the number of agent of type 2 and a the number of agents of type 1. The idea to take $t = a/b$ is because in this way we can construct a coalition containing exactly these number of each type. The geometric intuition is that if we don't cut any indifference curve then we are in a tangency point, and therefore we found a WE. Note that as we get closer to the WE, the set of points where we are better is shrinking down and therefore we will need more agents to form our coalition.

7 Core Convergence

Let us prove that the Core converge to WE.

Starting from our base economy E and $x \in X(N)$ such that $(x,...,x) \in Core(E^k), \forall k$.

We consider the set of improving trade vectors:

$$Q \subset R^l, Q = \{\sum^n \alpha_i z_i : z_i \in R^l, \sum^n \alpha_i = 1, \text{ for } \alpha_i \geq 0, \text{ s.t. } u_i(e_i + z_i) > u_i(x_i), \forall i\}$$
$$Q = \{z : \exists \alpha \in \triangle^n_+, (z_i)_i \in R^{ln}, z = \sum \alpha_i z_i, u_i(z_i + e_i) > u_i(x_i)\}$$

In words, Q is a set of possible movements from the endowment that would lead to an improvement of our original allocation x in the economy (Q represents the set of possible trades that make everyone happier). Note that $e_i + z_i$ may not be feasible; we only require that $z = \sum_i \alpha_i z_i$.

Hence, considering $x \in X(N)$ s.t. $(x,...,x) \in Core(E^k) \forall k \implies x \in WE(E)$.

We make the following claims:

- Q is convex (from quasi-concave utlity u_i)
- $0 \notin Q$, (not blocked by def)
- There exists p s.t. $p \cdot z \geq 0, \forall z \in Q$ (separating Hyperplane Thm)
- $p \geq 0$, (increasing and continuous utility)
- (x, p) is a W.E. (continuous utility)

Assumption 1 can be proved using convex definition, Assumption 2 can be proved by contradiction, Assumption 3 is based on separating hyperplane and Assumption 4 is based on the def of utility. So we still claim 5 to prove:

- If $u_i(y_i) > u_i(x_i)$, then $p \cdot y_i > p \cdot e_i \ \forall i$
- $p \cdot x_i \leq p \cdot e_i$, $\forall i$

For the first bullet point: Let us take x_i and y_i for which $u_i(y_i) > u_i(x_i)$, then we define $z_i = y_i - e_i$. Notice that $z = \{z_1, ..., z_n\} \in Q$. Now let $z_i^t = t \cdot y_i - e_i$ for $t \in (0,1)$. By continuity, this z_i^t is still in Q for some $t < 1$. Next,

$$p \cdot z_i \geq 0 \implies p \cdot y_i \geq p \cdot e_i > 0$$

Where the last inequality is because prices are not zero and $e_i \in R^l_{++}$. Therefore, for some t < 1,

$$p \cdot y_i > p \cdot t y_i \geq p \cdot e_i > 0$$

Now for the second bullet point, suppose that $p \cdot x_i > p \cdot e_i$ for some i and therefore there is another j for which $p \cdot x_j < p \cdot e_j$ (from feasibility constraint). Now:

$$y_i = (x_j + (\epsilon, \epsilon, ..., \epsilon)) \implies u_j(y_j) > u_j(x_j) \implies p \cdot y_j > p \cdot e_j$$

and taking $\epsilon \to 0$ we obtain that $p \cdot x_j \geq p \cdot e_j$ by continuity which contradict the assumption.

8 Cooperative game

8.1 NTU Game

Consider N agents. We define a mapping which describes the set of possible utilities that the subset of agents can get if they act together. $V : 2^N \implies R^{|c|}$ where $C \subset N$. Then, $V(C)$ is a concave function on a graph with u_1, u_2. Then, $V(C) \subset R^{|C|}, \forall c$.

Then the Core of an NTU game (N,v) can be described as a game in characteristic function form and is described:

$$Core(N,V) = \{u \in V(N) : \nexists C \subset N, v \in V(C) \text{ s.t. } v_i > u_i \forall i \in C\}$$

In words, $u \in V(N)$ if it is not blocked by any subset of agents.

Example : NTU game - 1 capitalist and many workers With N agents. Let us consider $n-1$ workers with person n the capitalist. Then , we define a function f that maps the number of work to production, $f : \{0, 1, ..., n-1\} \to R$, with $f(0) = 0$ and is increasing and concave (diminishing returns). Then we have two cases:

- If the capitalist does no show up we have $V(C) = \{(0, ..., 0)\}, n \notin C$
- If he is there, $V(C) = \{(u_i)_{i \in C} : \sum u_i \leq f(|C|-1)\}$ for $n \in C$

To find the core, we need to understand that all the worker are going to be used even though we have diminishing return, every extra worker is bring ϵ more production and this weakly increases the utility of everyone. So the optimal level would be $f(n-1)$.

Here the only blocking coalition need to include the capitalist. Hence, we must have that $u_i \leq f(n-1) - f(n-2)$ (sufficient due to concavity), and the capitalist gets the remainder.

For this game the Core looks like:

$$Core = \{u : u \in R^n_+, \sum_i^n u_i = f(n-1), n \geq i \geq 2 \implies u_i \leq f(n-1) - f(n-2)\}$$

8.2 TU Game

In a TU game we define the function $V(C)$: $V(C) = \{u \in R^{|C|}$ s.t. $\sum_{i \in S} u_i \leq v(C)\}$ where $v(C) \in R$ is the aggregate utility reachable by the coalition S^2. Note that TU games are a subset of NTU games. In addition TU game satisfy *super-additivity* property. In words, there are no disjoint subsets of agents that can do better than considering the whole set of agents working together.

Example : TU game - Party Game Consider 3 agents what represents 3 political parties.

Taking singleton, a party alone cannot do anything so we can't do much with $V(\{i\})$. We are looking at coalition we consider:

- $V(\{1,2\}) = \{u_1 + u_2 \leq 1 + y\}$
- $V(\{2,3\}) = \{u_2 + u_3 \leq 1\}$
- $V(\{1,3\}) = \{u_1 + u_3 \leq x < 1\}$
- Super Majority does not help: $V(\{1,2,3\}) = \{u_1 + u_2 + u_3 \leq 1 + y\}$

From the Core, we know that $U_i \geq 0, \forall i$. But given that party 1 and 2 can themselves use all the resource $1 + y$, then this means that party 3 does not get anything $u_3 = 0$. Also given that $u_2 + u_3 = 1$, we have $u_2 \geq 1$ and $u_1 \leq y$, so $\implies x \leq u_1 \leq y$. Then for the Core to be non empty then it must be that $y \geq x$.

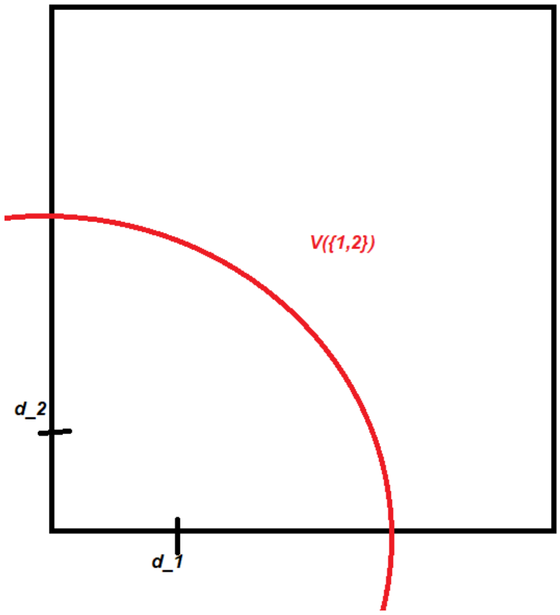
Example "Divide the dollar" Same game with y = 0 and x = 1

- $V(\{1,2\}) = \{u_1 + u_2 \leq 1\}$
- $V(\{2,3\}) = \{u_2 + u_3 \leq 1\}$
- $V(\{1,3\}) = \{u_1 + u_3 \leq x\}$

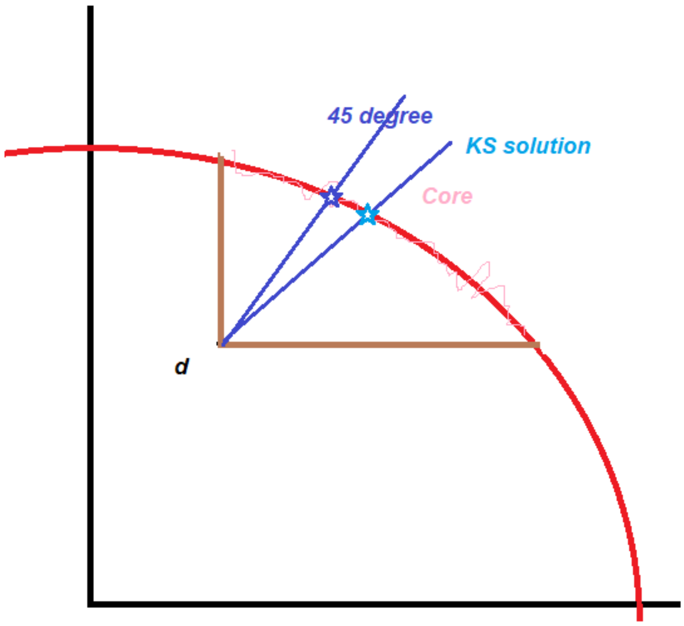
But then, $x > y$, so the core is empty. This means that synergy is required for the cooperation to work.

9 Core and Cooperation: Classic solution concept

When $N = 2$, we need to think of what $V(\{1,2\})$ can generate, and what $V(\{1\})$ and $V(\{2\})$ can generate alone. Now the point d_1, d_2 is the disagreement point. And the convex set is what they can achieve together by cooperation in Figure 9. We define a generic function $\Phi(v) \in \{u : \sum u_i = v(N)\}$. This function correspond to the different solution concepts we want to use.



Then the solution can give in proportion to each of the agents this is Kalai-Smnodmsky, or using the 45 degree line (Egalitarian solution). We can also use "society" indifference curves, that is $(u_1 - d_1)(u_2 - d_2)$ (Nash Bargaining). But this last method can be tricky because it might violate individual rationality constraint and give a point outside the Core.



In this world a solution can be define as a mapping $\Phi(V) \in V(\{1,2\})$. Then this solution needs to satisfied the following axioms:

- Pareto: this means that $u \in \Phi(V) \implies \nexists u' \in V(\{1,2\})$ s.t. $u_1' \geq u_1$ and $u_2' \geq u_2$ strict for at least one.
- Anonymity / Symmetry: Take V, V' s.t. $V(\{1\}) = V'(\{2\})$ and $V(\{2\}) = V'(\{1\})$, then $(u_1, u_2) \in V(\{1,2\}) \iff (u_2, u_1) \in V'(\{1,2\})$. Hence, $(u_1, u_2) = \Phi(V) \implies (u_2, u_1) = \Phi(V')$.
- Translation Invariant: Consider V, V' s.t. $\exists \alpha_i, \beta_i$ s.t. $d_i' = \alpha_i + \beta_i d_i$. If $u_i' = \alpha_i + \beta_i u_i$, $(u_1', u_2') \in V'(\{1,2\}) \iff (u_1, u_2) \in V(\{1,2\})$, then we have that $\Phi(V')_i = \alpha_i + \beta_i \Phi(V)_i$
- Nash IIA: For $V'(\{1,2\}) \subset V(\{1,2\})$ and $\Phi(V) \in V'(\{1,2\}) \implies \Phi(V') = \Phi(V)$ keeping d constant.

Theorem (Nash) if $\Phi(v)$ satisfies Pareto, Anonymity, Translation Invariance and NIIA iff. $\Phi = \Phi^{NashBarg.}$

Note that the Nash bargaining solution compensates for how much each agent brought to the table. This does not take into account possible coalitions $n \geq 2$, and therefore the Nash bargaining solution may not be in the core.

Theorem (KS) I $\Phi(v)$ satisfies Pareto, Anonymity, Translation Invariance and **Monotonicity** iff. $\Phi = \Phi^{KSBarg.}$

Expanding to a larger set: We have $N = \{1, ..., n\}, n \geq 2$. We could do the Nash solution which can be written as:

$$\max_{(u_1,..,u_n) \in V(N)} \prod_i (u_i - d_i) \text{ where } d_i \text{ is } \max_{u_i} V(\{i\})$$

But this solution does not account for the other potential condition in $V(C)$.

So we now consider **TU game**: $V(C) = \{(u_i)_{i \in C} \text{ s.t. } u_i \leq v(c)\}$ where $v(C)$ is a a number here.

Hence, we still have the our Nash solution:

$$\max_{u_1,..,u_n) : \sum u_i \leq v(N)} \prod_i (u_i - v(\{i\}))$$

We can also consider the Egalitarian Solution where every one get the same share:

$$\left(\frac{v(N)}{n}, ..., \frac{v(N)}{n} \right) \text{ with } v(\{i\}) = 0, \forall i$$

But with $v(\{i\}) > 0, \forall i$, the proportional egalitarian solution is $\frac{v(\{i\}) v(N)}{\sum_j v(\{j\})}$

Shapley Value

Consider the example with 3 agents. As such we have exactly 7 values for the core $v(\{1\})$, $v(\{2\})$, ..., $v(\{1,2\})$, ..., $v(\{1,2,3\})$.

The solution function $\Phi(v) \in R^n, \forall v \in TU(N), \forall N$:

$$\Phi_i^{SV}(v) = \sum_{C \subset N/\{i\}} \frac{|c|!(n-|c|-1)!}{n!}(v(c \cup \{i\}) - v(c))$$

This is the sum of the different value added of doing a coalition with person i times the number of different ways this coalition can be formed. The first term is the weight such that $\sum_i \Phi_i(v) = v(N)$ and the last term is the change in the value by adding person i to C.

For $n = 3$, this simplifies to:

$$\Phi_1(v) = \frac{V(\{1\})}{3} + \frac{1}{6}(v(\{1,2\}) - v(\{2\})) + \frac{1}{6}(v(\{1,3\}) - v(\{3\})) + \frac{1}{3}(v(\{1,2,3\}) - v(\{2,3\}))$$

Here Shapley values will always exists but may not be in the Core.

Recall our example:

- $V(\{1,2\}) = V(\{1,2,3\}) = 1 + y$
- $V(\{2,3\}) = 1$
- $V(\{1,3\}) = x < 1$
- $V(\{i\}) = 0$

In the Core, we know that $u_3 = 0$ and $u_2 \geq 1$, so we have that $u_1 \geq x$, and thus this is non-empty if $y \geq x$.

If we were to compute the Shapley value for $(u_1/u_2/u_3) = (y, 1, 0)$ and thus $\Phi_3^{SV}(v) = \frac{x+1}{6}$.

10 Core Consistency

$\Phi(v) \in Core$ if $Core(v) \neq \emptyset$

Characterization of Shapley Value

- A TU game: $\Phi(v, N) \in R^n$ s.t. $\sum \Phi_i(v) = v(N)$ this condition guarantee Pareto.

Properties Here, we already have *Symmetry / Anonymity*: $\pi : N \to N$ as a bijection (one to one mapping) $\implies v^{\pi}(c^{\pi}) = v(C)$ where $C^{\pi} = \{\pi(i) : i \in C\}$

+ *Carrier Axiom* For a game, with V and N, we can say that R is a carrier of V, for $R \subset N$, then $V(C) = V(C \cap R), \forall C \subset N$. Then $\phi_i(v) = 0$ if $i \notin R$. The carrier carry all the value that is $V(R) = V(N)$, for $i \in R$ and thus free-rider gets nothing.

+ *Additivity*

Theorem If Φ satisfies, Addi, Carrier, Anonymity, then $\Phi = \Phi^{SV}$

Proof

1. Φ^{SV} satisfies axioms. This is immediate and can be verified using the definition of the axioms

2. Axioms $\implies \Phi^{SV}$

Let us prove (2). Given a fixed N, we define V as the set for all TU game on N. Remember that this is just a vector of $2^n - 1$ number that represent all the possible coalition.

Then for $C \subset N$, we have $w^C(C') = 1$ if $C \subset C'$ and 0 otherwise. **Claim** $\{w^c\}_{c \subset N}$ from a basis from V.

Veto players $v(c) = 0$ if $i \notin C$. The core of a simple game is not empty if and only if there exists a veto player.

Example: UN Council $V(c) = \{0, 1\}$. Then we have that $v(c) = 1$ if $\{1, ..., 5\} \subset C$ and $|c| \geq 8$. Hence, the 5 are veto players but no a carrier. The Core would be $u_i = 0$ if $i \notin \{1, ..., 5\}$ and $\sum_i u_i = 1$ and $u_i \geq 0$, otherwise.

Part III

Week 5/6: Market Imperfections

REE $\implies$ Asymmetric Info $\implies$ Transaction between multiple players. So far, we have coalition manipulation which has nice properties but splitting values is not clear. We will now focus on smaller number of players and think about model interaction.

Hurwicz 72 We start form a an edgeworth box of 2 people and 2 goods with endowment $e_1 = (1,0)$ and $e_2 = (0,1)$. We consider the utility $u_2 = \min\{x_{21}, x_{22}\}$ and $u_1 = \min\{x_{11} + \epsilon, x_{12}\}$. In this economy agent 1 gets all of good 1 and good 2. But person 2 can lie about his own preference and get more good.

Thus, the model interaction might come from asymmetric information about characteristics of player and lead to adverse selection, the asymmetric information can also be about action and this leads to moral hazard.

Two set of problems:

- Principal-agent: presence of hidden actions that affect both agents payoffs.
- Adverse selection: presence of hidden information; for example, about types. For example, when offering insurance the hidden information is about the previous health history.

11 Moral Hazard

Interactions between system/contract and outcome from your contract will alter their possible set of outcome.

11.1 Model

- 1 buyer = principal
- 1 seller = agent that takes actions that are non-observable and non-contractable (can't see all the effort put into work) but we observe output or profit.

We consider two level of effort e_L, e_H which generate profits $F(\pi|e)$ (cumulative distribution) conditioned on effort. However, based on the payoffs we can not fully discriminate what was the level of effort; in other words, there is noise in the function that maps effort to production.

The firm is risk neutral $\rightarrow \max E[\pi -$ wage$]$ and $F(\pi|e_L) \geq F(\pi|e_H)$ (first order stochastic dominates). Worker is risk-adverse, he has $u(w,e) = v(w) - g(e)$ where v is concave and $g(\cdot)$ is increasing in effort.

Altogether, the optimal wage would be a constant wage, however this is not incentive compatible.

The firm problem is thus:

$$\max_{w()} \int \pi - w(\pi) dF(\pi|e)$$

s.t.

Individual Rationality: $E[v(w(\pi)) - g(e)] \geq \bar{u}$ given that the outside option is being unemployed

Incentive Compatible: $E_e(w(\pi)) - g(e)) \geq E_{e'}(w(\pi)) - g(e')), \forall e' \neq e$

Benchmark example: We will ignore IC, as if we were observing e and paying $w(\pi)$ only if e was provided.

Given e, we can find the $w(\cdot)$ that maximize the function and then pick the best e. As such we can rewrite the problem as:

$$\begin{aligned} &\min_{e,w()} \int w(\pi) dF(\pi|e) \\ &\text{s.t. IR:} E[v(w(\pi)) - g(e)] \geq \bar{u} \end{aligned}$$

We can then set up a Lagrangian:

$$\min_w \int w(\pi) dF(\pi|e) - \lambda(\int v(w(\pi)) dF(\pi|e) - g(e) - \bar{u})$$

The FOC gives us:

$$\begin{aligned} dF(\pi|e) &= \lambda v'(w(\pi)) dF(\pi|e) \\ 1 &= \lambda v'(w(\pi)) \text{ v is strictly concave} \\ \frac{1}{\lambda} &= v'(w(\pi)) \text{ constant in } \pi \\ \frac{1}{\lambda} &= \text{ constant in } w(\pi) \end{aligned}$$

Now that we know that wage is constant we can plug it back into the constraint which is $v(\bar{w}) - g(e) = \bar{u}$, this gives us a $\bar{w}_e$

We now give some structure to the distribution F, such that with high effort we have that the profit is 100 with prob 3/4 and 0 with prob 1/4, with low effort we have 100 with prob. 1/4 and 0 with prob 3/4. Then we have that $g(e_H) = 5$ and $g(e_L) = 0$ and $\bar{u} = 0$.

Then for low effort wage is 0 and for high effort wage is 25. Then we have that $\sqrt{\bar{w}_{e_H}} - 5 \geq 0$, but IC fails.

To make sure that the IC condition is not violated we need an extra condition, the LHS is tge worker's expected utility when $e = e_H$ and the RHS is the e_L:

$$\frac{3}{4}\sqrt{w_{100}} + \frac{1}{4}\sqrt{w_0} - 5 \geq \frac{1}{4}\sqrt{w_{100}} + \frac{3}{4}\sqrt{w_0} - 0 \implies \sqrt{w_{100}} \geq \sqrt{w_0} + 10$$

But from IC it follows that $\sqrt{w_0} = 0$, so we have that $W_{100} = 100$.

Hence, the profit is $3/4 \cdot 100 + 1/4 \cdot 0 - 3/4 \cdot 100 + 1/4 \cdot 0 = 0$

Claim Back to the general case: IC bind $\rightarrow$ getting e_H. Suppose it does not bind such that:

$$\begin{aligned} E_{e_H} v(w(\pi)) - g(e_H) &> E_{e_L}(w(\pi)) - g(e_L) \\ E_{e_H} v(w(\pi)) - E_{e_L}(w(\pi)) &> g(e_H) - g(e_L) \\ \int v(w(\pi))(dF(\pi|e_H) - dF(\pi|e_L)) &> g(e_H) - g(e_L) \end{aligned}$$

Let us prove that IR and IC are binding. Let us re-write the Lagrangian:

$$\begin{aligned} \max \int (\pi - w(\pi)) dF(\pi|e_H) - \lambda \Big(\int v(w(\pi)) dF(\pi|e_H) - g(e_h) - \bar{u} \Big) \\ - \mu \Big(\int v(w(\pi)) dF(\pi|e_H) - g(e_H) - \int v(w(\pi)) dF(\pi|e_L) - g(e_L) \Big) \end{aligned}$$

FOC (w):

$$\begin{aligned} -dF(\pi|e_H) - \lambda v'(w(\pi)) dF(\pi|e_H) - \mu v'(w(\pi))(dF(\pi|e_H) - dF(\pi|e_L)) = 0 \\ \implies -\left(\lambda + \mu \left(\frac{dF(\pi|e_L)}{dF(\pi|e_H)}\right)\right)^{-1} = v'(w(\pi)) \end{aligned}$$

We use to have just $v'(w(\pi)) = \frac{1}{\lambda}$ when we had full information.

Hence, if agent is more likely to be e_H, then $\frac{dF(\pi|e_L)}{dF(\pi|e_H)} < 1$, and thus the slope is flatter and thus the wage increase along the concave curve of $v(w(\pi))$.

We argued that when effort is not observable and when manager is risk averse both constraints bind. The argument is that is the second constraint does not bind, then the problem is equivalent to the previous problem (when effort is observable), and therefore the wage would be constant. Since it is not, then the constraint must be binding.

2 sources of inefficient: (1) we are overpaying to get the same utility and (2) we get underproduction.

- Scenario 1: e_L was optimal with observable effort. If this is true nothing changes, i.e., we have no incentives to try to make the manager work hard

- Scenario 2: e_H was optimal with observable effort. This is no longer optimal, because we get inefficient low profit and therefore there are no incentives for high effort. This is the case of inefficient production.
- Scenario 3: e_H is still optimal but now $w(\cdot)$ is non-constant, which implies that in order to obtain the same utility we have to pay workers more. In other words, we still get $\bar{u}$ but we pay more wage. In this world workers are indifferent (they get same $\bar{u}$) and firms are worst off (have to pay higher wages), and therefore we have inefficiency.

12 Adverse Selection: Hidden quality

Insurance, mortgages and others are examples of adverse selection. We assume workers are of type $\theta \in [0,1]$, distributed according to $F(\theta)$, and each type with reservation utility $r(\theta)$. Suppose we can observe the type of the worker, and assume we have to firms competing Bertrand to get the worker. Firms will maximize $\theta - w$ and therefore the optimal wage will be $w^*(\theta) = \theta$. The interesting case is when θ is not observable at the time of contracting, and not verifiable even ex-post, i.e., we can not deduce his quality based on the utility or production I have obtained so far.

- Buyer uninformed
- Seller is informed
- The value to the buyer is $\theta \in R_+$ where θ comes from $F(\theta)$ distribution
- The value to the seller (reserve value) is $r(\cdot)$ ans it can be < or > than θ
- Rational Expectation buyers
- We assume competition
- Because these is only one type we have that there is only one price
  $w^* = E[\theta | r(\theta) \leq w]$

Single type θ

Let us start with constant reserve rate $r(\theta) = r, \forall \theta \in \Theta$
2 possibilities in equilibrium we have that $w = E[(\theta)|r \leq w]$, bc everyone has the same r this is the unconditional expectation (bc firm compete a la Bertrand):

$$(1): w = E[\theta] \geq r \text{ everybody sales}$$
$$(2): w = E[\theta] < r \text{ no sales}$$

For $r(\theta) = r$, $F(\theta)$ has some $\theta > r$ should sale (over transacting) or if $\theta < r$ no sales (under-transacting).

In the case where $w = E(\theta) > r > \theta$. The value of the buyer is thus $\theta - E(\theta) < \theta$ and the seller is geting $E(\theta) - r > \theta$

Example - Akerloff $r(\theta) = \frac{2}{3}\theta$. Here we have $\theta > r(\theta)$ so we are in the case where all should transact and $\theta \sim U[0,1]$.
So $w = E[\theta|r(\theta) < w] = E[\theta|2/3 \cdot \theta < w] = E[\theta|\theta < 3/2w] = 3/4w$. Hence the only fixed point is when $w = 0$. Although all sales should happen none of them happen.

Different type θ_H and θ_L

We now consider the case where $r(\theta_H) < r(\theta_L)$, those that love their job θ_H are willing to do the job at lower price. As such this is a step function where you increase wage up until $r(\theta_L)$, then the low types enters, so the principal decrease the wage below $r(\theta_L)$, but then he is missing out some of the high type he could employ for lower wage. This keeps going such that there is no equilibrium in this economy.

Conclusion In case of adverse selection, we can end up with over-selling when price include inefficient sales. We can also have underselling, when the market collapse, and valuable transaction forgone (and equilibrium is non-existent).

We can then think of more complex contract, where we can only sell part of the good and buy insurance for it. If it going to be a multidimensional contract where we have how much is insured and up to what amount.

13 Insurance

We consider risk adverse buyers and risk neutral sellers in a competitive setting. There are two types split equality θ_H and θ_L.

In case 1, there is no accident and claim is 0, such that $u(1)$ and $u(0)$ is strictly concave. We consider the probability π that an accident happen, this probability is attached to the type of the agent π_L and π_H.

Fair Full Insurance

Consider that the firm is making 0 profit, which means that $p = p^{BE} = \frac{\pi_h}{2} + \frac{\pi_L}{2}$, for each type being equally represented in society. But here the "low" types (low accident risk) are overpaying.

Here, the PC can be written as $u(1-p) \geq (1-\pi)u(1) + \pi u(0)$ and this boils down to $u(1-p) \geq 1-\pi_L$.

Hence for $u(\cdot) = \sqrt{\cdot}$, we have that:

$$\begin{aligned} \sqrt{1-p} &\geq 1-\pi_L \\ 1-p &\geq (1-\pi_L)^2 \\ 1-(1-\pi_L)^2 \geq p &= \frac{\pi_h}{2} + \frac{\pi_L}{2} \end{aligned}$$

But if we consider $\pi_H = 3/4$ and $\pi_L = 1/4$, then we have that $p = 3/4$ and thus the low type are driven out of the market which makes this inefficient (missed opportunity of risk sharing).

Partial Insurance

Here, we are insuring $R = 1-c$, where c is like a co-payment.

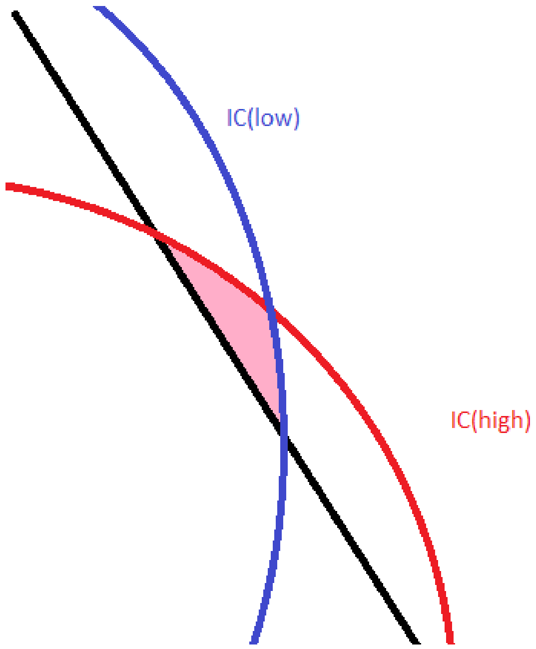
The new IC is : $(1-\pi)u(1-p) + \pi u(1-p-c) \geq (1-\pi)u(1) + \pi u(0)$

For the firm, we still consider the 0 profit condition: $p = (1-c)\left(\frac{\pi_h}{2} + \frac{\pi_L}{2}\right)$

From the 0-profit condition we can compute that $c = 1 - \frac{p}{\frac{\pi_h}{2}+\frac{\pi_L}{2}}$. pluggin this one in the IC, we get that $\frac{3}{4}\sqrt{1-p} + \frac{1}{4}\sqrt{p} \geq 3/4 = (1-\pi_L)$ Then we can simplify $1-2p$ for $\pi_H = 3/4$ and $\pi_L = 1/4$.

$$\begin{aligned} \frac{3}{4}\sqrt{1-p} + \frac{1}{4}\sqrt{p} &\geq 3/4 \\ \implies p &= 0.36 \end{aligned}$$

Pooling Equilibirum Let us try to find a zone where both type are happy to pay the same contract. Here the pink zone is below Indif Curve (ie more utility) and above BE line of the firm (+ve profit), so seems like a good candidate.



BE: 0 profit condition

BUT, there is no pooling equilibrium (think of Bertrand competition where you can always above cut by ϵ). Pooling equi requires:

- 1 single point
- Non-negative profit for the firm (above or on the BE line)
- Below or on both IC curves.

However, the problem is such that say low type will want to decrease the price such that it will be above the IC(high). But then, if one type is driven out of the market then the BE line also change because it must only count types operating in the market. Offering partial insurance to just the L type is profitable and thus nothing is stable.

But we can find a separating equilibrium where the high type buy full insurance, and for the low type we need to decrease the price such that the co-pay is so high that high type are not interested. Requirements:

- 2 points
- None negative profit for the firm (above or on the BE lines)
- Below or on both IC curves.
  - Make the high type IC binding

- Find any point preferred by low type but not by high type.
- It is possible that such point does not exists depending on IC shape.

Part IV

Week 7/8/9: Beyond Markets: Social Choice Theory and Voting

When considering preference we will consider aggregate of individual preference that are ordinal (ranking), and cardinal (utility measure). In this setting we will only consider complete and transitive and strict pref.

- $(\succ_i)_i \in L(a)^n$
- Social Ranking $R(\succ)$ social welfare ordering

14 For 2 Alternatives

May's Theorem We consider n voters, the set of alternative $A = \{a, b\}$ and we have strict individual preference $i : a \succ_i b$ or $b \succ_i a$. We have the social ranking possibilities as $R(\succ) \in \{(aR(\succ)b), (bR(\succ)a), (aR(\sim)b)\}$. Is it possibility that people have strict indiv pref but the aggregate gives indifference.

14.1 Majority Rule

- $N_a(\succ) = \{\#i \in N | a \succ_i b\}$ and $N_b = n - N_a(\succ)$
- Then we have $aR(\succ)b$ iff $N_a(\succ) \geq \frac{n}{2}$
- and $bR(\succ)a$ iff $N_b(\succ) \geq \frac{n}{2}$
- Here we are not counting preferences intensities we only consider the ranking ie we only have ordinal utility and thus $R(\succ)$ is also ordinal

Properties:

- Anonymity: everyone has the same weight and flipping electors' identity does not change the outcome of the election (in short "labels of indiv do not matter").

Formally, $\pi : N \to N$ permutation, for r $\succ$ and $\succ'$ such that $\succ_i = \succ'_{\pi(i)}, \forall i$, then $R(\succ) = R(\succ')$.

- Neutrality: same as anonymity but when flipping policies. This is violated as soon as the status quo is in the alternative because status quo is free compared to changing legislation for ex. (in short "labels of alternatives don't matter). Formally, for $\succ$ and $\succ'$ such that $\succ_i \neq \succ', \forall i$, (meaning that everyone's ranking is flipped), then we have that $aR(\succ)b \implies bR(\succ')a$ forall all alternatives a, b.

- Monotonicity: if someone change its prefers and now prefers a more, then the social ranking should also rank a as high or higher. Formally, for $\succ$ and $\succ'$ there exists i such that $\succ_j = \succ_i, \forall j \neq i$ and $a \succ_i b$ and $b \succ'_i a$, then $bR(\succ)a \implies bR(\succ)a$.

Theorem (preview of May's) For 2 alternatives, $R(\succ)$ (complete and transitive) satisfies A,N,M iff $\exists q \in [0, n/2]$ where q is an integer such that:

$$aR(\succ)b \iff N_a(\succ) \geq q \text{ equivalent to } |\{i : a \succ_i b\}| \geq q$$
$$bR(\succ)a \iff N_b(\succ) \geq q \text{ equivalent to } |\{i : b \succ_i a\}| \geq q$$

Proof: $\Leftarrow$ *Trivial*
$\Rightarrow$: Let use define $N_a(\succ) = |\{i : a \succ_i b\}|$ and $N_b(\succ) = n - N_a(\succ)$
Anonymity gives us that for $aR(\succ)b$ and $N_a(\succ) = N_a(\succ')$ and $N_b(\succ) = N_b(\succ')$ then we have that $aR(\succ')b$.
Monotonicity Suppose that $aR(\succ)b$ and $N_a(\succ') \geq N_a(\succ)$, then $aR(\succ')b$. This gives us the following property $q_a = \min_{\succ}\{N_a(\succ) : aR(\succ)b\}$.
Neutrality This is going to gives us that q_i as define just above has to be equal across alternative $q_a = q_b$.
Completeness It must be that $q_a = q_b = q \leq n/2$

But strict monotonicity is required in May's Theorem to pin down the value of q. We define **strong monotonicity** (SM), as :

$$\begin{gathered} \text{If } aR(\succ)b \text{ and } \succ' \text{ is such that } \succ_j = \succ'_j \ \forall j \neq i \\ \text{but } a \succ'_i b \text{ and } b \succ_i a \\ \text{i.e. } N_a(\succ') > N_a(\succ) \\ \text{then } aR(\succ)b \text{ and not } bR(\succ')a \end{gathered}$$

In words: If we use to like a, and more people like a then we can't be indifferent anymore and a is strictly preferred to b. We can now state May's theorem and pin down the value of q.

May's theorem: For 2 alternatives, $R(\succ)$ (complete and transitive) satisfies A,N, SM iff $\exists q = n/2$ (Majority Rule).

One drawback of this rule is that this is just an ordinal rule and throw away the intensity of preferences.

14.2 Intensity of Preference

So far, intensity of preferences have been discarded, nothing utilitarian. Indeed, majority rule ignores intensity.

When considering the cardinality of preference we want to distinguish intrapersonal which govern preference within a given a person and interpersonal prefer which concerns across people preference.

Casella - Storable Votes Voter are provided a "budget" of vote and than they can allocate more vote to election that they care more about.

15 For more than 2 alternatives & Arrow Theorem

We consider more than 3 alternative, with $\succ_i L(A)$ with complete and transitive preference.

An example from 1961, the alternative describe 3 different way to tax: W is a wealth tax, L Land tax, and N not tax to finance the war.

The *rural* population is in favor of W, then N and L. For the urban and pro-war, L is preferred to W which is preferred to N. Finally, for the *urban anti-war* N is preferred to L which is preferred to W.

Formally, we consider the following set-up: n agents, a finite set of alternatives A, and the preference profil $\succ_i$ which is complete, transitive and asymmetric (no indifference). We then define $L(A)$ a linear order which give SWO that satisfies completeness, transitivity and unanimity and **Arrow's Independence of Irrelevant Alternatives**.

15.1 Arrow's Theorem

For a finite set A with $|A| \geq 3$, we consider n agents, and $L(A)$ is a linear order. Then we can generate a Social Welfare Ordering $R(\cdot)$ which satisfies completeness, transitivity, unanimity and AIIA, then $\exists i$ s.t. $\forall \succ \in LA()^n, R(\succ) = \succ_i$

Arrow's Independence of Irrelevant alternative Consider the alternatives a and b, and we have two preference profile $\succ$ and $\succ'$ and $a \succ_i b$ iff $a \succ'_i b$ then we must have

$aR(\succ)b$ iff $aR(\succ')b$. This means that take any two alternatives in any set in any preference profile, the ordering of the preference should not change across preference profile.

Impossibility Theorem: Recall that $R(\cdot)$ is non-dictatorial is $\forall i, \exists$ some $\succ \in L(A)^n$ such that $R(\succ) \neq \succ_i$, However, there does not exsist a SWO $R(\cdot)$ that satifies C,T,U,AIIA *and* non-dictatorial.

Theorem: If we assume, complete, transitive, unanimous preference and AIIA, then there exist a dictator agent such that $\exists i$ s.t. $\forall \succ \in L(A)^n$ and $R(\succ) = \succ_i$.

Proof - Barbera / Geanakoploz 1980 We consider:

- Top alternative: $top(\succ_i) = a \succ_i b, \forall b \neq a$
- Bottom alternative: $bottom(\succ_i) = b$ s.t. $a \succ_i b, \forall a \neq b$

Step 1 For $\succ \in L(A)^n$ and b s.t. $\forall i$, either $b = top(\succ_i)$ or $b = bottom(\succ_i)$ then
either $b = top(R(\succ))$ and $bR(\succ)a$ $\forall a \neq b$
or $b = bottom(R(\succ))$ and $aR(\succ)b$ $\forall a \neq b$.

The idea is that people either hate or love and if people start switching pref, then their exists a pivotal agent which is the dictator of the game and who rules the social preference.

Now we consider $\succ$ and $\succ'$ and we observe that:

- $aR(\succ)bR(\succ)c$
- $aR(\succ')bR(\succ')c$ by AIIA
- $cR(\succ')a$ and not $aR(\succ')c$ by unanimity (which is contradict)

Here the idea is that $c \succ_i a, \forall i$, but alternative b keep on switching between bottom and top.

Step 2 We now consider the profiles $\succ^{bottom} \in L(A)$ and $\succ^{top} \in L(A)$. Then for n agents we have:

- $\succ_i^k = \succ_i^{bottom}, \forall i > k$
- $\succ_i^k = \succ_i^{top}, \forall i \leq k$

Then, $\exists i^*$ such that $i^*(b) = 1 + \arg\max_k : b = bottom[R(\succ^{(k)}]$ that is the first kth such that now $bR(\succ_{i^*})a, \forall a \neq b$.

Step 3 We can now say that $i^*(b)$ dictates on a, c for $a \neq b \neq c$ on all preference profile $\succ \in L(A)^n$ if $aR(\succ_{i^*})c \iff a \succ_{i^*} c$ and $cR(\succ_{i^*})a \iff c \succ_{i^*} a$

For $\succ$ s.t. $i^*(b)$, we have that
$a \succ_{i^*} c \implies aR(\succ)c$ and not $cR(\succ)a$. We define the profil $\succ' \in L(A)^n$ such that:

- $b = top(\succ'_i), \forall i < i^*(b)$

- $b = bottom(\succ'_i), \forall i \geq i^*(b)$

Then we have that $a \succ'_{i^*} b \succ'_{i^*} c$ and by transitivity we have that $aR(\succ')c$, and by AIIA, we know that $aR(\succ)c$, they alsways dictate.

Step 4 Suppose that $i^*(b)$ dictates over alternatives $a, c \neq b$ and $i^*(c)$ on alternative $a, b \neq c$. Now suppose that $i^*(b) \neq i^*(c)$. If $i^*(b)$ prefer a to c and c to b, and $i^*(c)$ prefers c to b and b to a, then if everybody else like $c \succ b \succ a$ but then the social pref must be $b \succ a \succ c$ because of the two dictators ... which is a contradiction because in all ranking by unanimity everyone prefers c to b but the social order reverses it.

15.2 Relaxing some assumptions

We first consider giving up on the transitive assumption, and consider *choice* $C(\succ, b) \in B \in A$ instead of ranking. We consider completeness $C(\succ, B) \in B, \forall B$ so that $C(\succ, B) \neq \emptyset$.

We can also give up in AIIA, and define Nash IIA such as $C(\succ, B), B' \subset B$ and $C(\succ, B) \cap B' \neq \emptyset$ then $C(\succ, B') = B' \cap C(\succ, B)$

Theorem If C satisfies NIIA, then $\exists R$ that is complete and transitive and satisfies $AIIA$ such that $C(\succ, B) = \arg\max_B R(\succ)$.

Note that if we were to add unanimity to the previous assumption when we would fall back into the case where we have a dictators.

16 Voting: Scoring rule and Condorcet Consistency

We will review two scoring rules: Borda and Plurality. We will also consider Condorcet Consistent rules: Copeland Rule and Simpson's.

Example of Scoring Rules

For a set of alternatives there are different alternatives and away of scoring different alternative.

Observe that in those example, R violate AIIA and C violates NIIA.

Table 1: Different Scoring Rules

Alternative	Borda	Plurality	Approval Voting
a	n-1	1	1
b	n-2	0	1
c	n-3	0	1
d	n-4	0	0
e	n-5	0	0

Table 2: Example

Scenario 1			**Plurality**	**Borda**
4x	3x	2x	**a**	**b (12)**
a	b	c	**b**	**a (8)**
b	c	b	**c**	**c (7)**
c	a	a		

16.1 Condorcet Consistent

Choose a Condorcet winner exists then there is an alternative majority preferred to all others. However, Borda rule is not Condorcet Consistent.

Copeland Rule Here we count how many alternatives are beaten in majority voting. If you beat an alternative this alternative is given +1 point, 0 if a tie, and −1 if losing (this is pairwise comparison).

Consistency If there is a Condorcet winner then this rule will pick it up.

Simpson's Rule We define this rule as $N_{>}(a,b) = \#\{i : a >_i b\}$. Hence, we have $Points_{>}(a) = \min_b N_{>}(a,b)$. Then we can rank alternative by this $Points_{>}$. This is also Condorcet consistent.

All theses rules assume: anonymity, unanimity, neutrality but will violate AIIA. But only Condorcet consistent pick a C-winner when it exists where as scoring rules do not. But scoring rule has a participation property.

17 Restricted preferences

We now restricted the preference where $> \subset L(A)^n$. We now consider single picked preferrences such that $A \subset R$, where $>_i$ is single picked (hence we eliminate v-shaped preference).

As such, $\forall i, \exists$ a peak $p(\succ_i)$ s.t. $p(\succ_i) \succ_i b, \forall b \neq p(\succ_i)$.
We can also defined $u_i(b) = -(b - p(\succ_i))^2$.

This is a nice property as it kills Condorcet cycles and thus, it buy us transitivity for the majority rule. This allows for a median voter which provide a "rational" society. It also buys us truthful announcement at preference is a dominant strategy.

18 Mechanism Design and Incentives

Let us define a set A with n agents. We are defining type Θ_i with $\theta_i \in \Theta_i$. And we define utility u_i as $u_i(a, \theta_i)$, where agent have a private values and θ_i is going to capture preferences $\succ_i$.

Let us define a Social Choice Function $f : \theta_i \times ... \times \theta_n \to A$. The types are actually characterize the picked.

When going into **Mechanism Design** we can also add up message M_i, where the function $g : M \to A$ is an outcome function. We can think about M as strategies and g is the outcome not necessarily the payoff because we are still missing the utility functions but more like allocation.

19 Dominant strategies and Revelation Principle

Given, N, u_i, Θ_i, A and mechanism g, M we define **dominant strategies** for $m_i \in M_i$ if:

$$u_i(g(m_i, m_{-1}), \theta_i) \geq u(g(m_i', m_{-i}), \theta_i) \forall m_i'$$

Revelation Principle Formally, if (g, M) is such that $\forall i, \theta_i \in Theta$ then $\exists$ a dominant strategy $m_i(\theta_i)$ and $\exists$ a direct mechanism f such that θ_i is a dominant strategy for f, and $f(\theta) = f(\Theta) = g(m_1(\theta_1), ..., m_n(\theta_n))$

Proof

$$\begin{aligned} u_i(f(\theta_i, \theta_{-i}), \theta_i) &= u_i(g(m_i(\theta_i), m_{-i}(\theta_{-i})), \theta_i) \\ &\geq u_i(g(m_i(\theta_i'), m_{-i}(\theta_{-i}))|\theta_i) \\ &= u_i(f(\theta_i', \theta_{-i}), \theta_i) \end{aligned}$$

Remark 1: We suppose that we have (g, M) such that $\forall i, \theta_i, \exists m_i(\theta_i)$ is a dominant strategy. Then we have that $f(\Theta) = g(m_1(\theta_1), ..., m_n(\theta_n))$ then this mechanism define by (f, Θ) is truth dominant $\forall i, \theta_i \in \Theta_i$.

Remark 2: If instead of working with g, M, we use an extensive form, then the revelation principal fails.

Example: Hurwict We consider $n = 2$ and $l = 2$, then we have $e_1 = (1,0)$ and $e_2 = (0,1)$ and utility are $u_2 = \min(x_{21}, x_{22}$ as he only has one type. As for agent 1, we have $u_1(x_1, \theta_1) = \min(x_{11} + \epsilon, x_{12})$ and $u_1(x_1, \theta_1') = \min(x_{11}, x_{12} + \epsilon)$

Here the WE are:

$$(\theta_1, \theta_2) = ((0,0),(1,1))(\theta_1, \theta_2) = ((1,1),(0,0))$$

Does there exists a M, g such that dominant strategy for each i, θ_i gives a WE? By the RP, we restrict attention ot Θ, f and the $WE(\theta)$ dominant strategy is not incentive compatible. Formally:

$$u_1(WE(\theta_i', \theta_2), \theta_1) > u_1(WE(\theta_i, \theta_2), \theta_1)$$

Thus, agent 1 is better off lying so the RP fails.

Example Borda It is also no incentive compatible. Example:

Table 3: Add caption

	Agent 1	**Agent 2**	**Agent 3**	***R***
3	a	b	a	***a***
2	b	c	b	***c***
1	c	d	c	***d***
0	d	a	d	***b***

If agent 1, change his vote he can lie and get a better outcomes for himself.

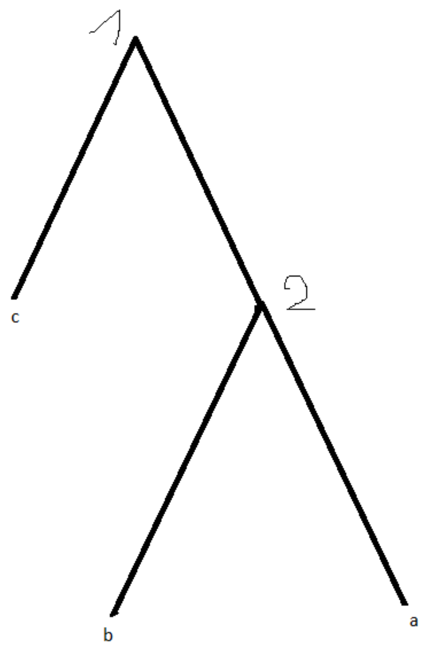
Single Peacked preferences For $A = R$, and the types $\Theta_i = SP(A)$, we can define $f(\theta) = med(p_i(\theta_i))_i$ now truth is a dominant strategy.

Example Let the set of alternative $A = \{a, b, c\}$ with 3 agents with utility u_i is given by 3 profile $\theta = a \succ b \succ c$ or $\hat{\theta} = b \succ a \succ c$ or $\tilde{\theta} = c \succ a \succ b$.
Now the message set is $M_i = \{a, b, c\}$ where $g(m)$ is the most frequent appearance of alternative (plurality rule).
Then everyone announce a vote and the mechanism g, M gives us the Nash Equi.
i. Say that all agent regardless of their type vote a and this is an equi, that is $f(\theta) = a, \forall \theta$.
ii. $\forall \theta_i$ vote for favorite (a,b) it is also a NE and $f(\theta)$ =majority winner of (a,b) and there must be several equilibrium amongst which one is truthful.

To conclude, take a Dominant Strategy Incentives Compatible (M,g) with $f = g(m(\theta))$ then the equilibrium given by (M,g) might not exactly coincide with the equilibrium given by f. It can go both way contracting and expanding the set. It is true that once you pick one of these equilibra there is a direct mechanism which is equivalent to that. In other words you have to tell me the mechanism + an equilibrium and then I can give you the equivalent direct mechanism. Conversely, we can have (g,M) with a unique equilibrium but f has multiple equilibria which is a superset. Also not that direct mechanism does not inherit all properties of the indirect mechanism.

Revelation fails for SPE

Consider $A = \{a,b,c\}$ and $n = 2$. Types are $\Theta_1 = \{\theta_1\} = a \succ c \succ b$ and $\Theta_2 = \{\theta_2, \hat{\theta}_2\} = \{c \succ a \succ b, c \succ b \succ a\}$. The game is



Then for $f(\theta_1,\theta_2) = a$ and $f(\theta_1,\hat{\theta}_2) = c$, given that agent 2 prefers c for both type then he has incentive to always play b and lie about his type to always adopt type $\hat{\theta}_2$.

Summary:

- It works for Dominant Strategy, Nash Equilibrium, Baysian Equilibrium: from f,Θ to M,g is WLOG when asking about what are the potential candidates for equi.
- but it does not work for SPE and Sequential equilibrium

20 Incentive Compatible

What are dominant strategy Incentive compatible ? Starting from $\Theta, f(\cdot)$ we have seen that WE and Borda does not work, but fixed prices and Single Peaked median voting works.

Let us take a step back and go back to the Arrow setting when $\Theta_i = L(A)$ with A finite.

20.1 Gibbard - Sathewait theorem

For more than 3 alternative, $|A| \geq 3$, we define Θ_i s.t. $\forall >_i \in L(A), \exists \theta_i in \Theta_i$ s.t $u_i(\cdot, \theta_i)$ generates this ranking and are the same for all $>_i$.
Then we have a function f such that the range of $f(\cdot) = A$ (which is weaker than unanimity), and such that this is a truth dominant strategy iff f is dictatorial and $\exists i$ s.t. $\forall \theta, f(\theta) = \arg\max u_i(\cdot, \theta_i)$.

Also, if preferences were single peaked and full range, then f is strategy proof if truth is a dominant strategy. Then this setting also satisfies unanimity and rationalizable by $R(>)$ (Social Welfare Ordering), and would also satisfy AIIA, transitive and complete preferences. Then we are back to the classic Arrow setting which gives the same results as Arrow.

Running example = public good Let us consider $x \in [0,1]$ as the amount to invest in a public good. Then we have $a \in A$ the allocation that specifies both how much to invest and the cost paid by each individual i.

Then, $u_i(a, \theta_i) = b_i(x, \theta_i) - c_i(c)$ (quasilinear function), where $c_i(x)$ is increasing and convex and $b_i(\cdot, \cdot)$ is increasing concave in x so that u_i is strictly concave and thus single peaked.

On strategy proof mechanism could be *median voting* or any statistic order like min or max.

20.2 Moulin 1980

Here we have a characterization of all the mechanism that are strategy-proof in the SP world. Using Phantom Voting Rules.

Consider n voters and $n+1$ phantom voters which have fixed preference given by society before individual votes. Hence, if $n = 1$ take 4 phantom fixed point and pick the median among the 7 (3 single-peaked maxima + 4 fixed points). This is non-manipulable.

20.3 Costs

Now for the costs, we consider fix costs (non transferable) say $c_i(x) = \frac{c(x)}{n}$, we reach efficient and thus Pareto Equilibrium. Hence, the total utility for the public good example is that for each type we have that the utility have the same rank as if we were ranking distance to median point.

Example Nonfixed Cost . Let us consider $n = 3$, $c(x) = x^2$ and allocation function is $b_1(x) = x, b_2(x) = 2x, b_3(x) = 4x$, hence type are $\theta_i = \{1, 2, 4\}$ and the generic utility is $\theta_i x - \frac{x^2}{3}$, FOC give $\theta_i = 2x/3$, to the peak is $p(\theta_i) = \frac{3\theta_i}{2}$ then:

$$\begin{aligned} u_1(x) = x - \frac{x^2}{3} &\implies \text{peak at } 3/2 \\ u_2(x) = 2x - \frac{x^2}{3} &\implies \text{peak at } 3 \\ u_3(x) = 4x - \frac{x^2}{3} &\implies \text{peak at } 6 \end{aligned}$$

Hence, the median is at $x = 3$, compared to the fixed costs $c(x)/n$, agent 1 is paying 1/8 extra, agent 2 is paying 1/4 and agent 3 5/8. The idea is that agent 3 is going to subsidy the other two agents. So the marginal gain (FOC of utility) is $\theta_i - 2xc_i$, for each agent this qtt is positive, which means that the median voting is not fully efficient (ie we have not reached the peaked and could do better).

Indeed, the problem is the incentive. If how much I pay depends on my type θ_i then i have an incentive to lie about my θ_i ($\rightarrow$ free rider problem).

21 Motivating example for the VCG mechanism

For $n = 2$, $x = \{0, 1\}$ (build or not build), we consider type $\theta_i \sim U[0, 2]$ and c the cost of the project and $c_i = c/2$. The efficient outcomes are defined as follows:

- build if $\sum \theta_i > c$
- no if $\sum \theta_i < c$

Analogue to the previous rule (median voting phantom). If they build than each pay $c/2$. Hence the decision rule is to build only if both agents say *yes*.

However, say that $\theta_1 = 1/3$ and $\theta_2 = 1$, then $\sum \theta_i = 4/3 > 1$ so all together they want to build but because $\theta_1 < 1/2$ they will not build according to the previous decision rule which is inefficient.

The puzzle comes from the different perspective:

- Indiv perspective: $\theta_i - \frac{c}{2} > 0$
- Society's perspective: $\theta_1 + \theta_j - c > 0$

To reconcile the two approaches we need to allow for transfers, so we can redefine individual's perspective as $\theta_i - \frac{c}{2} + t_i(\hat{\theta_j})$ where $t_i(\hat{\theta}_j) = \hat{\theta}_j - c/2$.
Now the new rule can be "build if $\hat{\theta}_i + \hat{\theta}_j - c > 0$ ".

This is convenient but now we hit the feasibility constraint $\sum t_i(\theta) \leq 0$.

To do so we can re-write the individual decision making process as :

- Build: $\theta_i - \frac{c}{2} + \hat{\theta}_j - \frac{c}{2} - \max\{\hat{\theta}_j - \frac{c}{2}, 0\}$
- Not Build: $0 + 0 - \max\{\hat{\theta}_j - \frac{c}{2}, 0\}$

This conserve the incentive but only if we burn this extra money.

So **upshot**:

- Voting $\rightarrow$ strategy-proof, inefficient decision, and balanced transfers
- Pivotal $\rightarrow$ strategy-proof, efficient decisions, but not efficient overall (burnt money), and feasible but not balanced.

21.1 General definition now: VCG Mechnism

Consider the following set up: $d \in A$ this is the decision, level of x, or who gets something, and $t_i \in R$ (transfers) and quasi-linear pref: $u_i = v_i(d(\theta), \theta_i) + t_i(\theta)$ where $d(\cdot)$ is the direct mechanism and θ_i the private value, finally t_i is the transfers to align incentive.

Ex-post the efficient decisions is given by: $\max_d \sum v_i(d(\theta), \theta_i)$. Then:

- *Efficiency*: $d(\hat{\theta}) \in \arg\max_x \sum_i v_i(x, \hat{\theta}_i)$
- Dominant Strategy Incentive compatible:
  $u_i(d(\theta_i, \hat{\theta}_{-i} + t_i(\theta_i, \hat{\theta}_i) \geq u_i(d(\tilde{\theta}_i, \hat{\theta}_{-i} + t_i(\tilde{\theta}_i, \hat{\theta}_i)$
- With balanced $\sum t_i(\hat{\theta}) = 0, \forall \hat{\theta}$ or feasible $\sum t_i(\hat{\theta}) \leq 0, \forall \hat{\theta}$

Theorem Grove, Green and Laffont.

1. If d is ex-post *efficient* and $\forall i, \exists x_i : \Theta_{-i} \rightarrow R$ and $t_i(\hat{\theta}) = x_i(\hat{\theta}_{-i}) + \sum_{j \neq i} v_j(d(\hat{\theta}), \hat{\theta}_j)$ then, (d, t) is *efficient* and Dominant Strategy Incentive Compatibility (DSIC) and thus strategy-proof.
2. If (d, t) is *strategy-proof / DSIC* and d is expost *efficient* and Θ is rich enough, then, $\forall i, \exists x_i(\hat{\theta}_{-i}$ s.t. $t_i(\hat{\theta})$ satisfy VCG (Groves mechanism) that is
   $t_i(\hat{\theta}) = x_i(\hat{\theta}_{-i}) + \sum_{j \neq i} v_j(d(\hat{\theta}), \hat{\theta}_j)$.

Proof: *Step 1:* $i : \max_{\hat{\theta}_i} u_i(d(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i) + t_i(\hat{\theta}_i, \hat{\theta}_{-i})$ given t
is equivalent to $\max_{\hat{\theta}_i} v_i(d(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i) + \sum_{j \neq j} v_j(d(\hat{\theta}_j, \hat{\theta}_{-j}))$.
Also by efficient $\hat{\theta}_i = \theta_i$ *Step 2:* We can also always write: $t_i(\hat{\theta}) = x_i(\hat{\theta}) + \sum_{j \neq i} v_j(d(\hat{\theta}), \hat{\theta}_{-i})$.
Here the goal is to show that $x_i(\hat{\theta})$ is independent of $\hat{\theta}_i$.
Let us argue by contradiction that $\exists i, \theta_i \theta_{-i}$ such that $x_i(\theta_i, \hat{\theta}_{-i}) \neq x_i(\hat{\theta}_i, \hat{\theta}_{-i})$.
$\implies$ WLOG: $x_i(\theta_i, \hat{\theta}_{-i}) - x_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \triangle > 0$ this means that $x_i(\theta_i, \hat{\theta}_{-i})$ is bigger transfers thus is much be that $d(\cdot)$ is also different (other it would violate strategy proof), thus $d(\theta_i, \hat{\theta}_{-i}) \neq d(\hat{\theta}_i, \hat{\theta}_{-i})$.

Using richness conditions $\implies \exists \tilde{\theta}_i$ such that :

$$v_i(d(\hat{\theta}_i, \hat{\theta}_{-i}), \tilde{\theta}_i) + \sum_j v_j(d(\hat{\theta}_i, \hat{\theta}_{-i}), \hat{\theta}_j) = \frac{\triangle}{2}$$
$$\text{and } v_i(d(\hat{\theta}_i, \hat{\theta}_{-i}), \tilde{\theta}_i) + \sum_j v_j(d(\hat{\theta}_i, \hat{\theta}_{-i}), \hat{\theta}_j) = 0 \; \forall \text{ other d}$$

As d is efficient $d(\tilde{\theta}_i, \hat{\theta}_{-i}) = d(\hat{\theta}_i, \hat{\theta}_{-i})$
then by strategy proofness $\implies x_i(\tilde{\theta}_i, \hat{\theta}_{-i}) = x_i(\hat{\theta}_i, \hat{\theta}_{-i})$

- truthful $\tilde{\theta}_i \implies: d(\hat{\theta})$ and utility is $\frac{\triangle}{2} + x_i(\hat{\theta}, \hat{\theta}_{-i})$
- lie $\theta_i \implies: d(\theta_i, \hat{\theta}_{-i})$ and utility is $0 + x_i(\theta, \hat{\theta}_{-i}) = x_i(\hat{\theta}, \hat{\theta}_{-i}) + \triangle$

Notice that this contradict strategy proofness because the incentive are set for me to lie about my true preference.

Upshot: In the quasi-linear world: *efficient* d + DSIC/strategy-proofness $\implies$ VCG. But Feasibility violates ex post individual rationality.

21.2 Clarck: Pivotal Mechanism special t_i

Here we consider:

$$t_i(\hat{\theta}) = \sum_{j \neq i} v_j(d(\hat{\theta}), \hat{\theta}_j) - \max_d \sum_{j \neq i} v_j(d, \hat{\theta}_j) \leq 0$$

The first term is what society does when i is present, whereas the second term is what they could do without i.

So now, why would this potentially violate balance and individual rationality (IR) ? the intuition is that the pivot mechanism is putting the externality of i on the rest of the society and charging them the difference. Hence we could violate IR because $t_i(\hat{\theta}) \leq 0$ and if $v_i(d) = 0$ then it could be that $u_i < 0$.

Example: Vickrey Auction We consider n agents: 1 seller and $2,...,n$ buyers. Type of the seller is $\Theta_i = \{0\}$ and for the buyers, their types are $\Theta_j = R_+$. Finally the set of alternatives is $A = \{1,2,...,n\}$ which is the label of who get the object. Then the utiliy are

$$u_i(d,\theta_i) = \begin{cases} \theta_i \text{ if } d = i \\ 0 \text{ if } d \neq i \end{cases}$$

In the pivotal mecanish, the efficient d solves $d \in \arg\max_i \theta_i$ and the transfers are given by $t_i(\hat{\theta} = \sum_{j\neq i} u_j(d(\hat{\theta}),\hat{\theta}_j) - \max \sum_{j\neq i} u_i(d,\hat{\theta}_j)$.

Hence, if $d(\hat{\theta}) = i$ and $u_i = 0$ and I get the object or I don't get the object $d(\hat{\theta}) \neq i$ and I pay zero and get $u_i = \max_{j\neq i} \hat{\theta}_j$

Summary With DSIC and ex-post efficient we have found a feasible pivotal mechanism $\sum_i t_i \leq 0$. But it can fail balance $\sum t_i = 0$ and fail individual rationality $v_i + t_i < 0$ sometimes. It is also group manipulable.

Example: Collusion

Table 4: Initial Set up

Agent	Build	Not Build	t_i
1	200	0	-150
2	100	0	-50
3	0	250	0

This mechanism is not efficient: not building would be better because $200 + 100 - 150 - 50 = 100$. Whereas not building it would have given a 250 utility.

Now, agent 1 and 2 decide to collude:

Table 5: With collusion between 1 and 2

Agent	Build	Not Build	t_i
1	250	0	-100
2	150	0	0
3	0	250	0

In this set up we still build and it also affect each others payment.

Example: VCG We have 2 agent, and a binary decisions. Cost is $c = 3/2$ and We need both agent to agree $d(1,1) = 1$ for the public good to be built. And type are $\theta_i\{0,1\}$.

A VCG mechanism under DSIC and efficient d + feasibility $\sum t_i \leq 0$. We know that $t_i = v_{-i}(d(\hat{\theta}, \hat{\theta}_{-i}) + x_i(\hat{\theta}_{-i}$.

The feasibility condition gives us that:

$$t_1(1,1) + t_2(1,1) \leq 0$$
$$t_1(0,0) + t_2(0,0) \leq 0$$

Then we know that $t_i(1,1) = v_{-i}(1,1) + x_i(1) = 1 - c_i + x_i(1)$. As such we have: $t_1(1,1) + t_2(1,1) = 2 - 3/2 + x_1(1) + x_2(1) \leq 0 \implies x_1(1) + x_2(1) \leq -1/2$

Hence, given that $d(0,0) = 0$ and $x_1(0) + x_2(0) \leq 0$ then either:

$$x_1(0) + x_2(1) \leq -1/4$$
$$x_1(1) + x_2(0) \leq -1/4$$

Then this means that:

$$t_1(1,0) + t_2(1,0) \leq -1/4$$
$$t_1(0,1) + t_2(0,1) \leq -1/4$$

22 Change assumptions to prevent collusion

One solution to prevent collusion is to drop dominant strategy incentives compatible and replace it with Baysian incentive compatible (BIC). This still satisfies d efficient decisions ex-post + t_i balanced but still might fail at IS and coalition manipulative.

Let us consider the following set-up: 2 agent a Buyer and a Seller. Now the set of alternative A is to either $Trade$ or Not. If agent decide to trade then there is a price at which they agree to trade and a transfers between agents happens (here we ensure that there is not waste and thus we are in the balanced scenario case).

Table 6: Outcomes and transfers

		Buyer	
		40	200
Sellers	160	No trade:	Trade: p = 180
		$t_b = -20$ and $t_s = -20$	$t_b = 0$ and $t_s = -80$
	0	Trade: $p = 20$	Trade: p = 100
		$t_b = -80$ and $t_s = 0$	$t_b = 0$ and $t_s = 0$

Let us consider the top right corner. The t_b are computed as the difference between what he gets which *Trade* with price of 180 minus the max of the given second column or the box depending on the exercise which is also *Trade* for a price of 180. That is why we have $t_b = 0$

Back to the definition of the VCG mechanism. The set of alternatives is $A = \{(NT), (T, 20), (T, 100), (T, 180)\}$ and the utility is:

$$u_i(d(\hat{\theta}), \theta_i) + u_j(d(\hat{\theta}), \theta_j) - \max_{d \in A} u_j(\hat{\theta}_j)$$

However, is it not incentive compatible de announce true type (even in the Bayesian sense). Running Clarck Pivot mechanism on to p and the t_i. In addition the prices need there transfers and thus a sizeable amount of utility is burnt.

Let us now drop the transfers and consider Bayesian instead of dominant strategy IC, this means that now we put a probability on trade. Also we assume that each agent has a 50-50 chaneg to be any of the two types.

Table 7: Bayesian

		Buyer	
		40	200
Sellers	160	No trade: 0	Trade: p = 180 π
	0	Trade: p = 20 π	Trade: p = 100 1

From there, we solve for the highest π.

If the Buyer tells the truth:

$$\begin{aligned} \text{Truth: } & \frac{\pi}{2}(200-180) + \frac{1}{2}(200-100) \\ \text{Lie: } & \frac{1}{2}0 + \frac{\pi}{2}(200-20) \\ & \implies \pi = \frac{5}{8} \end{aligned}$$

This results is inefficient but we got balance.

If instead we could adjust prices such that $\frac{1}{2}(200 - p(0,200)) + \frac{1}{2}(200-160) \geq \frac{1}{2}0 + \frac{1}{2}(200-40) \implies 80 \leq p(0,200) \leq 120$

Then we would have $\pi^* = 5/6$ which would be more efficient.

Table 8: Bayesian

		Buyer	
		40	200
Sellers	160	No trade: 0	Trade: p = **160**
			π
	0	Trade: p = **40**	Trade: p = 100
		π	1

Theorem: Myerson - Satterthwaite Consider a Buyer and a Seller, both risk neutral. And the value of the buyer are in $[c, d]$ and the cost for the seller are in $[a, b]$ with $a < c < b < d$ so there are gain from trade.

Then: there does not exist a Bayesian IC, interim IR and ex-post efficient (trade iff $v > c$ and no money burnt).