Interval Prediction for Continuous-Time Systems with Parametric Uncertainties

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Abstract— The problem of behaviour prediction for linear parameter-varying systems is considered in the interval framework. It is assumed that the system is subject to uncertain inputs and the vector of scheduling parameters is unmeasurable, but all uncertainties take values in a given admissible set. Then an interval predictor is designed and its stability is guaranteed applying Lyapunov function with a novel structure. The conditions of stability are formulated in the form of linear matrix inequalities. Efficiency of the theoretical results is demonstrated in the application to safe motion planning for autonomous vehicles.

I. INTRODUCTION

There are plenty of emerging application domains nowadays, where the decision algorithms have to operate in the conditions of a severe uncertainty. Therefore, the decision procedures need more information, then the estimation, identification and prediction algorithms come to the attention. In most of these applications, even the nominal simplified models are nonlinear, and in order to solve the problem of estimation and control in nonlinear and uncertain systems, a popular approach is based on the Linear Parameter-Varying (LPV) representation of their dynamics [1], [2], [3], [4], since it allows to reduce the problem to the linear context at the price of augmented parametric variation.

In the presence of uncertainty (unknown parameters or/and external disturbances) the design of a conventional estimator or predictor, approaching the ideal value of the state, can be realized under restrictive assumptions only. However, an interval estimation/prediction remains frequently feasible: using input-output information an algorithm evaluates the set of admissible values (interval) for the state at each instant of time [5], [6]. The interval length must be minimized via a parametric tuning of the system, and it is typically proportional to the size of the model uncertainty [7].

There are many approaches to design interval/set-membership estimators and predictors [8], [9], [10], [11], and this paper focuses on the design based on the monotone systems theory [10], [11], [12], [13], [14]. In such a way the main difficulty for synthesis consists in ensuring cooperativity of the interval error dynamics by a proper design

of the algorithm. As it has been shown in [15], [13], [16], such a complexity of the design can be handled by applying an additional transformation of coordinates to map a stable system into a stable and monotone one, see also [17], [7].

The objective of this paper is to propose an interval predictor for LPV systems. The main difficulty to overcome is the predictor stability, which contrarily to an observer cannot be imposed by a proper design of the gains. An interval inclusion of the uncertain components can be restrictive and transform an initially stable system to an unstable one. To solve this problem, a generic predictor is proposed for an LPV system. To analyze stability of the predictor, which is modeled by a nonlinear Lipschitz dynamics, a novel non-conservative Lyapunov function is developed, whose features can be verified through solution of linear matrix inequalities (LMIs). Finally, the utility of the developed theory is demonstrated on the problem of robust path planning for a self-driving car by making comparison with earlier results from [18].

II. PRELIMINARIES

We denote the real numbers by $\mathbb{R}$, the integers by $\mathbb{Z}$, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ and the sequence of integers $1, ..., k$ as $\overline{1, k}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}$ we denote its L_∞ norm by $||u||_{[t_0, t_1]} = \operatorname{ess\,sup}_{t \in [t_0, t_1)} |u(t)|$. If $t_1 = +\infty$ then we will simply write $||u||$. We will denote as $\mathcal{L}_\infty$ the set of all inputs u with the property $||u|| < \infty$.

The symbols I_n, $E_{n \times m}$ and E_p denote the identity matrix with dimension $n \times n$, and the matrices with all elements equal 1 with dimensions $n \times m$ and $p \times 1$, respectively.

For a matrix $A \in \mathbb{R}^{n \times n}$ the vector of its eigenvalues is denoted as $\lambda(A)$, $||A||_{max} = \max_{i=\overline{1,n}, j=\overline{1,n}} |A_{i,j}|$ (the elementwise maximum norm, it is not sub-multiplicative) and $||A||_2 = \sqrt{\max_{i=\overline{1,n}} \lambda_i(A^\top A)}$ (the induced L_2 matrix norm), the relation $||A||_{max} \leq ||A||_2 \leq n||A||_{max}$ is satisfied between these norms.

A. Interval arithmetic

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. The relation $P \prec 0$ ($P \succ 0$) means that a symmetric matrix $P \in \mathbb{R}^{n \times n}$ is negative (positive) definite. Given a matrix $A \in \mathbb{R}^{m \times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1. [19] *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \le x \le \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$.*

(1) If $A \in \mathbb{R}^{m \times n}$ is a constant matrix, then

$$A^+ \underline{x} - A^- \overline{x} \le Ax \le A^+ \overline{x} - A^- \underline{x}. \quad (1)$$

(2) If $A \in \mathbb{R}^{m \times n}$ is a matrix variable and $\underline{A} \le A \le \overline{A}$ for some $\underline{A}, \overline{A} \in \mathbb{R}^{m \times n}$, then

$$\underline{A}^+ \underline{x}^+ - \overline{A}^+ \underline{x}^- - \underline{A}^- \overline{x}^+ + \overline{A}^- \overline{x}^- \le Ax \quad (2)$$
$$\le \overline{A}^+ \overline{x}^+ - \underline{A}^+ \overline{x}^- - \overline{A}^- \underline{x}^+ + \underline{A}^- \underline{x}^-.$$

Furthermore, if $-\overline{A} = \underline{A} \le 0 \le \overline{A}$, then the inequality (2) can be simplified: $-\overline{A}(\overline{x}^+ + \underline{x}^-) \le Ax \le \overline{A}(\overline{x}^+ + \underline{x}^-)$.

B. Nonnegative systems

A matrix $A \in \mathbb{R}^{n \times n}$ is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of the linear system

$$\dot{x}(t) = Ax(t) + B\omega(t), \ t \ge 0, \quad (3)$$
$$y(t) = Cx(t) + D\omega(t),$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n \times n}$, is elementwise nonnegative for all $t \ge 0$ provided that $x(0) \ge 0$, $\omega : \mathbb{R}_+ \to \mathbb{R}_+^q$ and $B \in \mathbb{R}_+^{n \times q}$ [20], [21]. The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}_+^{p \times n}$ and $D \in \mathbb{R}_+^{p \times q}$. Such dynamical systems are called cooperative (monotone) or nonnegative if only initial conditions in $\mathbb{R}_+^n$ are considered [20], [21].

Lemma 2. [13] *Given the matrices $A \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{p \times n}$. If there is a matrix $L \in \mathbb{R}^{n \times p}$ such that the matrices $A - LC$ and Y have the same eigenvalues, then there is a matrix $S \in \mathbb{R}^{n \times n}$ such that $Y = S(A - LC)S^{-1}$ provided that the pairs $(A - LC, \chi_1)$ and (Y, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1 \times n}$, $\chi_2 \in \mathbb{R}^{1 \times n}$.*

This result allows to represent the system (3) in its non-negative form via a similarity transformation of coordinates.

Lemma 3. [17] *Let $D \in \Xi \subset \mathbb{R}^{n \times n}$ be a matrix variable satisfying the interval constraints $\Xi = \{D \in \mathbb{R}^{n \times n} : D_a - \Delta \le D \le D_a + \Delta\}$ for some $D_a^T = D_a \in \mathbb{R}^{n \times n}$ and $\Delta \in \mathbb{R}_+^{n \times n}$. If for some constant $\mu \in \mathbb{R}_+$ and a diagonal matrix $\Upsilon \in \mathbb{R}^{n \times n}$ the Metzler matrix $Y = \mu E_{n \times n} - \Upsilon$ has the same eigenvalues as the matrix D_a, then there is an orthogonal matrix $S \in \mathbb{R}^{n \times n}$ such that the matrices $S^T D S$ are Metzler for all $D \in \Xi$ provided that $\mu > n \|\Delta\|_{max}$.*

In the last lemma, the existence of similarity transformation is proven for an interval of matrices, *e.g.* LPV dynamics.

III. PROBLEM STATEMENT

Consider an LPV system:

$$\dot{x}(t) = A(\theta(t))x(t) + Bd(t), \ t \ge 0, \quad (4)$$

where $x(t) \in \mathbb{R}^n$ is the state, $\theta(t) \in \Pi \subset \mathbb{R}^r$ is the vector of scheduling parameters with a known set of admissible values Π, $\theta \in \mathcal{L}_\infty^r$; the signal $d : \mathbb{R}_+ \to \mathbb{R}^m$ is the external input.

The values of the scheduling vector $\theta(t)$ are not available for measurements, and only the set of admissible values Π is known. The matrix $B \in \mathbb{R}^{n \times m}$ is known, the matrix function $A : \Pi \to \mathbb{R}^{n \times n}$ is locally bounded (continuous) and known.

The following assumptions will be used in this work.

Assumption 1. *In the system (4), $x \in \mathcal{L}_\infty^n$. In addition, $x(0) \in [\underline{x}_0, \overline{x}_0]$ for some known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$.*

Assumption 2. *There exists known signals $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$ such that $\underline{d}(t) \le d(t) \le \overline{d}(t)$ for all $t \ge 0$.*

Assumption 1 means that the system (4) generates stable trajectories with a bounded state x for the applied class of inputs d, and the initial conditions $x(0)$ are constrained to belong to a given interval $[\underline{x}_0, \overline{x}_0]$. In Assumption 2, it is supposed that the input $d(t)$ belongs to a known bounded interval $[\underline{d}(t), \overline{d}(t)]$ for all $t \in \mathbb{R}_+$, which is the standard hypothesis for the interval estimation [5], [6].

Note that since the function A and the set Π are known, and $\theta \in \Pi$, then there exist matrices $\underline{A}, \overline{A} \in \mathbb{R}^{n \times n}$, which can be easily computed, such that

$$\underline{A} \le A(\theta) \le \overline{A}, \quad \forall \theta \in \Pi.$$

A. The goal

The objective of this work is to design an *interval predictor* for the system (4), which takes the information on the initial conditions $[\underline{x}_0, \overline{x}_0]$, the admissible bounds on the values of the exogenous input $[\underline{d}(t), \overline{d}(t)]$, the information about A and Π (*e.g.* the matrices $\underline{A}, \overline{A}$, but not the instant value of $\theta(t)$) and generates bounded interval estimates $\underline{x}(t), \overline{x}(t) \in \mathbb{R}^n$ such that

$$\underline{x}(t) \le x(t) \le \overline{x}(t), \quad \forall t \ge 0. \quad (5)$$

B. A motivation example

Following the result of Lemma 1, there is a straightforward solution to the problem used in [18]:

$$\begin{aligned}
\dot{\underline{x}}(t) &= \underline{A}^+ \underline{x}^+(t) - \overline{A}^+ \underline{x}^-(t) - \underline{A}^- \overline{x}^+(t) \\
&\quad + \overline{A}^- \overline{x}^-(t) + B^+ \underline{d}(t) - B^- \overline{d}(t), \quad (6) \\
\dot{\overline{x}}(t) &= \overline{A}^+ \overline{x}^+(t) - \underline{A}^+ \overline{x}^-(t) - \overline{A}^- \underline{x}^+(t) \\
&\quad + \underline{A}^- \underline{x}^-(t) + B^+ \overline{d}(t) - B^- \underline{d}(t), \\
\underline{x}(0) &= \underline{x}_0, \ \overline{x}(0) = \overline{x}_0,
\end{aligned}$$

then it is obvious to verify that the relations (5) are satisfied, but the stability analysis of the system (6) is more tricky. Indeed, (6) is a purely nonlinear system (since $\underline{x}^+$, $\underline{x}^-$, $\overline{x}^+$ and $\overline{x}^-$ are globally Lipschitz functions of the state $\underline{x}$ and $\overline{x}$), whose robust stability with respect to the bounded external inputs $\underline{d}$ and $\overline{d}$ can be assessed if a suitable Lyapunov function is found. And it is easy to find an example, where the matrices $\underline{A}$ and $\overline{A}$ are stable, but the system (6) is not:

Example (motivating). Consider a scalar system:

$$\dot{x}(t) = -\theta(t)x(t) + d(t), \ t \ge 0,$$

where $x(t) \in \mathbb{R}$ with $x(0) \in [\underline{x}_0, \overline{x}_0] = [1.0, 1.1]$, $\theta(t) \in \Pi = [\underline{\theta}, \overline{\theta}] = [0.5, 1.5]$ and $d(t) \in [\underline{d}, \overline{d}] = [-0.1, 0.1]$ for all
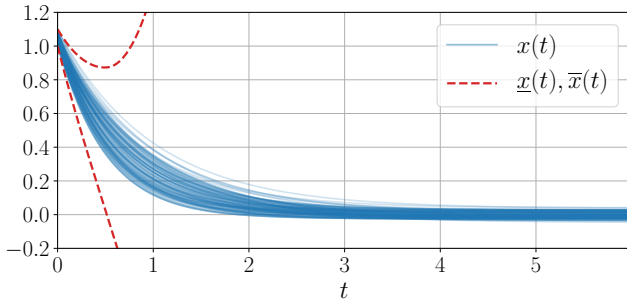
Fig. 1. The results of prediction by (6): even in such a simplistic setting, the predictor is unstable and diverges quickly.

$t \geq 0$. Obviously, assumptions 1 and 2 are satisfied, and this uncertain dynamics produces bounded trajectories (to prove this consider a Lyapunov function $V(x) = x^2$). Then the interval predictor (6) takes the form:

$$
\begin{aligned}
\dot{\underline{x}}(t) &= -\overline{\theta}\overline{x}^+(t) + \underline{\theta}\overline{x}^-(t) + \underline{d}, \\
\dot{\overline{x}}(t) &= -\underline{\theta}x^+(t) + \overline{\theta}\underline{x}^-(t) + \overline{d}.
\end{aligned}
$$

The results of simulation are shown in Fig. 1. As we can conclude, additional consideration and design are needed to properly solve the posed problem.

IV. INTERVAL PREDICTOR DESIGN

Note that, in related papers [22], [12], [23], [17], [24], [7], various interval observers for LPV systems have been proposed, but in those works the cooperativity and stability of the estimation error dynamics are ensured by a proper selection of observer gains and/or by design of control algorithms, which can be dependent on $\underline{x}$, $\overline{x}$ and guarantee the observer robust stability. For interval predictor there is no such a freedom, then a careful selection of hypotheses has to be made in order to provide a desired solution. We will additionally assume the following:

Assumption 3. *There exist a Metzler matrix $A_0 \in \mathbb{R}^{n \times n}$ and matrices $\Delta A_i \in \mathbb{R}^{n \times n}$, $i = \overline{1, N}$ for some $N \in \mathbb{Z}_+$ such that the following relations are satisfied for all $\theta \in \Pi$:*

$$
A(\theta) = A_0 + \sum_{i=1}^N \lambda_i(\theta) \Delta A_i,
$$

$$
\sum_{i=1}^N \lambda_i(\theta) = 1; \; \lambda_i(\theta) \in [0,1], \; i = \overline{1, N}.
$$

Therefore, it is assumed that the matrix $A(\theta)$ for any $\theta \in \Pi$ can be embedded in a polytope defined by N known vertices ΔA_i with the given center A_0, which admits already useful properties. According to the results of lemmas 2 and 3, the fulfillment of Assumption 3 can be imposed by applying a properly designed similarity transformation, which maps a matrix (interval of matrices) to a Metzler one. Design of such a transformation is not considered in this work, and we will just suppose in Assumption 3 that the system (4) has

been already put in the right form:

$$
\dot{x}(t) = [A_0 + \sum_{i=1}^N \lambda_i(\theta(t)) \Delta A_i] x(t) + B d(t).
$$

Denote $\Delta A_+ = \sum_{i=1}^N \Delta A_i^+$, $\Delta A_- = \sum_{i=1}^N \Delta A_i^-$, then the following interval predictor can be designed:

Theorem 1. *Let assumptions 1–3 be satisfied for the system (4), then an interval predictor*

$$
\begin{aligned}
\dot{\underline{x}}(t) &= A_0 \underline{x}(t) - \Delta A_+ \underline{x}^-(t) - \Delta A_- \overline{x}^+(t) \\
&\quad + B^+ \underline{d}(t) - B^- \overline{d}(t), \qquad (7) \\
\dot{\overline{x}}(t) &= A_0 \overline{x}(t) + \Delta A_+ \overline{x}^+(t) + \Delta A_- \underline{x}^-(t) \\
&\quad + B^+ \overline{d}(t) - B^- \underline{d}(t), \\
\underline{x}(0) &= \underline{x}_0, \; \overline{x}(0) = \overline{x}_0
\end{aligned}
$$

ensures the property (5). If there exist diagonal matrices P, Q, Q_+, Q_-, Z_+, Z_-, Ψ_+, Ψ_-, Ψ, $\Gamma \in \mathbb{R}^{2n \times 2n}$ such that the following LMIs are satisfied:

$$
P + \min\{Z_+, Z_-\} > 0, \; \Upsilon \preceq 0, \; \Gamma > 0,
$$

$$
Q + \min\{Q_+, Q_-\} + 2\min\{\Psi_+, \Psi_-\} > 0,
$$

where

$$
\Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & P \\ \Upsilon_{12}^\top & \Upsilon_{22} & \Upsilon_{23} & Z_+ \\ \Upsilon_{13}^\top & \Upsilon_{23}^\top & \Upsilon_{33} & Z_- \\ P & Z_+ & Z_- & -\Gamma \end{bmatrix},
$$

$$
\Upsilon_{11} = \mathcal{A}^\top P + P\mathcal{A} + Q, \; \Upsilon_{12} = \mathcal{A}^\top Z_+ + PR_+ + \Psi_+,
$$

$$
\Upsilon_{13} = \mathcal{A}^\top Z_- + PR_- + \Psi_-, \; \Upsilon_{22} = Z_+ R_+ + R_+^\top Z_+ + Q_+,
$$

$$
\Upsilon_{23} = Z_+ R_- + R_+^\top Z_- + \Psi, \; \Upsilon_{33} = Z_- R_- + R_-^\top Z_- + Q_-,
$$

$$
\mathcal{A} = \begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix}, \; R_+ = \begin{bmatrix} 0 & -\Delta A_- \\ 0 & \Delta A_+ \end{bmatrix}, \; R_- = \begin{bmatrix} \Delta A_+ & 0 \\ -\Delta A_- & 0 \end{bmatrix},
$$

then the predictor (7) is input-to-state stable with respect to the inputs $\underline{d}$, $\overline{d}$.

Note the requirement that the matrix P has to be diagonal is not restrictive, since for a Metzler matrix $\mathcal{A}$, its stability is equivalent to existence of a diagonal solution P of the Lyapunov equation $\mathcal{A}^\top P + P\mathcal{A} \prec 0$ [20].

Proof. First, let us demonstrate (5), to this end note that

$$
-\Delta A_+ \underline{x}^- - \Delta A_- \overline{x}^+ \leq \sum_{i=1}^N \lambda_i \Delta A_i x \leq \Delta A_+ \overline{x}^+ + \Delta A_- \underline{x}^-
$$

and introducing usual interval estimation errors $\underline{e} = x - \underline{x}$ and $\overline{e} = \overline{x} - x$ and calculating their dynamics we get:

$$
\begin{aligned}
\dot{\underline{e}}(t) &= A_0 \underline{e}(t) + \underline{r}_1(t) + \underline{r}_2(t), \\
\dot{\overline{e}}(t) &= A_0 \overline{e}(t) + \overline{r}_1(t) + \overline{r}_2(t),
\end{aligned}
$$

where

$$
\underline{r}_1 = \sum_{i=1}^N \lambda_i \Delta A_i x + \Delta A_+ \underline{x}^- + \Delta A_- \overline{x}^+,
$$

$$
\underline{r}_2 = Bd - B^+ \underline{d} + B^- \overline{d},
$$

$$
\overline{r}_1 = \Delta A_+ \overline{x}^+ + \Delta A_- \underline{x}^- - \sum_{i=1}^N \lambda_i \Delta A_i x,
$$

$$
\overline{r}_2 = B^+ \overline{d} - B^- \underline{d} - Bd.
$$

Non-negativity or $\underline{r}_2$ and $\overline{r}_2$ follows from Assumption 2 and Lemma 1. The signals $\underline{r}_1$ and $\overline{r}_1$ are also nonnegative provided that (5) holds and due to the calculations above. Note that the relations (5) are satisfied for $t = 0$ by construction and Assumption 1, then it is possible to show that $\underline{e}(t) \geq 0$ and $\overline{e}(t) \geq 0$ for all $t \geq 0$ [21], which confirms the relations (5).

Second, let us consider the stability of (7), and for this purpose define the extended state vector as $X = [\underline{x}^\top \ \overline{x}^\top]^\top$, whose dynamics admit the differential equation:

$$\dot{X}(t) = \mathcal{A}X(t) + R_+ X^+(t) - R_- X^-(t) + \delta(t),$$

where

$$\delta(t) = \begin{bmatrix} -B^- & B^+ \\ B^+ & -B^- \end{bmatrix} \begin{bmatrix} \overline{d}(t) \\ \underline{d}(t) \end{bmatrix}$$

is a bounded input vector, whose norm is proportional to $\underline{d}$, $\overline{d}$. Following [25], consider a candidate Lyapunov function:

$$V(X) = X^\top P X + X^\top Z_+ X^+ - X^\top Z_- X^-,$$

which is positive definite provided that

$$P + \min\{Z_+, Z_-\} > 0,$$

and whose derivative for the system dynamics takes the form, if $\Upsilon \preceq 0$:

$$\dot{V} \ \leq \ -X^\top \Omega X + \delta^\top \Gamma \delta,$$

where

$$\Omega = Q + \min\{Q_+, Q_-\} + 2\min\{\Psi_+, \Psi_-\} > 0$$

is a diagonal matrix. The substantiated properties of V and its derivative imply that (7) is input-to-state stable [26] with respect to the input δ (or, by its definition, with respect to $(\underline{d}, \overline{d})$). $\qquad\square$

Remark 1. The LMIs of the above theorem are not conservative, since the restriction on positive definiteness of involved matrix variables is not imposed on all of them separately, but on their combinations:

$$P + \min\{Z_+, Z_-\} > 0, \ \Gamma > 0,$$
$$Q + \min\{Q_+, Q_-\} + 2\min\{\Psi_+, \Psi_-\} > 0,$$

then some of them can be sign-indefinite or negative-definite ensuring the fulfillment of the last inequality: $\Upsilon \preceq 0$.

Example (motivating, continue). Let us apply the predictor (7) to the motivation example:

$$\begin{aligned}
\dot{\underline{x}}(t) &= -\overline{\theta}\underline{x}(t) - (\overline{\theta} - \underline{\theta})\underline{x}^-(t) + \underline{d}, \\
\dot{\overline{x}}(t) &= -\overline{\theta}\overline{x}(t) + (\overline{\theta} - \underline{\theta})\overline{x}^+(t) + \overline{d},
\end{aligned}$$

where $A_0 = -\overline{\theta}$ is chosen, then $\Delta A_+ = \overline{\theta} - \underline{\theta}$, $\Delta A_- = 0$ and all conditions of Theorem 1 are verified. The results of simulation are shown in Fig. 2. As we can see the new predictor generates very reasonable and bounded estimates.
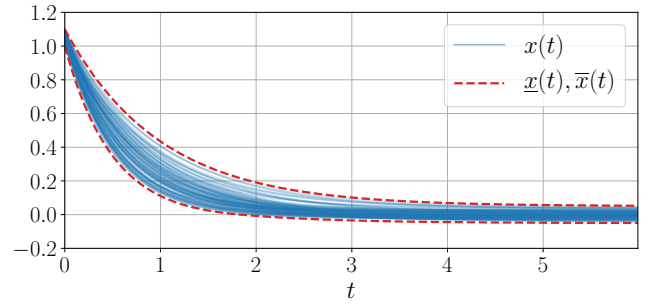

Fig. 2. The results of prediction by (7): the new predictor is stable and produces tight bounds.

V. PREDICTION FOR A SELF-DRIVING VEHICLE

We consider the problem of safe decision-making for autonomous highway driving [27]. The videos and source code of all experiments are available[1].

An autonomous vehicle is driving on a highway populated with N other agents, and uses Model Predictive Control to plan a sequence of decisions. To that end, it relies on parametrized dynamical model for each agent to predict the future trajectory of each traffic participant:

$$\dot{z}_i = f_i(Z, \theta_i), \ i = \overline{1, N},$$

where f_i are described below, $z_i \in \mathbb{R}^4$ is the state of an agent, $Z = [z_1, \ldots, z_N]^\top \in \mathbb{R}^{4N}$, and $\theta_i \in \mathbb{R}^5$ is the corresponding vector of unknown parameters. Crucially, this system describes the couplings and interactions between vehicles, so that the autonomous agent can properly anticipate their reactions. However, we assume that we do not have access to the true values of the behavioural parameters $\theta = [\theta_1, \ldots, \theta_N]^\top$; instead, we merely know that these parameters lie in a set of admissible values $\Pi \subset \mathbb{R}^{5N}$. In order to act safely in the face of uncertainty, we follow the framework of robust decision-making: the agent must consider all the possible trajectories in the space of Z that each vehicle can follow in order to take its decisions. In this work, we focus on how to compute these trajectory enclosures through interval prediction. In the following, we describe the system and its associated interval predictor.

A. Kinematics

The kinematics of any vehicle $i \in \overline{1, N}$ are represented by the Kinematic Bicycle Model [28]:

$$\begin{cases} \dot{x}_i = v_i \cos(\psi_i), & \dot{y}_i = v_i \sin(\psi_i), \\ \dot{v}_i = a_i, & \dot{\psi}_i = \dfrac{v_i}{l} tan(\beta_i) \end{cases}$$

where (x_i, y_i) is the vehicle position, v_i its forward velocity and ψ_i its heading, l its half-length, a_i is the acceleration command and β_i is the slip angle at the centre of gravity, used as a steering command, then $z_i = [x_i, y_i, v_i, \psi_i]^\top$.

[1]https://eleurent.github.io/interval-prediction/

B. Control

Longitudinal behaviour is modelled by a linear controller using three features inspired from the intelligent driver model (IDM) [29]: a desired velocity, a braking term to drive slower than the front vehicle, and a braking term to respect a safe distance to the front vehicle. Denoting f_i the index of the front vehicle preceding vehicle i, the acceleration command can be presented as follows:

$$a_i = \begin{bmatrix} \theta_{i,1} & \theta_{i,2} & \theta_{i,3} \end{bmatrix} \begin{bmatrix} v_0 - v_i \\ -(v_{f_i} - v_i)^- \\ -(x_{f_i} - x_i - (d_0 + v_i T))^- \end{bmatrix},$$

where v_0, d_0 and T respectively denote the speed limit, jam distance and time gap given by traffic rules.

The lane L_i with the lateral position y_{L_i} and heading ψ_{L_i} is tracked by a cascade controller of lateral position and heading β_i, which is selected in a way the closed-loop dynamics take the form:

$$\begin{cases} \dot{\psi}_i = \theta_{i,5} \left(\psi_{L_i} + \sin^{-1} \left(\frac{\tilde{v}_{i,y}}{v_i} \right) - \psi_i \right), \\ \tilde{v}_{i,y} = \theta_{i,4}(y_{L_i} - y_i). \end{cases} \quad (8)$$

We assume that the drivers choose their steering command β_i such that (8) is always achieved: $\beta_i = \tan^{-1}(\frac{l}{v_i} \dot{\psi}_i)$.

C. LPV formulation

The system presented so far is non-linear and must be cast into the LPV form. We approximate the non-linearities induced by the trigonometric operators through equilibrium linearisation around $y_i = y_{L_i}$ and $\psi_i = \psi_{L_i}$.

This yields the longitudinal dynamics: $\dot{x}_i = v_i$ and

$$\dot{v}_i = \theta_{i,1}(v_0 - v_i) + \theta_{i,2}(v_{f_i} - v_i) + \theta_{i,3}(x_{f_i} - x_i - d_0 - v_i T),$$

where $\theta_{i,2}$ and $\theta_{i,3}$ are set to 0 whenever the corresponding features are not active. It can be rewritten in the form

$$\dot{Z} = A(\theta)(Z - Z_c) + d.$$

For example, in the case of two vehicles only:

$$Z = \begin{bmatrix} x_i \\ x_{f_i} \\ v_i \\ v_{f_i} \end{bmatrix}, \quad Z_c = \begin{bmatrix} -d_0 - v_0 T \\ 0 \\ v_0 \\ v_0 \end{bmatrix}, \quad d = \begin{bmatrix} v_0 \\ v_0 \\ 0 \\ 0 \end{bmatrix}$$

$$A(\theta) = \begin{matrix} i \\ f_i \\ i \\ f_i \end{matrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\theta_{i,3} & \theta_{i,3} & -\theta_{i,1} - \theta_{i,2} - \theta_{i,3} & \theta_{i,2} \\ 0 & 0 & 0 & -\theta_{f_i,1} \end{bmatrix}$$
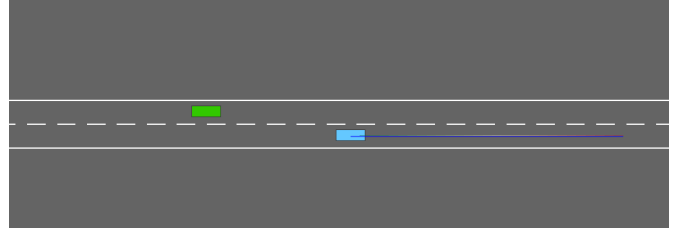
The lateral dynamics are in a similar form:

$$\begin{bmatrix} \dot{y}_i \\ \dot{\psi}_i \end{bmatrix} = \begin{bmatrix} 0 & v_i \\ -\frac{\theta_{i,4}\theta_{i,5}}{v_i} & -\theta_{i,5} \end{bmatrix} \begin{bmatrix} y_i - y_{L_i} \\ \psi_i - \psi_{L_i} \end{bmatrix} + \begin{bmatrix} v_i \psi_{L_i} \\ 0 \end{bmatrix}$$
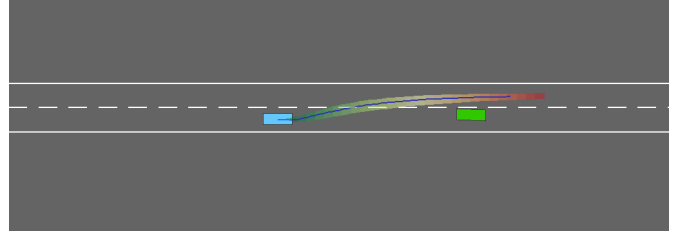
Here, the dependency in v_i is seen as an uncertain parametric dependency, *i.e.* $\theta_{i,6} = v_i$, with constant bounds assumed for v_i using an overset of the longitudinal interval predictor.
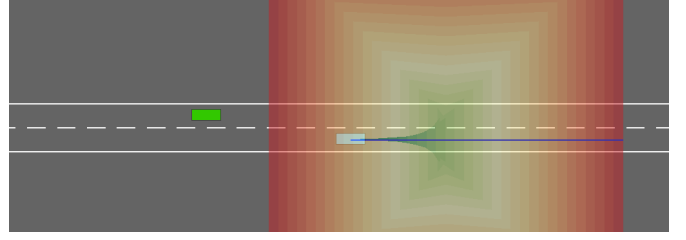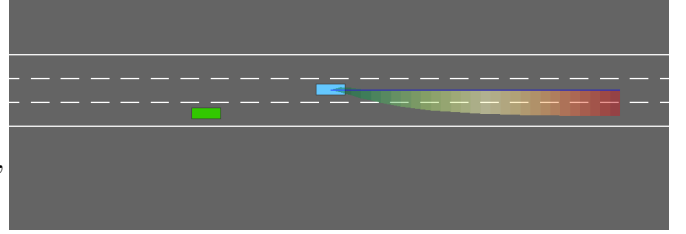

(a) The naive predictor (6) quickly diverges

(b) The proposed predictor (7) remains stable

(c) Prediction during a lane change maneuver

(d) Prediction with uncertainty in the followed lane L_i

Fig. 3. State intervals obtained by the two methods in different conditions.

D. Change of coordinates

In both cases, the obtained polytope centre A_0 is non-Metzler. We use lemmas 2 and 3 to compute a similarity transformation of coordinates. Precisely, we ensure that the polytope is chosen so that its centre A_0 is diagonalisable having real eigenvalues, and perform an eigendecomposition to compute its change of basis matrix S. The transformed system $Z' = S^{-1}(Z - Z_c)$ verifies Assumption 3 as required to apply the interval predictor of Theorem 1. Finally, the obtained predictor is transformed back to the original coordinates Z by using the interval arithmetic of Lemma 1.

E. Results

We show the resulting intervals in Fig. 3. The target vehicle with uncertain behaviour is in blue, while the ego-vehicle is in green. Its trajectory interval is computed over a duration of two seconds and represented by an area filled with a colour gradient representing time. The ground-truth trajectory is

shown in blue. In Fig. 3a, we observe that the direct predictor (6) is unstable and quickly diverges to cover the whole road, thus hindering any sensible decision-making. In [18], they had to circumvent this issue by subdividing Π and $[\underline{Z}, \overline{Z}]$ to reduce the initial overestimations and merely delay the divergence [30], at the price of a heavy computational load. In stark contrast, we see in Fig. 3b that the novel predictor (7) is stable even over long horizons, which allows the ego-vehicle to plan an overtaking maneuver. Until then, there was little uncertainty in the predicted trajectory for the target vehicle was isolated, but as the ego-vehicle cuts into its lane in Fig. 3c, we start seeing the effects of uncertain interactions between the two vehicles, in both longitudinal and lateral directions. Our framework is quite flexible in representing different assumptions on the admissible behaviours. For instance, we show in Fig. 3d a simulation in which we model a right-hand traffic where drivers are expected to keep to the rightmost lane. In such a situation, it is reasonable to assume that in the absence of any obstacle in front, a vehicle driving on the middle lane will either stay there or return to the right lane, but has no incentive to change to the left-lane. This simple assumption on L_i can easily be incorporated in the interval predictor, and enables the emergence of a realistic behaviour when running the robust decision-making procedure: the ego-vehicle cannot pass another vehicle by its right side, and can only overtake it by its left side. These behaviours displaying safe reasoning under uncertainty are showcased in the attached videos.

VI. Conclusion

The prediction problem for uncertain LPV systems is solved by designing an interval predictor, which is described by nonlinear differential equations, and whose stability is evaluated using a new Lyapunov function. The corresponding robust stability conditions are expressed in terms of LMIs. The proficiency of the method is demonstrated in application to a problem of safe motion planning for a self-driving car.

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