VNR VIGNANA JYOTHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

B.Tech. III Semester

(22PC1EC203) SIGNALS AND SYSTEMS

TEACHING SCHEME		
L	T/P	C
2	1	3

EVALUATION SCHEME				
SE	CA	ELA	SEE	TOTAL
30	5	5	60	100

COURSE PRE-REQUISITES: Ordinary Differential Equations and Vector Calculus

COURSE OBJECTIVES:
- To understand various fundamental characteristics of signals and systems
- To study the importance of transform domain
- To analyze and design various systems
- To study the operations of convolution, correlation and the effects of sampling
- To understand Laplace and Z-transforms properties for the analysis of signals and systems

COURSE OUTCOMES: After completion of the course, the student should be able to
CO-1: Classify signals and systems based on their characteristics
CO-2: Apply various transform techniques to analyze continuous time and discrete time signals
CO-3: Identify the conditions for transmission of signals through systems and conditions for physical realization of systems
CO-4: Apply convolution and correlation functions for various applications
CO-5: Analyze the sampling process and effects of various sampling rates

COURSE ARTICULATION MATRIX:
*(Correlation of Course Outcomes with Program Outcomes and Program Specific Outcomes using **mapping levels 1 = Slight, 2 = Moderate and 3 = Substantial**)*

CO	PROGRAM OUTCOMES (PO)												PROGRAM SPECIFIC OUTCOMES (PSO)		
	PO-1	PO-2	PO-3	PO-4	PO-5	PO-6	PO-7	PO-8	PO-9	PO-10	PO-11	PO-12	PSO-1	PSO-2	PSO-3
CO-1	3	2	-	-	-	-	-	-	-	-	-	-	3	-	-
CO-2	3	3	-	2	-	-	-	-	-	-	-	-	3	-	-
CO-3	3	2	-		-	-	-	-	-	-	-	-	3	-	-
CO-4	3	3	-	2	-	-	-	-	-	-	-	-	2	-	-
CO-5	3	3	-	2	-	-	-	-	-	-	-	-	2	-	-

UNIT-I

Representation of Signals: Continuous time and Discrete Time signals (41), Classification ofSignals – Periodic and aperiodic, even and odd, energy and power signals, deterministic and random signals, causal and non-causal signals (41-82), complex exponential (11-13) and sinusoidal signals (9-10). Concepts of standard signals (3-20). Various operations on Signals (20-41).

Signal Analysis: Analogy between vectors and signals (171-180), orthogonal signal space, Signal approximation using orthogonal functions (182-184), Closed or complete set of orthogonal functions (185-193).

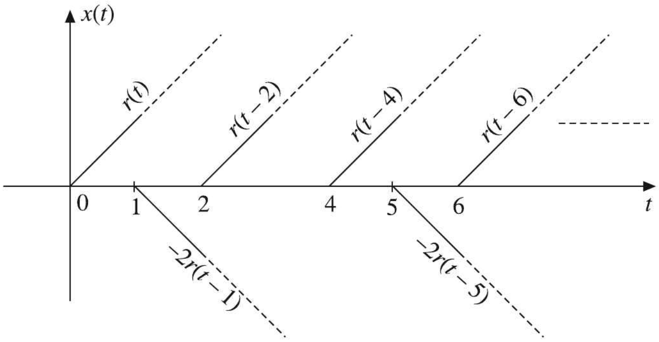


Figure 1.56 Example 1.8(f).

1.5 CLASSIFICATION OF SIGNALS

Based upon their nature and characteristics in the time domain, the signals may be broadly classified as under

(a) Continuous-time signals

(b) Discrete-time signals

Continuous-time signals

The signals that are defined for every instant of time are known as continuous-time signals. Continuous-time signals are also called analog signals. For continuous-time signals, the independent variable is time. They are denoted by $x(t)$. They are continuous in amplitude as well as in time. Most of the signals available are continuous-time signals. Figure 1.57(a) and (b) shows the graphical representation of continuous-time signals.

Discrete-time signals

The signals that are defined only at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude but discrete in time. For discrete-time signals, the amplitude between two time instants is just not defined. For discrete-time signals, the independent variable is time n. Since they are defined only at discrete instants of time, they are denoted by a sequence $x(nT)$ or simply by $x(n)$ where n is an integer.

The discrete-time signals may be inherently discrete or may be discrete versions of the continuous-time signals. Figure 1.57(c) and (d) show the graphical representation of discrete-time signals.

Both continuous-time and discrete-time signals may further be classified as under

1. Deterministic and random signals
2. Periodic and non-periodic signals
3. Energy and power signals
4. Causal and non-causal signals
5. Even and odd signals

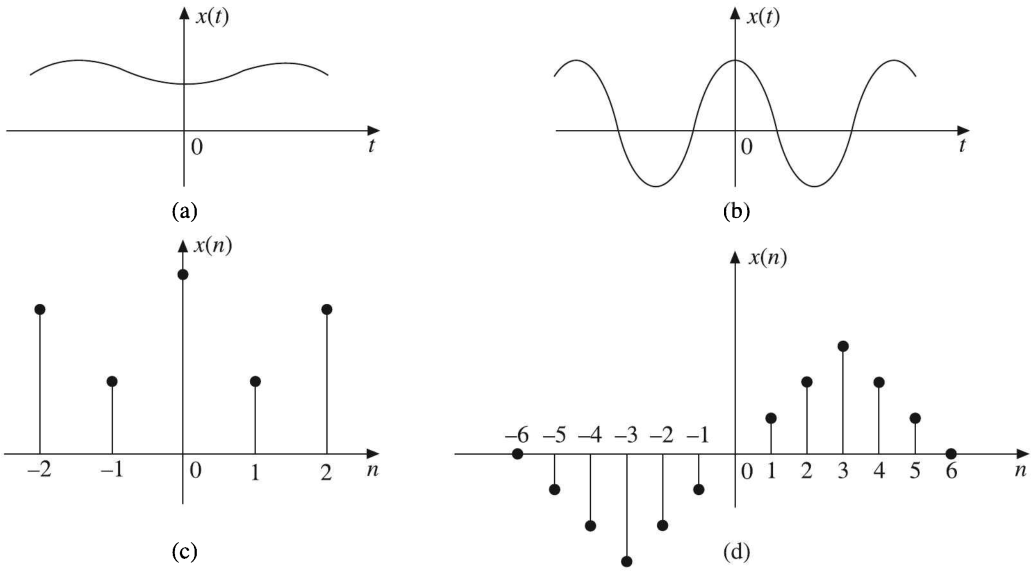


Figure 1.57 (a) and (b) Continuous-time signals, (c) and (d) Discrete-time signals.

1.5.1 Deterministic and Random Signals

A signal exhibiting no uncertainty of its magnitude and phase at any given instant of time is called deterministic signal. A deterministic signal has a regular pattern and can be completely represented by mathematical equation at any time. Its amplitude and phase at any time instant can be predicted in advance.

Examples: Sine wave, $x(t) = \cos \omega t$ or $x(n) = \cos \omega n$, Exponential signals, square wave, triangular wave, etc.

A signal characterized by uncertainty about its occurrence is called a random signal. A random signal cannot be represented by any mathematical equation. The pattern of such a signal is quite irregular. Its amplitude and phase at any time instant cannot be predicted in advance.

A typical example of non deterministic signals is thermal noise generated in an electric circuit. Such a noise has a probabilistic nature.

Figure 1.58 shows the graphical representation of deterministic and random signals.

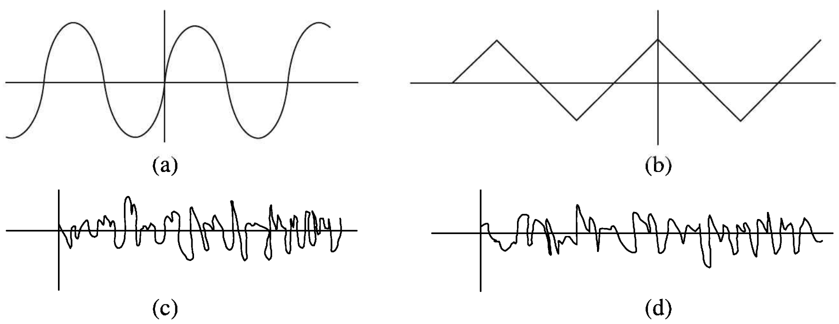


Figure 1.58 (a) and (b) Deterministic signals, (c) and (d) Random signals.

1.5.2 Periodic and Non-periodic Signals

A signal which has a definite pattern and which repeats itself at regular intervals of time is called a periodic signal, and a signal which does not repeat at regular intervals of time is called a non-periodic or aperiodic signal.

Mathematically, a continuous-time signal $x(t)$ is called periodic if and only if

$$x(t + T) = x(t) \text{ for all } t, \text{ i.e. for } -\infty < t < \infty$$

where t denotes time and T is a constant representing the period. The smallest value of T which satisfies the above condition is called the fundamental period or simply period of $x(t)$. A signal is aperiodic if the above condition is not satisfied even for one value of t. Sometimes aperiodic signals are said to have a period equal to infinity.

The reciprocal of fundamental period T is called the fundamental frequency f of $x(t)$,

i.e. $$f = \frac{1}{T}$$

The angular frequency is given by

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$\therefore$ Fundamental period $T = \dfrac{2\pi}{\omega}$

Similarly, a discrete-time signal $x(n)$ is said to be periodic if it satisfies the condition $x(n) = x(n + N)$ for all integers n.

The smallest value of N which satisfies the above condition is known as fundamental period.

If the above condition is not satisfied even for one value of n, then the discrete-time signal is aperiodic.

The angular frequency is given by

$$\omega = \frac{2\pi}{N}$$

$\therefore$ Fundamental period $N = \dfrac{2\pi}{\omega}$

1. The sum of two continuous-time periodic signals $x_1(t)$ and $x_2(t)$ with periods T_1 and T_2 may or may not be periodic depending on the relation between T_1 and T_2.
2. The sum of two periodic signals is periodic only if the ratio of their respective periods T_1/T_2 is a rational number or ratio of two integers.
3. The fundamental period is the least common multiple (LCM) of T_1 and T_2.
4. If the ratio T_1/T_2 is an irrational number, then the signals $x_1(t)$ and $x_2(t)$ do not have a common period and $x(t)$ cannot be periodic.
5. The sum of two discrete-time periodic sequences is always periodic.

Some examples of continuous-time and discrete-time periodic/non-periodic signals are shown in Figure 1.59.

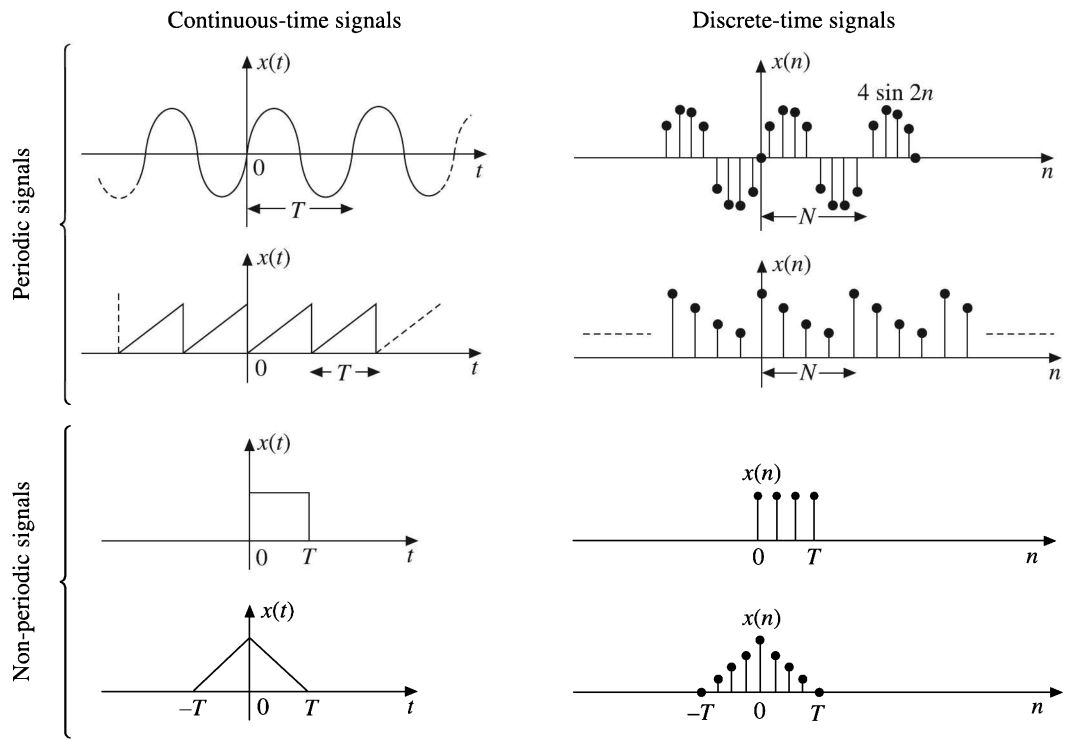


Figure 1.59 Examples of periodic and non-periodic signals.

EXAMPLE 1.9 Show that the complex exponential signal $x(t) = e^{j\omega_0 t}$ is periodic with period $2\pi/\omega_0$.

Solution: Given $$x(t) = e^{j\omega_0 t}$$

$x(t)$ will be periodic if $$x(t + T) = x(t)$$

i.e. if $$e^{j\omega_0(t+T)} = e^{j\omega_0 t}$$

i.e. if $$e^{j\omega_0 t}e^{j\omega_0 T} = e^{j\omega_0 t}$$

i.e. if $$e^{j\omega_0 T} = 1$$

This is valid if $$\omega_0 T = n(2\pi)$$
where n is an integer.

$\therefore$ $$T = \frac{2\pi n}{\omega_0}$$

Thus, the fundamental period, i.e. the smallest positive T of $x(t)$ is given by

$$T = \frac{2\pi}{\omega_0}$$

EXAMPLE 1.10 Show that the sinusoidal signal $x(t) = \sin(\omega_0 t + \theta)$ is periodic with period $2\pi/\omega_0$.

Solution: Given $$x(t) = \sin(\omega_0 t + \theta)$$

$x(t)$ will be periodic if $$x(t) = x(t + T)$$

But $$x(t+T) = \sin[\omega_0(t+T) + \theta] = \sin[\omega_0 t + \theta + \omega_0 T]$$

For $\sin(\omega_0 t + \theta)$ to be equal to $\sin(\omega_0 t + \theta + \omega_0 T)$, $\omega_0 T$ must be equal to $2n\pi$, where n is a positive integer.

Therefore, $$T = \frac{2\pi n}{\omega_0}$$

This shows that the sinusoidal signal $x(t) = \sin(\omega_0 t + \theta)$ is periodic with fundamental period $T = (2\pi/\omega_0)$.

EXAMPLE 1.11 Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2 respectively. Under what conditions is the sum $x(t) = x_1(t) + x_2(t)$ periodic and what is the fundamental period of $x(t)$ if it is periodic?

Or

Show that a composite signal is periodic if the ratio of their fundamental periods is a rational number.

Solution: Given signals $x_1(t)$ and $x_2(t)$ are periodic with fundamental periods T_1 and T_2 respectively. So

$$x_1(t) = x_1(t + T_1) = x_1(t + mT_1), \ m \text{ is a positive integer}$$

$$x_2(t) = x_2(t + T_2) = x_2(t + kT_2), \ k \text{ is a positive integer}$$

Given signal is $$x(t) = x_1(t) + x_2(t)$$

$\therefore$ $$x(t) = x_1(t + mT_1) + x_2(t + kT_2)$$

For $x(t)$ to be periodic with period T, we require

$$x(t) = x(t+T) = x_1(t+T) + x_2(t+T)$$
$$= x_1(t + mT_1) + x_2(t + kT_2)$$

$\therefore$ $$mT_1 = kT_2 = T$$

$\therefore$ $$\frac{T_1}{T_2} = \frac{k}{m} = \text{rational number}$$

Then the fundamental period T of $x(t)$ is the LCM of T_1 and T_2. Therefore,

$$T = mT_1 = kT_2$$

So we can say that, the sum of two periodic signals is periodic only if the ratio of their respective periods is a rational number. Then the fundamental period is the LCM of the respective fundamental periods.

Note: The same condition holds for discrete-time signals also.

EXAMPLE 1.12 Examine whether the following signals are periodic or not? If periodic determine the fundamental period.

(a) $\sin 12\pi t$
(b) $e^{j4\pi t}$
(c) $\sin \pi t\, u(t)$
(d) $e^{-|t|}$
(e) $\cos 2t + \sin \sqrt{3}t$
(f) $3 \sin 200\pi t + 4 \cos 100t$
(g) $\sin 10\pi t + \cos 20\pi t$
(h) $\sin(10t + 1) - 2\cos(5t - 2)$
(i) je^{j6t}
(j) $3u(t) + 2 \sin 2t$
(k) $6e^{j[4t+(\pi/3)]} + 8e^{j[3\pi t+(\pi/4)]}$
(l) $u(t) - 2u(t - 5)$
(m) $2 + \cos 2\pi t$

Solution:

(a) Given $\qquad x(t) = \sin 12\pi t$

Comparing it with $\sin \omega t$, we have

$$\omega = 12\pi \quad \text{or} \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{12\pi} = \frac{1}{6}$$

Since T is a ratio of two integers, $x(t)$ is periodic with fundamental period $T = 1/6$ sec.

(b) Given $\qquad x(t) = e^{j4\pi t}$

Comparing it with $e^{j\omega t}$, we have

$$\omega = 4\pi \quad \text{or} \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

Since T is a ratio of two integers, $x(t)$ is periodic with fundamental period $T = 1/2$ sec.

(c) Given $\qquad x(t) = \sin \pi t\, u(t)$

$\sin \pi t$ is periodic with period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\text{ sec}$. $u(t)$ exists only between $t = 0$ to $t = \infty$. Hence it is not periodic.

Therefore, $\sin \pi t\, u(t)$ which is the product of a periodic and a non-periodic signal is not periodic.

(d) Given $\qquad x(t) = e^{-|t|}$

The plot of $x(t) = e^{-|t|}$ versus t is shown in Figure 1.60. It does not repeat at all. So it is aperiodic.

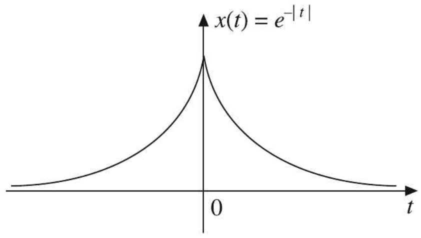


Figure 1.60 Plot of $x(t) = e^{-|t|}$.

(e) Given $$x(t) = \cos 2t + \sin \sqrt{3}t$$

Let $$x(t) = x_1(t) + x_2(t)$$

where $$x_1(t) = \cos 2t \quad \text{and} \quad x_2(t) = \sin \sqrt{3}t$$

Comparing $x_1(t) = \cos 2t$ with $\cos \omega_1 t$, we have

$$\omega_1 = 2, \text{ i.e. } 2\pi f_1 = 2 \quad \text{or} \quad f_1 = \frac{1}{\pi}$$

$\therefore$ Period of $x_1(t)$ is $$T_1 = \frac{1}{f_1} = \pi$$

Comparing $x_2(t) = \sin \sqrt{3}t$ with $\sin \omega_2 t$, we have

$$\omega_2 = \sqrt{3}, \text{ i.e. } 2\pi f_2 = \sqrt{3} \quad \text{or} \quad f_2 = \frac{\sqrt{3}}{2\pi}$$

$\therefore$ Period of $x_2(t)$ is $$T_2 = \frac{1}{f_2} = \frac{2\pi}{\sqrt{3}}$$

The ratio of two periods $= \dfrac{T_1}{T_2} = \dfrac{\pi}{2\pi/\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

Since T_1/T_2 is not a ratio of two integers (i.e. not a rational number), the given signal $x(t)$ is non-periodic.

(f) Given $$x(t) = 3 \sin 200\pi t + 4 \cos 100t$$

Let $$x(t) = x_1(t) + x_2(t)$$

where $$x_1(t) = 3 \sin 200\pi t \quad \text{and} \quad x_2(t) = 4 \cos 100t$$

Comparing $x_1(t) = 3 \sin 200\pi t$ with $A \sin \omega_1 t$, we have

$$\omega_1 = 200\pi, \text{ i.e. } 2\pi f_1 = 200\pi \quad \text{or} \quad f_1 = 100$$

$\therefore$ Period of $x_1(t)$ is $$T_1 = \frac{1}{f_1} = \frac{1}{100}$$

Comparing $x_2(t) = 4 \cos 100t$ with $B \cos \omega_2 t$, we have

$$\omega_2 = 100, \text{ i.e. } 2\pi f_2 = 100 \quad \text{or} \quad f_2 = \frac{50}{\pi}$$

$\therefore$ Period of $x_2(t)$ is $\quad T_2 = \dfrac{1}{f_2} = \dfrac{\pi}{50}$

The ratio of two periods $= \dfrac{T_1}{T_2} = \dfrac{1/100}{\pi/50} = \dfrac{1}{2\pi}$

Since T_1/T_2 is not a ratio of two integers (i.e. not a rational number), the given signal $x(t)$ is non-periodic.

(g) Given $\quad x(t) = \sin 10\pi t + \cos 20\pi t$

Let $\quad x(t) = x_1(t) + x_2(t)$

where $\quad x_1(t) = \sin 10\pi t \quad$ and $\quad x_2(t) = \cos 20\pi t$

Comparing $x_1(t) = \sin 10\pi t$ with $\sin \omega_1 t$, we have

$$\omega_1 = 10\pi, \text{ i.e. } 2\pi f_1 = 10\pi \quad \text{or} \quad f_1 = 5$$

$\therefore$ Period of $x_1(t)$ is $\quad T_1 = \dfrac{1}{f_1} = \dfrac{1}{5}$

Comparing $x_2(t) = \cos 20\pi t$ with $\cos \omega_2 t$, we have

$$\omega_2 = 20\pi, \text{ i.e. } 2\pi f_2 = 20\pi \quad \text{or} \quad f_2 = 10$$

$\therefore$ Period of $x_2(t)$ is $\quad T_2 = \dfrac{1}{f_2} = \dfrac{1}{10}$

The ratio of two periods $= \dfrac{T_1}{T_2} = \dfrac{1/5}{1/10} = 2$

$\therefore \quad T_1 = 2T_2$

Since T_1/T_2 is a rational number (ratio of two integers 2 and 1), the given signal $x(t)$ is periodic.

The fundamental period $\quad T = T_1 = 2T_2 = \dfrac{1}{5}$ sec

(h) Given $\quad x(t) = \sin(10t + 1) - 2\cos(5t - 2)$

Let $\quad x(t) = x_1(t) + x_2(t)$

where $\quad x_1(t) = \sin(10t + 1) \quad$ and $\quad x_2(t) = 2\cos(5t - 2)$

Comparing $x_1(t) = \sin(10t + 1)$ with $\sin(\omega_1 t + \theta_1)$, we have

$$\omega_1 = 10, \text{ i.e. } 2\pi f_1 = 10 \quad \text{or} \quad f_1 = \frac{10}{2\pi} = \frac{5}{\pi}$$

The period of $x_1(t)$ is:

$$T_1 = \frac{1}{f_1} = \frac{\pi}{5} \text{ sec}$$

Comparing $x_2(t) = 2\cos(5t - 2)$ with $A\cos(\omega_2 t + \theta_2)$, we have

$$\omega_2 = 5, \text{ i.e. } 2\pi f_2 = 5 \quad \text{or} \quad f_2 = \frac{5}{2\pi}$$

The period of $x_2(t)$ is:

$$T_2 = \frac{1}{f_2} = \frac{2\pi}{5} \text{ sec}$$

The ratio of two periods $= \dfrac{T_1}{T_2} = \dfrac{\pi/5}{2\pi/5} = \dfrac{1}{2}$

$\therefore \qquad 2T_1 = T_2$

Since T_1/T_2 is a ratio of two integers (i.e. a rational number), the given signal $x(t)$ is periodic.

$$\text{Fundamental period } T = 2T_1 = T_2 = \frac{2\pi}{5} \text{ sec}$$

(i) Given $\qquad x(t) = je^{j6t}$

Comparing this with the standard form $x(t) = e^{j\omega t}$, we have

$$\omega = 6, \text{ i.e. } 2\pi f = 6 \quad \text{or} \quad f = \frac{3}{\pi}$$

$\therefore$ Time period $T = \dfrac{1}{f} = \dfrac{\pi}{3}$

Since T is not a rational number (i.e. not a ratio of integers), $x(t)$ is not periodic.

(j) Given $x(t) = 3u(t) + 2\sin 2t$ (Figure 1.61).

Period of $3u(t)$ is zero, i.e. it is aperiodic. Period of 2 sin $2t$ is $T = (2\pi/2) = \pi$ sec. 2 sin $2t$ is periodic. Therefore, $x(t)$, the sum of an aperiodic signal and a periodic signal, is aperiodic.

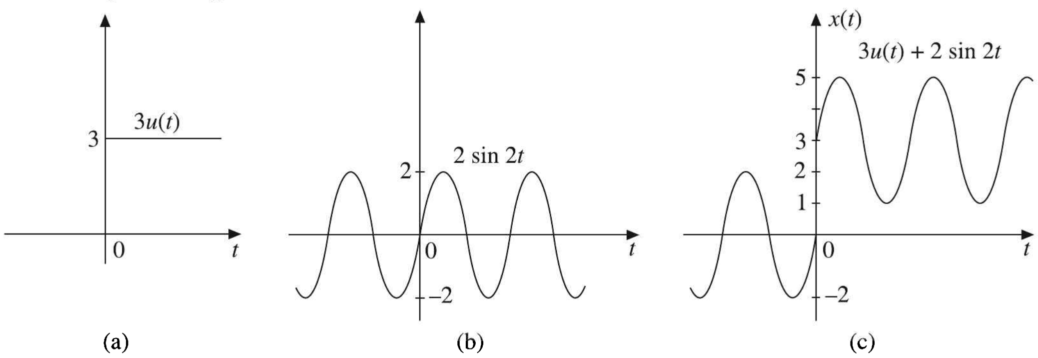


Figure 1.61 Signal for Example 1.12(j).

(k) Given $$x(t) = 6e^{j[4t+(\pi/3)]} + 8e^{j[3\pi t+(\pi/4)]}$$

Let $$x(t) = x_1(t) + x_2(t)$$

where $x_1(t) = 6e^{j[4t+(\pi/3)]}$ and $x_2(t) = 8e^{j[3\pi t+(\pi/4)]}$

Comparing $x_1(t) = 6e^{j[4t+(\pi/3)]}$ with $Ae^{j(\omega_1 t+\theta_1)}$, we have

$$\omega_1 = 4, \text{ i.e. } 2\pi f_1 = 4 \quad \text{or} \quad f_1 = \frac{2}{\pi}$$

$\therefore$ Time period of $6e^{j[4t+(\pi/3)]}$ is $T_1 = \dfrac{1}{f_1} = \dfrac{\pi}{2}$.

Comparing $x_2(t) = 8e^{j[3\pi t+(\pi/4)]}$ with $Be^{j(\omega_2 t+\theta_2)}$, we have

$$\omega_2 = 3\pi; \text{ i.e. } 2\pi f_2 = 3\pi \quad \text{or} \quad f_2 = \frac{3}{2}$$

$\therefore$ Time period of $8e^{j[3\pi t+(\pi/4)]}$ is $T_2 = \dfrac{1}{f_2} = \dfrac{2}{3}$.

The ratio of two periods $= \dfrac{T_1}{T_2} = \dfrac{\pi/2}{2/3} = \dfrac{3\pi}{4}$.

Since the ratio T_1/T_2 is not rational, the given signal $x(t)$ is not periodic.

(l) Given $$x(t) = u(t) - 2u(t-5)$$

Period of $u(t)$ as well as $u(t - 5)$ is zero. So both $u(t)$ and $u(t - 5)$ are aperiodic. Therefore, $x(t)$, the sum of two aperiodic signals, is aperiodic as shown in Figure 1.62.

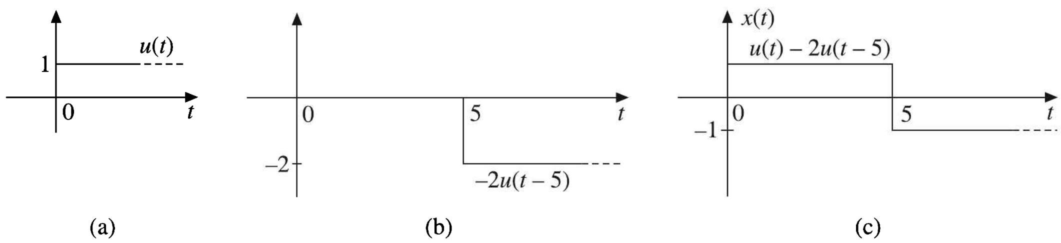


Figure 1.62 Signal for Example 1.12(l).

(m) Given $$x(t) = 2 + \cos 2\pi t$$

2 is a dc signal extending from $-\infty$ to ∞. The time period of $\cos 2\pi t$ is $T = (2\pi/2\pi) = 1$. Since it is a rational number, $\cos 2\pi t$ is periodic. The signal $x(t) = 2 + \cos 2\pi t$ is nothing but $\cos 2\pi t$ shifted upwards by 2 as shown in Figure 1.63. So $x(t)$ is also periodic with a fundamental period $T = 1$ sec.

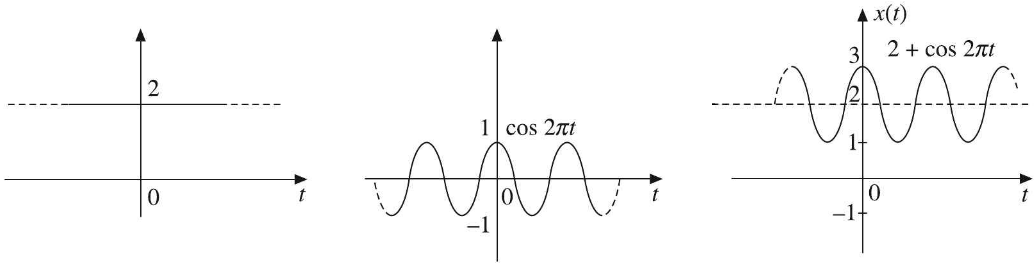


Figure 1.63 Signal for Example 1.12(m).

EXAMPLE 1.13 Show that the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic only if $\omega_0/2\pi$ is a rational number.

Solution: Given $\qquad x(n) = e^{j\omega_0 n}$

$x(n)$ will be periodic if $\qquad x(n + N) = x(n)$

i.e. $\qquad e^{j[\omega_0(n+N)]} = e^{j\omega_0 n}$

i.e. $\qquad e^{j\omega_0 N} e^{j\omega_0 n} = e^{j\omega_0 n}$

This is possible only if $\qquad e^{j\omega_0 N} = 1$

This is true only if $\qquad \omega_0 N = 2\pi k$

where k is a positive integer.

$$\therefore \qquad \frac{\omega_0}{2\pi} = \frac{k}{N} = \text{Rational number}$$

This shows that the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number.

EXAMPLE 1.14 Let $x(t)$ be the complex exponential signal, $x(t) = e^{j\omega_0 t}$ with radian frequency ω_0 and fundamental period $T = 2\pi/\omega_0$. Consider the discrete-time sequence $x(n)$ obtained by the uniform sampling of $x(t)$ with sampling interval T_s, i.e.

$$x(n) = x(nT_s) = e^{jn\omega_0 T_s}$$

Show that $x(n)$ is periodic if the ratio of the sampling interval T_s to the fundamental period T of $x(t)$, i.e. T_s/T is a rational number.

Solution: Given $\qquad x(t) = e^{j\omega_0 t}$

$$x(n) = x(nT_s) = e^{jn\omega_0 T_s}$$

T_s is the sampling interval.

Now, fundamental period $T = \dfrac{2\pi}{\omega_0}$

$$\therefore \qquad \omega_0 = \frac{2\pi}{T}$$

If $x(n)$ is periodic with the fundamental period N, then

$$x(n+N) = x(n)$$

i.e. $$e^{j(n+N)\omega_0 T_s} = e^{jn\omega_0 T_s}$$

i.e. $$e^{jn\omega_0 T_s} e^{jN\omega_0 T_s} = e^{jn\omega_0 T_s}$$

This is true only if $$e^{jN\omega_0 T_s} = 1$$

i.e. $$N\omega_0 T_s = 2\pi m$$

where m is a positive integer.

i.e. $$N\frac{2\pi}{T}T_s = 2\pi m$$

$$\therefore \qquad \frac{T_s}{T} = \frac{m}{N} = \text{Rational number}$$

This shows that $x(n)$ is periodic if the ratio of the sampling interval to the fundamental period of $x(t)$, T_s/T is a rational number.

EXAMPLE 1.15 Obtain the condition for discrete-time sinusoidal signal to be periodic.

Solution: In case of continuous-time signals, all sinusoidal signals are periodic. But in discrete-time case, not all sinusoidal sequences are periodic.
Consider a discrete-time signal given by

$$x(n) = A\sin(\omega_0 n + \theta)$$

where A is amplitude, ω_0 is frequency and θ is phase shift.

A discrete-time signal is periodic if and only if

$$x(n) = x(n+N) \text{ for all } n$$

Now, $$x(n+N) = A\sin[\omega_0(n+N)+\theta)] = A\sin(\omega_0 n + \theta + \omega_0 N)$$

Therefore, $x(n)$ and $x(n+N)$ are equal if $\omega_0 N = 2\pi m$. That is, there must be an integer m such that

$$\omega_0 = \frac{2\pi m}{N} = 2\pi\left[\frac{m}{N}\right]$$

or $$N = 2\pi\left[\frac{m}{\omega_0}\right]$$

From the above equation we find that, for the discrete-time signal to be periodic, the fundamental frequency ω_0 must be a rational multiple of 2π. Otherwise the discrete-time signal is aperiodic. The smallest value of positive integer N, for some integer m, is the fundamental period.

EXAMPLE 1.16 Determine whether the following discrete-time signals are periodic or not? If periodic, determine the fundamental period.

(a) $\sin(0.02\pi n)$ (b) $\sin(5\pi n)$

(c) $\cos 4n$ (d) $\sin\dfrac{2\pi n}{3}+\cos\dfrac{2\pi n}{5}$

(e) $\cos\left(\dfrac{n}{6}\right)\cos\left(\dfrac{n\pi}{6}\right)$ (f) $\cos\left(\dfrac{\pi}{2}+0.3n\right)$

(g) $e^{j(\pi/2)n}$ (h) $1+e^{j2\pi n/3}-e^{j4\pi n/7}$

Solution:

(a) Given $x(n)=\sin(0.02\pi n)$

Comparing it with $x(n)=\sin(2\pi fn)$,

we have $0.02\pi = 2\pi f$ or $f=\dfrac{0.02\pi}{2\pi}=0.01=\dfrac{1}{100}=\dfrac{k}{N}$

Here f is expressed as a ratio of two integers with k = 1 and N = 100. So it is rational. Hence the given signal is periodic with fundamental period N = 100.

(b) Given $x(n)=\sin(5\pi n)$

Comparing it with $x(n)=\sin(2\pi fn)$,

we have $2\pi f = 5\pi$ or $f=\dfrac{5}{2}=\dfrac{k}{N}$

Here f is a ratio of two integers with k = 5 and N = 2. Hence it is rational. Therefore, the given signal is periodic with fundamental period N = 2.

(c) Given $x(n) = \cos 4n$

Comparing it with $x(n) = \cos 2\pi fn$,

we have $2\pi f = 4$ or $f=\dfrac{2}{\pi}$

Since $f = (2/\pi)$ is not a rational number, $x(n)$ is not periodic.

(d) Given $x(n)=\sin\dfrac{2\pi n}{3}+\cos\dfrac{2\pi n}{5}$

Comparing it with $x(n)=\sin 2\pi f_1 n+\cos 2\pi f_2 n$

we have $2\pi f_1=\dfrac{2\pi}{3}$ or $f_1=\dfrac{1}{3}=\dfrac{k_1}{N_1}$

$\therefore$ $N_1 = 3$

and $2\pi f_2=\dfrac{2\pi}{5}$ or $f_2=\dfrac{1}{5}$

$\therefore$ $N_2 = 5$

Since $\frac{N_1}{N_2}=\frac{3}{5}$ is a ratio of two integers, the sequence $x(n)$ is periodic. The period of $x(n)$ is the LCM of N_1 and N_2. Here LCM of $N_1 = 3$ and $N_2 = 5$ is 15. Therefore, the given sequence is periodic with fundamental period $N = 15$.

(e) Given $$x(n)=\cos\left(\frac{n}{6}\right)\cos\left(\frac{n\pi}{6}\right)$$

Comparing it with $x(n)=\cos(2\pi f_1 n)\cos(2\pi f_2 n)$,

we have $$2\pi f_1 n=\frac{n}{6} \quad \text{or} \quad f_1=\frac{1}{12\pi}$$

which is not rational

and $$2\pi f_2 n=\frac{n\pi}{6} \quad \text{or} \quad f_2=\frac{1}{12}$$

which is rational

Thus, $\cos(n/6)$ is non-periodic and $\cos(n\pi/6)$ is periodic. $x(n)$ is non-periodic because it is the product of periodic and non-periodic signals.

(f) Given $$x(n)=\cos\left(\frac{\pi}{2}+0.3n\right)$$

Comparing it with $x(n)=\cos(2\pi fn+\theta)$,

we have $2\pi fn = 0.3n$ and phase shift $\theta=\frac{\pi}{2}$

$$\therefore \quad f=\frac{0.3}{2\pi}=\frac{3}{20\pi}$$

which is not rational.

Hence, the signal $x(n)$ is non-periodic.

(g) Given $$x(n)=e^{j(\pi/2)n}$$

Comparing it with $x(n)=e^{j2\pi fn}$,

we have $$2\pi f=\frac{\pi}{2} \quad \text{or} \quad f=\frac{1}{4}=\frac{k}{N}$$

which is rational.

Hence, the given signal $x(n)$ is periodic with fundamental period $N = 4$.

(h) Given $$x(n)=1+e^{j2\pi n/3}-e^{j4\pi n/7}$$

Let $$x(n)=1+e^{j2\pi n/3}-e^{j4\pi n/7}=x_1(n)+x_2(n)+x_3(n)$$

where $x_1(n)=1,\ x_2(n)=e^{j2\pi n/3}$ and $x_3(n)=e^{j4\pi n/7}$

$x_1(n) = 1$ is a dc signal with an arbitrary period $N_1 = 1$

$$x_2(n) = e^{j2\pi n/3} = e^{j2\pi f_2 n}$$

$$\therefore \qquad \frac{2\pi n}{3} = 2\pi f_2 n \quad \text{or} \quad f_2 = \frac{1}{3} = \frac{k_2}{N_2} \quad \text{where } N_2 = 3.$$

Hence $x_2(n)$ is periodic with period $N_2 = 3$.

$$x_3(n) = e^{j4\pi n/7} = e^{j2\pi f_3 n}$$

$$\therefore \qquad \frac{4\pi n}{7} = 2\pi f_3 n \quad \text{or} \quad f_3 = \frac{2}{7} = \frac{k_3}{N_3} \quad \text{where } N_3 = \frac{7}{2}$$

Now,
$$\frac{N_1}{N_2} = \frac{1}{3} = \text{Rational number}$$

$$\frac{N_1}{N_3} = \frac{1}{7/2} = \frac{2}{7} = \text{Rational number}$$

The LCM of
$$N_1, N_2, N_3 = \frac{7}{2} \times 3 = \frac{21}{2}$$

The given signal $x(n)$ is periodic with fundamental period $N = 10.5$.

1.5.3 Energy and Power Signals

In electrical systems, signals may represent voltage or current. Consider a voltage signal $v(t)$ across a resistance R producing a current $i(t)$ as shown in Figure 1.64.

The instantaneous power developed in R is given by

$$p(t) = v(t)\, i(t)$$

$$= v(t)\frac{v(t)}{R} = \frac{v^2(t)}{R}$$

$$= i(t)\, Ri(t) = i^2(t)R$$

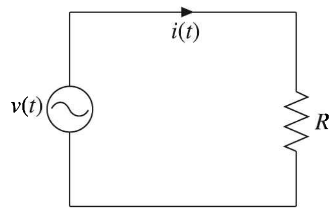


Figure 1.64 Simple resistive circuit.

When $R = 1$ ohm, the power dissipated is called *normalized power.*

So Normalized power $p(t) = v^2(t)$ or $i^2(t)$

If $v(t)$ or $i(t)$ is denoted by a signal $x(t)$, then the instantaneous power is equal to the square of the amplitude of the signal.

i.e. $p(t) = x^2(t)$

Thus, for current as well as voltage, the equation for normalized power is same.
The total energy or normalized energy of a continuous-time signal $x(t)$ is given by

$$E = \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} x^2(t)\, dt \text{ joules}$$

The average power or normalized average power of a continuous-time signal $x(t)$ is given by

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt \text{ watts}$$

The square root of P is known as rms value of the signal $x(t)$.

In the case of discrete-time signal $x(n)$, the integrals are replaced by summation. So the total energy E of a discrete-time signal $x(n)$ is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

and the average power P of a discrete-time signal $x(n)$ is defined as:

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

Signals may also be classified as energy signals and power signals. However there are some signals which can neither be classified as energy signals nor power signals.

- A signal is said to be an energy signal if and only if its total energy E is finite (i.e. $0 < E < \infty$). For an energy signal, average power $P = 0$. Non-periodic signals are examples of energy signals.
- A signal is said to be a power signal if its average power P is finite (i.e. $0 < P < \infty$). For a power signal, total energy $E = \infty$. Periodic signals are examples of power signals.
- Both energy and power signals are mutually exclusive, i.e. no signal can be both energy signal and power signal.
- The signals that do not satisfy the above properties are neither energy signals nor power signals.

EXAMPLE 1.17 Determine the power and rms value of the signal $x(t) = A \sin(\omega_0 t + \theta)$

Solution: Given $x(t) = A\sin(\omega_0 t + \theta)$

Then Average power

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |A \sin(\omega_0 t + \theta)|^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{A^2}{2T} \int_{-T}^{T} \left[\frac{1 - \cos(2\omega_0 t + 2\theta)}{2}\right] dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{A^2}{4T} \int_{-T}^{T} dt - \underset{T\to\infty}{\text{Lt}} \frac{A^2}{4T} \int_{-T}^{T} \cos(2\omega_0 t + 2\theta)\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{A^2}{4T} \int_{-T}^{T} dt - 0$$

[Integration of cosine function over one full cycle is always zero.]

$$= \underset{T\to\infty}{\text{Lt}} \frac{A^2}{4T}[T+T]$$

$$= \frac{A^2}{2}$$

$$\text{rms value} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}}$$

So we can conclude that the power of a sinusoidal signal with amplitude A is equal to $A^2/2$ and rms value is equal to $A/\sqrt{2}$.

$$\text{Normalized energy } E = \int_{-\infty}^{\infty} |A \sin(\omega_0 t + \theta)|^2\, dt$$

$$= \int_{-\infty}^{\infty} \frac{A^2[1 - \cos 2(\omega_0 t + \theta)]\, dt}{2}$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty} dt - \frac{A^2}{2}\int_{-\infty}^{\infty} [\cos(2\omega_0 t + 2\theta)]\, dt = \frac{A^2}{2}[t]_{-\infty}^{\infty} - 0$$

$$= \infty$$

$\therefore$ Energy $E = \infty$

EXAMPLE 1.18 Prove the following:

(a) The power of the energy signal is zero over infinite time.
(b) The energy of the power signal is infinite over infinite time.

Solution:

(a) **Power of the energy signal**

Let $x(t)$ be an energy signal, i.e. $E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$ is finite

$$\text{Power of the signal } P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\left[\underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} |x(t)|^2\, dt\right] = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\left[\int_{-\infty}^{\infty} |x(t)|^2\, dt\right]$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}[E] \qquad \left[\text{since } E = \int_{-\infty}^{\infty} |x(t)|^2\, dt\right]$$

$$= 0 \times E = 0 \qquad \left[\text{since } \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} = 0\right]$$

Thus, the power of the energy signal is zero over infinite time.

(b) **Energy of the power signal**

Let $x(t)$ be a power signal, i.e. $P = \underset{T\to\infty}{\text{Lt}} \dfrac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt$ is finite.

Energy of the signal $\quad E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$

Let us change the limits of integration as $-T$ to T and take limit $T \to \infty$. This will not change the meaning of above equation for E.

i.e. $$E = \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} |x(t)|^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \left[2T \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt\right] = \underset{T\to\infty}{\text{Lt}}\ 2T\left[\underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt\right]$$

$$= \underset{T\to\infty}{\text{Lt}}\ 2TP \qquad \left[\text{since } P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt\right]$$

$$= \infty$$

Thus, the energy of the power signal is infinite over infinite time.

Important tips to determine whether the given signal is an energy signal or power signal

- Observe the signal carefully. If it is periodic and of infinite duration, then it can be a power signal. Hence calculate its power directly (Its $E = \infty$).
- If the signal is periodic only over a finite duration or not periodic at all, then it can be energy signal. Hence calculate its energy (Its $P = 0$).

EXAMPLE 1.19 Determine the power and rms value of the following signals:

(a) $x(t) = 7\cos\left(20t + \dfrac{\pi}{2}\right)$

(b) $x(t) = 12\cos\left(20t + \dfrac{\pi}{3}\right) + 16\sin\left(30t + \dfrac{\pi}{2}\right)$

(c) $x(t) = 8\cos 4t \cos 6t$

(d) $x(t) = e^{j2t}\cos 10t$

(e) $x(t) = Ae^{j5t}$

Solution:

(a) Given $\quad x(t) = 7\cos\left(20t + \dfrac{\pi}{2}\right)$

It is of the form $\quad A\cos(\omega_0 t + \theta)$

Then $\quad$ Power of the signal $P = \dfrac{7^2}{2} = 24.5 \text{ W}$

The rms value of the signal is $\sqrt{24.5}$.

(b) Given $\quad x(t) = 12 \cos\left(20t + \dfrac{\pi}{3}\right) + 16 \sin\left(30t + \dfrac{\pi}{2}\right)$

Then $\quad$ Power of the signal $P = \dfrac{12^2}{2} + \dfrac{16^2}{2} = 200 \text{ W}$

and $\quad$ rms value of the signal $= \sqrt{200}$

(c) Given

$$x(t) = 8 \cos 4t \cos 6t$$

$$= 8 \frac{[\cos 10t + \cos 2t]}{2}$$

$$= 4 \cos 10t + 4 \cos 2t$$

Then $\quad$ Power of the signal $P = \dfrac{4^2}{2} + \dfrac{4^2}{2} = 16 \text{ W}$

The rms value of the signal is $\sqrt{16} = 4$.

(d) Given

$$x(t) = e^{j2t} \cos 10t$$

$$= (\cos 2t + j \sin 2t) \cos 10t$$

$$= \cos 2t \cos 10t + j \sin 2t \cos 10t$$

$$= \frac{(\cos 12t + \cos 8t)}{2} + \frac{j(\sin 12t - \sin 8t)}{2}$$

Then $\quad$ Power of the signal $= \dfrac{\left(\frac{1}{2}\right)^2}{2} + \dfrac{\left(\frac{1}{2}\right)^2}{2} + \dfrac{\left(\frac{1}{2}\right)^2}{2} + \dfrac{\left(\frac{1}{2}\right)^2}{2} = \dfrac{1}{2}$

and $\quad$ rms value $= \sqrt{\dfrac{1}{2}}$

(e) Given

$$x(t) = Ae^{j5t}$$

$$= A \cos 5t + jA \sin 5t$$

Then $\quad$ Power of the signal $= \dfrac{A^2}{2} + \dfrac{A^2}{2} = A^2$

and $\quad$ rms value of the signal $= \sqrt{A^2} = A$

EXAMPLE 1.20 Determine whether the following signals are energy signals or power signals and calculate their energy or power:

(a) $x(t) = \sin^2 \omega_0 t$ (b) $x(t) = \text{rect}\,(t/\tau)$

(c) $x(t) = \text{rect}\,(t/\tau) \sin \omega_0 t$ (d) $x(t) = Ae^{-at}u(t),\ a > 0$

(e) $x(t) = u(t)$ (f) $x(t) = tu(t)$

(g) $x(t) = e^{j[3t+(\pi/2)]}$

Solution:

(a) Given $\quad x(t) = \sin^2 \omega_0 t$

This is a squared sine wave. Hence it is periodic signal. So it can be a power signal and calculate the power directly.

The normalized average power is:

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

Here $[\sin^2 \omega_0 t]^2 = \sin^4 \omega_0 t$ has some period T and it is real. Hence, the above equation will be

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} \sin^4 \omega_0 t\, dt$$

It can be expanded by standard trigonometric relations as follows

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} \frac{1}{8}[3 - 4\cos 2\omega_0 t + \cos 4\omega_0 t]\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} \frac{3}{8} dt - \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} \frac{4}{8} \cos 2\omega_0 t\, dt + \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} \frac{1}{8} \cos 4\omega_0 t\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \frac{3}{8} [t]_{-T}^{T} - 0 + 0 = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \frac{3}{8} [2T] = \frac{3}{8}$$

The power of the signal is finite and non-zero.

Now, Normalized energy of signal $E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$

$$= \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} (\sin^2 \omega_0 t)^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} \frac{1}{8}[3 - 4\cos 2\omega_0 t + \cos 4\omega_0 t]\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{3}{8} [t]_{-T}^{T} = \underset{T\to\infty}{\text{Lt}} \frac{3}{8} 2T = \infty$$

Hence it is a power signal with $P = 3/8$ watts.

(b) Given $$x(t) = \text{rect}\left(\frac{t}{\tau}\right)$$

The rect (t/τ) function is given as:

$$\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } -\dfrac{\tau}{2} < t < \dfrac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

As shown in Figure 1.65, it is a non-periodic function.

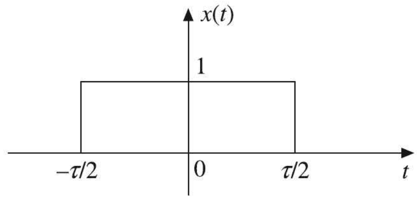


Figure 1.65 rect (t/τ).

So it can be an energy signal and calculate the energy directly.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\tau/2}^{\tau/2} (1)^2 \, dt$$

$$= \frac{\tau}{2} + \frac{\tau}{2} = \tau$$

The energy of the signal is finite and non zero.

Now, Normalized power $$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-\tau/2}^{\tau/2} (1)^2 \, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} [\tau]$$

$$= 0$$

Hence it is an energy signal with $E = \tau$ joules.

(c) Given signal is $x(t)$ = rect (t/τ) sin $\omega_0 t$ and is shown in Figure 1.66.

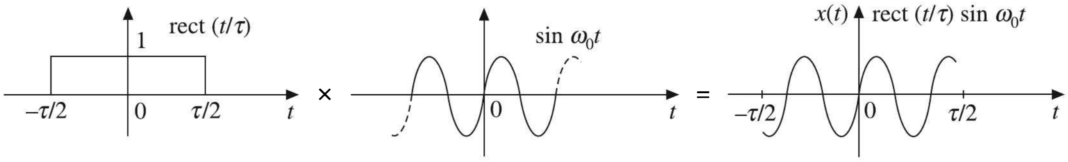


Figure 1.66 Signal for Example 1.20(c).

The given signal is the product of a rect. function and a sine wave. Figure 1.66 shows how $x(t)$ is derived.

- sin $\omega_0 t$ is periodic and infinite duration signal. So basically it is a power signal.
- rect (t/τ) is non-periodic and finite duration signal. So basically it is an energy signal.
- rect (t/τ) sin $\omega_0 t$ is product of the above two. Hence it is a sine wave of duration $-\frac{\tau}{2} < t < \frac{\tau}{2}$, i.e. it is periodic over a finite interval only. So it is an energy signal. Calculate energy directly.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\tau/2}^{\tau/2} [\sin \omega_0 t]^2 \, dt = \int_{-\tau/2}^{\tau/2} \left(\frac{1 - \cos 2\omega_0 t}{2} \right) dt$$

$$= \frac{1}{2} \int_{-\tau/2}^{\tau/2} dt - \frac{1}{2} \int_{-\tau/2}^{\tau/2} \cos 2\omega_0 t \, dt = \frac{\tau}{2} - 0$$

$$= \frac{\tau}{2} \text{ joules}$$

Here energy is finite and non-zero. Hence it is energy signal with $E = \tau/2$ joules.

Now, Normalized power $P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-\tau/2}^{\tau/2} \sin^2 \omega_0 t \, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-\tau/2}^{\tau/2} \left(\frac{1 - \cos 2\omega_0 t}{2} \right) dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \left[\frac{\tau}{2} \right] - 0$$

$$= 0$$

(d) Given $x(t) = Ae^{-at}u(t),\ a > 0$ (Figure 1.67).

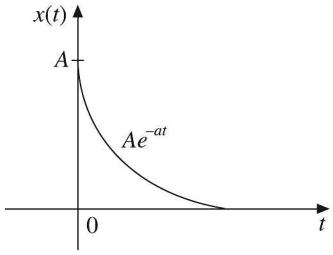


Figure 1.67 Signal for Example 1.20(d).

$u(t) = 1$ exists only for $0 < t < \infty$. The given signal is non-periodic and of finite duration. So it can be an energy signal.

$$x(t) = Ae^{-at}\,u(t) = \begin{cases} Ae^{-at}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$\therefore$
$$|x(t)|^2 = A^2 e^{-2at} \quad \text{for } t \ge 0$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$= \int_0^{\infty} \left|A \quad e^{-at}\right|^2 dt = A^2 \int_0^{\infty} e^{-2at}\, dt$$

$$= A^2 \left[\frac{e^{-2at}}{-2a}\right]_0^{\infty}$$

$$= \frac{A^2}{2a}$$

The energy is finite and non-zero. Hence the given signal is energy signal with $E = A^2/2a$.

Now, Normalized power $P = \underset{T\to\infty}{\text{Lt}} \dfrac{1}{2T} \displaystyle\int_{-T}^{T} |x(t)|^2\, dt$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_0^{T} (Ae^{-at})^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{A^2}{2T} [e^{-2at}]_0^T$$

$$= 0$$

(e) Given $x(t) = u(t)$ (Figure 1.68)

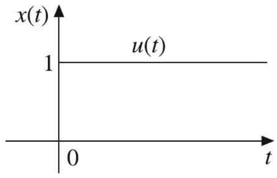


Figure 1.68 Signal for Example 1.20(e).

It is a non-periodic signal extending from $t = 0$ to $t = \infty$ with amplitude remaining constant.
Hence it is a power signal with finite power and infinite energy

Then Normalized average power $P = \underset{T\to\infty}{\text{Lt}} \dfrac{1}{2T} \displaystyle\int_{-T}^{T} |x(t)|^2\, dt$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_0^T (1)^2\, dt = \frac{1}{2}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \int_0^T (1)^2\, dt = \underset{T\to\infty}{\text{Lt}} [T] = \infty$$

Here $P = 1/2$ and $E = \infty$. Hence it is a power signal.

(f) Given $x(t) = tu(t)$

It is a non-periodic signal with magnitude increasing linearly from $t = 0$ to $t = \infty$ as shown in Figure 1.69.

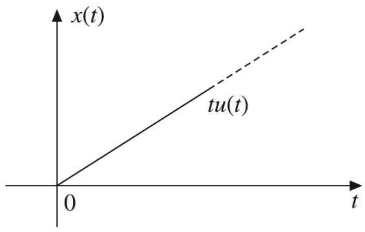


Figure 1.69 Plot of $x(t) = tu(t)$.

$$x(t) = tu(t) = \begin{cases} t, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$\therefore$ $|x(t)|^2 = t^2$ for $t \ge 0$

Energy of the signal $E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$

$$= \underset{T\to\infty}{\text{Lt}} \int_0^T (t)^2\, dt = \underset{T\to\infty}{\text{Lt}} \left[\frac{t^3}{3}\right]_0^T = \underset{T\to\infty}{\text{Lt}} \frac{T^3}{3} = \infty$$

$\therefore \qquad E = \infty$

Average power of the signal

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_0^T t^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \left[\frac{t^3}{3}\right]_0^T = \infty$$

$\therefore \qquad P = \infty$

In this case, $E = \infty$ and $P = \infty$. Hence the given signal is neither energy signal nor power signal.

(g) Given $\qquad x(t) = e^{j[3t+(\pi/2)]}$

The given signal is an infinite duration periodic signal which is a combination of sine and cosine signals. So it can be a power signal.

Then $\qquad$ Energy of the signal $E = \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} |x(t)|^2\, dt$

$$= \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} \left|e^{j[3t+(\pi/2)]}\right|^2 dt$$

$$= \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} (1)^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} [2T] = \infty$$

The average power of the signal is:

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} \left|e^{j[3t+(\pi/2)]}\right|^2 dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} [2T] = 1$$

The average power of the signal is finite and the total energy of the signal is infinite. Therefore, the given signal is a power signal.

EXAMPLE 1.21 Sketch the following signals and say whether they are energy signals or power signals:

(a) $e^{-5t}\,u(t)$ (b) $u(t) - u(t-4)$

(c) $\sin\omega t\; u(t-1)\, u(9-t)$ (d) $u(t) + u(t-2)$

(e) $r(t) - r(t-3)$ (f) $(2 + e^{-6t})\, u(t)$

(g) $t^2\, u(t)$ (h) $(2 + e^{4t})\, u(t)$

Solution: The given signals are plotted as shown in Figure 1.70.

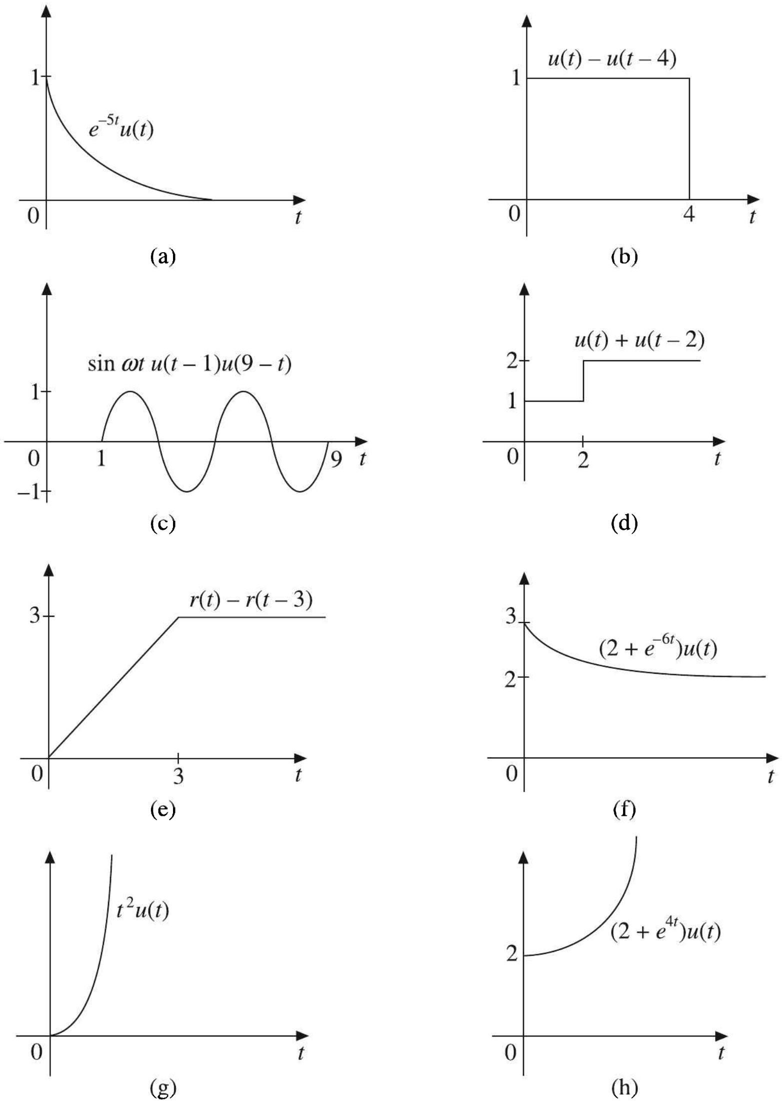


Figure 1.70 Signals for Example 1.21.

From Figure 1.70 we can conclude the following about the given signals:

(a) It is an aperiodic signal with magnitude dropping from 1 to 0 exponentially. So its energy E = Finite and average power $P = 0$. Hence it is an energy signal.

$$E = \int_0^{\infty} \left|e^{-5t}\right|^2 dt = \frac{1}{10} \text{ joules}$$

(b) It is an aperiodic signal with constant magnitude of unity from $t = 0$ to $t = 4$. So its energy E is finite and average power $P = 0$. Hence it is an energy signal.

$$E = \int_0^{4} (1)^2 \, dt = 4 \text{ joules}$$

(c) It is periodic only between $t = 1$ to $t = 9$. So it is an aperiodic signal. So its energy E is finite and average power $P = 0$. Hence it is an energy signal.

$$E = \int_1^{9} (\sin \omega t)^2 \, dt = 8 \text{ joules}$$

(d) It is a non-periodic infinite duration signal with constant amplitude of 1 from $t = 0$ to $t = 2$ and 2 between $t = 2$ to $t = \infty$. So its energy $E = \infty$ and average power is finite. So it is a power signal.

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\left[\int_0^{2} (1)^2 \, dt + \int_2^{T} (2)^2 \, dt\right] = 2 \text{ watts}$$

(e) It is a non-periodic infinite duration signal with linearly rising amplitude from $t = 0$ to $t = 3$ and with constant amplitude of 3 between $t = 3$ to $t = \infty$. So its energy $E = \infty$ and average power is finite. So it is a power signal.

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\left[\int_0^{3} (t)^2 \, dt + \int_3^{T} (3)^2 \, dt\right] = 4.5 \text{ watts}$$

(f) It is a non-periodic infinite duration signal with amplitude falling from 3 to 2 and then remaining constant. So its energy $E = \infty$ and average power is finite. So it is a power signal.

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\left[\int_0^{T} (2 + e^{-6t})^2 \, dt\right] = 2 \text{ watts}$$

(g) It is a non-periodic infinite duration signal with exponentially increasing amplitude between 0 to ∞. So its energy $E = \infty$ and average power $P = \infty$. Hence, it is neither power signal nor energy signal.

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\left[\int_0^{T} (t^2)^2 \, dt\right] = \infty$$

(h) It is a non-periodic infinite duration signal with exponentially increasing amplitude between 0 to ∞. So its energy $E = \infty$ and average power $P = \infty$. Hence, it is neither power signal nor energy signal.

EXAMPLE 1.22 Find which of the following signals are energy signals, power signals, neither energy nor power signals:

(a) $\left(\frac{1}{2}\right)^n u(n)$ (b) $e^{j[(\pi/3)n+(\pi/2)]}$

(c) $\sin\left(\frac{\pi}{3}n\right)$ (d) $u(n) - u(n-6)$

(e) $nu(n)$ (f) $r(n) - r(n-4)$

Solution:

(a) Given $$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

Energy of the signal $$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \left[\left(\frac{1}{2}\right)^n\right]^2 u(n)$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-(1/4)} = \frac{4}{3} \text{ joules}$$

Power of the signal $$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \left[\frac{1-(1/4)^{N+1}}{1-(1/4)}\right]$$

$$= 0$$

The energy is finite and power is zero. Therefore, $x(n)$ is an energy signal.

(b) Given $$x(n) = e^{j[(\pi/3)n + (\pi/2)]}$$

Energy of the signal $$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \left| e^{j[(\pi/3)n + (\pi/2)]} \right|^2$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} 1$$

$$= \underset{N\to\infty}{\text{Lt}} [2N+1] = \infty$$

Power of the signal $$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j[(\pi/3)n + (\pi/2)]} \right|^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} [2N+1] = 1 \text{ watt}$$

The energy is infinite and the power is finite. Therefore, it is a power signal.

(c) Given $$x(n) = \sin\left(\frac{\pi}{3} n\right)$$

Energy of the signal $$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \sin^2\left(\frac{\pi}{3} n\right)$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \frac{1 - \cos[(2\pi/3)n]}{2}$$

$$= \frac{1}{2} \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \left(1 - \cos\frac{2\pi}{3} n\right)$$

$$= \infty$$

Power of the signal $$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} \sin^2\left(\frac{\pi}{3} n\right)$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1 - \cos[(2\pi/3)n]}{2}$$

$$= \frac{1}{2} \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} [2N+1] = \frac{1}{2} \text{ watt}$$

The energy is infinite and power is finite. Therefore, it is a power signal.

(d) Given $$x(n) = u(n) - u(n-6)$$

$$\text{Energy of the signal } E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} [u(n) - u(n-6)]^2$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=0}^{5} 1 = 6 \text{ joules}$$

$$\text{Power of the signal } P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} [u(n) - u(n-6)]^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=0}^{5} 1 = 0$$

Energy is finite and power is zero. Thus, it is an energy signal.

(e) Given $$x(n) = nu(n)$$

$$\text{Energy of the signal } E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} [n]^2 u(n)$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=0}^{N} [n^2]$$

$$= \infty$$

$$\text{Power of the signal } P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} [n]^2 u(n)$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} n^2$$

$$= \infty$$

Energy is infinite and power is also infinite. Therefore, it is neither energy signal nor power signal.

(f) Given $$x(n) = r(n) - r(n-4)$$

$$\text{Energy of the signal } E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} [r(n) - r(n-4)]^2$$

$$= \underset{N\to\infty}{\text{Lt}} \left[\sum_{n=0}^{4} n^2 + \sum_{n=5}^{N} (4)^2 \right]$$

$$= \infty$$

$$\text{Power of the signal } P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} [r(n) - r(n-4)]^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \left[\sum_{n=0}^{4} n^2 + \sum_{n=5}^{N} (4)^2 \right]$$

$$= 8 \text{ watts}$$

Energy is infinite and power is finite. Thus, it is a power signal.

EXAMPLE 1.23 Find whether the signal

$$x(t) = \begin{cases} t-2 & -2 \le t \le 0 \\ 2-t & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

is energy signal or power signal. Also find the energy and power of the signal.

Solution: The given signal is a non-periodic finite duration signal. So it has finite energy and zero average power. So it is an energy signal.

$$\text{Energy of the signal } E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \left[\int_{-2}^{0} (t-2)^2 \, dt + \int_{0}^{2} (2-t)^2 \, dt \right]$$

$$= \int_{-2}^{0} (t^2 - 4t + 4) \, dt + \int_{0}^{2} (4 + t^2 - 4t) \, dt$$

$$= \left[\frac{t^3}{3} - \frac{4t^2}{2} + 4t \right]_{-2}^{0} + \left[4t + \frac{t^3}{3} - \frac{4t^2}{2} \right]_{0}^{2}$$

$$= \frac{64}{3} \text{ joules}$$

$$\text{Power of the signal } P = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \left[\int_{-2}^{0} (t-2)^2 \, dt + \int_{0}^{2} (2-t)^2 \, dt \right]$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \left[\frac{64}{3} \right] = 0$$

Since energy is finite and power is zero, it is an energy signal.

EXAMPLE 1.24 Find whether the signal

$$x(n) = \begin{cases} n^2 & 0 \le n \le 3 \\ 10 - n & 4 \le n \le 6 \\ n & 7 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

is a power signal or energy signal. Also find the energy and power of the signal.

Solution: The given signal is a non-periodic finite duration signal. So it has finite energy and zero average power. So it is an energy signal.

$$\text{Energy of the signal } E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{3} (n^2)^2 + \sum_{n=4}^{6} (10-n)^2 + \sum_{n=7}^{9} (n)^2$$

$$= \sum_{n=0}^{3} n^4 + \sum_{n=4}^{6} (100 + n^2 - 20n) + \sum_{n=7}^{9} n^2$$

$$= (0 + 1 + 16 + 81) + (36 + 25 + 16) + (49 + 64 + 81)$$

$$= 369 < \infty \text{ joules}$$

$$\text{Power of the signal } P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \left[\sum_{n=0}^{3} (n^2)^2 + \sum_{n=4}^{6} (10-n)^2 + \sum_{n=7}^{9} (n)^2 \right]$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} [369] = 0$$

Here energy is finite and power is zero. So it is an energy signal.

EXAMPLE 1.25 Find the energy of the signals shown in Figure 1.71.

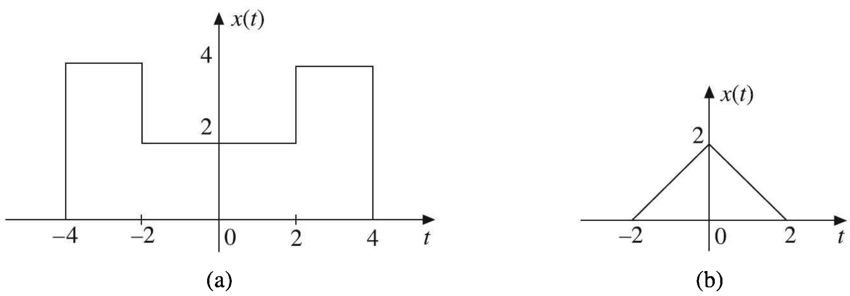


Figure 1.71 Signals for Example 1.25.

Solution:

(a) The signal shown in Figure 1.71(a) can be expressed as:

$$x(t) = \begin{cases} 4 & -4 \le t \le -2 \\ 2 & -2 \le t \le 2 \\ 4 & 2 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Energy of the signal
$$E = \int_{-4}^{-2} [4]^2\, dt + \int_{-2}^{2} [2]^2\, dt + \int_{2}^{4} [4]^2\, dt$$

$$= 16[t]_{-4}^{-2} + 4[t]_{-2}^{2} + 16[t]_{2}^{4}$$

$$= 16(-2+4) + 4(2+2) + 16(4-2)$$

$$= 80 \text{ joules}$$

(b) The signal shown in Figure 1.71(b) can be expressed as:

$$x(t) = \begin{cases} t+2 & -2 \le t \le 0 \\ 2-t & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Energy of the signal
$$E = \int_{-2}^{0} [t+2]^2\, dt + \int_{0}^{2} [2-t]^2\, dt$$

$$= \int_{-2}^{0} [t^2 + 4t + 4] dt + \int_{0}^{2} [4 + t^2 - 4t]\, dt$$

$$= \frac{16}{3} \text{ joules}$$

1.5.4 Causal and Non-causal Signals

A continuous-time signal $x(t)$ is said to be causal if $x(t) = 0$ for $t < 0$, otherwise the signal is non-causal. A continuous-time signal $x(t)$ is said to be anti-causal if $x(t) = 0$ for $t > 0$.

A causal signal does not exist for negative time and an anti-causal signal does not exist for positive time. A signal which exists in positive as well as negative time is neither causal nor anti-causal. It is non-causal. $u(t)$ is a causal signal and $u(-t)$ is anti-causal signal.

Similarly, a discrete-time signal $x(n)$ is said to be causal if $x(n) = 0$ for $n < 0$, otherwise the signal is non-causal. A discrete-time signal $x(n)$ is said to be anti-causal if $x(n) = 0$ for $n > 0$.

EXAMPLE 1.26 Find which of the following signals are causal or non-causal.

(a) $x(t) = e^{2t} u(t-1)$

(b) $x(t) = e^{-3t} u(-t+2)$

(c) $x(t) = 3 \text{ sinc } 2t$

(d) $x(t) = u(t+2) - u(t-2)$

(e) $x(t) = \cos 2t$

(f) $x(t) = \sin 2t \, u(t)$

(g) $x(n) = u(n+4) - u(n-2)$

(h) $x(n) = \left(\frac{1}{4}\right)^n u(n+2) - \left(\frac{1}{2}\right)^n u(n-4)$

(i) $x(t) = 2u(-t)$

(j) $x(n) = u(-n)$

Solution:

(a) Given $x(t) = e^{2t} u(t-1)$
The signal $x(t)$ is causal because $x(t) = 0$ for $t < 0$.

(b) Given $x(t) = e^{-3t} u(-t+2)$
The given signal $x(t)$ is non-causal because $x(t) \neq 0$ for $t < 0$.

(c) Given $x(t) = 3 \text{ sinc } 2t$
A sinc signal exists for $t < 0$ also. So the given signal is non-causal.

(d) Given $x(t) = u(t+2) - u(t-2)$
The given signal exists from $t = -2$ to $t = +2$. Since $x(t) \neq 0$ for $t < 0$ it is non-causal.

(e) Given $x(t) = \cos 2t$
The given signal exists from $-\infty$ to ∞. Since $x(t) \neq 0$ for $t < 0$, the signal is non-causal.

(f) Given $x(t) = \sin 2t \, u(t)$
The given signal is causal because $x(t) = 0$ for $t < 0$.

(g) Given $x(n) = u(n+4) - u(n-2)$
The given signal exists from $n = -4$ to $n = 1$. Since $x(n) \neq 0$ for $n < 0$, it is non-causal.

(h) Given $x(n) = \left(\frac{1}{4}\right)^n u(n+2) - \left(\frac{1}{2}\right)^n u(n-4)$
The given signal exists for $n < 0$ also. So it is non-causal.

(i) Given $x(t) = 2u(-t)$
The given signal exists only for $t < 0$. So it is anti-causal. It can be called non-causal also.

(j) Given $x(n) = u(-n)$
The given signal exists only for $n < 0$. So it is anti-causal. It can be called non-causal also.

1.5.5 Even and Odd Signals

Even (symmetric) signal

A continuous-time signal $x(t)$ is said to be an even (symmetric) signal if it satisfies the condition

$$x(t) = x(-t) \text{ for all } t$$

A discrete-time signal $x(n)$ is said to be an even (symmetric) signal if it satisfies the condition

$$x(n) = x(-n) \text{ for all } n$$

Even signals are symmetrical about the vertical axis or time origin. Hence they are also called symmetric signals: cosine wave is an example of an even signal. Some even signals are shown in Figure 1.72(a).

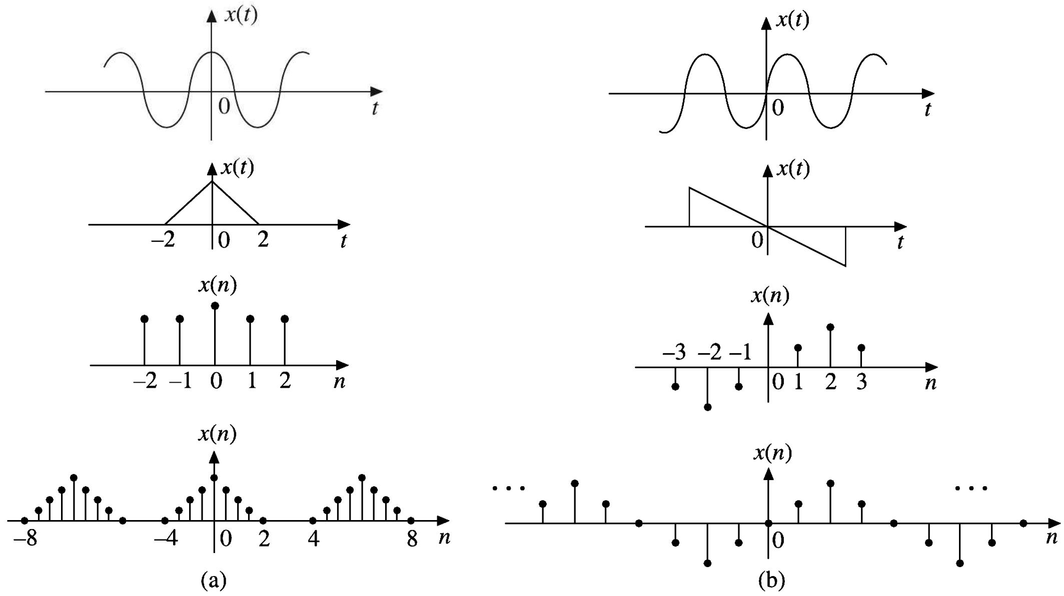


Figure 1.72 (a) Even signals, (b) Odd signals.

Odd (antisymmetric) signal

A continuous-time signal $x(t)$ is said to be an odd (antisymmetric) signal if it satisfies the condition

$$x(-t) = -x(t) \text{ for all } t$$

A discrete-time signal $x(n)$ is said to be an odd (antisymmetric) signal if it satisfies the condition

$$x(-n) = -x(n) \text{ for all } n$$

Odd signals are antisymmetrical about the vertical axis. Hence they are called antisymmetric signals. Sine wave is an example of an odd signal. For an odd signal $x(t) = 0$, $x(n) = 0$. Some odd signals are shown in Figure 1.72(b).

Any signal $x(t)$ can be expressed as sum of even and odd components. That is

$$x(t) = x_e(t) + x_o(t)$$

where $x_e(t)$ is even components and $x_o(t)$ is odd components of the signal.

Evaluation of even and odd parts of a signal

We have $$x(t) = x_e(t) + x_o(t)$$

$\therefore$ $$x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t)$$

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t) = 2x_e(t)$$

$\therefore$ $$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x(t) - x(-t) = [x_e(t) + x_o(t)] - [x_e(t) - x_o(t)] = 2x_o(t)$$

$\therefore$ $$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Similarly, the even and odd parts of a discrete-time signal, $x_e(n)$ and $x_o(n)$ are given by

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

and

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

The product of two even or odd signals is an even signal and the product of even signal and odd signal is an odd signal.

We can prove this as follows:

Let $$x(t) = x_1(t)\, x_2(t)$$

(i) If $x_1(t)$ and $x_2(t)$ are both even, i.e.

$$x_1(-t) = x_1(t)$$

and $$x_2(-t) = x_2(t)$$

Then $$x(-t) = x_1(-t)\, x_2(-t) = x_1(t)\, x_2(t) = x(t)$$

Therefore, $x(t)$ is an even signal.

If $x_1(t)$ and $x_2(t)$ are both odd, i.e.

$$x_1(-t) = -x_1(t)$$

and $$x_2(-t) = -x_2(t)$$

Then $$x(-t) = x_1(-t)\, x_2(-t) = [-x_1(t)][-x_2(t)] = x_1(t)\, x_2(t) = x(t)$$

Therefore, $x(t)$ is an even signal.

(ii) If $x_1(t)$ is even and $x_2(t)$ is odd, i.e.

$$x_1(-t) = x_1(t)$$

and $$x_2(-t) = -x_2(t)$$

Then $$x(-t) = x_1(-t)\, x_2(-t) = -x_1(t)\, x_2(t) = -x(t)$$

Therefore $x(t)$ is an odd signal.

Thus, the product of two even signals or of two odd signals is an even signal and the product of even and odd signals is an odd signal.

Every signal need not be either purely even signal or purely odd signal, but every signal can be decomposed into sum of even and odd parts.

EXAMPLE 1.27 Find the even and odd components of the following signals:

(a) $x(t) = e^{j2t}$

(b) $x(t) = \cos\left(\omega_0 t + \dfrac{\pi}{3}\right)$

(c) $x(t) = (1 + t^2 + t^3)\cos^2 10t$

(d) $x(t) = \sin 2t + \sin 2t \cos 2t + \cos 2t$

(e) $x(t) = 1 + 2t + 3t^2 + 4t^3$

Solution:

(a) Given $$x(t) = e^{j2t}$$

$\therefore$ $$x(-t) = e^{-j2t}$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[e^{j2t} + e^{-j2t}] = \cos 2t$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[e^{j2t} - e^{-j2t}] = j\sin 2t$$

(b) Given $$x(t) = \cos\left(\omega_0 t + \frac{\pi}{3}\right)$$

$\therefore$ $$x(-t) = \cos\left(-\omega_0 t + \frac{\pi}{3}\right) = \cos\left(\omega_0 t - \frac{\pi}{3}\right)$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}\left[\cos\left(\omega_0 t + \frac{\pi}{3}\right) + \cos\left(\omega_0 t - \frac{\pi}{3}\right)\right]$$

$$= \cos \omega_0 t \cos\frac{\pi}{3} = \frac{1}{2}\cos \omega_0 t$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}\left[\cos\left(\omega_0 t + \frac{\pi}{3}\right) - \cos\left(\omega_0 t - \frac{\pi}{3}\right)\right]$$

$$= -\sin \omega_0 t \sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}\sin \omega_0 t$$

(c) Given $$x(t) = (1 + t^2 + t^3)\cos^2 10t$$

$\therefore$ $$x(-t) = (1 + t^2 - t^3)\cos^2 10t$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[(1 + t^2 + t^3)\cos^2 10t + (1 + t^2 - t^3)\cos^2 10t]$$

$$= \frac{1}{2}[2(1 + t^2)\cos^2 10t] = (1 + t^2)\cos^2 10t$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[(1 + t^2 + t^3)\cos^2 10t - (1 + t^2 - t^3)\cos^2 10t]$$

$$= \frac{1}{2}[2t^3\cos^2 10t] = t^3\cos^2 10t$$

(d) Given $$x(t) = \sin 2t + \sin 2t \cos 2t + \cos 2t$$

$\therefore$ $$x(-t) = \sin(-2t) + \sin(-2t)\cos(-2t) + \cos(-2t)$$

$$= -\sin 2t - \sin 2t \cos 2t + \cos 2t$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[(\sin 2t + \sin 2t\cos 2t + \cos 2t) + (-\sin 2t - \sin 2t\cos 2t + \cos 2t)]$$

$$= \frac{1}{2}[2\cos 2t] = \cos 2t$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[(\sin 2t + \sin 2t\cos 2t + \cos 2t) - (-\sin 2t - \sin 2t\cos 2t + \cos 2t)]$$

$$= \frac{1}{2}2[\sin 2t + \sin 2t\cos 2t] = \sin 2t + \sin 2t\cos 2t$$

(e) Given $$x(t) = 1 + 2t + 3t^2 + 4t^3$$

$\therefore$ $$x(-t) = 1 - 2t + 3t^2 - 4t^3$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[(1 + 2t + 3t^2 + 4t^3) + (1 - 2t + 3t^2 - 4t^3)]$$

$$= \frac{1}{2}[2(1 + 3t^2)] = 1 + 3t^2$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[(1 + 2t + 3t^2 + 4t^3) - (1 - 2t + 3t^2 - 4t^3)]$$

$$= \frac{1}{2}2[2t + 4t^3] = 2t + 4t^3$$

EXAMPLE 1.28 Find whether the following signals are even or odd.

(a) $x(t) = e^{-3t}$ (b) $x(t) = u(t+2)$

(c) $x(t) = 3e^{j4\pi t}$ (d) $x(t) = u(t+4) - u(t-2)$

Solution:

(a) Given
$$x(t) = e^{-3t}$$
$$x(-t) = e^{3t}$$
$$-x(t) = -e^{-3t}$$

Since $x(-t) \neq x(t)$ and $x(-t) \neq -x(t)$, the given signal is neither even signal nor odd signal.

(b) Given
$$x(t) = u(t + 2)$$
$$x(-t) = u(-t + 2)$$
$$-x(t) = -u(t + 2)$$

Then
$$u(t+2) = \begin{cases} 1 & \text{for } t \geq -2 \\ 0 & \text{for } t < -2 \end{cases}$$
$$u(-t+2) = \begin{cases} 1 & \text{for } t \leq 2 \\ 0 & \text{for } t > 2 \end{cases}$$
$$-u(t+2) = \begin{cases} -1 & \text{for } t > -2 \\ 0 & \text{for } t < -2 \end{cases}$$

Since $x(-t) \neq x(t)$ and $x(-t) \neq -x(t)$, the given signal is neither even signal nor odd signal.

(c) Given
$$x(t) = 3e^{j4\pi t}$$
$$x(-t) = 3e^{-j4\pi t}$$
$$-x(t) = -3e^{j4\pi t}$$

Since $x(-t) \neq x(t)$ and $x(-t) \neq -x(t)$, the given signal is neither even signal nor odd signal.

(d) Given
$$x(t) = u(t+4) - u(t-2)$$
$$x(-t) = u(-t+4) - u(-t-2)$$
$$-x(t) = -u(t+4) + u(t-2)$$

Since $x(-t) \neq x(t)$ and $x(-t) \neq -x(t)$, the given signal is neither even signal nor odd signal.

EXAMPLE 1.29 Find the even and odd components of the following signals.

(a) $x(n)=\left\{-3, 1, \underset{\uparrow}{2}, -4, 2\right\}$ (b) $x(n)=\left\{-2, 5, \underset{\uparrow}{1}, -3\right\}$

(c) $x(n)=\left\{\underset{\uparrow}{5}, 4, 3, 2, 1\right\}$ (d) $x(n)=\left\{5, 4, 3, 2, \underset{\uparrow}{1}\right\}$

Solution:

(a) Given

$$x(n)=\left\{-3, 1, \underset{\uparrow}{2}, -4, 2\right\}$$

$$x(-n)=\left\{2, -4, \underset{\uparrow}{2}, 1, -3\right\}$$

$$\therefore \qquad x_e(n)=\frac{1}{2}[x(n)+x(-n)]$$

$$=\frac{1}{2}\left[-3+2, 1-4, 2+2, -4+1, 2-3\right]$$

$$=\left\{-0.5, -1.5, \underset{\uparrow}{2}, -1.5, -0.5\right\}$$

$$x_o(n)=\frac{1}{2}[x(n)-x(-n)]$$

$$=\frac{1}{2}\left[-3-2, 1+4, 2-2, -4-1, 2+3\right]$$

$$=\left\{-2.5, 2.5, \underset{\uparrow}{0}, -2.5, 2.5\right\}$$

(b) Given

$$x(n)=-2, 5, \underset{\uparrow}{1}, -3$$

$$x(-n)=\quad -3, \underset{\uparrow}{1}, 5, -2$$

$$\therefore \qquad x_e(n)=\frac{1}{2}[x(n)+x(-n)]$$

$$=\frac{1}{2}\left[-2+0, 5-3, 1+1, -3+5, 0-2\right]$$

$$=-1,\ 1,\ \underset{\uparrow}{1},\ 1,\ -1$$

$$x_o(n)=\frac{1}{2}[x(n)-x(-n)]$$

$$=\frac{1}{2}\left[-2-0, 5+3, 1-1, -3-5, 0+2\right]$$

$$=-1, 4, \underset{\uparrow}{0}, -4, 1$$

(c) Given $x(n) = \left\{\underset{\uparrow}{5}, 4, 3, 2, 1\right\}$

$n = 0, 1, 2, 3, 4$

$\therefore$ $x(n) = \underset{\uparrow}{5}, 4, 3, 2, 1$

$x(-n) = 1, 2, 3, 4, \underset{\uparrow}{5}$

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[1, 2, 3, 4, 5 + 5, 4, 3, 2, 1]$$

$$= \left\{0.5, 1, 1.5, 2, \underset{\uparrow}{5}, 2, 1.5, 1, 0.5\right\}$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}[-1, -2, -3, -4, 0, 4, 3, 2, 1]$$

$$= \left\{-0.5, -1, -1.5, -2, \underset{\uparrow}{0}, 2, 1.5, 1, 0.5\right\}$$

(d) Given $x(n) = \left\{5, 4, 3, 2, \underset{\uparrow}{1}\right\}$

$n = -4, -3, -2, -1, 0$

$\therefore$ $x(n) = 5, 4, 3, 2, \underset{\uparrow}{1}$

$x(-n) = \underset{\uparrow}{1}, 2, 3, 4, 5$

$\therefore$
$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[5, 4, 3, 2, 1 + 1, 2, 3, 4, 5]$$

$$= [2.5, 2, 1.5, 1, 1, 1, 1.5, 2, 2.5]$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}[5, 4, 3, 2, 0, -2, -3, -4, -5]$$

$$= \frac{1}{2}[2.5, 2, 1.5, 1, 0, -1, -1.5, -2, -2.5]$$

When the signal is given as a waveform

The even part of the signal can be found by folding the signal about the y-axis and adding the folded signal to the original signal and dividing the sum by two. The odd part of the signal can be found by folding the signal about y-axis and subtracting the folded signal from the original signal and dividing the difference by two as illustrated in Figure 1.73.

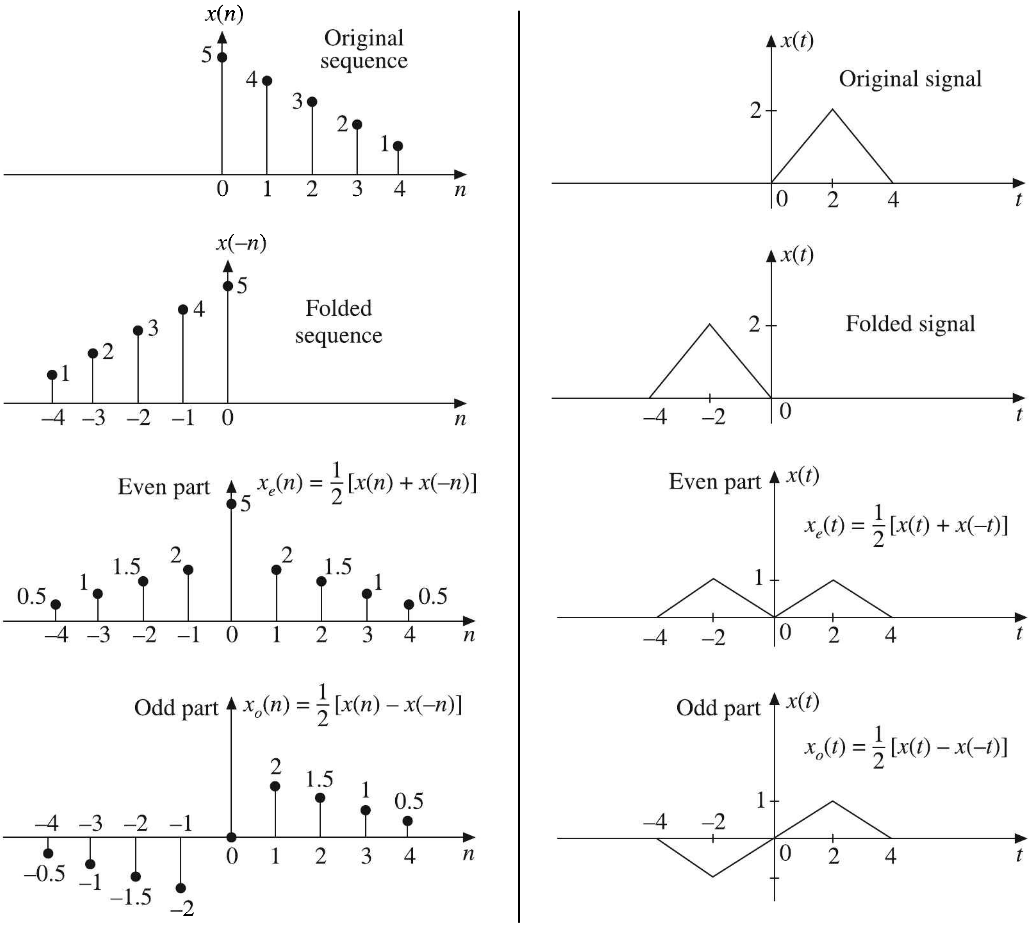


Figure 1.73 Graphical evaluation of even and odd parts.

MATLAB PROGRAMS

Program 1.1

```
% Generation of elementary signals in continuous-time
clc; close all; clear all;
% Unit Impulse signal
t=-1:0.001:1;
```

1.2.4 Sequence Representation

A finite duration sequence given in 1.2.1 can be represented as:

$$x(n) = \left\{ \begin{matrix} -3, 2, 0, 3, 1, 2 \\ \uparrow \end{matrix} \right\}$$

Another example is

$$x(n) = \left\{ \begin{matrix} \ldots 2, 3, 0, 1, -2, \ldots \\ \uparrow \end{matrix} \right\}$$

The arrow mark $\uparrow$ denotes the $n = 0$ term. When no arrow is indicated, the first term corresponds to $n = 0$

So a finite duration sequence, that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as $x(n) = \{3, 5, 2, 1, 4, 7\}$.

Sum and product of discrete-time sequences

The sum of two discrete-time sequences is obtained by adding the corresponding elements of sequences

$$\{C_n\} = \{a_n\} + \{b_n\} \quad \rightarrow \quad C_n = a_n + b_n$$

The product of two discrete-time sequences is obtained by multiplying the corresponding elements of the sequences.

$$\{C_n\} = \{a_n\}\{b_n\} \quad \rightarrow \quad C_n = a_n b_n$$

The multiplication of a sequence by a constant k is obtained by multiplying each element of the sequence by that constant.

$$\{C_n\} = k\{a_n\} \quad \rightarrow \quad C_n = k a_n$$

1.3 ELEMENTARY SIGNALS

There are several elementary signals which play vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. Infact, these elementary signals may be used to model a large number of physical signals which occur in nature. These elementary signals are also called standard signals.

The standard signals are:

1. Unit step function
2. Unit ramp function
3. Unit parabolic function
4. Unit impulse function
5. Sinusoidal function
6. Real exponential function
7. Complex exponential function, etc.

1.3.1 Unit Step Function

The step function is an important signal used for analysis of many systems. The step function is that type of elementary function which exists only for positive time and is zero for negative time. It is equivalent to applying a signal whose amplitude suddenly changes and remains constant forever after application.

If a step function has unity magnitude, then it is called unit step function. The usefulness of the unit-step function lies in the fact that if we want a signal to start at $t = 0$, so that it may have a value of zero for $t < 0$, we only need to multiply the given signal with unit step function $u(t)$. A unit step function is useful as a test signal because the response of the system for a unit step reveals a great deal about how quickly the system responds to a sudden change in the input signal.

The continuous-time unit step function $u(t)$ is defined as:

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

From the above equation for $u(t)$, we can observe that when the argument t in $u(t)$ is less than zero, then the unit step function is zero, and when the argument t in $u(t)$ is greater than or equal to zero, then the unit step function is unity.

The shifted unit step function $u(t - a)$ is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

It is zero if the argument $(t - a) < 0$ and equal to 1 if the argument $(t - a) \geq 0$.
The graphical representations of $u(t)$ and $u(t - a)$ are shown in Figure 1.2[(a) and (b)].

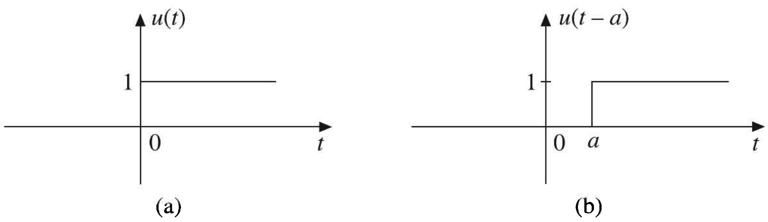


Figure 1.2 (a) Unit step function, (b) Delayed unit step function.

The discrete-time unit step sequence $u(n)$ is defined as:

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

The shifted version of the discrete-time unit step sequence $u(n - k)$ is defined as

$$u(n-k) = \begin{cases} 1 & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

The graphical representations of $u(n)$ and $u(n - k)$ are shown in Figure 1.3[(a) and (b)].

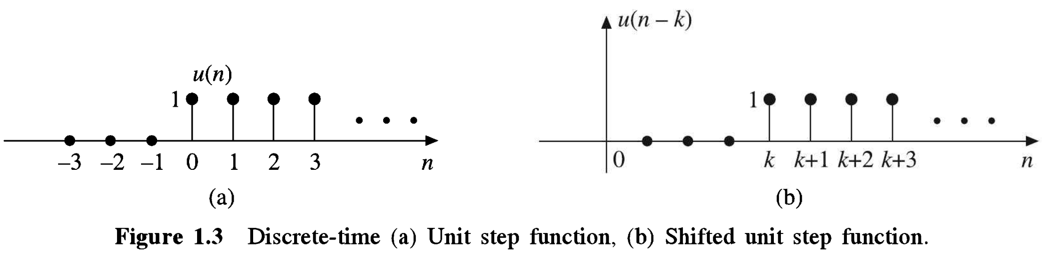


Figure 1.3 Discrete-time (a) Unit step function, (b) Shifted unit step function.

1.3.2 Unit Ramp Function

The continuous-time unit ramp function $r(t)$ is that function which starts at $t = 0$ and increases linearly with time and is defined as:

$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or $$r(t) = t\,u(t)$$

The unit ramp function has unit slope. It is a signal whose amplitude varies linearly. It can be obtained by integrating the unit step function. That means, a unit step signal can be obtained by differentiating the unit ramp signal.

i.e. $$r(t) = \int u(t)\,dt = \int dt = t \quad \text{for } t \geq 0$$

$$u(t) = \frac{d}{dt} r(t)$$

The delayed unit ramp signal $r(t - a)$ is given by

$$r(t-a) = \begin{cases} t-a & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

or $$r(t-a) = (t-a)\,u(t-a)$$

The graphical representations of $r(t)$ and $r(t - a)$ are shown in Figure 1.4[(a) and (b)].

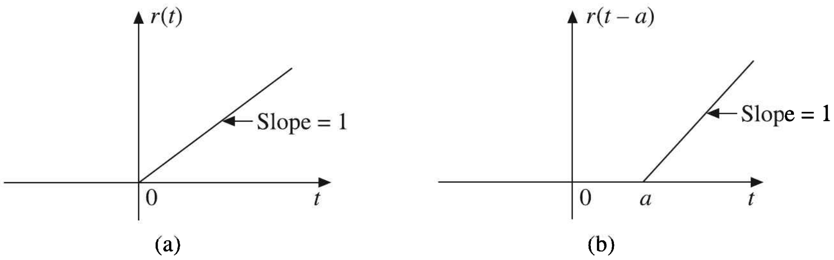


Figure 1.4 (a) Unit ramp signal, (b) Delayed unit ramp signal.

The discrete-time unit ramp sequence $r(n)$ is defined as

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$r(n) = n\,u(n)$$

The shifted version of the discrete-time unit-ramp sequence $r(n - k)$ is defined as

$$r(n-k) = \begin{cases} n-k & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

or $$r(n-k) = (n-k)\,u(n-k)$$

The graphical representations of $r(n)$ and $r(n-2)$ are shown in Figure 1.5[(a) and (b)].

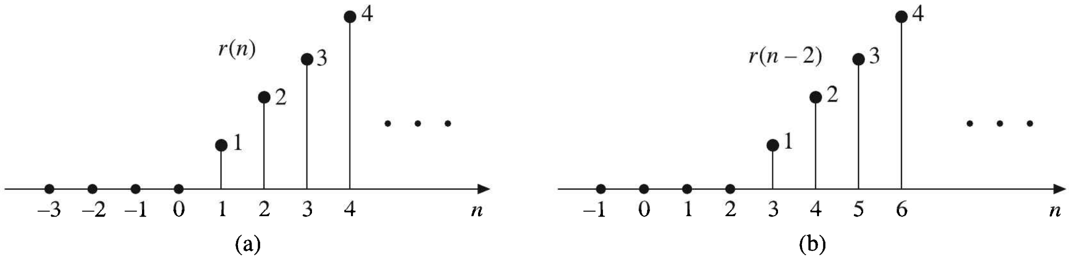


Figure 1.5 Discrete-time (a) Unit-ramp sequence, (b) Shifted-ramp sequence.

1.3.3 Unit Parabolic Function

The continuous-time unit parabolic function $p(t)$, also called unit acceleration signal starts at $t = 0$, and is defined as:

$$p(t) = \begin{cases} \dfrac{t^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or $$p(t) = \frac{t^2}{2} u(t)$$

The shifted version of the unit parabolic sequence $p(t - a)$ is given by

$$p(t-a) = \begin{cases} \dfrac{(t-a)^2}{2} & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

or $$p(t-a) = \frac{(t-a)^2}{2} u(t-a)$$

The graphical representations of $p(t)$ and $p(t - a)$ are shown in Figure 1.6[(a) and (b)].

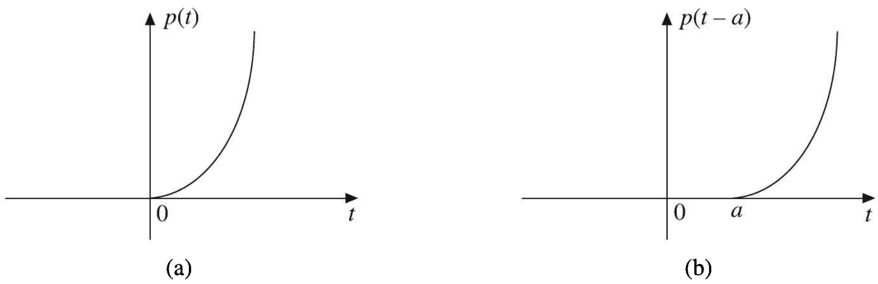


Figure 1.6 (a) Unit parabolic signal, (b) Delayed parabolic signal.

The unit parabolic function can be obtained by integrating the unit ramp function or double integrating the unit step function.

$$p(t) = \iint u(t)\, dt = \int r(t)\, dt = \int t\, dt = \frac{t^2}{2} \quad \text{for } t \geq 0$$

The ramp function is derivative of parabolic function and step function is double derivative of parabolic function

$$r(t) = \frac{d}{dt}\,p(t); \qquad u(t) = \frac{d^2}{dt^2}\,p(t)$$

The discrete-time unit parabolic sequence $p(n)$ is defined as:

$$p(n) = \begin{cases} \dfrac{n^2}{2} & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$p(n) = \frac{n^2}{2}\,u(n)$$

The shifted version of the discrete-time unit parabolic sequence $p(n - k)$ is defined as:

$$p(n-k) = \begin{cases} \dfrac{(n-k)^2}{2} & \text{for } n \ge k \\ 0 & \text{for } n < k \end{cases}$$

or $$p(n-k) = \frac{(n-k)^2}{2}\,u(n-k)$$

The graphical representations of $p(n)$ and $p(n - 3)$ are shown in Figure 1.7[(a) and (b)].

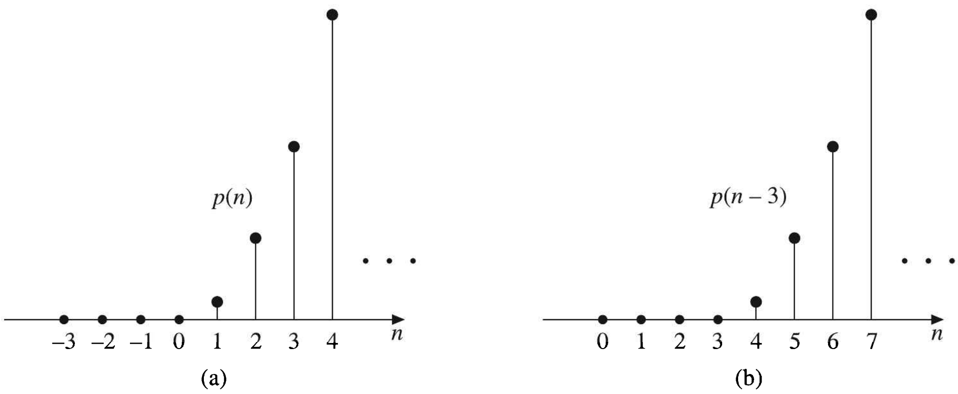


Figure 1.7 Discrete-time (a) Parabolic sequence, (b) Shifted parabolic sequence.

1.3.4 Unit Impulse Function

The unit impulse function is the most widely used elementary function used in the analysis of signals and systems. The continuous-time unit impulse function $\delta(t)$, also called Dirac delta function, plays an important role in signal analysis. It is defined as:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

and $$\delta(t) = 0 \quad \text{for } t \neq 0$$

i.e. as $$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

That is, the impulse function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude is infinity so that the area under the curve is unity. $\delta(t)$ can be represented as a limiting case of a rectangular pulse function.

As shown in Figure 1.8(a),

$$x(t) = \frac{1}{\Delta}[u(t) - u(t - \Delta)]$$

$$\delta(t) = \underset{\Delta \to 0}{\text{Lt}}\ x(t) = \underset{\Delta \to 0}{\text{Lt}}\ \frac{1}{\Delta}[u(t) - u(t - \Delta)]$$

A delayed unit impulse function $\delta(t - a)$ is defined as:

$$\delta(t - a) = \begin{cases} 1 & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

The graphical representations of $\delta(t)$ and $\delta(t - a)$ are shown in Figure 1.8[(b) and (c)].

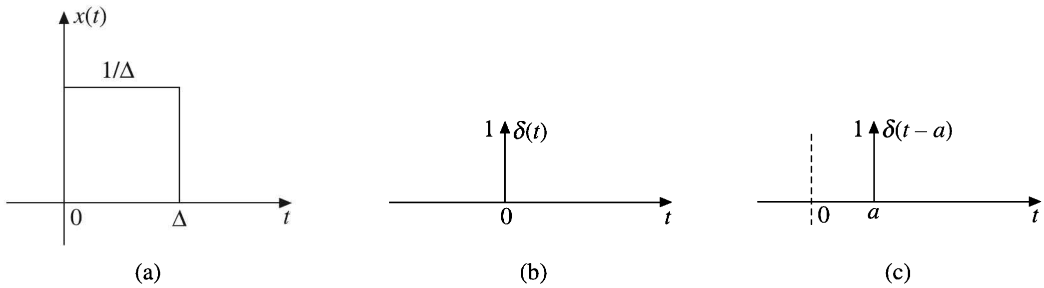


Figure 1.8 (a) $\delta(t)$ as limiting case of a pulse, (b) Unit impulse, (c) Delayed unit impulse.

If unit impulse function is assumed in the form of a pulse, then the following points may be observed about a unit impulse function.

(i) The width of the pulse is zero. This means the pulse exists only at $t = 0$.
(ii) The height of the pulse goes to infinity.
(iii) The area under the pulse curve is always unity.
(iv) The height of arrow indicates the total area under the impulse.

The integral of unit impulse function is a unit step function and the derivate of unit step function is a unit impulse function.

$$u(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

and $$\delta(t) = \frac{d}{dt} u(t)$$

Properties of continuous-time unit impulse function

1. It is an even function of time t, i.e. $\delta(t) = \delta(-t)$
2. $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$; $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$
3. $\delta(at) = \frac{1}{|a|}\delta(t)$
4. $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0) = x(t_0)$; $x(t)\,\delta(t) = x(0)\,\delta(t) = x(0)$
5. $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

The discrete-time unit impulse function $\delta(n)$, also called unit sample sequence, is defined as:

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

The shifted unit impulse function $\delta(n-k)$ is defined as:

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$$

The graphical representations of $\delta(n)$ and $\delta(n-3)$ are shown in Figure 1.9[(a) and (b)].

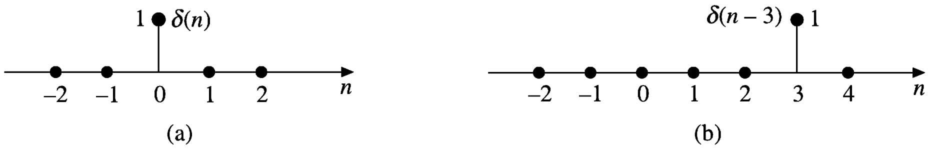


Figure 1.9 Discrete-time (a) Unit sample sequence, (b) Delayed unit sample sequence.

Properties of discrete-time unit sample sequence

1. $\delta(n) = u(n) - u(n-1)$
2. $\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$
3. $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$
4. $\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$

1.3.5 Sinusoidal Signal

A continuous-time sinusoidal signal in its most general form is given by

$$x(t) = A\sin(\omega t + \phi)$$

where

A = Amplitude

ω = Angular frequency in radians

ϕ = Phase angle in radians

Figure 1.10 shows the waveform of a sinusoidal signal. A sinusoidal signal is an example of a periodic signal. The time period of a continuous-time sinusoidal signal is given by

$$T = \frac{2\pi}{\omega}$$

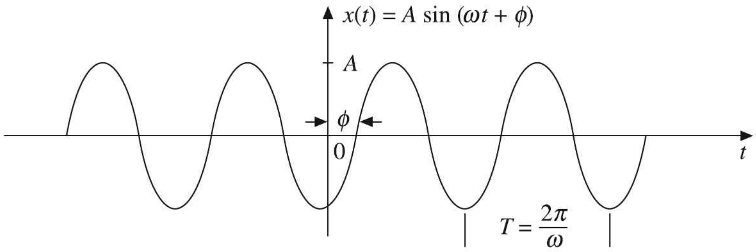


Figure 1.10 Sinusoidal waveform.

The discrete-time sinusoidal sequence is given by

$$x(n) = A \sin(\omega n + \phi)$$

where A is the amplitude, ω is angular frequency, ϕ is phase angle in radians and n is an integer.

The period of the discrete-time sinusoidal sequence is:

$$N = \frac{2\pi}{\omega} m$$

where N and m are integers.

All continuous-time sinusoidal signals are periodic but discrete-time sinusoidal sequences may or may not be periodic depending on the value of ω.

For a discrete-time signal to be periodic, the angular frequency ω must be a rational multiple of 2π.

The graphical representation of a discrete-time sinusoidal signal is shown in Figure 1.11.

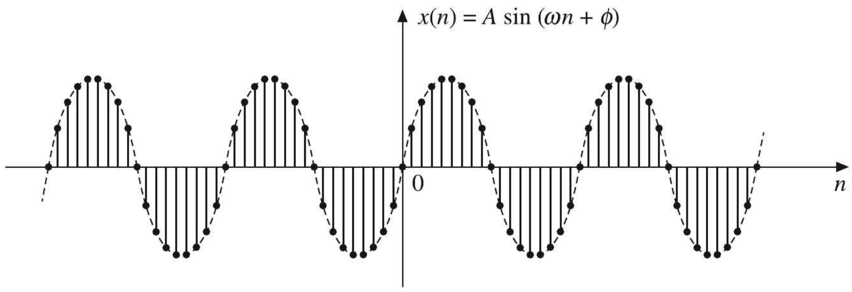


Figure 1.11 Discrete-time sinusoidal signal.

1.3.6 Real Exponential Signal

A continuous-time real exponential signal has the general form as:

$$x(t) = Ae^{\alpha t}$$

where both A and α are real.

The parameter A is the amplitude of the exponential measured at $t = 0$. The parameter α can be either positive or negative. Depending on the value of α, we get different exponentials.

1. If $\alpha = 0$, the signal $x(t)$ is of constant amplitude for all times.
2. If α is positive, i.e. $\alpha > 0$, the signal $x(t)$ is a growing exponential signal.
3. If α is negative, i.e. $\alpha < 0$, the signal $x(t)$ is a decaying exponential signal.

These three waveforms are shown in Figure 1.12[(a), (b) and (c)].

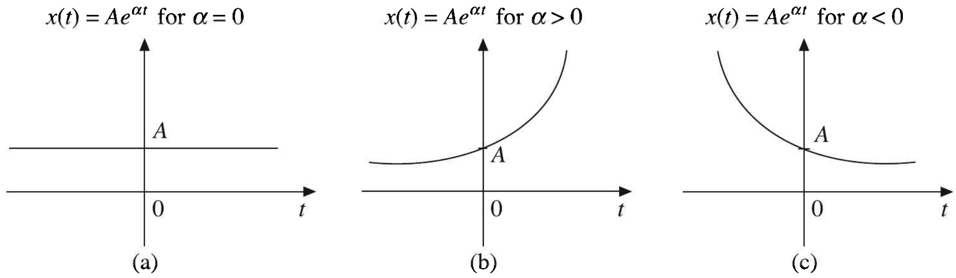


Figure 1.12 Continuous-time real exponential signals $x(t) = Ae^{\alpha t}$ for (a) $\alpha = 0$, (b) $\alpha > 0$, (c) $\alpha < 0$.

The discrete-time real exponential sequence a^n is defined as:

$$x(n) = a^n \quad \text{for all } n$$

Figure 1.13 illustrates different types of discrete-time exponential signals.

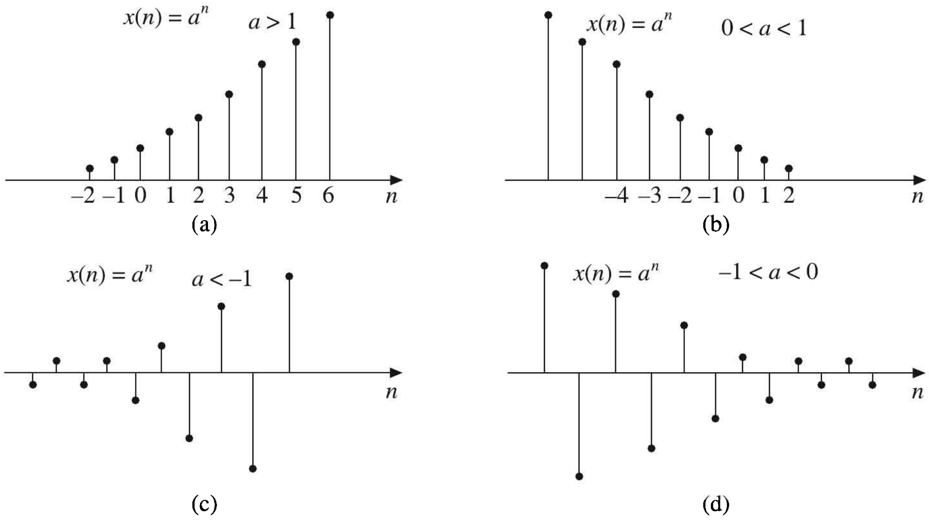


Figure 1.13 Discrete-time exponential signal a^n for (a) $a > 1$, (b) $0 < a < 1$, (c) $a < -1$, (d) $-1 < a < 0$.

When $a > 1$, the sequence grows exponentially as shown in Figure 1.13(a).

When $0 < a < 1$, the sequence decays exponentially as shown in Figure 1.13(b).

When $a < 0$, the sequence takes alternating signs as shown in Figure 1.13[(c) and (d)].

1.3.7 Complex Exponential Signal

The complex exponential signal has a general form as

$$x(t) = Ae^{st}$$

where A is the amplitude and s is a complex variable defined as

$$s = \sigma + j\omega$$

Therefore,
$$x(t) = Ae^{st} = Ae^{(\sigma + j\omega)t} = Ae^{\sigma t}e^{j\omega t}$$
$$= Ae^{\sigma t}[\cos \omega t + j \sin \omega t]$$

Depending on the values of σ and ω, we get different waveforms as shown in Figure 1.14.

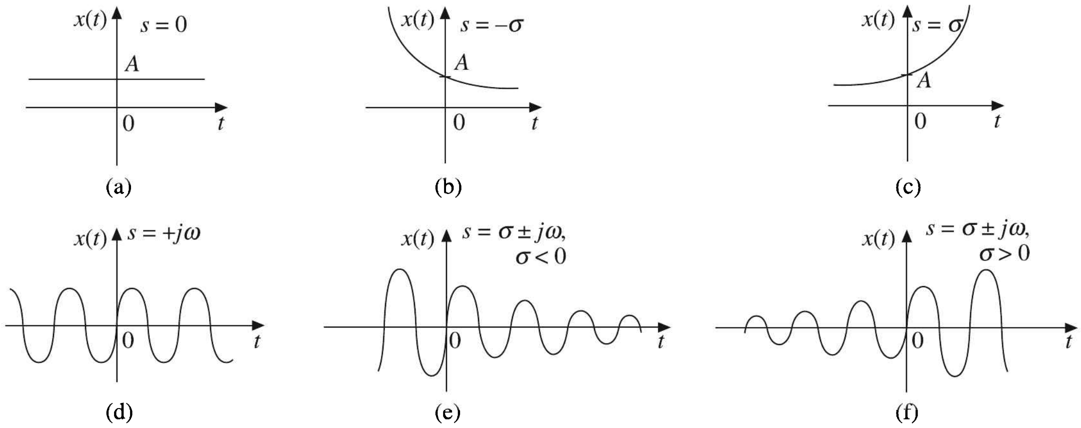


Figure 1.14 Complex exponential signals.

The discrete-time complex exponential sequence is defined as

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$
$$= a^n \cos(\omega_0 n + \phi) + ja^n \sin(\omega_0 n + \phi)$$

For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.

For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown in Figure 1.15(a).

For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially decays as shown in Figure 1.15(b)

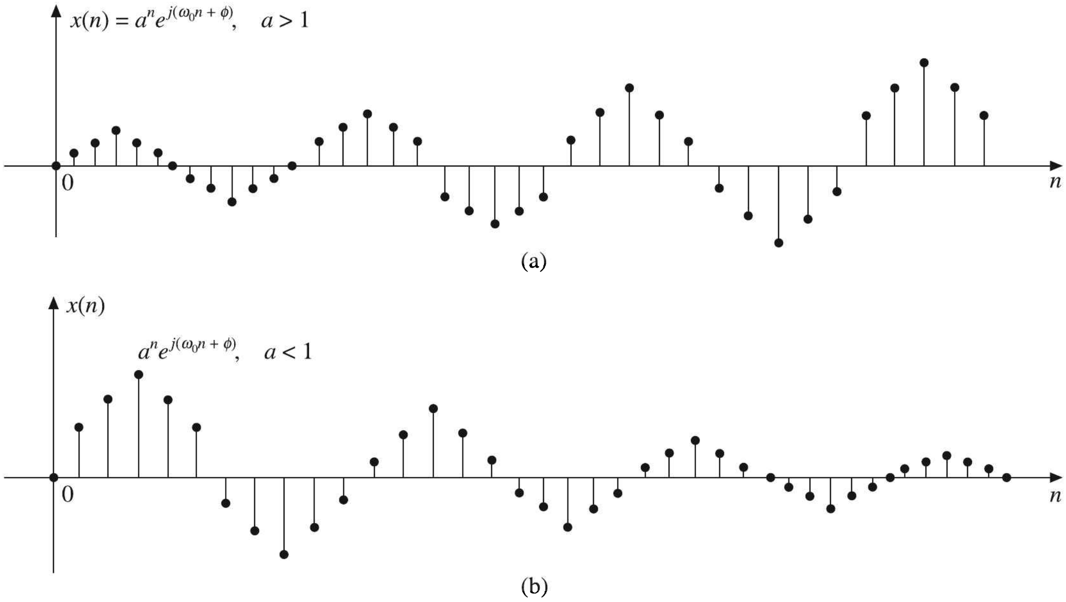


Figure 1.15 Complex exponential sequence $x(n) = a^n e^{j(\omega_0 n + \phi)}$ for (a) $a > 1$, (b) $a < 1$.

1.3.8 Rectangular Pulse Function

The unit rectangular pulse function $\Pi(t/\tau)$ shown in Figure 1.16 is defined as

$$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \le \dfrac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

It is an even function of t.

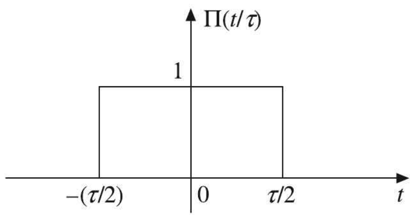


Figure 1.16 Rectangular pulse function.

1.3.9 Triangular Pulse Function

The unit triangular pulse function $\Delta(t/\tau)$ shown in Figure 1.17 is defined as:

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - (2|t|/\tau) & \text{for } |t| < (\tau/2) \\ 0 & \text{for } |t| > (\tau/2) \end{cases}$$

It is an even function of t.

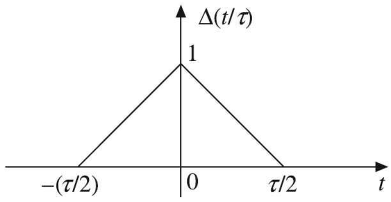


Figure 1.17 Triangular pulse function.

1.3.10 Signum Function

The unit signum function sgn (t) shown in Figure 1.18 is defined as:

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

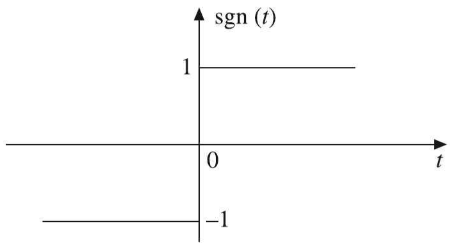


Figure 1.18 Signum function.

The signum function can be expressed in terms of unit step function as:

$$\operatorname{sgn}(t) = -1 + 2u(t)$$

1.3.11 Sinc Function

The sinc function sinc (t) shown in Figure 1.19 is defined as:

$$\operatorname{sinc}(t) = \frac{\sin t}{t} \quad \text{for } -\infty < t < \infty$$

The sinc function oscillates with period 2π and decays with increasing t. Its value is zero at $n\pi$, $n = \pm 1, \pm 2, \ldots$. It is an even function of t.

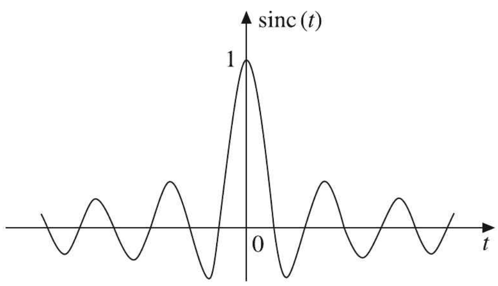


Figure 1.19 Sinc function.

1.3.12 Gaussian Function

The Gaussian function $g_a(t)$ shown in Figure 1.20 is defined as:

$$g_a(t) = e^{-at^2} \quad \text{for } -\infty < t < \infty$$

This function is extremely useful in probability theory.

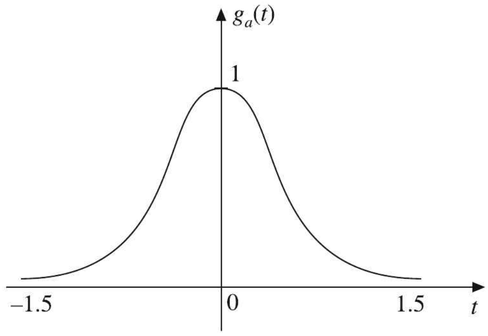


Figure 1.20 Gaussian function.

EXAMPLE 1.1 Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} e^{-at^2} \delta(t-5)\, dt$ (b) $\int_{0}^{\infty} t^2 \delta(t-6)\, dt$

(c) $\int_{0}^{3} \delta(t) \sin 5\pi t\, dt$ (d) $\int_{-\infty}^{\infty} \delta(t+2) e^{-2t}\, dt$

(e) $\int_{-\infty}^{\infty} (t-2)^3 \delta(t-2)\, dt$ (f) $\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt$

(g) $\int_{-\infty}^{\infty} [\delta(t) \cos 2t + \delta(t-2) \sin 2t]\, dt$

Solution:

(a) Given $$\int_{-\infty}^{\infty} e^{-at^2} \delta(t-5)\, dt$$

We know that $$\delta(t-5) = \begin{cases} 1 & \text{for } t = 5 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-at^2} \delta(t-5)\, dt = \left[e^{-at^2} \right]_{t=5} = e^{-25a}$$

(b) Given $\int_0^{\infty} t^2 \delta(t-6)\, dt$

We know that $\delta(t-6) = \begin{cases} 1 & \text{for } t = 6 \\ 0 & \text{elsewhere} \end{cases}$

$$\therefore \quad \int_0^{\infty} t^2 \delta(t-6)\, dt = [t^2]_{t=6} = 36$$

(c) Given $\int_0^{3} \delta(t) \sin 5\pi t\, dt$

We know that $\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\therefore \quad \int_0^{3} \delta(t) \sin 5\pi t\, dt = [\sin 5\pi t]_{t=0} = 0$$

(d) Given $\int_{-\infty}^{\infty} \delta(t+2) e^{-2t}\, dt$

We know that $\delta(t+2) = \begin{cases} 1 & \text{for } t = -2 \\ 0 & \text{elsewhere} \end{cases}$

$$\therefore \quad \int_{-\infty}^{\infty} \delta(t+2)\, e^{-2t} dt = [e^{-2t}]_{t=-2} = e^4$$

(e) Given $\int_{-\infty}^{\infty} (t-2)^3 \delta(t-2)\, dt$

We know that $\delta(t-2) = \begin{cases} 1 & \text{for } t = 2 \\ 0 & \text{elsewhere} \end{cases}$

$$\therefore \quad \int_{-\infty}^{\infty} (t-2)^3\, \delta(t-2)\, dt = [(t-2)^3]_{t=2} = 0$$

(f) Given $\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt$

We know that $\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\therefore \qquad \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = [e^{-j\omega t}]_{t=0} = 1$$

(g) Given $$\int_{-\infty}^{\infty} [\delta(t) \cos 2t + \delta(t-2) \sin 2t]\, dt$$

We know that $\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{elsewhere} \end{cases}$ and $\delta(t-2) = \begin{cases} 1 & \text{for } t = 2 \\ 0 & \text{elsewhere} \end{cases}$

$$\therefore \qquad \int_{-\infty}^{\infty} [\delta(t) \cos 2t + \delta(t-2) \sin 2t]\, dt = [\cos 2t]_{t=0} + [\sin 2t]_{t=2} = 1 + \sin 4t$$

EXAMPLE 1.2 Find the following summations

(a) $\sum_{n=-\infty}^{\infty} e^{3n}\, \delta(n-3)$

(b) $\sum_{n=-\infty}^{\infty} \delta(n-2) \cos 3n$

(c) $\sum_{n=-\infty}^{\infty} n^2 \delta(n+4)$

(d) $\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{n^2}$

(e) $\sum_{n=0}^{\infty} \delta(n+1)\, 4^n$

Solution:

(a) Given $$\sum_{n=-\infty}^{\infty} e^{3n}\, \delta(n-3)$$

We know that $$\delta(n-3) = \begin{cases} 1 & \text{for } n = 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} e^{3n} \delta(n-3) = [e^{3n}]_{n=3} = e^9$$

(b) Given $$\sum_{n=-\infty}^{\infty} \delta(n-2) \cos 3n$$

We know that $$\delta(n-2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} \delta(n-2) \cos 3n = [\cos 3n]_{n=2} = \cos 6$$

(c) Given $$\sum_{n=-\infty}^{\infty} n^2\delta(n+4)$$

We know that $$\delta(n+4)=\begin{cases}1 & \text{for } n=-4\\ 0 & \text{elsewhere}\end{cases}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} n^2\delta(n+4)=[n^2]_{n=-4}=16$$

(d) Given $$\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{n^2}$$

We know that $$\delta(n-2)=\begin{cases}1 & \text{for } n=2\\ 0 & \text{elsewhere}\end{cases}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{n^2}=[e^{n^2}]_{n=2}=e^{2^2}=e^4$$

(e) Given $$\sum_{n=0}^{\infty} \delta(n+1)\, 4^n$$

We know that $$\delta(n+1)=\begin{cases}1 & \text{for } n=-1\\ 0 & \text{for } n\neq -1\end{cases}$$

$\therefore$ $$\sum_{n=0}^{\infty} \delta(n+1)\, 4^n=0$$

EXAMPLE 1.3 Prove the properties of impulse function.

Solution: First property $$\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt=x(0)$$

Proof: Let $x(t)$ be continuous at $t = 0$; the value of $x(t)$ at $t = 0$ is $x(0)$. The impulse $\delta(t)$ exists only at $t = 0$. For all other time $\delta(t) = 0$. Therefore, the integration of $x(t)\,\delta(t)$ from $t = -\infty$ to ∞ has a value only at $t = 0$.

$\therefore$ $$\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt=\int_{-\infty}^{\infty} x(0)\,\delta(t)\,dt=x(0)\int_{-\infty}^{\infty}\delta(t)\,dt=x(0)$$

Second property $$x(t)\,\delta(t-t_0)=x(t_0)\,\delta(t-t_0)$$

Proof: Let the signal $x(t)$ be continuous at $t = t_0$. The value of $x(t)$ at $t = t_0$ is $x(t_0)$. $\delta(t-t_0)$ is an impulse function that exists only at $t = t_0$. Therefore,

$$x(t)\,\delta(t-t_0)=x(t_0)\,\delta(t-t_0)$$

Third property $$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt=x(t_0)$$

Proof: Let $x(t)$ be continuous at $t = t_0$ and let its value at $t = t_0$ be $x(t_0)$. We know that $\delta(t - t_0) = 1$ only for $t = t_0$. For all other time it is equal to zero. Therefore, the integration of the product term $x(t)\ \delta(t - t_0)$ from $-\infty$ to ∞ has a value only at $t = t_0$.

$$\therefore \qquad \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = x(t_0)$$

Fourth property $$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$$

Proof: Let $x(\tau)$ be continuous at $\tau = t$. Let the value of $x(\tau)$ at $\tau = t$ be $x(t)$. We know that $\delta(t - \tau) = 1$ only at $\tau = t$. For all other τ it is zero. So the integration of the product $x(\tau)\ \delta(t - \tau)$ has a value only at $\tau = t$.

$$\therefore \qquad \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(\tau)\big|_{\tau=t} = x(t)$$

This is the formula for convolution of $x(t)$ with $\delta(t)$. This property says that the convolution of any signal with an impulse results in the original signal itself.

Fifth property $$\delta(at) = \frac{1}{|a|}\delta(t) \qquad \text{[Scaling property]}$$

Proof: Let $x(t)$ be some function. Consider the integral

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt \quad \text{for } a > 0$$

Let $$at = \tau$$

$$\therefore \qquad t = \frac{\tau}{a} \quad \text{and} \quad dt = \frac{d\tau}{a}$$

If $a > 0$ $$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x\left(\frac{\tau}{a}\right)\delta(\tau)\,d\tau$$

$$= \left[\frac{1}{a}x\left(\frac{\tau}{a}\right)\right]_{\tau=0}$$

$$= \frac{1}{a}x(0)$$

Similarly, for $a < 0$ $$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{-a}x(0)$$

$$\therefore \qquad \int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{|a|}x(0)$$

Now, consider $\frac{1}{|a|}x(0)$

We know that $x(0) = \int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt$

$$\therefore \quad \frac{1}{|a|}x(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\,\frac{1}{|a|}\,\delta(t)\,dt$$

which indicates that $\delta(at) = \frac{1}{|a|}\delta(t)$

Sixth property $\delta(t) = \delta(-t)$

i.e. impulse function is an even function.

Proof: Consider the scaling property,

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Let $a = -1$

$$\therefore \quad \delta(-t) = \frac{1}{|-1|}\delta(t) = \delta(t)$$

1.4 BASIC OPERATIONS ON SIGNALS

When we process a signal, this signal may undergo several manipulations involving the independent variable or the amplitude of the signal. The basic operations on signals are as follows:

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

The first three operations correspond to transformation in independent variable t or n of a signal. The last three operations correspond to transformation on amplitude of a signal.

1.4.1 Time Shifting

Mathematically, the time shifting of a continuous-time signal $x(t)$ can be represented by

$$y(t) = x(t - T)$$

The time shifting of a signal may result in time delay or time advance. In the above equation if T is positive the shift is to the right and then the shifting delays the signal, and if T is negative the shift is to the left and then the shifting advances the signal. An arbitrary signal $x(t)$, its delayed version and advanced version are shown in Figure 1.21[(a), (b) and (c)]. Shifting a signal in time means that a signal may be either advanced in the time axis or delayed in the time axis.

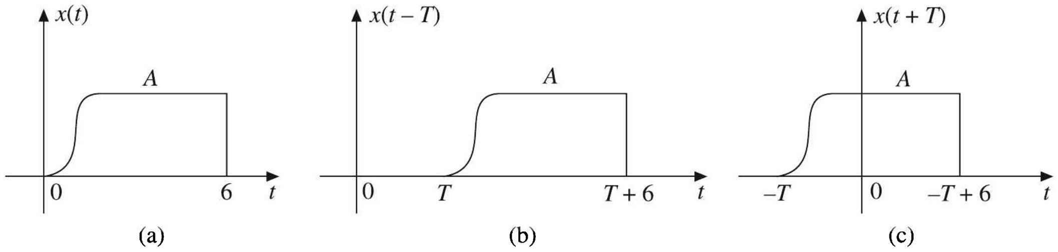


Figure 1.21 (a) Signal, (b) Its delayed version, (c) Its time advanced version.

Similarly, the time shifting operation of a discrete-time signal $x(n)$ can be represented by

$$y(n) = x(n - k)$$

This shows that the signal $y(n)$ can be obtained by time shifting the signal $x(n)$ by k units. If k is positive, it is delay and the shift is to the right, and if k is negative, it is advance and the shift is to the left.

An arbitrary signal $x(n)$ is shown in Figure 1.22(a). $x(n - 3)$ which is obtained by shifting $x(n)$ to the right by 3 units [i.e. delay $x(n)$ by 3 units] is shown in Figure 1.22(b). $x(n + 2)$ which is obtained by shifting $x(n)$ to the left by 2 units (i.e. advancing $x(n)$ by 2 units) is shown in Figure 1.22(c).

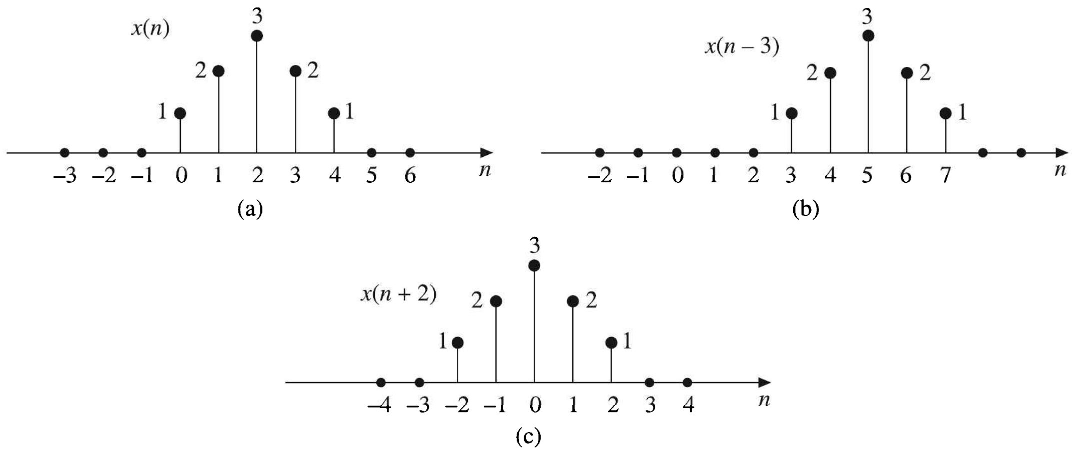


Figure 1.22 (a) Sequence $x(n)$, (b) $x(n - 3)$, (c) $x(n + 2)$.

1.4.2 Time Reversal

The time reversal, also called time folding of a signal $x(t)$ can be obtained by folding the signal about $t = 0$. This operation is very useful in convolution. It is denoted by $x(-t)$. It is obtained by replacing the independent variable t by $(-t)$. Folding is also called as the reflection of the signal about the time origin $t = 0$. Figure 1.23(a) shows an arbitrary signal $x(t)$, and Figure 1.23(b) shows its reflection $x(-t)$.

The signal $x(-t + 3)$ obtained by shifting the reversed signal $x(-t)$ to the right by 3 units (delay by 3 units) is shown in Figure 1.23(c). The signal $x(-t - 3)$ obtained by shifting the reversed signal $x(-t)$ to the left by 3 units (advance by 3 units) is shown in Figure 1.23(d).

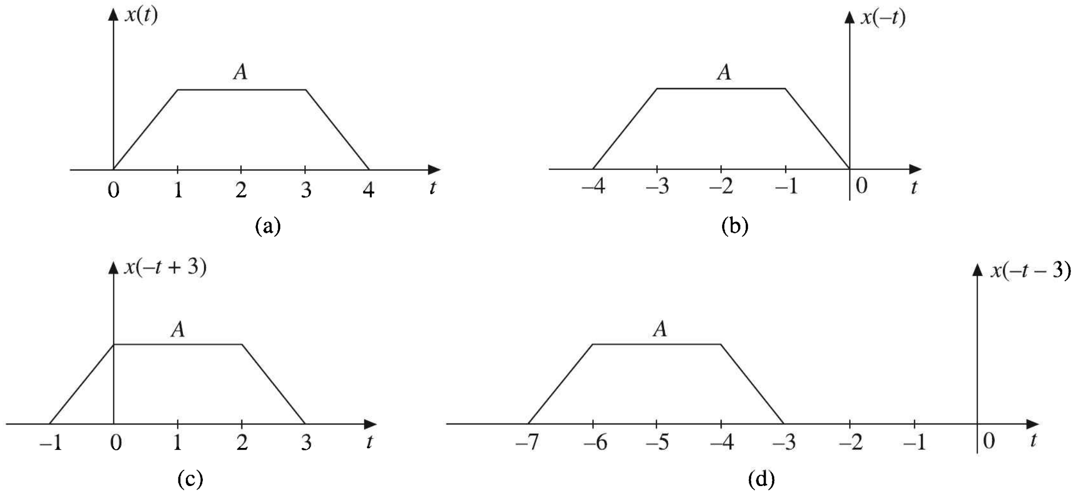


Figure 1.23 (a) An arbitrary signal $x(t)$, (b) Time reversed signal $x(-t)$, (c) Time reversed and delayed signal $x(-t + 3)$, (d) Time reversed and advanced signal $x(-t - 3)$.

Other examples for time reversal operation are shown in Figure 1.24.

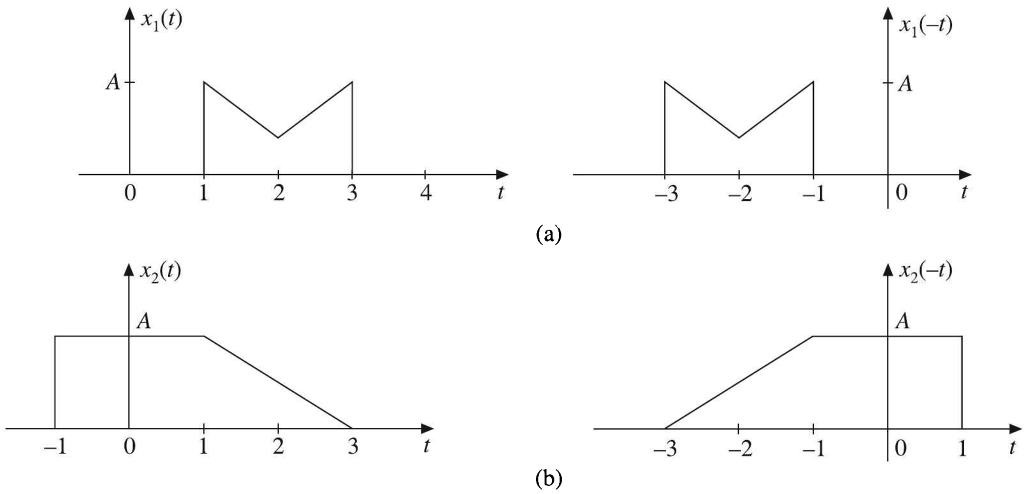


Figure 1.24 Time reversal operations.

The time reversal of a discrete-time signal $x(n)$ can be obtained by folding the sequence about $n = 0$. Figure 1.25(a) shows an arbitrary discrete-time signal $x(n)$ and its time reversed version $x(-n)$ is shown in Figure 1.25(b). Figure 1.25[(c) and (d)] shows the delayed and advanced versions of reversed signal $x(-n)$.

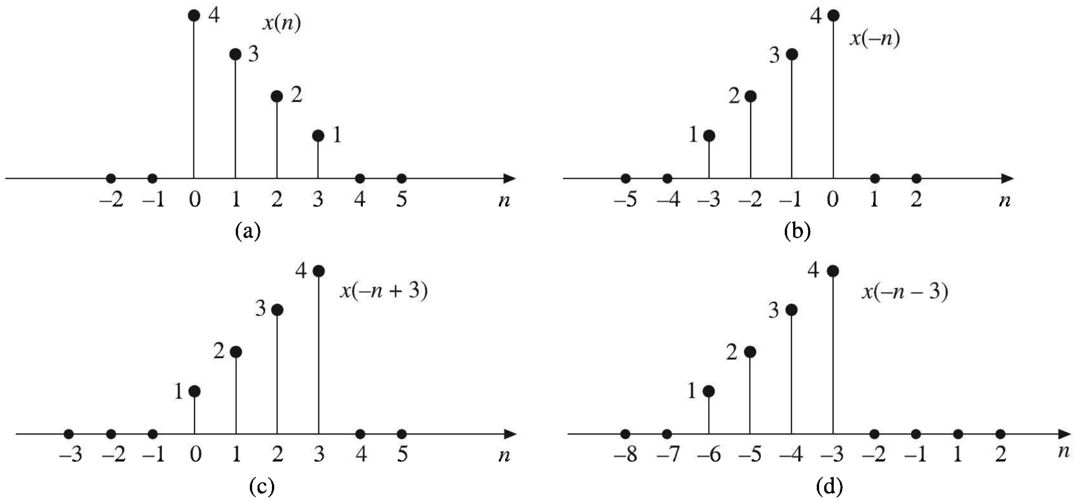


Figure 1.25 (a) Original signal $x(n)$, (b) Time reversed signal $x(-n)$, (c) Time reversed and delayed signal $x(-n + 3)$, (d) Time reversed and advanced signal $x(-n - 3)$.

The signal $x(-n + 3)$ is obtained by delaying (shifting to the right) the time reversed signal $x(-n)$ by 3 units of time. The signal $x(-n - 3)$ is obtained by advancing (shifting to the left) the time reversed signal $x(-n)$ by 3 units of time.

Figure 1.26 shows other examples for time reversal of signals.

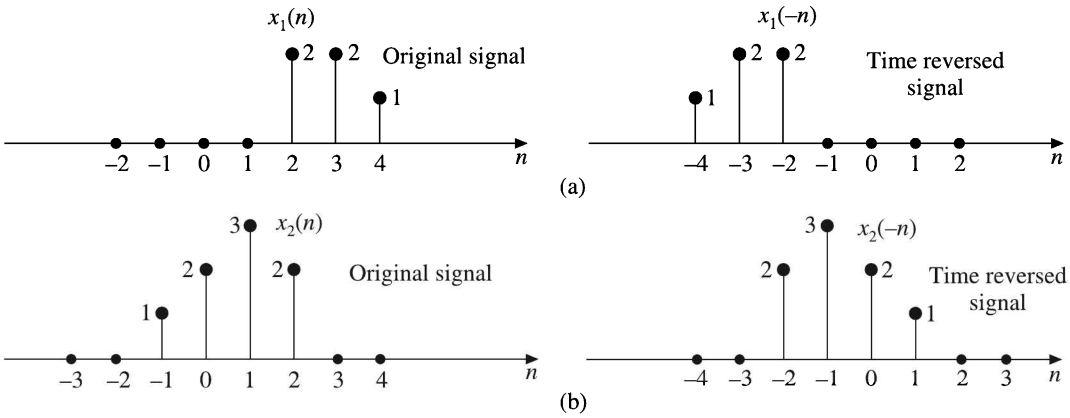


Figure 1.26 Time reversal operations.

EXAMPLE 1.4 Sketch the following signals

(a) $u(-t + 2)$ (b) $-2u(t + 2)$
(c) $-4r(t)$ (d) $2r(t - 2)$
(e) $r(-t + 3)$ (f) $\Pi(t - 2)$

Solution:

(a) Given $x(t) = u(-t + 2)$

The signal $u(-t + 2)$ can be obtained by first drawing the unit step signal $u(t)$ as shown in Figure 1.27(a), then time reversing the signal $u(t)$ about $t = 0$ to obtain

$u(-t)$ as shown in Figure 1.27(b), and then shifting the reversed signal to the right by 2 units of time to obtain $u(-t + 2)$ as shown in Figure 1.27(c).

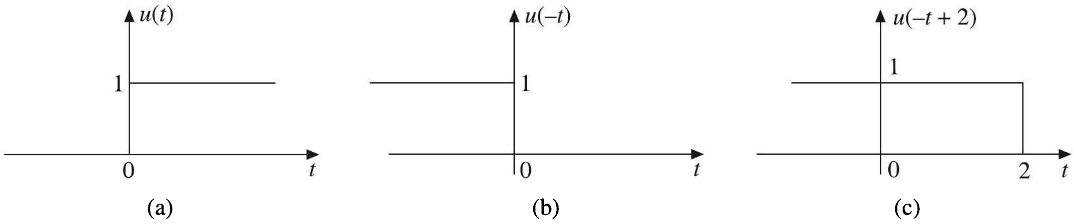


Figure 1.27 (a) Unit step signal, (b) Folded unit step signal, (c) Delayed folded signal.

(b) Given $$x(t) = -2u(t + 2)$$

The signal $-2u(t + 2)$ can be obtained by first drawing the unit step signal $u(t)$ as shown in Figure 1.28(a), then shifting the signal $u(t)$ to the left by 2 units of time to obtain $u(t + 2)$ as shown in Figure 1.28(b), and then multiplying that signal $u(t + 2)$ by -2 to obtain $-2u(t + 2)$ as shown in Figure 1.28(c).

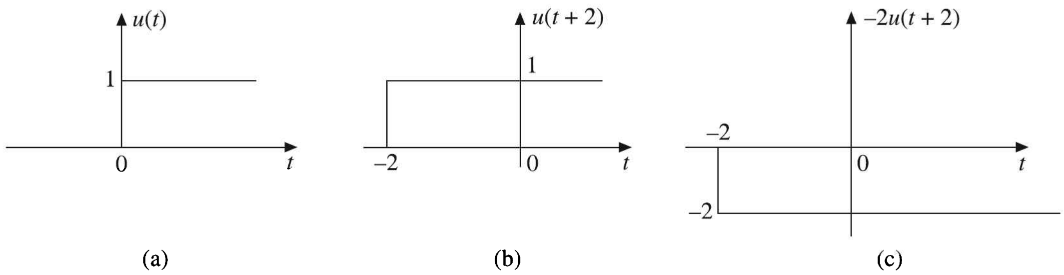


Figure 1.28 (a) Unit step signal $u(t)$, (b) Shifted signal $u(t + 2)$, (c) Scaled signal $-2u(t + 2)$.

(c) Given $$x(t) = -4r(t)$$

The signal $x(t)$ is a ramp signal with a slope of -4 as shown in Figure 1.29.

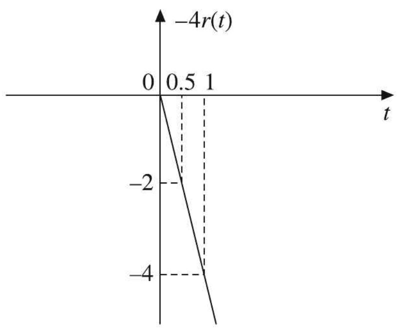


Figure 1.29 Ramp signal $x(t) = -4r(t)$.

(d) Given $$x(t) = 2r(t - 2)$$

The signal $2r(t - 2)$ can be obtained by first drawing the ramp signal $2r(t)$ with slope of 2 as shown in Figure 1.30(a), and then shifting it to the right by 2 units to obtain $2r(t - 2)$ as shown in Figure 1.30(b).

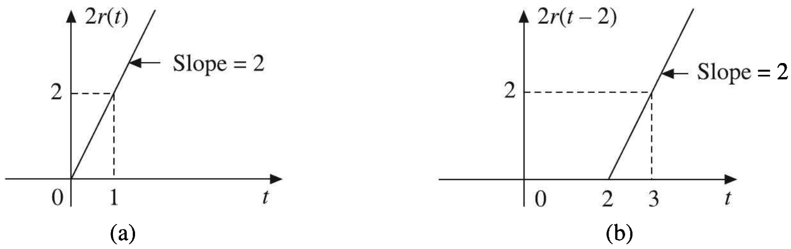


Figure 1.30 (a) Scaled ramp, (b) Delayed scaled ramp.

(e) Given $$x(t) = r(-t + 3)$$

The signal $r(-t + 3)$ can be obtained by first drawing the unit ramp signal $r(t)$ as shown in Figure 1.31(a), folding the signal $r(t)$ about $t = 0$ to obtain $r(-t)$ as shown in Figure 1.31(b) and then shifting it to the right (delaying) by 3 units of time to obtain $r(-t + 3)$ as shown in Figure 1.31(c).

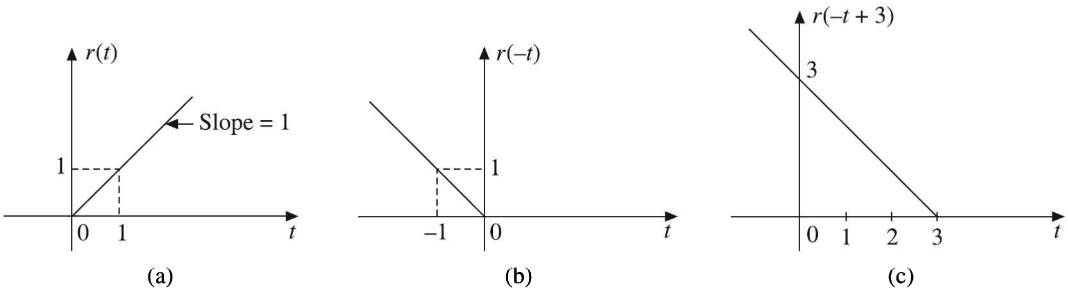


Figure 1.31 (a) Ramp signal, (b) Folded ramp, (c) Delayed folded ramp.

(f) Given $$x(t) = \Pi(t - 2)$$

The signal $\Pi(t - 2)$ can be obtained by first drawing $\Pi(t)$ as shown in Figure 1.32(a) and then shifting it to the right by 2 units to obtain $\Pi(t - 2)$ as shown in Figure 1.32(b).

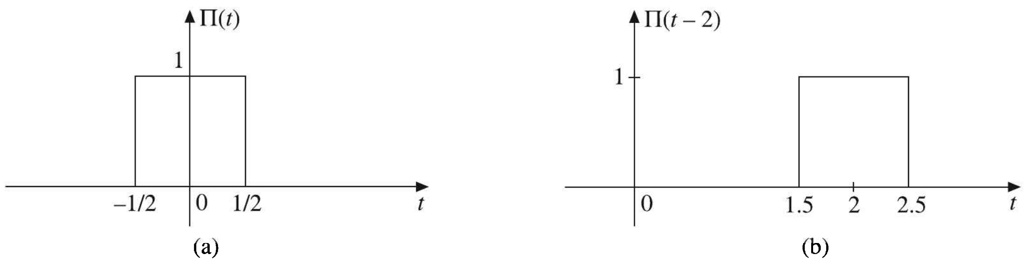


Figure 1.32 (a) Signal $\Pi(t)$, (b) Signal $\Pi(t - 2)$.

EXAMPLE 1.5 Sketch the following signals

(a) $u(n+2)\,u(-n+3)$ (b) $x(n) = u(n+4) - u(n-2)$

Solution:

(a) Given $$x(n) = u(n+2)\,u(-n+3)$$

The signal $u(n + 2)\,u(-n + 3)$ can be obtained by first drawing the signal $u(n + 2)$ as shown in Figure 1.33(a), then drawing $u(-n + 3)$ as shown in Figure 1.33(b) and then multiplying these sequences element by element to obtain $u(n + 2)\,u(-n + 3)$ as shown in Figure 1.33(c).

$x(n) = 0$ for $n < -2$ and $n > 3$; $x(n) = 1$ for $-2 < n < 3$

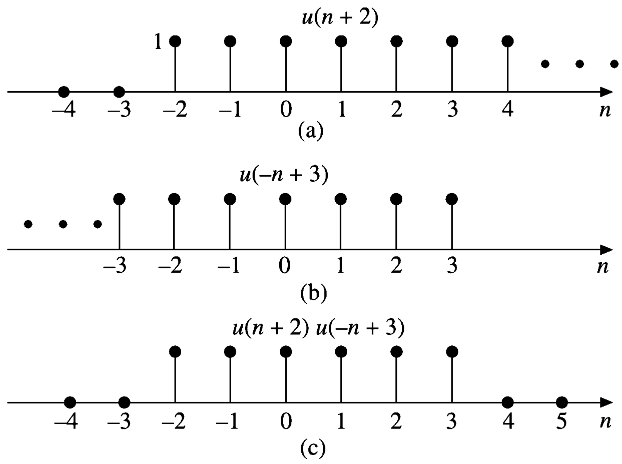


Figure 1.33 Plots of (a) $u(n + 2)$, (b) $u(-n + 3)$, (c) $u(n + 2)\,u(-n + 3)$.

(b) Given $$x(n) = u(n + 4) - u(n - 2)$$

The signal $u(n + 4) - u(n - 2)$ can be obtained by first plotting $u(n + 4)$ as shown in Figure 1.34(a), then plotting $u(n - 2)$ as shown in Figure 1.34(b) and then subtracting each element of $u(n - 2)$ from the corresponding element of $u(n + 4)$ to obtain the result shown in Figure 1.34(c).

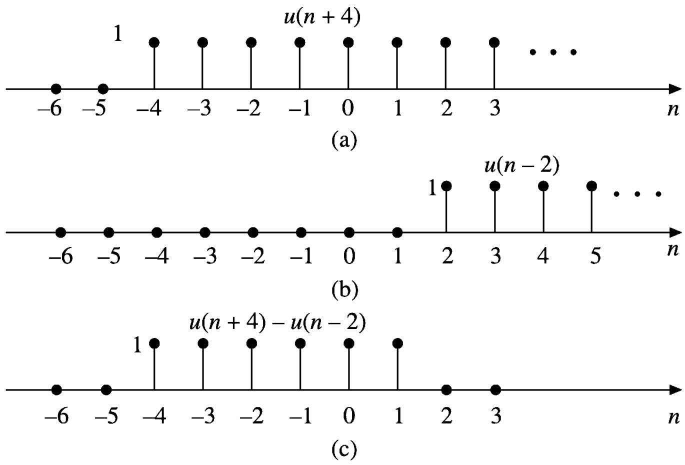


Figure 1.34 Plots of (a) $u(n + 4)$, (b) $u(n - 2)$, (c) $u(n + 4) - u(n - 2)$.

1.4.3 Amplitude Scaling

The amplitude scaling of a continuous-time signal $x(t)$ can be represented by

$$y(t) = Ax(t)$$

where A is a constant.

The amplitude of $y(t)$ at any instant is equal to A times the amplitude of $x(t)$ at that instant, but the shape of $y(t)$ is same as the shape of $x(t)$. If $A > 1$, it is amplification and if $A < 1$, it is attenuation.

Here the amplitude is rescaled. Hence the name amplitude scaling. Figure 1.35(a) shows an arbitrary signal $x(t)$ and Figure 1.35(b) shows $y(t) = 2x(t)$.

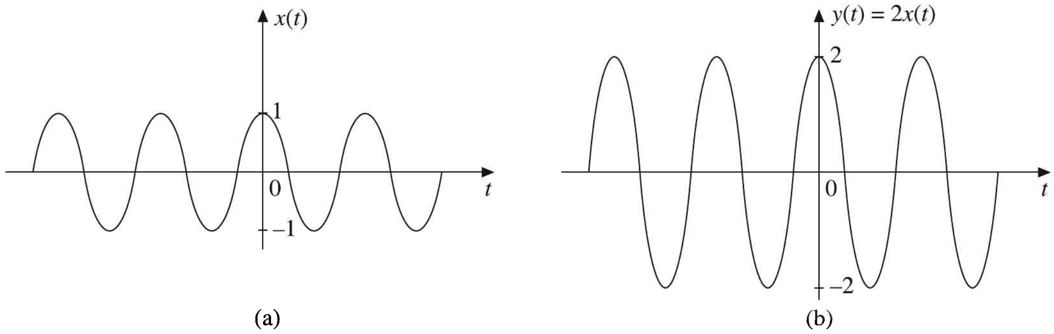


Figure 1.35 Plots of (a) $x(t) = \cos \omega t$, (b) $y(t) = 2x(t)$.

Similarly, the amplitude scaling of a discrete-time signal can be represented by

$$y(n) = ax(n)$$

where a is a constant.

Figure 1.36(a) shows a signal $x(n)$ and Figure 1.36(b) shows a scaled signal $y(n) = 2x(n)$.

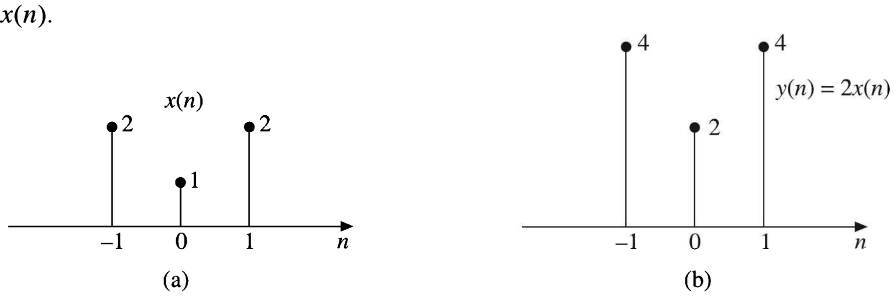


Figure 1.36 Plots of (a) Signal $x(n)$, (b) $y(n) = 2x(n)$.

1.4.4 Time Scaling

Time scaling may be time expansion or time compression. The time scaling of a signal $x(t)$ can be accomplished by replacing t by at in it. Mathematically, it can be expressed as:

$$y(t) = x(at)$$

If $a > 1$, it results in time compression by a factor a and if $a < 1$, it results in time expansion by a factor a because with that transformation a point at 'at' in signal $x(t)$ becomes a point at 't' in $y(t)$.

Consider a signal shown in Figure 1.37(a). For a transformation $y(t) = x(2t)$, the time compressed signal is as shown in Figure 1.37(b) and for a transformation $y(t) = x(t/2)$ the time expanded signal is as shown in Figure 1.37(c).

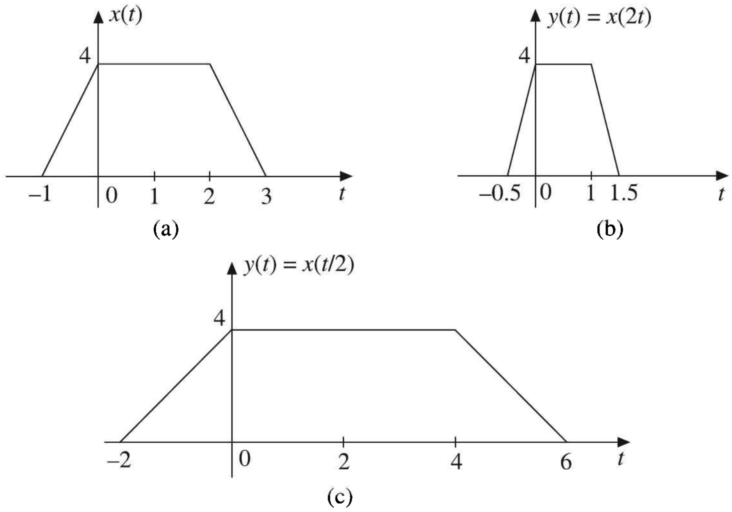


Figure 1.37 (a) Original signal, (b) Compressed signal, (c) Enlarged signal.

Observe that in Figure 1.37(a), $x(t)$ is increasing linearly from 0 to 4 in the interval $t = -1$ to $t = 0$ and remaining constant at 4 in the interval $t = 0$ to $t = 2$ and decreasing linearly from 4 to 0 in the interval $t = 2$ to $t = 3$.

In Figure 1.37(b), the time scaled (compressed) signal $x(2t)$ increases linearly from 0 to 4 in the interval $t = -(1/2)$ to $t = (0/2)$, remains constant at 4 from $t = (0/2)$ to $t = (2/2)$ and then decreases linearly from 4 to 0 in the interval $t = (2/2)$ to $t = (3/2)$.

In Figure 1.37(c), the time scaled (expanded) signal $x(t/2)$ increases linearly from 0 to 4, in the interval $t = -1 \times 2$ to $t = 0 \times 2$, remains constant at 4 in the interval $t = 0 \times 2$ to $t = 2 \times 2$ and then decreases linearly from 4 to 0 in the interval $t = 2 \times 2$ to $t = 2 \times 3$.

In the discrete-time case, we can write the time scaling as follows.

$$y(n) = x(an)$$

again when $a > 1$, it is time compression and when $a < 1$, it is time expansion.

Let $x(n)$ be a sequence as shown in Figure 1.38(a). If $a = 2$, $y(n) = x(2n)$. Then we can plot the time scaled signal $y(n)$ by substituting different values for n as shown in Figure 1.38(b). Here the signal is compressed by 2.

$$y(0) = x(0) = 1$$
$$y(-1) = x(-2) = 3$$
$$y(-2) = x(-4) = 0$$
$$y(1) = x(2) = 3$$
$$y(2) = x(4) = 0$$

and so on.

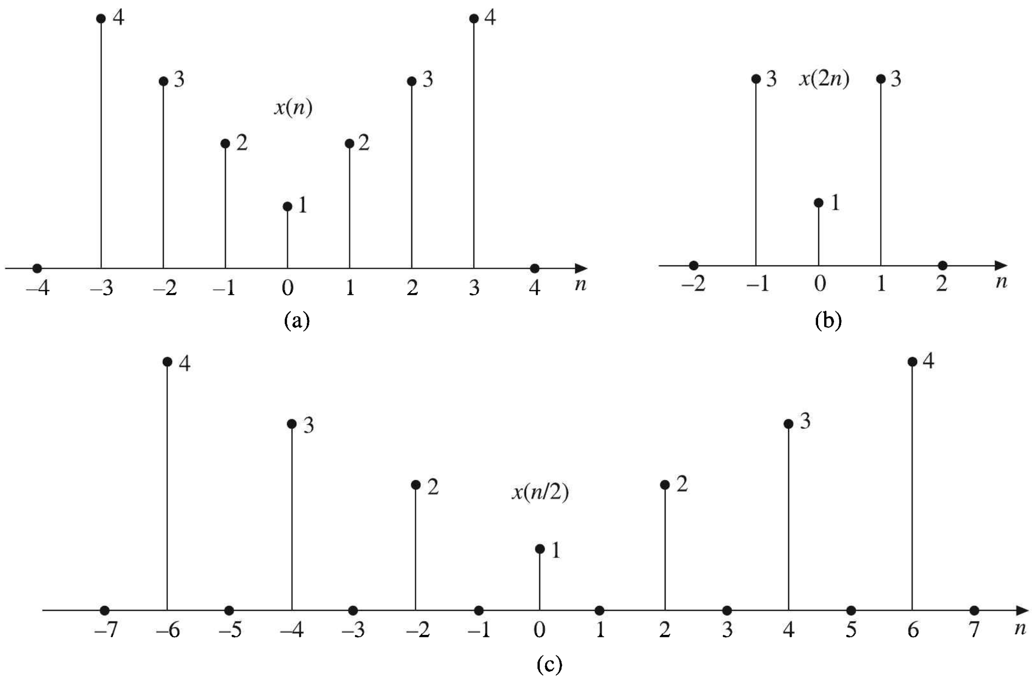


Figure 1.38 Discrete-time scaling (a) Plot of $x(n)$, (b) Plot of $x(2n)$, (c) Plot of $x(n/2)$.

So to plot $x(2n)$ we have to skip odd numbered samples in $x(n)$.

If $a = (1/2)$, $y(n) = x(n/2)$, then

$$y(0) = x(0) = 1$$

$$y(2) = x(1) = 2$$

$$y(4) = x(2) = 3$$

$$y(6) = x(3) = 4$$

$$y(8) = x(4) = 0$$

$$y(-2) = x(-1) = 2$$

$$y(-4) = x(-2) = 3$$

$$y(-6) = x(-3) = 4$$

$$y(-8) = x(-4) = 0$$

We can plot $y(n) = x(n/2)$ as shown in Figure 1.38(c). Here the signal is expanded by 2. All odd components in $x(n/2)$ are zero because $x(n)$ does not have any value in between the sampling instants.

Time scaling is very useful when data is to be fed at some rate and is to be taken out at a different rate.

1.4.5 Signal Addition

The sum of two continuous-time signals $x_1(t)$ and $x_2(t)$ can be obtained by adding their values at every instant of time. Similarly, the subtraction of one continuous-time signal $x_2(t)$ from another signal $x_1(t)$ can be obtained by subtracting the value of $x_2(t)$ from that of $x_1(t)$ at every instant. Consider two signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.39[(a) and (b)].

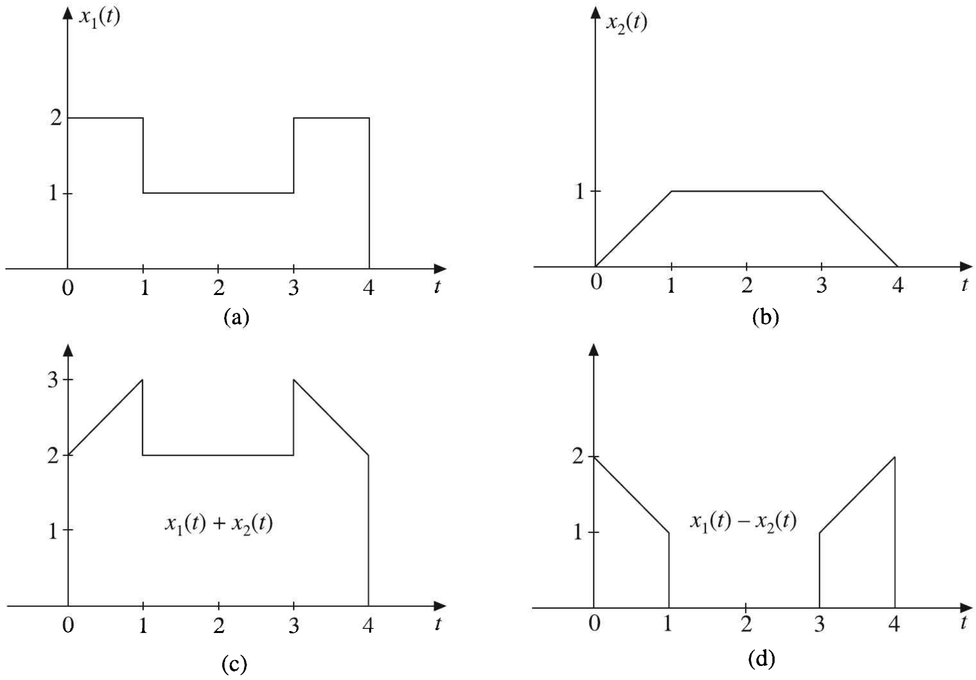


Figure 1.39 Addition and subtraction of continuous-time signals.

The addition of those two signals $x_1(t)$ and $x_2(t)$ can be obtained by considering different time intervals as follows:

For $0 \le t \le 1$ $\quad x_1(t) = 2$ and $x_2(t)$ is rising linearly from 0 to 1.
Hence $x_1(t) + x_2(t)$ will rise linearly from $2 + 0 = 2$ to $2 + 1 = 3$.

For $1 \le t \le 3$ $\quad x_1(t) = 1$ and $x_2(t) = 1$
Hence $x_1(t) + x_2(t)$ will be equal to $1 + 1 = 2$.

For $3 \le t \le 4$ $\quad x_1(t) = 2$ and $x_2(t)$ falls linearly from 1 to 0.
Hence $x_1(t) + x_2(t)$ will fall linearly from $2 + 1 = 3$ to $2 + 0 = 2$.

The sum $x_1(t) + x_2(t)$ is as shown in Figure 1.39(c).

The subtraction of signal $x_2(t)$ from $x_1(t)$ can be performed by considering different time intervals as follows:

For $0 \le t \le 1$ $\quad x_1(t) = 2$ and $x_2(t)$ rises linearly from 0 to 1.
Hence $x_1(t) - x_2(t)$ falls linearly from $2 - 0 = 2$ to $2 - 1 = 1$.

For $1 \le t \le 3$ $x_1(t) = 1$ and $x_2(t) = 1$

Hence $x_1(t) - x_2(t) = 1 - 1 = 0$

For $3 \le t \le 4$ $x_1(t) = 2$ and $x_2(t)$ is falling linearly from 1 to 0.

Hence $x_1(t) - x_2(t)$ rises linearly from $2 - 1 = 1$ to $2 - 0 = 2$.

The difference $x_1(t) - x_2(t)$ is as shown in Figure 1.39(d).

In discrete-time domain, the sum of two signals $x_1(n)$ and $x_2(n)$ can be obtained by adding the corresponding sample values and the subtraction of $x_2(n)$ from $x_1(n)$ can be obtained by subtracting each sample of $x_2(n)$ from the corresponding sample of $x_1(n)$ as illustrated below.

If $x_1(n) = \{1, 2, 3, 1, 5\}$

and $x_2(n) = \{2, 3, 4, 1, -2\}$

Then $x_1(n) + x_2(n) = \{1 + 2, 2 + 3, 3 + 4, 1 + 1, 5 - 2\} = \{3, 5, 7, 2, 3\}$

and $x_1(n) - x_2(n) = \{1 - 2, 2 - 3, 3 - 4, 1 - 1, 5 + 2\} = \{-1, -1, -1, 0, 7\}$

1.4.6 Signal Multiplication

The multiplication of two continuous-time signals can be performed by multiplying their values at every instant. Two continuous-time signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.40[(a) and (b)] are multiplied as shown below to obtain $x_1(t)\, x_2(t)$ shown in Figure 1.40(c).

For $0 \le t \le 1$ $x_1(t) = 2$ and $x_2(t) = 1$

Hence $x_1(t)\, x_2(t) = 2 \times 1 = 2$

For $1 \le t \le 2$ $x_1(t) = 1$ and $x_2(t) = 1 + (t - 1)$

Hence $x_1(t)\, x_2(t) = (1)[1 + (t - 1)] = 1 + (t - 1)$

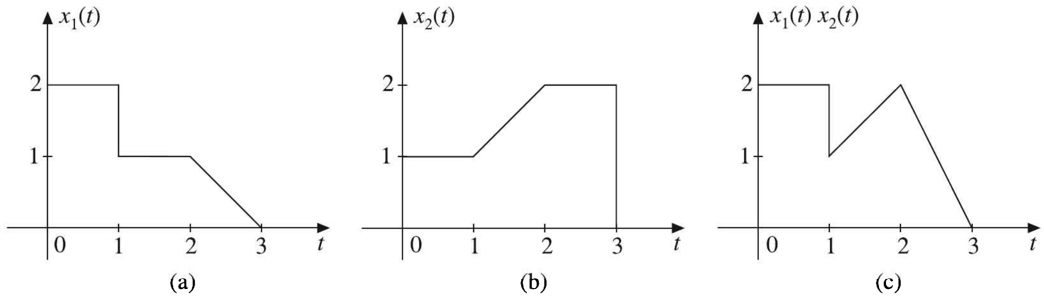


Figure 1.40 Multiplication of continuous-time signals.

For $2 \le t \le 3$ $x_1(t) = 1 - (t - 2)$ and $x_2(t) = 2$

Hence $x_1(t)\, x_2(t) = [1 - (t - 2)]\, 2 = 2 - 2(t - 2)$

Multiplication of two discrete-time sequences can be performed by multiplying their values at the sampling instants.

If $\quad x_1(n) = \{1, -3, 2, 4, 1.5\}$ and $x_2(n) = \{2, -1, 3, 1.5, 2\}$

Then
$$x_1(n)\,x_2(n) = \{1 \times 2, -3 \times -1, 2 \times 3, 4 \times 1.5, 1.5 \times 2\}$$
$$= \{2, 3, 6, 6, 3\}$$

EXAMPLE 1.6 For the signal $x(t)$ shown in Figure 1.41, find the signals.

(a) $x(t - 3)$ and $x(t + 3)$ (b) $x(2t + 2)$ and $x\left(\frac{1}{2}t - 2\right)$

(c) $x\left(\frac{5}{3}t\right)$ and $x\left(\frac{3}{5}t\right)$ (d) $x(-t + 2)$ and $x(-t - 2)$

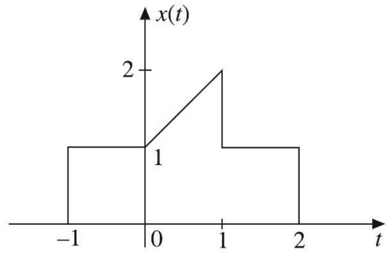


Figure 1.41 Signal for Example 1.6.

Solution:

(a) The signal $x(t - 3)$ can be obtained by shifting $x(t)$ to the right (delay) by 3 units as shown in Figure 1.42(a).

The signal $x(t + 3)$ can be obtained by shifting $x(t)$ to the left (advance) by 3 units as shown in Figure 1.42(b).

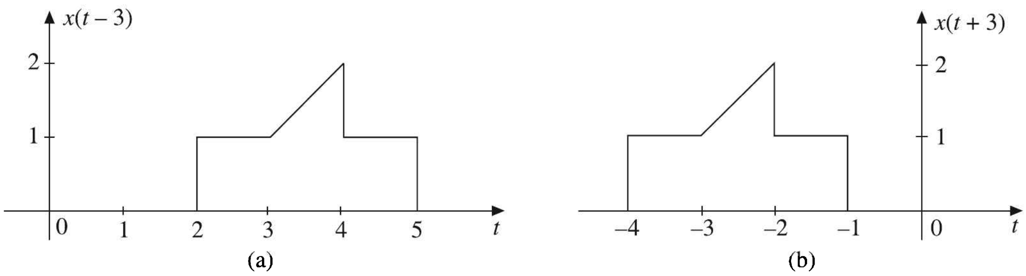


Figure 1.42 (a) Time delay, (b) Time advance.

(b) The signal $x(2t + 2)$ can be obtained by first advancing $x(t)$ by two units to get $x(t + 2)$ as shown in Figure 1.43(a) and then time scaling (compression) $x(t + 2)$ by a factor of 2 as shown in Figure 1.43(b).

The signal $x\left(\frac{1}{2}t - 2\right)$ can be obtained by first time delaying $x(t)$ by 2 units to obtain $x(t - 2)$ as shown in Figure 1.43(c) and then time scaling (expanding) $x(t - 2)$ by a factor of (1/2) as shown in Figure 1.43(d).

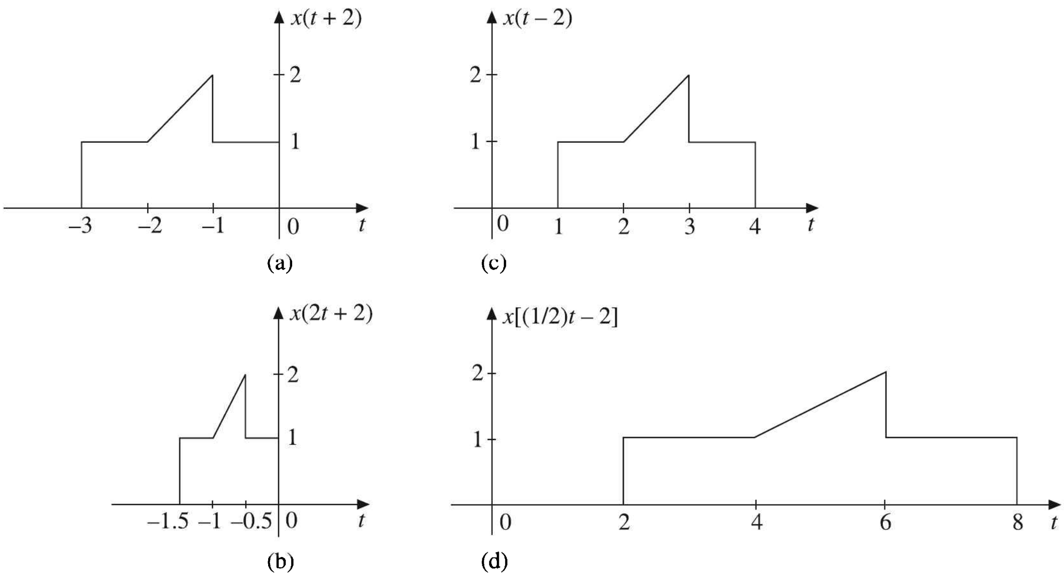


Figure 1.43 (a) Time advance, (b) Time compression, (c) Time delay, (d) Time expansion.

(c) The signal $x[(5/3)t]$ can be obtained by time scaling $x(t)$ by a factor of (5/3) i.e. by compressing the signal $x(t)$ by (3/5) times as shown in Figure 1.44(a).

The signal $x[(3/5)t]$ can be obtained by time scaling $x(t)$ by a factor of (3/5), i.e. by expanding $x(t)$ by (5/3) times as shown in Figure 1.44(b). The zero point remains as it is because $0 \times a = 0$ itself.

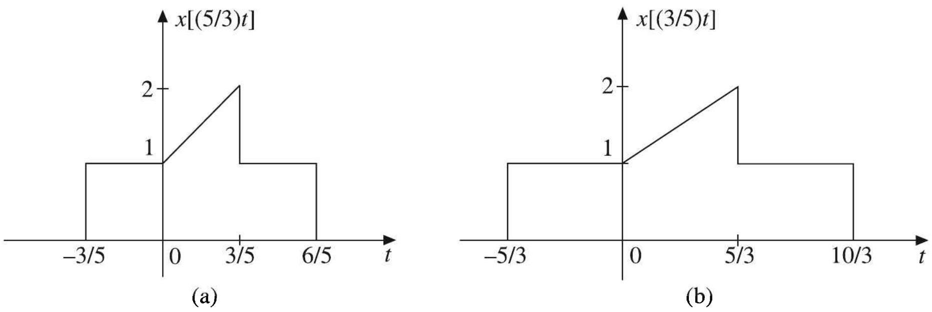


Figure 1.44 (a) Time compression, (b) Time expansion.

(d) The signal $x(-t + 2)$ can be obtained by folding the signal $x(t)$ about $t = 0$ to obtain the time reversed signal $x(-t)$ as shown in Figure 1.45(a) and then shifting $x(-t)$ to the right by 2 units as shown in Figure 1.45(b).

The signal $x(-t - 2)$ can be obtained by shifting $x(-t)$ to the left by two units as shown in Figure 1.45(c).

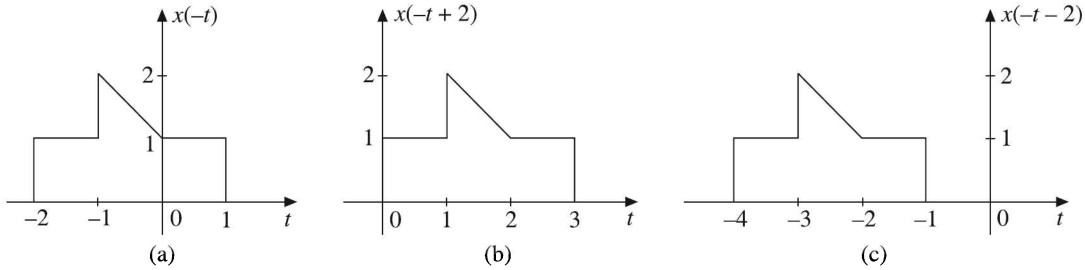


Figure 1.45 (a) Time reversal, (b) Time delay, and (c) Time advance operations on $x(t)$.

EXAMPLE 1.7 Sketch the following signals

(a) $2u(t + 2) - 2u(t - 3)$ (b) $u(t + 4)\, u(-t + 4)$

(c) $\Pi\left(\dfrac{t-2}{2}\right) + \Pi(2t - 3.5)$ (d) $r(t) - r(t-1) - r(t-3) + r(t-4)$

(e) $r(-t)\, u(t+2)$ (f) $r(-0.25t + 1)$

Solution:

(a) Given $2u(t + 2) - 2u(t - 3)$

The signal $u(t)$ is shown in Figure 1.46(a). The signal $2u(t + 2)$ is obtained by shifting $u(t)$ to the left by 2 units and multiplying by 2 as shown in Figure 1.46(b).

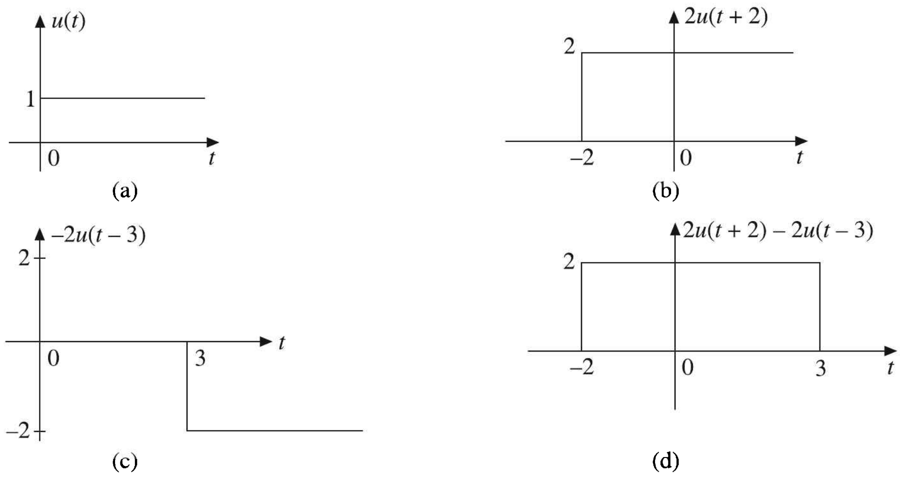


Figure 1.46 Waveforms for Example 1.7(a).

The signal $-2u(t - 3)$ is obtained by shifting $u(t)$ to the right by 3 units and multiplying by -2 as shown in Figure 1.46(c). The signal $2u(t + 2) - 2u(t - 3)$ obtained by adding signals $2u(t + 2)$ and $-2u(t - 3)$ is shown in Figure 1.46(d).

$$2u(t+2) = 2 \quad \text{for } t \geq -2$$

and $$-2u(t-3) = -2 \quad \text{for } t \geq 3$$

Therefore, $$2u(t+2) - 2u(t-3) = \begin{cases} 2 & \text{for } -2 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) Given $$u(t + 4)\, u(-t + 4)$$

The signal $u(t)$ is shown in Figure 1.47(a). The signal $u(t + 4)$ is obtained by shifting $u(t)$ to the left by 4 units as shown in Figure 1.47(b). The signal $u(-t + 4)$ is obtained by reversing $u(t)$ to obtain $u(-t)$ and then shifting $u(-t)$ to the right by 4 units as shown in Figure 1.47(c). The signal $u(t + 4)\, u(-t + 4)$ obtained by multiplying the signals $u(t + 4)$ and $u(-t + 4)$ is shown in Figure 1.47(d).

$$u(t+4) = \begin{cases} 1 & \text{for } t \ge -4 \\ 0 & \text{elsewhere} \end{cases}$$

and $$u(-t+4) = \begin{cases} 1 & \text{for } t \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore, $$u(t+4)\, u(-t+4) = \begin{cases} 1 & \text{for } -4 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$$

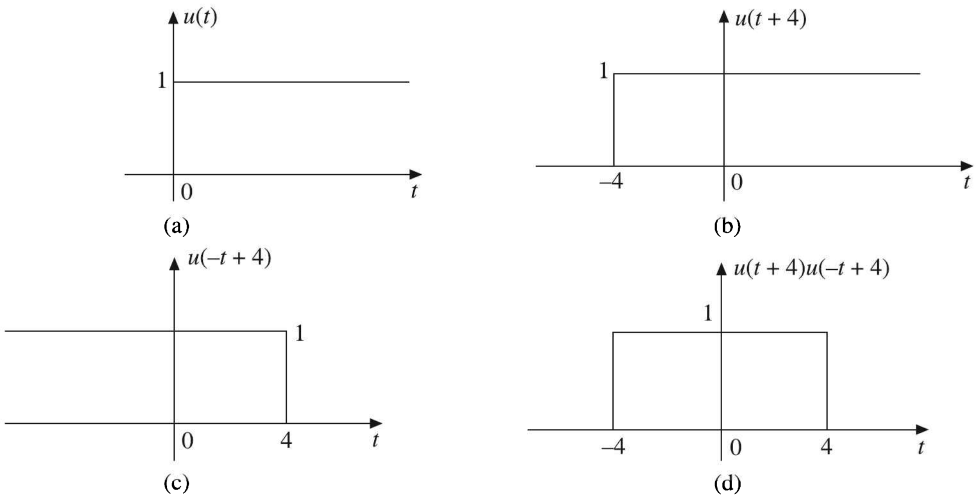


Figure 1.47 Waveforms for Example 1.7(b).

(c) Given $$\Pi\left(\frac{t-2}{2}\right) + \Pi(2t - 3.5)$$

The signal $\Pi(t)$ is shown in Figure 1.48(a). The signal $\Pi[(t - 2)/2] = \Pi(0.5t - 1)$ is obtained by first shifting $\Pi(t)$ by 1 unit right and then expanding the time scale by 2 as shown in Figure 1.48(b). The signal $\Pi(2t - 3.5)$ is obtained by first shifting $\Pi(t)$ by 3.5 units right and then compressing the time sale by 2 as shown in Figure 1.48(c). The signal $\Pi[(t - 2)/2] + \Pi(2t - 3.5)$ is obtained by adding the signals $\Pi[(t - 2)/2]$ and $\Pi(2t - 3.5)$ as shown in Figure 1.48(d).

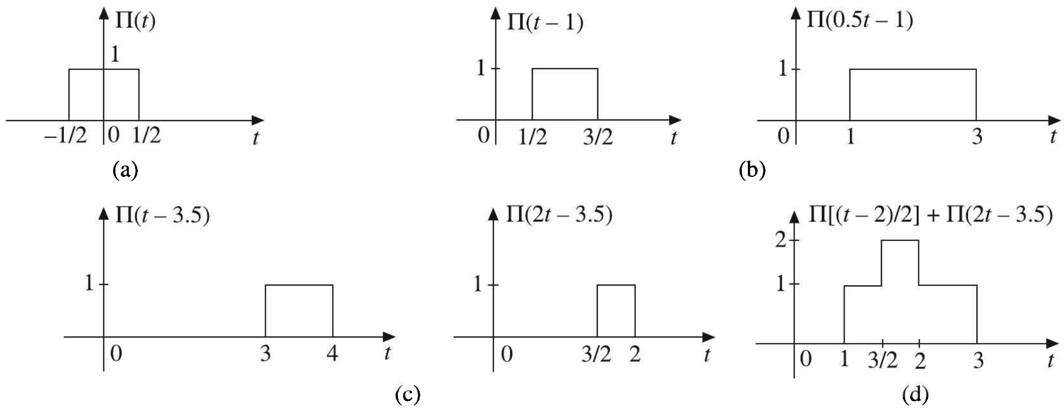


Figure 1.48 Waveforms for Example 1.7(c).

$$\Pi\left(\frac{t-2}{2}\right)=1 \quad \text{for } 1\le t\le 3$$

$$\Pi(2t-3.5)=1 \quad \text{for } 1.5\le t\le 2$$

$$\therefore \qquad \Pi\left(\frac{t-2}{2}\right)+\Pi(2t-3.5)=\begin{cases}1 & \text{for } 1\le t\le 1.5\\ 2 & \text{for } 1.5\le t\le 2\\ 1 & \text{for } 2\le t\le 3\end{cases}$$

(d) Given $\qquad r(t) - r(t-1) - r(t-3) + r(t-4)$

The signal $r(t)$ is drawn as shown in Figure 1.49(a). It is a straight line starting at origin with a slope 1. $-r(t-1)$ is a straight line starting at $t = 1$ with a slope -1 as shown in Figure 1.49(a). $-r(t-3)$ is a straight line starting at $t = 3$ with a

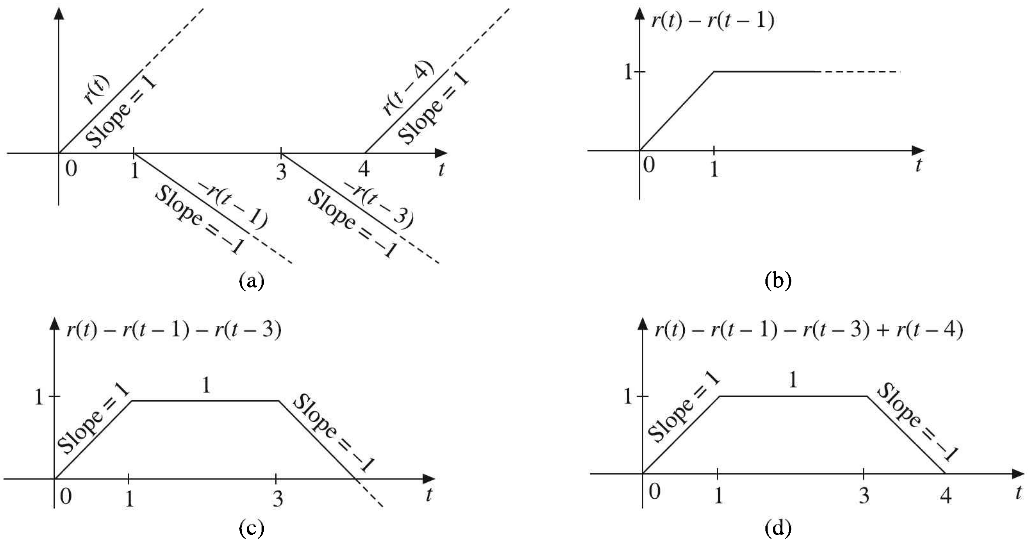


Figure 1.49 Waveforms for Example 1.7(d).

slope –1 as shown in Figure 1.49(a) and $r(t - 4)$ is a straight line starting at $t = 4$ with a slope 1 as shown in Figure 1.49(a). All these straight lines extend upto ∞.

The sum $r(t) - r(t - 1) - r(t - 3) + r(t - 4)$ is a trapezoidal signal as shown in Figure 1.49(d).

In the interval $0 \le t < 1$, only $r(t)$ is existing and the amplitudes of $-r(t - 1)$, $-r(t - 3)$ and $r(t - 4)$ are zero. Therefore, the resultant output is $r(t)$ alone. In the interval $1 \le t \le 3$, the amplitudes of $-r(t - 3)$ and $r(t - 4)$ are zero. Therefore, the resultant output is the sum of $r(t)$ and $-r(t - 1)$. That is, the slope is changed from 1 to 0. In the interval $3 \le t \le 4$, the amplitude of $r(t - 4)$ is zero. Therefore, the resultant output is the sum of $r(t)$, $-r(t - 1)$ and $-r(t - 3)$. That is, the slope is changed from 0 to –1. For $4 \le t \le \infty$, the amplitude of resultant output is the sum of $r(t)$, $-r(t - 1)$, $-r(t - 3)$ and $r(t - 4)$. It is equal to zero.

(e) Given $$r(-t)\, u(t + 2)$$

$r(-t)$ is time reversal of $r(t)$ as shown in Figure 1.50(a).

$$r(-t) = \begin{cases} -t & \text{for } t < 0 \\ 0 & \text{for } t > 0 \end{cases}$$

$$u(t+2) = \begin{cases} 1 & \text{for } t > -2 \\ 0 & \text{for } t < -2 \end{cases}$$

as shown in Figure 1.50(a).

The product

$$r(-t)u(t+2) = \begin{cases} r(-t) = -t & \text{for } -2 \le t < 0 \\ 0 & \text{for } t < -2 \\ 0 & \text{for } t > 0 \end{cases}$$

as shown in Figure 1.50(b).

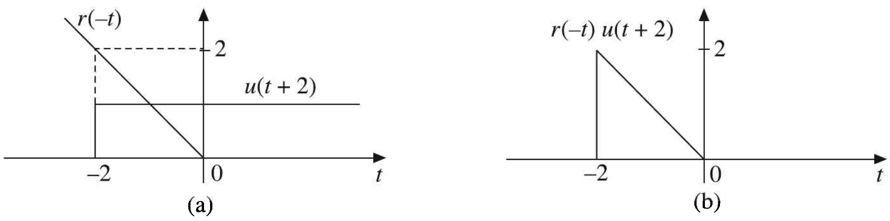


Figure 1.50 Waveforms for Example 1.7(e).

(f) Given $$r(-0.25t + 1)$$

$r(-0.25t + 1)$ is obtained by time reversing the signal $r(t)$ to obtain $r(-t)$ delaying it (moving to right) by 1 unit to obtain $r(-t + 1)$ and then expanding the signal $r(-t + 1)$ by 4 times. The entire operation is shown in Figure 1.51.

$$r(-0.25t+1) = \begin{cases} -0.25t + 1 & \text{for } -0.25t + 1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $$r(-0.25t+1) = \begin{cases} -0.25t + 1 & \text{for } t < 4 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $\quad r(-0.25t+1) = 0 \quad$ for $t \geq 4$

$= 0.25 \quad$ for $t = 3$

$= 0.75 \quad$ for $t = 1$

$= 1 \quad$ for $t = 0$

$= 1.5 \quad$ for $t = -2$

and so on.

This can be plotted as shown in Figure 1.51(c).

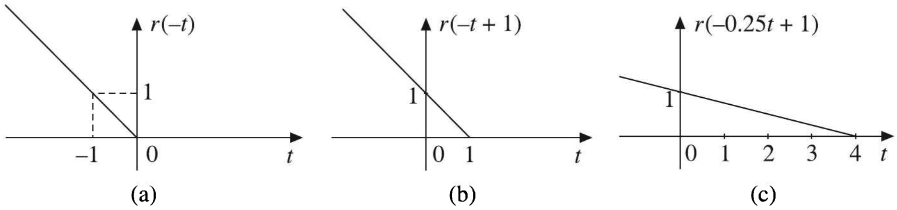


Figure 1.51 Waveforms for Example 1.7(f).

EXAMPLE 1.8 Express the following signals as sum of singular functions:

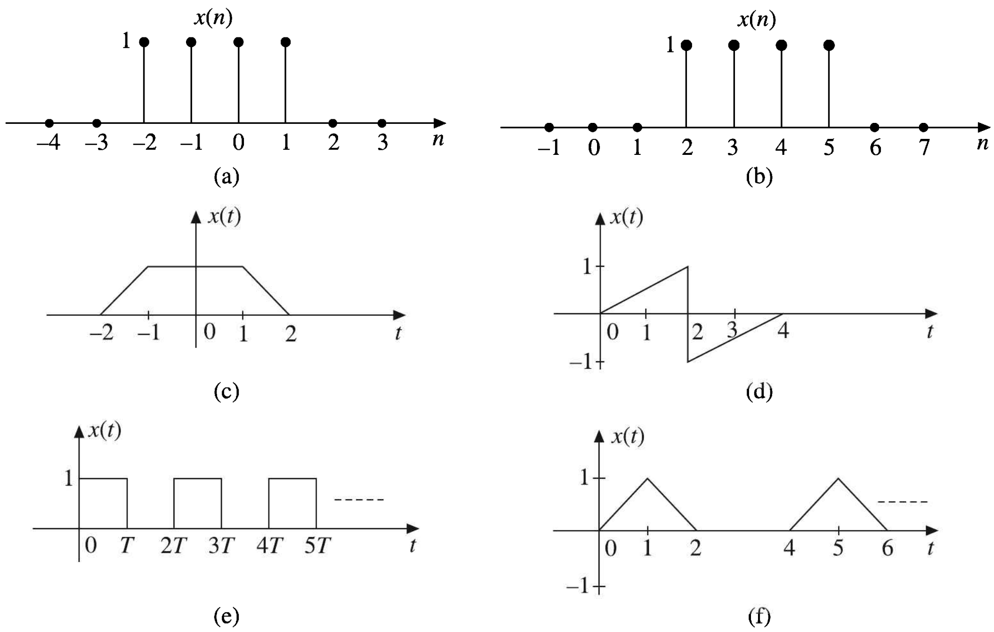


Figure 1.52 Waveforms for Example 1.8.

Solution:

(a) The given signal shown in Figure 1.52(a) is:

$$x(n) = \delta(n+2) + \delta(n+1) + \delta(n) + \delta(n-1)$$

$$x(n) = \begin{cases} 0 & \text{for } n \leq -3 \\ 1 & \text{for } -2 \leq n \leq 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

$\therefore \quad x(n) = u(n+2) - u(n-2)$

(b) The signal shown in Figure 1.52(b) is:

$$x(n) = \delta(n-2) + \delta(n-3) + \delta(n-4) + \delta(n-5)$$

$$x(n) = \begin{cases} 0 & \text{for } n \le 1 \\ 1 & \text{for } 2 \le n \le 5 \\ 0 & \text{for } n \ge 6 \end{cases}$$

$$\therefore \qquad x(n) = u(n-2) - u(n-6)$$

(c) The signal shown in Figure 1.52(c) is starting at $t = -2$ with a slope 1 and extends up to $t = -1$. Therefore

From $-2 \le t \le -1$ $\qquad x(t) = r(t + 2)$

At $t = -1$, the slope is changing from 1 to 0. So at $t = 1$, we have to add a ramp with slope -1. This 0 slope is maintained up to $t = 1$. Therefore

From $-2 \le t \le 1$ $\qquad x(t) = r(t + 2) - r(t + 1)$

At $t = 1$, the slope is changing from 0 to -1. So at $t = 1$, we have to add a ramp with a slope -1. This -1 slope is maintained up to $t = 2$. Therefore

From $-2 \le t \le 2$ $\qquad x(t) = r(t + 2) - r(t + 1) - r(t - 1)$

At $t = 2$, the slope is changing from -1 to 0. So at $t = 2$, we have to add a ramp with a slope 1. This 0 slope is maintained up to $t = \infty$. Therefore

From $-2 \le t \le \infty$ $\qquad x(t) = r(t + 2) - r(t + 1) - r(t - 1) + r(t - 2)$

$\therefore \qquad x(t) = r(t + 2) - r(t + 1) - r(t - 1) + r(t - 2)$

The analysis is shown in Figure 1.53.

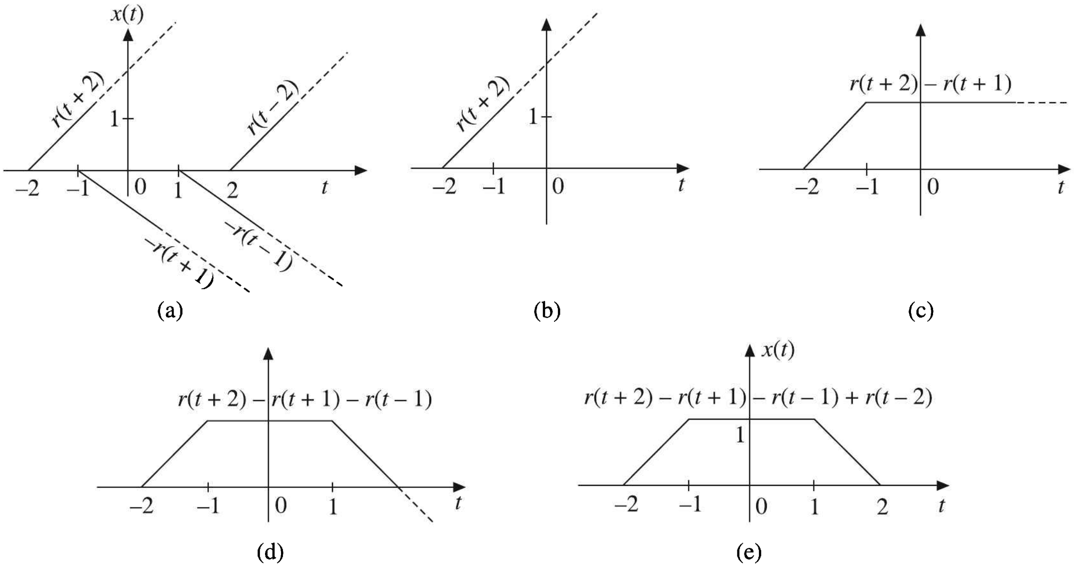


Figure 1.53 Waveforms for Example 1.8(c).

(d) The signal shown in Figure 1.52(d) is starting from $t = 0$ with a slope 1/2. So it is ramp $(1/2)r(t)$. At $t = 2$, the value of $(1/2)r(t) = 1$. At $t = 2$, the amplitude is suddenly falling to -1 and then rising linearly with same slope 1/2. Hence at $t = 2$, we have to add a step of -2 amplitude, i.e. $-2u(t - 2)$. At $t = 4$, the signal terminates. So at

$t = 4$, we have to add a ramp with a slope $-(1/2)$, i.e. $-(1/2)r(t-4)$. The analysis is shown in Figure 1.54.

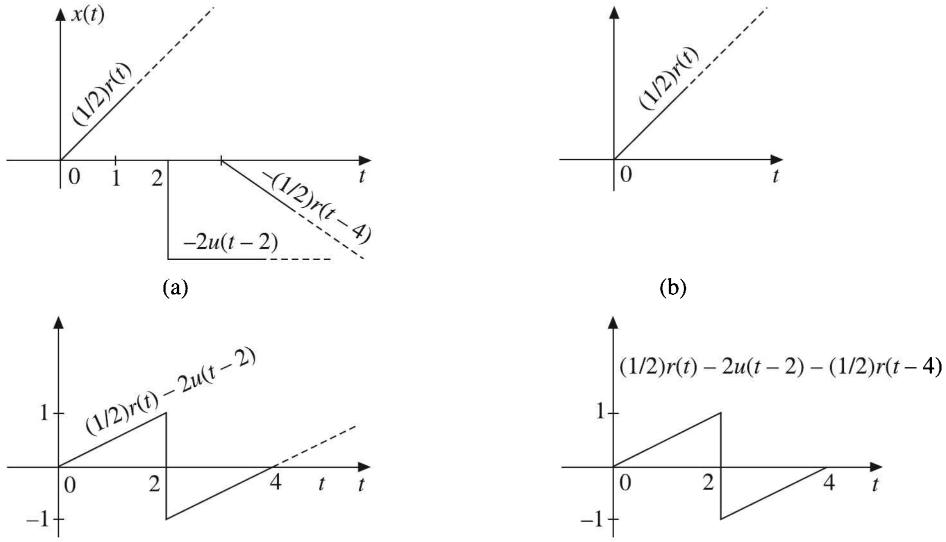


Figure 1.54 Waveforms for Example 1.8(d).

(e) Each rectangular pulse in the given signal of Figure 1.52(e) can be expressed as the sum of two step functions as shown in Figure 1.55. The pulse train extends up to infinity.

$$\therefore \qquad x(t) = u(t) - u(t-T) + u(t-2T) - u(t-3T) + u(t-4T) - u(t-5T) + \cdots$$

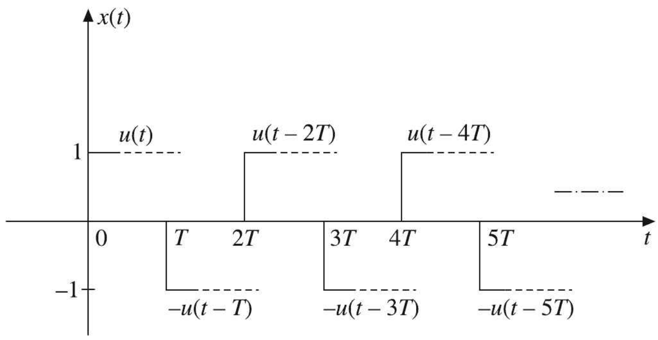


Figure 1.55 Waveforms for Example 1.8(e).

(f) Each triangular pulse in the given signal of Figure 1.52(f) can be expressed as the sum of 3 ramp functions as shown in Figure 1.56. The pulse train extends upto infinity.

$$\therefore \qquad x(t) = r(t) - 2r(t-1) + r(t-2) + r(t-4) - 2r(t-5) + r(t-6) + \cdots$$

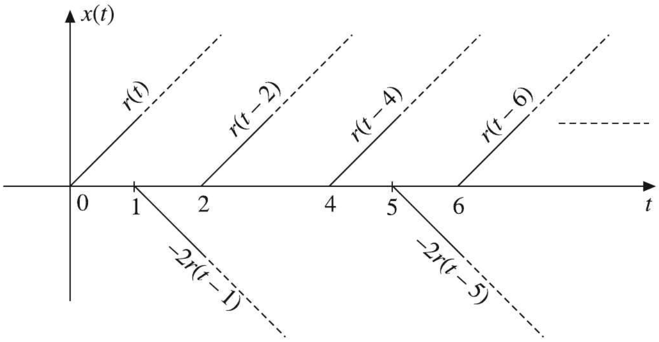


Figure 1.56 Example 1.8(f).

1.5 CLASSIFICATION OF SIGNALS

Based upon their nature and characteristics in the time domain, the signals may be broadly classified as under

(a) Continuous-time signals

(b) Discrete-time signals

Continuous-time signals

The signals that are defined for every instant of time are known as continuous-time signals. Continuous-time signals are also called analog signals. For continuous-time signals, the independent variable is time. They are denoted by $x(t)$. They are continuous in amplitude as well as in time. Most of the signals available are continuous-time signals. Figure 1.57(a) and (b) shows the graphical representation of continuous-time signals.

Discrete-time signals

The signals that are defined only at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude but discrete in time. For discrete-time signals, the amplitude between two time instants is just not defined. For discrete-time signals, the independent variable is time n. Since they are defined only at discrete instants of time, they are denoted by a sequence $x(nT)$ or simply by $x(n)$ where n is an integer.

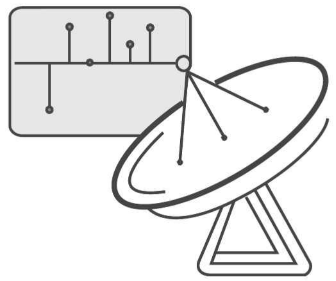
The discrete-time signals may be inherently discrete or may be discrete versions of the continuous-time signals. Figure 1.57(c) and (d) show the graphical representation of discrete-time signals.

Both continuous-time and discrete-time signals may further be classified as under

1. Deterministic and random signals
2. Periodic and non-periodic signals
3. Energy and power signals
4. Causal and non-causal signals
5. Even and odd signals

3

Signal Analysis



3.1 INTRODUCTION

It is very easy to understand or remember a new problem when it is associated with a familiar phenomenon. So while studying a new problem, we always search for analogies between that problem and a familiar phenomenon. Some insight into the new problem can be gained from the knowledge of the analogous phenomenon. For example, there is a perfect analogy between electrical and mechanical systems. If we are familiar with electrical systems, mechanical systems can be easily understood by using the analogy between them. We are familiar with vectors and in this book we want to discuss signal analysis. There is a perfect analogy between vectors and signals. In this chapter, using the analogy between vectors and signals, we try to understand signal analysis in a better way.

3.2 ANALOGY BETWEEN VECTORS AND SIGNALS

Vectors

A vector is specified by magnitude and direction. We shall denote all vectors by boldface type and their magnitudes by light face type. Consider two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ as shown in Figure 3.1. The vector $\mathbf{V}_1$ can be expressed in terms of vector $\mathbf{V}_2$ in infinite ways by drawing a line from the end of $\mathbf{V}_1$ on to $\mathbf{V}_2$. In each representation, $\mathbf{V}_1$ is represented in terms of $\mathbf{V}_2$ plus another vector which will be called the *error vector*. In Figure 3.1(a), $\mathbf{V}_1$ is approximated by $C_1\mathbf{V}_2$ with an error $\mathbf{V}_{e_1}$, i.e. $\mathbf{V}_1 = C_1\mathbf{V}_2 + \mathbf{V}_{e_1}$. In Figure 3.1(b), $\mathbf{V}_1$ is approximated by $C_2\mathbf{V}_2$ with an error $\mathbf{V}_{e_2}$ as $\mathbf{V}_1 = C_2\mathbf{V}_2 + \mathbf{V}_{e_2}$. In Figure 3.1(c), $\mathbf{V}_1$ is approximated by $C_{12}\mathbf{V}_2$ with an error $\mathbf{V}_e$ as $\mathbf{V}_1 = C_{12}\mathbf{V}_2 + \mathbf{V}_e$. Geometrically the component of a vector $\mathbf{V}_1$ along vector $\mathbf{V}_2$ is obtained by drawing a perpendicular from the end of $\mathbf{V}_1$ onto $\mathbf{V}_2$. In this case, error vector $\mathbf{V}_e$ is the minimum. The component of a vector $\mathbf{V}_1$ along another vector $\mathbf{V}_2$ is given by $C_{12}\mathbf{V}_2$, where C_{12} is chosen such that the error vector is minimum. The magnitude of C_{12} is an indication of the similarity of the two vectors. If C_{12} is zero, then vector $\mathbf{V}_1$ has no component along the other vector $\mathbf{V}_2$, and hence two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are mutually perpendicular. The vectors which are mutually perpendicular to each other are called *orthogonal vectors*. Orthogonal vectors are thus independent vectors. If the vectors are orthogonal, then the parameter C_{12} is zero.

The component
$$\mathbf{V}_1 \text{ along } \mathbf{V}_2 = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_2} = C_{12}V_2$$

Therefore,
$$C_{12} = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_2^2} = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{\mathbf{V}_2 \cdot \mathbf{V}_2}$$

When vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal, their dot product

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$$

and
$$C_{12} = 0$$

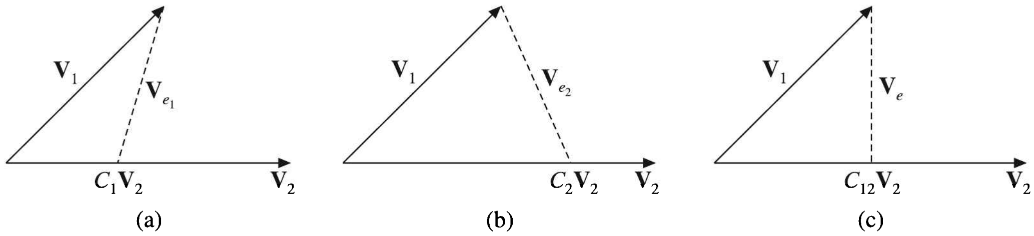


Figure 3.1 One vector in terms of another.

Signals

The concept of vector comparison and orthogonality can be extended to signals. Consider two signals $x_1(t)$ and $x_2(t)$. We can approximate $x_1(t)$ in terms of $x_2(t)$ over a certain interval $(t_1 < t < t_2)$ as follows:

$$x_1(t) \approx C_{12}x_2(t) \qquad \text{for } (t_1 < t < t_2)$$

[This is similar to $\mathbf{V}_1 \approx C_{12}\mathbf{V}_2$ in the case of vectors.]

Like in vectors, the main criterion in selecting C_{12} is to minimise the error between the actual function and the approximated function over the interval $(t_1 < t < t_2)$.

The error function $x_e(t)$ is defined as:

$$x_e(t) = x_1(t) - C_{12}x_2(t)$$

If we choose to minimise the error $x_e(t)$ over the interval t_1 to t_2 by minimising the average value of $x_e(t)$ over this interval, i.e. by minimising

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_e(t)\,dt, \quad \text{i.e. } \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \left[x_1(t) - C_{12}x_2(t)\right] dt$$

it may give wrong results.

This is because there can be large positive and negative errors present in interval t_1 to t_2 that may cancel one another in the process of averaging and give the false indication that the error is zero. To overcome this we choose to minimise the mean square of the error instead of the error itself. Let us designate the average of squared error $[x_e^2(t)]$ by ε.

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_e^2(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[x_1(t) - C_{12}x_2(t)\right]^2 dt$$

The value of C_{12} which minimises ε can be found from $(d\varepsilon/dC_{12}) = 0$.

$$\frac{d\varepsilon}{dC_{12}} = \frac{d}{dC_{12}}\left[\frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2}\left[x_1(t) - C_{12}x_2(t)\right]^2 dt\right] = 0$$

i.e.
$$\frac{d}{dC_{12}}\left\{\frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2}[x_1^2(t) - 2C_{12}x_1(t)\,x_2(t) + C_{12}^2x_2^2(t)]\,dt\right]\right\} = 0$$

Changing the order of integration and differentiation, we get

$$\frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2}\frac{d}{dC_{12}}[x_1^2(t)]\,dt - 2\int_{t_1}^{t_2}\frac{d}{dC_{12}}[C_{12}x_1(t)\,x_2(t)]\,dt + \int_{t_1}^{t_2}\frac{d}{dC_{12}}[C_{12}^2x_2^2(t)]\,dt\right] = 0$$

i.e.
$$\frac{1}{(t_2 - t_1)}\left[0 - 2\int_{t_1}^{t_2}x_1(t)\,x_2(t)\,dt + \int_{t_1}^{t_2}2C_{12}x_2^2(t)\,dt\right] = 0$$

$\therefore$
$$\int_{t_1}^{t_2}x_1(t)\,x_2(t)\,dt = C_{12}\int_{t_1}^{t_2}x_2^2(t)\,dt$$

$\therefore$
$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\,x_2(t)\,dt}{\int_{t_1}^{t_2} x_2^2(t)\,dt}$$

For vectors, we had	$C_{12} = \dfrac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_2^2}$

By analogy with vectors, we say that $x_1(t)$ has a component of waveform $x_2(t)$ [i.e. $C_{12}x_2(t)$] and this component has a magnitude C_{12}. If $C_{12} = 0$, then the signal $x_1(t)$ contains no component of the signal $x_2(t)$, and we say that the two functions are orthogonal over the interval (t_1, t_2). So we can say that the two functions $x_1(t)$ and $x_2(t)$ are orthogonal over an interval (t_1, t_2) if

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

[Two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal if $\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$]
Thus, we can conclude that in the case of vectors,

$$C_{12} = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_2^2}$$

and in the case of signals,

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

In both cases for orthogonality, $C_{12} = 0$.

Two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal if their dot product is zero, i.e. $\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$. Two signals $x_1(t)$ and $x_2(t)$ are orthogonal if the integral of the product of those signals is zero, i.e.

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

In the analogy of vectors and signals, the dot product of two vectors is analogous to the integral of the product of two signals, that is

$$\mathbf{A} \cdot \mathbf{B} \sim \int_{t_1}^{t_2} x_A(t)\, x_B(t)\, dt$$

The square of the length A of vector $\mathbf{A}$ is analogous to the integral of the square of a function, that is

$$\mathbf{A} \cdot \mathbf{A} = A^2 \sim \int_{t_1}^{t_2} x_A^2(t)\, dt$$

We can easily show that the functions $\sin n\omega_0 t$, $\sin m\omega_0 t$; $\cos n\omega_0 t$, $\cos m\omega_0 t$; and $\sin n\omega_0 t$, $\cos m\omega_0 t$ are orthogonal over any interval $[t_0, t_0 + (2\pi/\omega_0)]$ for integral values of m and n.

EXAMPLE 3.1 Show that the following signals are orthogonal over an interval [0, 1]:

$$x_1(t) = 2$$

$$x_2(t) = \sqrt{3}(1 - 2t)$$

Solution: We know that two signals $x_1(t)$ and $x_2(t)$ are orthogonal over an interval (t_1, t_2), if

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

In this case,

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = \int_0^1 2[\sqrt{3}(1-2t)]\, dt$$

$$= 2\sqrt{3}\int_0^1 dt - 4\sqrt{3}\int_0^1 t\, dt = 2\sqrt{3}\,[t]_0^1 - 4\sqrt{3}\left[\frac{t^2}{2}\right]_0^1$$

$$= 2\sqrt{3}\,[1-0] - 4\sqrt{3}\left[\frac{1}{2}-0\right]$$

$$= 0$$

This shows that the two given signals are orthogonal over an interval [0, 1].

EXAMPLE 3.2 Show that the functions sin $n\omega_0 t$ and cos $m\omega_0 t$ are orthogonal over any interval $\{t_0$ to $[t_0 + (2\pi/\omega_0)]\}$ for integral values of n and m.

Solution: To show that the functions sin $n\omega_0 t$ and cos $m\omega_0 t$ are orthogonal over any interval $\{t_0$ to $[t_0 + (2\pi/\omega_0)]\}$ for integral values of n and m, we have to show that the integral

$$I = \int_{t_0}^{t_0+(2\pi/\omega_0)} \sin n\omega_0 t \cos m\omega_0 t\, dt = 0$$

$$I = \int_{t_0}^{t_0+(2\pi/\omega_0)} \sin n\omega_0 t \cos m\omega_0 t\, dt$$

$$= \int_{t_0}^{t_0+(2\pi/\omega_0)} \frac{1}{2}\left[\sin\,(n+m)\,\omega_0 t + \sin\,(n-m)\,\omega_0 t\right] dt$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)\,\omega_0 t}{n+m} - \frac{\cos(n-m)\,\omega_0 t}{n-m}\right]_{t_0}^{t_0+(2\pi/\omega_0)}$$

$$= -\frac{1}{2}\left\{\begin{aligned}&\left[\frac{\cos(n+m)\,\omega_0[t_0+(2\pi/\omega_0)] - \cos(n+m)\,\omega_0 t_0}{n+m}\right]\\ &+\left[\frac{\cos(n-m)\,\omega_0[t_0+(2\pi/\omega_0)] - \cos(n-m)\,\omega_0 t_0}{n-m}\right]\end{aligned}\right\}$$

$$= -\frac{1}{2}\left\{\left[\frac{\cos(n+m)\,\omega_0 t_0 - \cos(n+m)\,\omega_0 t_0}{n+m}\right] + \left[\frac{\cos(n-m)\,\omega_0 t_0 - \cos(n-m)\,\omega_0 t_0}{n-m}\right]\right\}$$

$$= 0$$

This shows that the functions sin $n\omega_0 t$ and cos $m\omega_0 t$ are mutually orthogonal.

EXAMPLE 3.3 Prove that the complex exponential signals are orthogonal functions.

Solution: Consider two complex exponential signals,

$$x_1(t) = e^{jn\omega_0 t} \quad \text{and} \quad x_2(t) = e^{jm\omega_0 t}$$

Let the interval be t_0 to $t_0 + T$, i.e. from t_0 to $t_0 + (2\pi/\omega_0)$. $x_1(t)$ and $x_2(t)$ are orthogonal over the interval t_0 to $t_0 + (2\pi/\omega_0)$, if $I = \int_{t_0}^{t_0+(2\pi/\omega_0)} x_1(t)\, x_2^*(t)\, dt = 0$.

Here $x_1(t) = e^{jn\omega_0 t}$ and $x_2^*(t) = [e^{jm\omega_0 t}]^* = e^{-jm\omega_0 t}$

$$\therefore \qquad I = \int_{t_0}^{t_0+(2\pi/\omega_0)} e^{jn\omega_0 t}\, e^{-jm\omega_0 t}\, dt$$

$$= \int_{t_0}^{t_0+(2\pi/\omega_0)} e^{j(n-m)\,\omega_0 t}\, dt = \left[\frac{e^{j(n-m)\,\omega_0 t}}{j(n-m)\,\omega_0}\right]_{t_0}^{t_0+(2\pi/\omega_0)}$$

$$= \left[\frac{e^{j(n-m)\,\omega_0 [t_0+(2\pi/\omega_0)]} - e^{j(n-m)\,\omega_0 t_0}}{j(n-m)\,\omega_0}\right] = \frac{e^{j(n-m)\,\omega_0 t_0}\, e^{j(n-m)2\pi} - e^{j(n-m)\,\omega_0 t_0}}{j(n-m)\,\omega_0}$$

$$= \frac{e^{j(n-m)\,\omega_0 t_0} - e^{j(n-m)\,\omega_0 t_0}}{j(n-m)\omega_0} \qquad \boxed{e^{j(n-m)2\pi} = \cos(n-m)\,2\pi + j\sin(n-m)\,2\pi = 1}$$

$$= 0$$

$$\therefore \qquad \int_{t_0}^{t_0+T} e^{jn\omega_0 t}\, (e^{jm\omega_0 t})^*\, dt = 0$$

This proves that the complex exponential signals are orthogonal functions.

EXAMPLE 3.4 Prove that the functions $x_p(t)$ and $x_q(t)$ where $x_r(t) = \frac{1}{\sqrt{T}}(\cos r\omega_0 t + \sin r\omega_0 t)$, $T = (2\pi/\omega_0)$ are orthogonal over the period (0 to T).

Solution: Given
$$x_r(t) = \frac{1}{\sqrt{T}}(\cos r\omega_0 t + \sin r\omega_0 t)$$

where
$$T = \frac{2\pi}{\omega_0}, \quad \text{i.e. } \omega_0 = \frac{2\pi}{T}$$

$$\therefore \qquad x_p(t) = \frac{1}{\sqrt{T}}(\cos p\omega_0 t + \sin p\omega_0 t)$$

and
$$x_q(t) = \frac{1}{\sqrt{T}}(\cos q\omega_0 t + \sin q\omega_0 t)$$

We know that the signals $x_p(t)$ and $x_q(t)$ are orthogonal over the period (0 to T), if

$$\int_0^T x_p(t)\, x_q(t)\, dt = 0$$

In this case,

$$x_p(t)\, x_q(t) = \frac{1}{\sqrt{T}}[\cos p\omega_0 t + \sin p\omega_0 t]\frac{1}{\sqrt{T}}[\cos q\omega_0 t + \sin q\omega_0 t]$$

$$= \frac{1}{T}[\cos p\omega_0 t \cos q\omega_0 t + \sin p\omega_0 t \sin q\omega_0 t + \cos p\omega_0 t \sin q\omega_0 t + \sin p\omega_0 t \cos q\omega_0 t]$$

$$= \frac{1}{T}[\cos(p-q)\,\omega_0 t + \sin(p+q)\,\omega_0 t]$$

$$\therefore \qquad I = \int_{t_0}^{t_0+T} x_p(t)\, x_q(t)\, dt$$

$$= \int_0^T \frac{1}{T}[\cos(p-q)\,\omega_0 t + \sin(p+q)\,\omega_0 t]\, dt$$

$$= \frac{1}{T}\left[\frac{\sin(p-q)\,\omega_0 t}{(p-q)\,\omega_0} - \frac{\cos(p+q)\,\omega_0 t}{(p+q)\,\omega_0}\right]_0^T$$

$$= \frac{1}{T}\left[\frac{\sin(p-q)\,\omega_0 T - \sin 0}{(p-q)\,\omega_0} - \frac{\cos(p+q)\,\omega_0 T - \cos 0}{(p+q)\,\omega_0}\right]$$

$$= \frac{1}{T}\left[\frac{\sin(p-q)\,\omega_0(2\pi/\omega_0) - \sin 0}{(p-q)\,\omega_0} - \frac{\cos(p+q)\,\omega_0(2\pi/\omega_0) - \cos 0}{(p+q)\,\omega_0}\right]$$

$$= \frac{1}{T}\left[\frac{\sin(p-q)2\pi - \sin 0}{(p-q)\omega_0} - \frac{\cos(p+q)2\pi - \cos 0}{(p+q)\omega_0}\right]$$

$$= \frac{1}{T}\left[\frac{0-0}{(p-q)\omega_0} - \frac{1-1}{(p+q)\omega_0}\right]$$

$$= 0$$

$$\therefore \qquad \int_0^T x_p(t)\, x_q(t) = 0$$

Thus, the functions $x_p(t)$ and $x_q(t)$ are orthogonal over the period [0 to T].

EXAMPLE 3.5 Prove that the signals $x(t)$ and $y(t)$ shown in Figure 3.2 are orthogonal over the interval [0, 4].

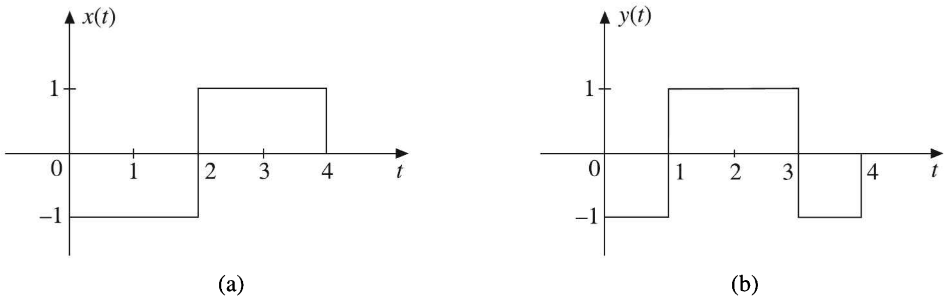


Figure 3.2 Waveforms for Example 3.5.

Solution: The signals $x(t)$ and $y(t)$ shown in Figure 3.2 can be mathematically expressed as:

$$x(t) = \begin{cases} -1 & \text{for } 0 < t < 2 \\ 1 & \text{for } 2 < t < 4 \end{cases}$$

$$y(t) = \begin{cases} -1 & \text{for } 0 < t < 1 \\ 1 & \text{for } 1 < t < 3 \\ -1 & \text{for } 3 < t < 4 \end{cases}$$

We know that $x(t)$ and $y(t)$ are orthogonal over the interval [0, 4] if $\int_0^4 x(t)\,y(t)\,dt = 0$.

In this case,

$$\begin{aligned}
\int_0^4 x(t)\,y(t)\,dt &= \int_0^1 x(t)\,y(t)\,dt + \int_1^2 x(t)\,y(t)\,dt + \int_2^3 x(t)\,y(t)\,dt + \int_3^4 x(t)\,y(t)\,dt \\
&= \int_0^1 (-1)(-1)\,dt + \int_1^2 (-1)(1)\,dt + \int_2^3 (1)(1)\,dt + \int_3^4 (1)(-1)\,dt \\
&= \int_0^1 dt - \int_1^2 dt + \int_2^3 dt - \int_3^4 dt \\
&= [t]_0^1 - [t]_1^2 + [t]_2^3 - [t]_3^4 \\
&= [1-0] - [2-1] + [3-2] - [4-3] \\
&= 1 - 1 + 1 - 1 \\
&= 0
\end{aligned}$$

$$\therefore \qquad \int_0^4 x(t)\,y(t)\,dt = 0$$

This proves that the given signals are orthogonal over the interval [0, 4].

EXAMPLE 3.6 A rectangular function is defined as:

$$x(t) = \begin{cases} A & \text{for } 0 < t < \dfrac{\pi}{2} \\ -A & \text{for } \dfrac{\pi}{2} < t < \dfrac{3\pi}{2} \\ A & \text{for } \dfrac{3\pi}{2} < t < 2\pi \end{cases}$$

Approximate the above function by $A \cos t$ between the intervals $(0, 2\pi)$ such that the mean square error is minimum.

Solution: The given function $x(t)$ can be approximated by $A \cos t$ over the interval $(0, 2\pi)$ as:

$$x(t) \approx C_{12}\, A \cos t$$

C_{12} has to be found such that the mean square error is minimum over the interval $(0, 2\pi)$ as:

$$C_{12} = \frac{\int_0^{2\pi} x(t)\, A \cos t\, dt}{\int_0^{2\pi} [A \cos t]^2\, dt}$$

$$= \frac{\int_0^{\pi/2} (A)\, A \cos t\, dt + \int_{\pi/2}^{3\pi/2} (-A)\, A \cos t\, dt + \int_{3\pi/2}^{2\pi} (A)\, A \cos t\, dt}{\int_0^{2\pi} A^2 \dfrac{[1 + \cos 2t]}{2}\, dt}$$

$$= \frac{A^2 \left[\int_0^{\pi/2} \cos t\, dt - \int_{\pi/2}^{3\pi/2} \cos t\, dt + \int_{3\pi/2}^{2\pi} \cos t\, dt \right]}{\dfrac{A^2}{2} \left[\int_0^{2\pi} (1 + \cos 2t)\, dt \right]} = \frac{[\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{3\pi/2} + [\sin t]_{3\pi/2}^{2\pi}}{\dfrac{1}{2} \left[t + \dfrac{\sin 2t}{2} \right]_0^{2\pi}}$$

$$= \frac{\left[\sin \dfrac{\pi}{2} - \sin 0 \right] - \left[\sin \dfrac{3\pi}{2} - \sin \dfrac{\pi}{2} \right] + \left[\sin 2\pi - \sin \dfrac{3\pi}{2} \right]}{\dfrac{1}{2} \left[(2\pi - 0) + \dfrac{\sin 4\pi - \sin 0}{2} \right]}$$

$$= \frac{(1 - 0) - (-1 - 1) + [0 - (-1)]}{\pi + \dfrac{0 - 0}{2}} = \frac{4}{\pi}$$

$$\therefore \quad C_{12} = \frac{4}{\pi}$$

$$\therefore \qquad x(t) \approx \frac{4}{\pi} A \cos t$$

So the given function $x(t)$ can be approximated by $A \cos t$ as $x(t) \approx (4/\pi) A \cos t$ over the interval $(0, 2\pi)$ such that the mean square error is minimum. The rectangular function $x(t)$, the cosine waveform $A \cos t$, and the approximation of the rectangular function in terms of the cosine waveform are shown in Figure 3.3.

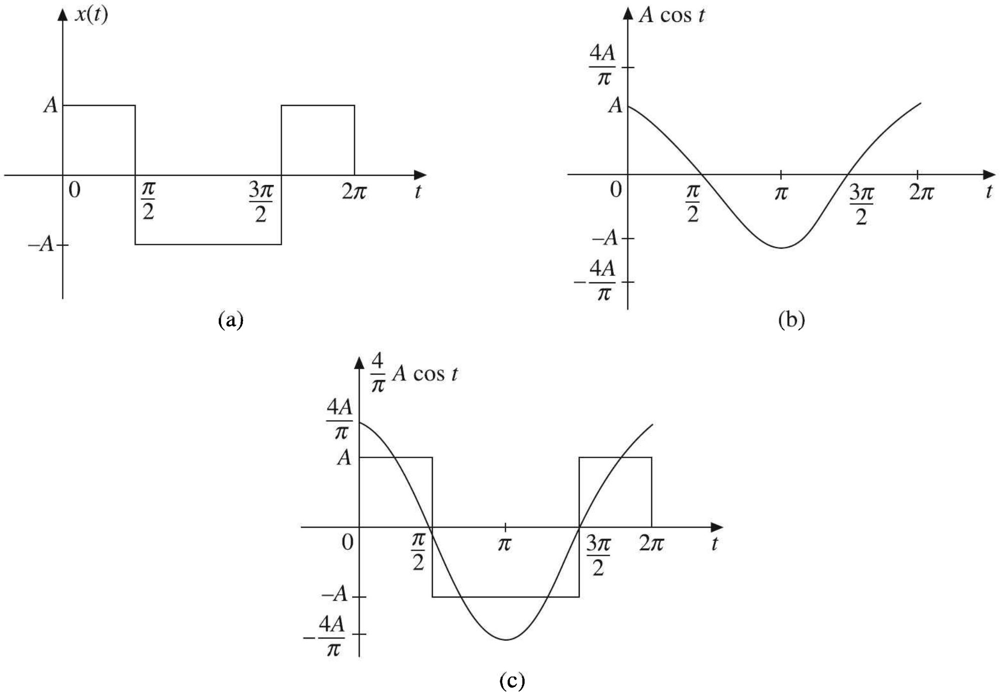


Figure 3.3 (a) Rectangular function, (b) Cosine wave, (c) Approximation.

3.3 GRAPHICAL EVALUATION OF A COMPONENT OF ONE FUNCTION IN THE OTHER

In the previous section, we discussed that the component of a function $x_1(t)$ in another function $x_2(t)$ can be evaluated analytically using the equation

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

In this section, we discuss how to evaluate the component of a function in the other function by graphical means using the same equation.

The constants C_1, C_2, C_3, ..., C_n in the equation for **A** represent the magnitudes of the components of **A** along the vectors x_1, x_2, x_3, ..., x_n respectively.

The dot product of both sides of the equation for **A** with vector x_r gives

$$\mathbf{A} \cdot x_r = C_1 x_1 \cdot x_r + C_2 x_2 \cdot x_r + \cdots + C_r x_r \cdot x_r + \cdots + C_n x_n \cdot x_r$$

Since $x_m \cdot x_n = 0$ for $m \neq n$ and $x_m \cdot x_n = 1$ for $m = n$, we get all the terms of the form $C_j x_j \cdot x_r (j \neq r)$ are zero. Therefore,

$$\mathbf{A} \cdot x_r = C_r x_r \cdot x_r = C_r$$

This set of vectors (x_1, x_2, ..., x_n) which are mutually perpendicular to each other are called an *orthogonal vector space*. In general, the product $x_m \cdot x_n$ can be some constant k_m instead of unity. When k_m is unity, the set is called *normalized orthogonal set*, or *orthonormal vector space*. Therefore, in general, for orthogonal vector space $\{x_r\}$ ... (r = 1, 2, ..., n), we have

$$x_m \cdot x_r = \begin{cases} 0 & m \neq n \\ k_m & m = n \end{cases}$$

For an orthogonal vector space, the equation for $\mathbf{A} \cdot x_r$ is modified to

$$\mathbf{A} \cdot x_r = C_r x_r \cdot x_r = C_r k_r$$

and

$$C_r = \frac{\mathbf{A} \cdot x_r}{k_r}$$

The *summary* of the above discussion for vector space is:

For an orthogonal vector space $\{x_r\}$ ... (r = 1, 2, ..., n)

$$x_m \cdot x_n = \begin{cases} 0 & m \neq n \\ k_m & m = n \end{cases}$$

If this vector space is complete, then any vector **F** can be expressed as:

$$\mathbf{F} = C_1 x_1 + C_2 x_2 + \cdots + C_r x_r$$

where

$$C_r = \frac{\mathbf{F} \cdot x_r}{k_r} = \frac{\mathbf{F} \cdot x_r}{x_r \cdot x_r}$$

3.5 ORTHOGONAL SIGNAL SPACE

In the case of vectors, we have discussed that any vector **A** can be expressed as a sum of its components along a set of n mutually orthogonal vectors, provided these vectors form a complete set of co-ordinate system. Similarly, in the case of signals, any signal $x(t)$ can be expressed as a sum of its components along a set of n mutually orthogonal functions if these functions form a complete set.

3.5.1 Approximation of a Function by a Set of Mutually Orthogonal Functions

Consider a set of n functions $g_1(t)$, $g_2(t)$, ..., $g_n(t)$ which are orthogonal to one another over an interval t_1 to t_2, that is

$$\int_{t_1}^{t_2} g_j(t)\, g_k(t)\, dt = 0 \qquad j \neq k$$

and let

$$\int_{t_1}^{t_2} g_j^2(t)\, dt = K_j$$

Let an arbitrary signal $x(t)$ be approximated over an interval (t_1, t_2) by a linear combination of these n mutually orthogonal signals.

$$x(t) \approx C_1 g_1(t) + C_2 g_2(t) + \cdots + C_k g_k(t) + \cdots + C_n g_n(t)$$

$$= \sum_{r=1}^{n} C_r g_r(t)$$

For the best approximation we must find the proper values of constants C_1, C_2, ..., C_n such that ε, the mean square of error $x_e(t)$ is minimised.
By definition,

$$x_e(t) = x(t) - \sum_{r=1}^{n} C_r g_r(t)$$

and

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

This equation shows that ε is a function of C_1, C_2, ..., C_n and to minimise ε, we must have

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \cdots = \frac{\partial \varepsilon}{\partial C_j} = \cdots = \frac{\partial \varepsilon}{\partial C_n} = 0$$

Consider the equation,

$$\frac{\partial \varepsilon}{\partial C_j} = 0$$

Since $(t_2 - t_1)$ is a constant, we can write

$$\frac{\partial}{\partial C_j} \left\{ \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt \right\} = 0$$

i.e.

$$\frac{\partial}{\partial C_j} \left\{ \int_{t_1}^{t_2} \left[x^2(t) - 2 \sum_{r=1}^{n} x(t)\, C_r g_r(t) + \sum_{r=1}^{n} C_r^2 g_r^2(t) \right] dt \right\} = 0$$

In this expansion, all the terms arising due to the cross product of the orthogonal functions are zero by virtue of orthogonality, that is, all the terms of the form $\int g_j(t)\, g_k(t)\, dt$ are zero. Similarly, the derivative with respect to C_j of all the terms that do not contain C_j are zero, that is

$$\frac{\partial}{\partial C_j}\int_{t_1}^{t_2} x^2(t)\, dt = \frac{\partial}{\partial C_j}\int_{t_1}^{t_2} C_r^2 g_r^2(t)\, dt = \frac{\partial}{\partial C_j}\int_{t_1}^{t_2} C_r x(t)\, g_r(t)\, dt = 0$$

This leaves only two non-zero terms in the equation $\partial\varepsilon/\partial C_j = 0$ as follows:

$$\frac{\partial}{\partial C_j}\int_{t_1}^{t_2} [-2C_j x(t)\, g_j(t) + C_j^2 g_j^2(t)]\, dt = 0$$

Changing the order of differentiation and integration, we get

$$2\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt = 2C_j \int_{t_1}^{t_2} g_j^2(t)\, dt$$

Therefore,

$$C_j = \frac{\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt}$$

$$= \frac{1}{K_j}\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt$$

This result may be summarised as follows:

Given a set of n functions $g_1(t)$, $g_2(t)$, ..., $g_n(t)$ mutually orthogonal over the interval (t_1, t_2), it is possible to approximate an arbitrary function $x(t)$ over this interval by a linear combination of these n functions

$$x(t) \approx C_1 g_1(t) + C_2 g_2(t) + \cdots + C_n g_n(t)$$

$$= \sum_{r=1}^{n} C_r g_r(t)$$

For the best approximation, that is, the one that will minimize the mean of the square error over the interval, we must choose the coefficients C_1, C_2, ..., C_n as given by

$$C_j = \frac{\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt} = \frac{1}{K_j}\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt$$

3.5.2 Evaluation of Mean Square Error

Let us now find the values of ε when optimum values of coefficients C_1, C_2, ..., C_n are chosen according to the equation for C_j given in section 3.5.1.
By definition,

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t)\, dt + \sum_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} g_r^2(t)\, dt - 2 \sum_{r=1}^{n} C_r \int_{t_1}^{t_2} x(t)\, g_r(t)\, dt \right]$$

But from the equation for C_j, we have

$$\int_{t_1}^{t_2} x(t)\, g_r(t)\, dt = C_r \int_{t_1}^{t_2} g_r^2(t)\, dt = C_r K_r$$

Substituting this in the equation for ε, we have

$$\varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t)\, dt + \sum_{r=1}^{n} C_r^2 K_r - 2 \sum_{r=1}^{n} C_r^2 K_r \right]$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t)\, dt - \sum_{r=1}^{n} C_r^2 K_r \right]$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t)\, dt - (C_1^2 K_1 + C_2^2 K_2 + \cdots + C_n^2 K_n) \right]$$

The mean square error can therefore be evaluated using this equation for ε.

3.5.3 Representation of a Function by a Closed or a Complete Set of Mutually Orthogonal Functions

From the equation for mean square error ε, i.e. from the equation

$$\varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t)\, dt - (C_1^2 K_1 + C_2^2 K_2 + \cdots + C_n^2 K_n) \right]$$

We can observe that if we increase n, that is, if we approximate $x(t)$ by a larger number of orthogonal functions, the error will become smaller. But ε, which is mean of squared error is a positive quantity; hence in the limit as the number of terms is made infinity, the sum $\sum_{r=1}^{n} C_r^2 K_r$ may converge to the integral

$$\int_{t_1}^{t_2} x^2(t)\,dt$$

and then ε vanishes. Thus,

$$\int_{t_1}^{t_2} x^2(t)\,dt = \sum_{r=1}^{\infty} C_r^2 K_r$$

Under these conditions, $x(t)$ is represented by the infinite series

$$x(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_r g_r(t) + \cdots$$

The infinite series on the right-hand side of this equation for $x(t)$ converges to $x(t)$ such that the mean square of the error is zero. The series is said to converge in the mean. The representation of $x(t)$ is now exact.

A set of functions $g_1(t)$, $g_2(t)$, ..., $g_r(t)$ mutually orthogonal over the interval (t_1, t_2) is said to be a complete or a closed set if there exists no function $y(t)$ for which it is true that

$$\int_{t_1}^{t_2} y(t)\, g_k(t)\,dt = 0 \qquad \text{for } k = 1, 2, \ldots$$

If a function $y(t)$ could be found such that the above integral is zero, then obviously $y(t)$ is orthogonal to each member of the set $\{g_r(t)\}$ and consequently, is itself a member of the set. So the set cannot be complete without $y(t)$ being its member.

Summary of the results of the discussion

For a set $\{g_r(t)\}(r = 1, 2, \ldots)$ mutually orthogonal over the interval (t_1, t_2),

$$\int_{t_1}^{t_2} g_m(t)\, g_n(t)\,dt = \begin{cases} 0 & \text{if } m \neq n \\ K_m & \text{if } m = n \end{cases}$$

If this function set is complete, then any function $x(t)$ can be expressed as:

$$x(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_r g_r(t) + \cdots$$

where
$$C_r = \frac{\int_{t_1}^{t_2} x(t)\, g_r(t)\,dt}{K_r} = \frac{\int_{t_1}^{t_2} x(t)\, g_r(t)\,dt}{\int_{t_1}^{t_2} g_r^2(t)\,dt}$$

Representation of a function $x(t)$ by a set of infinite mutually orthogonal functions is called *generalised Fourier series representation of* $x(t)$.

3.6 ORTHOGONALITY IN COMPLEX FUNCTIONS

Till now we have considered only real functions of real variable. If $x_1(t)$ and $x_2(t)$ are complex functions of real variable t, $x_1(t)$ can be approximated by $C_{12}x_2(t)$ over an interval (t_1, t_2)

$$x_1(t) \approx C_{12}x_2(t)$$

The optimum value of C_{12} to minimise the mean square error magnitude is given by

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2^*(t)\, dt}{\int_{t_1}^{t_2} x_2(t)\, x_2^*(t)\, dt}$$

where $x_2^*(t)$ is complex conjugate of $x_2(t)$.
For the complex functions to be orthogonal over the interval (t_1, t_2), $C_{12} = 0$.

$$\therefore \qquad \int_{t_1}^{t_2} x_1(t)\, x_2^*(t)\, dt = \int_{t_1}^{t_2} x_1^*(t)\, x_2(t)\, dt = 0$$

For a set of complex functions $\{g_r(t)\}$, $(r = 1, 2, \ldots)$ mutually orthogonal over the interval (t_1, t_2),

$$\int_{t_1}^{t_2} g_m(t)\, g_n^*(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ K_m & \text{if } m = n \end{cases}$$

If this set of functions is complete, then any function $x(t)$ can be expressed as:

$$x(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_r g_r(t) + \cdots$$

where
$$C_r = \frac{1}{K_r} \int_{t_1}^{t_2} x(t)\, g_r^*(t)\, dt$$

EXAMPLE 3.7 A rectangular function $x(t)$ is defined by

$$x(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ -1 & \text{for } \pi < t < 2\pi \end{cases}$$

Approximate the above rectangular function by a single sinusoid sin t over the interval $[0, 2\pi]$ such that the mean square error is minimum. Evaluate the mean square error in this approximation. Also show what happens when more number of sinusoids is used for approximation.

Solution: The given rectangular function $x(t)$ is shown in Figure 3.5(a). This function $x(t)$ can be approximated over the interval $[0, 2\pi]$ as:

$$x(t) \approx C_{12} \sin t$$

We shall find the optimum value of C_{12} which will minimise the mean square error in this approximation.

In general, the optimum value of C_{12} is:

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

In this case,

$$x_1(t) = x(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ -1 & \text{for } \pi < t < 2\pi \end{cases}$$

as shown in Figure 3.5(a) and $x_2(t) = \sin t$ as shown in Figure 3.5(b).

$$\therefore \qquad C_{12} = \frac{\int_0^{2\pi} x(t) \sin t\, dt}{\int_0^{2\pi} \sin^2(t)\, dt}$$

$$= \frac{\int_0^{\pi} (1) \sin t\, dt + \int_{\pi}^{2\pi} (-1) \sin t\, dt}{\int_0^{2\pi} \left[\frac{1 - \cos 2t}{2}\right] dt} = \frac{[-\cos t]_0^{\pi} - [-\cos t]_{\pi}^{2\pi}}{\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}}$$

$$= \frac{-[\cos \pi - \cos 0] + [\cos 2\pi - \cos \pi]}{\frac{1}{2}\left[(2\pi - 0 - \frac{\sin 4\pi - \sin 0}{2}\right]} = \frac{-[-1 - 1] + [1 - (-1)]}{\frac{1}{2}[2\pi]}$$

$$= \frac{4}{\pi}$$

Thus, $\qquad x(t) \approx \frac{4}{\pi} \sin t$

Thus, the best approximation of the rectangular function $x(t)$ by a sinusoidal function $\sin t$ with minimum mean square error is shown in Figure 3.5(c) and is given as:

$$x(t) \approx \frac{4}{\pi} \sin t$$

The mean square error can be evaluated using the formula

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t)\, dt - C_{12}^2 \int_{t_1}^{t_2} \sin^2(t)\, dt\right]$$

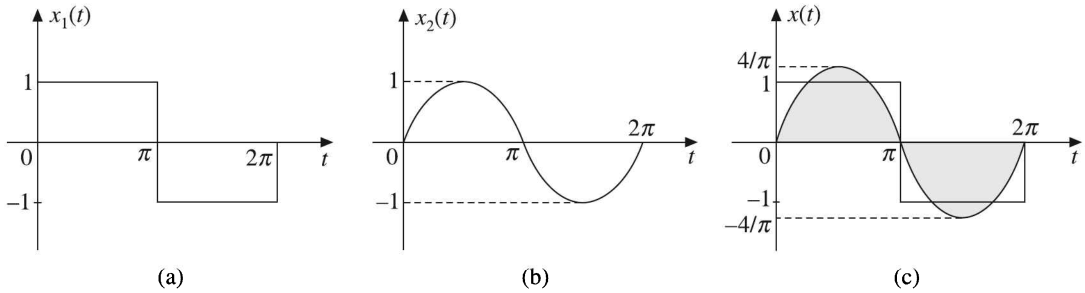


Figure 3.5 (a) Rectangular function, (b) Sine wave, (c) Approximation.

Here $t_1 = 0$ and $t_2 = 2\pi$.

$$\therefore \qquad \varepsilon = \frac{1}{2\pi - 0}\left[\int_0^{2\pi} 1\,dt - \left(\frac{4}{\pi}\right)^2 \int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)dt\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} dt - \frac{16}{\pi^2}\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)dt\right]$$

$$= \frac{1}{2\pi}\left[[t]_0^{2\pi} - \frac{16}{2\pi^2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left\{(2\pi - 0) - \frac{8}{\pi^2}\left[(2\pi - 0) - \frac{(\sin 4\pi - \sin 0)}{2}\right]\right\}$$

$$= \frac{1}{2\pi}\left[2\pi - \frac{16}{\pi}\right]$$

$$= 1 - \frac{8}{\pi^2}$$

$$= 0.189$$

$\therefore$ Mean square error $\varepsilon = 0.189 = 18.9\%$.

Approximation using a finite series of sinusoids

When the given rectangular function $x(t)$ is approximated using a finite series of sinusoids, we get

$$x(t) = \sum_{r=1}^{n} C_r \sin rt$$

where
$$C_r = \frac{\int_0^{2\pi} x(t)\sin rt\,dt}{\int_0^{2\pi} \sin^2 rt\,dt}$$

$$= \frac{\int_0^{\pi} (1)\sin rt\,dt + \int_{\pi}^{2\pi} (-1)\sin rt\,dt}{\int_0^{2\pi} \left(\frac{1-\cos 2rt}{2}\right) dt}$$

$$= \frac{\int_0^{\pi} \sin rt\,dt - \int_{\pi}^{2\pi} \sin rt\,dt}{\frac{1}{2}\left\{\int_0^{2\pi} (1-\cos 2rt)\,dt\right\}} = \frac{\left[-\frac{\cos rt}{r}\right]_0^{\pi} + \left[\frac{\cos rt}{r}\right]_{\pi}^{2\pi}}{\frac{1}{2}\left[t - \frac{\sin 2rt}{2r}\right]_0^{2\pi}}$$

$$= \frac{\frac{1}{r}[-\cos r\pi + \cos 0 + \cos 2r\pi - \cos r\pi]}{\frac{1}{2}\left[2\pi - 0 - \frac{\sin 4\pi r - \sin 0}{2r}\right]}$$

We have
$$\cos r\pi = (-1)^r$$
$$\cos 2r\pi = 1$$
$$\sin 4r\pi = 0$$

$\therefore$
$$C_r = \frac{\frac{1}{r}[-(-1)^r + 1 + 1 - (-1)^r]}{\frac{1}{2}[2\pi - 0]}$$

$$= \frac{2}{r\pi}[1 - (-1)^r]$$

$$C_r = \frac{2}{r\pi}[1 + (-1)^{r+1}]$$

$$C_r = \begin{cases} 4/r\pi & r = \text{odd} \\ 0 & r = \text{even} \end{cases}$$

Thus, the approximated function is given by

$$x(t) = \sum_{r=1}^{n} C_r \sin rt = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \cdots$$

Now, to evaluate the MSE, we have

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t)\,dt - \sum_{r=1}^{n} C_r^2 k_r\right]$$

where $$k_r = \int_{t_1}^{t_2} \sin^2 rt\,dt$$

$\therefore$ $$\varepsilon = \frac{1}{2\pi}\left[\int_0^{2\pi} x^2(t)\,dt - \sum_{r=1}^{n} C_r^2 k_r\right]$$

$$x^2(t) = 1 \qquad \text{for } 0 < t < 2\pi$$

$\therefore$ $$\int_0^{2\pi} x^2(t)\,dt = \int_0^{2\pi} dt = [t]_0^{2\pi} = 2\pi$$

$$C_r = \begin{cases} 4/r\pi & \text{for odd } r \\ 0 & \text{for even } r \end{cases}$$

$$k_r = \int_0^{2\pi} \sin^2 rt\,dt = \int_0^{2\pi}\left(\frac{1 - \cos 2rt}{2}\right) dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2rt}{2r}\right]_0^{2\pi} = \frac{1}{2}\left[2\pi - 0 - \frac{\sin 4\pi r - \sin 0}{2r}\right]$$

$$= \frac{1}{2}[2\pi - 0]$$

$\therefore$ $$k_r = \pi$$

First approximate $x(t)$ by a single sinusoid as:

$$x(t) = \frac{4}{\pi}\sin t$$

and $$\text{MSE}_1 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \pi\right] = \frac{1}{2\pi}\left[2\pi - \frac{16}{\pi}\right] = 0.189$$

Now, approximate $x(t)$ by two sinusoids. Therefore,

$$x(t) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t$$

and

$$\text{MSE}_2 = \frac{1}{2\pi}\left\{2\pi - \left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2\right]\pi\right\}$$

$$= \frac{1}{2\pi}\left[2\pi - \left(\frac{16}{\pi} + \frac{16}{9\pi}\right)\right]$$

$\therefore$ $$\text{MSE}_2 = 0.099$$

Similarly for three sinusoids, the approximation becomes

$$x(t) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin t + \frac{4}{5\pi}\sin 5t$$

and

$$\text{MSE}_3 = \frac{1}{2\pi}\left\{2\pi - \left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2\right]\pi\right\}$$

$$= \frac{1}{2\pi}\left[2\pi - \left(\frac{16}{\pi} + \frac{16}{9\pi} + \frac{16}{25\pi}\right)\right]$$

$\therefore$ $$\text{MSE}_3 = 0.066$$

We observe that as the number of sinusoids in the approximation increases, the MSE decreases. If the number of sinusoids is increased to infinity, then the MSE becomes zero.

EXAMPLE 3.8 Discuss how an unknown function $x(t)$ can be expressed using infinite mutually orthogonal functions. Hence show the representation of a waveform $x(t)$ using trigonometric Fourier series.

Solution:

Expression of function x(t) in terms of infinite mutually orthogonal functions

We know that an arbitrary function $x(t)$ can be represented as a sum of its components along a set of mutually orthogonal functions if these functions form a complete set.

Consider a set of infinite mutually orthogonal functions $x_1(t)$, $x_2(t)$..., $x_n(t)$.... Then the condition for them to be orthogonal over the interval (t_1 to t_2) is:

$$\int_{t_1}^{t_2} x_p(t)\, x_q(t) = \begin{cases} 0 & \text{if } p \neq q \\ k_p & \text{if } p = q \end{cases}$$

If $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ is complete, then any unknown function $x(t)$ can be expressed in terms of infinite sum of orthogonal functions as:

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t) + \cdots$$

where

$$C_n = \frac{\int_{t_1}^{t_2} x(t)\, x_n(t)\, dt}{\int_{t_1}^{t_2} x_n^2(t)\, dt}$$

Representation of a waveform x(t) using trigonometric Fourier series

Trigonometric Fourier series is the representation of any unknown signal by cosine and sine terms.

We know that the functions $\sin \omega_0 t$, $\sin 2\omega_0 t$, ... form an orthogonal set over any interval $[t_0,\ t_0 + (2\pi/\omega_0)]$.

The set, however, is not complete because there exists a function $\cos n\omega_0 t$, $n = 1, 2, \ldots$ orthogonal to $\sin m\omega_0 t$, $m = 1, 2, \ldots$.

So now the complete set of orthogonal functions is 1, $\cos \omega_0 t$, $\cos 2\omega_0 t$, ..., $\sin \omega_0 t$, $\sin 2\omega_0 t$, ... etc.

Any arbitrary function $x(t)$ can be expressed by the sum of its components along this set of mutually orthogonal functions, i.e.

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \cdots + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t; \qquad t_0 \le t \le t_0 + \frac{2\pi}{\omega_0};\ T = \frac{2\pi}{\omega_0}$$

where
$$a_n = \frac{\int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\, dt} = \frac{2}{T}\int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t\, dt$$

and
$$b_n = \frac{\int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \sin^2 n\omega_0 t\, dt} = \frac{2}{T}\int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t\, dt$$

and
$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, dt$$

MATLAB PROGRAMS

Program 3.1

```
% Testing orthogonality between two signals
clc;clear;
syms t
x1=input('Enter the first signal =');
x2=input('Enter the second signal =');
t1=input('tmin =');
```

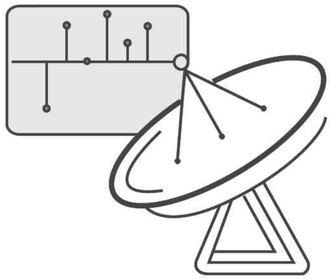
UNIT-II

Fourier Series: Representation of Continuous time periodic signals using Fourier series (203-211), Dirichlet's conditions (204), Trigonometric Fourier Series (204-213) and Exponential Fourier Series (229-233), Complex Fourier spectrum (233-242). Problems (250-276)

Fourier Transforms: Deriving Fourier Transform from Fourier series (298-300), Fourier Transform of standard signals (300-311), Fourier Transform of Periodic Signals (325-326), Properties of Fourier Transform (312-325), Inverse Fourier Transform (You will get in the problems), Introduction to Hilbert Transform (380-382). Problems (326-371)

4

Fourier Series Representation of Periodic Signals



4.1 INTRODUCTION

A signal is said to be a continuous-time signal if it is available at all instants of time. A real time naturally available signal is in the form of time domain. However, the analysis of a signal is far more convenient in the frequency domain. There are three important classes of transformation methods available for continuous-time systems. They are:

1. Fourier series
2. Fourier transform
3. Laplace transform

Out of these three methods, the Fourier series is applicable only to periodic signals, i.e. signals which repeat periodically over $-\infty < t < \infty$. Not all periodic signals can be represented by Fourier series. In this chapter, we discuss the conditions to be satisfied for a periodic signal to be represented by Fourier series, different types of Fourier series representations, conversion from one type to other and analysis of signals using these representations. Different types of symmetry present in the waveforms are also discussed and their utilisation in simplifying the computations is illustrated.

4.2 REPRESENTATION OF FOURIER SERIES

The representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions is called *Fourier series*. The Fourier analysis is also sometimes called the harmonic analysis. Fourier series is applicable only for periodic signals. It cannot be applied to non-periodic signals. A periodic signal is one which repeats itself at regular intervals of time, i.e. periodically over $-\infty$ to ∞. Three important classes of Fourier series methods are available. They are:

1. Trigonometric form
2. Cosine form
3. Exponential form

In the representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions, if the orthogonal functions are exponential functions, then it is called *exponential Fourier series*.

Similarly, in the representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions, if the orthogonal functions are trigonometric functions, then it is called *trigonometric Fourier series*.

4.3 EXISTENCE OF FOURIER SERIES

For the Fourier series to exist for a periodic signal, it must satisfy certain conditions.

The conditions under which a periodic signal can be represented by a Fourier series are known as *Dirichlet's conditions* named after the mathematician Dirichlet who first found them. They are as follows:

In each period,

1. The function $x(t)$ must be a single valued function.
2. The function $x(t)$ has only a finite number of maxima and minima.
3. The function $x(t)$ has a finite number of discontinuities.
4. The function $x(t)$ is absolutely integrable over one period, that is $\int_0^T |x(t)|\,dt < \infty$.

These are the sufficient but not necessary conditions for the existence of the Fourier series of a periodic function $x(t)$. Condition 4 is known as the *weak* Dirichlet condition. If a function satisfies the weak Dirichlet condition, the existence of Fourier series is guaranteed, but the series may not converge at every point. Conditions 2 and 3 are known as *strong Dirichlet conditions*. If these are satisfied, the convergence is also guaranteed.

4.4 TRIGONOMETRIC FORM OF FOURIER SERIES

A sinusoidal signal, $x(t) = A \sin \omega_0 t$ is a periodic signal with period $T = 2\pi/\omega_0$. Also, the sum of two sinusoids is periodic provided that their frequencies are integral multiples of a fundamental frequency ω_0. We can show that a signal $x(t)$, a sum of sine and cosine functions whose frequencies are integral multiples of ω_0, is a periodic signal.

Let the signal $x(t)$ be

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \cdots + a_k \cos k\omega_0 t$$

$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \cdots + b_k \sin k\omega_0 t$$

i.e.

$$x(t) = a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 nt + b_n \sin \omega_0 nt$$

where $a_0, a_1, a_2, \ldots, a_k$ and $b_0, b_1, \ldots, b_k$ are constants, and ω_0 is the fundamental frequency.

For the signal $x(t)$ to be periodic, it must satisfy the condition $x(t) = x(t + T)$ for all t, i.e.

$$x(t+T) = a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n(t+T) + b_n \sin \omega_0 n(t+T)$$

$$= a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 n\left(t + \frac{2\pi}{\omega_0}\right) + b_n \sin \omega_0 n\left(t+\frac{2\pi}{\omega_0}\right)$$

$$= a_0 + \sum_{n=1}^{k} a_n \cos(\omega_0 nt + 2n\pi) + b_n \sin(\omega_0 nt + 2n\pi)$$

$$= a_0 + \sum_{n=1}^{k} a_n \cos \omega_0 nt + b_n \sin \omega_0 nt$$

$$= x(t)$$

This proves that the signal $x(t)$, which is a summation of sine and cosine functions of frequencies 0, ω_0, $2\omega_0$, ..., $k\omega_0$, is a periodic signal with period T. By changing a_ns and b_ns, we can construct any periodic signal with period T. If $k \rightarrow \infty$ in the expression for $x(t)$, we obtain the Fourier series representation of any periodic signal $x(t)$. That is, any periodic signal can be represented as an infinite sum of sine and cosine functions which themselves are periodic signals of angular frequencies 0, ω_0, $2\omega_0$, ..., $k\omega_0$. This set of harmonically related sine and cosine functions, i.e. $\sin n\omega_0 t$ and $\cos n\omega_0 t$, $n = 0, 1, \ldots$ forms a complete orthogonal set over the interval t_0 to $t_0 + T$ where $T = 2\pi/\omega_0$.

The infinite series of sine and cosine terms of frequencies 0, ω_0, $2\omega_0$, ..., $k\omega_0$ is known as *trigonometric form of Fourier series* and can be written as:

$$x(t) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

or

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

where a_n and b_n are constants; the coefficient a_0 is called the dc component; $a_1 \cos \omega_0 t + b_1 \sin \omega_0 t$ the first harmonic, $a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t$ the second harmonic and $a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$ the nth harmonic. The constant $b_0 = 0$ because $\sin n\omega_0 t = 0$ for $n = 0$.

4.4.1 Evaluation of Fourier Coefficients of the Trigonometric Fourier Series

The constants a_0, a_1, ..., a_n, b_0, b_1, ..., b_n are called Fourier coefficients. To evaluate a_0, we shall integrate both sides of the equation for $x(t)$ over one period (t_0 to $t_0 + T$) at an arbitrary time t_0. Thus,

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \left[\sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]\,dt\right]$$

$$= a_0 T + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t\,dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t\,dt$$

We know that $\int_{t_0}^{t_0+T} \cos n\omega_0 t\,dt = 0$ and $\int_{t_0}^{t_0+T} \sin n\omega_0 t\,dt = 0$, since the net areas of sinusoids over complete periods are zero for any nonzero integer n and any time t_0. Hence, each of the integrals in the above summation is zero.

Thus, we obtain

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 T$$

or

$$\boxed{a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt}$$

To evaluate a_n and b_n, we can use the following results:

$$\int_{t_0}^{t_0+T} \cos n\omega_0 t \cos m\omega_0 t\,dt = \begin{cases} 0 & \text{for } m \neq n \\ T/2 & \text{for } m = n \neq 0 \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \sin m\omega_0 t\,dt = \begin{cases} 0 & \text{for } m \neq n \\ T/2 & \text{for } m = n \neq 0 \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \cos m\omega_0 t\,dt = 0 \quad \text{for all } m \text{ and } n$$

To find Fourier coefficients a_n, multiply the equation for $x(t)$ by $\cos m\omega_0 t$ and integrate over one period. That is,

$$\int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t\,dt = a_0 \int_{t_0}^{t_0+T} \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \cos m\omega_0 t\,dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \cos m\omega_0 t\,dt$$

The first and third integrals in the above equation are equal to zero and the second integral is equal to $T/2$ when $m = n$. Therefore,

$$\int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt = a_m \frac{T}{2}$$

i.e.

$$a_m = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt$$

or

$$\boxed{a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t \, dt}$$

To find b_n, multiply both sides of equation for $x(t)$ by $\sin m\omega_0 t$ and integrate over one period. Then

$$\int_{t_0}^{t_0+T} x(t) \sin m\omega_0 t \, dt = a_0 \int_{t_0}^{t_0+T} \sin m\omega_0 t \, dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \sin m\omega_0 t \, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \sin m\omega_0 t \, dt$$

The first and second integrals in the above equation are zero, and the third integral is equal to $T/2$ when $m = n$. Thus, we have

$$\int_{t_0}^{t_0+T} x(t) \sin m\omega_0 t \, dt = b_m \frac{T}{2}$$

$\therefore$

$$b_m = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin m\omega_0 t \, dt$$

or

$$\boxed{b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t \, dt}$$

a_0, a_n and b_n are called *trigonometric Fourier series coefficients.*

A periodic signal has the same Fourier series for the entire interval $-\infty$ to ∞ as for the interval t_0 to $t_0 + T$, since the same function repeats after every T seconds. The Fourier series expansion of a periodic function is unique irrespective of the location of t_0 of the signal.

EXAMPLE 4.1 Find the Fourier series expansion of the half wave rectified sine wave shown in Figure 4.1.

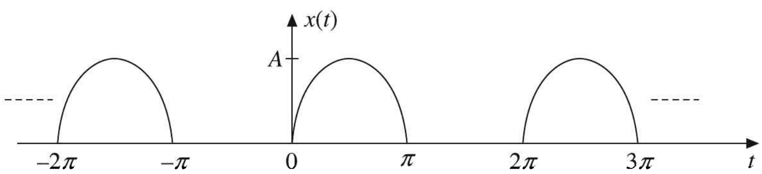


Figure 4.1 Waveform for Example 4.1.

Solution: The periodic waveform shown in Figure 4.1 with period 2π is half of a sine wave with period 2π.

$$x(t) = \begin{cases} A \sin \omega t = A \sin \dfrac{2\pi}{2\pi} t = A \sin t & 0 \le t \le \pi \\ 0 & \pi \le t \le 2\pi \end{cases}$$

Now the fundamental period $T = 2\pi$

$$\text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Let $\quad t_0 = 0, \; t_0 + T = T = 2\pi$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} A \sin t\,dt$$

$$= \frac{A}{2\pi}\left[-\cos t\right]_0^{\pi} = \frac{A}{2\pi}\left[-(\cos \pi - \cos 0)\right] = \frac{2A}{2\pi} = \frac{A}{\pi}$$

$\therefore$
$$a_0 = \frac{A}{\pi}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} A \sin t \cos nt\,dt = \frac{A}{\pi}\int_0^{\pi} \sin t \cos nt\,dt$$

$$= \frac{A}{2\pi}\int_0^{\pi} \sin(1+n)t + \sin(1-n)t = \frac{A}{2\pi}\left[-\frac{\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$$

$$= -\frac{A}{2\pi}\left[\frac{\cos(1+n)\pi - \cos 0}{1+n} + \frac{\cos(1-n)\pi - \cos 0}{1-n}\right]$$

$$= -\frac{A}{2\pi}\left\{\left[\frac{(-1)^{n+1} - 1}{1+n}\right] + \frac{(-1)^{n-1} - 1}{1-n}\right\}$$

For odd n,
$$a_n = -\frac{A}{2\pi}\left[\frac{1-1}{1+n} + \frac{1-1}{1-n}\right] = 0$$

For even n,
$$a_n = -\frac{A}{2\pi}\left[\frac{-1-1}{1+n} + \frac{-1-1}{1-n}\right] = \frac{A}{2\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right] = -\frac{2A}{\pi(n^2-1)}$$

$\therefore$
$$a_n = -\frac{2A}{\pi(n^2-1)} \qquad \text{(for even } n\text{)}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\sin nt\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} A\sin t\sin nt\,dt = \frac{A}{\pi}\int_0^{\pi}\sin t\sin nt\,dt$$

$$= \frac{A}{2\pi}\int_0^{\pi}\left[\cos(n-1)t - \cos(n+1)t\right]dt$$

$$= \frac{A}{2\pi}\left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right]_0^{\pi}$$

This is zero for all values of n except for $n = 1$.

For $n = 1$, $$b_1 = \frac{A}{2\pi}$$

Therefore, the trigonometric Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= a_0 + b_1 \sin t + \sum_{n=1}^{\infty} a_n \cos nt$$

$$= \frac{A}{\pi} + \frac{A}{2\pi}\sin t - \sum_{n=\text{even}}^{\infty} \frac{2A}{\pi(n^2-1)}\cos nt$$

EXAMPLE 4.2 Obtain the trigonometric Fourier series for the waveform shown in Figure 4.2

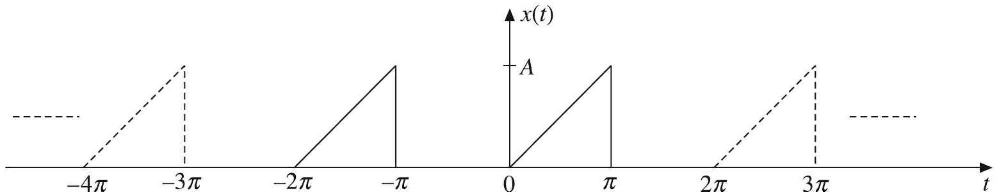


Figure 4.2 Waveform for Example 4.2.

Solution: The waveform shown in Figure 4.2 is periodic with a period $T = 2\pi$.

Let $t_0 = 0,\ t_0 + T = 2\pi$

Then, Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

The waveform is described by

$$x(t) = \begin{cases} (A/\pi)t & \text{for } 0 \le t \le \pi \\ 0 & \text{for } \pi \le t \le 2\pi \end{cases}$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} \frac{A}{\pi} t\, dt = \frac{A}{2\pi^2}\left[\frac{t^2}{2}\right]_0^{\pi} = \frac{A}{4}$$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} \frac{A}{\pi} t \cos nt\, dt = \frac{A}{\pi^2}\int_0^{\pi} t \cos nt\, dt$$

$$= \frac{A}{\pi^2}\left[\left[\frac{t \sin nt}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin nt}{n}\, dt\right] = \frac{A}{\pi^2}\left[\frac{0-0}{n} + \left(\frac{\cos nt}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{A}{\pi^2 n^2}(\cos n\pi - \cos 0)$$

$$\therefore \quad a_n = \begin{cases} -(2A/\pi^2 n^2) & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

$$b_n = \frac{2}{T}\int_0^T x(t) \sin n\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x(t) \sin nt\, dt = \frac{1}{\pi}\int_0^{\pi} \frac{A}{\pi} t \sin nt\, dt$$

$$= \frac{A}{\pi^2}\int_0^{\pi} t \sin nt\, dt = \frac{A}{\pi^2}\left[\left[\frac{t(-\cos nt)}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nt}{n}\, dt\right]$$

$$= \frac{A}{\pi^2}\left[-\pi\frac{\cos n\pi}{n} + \left(\frac{\sin nt}{n^2}\right)_0^{\pi}\right]$$

$$= -\frac{A}{n\pi}\cos n\pi = \frac{A}{n\pi}(-1)^{n+1}$$

$$\therefore \quad b_n = \begin{cases} A/n\pi & \text{for odd } n \\ -(A/n\pi) & \text{for even } n \end{cases}$$

The trigonometric Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= \frac{A}{4} - \frac{2A}{\pi^2} \sum_{n=\text{odd}}^{\infty} \frac{\cos nt}{n^2} + \frac{A}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$

$$= \frac{A}{4} - \frac{2A}{\pi^2} \left[\cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \cdots \right]$$

$$+ \frac{A}{\pi} \left[\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \frac{1}{4} \sin 4t + \cdots \right]$$

4.5 COSINE REPRESENTATION (ALTERNATE FORM OF THE TRIGONOMETRIC REPRESENTATION)

The trigonometric Fourier series of $x(t)$ contains sine and cosine terms of the same frequency.

i.e. $$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\omega_0 t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin n\omega_0 t \right)$$

Substituting the values $A_0 = a_0$, $A_n = \sqrt{a_n^2 + b_n^2}$, $\cos \theta_n = \dfrac{a_n}{\sqrt{a_n^2 + b_n^2}}$ and $\sin \theta_n = -\dfrac{b_n}{\sqrt{a_n^2 + b_n^2}}$,

$\left(\text{i.e. } \theta_n = -\tan^{-1} \dfrac{b_n}{a_n} \right)$, we have

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \left[\cos \theta_n \cos n\omega_0 t - \sin \theta_n \sin n\omega_0 t \right]$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \left[\cos (n\omega_0 t + \theta_n) \right]$$

This is the cosine representation of $x(t)$ which contains sinusoids of frequencies ω_0, $2\omega_0$, ...

The term A_0 is called the dc component, and the term $A_n \cos (n\omega_0 t + \theta_n)$ is called the nth harmonic component of $x(t)$. The first harmonic component $A_1 \cos (\omega_0 t + \theta_1)$ is commonly called the fundamental component as it has the same fundamental period as $x(t)$.

The number A_n represents *amplitude coefficients* or *harmonic amplitudes* or *spectral amplitudes* of the Fourier series and the number θ_n represents the *phase coefficients* or *phase angles* or *spectral phase* of the Fourier series.

The cosine form is also called the *Harmonic form Fourier series* or *Polar form Fourier series*.

EXAMPLE 4.3 Find the cosine Fourier series for the waveform shown in Figure 4.3.

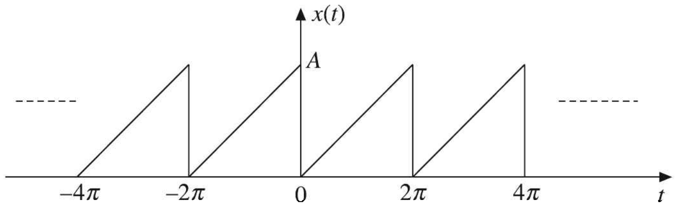


Figure 4.3 Waveform for Example 4.3.

Solution: The waveform shown in Figure 4.3 is given by

$$x(t) = \frac{A}{2\pi} t, \quad \text{for } 0 \le t \le 2\pi$$

Let $$t_0 = 0$$

$\therefore$ $$t_0 + T = T = 2\pi$$

and $$\text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Trigonometric Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

The coefficients are evaluated as follows:

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt = \frac{1}{T} \int_0^T x(t)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} t\, dt = \frac{A}{(2\pi)^2} \left[\frac{t^2}{2}\right]_0^{2\pi} = \frac{A}{2}$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t\, dt = \frac{2}{2\pi} \int_0^{2\pi} \left(\frac{A}{2\pi} t\right) \cos nt\, dt$$

$$= \frac{2A}{(2\pi)^2} \int_0^{2\pi} t \cos nt\, dt = \frac{2A}{(2\pi)^2} \left\{\left[\frac{t(\sin nt)}{n}\right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin nt}{n} dt\right\}$$

$$= \frac{2A}{(2\pi)^2} \left[0 - 0 - \frac{1}{n}\left(-\frac{\cos\ nt}{n}\right)_0^{2\pi}\right] = \frac{2A}{(2\pi)^2} \left[0 - 0 + \frac{1}{n} \frac{(1-1)}{n}\right] = 0$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t\, dt = \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{A}{2\pi}t\right)\sin nt\, dt$$

$$= \frac{2A}{(2\pi)^2}\int_0^{2\pi} t\sin nt\, dt = \frac{2A}{(2\pi)^2}\left\{\left[\frac{t(-\cos nt)}{n}\right]_0^{2\pi} - \int_0^{2\pi}\left(-\frac{\cos nt}{n}\right)dt\right\}$$

$$= \frac{2A}{(2\pi)^2}\left\{\left[\frac{t(-\cos nt)}{n}\right]_0^{2\pi} + \frac{1}{n}\left(\frac{\sin nt}{n}\right)_0^{2\pi}\right\} = \frac{2A}{(2\pi)^2}\left[-2\pi\frac{[1]}{n} - 0 + \frac{1}{n}\frac{(0-0)}{n}\right]$$

$$= \frac{2A}{(2\pi)^2}\left(-\frac{2\pi}{n}\right) = -\frac{A}{\pi n}$$

$$\therefore \qquad a_0 = \frac{A}{2};\ a_n = 0 \text{ and } b_n = -\frac{A}{\pi n}$$

are the trigonometric Fourier series coefficients.

$$\therefore \qquad x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} -\frac{A}{n\pi}\sin nt = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin nt}{n} \qquad 0 \le t \le 2\pi$$

is the trigonometric Fourier series representation.

Cosine form of representation

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

$$A_0 = a_0 = \frac{A}{2}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{0^2 + \left(-\frac{A}{\pi n}\right)^2} = \frac{A}{n\pi}$$

$$\theta_n = -\tan^{-1}\frac{b_n}{a_n} = -\tan^{-1}\left(\frac{-A/n\pi}{0}\right) = -\frac{\pi}{2}$$

$$\therefore \qquad x(t) = \frac{A}{2} + \sum_{n=1}^{\infty}\frac{A}{n\pi}\cos\left(nt - \frac{\pi}{2}\right) \text{ is the cosine representation}$$

4.6 WAVE SYMMETRY

If the periodic signal $x(t)$ has some type of symmetry, then some of the trigonometric Fourier series coefficients may become zero and calculation of the coefficients becomes simple.

There are following four types of symmetry a function $x(t)$ can have:

1. Even symmetry
2. Odd symmetry
3. Half wave symmetry
4. Quarter wave symmetry

Then, $b_n = \dfrac{8}{T}\displaystyle\int_0^{T/4} x(t)\sin n\omega_0 t\, dt$ (for odd n)

$$= \frac{8}{2\pi}\int_0^{\pi/2} \frac{2A}{\pi} t\sin nt\, dt = \frac{8A}{\pi^2}\left\{\left[t\frac{(-\cos nt)}{n}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{(-\cos nt)}{n}dt\right\}$$

$$= \frac{8A}{\pi^2}\left[\frac{1}{n}\left(-\frac{\pi}{2}\cos\frac{n\pi}{2}\right)+\left(\frac{\sin nt}{n^2}\right)_0^{\pi/2}\right] = \frac{8A}{\pi^2}\left[-\frac{\pi}{2n}(0)+\frac{\sin n(\pi/2)}{n^2}\right] \quad \text{(for odd } n\text{)}$$

$$= \frac{8A}{\pi^2 n^2}\sin\frac{n\pi}{2}$$

Hidden symmetry

Sometimes we come across a periodic function without any apparent form of symmetry, as it possesses a Fourier series containing only sine or cosine terms. Example 4.3 (Figure 4.3) illustrates this point. Here the periodic function does not satisfy either of the symmetry conditions, yet the Fourier series consists of a dc term and sine terms alone. The reason for this behaviour is that the symmetry of the function is obscured by the dc term. If we subtract a constant $A/2$ from this function, the new function $x_n(t)$ is antisymmetrical, i.e.

$$x_n(t) = x(t) - \frac{A}{2}$$

The subtraction of a constant term from $x(t)$ merely shifts the horizontal axis upwards by the amount $A/2$. It is evident that $x_n(t)$ is an odd periodic function, and hence the Fourier series of $x_n(t)$ consists entirely of sine terms. Therefore, the Fourier series for $x(t)$ consists of a dc term ($A/2$) and sine terms alone.

4.7 EXPONENTIAL FOURIER SERIES

The exponential Fourier series is the most widely used form of Fourier series. In this, the function $x(t)$ is expressed as a weighted sum of the complex exponential functions. Although the trigonometric form and the cosine representation are the common forms of Fourier series, the complex exponential form is more general and usually more convenient and more compact. So, it is used almost exclusively, and it finds extensive application in communication theory.

The set of complex exponential functions

$$\{e^{jn\omega_0 t}, n = 0, \pm 1, \pm 2, \ldots\}$$

forms a closed orthogonal set over an interval $(t_0, t_0 + T)$ where $T = (2\pi/\omega_0)$ for any value of t_0, and therefore it can be used as a Fourier series. Using Euler's identity, we can write

$$A_n \cos(n\omega_0 t + \theta_n) = A_n\left[\frac{e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}}{2}\right]$$

Substituting this in the definition of the cosine Fourier representation, we obtain

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2}\left[e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}\right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2}\left[e^{jn\omega_0 t}e^{j\theta_n} + e^{-jn\omega_0 t}e^{-j\theta_n}\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j\omega_0 nt}e^{j\theta_n}\right) + \left(\sum_{n=1}^{\infty}\frac{A_n}{2} e^{-j\omega_0 nt}\, e^{-j\theta_n}\right)$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j\theta_n}\right)e^{j\omega_0 nt} + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j(-\theta_n)}\right)e^{-j\omega_0 nt}$$

Letting $n = -k$ in the second summation of the above equation, we have

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j\theta_n}\right)e^{j\omega_0 nt} + \sum_{k=-1}^{-\infty}\left(\frac{A_k}{2} e^{j\theta_k}\right)e^{jk\omega_0 t}$$

On comparing the above two equations for $x(t)$, we get

$$A_n = A_k; \quad (-\theta_n) = \theta_k \quad \begin{matrix} n > 0 \\ k < 0 \end{matrix}$$

Let us define

$$C_0 = A_0; \quad C_n = \frac{A_n}{2}e^{j\theta_n}, \quad n > 0$$

$\therefore$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{j\theta_n}\, e^{j\omega_0 nt} + \sum_{n=-1}^{-\infty} \frac{A_n}{2} e^{j\theta_n}\, e^{jn\omega_0 t}$$

i.e.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

This is known as exponential form of Fourier series. The above equation is called the *synthesis equation.*

So the exponential series from cosine series is:

$$C_0 = A_0$$

$$C_n = \frac{A_n}{2} e^{j\theta_n}$$

4.7.1 Determination of the Coefficients of Exponential Fourier Series

We have

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}; \; \omega_0 = \frac{2\pi}{T}$$

Multiplying both sides by $e^{-jk\omega_0 t}$ and integrating over one period, we get

$$\int_{t_0}^{t_0+T} x(t)\, e^{-jk\omega_0 t}\, dt = \int_{t_0}^{t_0+T} \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \right) e^{-jk\omega_0 t}\, dt = \sum_{n=-\infty}^{\infty} C_n \int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t}\, dt$$

We know that

$$\int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t}\, dt = \begin{cases} 0 & \text{for } k \neq n \\ T & \text{for } k = n \end{cases}$$

$$\therefore \qquad \int_{t_0}^{t_0+T} x(t)\, e^{-jk\omega_0 t}\, dt = TC_k$$

or

$$C_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, e^{-jk\omega_0 t}\, dt$$

or

$$\boxed{C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt}$$

where C_n are the Fourier coefficients of the exponential series. The above equation is called the *analysis equation.*

$$\boxed{C_0 = A_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt}$$

The Fourier coefficients of $x(t)$ have only a discrete spectrum because values of C_n exist only for discrete values of n. As it represents a complex spectrum, it has both magnitude and phase spectra. The following points may be noted:

1. The magnitude line spectrum is always an even function of n.
2. The phase line spectrum is always an odd function of n.

4.7.2 Trigonometric Fourier Series from Exponential Fourier Series

The complex exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = C_0 + \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} (C_{-n} e^{-jn\omega_0 t} + C_n e^{jn\omega_0 t})$$

$$= C_0 + \sum_{n=1}^{\infty} C_{-n} (\cos n\omega_0 t - j \sin n\omega_0 t) + C_n (\cos n\omega_0 t + j \sin n\omega_0 t)$$

$$\therefore \qquad x(t) = C_0 + \sum_{n=1}^{\infty} (C_n + C_{-n}) \cos n\omega_0 t + j(C_n - C_{-n}) \sin n\omega_0 t$$

Comparing this $x(t)$ with the standard trigonometric Fourier series, i.e. with

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

we get the formulae for conversion of exponential Fourier series to trigonometric Fourier series as:

$$a_0 = C_0$$

$$a_n = C_n + C_{-n}$$

$$b_n = j(C_n - C_{-n})$$

Similarly, an exponential Fourier series can be derived from the trigonometric Fourier series (section 4.7.3).

4.7.3 Exponential Fourier Series from Trigonometric Fourier Series

From the exponential Fourier series, we know that

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_0^T x(t)\,[\cos n\omega_0 t - j \sin n\omega_0 t]\, dt$$

$$= \frac{1}{2}\left(\frac{2}{T}\int_0^T x(t) \cos n\omega_0 t\, dt - j\frac{2}{T}\int_0^T x(t) \sin n\omega_0 t\, dt\right) = \frac{1}{2}[a_n - jb_n]$$

$$C_{-n} = \frac{1}{T}\int_0^T x(t) e^{jn\omega_0 t} dt = \frac{1}{T}\int_0^T x(t)\,[\cos n\omega_0 t + j \sin n\omega_0 t]\, dt$$

$$= \frac{1}{2}\left(\frac{2}{T}\int_0^T x(t) \cos n\omega_0 t\, dt + j\frac{2}{T}\int_0^T x(t) \sin n\omega_0 t\, dt\right) = \frac{1}{2}[a_n + jb_n]$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\, dt = a_0$$

So, the formulae for conversion of trigonometric series to exponential series are:

$$C_0 = a_0$$

$$C_n = \frac{1}{2}(a_n - jb_n)$$

$$C_{-n} = \frac{1}{2}(a_n + jb_n)$$

4.7.4 Cosine Fourier Series from Exponential Fourier Series

We know that

$$A_0 = a_0 = C_0$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{(C_n + C_{-n})^2 + [j(C_n - C_{-n})]^2}$$

$$= \sqrt{(C_n^2 + C_{-n}^2 + 2C_nC_{-n}) - (C_n^2 + C_{-n}^2 - 2C_nC_{-n})}$$

$$= \sqrt{4C_nC_{-n}}$$

$$= 2|C_n|$$

$$\therefore \qquad A_0 = C_0$$

$$A_n = 2\ |C_n|$$

4.8 FOURIER SPECTRUM

The Fourier spectrum of a periodic signal $x(t)$ is a plot of its Fourier coefficients versus frequency ω. It is in two parts: (a) Amplitude spectrum and (b) Phase spectrum. The plot of the amplitude of Fourier coefficients versus frequency is known as the *amplitude spectra*, and the plot of the phase of Fourier coefficients versus frequency is known as the *phase spectra*. The two plots together are known as Fourier frequency spectra of $x(t)$. This type of representation is also called frequency domain representation. The Fourier spectrum exists only at discrete frequencies $n\omega_0$, where $n = 0, 1, 2, \ldots$. Hence it is known as discrete spectrum or line spectrum. The envelope of the spectrum depends only upon the pulse shape, but not upon the period of repetition.

The trigonometric representation of a periodic signal $x(t)$ contains both sine and cosine terms with positive and negative amplitude coefficients (a_n and b_n) but with no phase angles.

The cosine representation of a periodic signal contains only positive amplitude coefficients with phase angle θ_n. Therefore, we can plot amplitude spectra (A_n versus ω) and phase spectra (θ_n versus ω). Since, in this representation, Fourier coefficients exist only for positive frequencies, this spectra is called *single-sided spectra*.

The exponential representation of a periodic signal $x(t)$ contains amplitude coefficients C_n which are complex. Hence, they can be represented by magnitude and phase. Therefore, we can plot two spectra; the magnitude spectrum ($|C_n|$ versus ω) and phase spectrum ($\angle C_n$ versus ω). The spectra can be plotted for both positive and negative frequencies. Hence, it is called a *two-sided spectra*.

Figure 4.15(a) represents the spectrum of a trigonometric Fourier series extending from 0 to ∞, producing a one-sided spectrum as no negative frequencies exist here. Figure 4.15(b) represents the spectrum of a complex exponential Fourier series extending from $-\infty$ to ∞, producing a two-sided spectrum.

The amplitude spectrum of the exponential Fourier series is symmetrical about the vertical axis. This is true for all periodic functions.

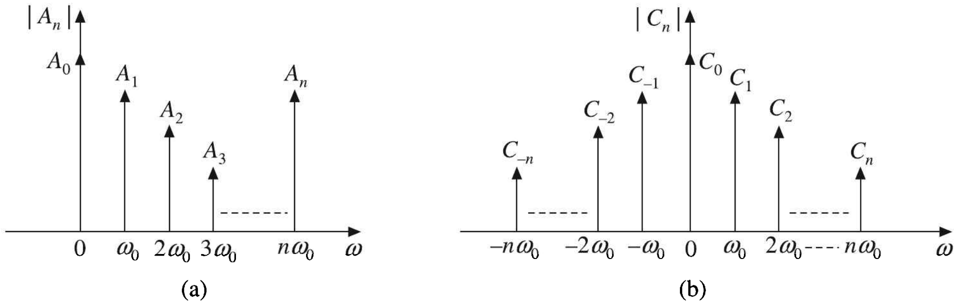


Figure 4.15 Complex frequency spectrum for (a) Trigonometric Fourier series and (b) Complex exponential Fourier series.

If C_n is a general complex number, then

$$C_n = |C_n|e^{j\theta_n}$$

$$C_{-n} = |C_n|e^{-j\theta_n}$$

$$|C_n| = |C_{-n}|$$

The magnitude spectrum is symmetrical about the vertical axis passing through the origin, and the phase spectrum is antisymmetrical about the vertical axis passing through the origin. So the magnitude spectrum exhibits even symmetry and phase spectrum exhibits odd symmetry.

When $x(t)$ is real, then $C_{-n} = C_n^*$, the complex conjugate of C_n.

EXAMPLE 4.11 Obtain the exponential Fourier series for the waveform shown in Figure 4.16. Also draw the frequency spectrum.

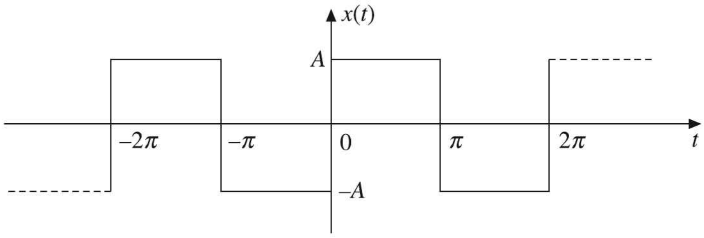


Figure 4.16 Waveform for Example 4.11.

Solution: The periodic waveform shown in Figure 4.16 with a period $T = 2\pi$ can be expressed as:

$$x(t) = \begin{cases} A & 0 \le t \le \pi \\ -A & \pi \le t \le 2\pi \end{cases}$$

Let $\qquad t_0 = 0$

$\therefore$ $$t_0 + T = 2\pi$$

and Fundamental frequency $\omega_0 = \dfrac{2\pi}{T} = \dfrac{2\pi}{2\pi} = 1$

Exponential Fourier series

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\left(\int_0^{\pi} A\,dt + \int_{\pi}^{2\pi} -A\,dt\right) = \frac{A}{2\pi}\left[(t)_0^{\pi} - (t)_{\pi}^{2\pi}\right] = 0$$

$$C_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi} Ae^{-jnt}\,dt + \int_{\pi}^{2\pi} -Ae^{-jnt}\,dt\right) = \frac{A}{2\pi}\left[\left(\frac{e^{-jnt}}{-jn}\right)_0^{\pi} - \left(\frac{e^{-jnt}}{-jn}\right)_{\pi}^{2\pi}\right]$$

$$= -\frac{A}{j2n\pi}\left[(e^{-jn\pi} - e^0) - (e^{-j2n\pi} - e^{-jn\pi})\right]$$

$$= -\frac{A}{j2n\pi}\left[(-1)^n - 1\right] - \left[1 - (-1)^n\right] = -j\frac{2A}{n\pi}$$

$$\therefore \qquad C_n = \begin{cases} -j\dfrac{2A}{n\pi} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

$$\therefore \qquad x(t) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} -j\frac{2A}{n\pi}e^{jnt} \quad \text{for odd } n$$

$$\therefore \qquad C_0 = 0,\; C_1 = C_{-1} = \frac{2A}{\pi},\quad C_3 = C_{-3} = \frac{2A}{3\pi},\quad C_5 = C_{-5} = \frac{2A}{5\pi}$$

The frequency spectrum is shown in Figure 4.17.

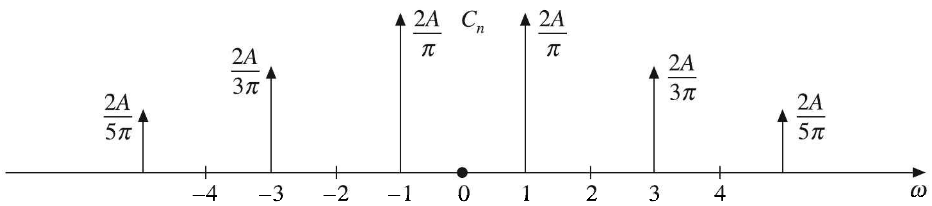


Figure 4.17 Frequency spectrum.

EXAMPLE 4.12 Find the exponential Fourier series and plot the frequency spectrum for the full wave rectified sine wave given in Figure 4.18.

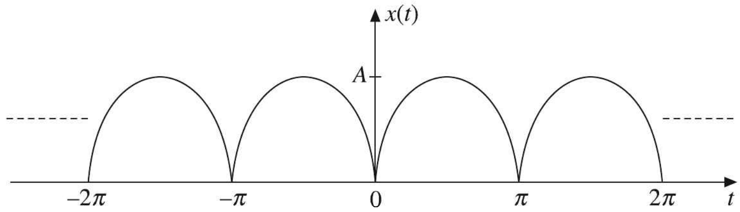


Figure 4.18 Waveform for Example 4.12.

Solution: The waveform shown in Figure 4.18 can be expressed over one period (0 to π) as:

$$x(t) = A \sin \omega t \quad \text{where } \omega = \frac{2\pi}{2\pi} = 1$$

because it is part of a sine wave with period = 2π.

$$x(t) = A \sin t \quad 0 \le t \le \pi$$

The full wave rectified sine wave is periodic with period $T = \pi$.

Let $$t_0 = 0$$

$\therefore$ $$t_0 + T = 0 + \pi = \pi$$

and $$\text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$$

The exponential Fourier series is:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2nt}$$

where $$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} A \sin t\, e^{-j2nt}\, dt = \frac{A}{\pi}\int_0^{\pi} \sin te^{-j2nt}\, dt$$

$$= \frac{A}{\pi}\int_0^{\pi}\left(\frac{e^{jt} - e^{-jt}}{2j}\right) e^{-j2nt}\, dt = \frac{A}{j2\pi}\left(\int_0^{\pi} e^{j(1-2n)t} - e^{-j(1+2n)t}\right) dt$$

$$= \frac{A}{j2\pi}\left(\frac{e^{j(1-2n)t}}{j(1-2n)} - \frac{e^{-j(1+2n)t}}{-j(1+2n)}\right)_0^{\pi}$$

$$= \frac{A}{j2\pi}\left[\frac{e^{j(1-2n)\pi} - e^0}{j(1-2n)} - \frac{e^{-j(1+2n)\pi} - e^0}{-j(1+2n)}\right]$$

$\therefore$ $$C_n = \frac{2A}{\pi(1-4n^2)}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{\pi}\int_0^{\pi} A\sin t\,dt = \frac{A}{\pi}[-\cos t]_0^{\pi} = \frac{A}{\pi}[-(\cos\pi - \cos 0)] = \frac{2A}{\pi}$$

The exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{j2nt} = \frac{2A}{\pi} + \frac{2A}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{e^{j2nt}}{1-4n^2}\right)$$

$\therefore$

$$C_0 = \frac{2A}{\pi}$$

$$C_1 = C_{-1} = \frac{2A}{\pi[1-4(1)^2]} = -\frac{2A}{3\pi}$$

$$C_2 = C_{-2} = \frac{2A}{\pi[1-4(2)^2]} = \frac{-2A}{15\pi}$$

$$C_3 = C_{-3} = \frac{2A}{\pi[1-4(3)^2]} = \frac{-2A}{35\pi}$$

The frequency spectrum is shown in Figure 4.19.

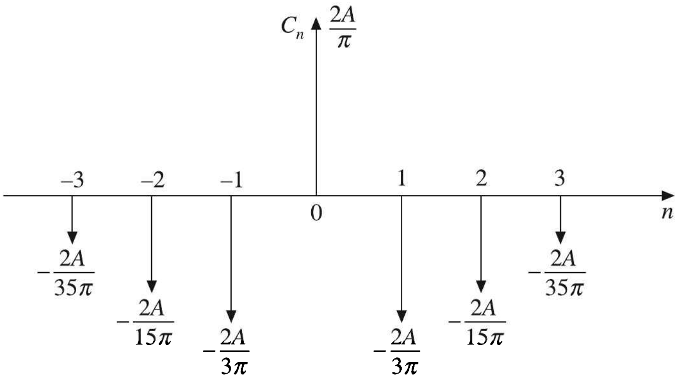


Figure 4.19 Frequency spectrum.

EXAMPLE 4.13 Find the exponential Fourier series for the rectified sine wave shown in Figure 4.20

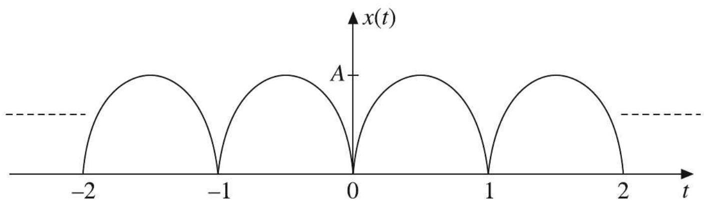


Figure 4.20 Waveform for Example 4.13.

Solution: The waveform shown in Figure 4.20 is a part of a sine wave with period = 2.

$\therefore$
$$x(t) = A\sin\omega t \quad 0 \le t \le 1, \text{ i.e. } \omega = \frac{2\pi}{2} = \pi$$

Hence

$$x(t) = A\sin\pi t \quad \text{for } 0 \le t \le 1$$

The period of the rectified sine wave is $T = 1$.

Let
$$t_0 = 0$$

$\therefore$
$$t_0 + T = 1$$

The fundamental frequency of the rectified sine wave is:

$$\omega_0 = \frac{2\pi}{T} = 2\pi$$

The exponential Fourier series is:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2n\pi t}$$

where

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{1}\int_0^1 A\sin\pi t\, e^{-j2n\pi t}\, dt$$

$$= \frac{A}{2j}\int_0^1 \left[e^{j\pi t}e^{-j2n\pi t} - e^{-j\pi t}e^{-j2n\pi t}\right] = \frac{A}{2j}\left[\int_0^1 e^{j\pi(1-2n)t}\, dt - \int_0^1 e^{-j\pi(1+2n)t}\, dt\right]$$

$$= \frac{A}{2j}\left[\left(\frac{e^{j\pi(1-2n)t}}{j\pi(1-2n)}\right)_0^1 - \left(\frac{e^{-j\pi(1+2n)t}}{-j\pi(1+2n)}\right)_0^1\right]$$

$$= \frac{2A}{\pi(1-4n^2)}$$

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, dt = \frac{1}{1}\int_0^1 A\sin\pi t\, dt = A\left[\frac{-\cos\pi t}{\pi}\right]_0^1 = \frac{2A}{\pi}$$

The exponential Fourier series is:

$$x(t) = \frac{2A}{\pi} + \sum_{\substack{n=-\infty \\ n \ne 0}}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{j2n\pi t}$$

EXAMPLE 4.14 Derive the exponential Fourier series from the trigonometric Fourier series for the waveform shown in Figure 4.2 (Example 4.2). Also find the exponential Fourier series directly.

Solution: *Derivation of exponential series from trigonometric series*

In Example 4.2, we got the trigonometric Fourier coefficients as:

$$a_0 = \frac{A}{4},\ a_n = -\frac{2A}{n^2\pi^2} \quad \text{(for odd } n) \quad \text{and} \quad b_n = \frac{A}{n\pi}(-1)^{n+1}$$

$$C_0 = a_0 = \frac{A}{4}$$

$$C_n = \frac{1}{2}\left[a_n - jb_n\right] = \frac{1}{2}\left[-\frac{2A}{n^2\pi^2} - j\frac{A}{n\pi}(-1)^{n+1}\right]$$

Direct derivation of exponential Fourier series

$$x(t) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{jn\omega_0 t} = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{jnt}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\int_0^{\pi}\frac{A}{\pi}t\,dt = \frac{A}{2\pi^2}\left(\frac{t^2}{2}\right)_0^{\pi} = \frac{A}{4}$$

$$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-jnt}\,dt = \frac{1}{2\pi}\int_0^{\pi}\frac{A}{\pi}te^{-jnt}dt$$

$$= \frac{A}{2\pi^2}\left[\left(t\frac{e^{-jnt}}{-jn}\right)_0^{\pi} - \int_0^{\pi}\frac{e^{-jnt}}{-jn}dt\right] = \frac{A}{2\pi^2}\left[\left(\frac{\pi e^{-jn\pi}}{-jn} - 0\right) + \frac{1}{jn}\left(\frac{e^{-jnt}}{-jn}\right)_0^{\pi}\right]$$

$$= \frac{A}{2\pi^2}\left[-\frac{\pi}{jn}(-1)^n - \frac{1}{(jn)^2}(e^{-jn\pi} - 1)\right]$$

$$= \frac{A}{2\pi^2}\left\{-\frac{\pi}{jn}(-1)^n - \frac{1}{(jn)^2}[(-1)^n - 1]\right\}$$

$$\therefore \qquad x(t) = \frac{A}{4} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\frac{A}{2\pi^2}\left[-\frac{\pi}{jn}(-1)^n - \frac{1}{(jn)^2}[(-1)^n - 1]\right]e^{-jnt}$$

EXAMPLE 4.15 Derive the exponential Fourier series from the trigonometric Fourier series for the waveform shown in Figure 4.3 (Example 4.3). Also determine the exponential series directly.

Solution: In Example 4.3, we got the trigonometric Fourier coefficients as:

$$a_0 = \frac{A}{2},\ a_n = 0 \quad \text{and} \quad b_n = -\frac{A}{\pi n}$$

Determination of exponential Fourier series from trigonometric Fourier series

$$C_0 = a_0 = \frac{A}{2}$$

$$C_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}\left[0 - j\left(-\frac{A}{\pi n}\right)\right] = j\frac{A}{2\pi n}$$

$$C_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{2}\left[0 + j\left(-\frac{A}{\pi n}\right)\right] = \frac{-jA}{2\pi n}$$

$$\therefore \qquad x(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \frac{A}{2} + \sum_{n=-\infty}^{\infty} \frac{jA}{2\pi n} e^{jnt}$$

$$= \frac{A}{2} + j\frac{A}{2\pi}\sum_{n=1}^{\infty} \frac{e^{jnt}}{n}$$

Direct determination of exponential Fourier series

$$x(t) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{jn\omega_0 t} = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jnt}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{A}{2\pi}t\right) dt = \frac{A}{(2\pi)^2}\left(\frac{t^2}{2}\right)_0^{2\pi} = \frac{A}{2}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{A}{2\pi} te^{-jnt}\, dt$$

$$= \frac{A}{(2\pi)^2}\left[\left(t\,\frac{e^{-jnt}}{-jn}\right)_0^{2\pi} - \int_0^{2\pi} \frac{e^{-jnt}}{-jn}\, dt\right]$$

$$= \frac{A}{(2\pi)^2}\left\{-\frac{1}{jn}\left[2\pi e^{-j2n\pi} - 0 + \left(\frac{e^{-jnt}}{jn}\right)_0^{2\pi}\right]\right\}$$

$$= \frac{A}{(2\pi)^2}\left\{-\frac{1}{jn}\left[2\pi(1) + \frac{1}{jn}(e^{-j2n\pi} - e^0)\right]\right\}$$

$$= \frac{A}{(2\pi)^2}\left[-\frac{1}{jn}2\pi + \frac{1}{jn}(1-1)\right] = j\frac{A}{2\pi n}$$

$$\therefore \qquad x(t) = \frac{A}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{jA}{2\pi n} e^{jnt}$$

$$= \frac{A}{2} + j\frac{A}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{jnt}}{n}$$

is the exponential Fourier series.

EXAMPLE 4.16 Derive the exponential Fourier series from the trigonometric Fourier series for the waveform shown in Figure 4.8 (Example 4.6). Also determine the exponential series directly.

Solution: In Example 4.6, we got the trigonometric Fourier coefficients as:

$$a_0 = 0,\ a_n = 0 \quad \text{and} \quad b_n = \frac{2A}{n\pi}(-1)^{n+1}$$

$$\therefore \qquad C_0 = a_0 = 0,\ C_n = \frac{1}{2}[a_n - jb_n] = \frac{1}{2}\left[0 - j\frac{2A}{n\pi}(-1)^{n+1}\right] = -j\frac{A}{n\pi}(-1)^{n+1}$$

Exponential Fourier series

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \frac{2A}{T} t\, dt$$

$$= \frac{2A}{T^2}\left(\frac{t^2}{2}\right)_{-T/2}^{T/2} = \frac{2A}{2T^2}\left[\left(\frac{T}{2}\right)^2 - \left(-\frac{T}{2}\right)^2\right] = 0$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{jn\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \frac{2A}{T} te^{jn\omega_0 t}\, dt$$

$$= \frac{2A}{T^2}\left[\left(\frac{te^{jn\omega_0 t}}{jn\omega_0}\right)_{-T/2}^{T/2} - \int_{-T/2}^{T/2} \frac{e^{jn\omega_0 t}}{jn\omega_0}\, dt\right]$$

$$= \frac{2A}{T^2}\left\{\frac{\frac{T}{2}e^{jn\pi} + \frac{T}{2}e^{-jn\pi}}{jn\omega_0} - \left[\frac{e^{jn\omega_0 t}}{(jn\omega_0)^2}\right]_{-T/2}^{T/2}\right\}$$

$$= \frac{2A}{T^2}\left\{\frac{T[\cos n\pi]}{jn(2\pi/T)} - \frac{e^{jn\pi} - e^{-jn\pi}}{[jn(2\pi/T)]^2}\right\}$$

$$= \frac{A}{jn\pi}(-1)^n = j\frac{A}{n\pi}(-1)^{n+1}$$

The exponential Fourier series is:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} j\frac{A}{n\pi}(-1)^{n+1} e^{jn(2\pi/T)t}$$

4.9 POWER REPRESENTATION USING THE FOURIER SERIES

If I is the current flowing through a resistance of R ohms and V is the voltage across it, then the power dissipated in the resistor is given by I^2R and V^2/R. If the current signal is not constant, then the power varies at every instant, and the expressions for instantaneous power become $i^2(t)R$ and $v^2(t)/R$, where $i(t)$ and $v(t)$ are the corresponding instantaneous values. If the resistance R is equal to 1 ohm, then the instantaneous power can be represented as $i^2(t)$ and $v^2(t)$. Therefore, the instantaneous power of a signal $x(t)$ can be represented by $x^2(t)$. The mean (average) power over a given time interval (period) is given by

$$\text{Mean power} = \frac{1}{T}\int_0^T x^2(t)\, dt$$

Using Parseval's theorem, we have

$$\frac{1}{T}\int_0^T |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = C_0^2 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n^2$$

$$= C_0^2 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n C_n^* = a_0^2 + \sum_{n=1}^{\infty} 2\left[\text{Re}\,(C_n^2) + \text{Im}\,(C_n^2)\right]$$

$$= a_0^2 + \sum_{n=1}^{\infty} \frac{a_n^2}{2} + \frac{b_n^2}{2}$$

or Mean power over a period of time

= (dc term)^2 + Sum of mean square values of cosine terms
+ Sum of mean square values of sine terms

EXAMPLE 4.17 Find the average power of the signal.

$$x(t) = \cos^2(5000\pi t)\sin(20000\pi t)$$

If this signal is transmitted through a telephone system which blocks dc and frequencies above 14 kHz, then compute the ratio of received power to transmitted power.

Solution:

$$\boxed{\begin{aligned} \cos^2\theta &= \frac{1+\cos 2\theta}{2} \\ \sin A \cos B &= \left[\frac{\sin(A+B) + \sin(A-B)}{2}\right] \end{aligned}}$$

and Parseval's identity states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad \text{if } x_1(t) = x_2(t) = x(t)$$

Proof: Parseval's relation

$$\text{LHS} = \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\, x_2^*(t)\, dt = \frac{1}{T}\int_{t_0}^{t_0+T} \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right) x_2^*(t)\, dt$$

Interchanging the order of integration and summation in the RHS, we have

$$\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\, x_2^*(t)\, dt = \sum_{n=-\infty}^{\infty} C_n \left[\frac{1}{T}\int_{t_0}^{t_0+T} x_2^*(t)\, e^{jn\omega_0 t}\, dt\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n \left[\frac{1}{T}\int_{t_0}^{t_0+T} x_2(t)\, e^{-jn\omega_0 t}\, dt\right]^* = \sum_{n=-\infty}^{\infty} C_n [D_n]^*$$

$$\therefore \quad \boxed{\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\, x_2^*(t)\, dt = \sum_{n=-\infty}^{\infty} C_n D_n^*} \quad \text{proved.}$$

Parseval's identity

If $x_1(t) = x_2(t) = x(t)$, then the relation changes to

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, x^*(t)\, dt = \sum_{n=-\infty}^{\infty} C_n C_n^*$$

Since $|x(t)|^2 = x(t)\, x^*(t)$ and $|C_n|^2 = C_n C_n^*$, substituting these values in above equation, we get

$$\boxed{\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2} \quad \text{proved.}$$

The summary of the properties of continuous-time Fourier series is given in Appendix B (Table B.1).

EXAMPLE 4.18 Show that the magnitude spectrum of every periodic function is symmetrical about the vertical axis passing through the origin and the phase spectrum is antisymmetrical about the vertical axis passing through the origin.

Solution: The coefficient C_n of exponential Fourier series is given by

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

and $$C_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{jn\omega_0 t}\, dt$$

It is evident from these equations that the coefficients C_n and C_{-n} are complex conjugate of each other, that is

$$C_n = C_{-n}^*$$

Hence $$|C_n| = |C_{-n}|$$

It, therefore, follows that the magnitude spectrum is symmetrical about the vertical axis passing through the origin, and hence is an even function of ω.

If C_n is real, then C_{-n} is also real and C_n is equal to C_{-n}. If C_n is complex, let

$$C_n = |C_n|\, e^{j\theta_n}$$

then $$C_{-n} = |C_n|\, e^{-j\theta_n}$$

The phase of C_n is θ_n; however, the phase of C_{-n} is $-\theta_n$. Hence, it is obvious that the phase spectrum is antisymmetrical (an odd function), and the magnitude spectrum is symmetrical (an even function) about the vertical axis passing through the origin.

EXAMPLE 4.19 With regard to Fourier series representation, justify the following statements:

(a) Odd functions have only sine terms.

(b) Even functions have no sine terms.

(c) Functions with half wave symmetry have only odd harmonics.

Solution: We know that the trigonometric Fourier series of a periodic function $x(t)$ in any interval t_0 to $t_0 + T$ or 0 to T or $-T/2$ to $T/2$ is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \cos n\omega_0 t\, dt$$

and $$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \sin n\omega_0 t\, dt$$

(a) For an odd function

$$x(t) = -x(-t)$$

Also even part is zero, i.e. $x_e(t) = 0$ and $x(t) = x_o(t)$

$$\therefore \qquad a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x_o(t)\,dt = 0$$

since the integration of an odd function (antisymmetrical about the origin), over one cycle is always zero.

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x_o(t)\cos n\omega_0 t\,dt$$

Here $x_0(t)$ is odd and $\cos n\omega_0 t$ is even. So $x_0(t)\cos n\omega_0 t$, i.e. the product of an odd function and even function is odd. So integration over one cycle is zero. Therefore,

$$a_n = 0$$

Now,
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x_o(t)\sin n\omega_0 t\,dt$$

Now, $x_o(t)$ is odd and $\sin n\omega_0 t$ is also odd. Therefore, the product $x_o(t)\sin n\omega_0 t$ is even. So we have to evaluate the integral:

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt = \frac{4}{T}\int_{0}^{T/2} x(t)\sin n\omega_0 t\,dt$$

$$\therefore \qquad x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Thus, the Fourier series of odd functions contains only sine terms.

(b) For an even function,

$$x(t) = x(-t)$$

Also odd part is zero, i.e. $\qquad x(t) = x_e(t)$

$$\therefore \qquad a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x_e(t)\,dt = \frac{2}{T}\int_{0}^{T/2} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x_e(t)\cos n\omega_0 t\,dt$$

$x_e(t)$ is even and also $\cos n\omega_0 t$ is even. So $x_e(t)\cos n\omega_0 t$ i.e. the product of two even functions is even, and we have to evaluate the integral:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x_e(t)\cos n\omega_0 t\,dt = \frac{4}{T}\int_{0}^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x_e(t)\sin n\omega_0 t\,dt$$

Here $x_e(t)$ is even and $\sin n\omega_0 t$ is odd. So the product $x_e(t) \sin n\omega_0 t$ is odd. Thus, the integral over one complete cycle is zero.

$$\therefore \qquad b_n = 0$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

Thus, the Fourier series of even function contains no sine terms.

(c) For a half wave symmetric function,

$$x(t) = -x\left(t \pm \frac{T}{2}\right)$$

Therefore, $\quad a_0 = 0$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t \, dt$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(t) \cos n\omega_0 t \, dt + \int_{T/2}^{T} x(t) \cos n\omega_0 t \, dt\right]$$

To have the limits of second integration also from 0 to $T/2$, change the variable t by $t + (T/2)$ in the second integration.

$$\therefore \qquad a_n = \frac{2}{T}\left[\int_0^{T/2} x(t) \cos n\omega_0 t \, dt + \int_0^{T/2} x\left(t + \frac{T}{2}\right) \cos n\omega_0 \left(t + \frac{T}{2}\right) dt\right]$$

But $\quad x(t) = -x\left(t \pm \frac{T}{2}\right)$

$$\therefore \qquad a_n = \frac{2}{T}\left[\int_0^{T/2} x(t) \cos n\omega_0 t \, dt + \int_0^{T/2} -x(t) \cos (n\omega_0 t + n\pi) \, dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(t) \cos n\omega_0 t \, dt - x(t) \cos n\omega_0 t \cos n\pi\right] dt$$

$$\therefore \qquad a_n = \begin{cases} 0 & \text{for even } n \\ \dfrac{4}{T}\displaystyle\int_0^{T/2} x(t) \cos n\omega_0 t & \text{for odd } n \end{cases}$$

Now, $\quad b_n = \dfrac{2}{T}\displaystyle\int_0^T x(t) \sin n\omega_0 t \, dt$

$$= \frac{2}{T}\left[\int_0^{T/2} x(t) \sin n\omega_0 t \, dt + \int_{T/2}^{T} x(t) \sin n\omega_0 t \, dt\right]$$

To have the limits of second integration also from 0 to $T/2$ change the variable t by $t + (T/2)$ in the second integration.

$$\therefore \qquad b_n = \frac{2}{T}\left[\int_0^{T/2} x(t)\sin n\omega_0 t\, dt + \int_0^{T/2} x\left(t+\frac{T}{2}\right)\sin n\omega_0\left(t+\frac{T}{2}\right)\right]$$

But $\qquad x(t) = -x\left(t \pm \frac{T}{2}\right)$

$$\therefore \qquad b_n = \frac{2}{T}\left[\int_0^{T/2} x(t)\sin n\omega_0 t\, dt + \int_0^{T/2} -x(t)\sin(n\omega_0 t + n\pi)\, dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(t)\sin n\omega_0 t\, dt - \int_0^{T/2} x(t)\sin n\omega_0 t\cos n\pi\, dt\right]$$

$$\therefore \qquad b_n = \begin{cases} 0 & \text{for } n = \text{even} \\ \dfrac{4}{T}\displaystyle\int_0^{T/2} x(t)\sin n\omega_0 t\, dt & \text{for } n = \text{odd} \end{cases}$$

Thus, the Fourier series of half wave symmetric function consists of only odd harmonics.

EXAMPLE 4.20 Find the trigonometric Fourier series and the complex exponential Fourier series for the waveform shown in Figure 4.22.

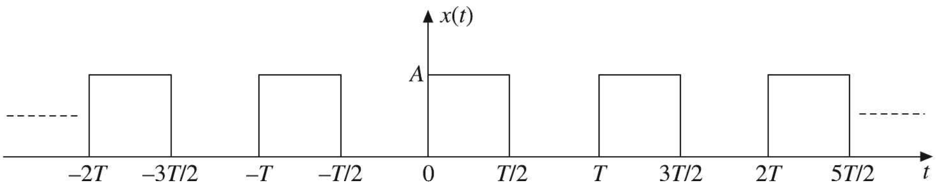


Figure 4.22 Waveform for Example 4.20.

Solution: The waveform shown in Figure 4.22 is periodic with period = T and can be expressed as:

$$x(t) = A \quad \text{for } 0 \le t \le \frac{T}{2}$$

$$= 0 \quad \text{for } \frac{T}{2} \le t \le T$$

Fundamental frequency $\omega_0 = (2\pi/T)$

Trigonometric Fourier series:

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt = \frac{1}{T}\int_0^{T/2} A\, dt = \frac{A}{T}[t]_0^{T/2} = \frac{A}{T}\left(\frac{T}{2} - 0\right) = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_0^{T/2} A\cos n\frac{2\pi}{T}t\,dt$$

$$= \frac{2A}{T}\left(\frac{\sin n(2\pi/T)t}{2n(\pi/T)}\right)_0^{T/2} = \frac{A}{n\pi}[\sin n\pi - \sin 0] = 0$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_0^{T/2} A\sin\frac{n2\pi t}{T}\,dt$$

$$= \frac{2A}{T}\left(\frac{-\cos n(2\pi/T)t}{2n(\pi/T)}\right)_0^{T/2} = \frac{A}{n\pi}[-(\cos n\pi - \cos 0)]$$

$$= -\frac{A}{n\pi}[(-1)^n - 1]$$

$$\therefore \qquad b_n = \begin{cases} 2A/n\pi & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

The trigonometric Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos n\frac{2\pi}{T}t + b_n\sin\frac{n2\pi}{T}t$$

$$= \frac{A}{2} + \sum_{n=1}^{\infty}\frac{2A}{n\pi}\sin\frac{n2\pi}{T}t \qquad \text{for odd } n$$

Exponential Fourier series

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\int_0^{T/2} A\,dt = \frac{A}{T}[t]_0^{T/2} = \frac{A}{T}\left(\frac{T}{2} - 0\right) = \frac{A}{2}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_0^{T/2} Ae^{-jn(2\pi/T)t}\,dt$$

$$= \frac{A}{T}\left(\frac{e^{-jn(2\pi/T)t}}{-jn(2\pi/T)}\right)_0^{T/2} = -\frac{A}{jn2\pi}[e^{-jn\pi} - 1]$$

$$= \frac{A}{jn2\pi}[1 - e^{-jn\pi}] = \frac{A}{jn2\pi}[1 - (-1)^n]$$

$$\therefore \qquad C_n = \begin{cases} 0 & \text{for even } n \\ A/jn\pi & \text{for odd } n \end{cases}$$

$$\therefore \qquad x(t) = \frac{A}{2} + \sum_{n=\text{odd}}^{\infty}\left(\frac{A}{jn\pi}\right)e^{-j2n\pi/T}$$

EXAMPLE 4.21 Find the trigonometric and exponential Fourier series for the waveform shown in Figure 4.23.

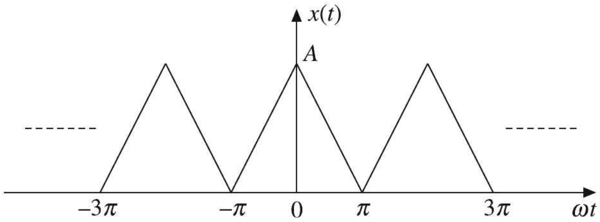


Figure 4.23 Waveform for Example 4.21.

Solution: The waveform shown in Figure 4.23 is periodic with period $T = 2\pi$, and it can be expressed as:

$$x(t) = \frac{A}{\pi}(\omega t + \pi) \quad \text{for } -\pi \le \omega t \le 0$$

$$= -\frac{A}{\pi}(\omega t - \pi) \quad \text{for } 0 \le \omega t \le \pi$$

Let $t_0 = -\pi$, then $\qquad t_0 + T = \pi$

Fundamental frequency $\qquad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

Trigonometric Fourier series

The independent variable is not t. It is ωt. The given waveform has even symmetry because

$$x(\omega t) = x(-\omega t)$$

$\therefore \qquad b_n = 0$

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\, d\omega t \quad \text{and} \quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 \omega t\, d\omega t$$

Now, $\qquad a_0 = \frac{2}{T}\int_0^{T/2} x(t)\, d\omega t = \frac{2}{2\pi}\int_{-\pi}^{0} \frac{A}{\pi}(\omega t + \pi)\, d\omega t = \frac{A}{\pi}\int_{-\pi}^{0}\left(\frac{1}{\pi}\omega t + 1\right) d\omega t$

$$= \frac{A}{\pi}\left\{\frac{1}{\pi}\left[\frac{(\omega t)^2}{2}\right]_{-\pi}^{0} + [\omega t]_{-\pi}^{0}\right\} = \frac{A}{\pi}\left[\frac{1}{\pi}\left(0 - \frac{\pi^2}{2}\right) + (\pi)\right] = \frac{A}{2}$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0\, \omega t\, d\omega t = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega t\, d\omega t$$

$$= \frac{4}{2\pi}\int_{-\pi}^{0} \frac{A}{\pi}(\omega t + \pi)\cos n\omega t\, d\omega t = \frac{2A}{\pi}\int_{-\pi}^{0}\left(\frac{1}{\pi}\omega t\cos n\omega t + \cos n\omega t\right) d\omega t$$

$$= \frac{2A}{\pi}\left\{\frac{1}{\pi}\left[\left(\omega t\,\frac{\sin n\omega t}{n}\right)_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin n\omega t}{n}\,d\omega t\right] + \left(\frac{\sin n\omega t}{n}\right)_{-\pi}^{0}\right\}$$

$$= \frac{2A}{\pi}\left[\frac{1}{\pi}\left(\frac{\cos n\omega t}{n^2}\right)_{-\pi}^{0}\right] = \frac{2A}{\pi^2 n^2}[\cos 0 - \cos n\pi] = \frac{2A}{\pi^2 n^2}[1-(-1)^n]$$

$$\therefore \qquad a_n = \begin{cases} \dfrac{4A}{\pi^2 n^2} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

The trigonometric Fourier series is:

$$x(t) = \frac{A}{2} + \sum_{n=\text{odd}}^{\infty}\frac{4A}{\pi^2 n^2}\cos n\omega_0 \omega t = \frac{A}{2} + \sum_{n=\text{odd}}^{\infty}\frac{4A}{\pi^2 n^2}\cos n\omega t$$

Exponential Fourier series

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\left(\int_{-\pi}^{0}\frac{A}{\pi}(\omega t+\pi)\,d\omega t + \int_0^{\pi} -\frac{A}{\pi}(\omega t-\pi)\,d\omega t\right)$$

$$= \frac{A}{2\pi}\left[\int_{-\pi}^{0}\left(\frac{1}{\pi}(\omega t)\,d\omega t + d\omega t\right) + \int_0^{\pi}\left(-\frac{1}{\pi}(\omega t)\,d\omega t + d\omega t\right)\right]$$

$$= \frac{A}{2\pi}\left[\left\{\frac{1}{\pi}\left[\frac{(\omega t)^2}{2}\right]_{-\pi}^{0} + [\omega t]_{-\pi}^{0}\right\} + \left\{-\frac{1}{\pi}\left[\frac{(\omega t)^2}{2}\right]_0^{\pi} + [\omega t]_0^{\pi}\right\}\right]$$

$$= \frac{A}{2\pi}\left[\left\{\frac{1}{\pi}\left(\frac{0-\pi^2}{2}\right)+\pi\right\} + \left\{-\frac{1}{\pi}\left(\frac{\pi^2-0}{2}\right)+\pi\right\}\right] = \frac{A}{2}$$

By inspection also we can say that the average value of the waveform over one cycle is:

$$a_0 = C_0 = \frac{A}{2}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 \omega t}\,d\omega t$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0}\frac{A}{\pi}(\omega t+\pi)\,e^{-jn\omega t}\,d\omega t + \int_0^{\pi} -\frac{A}{\pi}(\omega t-\pi)\,e^{-jn\omega t}\,d\omega t\right)$$

$$= \frac{A}{2\pi}\left(\int_{-\pi}^{0}\frac{1}{\pi}(\omega t)e^{-jn\omega t}\,d\omega t + \int_{-\pi}^{0} e^{-jn\omega t}\,d\omega t + \int_0^{\pi} -\frac{1}{\pi}(\omega t)e^{-jn\omega t}\,d\omega t + \int_0^{\pi} e^{-jn\omega t}\,d\omega t\right)$$

$$= \frac{A}{2\pi^2}\left(\int_{-\pi}^{0} (\omega t) e^{-jn\omega t}\, d\omega t - \int_{0}^{\pi} (\omega t) e^{-jn\omega t}\, d\omega t\right) + \frac{A}{2\pi}\int_{-\pi}^{\pi} e^{-jn\omega t}\, d\omega t$$

$$= \frac{A}{2\pi^2}\left\{\left[\frac{e^{-jn\omega t}(-jn\omega t - 1)}{(-jn)^2}\right]_{-\pi}^{0} - \left[\frac{e^{-jn\omega t}(-jn\omega t - 1)}{(-jn)^2}\right]_{0}^{\pi}\right\} + 0$$

$\therefore$
$$C_n = \frac{A}{n^2\pi^2}(1 - e^{jn\pi})$$

$$\boxed{\int te^{-nt}\, dt = \frac{e^{-nt}(-nt-1)}{(-n)^2}}$$

$\therefore$
$$C_n = \begin{cases} 0 & \text{for even } n \\ \dfrac{2A}{n^2\pi^2} & \text{for odd } n \end{cases}$$

$\therefore$
$$x(t) = \frac{A}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2A}{n^2\pi^2} e^{jn\omega t}$$

EXAMPLE 4.22 (a) Determine the trigonometric Fourier series of a full wave rectified cosine function shown in Figure 4.24.

(b) Derive the corresponding exponential Fourier series.
(c) Draw the complex Fourier spectrum.
(d) Find the exponential Fourier series directly.

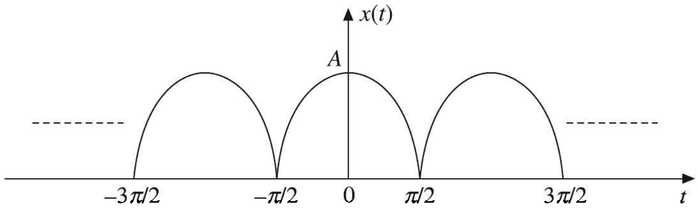


Figure 4.24 Waveform for Example 4.22.

Solution: The periodic waveform shown in Figure 4.24 is half of a cosine wave with period $= 2\pi$. Therefore,

$$x(t) = A\cos\omega t = A\cos\frac{2\pi}{2\pi}t = A\cos t$$

Let
$$t_0 = -\frac{\pi}{2}$$

then
$$t_0 + T = \frac{\pi}{2}$$

The period of the given waveform is from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$. Therefore, $T = \pi$.

So Fundamental frequency $\omega_0 = \dfrac{2\pi}{T} = \dfrac{2\pi}{\pi} = 2$

(a) *Trigonometric Fourier series*

The given function $x(t)$ is even, So $b_n = 0$.

$$\therefore \quad x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} A\cos t\,dt$$

$$= \frac{A}{\pi}[\sin t]_{-\pi/2}^{\pi/2} = \frac{A}{\pi}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{2A}{\pi}\sin\frac{\pi}{2} = \frac{2A}{\pi}$$

$$\therefore \quad a_0 = \frac{2A}{\pi}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} A\cos t\cos 2nt\,dt = \frac{A}{\pi}\int_{-\pi/2}^{\pi/2}[\cos(2n+1)t + \cos(2n-1)t]\,dt$$

$$= \frac{A}{\pi}\left\{\left[\frac{\sin(2n+1)t}{2n+1}\right]_{-\pi/2}^{\pi/2} + \left[\frac{\sin(2n-1)t}{2n-1}\right]_{-\pi/2}^{\pi/2}\right\}$$

$$= \frac{A}{\pi}\left[\frac{\sin(2n+1)\frac{\pi}{2} - \sin(2n+1)\left(-\frac{\pi}{2}\right)}{2n+1} + \frac{\sin(2n-1)\frac{\pi}{2} - \sin(2n-1)\left(-\frac{\pi}{2}\right)}{2n-1}\right]$$

$$= \frac{A}{\pi}\left[\frac{2\sin(2n+1)\frac{\pi}{2}}{(2n+1)} + \frac{2\sin(2n-1)\frac{\pi}{2}}{(2n-1)}\right] = \frac{2A}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]$$

$$\therefore \quad a_n = \frac{2A}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]$$

$$\therefore \quad x(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty}\frac{2A}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]\cos 2nt$$

(b) *Derivation of exponential Fourier series from trigonometric Fourier series*
We know that

$$C_0 = a_0$$

Also, we know that

$$C_n = \frac{1}{2}(a_n - jb_n)$$

In this case,

$$b_n = 0$$

$\therefore$

$$C_n = \frac{a_n}{2}$$

and

$$C_n = \frac{1}{2}\left\{\frac{2A}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]\right\}$$

$$= \frac{A}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]$$

So the exponential Fourier series is:

$$x(t) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{jn\omega_0 t}$$

i.e.

$$x(t) = \frac{2A}{\pi} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{A}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right] e^{j2nt}$$

To draw the frequency spectrum,

$$C_0 = \frac{2A}{\pi}$$

$$C_1 = C_{-1} = \frac{A}{\pi}\left[\frac{(-1)^1}{2\times 1+1} + \frac{(-1)^{1+1}}{2\times 1-1}\right] = \frac{A}{\pi}\left(\frac{-1}{3} + \frac{1}{1}\right) = \frac{2A}{3\pi}$$

$$C_2 = C_{-2} = \frac{A}{\pi}\left[\frac{(-1)^2}{2\times 2+1} + \frac{(-1)^{2+1}}{2\times 2-1}\right] = \frac{A}{\pi}\left(\frac{1}{5} - \frac{1}{3}\right) = -\frac{2A}{15\pi}$$

$$C_3 = C_{-3} = \frac{A}{\pi}\left[\frac{(-1)^3}{2\times 3+1} + \frac{(-1)^{3+1}}{2\times 3-1}\right] = \frac{A}{\pi}\left(\frac{-1}{7} + \frac{1}{5}\right) = \frac{2A}{35\pi}$$

(c) *Complex Fourier spectrum*
It is a plot between C_n and $n\omega_0 = \omega$. As C_n is real, only amplitude plot is sufficient. The amplitude versus frequency plot, called the amplitude spectrum, is shown in Figure 4.25.

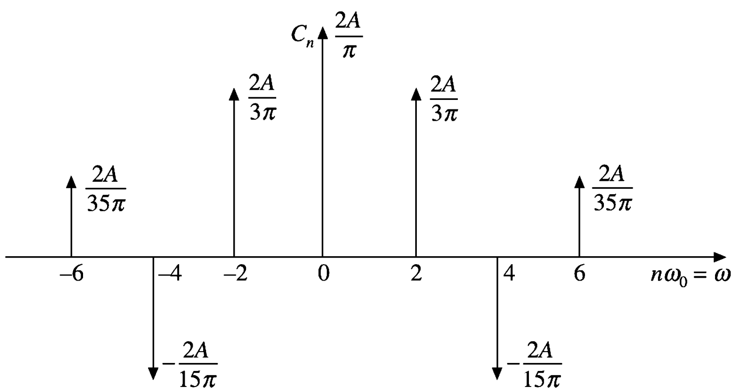


Figure 4.25 Amplitude spectrum.

(d) *Direct determination of exponential Fourier series*

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} A\cos t\,dt = \frac{A}{\pi}\left[\sin t\right]_{-\pi/2}^{\pi/2} = \frac{2A}{\pi}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,e^{jn\omega_0 t}\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} A\cos t e^{j2nt}\,dt$$

$$= \frac{A}{\pi}\int_{-\pi/2}^{\pi/2} \frac{e^{jt}+e^{-jt}}{2}e^{j2nt}\,dt = \frac{A}{2\pi}\left[\int_{-\pi/2}^{\pi/2} e^{j(2n+1)t}\,dt + \int_{-\pi/2}^{\pi/2} e^{j(2n-1)t}\,dt\right]$$

$$= \frac{A}{2\pi}\left\{\left[\frac{e^{j(2n+1)t}}{j(2n+1)}\right]_{-\pi/2}^{\pi/2} + \left[\frac{e^{j(2n-1)t}}{j(2n-1)}\right]_{-\pi/2}^{\pi/2}\right\}$$

$$= \frac{A}{2\pi}\left[\frac{2j\sin(2n+1)\frac{\pi}{2}}{j(2n+1)} + \frac{2j\sin(2n-1)\frac{\pi}{2}}{j(2n-1)}\right] = \frac{A}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]$$

$$\therefore \qquad x(t) = \frac{2A}{\pi} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{A}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right]$$

EXAMPLE 4.23 For the periodic gate function shown in Figure 4.26,

(a) Find the trigonometric Fourier series.
(b) Derive the corresponding exponential Fourier series.
(c) Find the exponential Fourier series directly.
(d) Plot the magnitude and phase spectra.

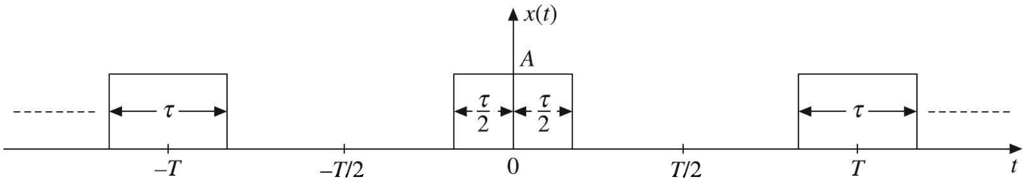


Figure 4.26 Waveform for Example 4.23.

Solution: The waveform shown in Figure 4.26 is a periodic gate function with period T. It is given by

$$x(t) = A\Pi\left(\frac{t}{\tau}\right) = A\,\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} A & \text{for } -(\tau/2) \le t \le (\tau/2) \\ 0 & \text{elsewhere} \end{cases}$$

Let $$t_0 = -\frac{T}{2}$$

Then $$t_0 + T = \frac{T}{2}$$

Fundamental frequency $\omega_0 = (2\pi/T)$

(a) *Trigonometric Fourier series*

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

The waveform has even symmetry, therefore, $b_n = 0$.

$$\therefore \qquad x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\,dt = \frac{A}{T}\left[t\right]_{-\tau/2}^{\tau/2} = \frac{A}{T}\left[\frac{\tau}{2} - \left(-\frac{\tau}{2}\right)\right] = \frac{A\tau}{T}$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_{-\tau/2}^{\tau/2} A\cos n\omega_0 t\,dt$$

$$= \frac{2A}{T}\left[\frac{\sin n\omega_0 t}{n\omega_0}\right]_{-\tau/2}^{\tau/2} = \frac{2A}{T}\left\{\frac{\sin n\omega_0 \frac{\tau}{2} - \sin[-n\omega_0(\tau/2)]}{n\omega_0}\right\}$$

$$= \frac{2A}{T}\left[\frac{2\sin n\omega_0(\tau/2)}{n\omega_0(\tau/2)}\right]\frac{\tau}{2} = \frac{2A}{T}\tau\,\text{sinc}\left(n\omega_0\frac{\tau}{2}\right)$$

$$\therefore\ x(t) = \frac{A\tau}{T} + \sum_{n=1}^{\infty}\frac{2A}{T}\tau\,\text{sinc}\left(n\omega_0\frac{\tau}{2}\right)\cos n\omega_0 t$$

(b) *Exponential Fourier series from the trigonometric Fourier series*

$$C_0 = a_0 = \frac{A\tau}{T}$$

$$C_n = \frac{1}{2}(a_n - jb_n) = \frac{a_n}{2} = \frac{1}{2}\left[\frac{2A}{T}\tau \operatorname{sinc}\left(n\omega_0 \frac{\tau}{2}\right)\right] = \frac{A}{T}\tau \operatorname{sinc}\left(n\omega_0 \frac{\tau}{2}\right)$$

$$\therefore \quad x(t) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \frac{A\tau}{T} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{A}{T}\tau \operatorname{sinc}\left(n\omega_0 \frac{\tau}{2}\right) e^{jn\omega_0 t}$$

$$= \frac{A\tau}{T} + \frac{A}{T}\tau \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \operatorname{sinc}\left(n\omega_0 \frac{\tau}{2}\right) e^{jn\omega_0 t}$$

(c) *Direct determination of exponential Fourier series*

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\,dt = \frac{A}{T}\left[t\right]_{-\tau/2}^{\tau/2} = \frac{A\tau}{T}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-\tau/2}^{\tau/2} = \frac{A}{T}\left[\frac{e^{-jn\omega_0(\tau/2)} - e^{jn\omega_0(\tau/2)}}{-jn\omega_0}\right]$$

$$= \frac{2A}{n\omega_0 T}\left[\frac{e^{jn\omega_0(\tau/2)} - e^{-jn\omega_0(\tau/2)}}{2j}\right] = \frac{A}{T}\left[\frac{\sin n\omega_0(\tau/2)}{n\omega_0(\tau/2)}\right]\tau$$

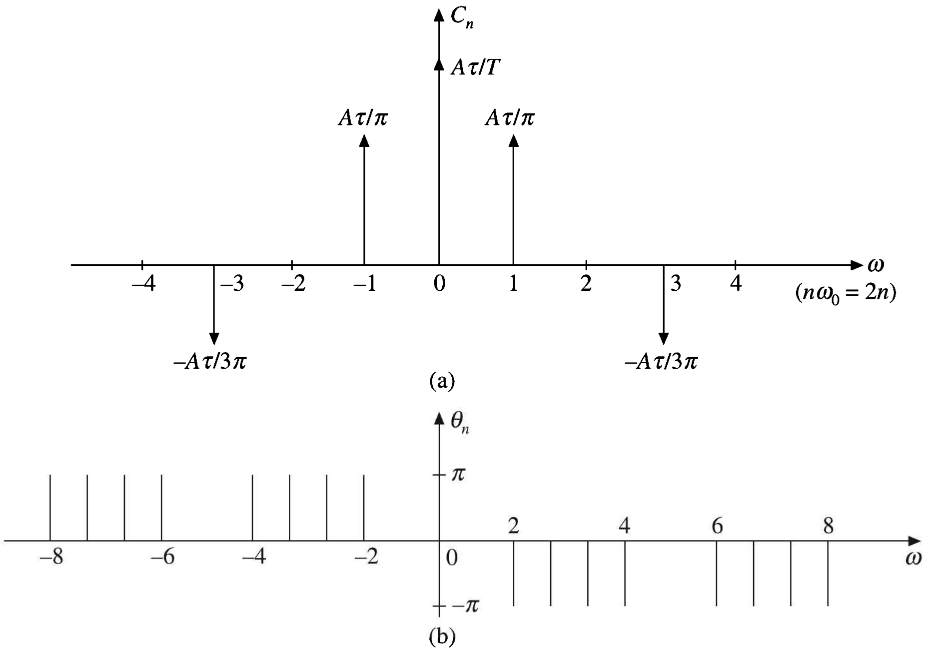
$$= \frac{A}{T}\tau \operatorname{sinc}\left(n\omega_0 \frac{\tau}{2}\right)$$

The exponential Fourier series is:

$$x(t) = \frac{A}{T}\tau + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{A\tau}{T} \operatorname{sinc}\left(n\omega_0 \frac{\tau}{2}\right) e^{jn\omega_0 t}$$

(d) *Amplitude and phase spectra*

The amplitude and phase spectra are shown in Figures 4.27(a) and (b) respectively.



Figures 4.27 (a) Magnitude spectrum, (b) Phase spectrum.

EXAMPLE 4.24 Find the complex exponential Fourier series and the trigonometric Fourier series of unit impulse train $\delta_T(t)$ shown in Figure 4.28.

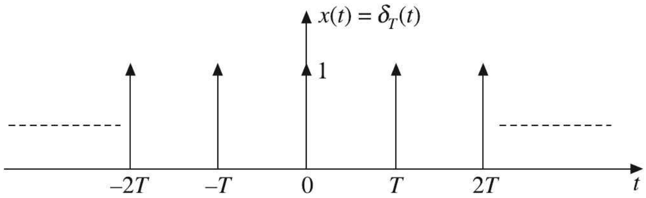


Figure 4.28 Waveform for Example 4.24.

Solution: The periodic waveform shown in Figure 4.28 with period T can be expressed as:

$$x(t) = \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Let $$t_0 = -\frac{T}{2}$$

So $$t_0 + T = \frac{T}{2}$$

In one period, only $\delta(t)$ exists.

$\therefore$ Fundamental frequency $\omega_0 = (2\pi/T)$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_0 t}\, dt \qquad \boxed{\begin{aligned}\delta(t) &= 1 \quad \text{at } t=0\\ &= 0 \quad \text{elsewhere}\end{aligned}}$$

$$= \frac{1}{T}e^{0} = \frac{1}{T}$$

$$\therefore \qquad C_n = \frac{1}{T}$$

The exponential Fourier series is:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn(2\pi/T)t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn(2\pi/T)t}$$

To get the trigonometric Fourier coefficients, we have

$$a_0 = C_0 = \frac{1}{T}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\, dt = \frac{2}{T}\int_0^T \delta(t)\cos n\omega_0 t\, dt = \frac{2}{T}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\, dt = \frac{2}{T}\int_0^T \delta(t)\sin n\omega_0 t\, dt = 0$$

Therefore, the trigonometric Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = \frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty} \cos n\omega_0 t$$

The complex Fourier series coefficients are:

$$C_n = \frac{1}{T} \qquad \text{for all } n$$

The magnitude and phase spectrums are:

$$|C_n| = \left|\frac{1}{T}\right| \qquad \text{for all } n$$

$$\angle C_n = 0^\circ \qquad \text{for all } n$$

The frequency spectra are plotted in Figure 4.29.

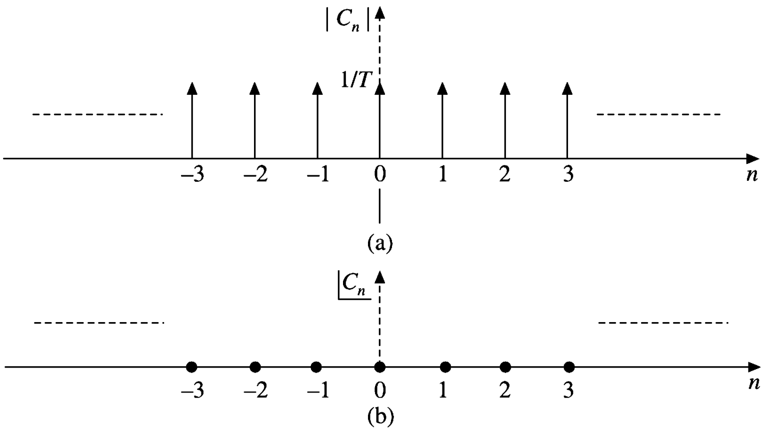


Figure 4.29 (a) Magnitude spectrum, (b) Phase spectrum.

EXAMPLE 4.25 Find the complex exponential Fourier series representation of the following signals:

(a) $x(t) = 4\cos 2\omega_0 t$ (b) $x(t) = 3\sin 4\omega_0 t$

(c) $x(t) = \sin\left(2t + \dfrac{\pi}{4}\right)$ (d) $x(t) = \cos^2 t$

Solution:

(a) Given $$x(t) = 4\cos 2\omega_0 t = 4\left[\frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2}\right]$$

$\therefore$ $$x(t) = 2e^{j2\omega_0 t} + 2e^{-j2\omega_0 t}$$

is the complex exponential Fourier series representation. Comparing this with the general complex exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

we get the exponential Fourier series coefficients

$$C_2 = 2 = C_{-2},\ C_n = 0 \quad \text{for } n \neq 2$$

(b) Given $$x(t) = 3\sin 4\omega_0 t = 3\left[\frac{e^{j4\omega_0 t} - e^{-j4\omega_0 t}}{2j}\right]$$

$\therefore$ $$x(t) = \frac{3}{2j}e^{j4\omega_0 t} - \frac{3}{2j}e^{-j4\omega_0 t}$$

is the complex exponential Fourier series representation. Comparing this with the general complex exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

we get the exponential Fourier series coefficients

$$C_4 = \frac{3}{2j},\ C_{-4} = \frac{-3}{2j},\ C_n = 0 \qquad \text{for } n \neq 4$$

(c) Given
$$x(t) = \sin\left(2t + \frac{\pi}{4}\right) = \frac{e^{j[2t+(\pi/4)]} - e^{-j[2t+(\pi/4)]}}{2j}$$

$\therefore$
$$x(t) = \left[\frac{1}{j2}e^{j(\pi/4)}\right]e^{j2t} + \left[-\frac{1}{j2}e^{-j(\pi/4)}\right]e^{-j2t}$$

is the complex exponential form.

Comparing it with $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$, we have the complex exponential Fourier coefficients

$$C_2 = \frac{1}{j2}e^{j(\pi/4)} = -0.5je^{j(\pi/4)},\ C_{-2} = -\frac{1}{j2}e^{-j(\pi/4)} = 0.5je^{-j(\pi/4)}$$

and
$$C_n = 0 \qquad \text{for } |n| \neq 2$$

(d) Given
$$x(t) = \cos^2 t = \left[\frac{e^{jt} + e^{-jt}}{2}\right]^2 = \frac{1}{4}\left[e^{j2t} + 2 + e^{-j2t}\right]$$

$\therefore$
$$x(t) = \frac{1}{4}e^{-j2t} + \frac{1}{2} + \frac{1}{4}e^{j2t}$$

is the complex exponential form.

Comparing it with $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$, we have the complex exponential Fourier coefficients

$$C_{-2} = \frac{1}{4},\ C_0 = \frac{1}{2},\ C_2 = \frac{1}{4} \text{ and } C_n = 0, \text{ when } n \neq 0 \text{ and } n \neq 2$$

EXAMPLE 4.26 For the continuous-time periodic signal $x(t) = 2 + \cos 2t + \sin 4t$, determine the fundamental frequency ω_0 and the Fourier series coefficients C_n such that

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Solution: Given
$$x(t) = 2 + \cos 2t + \sin 4t$$

The time period of the signal cos $2t$ is:

$$T_1 = \frac{2\pi}{2} = \pi \text{ sec}$$

The time period of the signal sin $4t$ is:

$$T_2 = \frac{2\pi}{4} = \frac{\pi}{2} \text{ sec}$$

$$\therefore \qquad \frac{T_1}{T_2} = \frac{\pi}{\pi/2} = 2$$

$$\therefore \qquad T_1 = 2T_2$$

The fundamental period of the signal $x(t)$ is:

$$T = T_1 = 2T_2 = \pi \text{ sec}$$

and Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

Now,

$$x(t) = 2 + \cos 2t + \sin 4t$$

$$= 2 + \frac{e^{j2t} + e^{-j2t}}{2} + \frac{e^{j4t} - e^{-j4t}}{2j}$$

$$= -\frac{1}{2j}e^{-j4t} + \frac{1}{2}e^{-j2t} + 2 + \frac{1}{2}e^{j2t} + \frac{1}{2j}e^{j4t}$$

$$= -\frac{1}{2j}e^{-j2(2)t} + \frac{1}{2}e^{-j1(2)t} + 2 + \frac{1}{2}e^{j1(2)t} + \frac{1}{2j}e^{j2(2)t}$$

$$= C_{-2}e^{-j2(\omega_0)t} + C_{-1}e^{-j1(\omega_0)t} + C_0 + C_1e^{j1(\omega_0)t} + C_2e^{j2(\omega_0)t}$$

The exponential Fourier coefficients are:

$$C_{-2} = -\frac{1}{2j}, \; C_{-1} = \frac{1}{2}, \; C_0 = 2, \; C_1 = \frac{1}{2}, \; C_2 = \frac{1}{2j}$$

EXAMPLE 4.27 Determine the Fourier series representation of the signal $x(t) = 3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$ using the method of inspection.

Solution: Looking at the given signal $x(t) = 3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$, by inspection, we can say that, it is cosine form of representation. Comparing it with the cosine representation

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

we get

$$A_0 = 0, \; A_1 = 3, \; \omega_0 = \frac{\pi}{2}, \; \theta_1 = \frac{\pi}{4}$$

$$\therefore \qquad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/2} = 4$$

Trigonometric form

Given $$x(t) = 3\cos\left[\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)\right] = 3\left[\cos\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi t}{2}\right)\sin\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{3}{\sqrt{2}}\left[\cos\left(\frac{\pi t}{2}\right) - \sin\left(\frac{\pi t}{2}\right)\right]$$

Comparing this with the trigonometric Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

we get $a_0 = 0$, $a_1 = \frac{3}{\sqrt{2}}$, $b_1 = -\frac{3}{\sqrt{2}}$, other coefficients are zero

$$\omega_0 = \frac{\pi}{2}$$

$$\therefore \qquad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/2} = 4$$

EXAMPLE 4.28 Find the Fourier series of the signal $x(t) = e^{-t}$ with $T = 1$ sec as shown in Figure 4.30. Draw its magnitude and phase spectra.

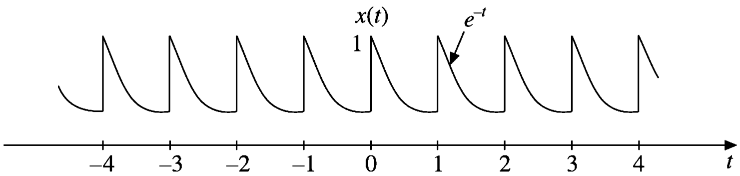


Figure 4.30 Waveform for Example 4.28.

Solution: Given signal $$x(t) = e^{-t}$$

Period $T = 1$ sec

$$\therefore \qquad \text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

The exponential Fourier series is:

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{1}\int_0^1 e^{-t}\, e^{-jn2\pi t}\, dt$$

$$= \int_0^1 e^{-(1+j2n\pi)t}\, dt = \left[\frac{e^{-(1+j2n\pi)t}}{-(1+j2n\pi)}\right]_0^1$$

$$= \frac{e^{-(1+j2n\pi)} - 1}{-(1+j2n\pi)} = \frac{1 - e^{-1}e^{-j2n\pi}}{1 + j2n\pi} = \frac{1 - e^{-1}}{1 + j2n\pi}$$

$$\therefore \qquad x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \left(\frac{1-e^{-1}}{1+j2n\pi} \right) e^{jn2\pi t}$$

The spectra are shown in Figure 4.31.

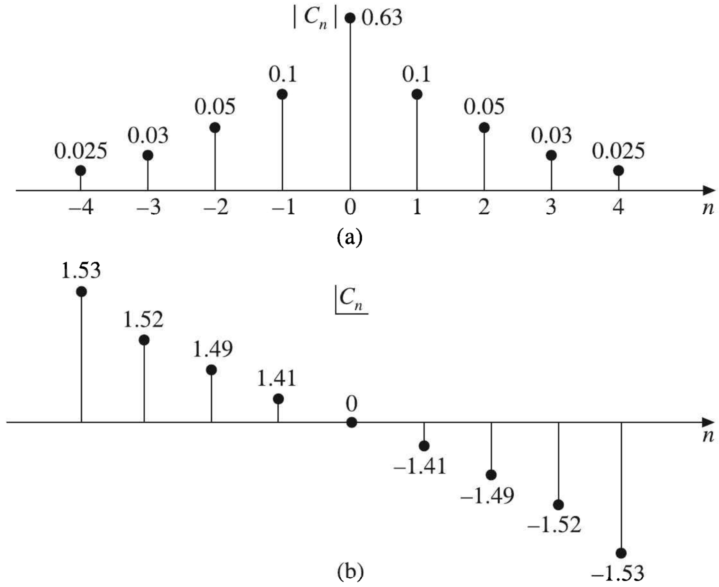


Figure 4.31 (a) Amplitude spectrum, (b) Phase spectrum for Example 4.28.

EXAMPLE 4.29 Find the Fourier series for the periodic signal $x(t) = t$ for $0 \le t \le 1$ so that it repeats every 1 second.

Solution: Given $\qquad x(t) = t \quad \text{for } 0 \le t \le 1$

Period $T = 1$ sec

$$\therefore \qquad \text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = 2\pi$$

To get the exponential Fourier series, we have

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{1}\int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt = \frac{1}{1}\int_0^1 t e^{-jn2\pi t}\,dt = \left[t\frac{e^{-j2n\pi t}}{-j2n\pi}\right]_0^1 - \int_0^1 \frac{e^{-j2n\pi t}}{-j2n\pi}\,dt$$

$$= \frac{e^{-j2n\pi} - 0}{-j2n\pi} + \frac{1}{j2n\pi}\left[\frac{e^{-j2n\pi t}}{-j2n\pi}\right]_0^1 = -\frac{1}{j2n\pi} + \frac{1}{4n^2\pi^2}[e^{-j2n\pi} - e^0]$$

$$= -\frac{1}{j2n\pi} = j\frac{1}{2n\pi}$$

Therefore, the exponential Fourier series is:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(j\frac{1}{2n\pi} \right) e^{jn\omega_0 t}$$

$$= -j\frac{1}{4\pi} e^{-j2\omega_0 t} - j\frac{1}{2\pi} e^{-j\omega_0 t} + \frac{1}{2} + j\frac{1}{2\pi} e^{j\omega_0 t} + j\frac{1}{4\pi} e^{j2\omega_0 t} + \cdots$$

For the trigonometric Fourier series, we have

$$a_0 = C_0 = \frac{1}{2}$$

$$a_n = C_n + C_{-n} = j\frac{1}{2n\pi} + \left(-j\frac{1}{2n\pi} \right) = 0$$

$$b_n = j[C_n - C_{-n}] = j\left[j\frac{1}{2n\pi} - j\frac{1}{-2n\pi} \right] = -\frac{1}{n\pi}$$

Therefore, the trigonometric Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left(-\frac{1}{n\pi} \right) \sin n\omega_0 t$$

$$= \frac{1}{2} - \frac{1}{\pi} \sin (\omega_0 t) - \frac{1}{2\pi} \sin (2\omega_0 t) - \frac{1}{3\pi} \sin (3\omega_0 t) - \frac{1}{4\pi} \sin (4\omega_0 t) - \cdots$$

EXAMPLE 4.30 Determine the time signal corresponding to the magnitude and phase spectra shown in Figure 4.32 with $\omega_0 = \pi$.

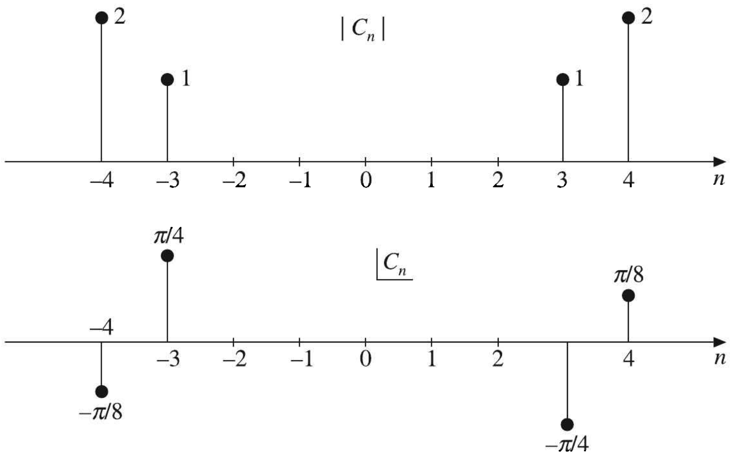


Figure 4.32 Spectra for Example 4.30.

Solution: From Figure 4.32, we have

$$C_3 = 1\angle\frac{-\pi}{4} = 1e^{-j\pi/4}$$

$$C_{-3} = 1\angle\frac{\pi}{4} = 1e^{j\pi/4}$$

$$C_4 = 2\angle\frac{\pi}{8} = 2e^{j\pi/8}$$

$$C_{-4} = 2\angle\frac{-\pi}{8} = 2e^{-j\pi/8}$$

By definition, the exponential Fourier series is:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= 2e^{-j\pi/8}e^{-j4\omega_0 t} + 1e^{j\pi/4}e^{-j3\omega_0 t} + e^{-j\pi/4}e^{j3\omega_0 t} + 2e^{j\pi/8}e^{j4\omega_0 t}$$

$$= 2\left\{e^{-j[(\pi/8)+4\pi t]} + e^{j[(\pi/8)+4\pi t]}\right\} + \left\{e^{j[(\pi/4)-3\pi t]} + e^{-j[(\pi/4)-3\pi t]}\right\}$$

$$= 4\cos\left(4\pi t + \frac{\pi}{8}\right) + 2\cos\left(3\pi t - \frac{\pi}{4}\right)$$

EXAMPLE 4.31 Find the time domain signal whose Fourier series coefficient is given by

$$C_n = j\delta(n-1) - j\delta(n+1) + \delta(n-3) + \delta(n+3),\ \omega_0 = \pi$$

Solution: We have

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$\therefore$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\pi t}$$

$$= je^{j\pi t} - je^{-j\pi t} + e^{j3\pi t} + e^{-j3\pi t}$$

$$= 2\cos 3\pi t - 2\sin \pi t$$

EXAMPLE 4.32 Find the Fourier series coefficient C_n for the signal

$$x(t) = \sum_{n=-\infty}^{\infty}\left[\delta\left(t - \frac{1}{2}n\right) + \delta\left(t - \frac{3}{2}n\right)\right]$$

Also sketch the amplitude and phase spectra.

Solution: The sketch of $x(t)$ is shown in Figure 4.33.

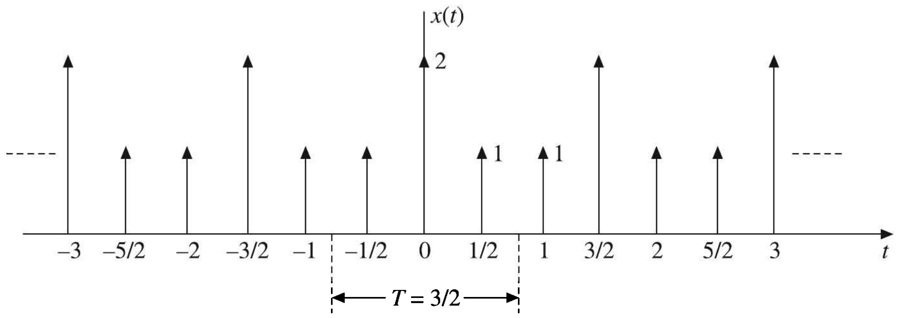


Figure 4.33 Signal for Example 4.32.

From Figure 4.33, we observe that $T = 3/2$.

$$\therefore \qquad \omega_0 = \frac{2\pi}{T} = \frac{4\pi}{3}$$

The expression for $x(t)$ over one period is:

$$x(t) = \delta\left(t + \frac{1}{2}\right) + 2\delta(t) + \delta\left(t - \frac{1}{2}\right)$$

We have $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{3/2}\int_{-3/4}^{3/4}\left[\delta\left(t + \frac{1}{2}\right) + 2\delta(t) + \delta\left(t - \frac{1}{2}\right)\right] e^{-jn(4\pi/3)t}$$

$$\therefore \qquad C_n = \frac{2}{3}\left\{e^{-jn(4\pi/3)[-(1/2)]} + 2e^{-jn(4\pi/3)(0)} + e^{-jn(4\pi/3)[(1/2)]}\right\}$$

$$= \frac{2}{3}\left[e^{jn(2\pi/3)} + e^{-jn(2\pi/3)} + 2\right]$$

$$= \frac{4}{3}\left[1 + \cos\frac{2n\pi}{3}\right]$$

The magnitude spectrum is shown in Figure 4.34. Since the given signal $x(t)$ is real and even, the phase spectrum is null spectrum.

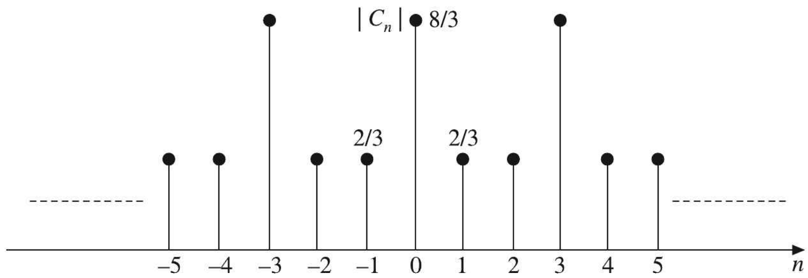


Figure 4.34 Spectrum for Example 4.32.

EXAMPLE 4.33 Find the time domain signal corresponding to

$$C_n = \left(-\frac{1}{2}\right)^{|n|}; \ \omega_0 = 1$$

Solution: We have $$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

$$\therefore \qquad x(t) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{|n|} e^{jnt}$$

$$= \sum_{n=-\infty}^{-1} \left(-\frac{1}{2}\right)^{-n} e^{jnt} + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} e^{jnt}$$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n} e^{-jnt} + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} e^{jnt}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} e^{-jnt} - \left[\left(-\frac{1}{2}\right)^{n} e^{-jnt}\right]_{n=0} + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} e^{jnt}$$

$$= \frac{-(1/2)\, e^{-jt}}{1+(1/2)e^{-jt}} + \frac{1}{1+(1/2)e^{jt}} = \frac{3/4}{(5/4)+\cos t}$$

EXAMPLE 4.34 For the signal shown in Figure 4.35 obtain the Fourier series representation using time differentiation property.

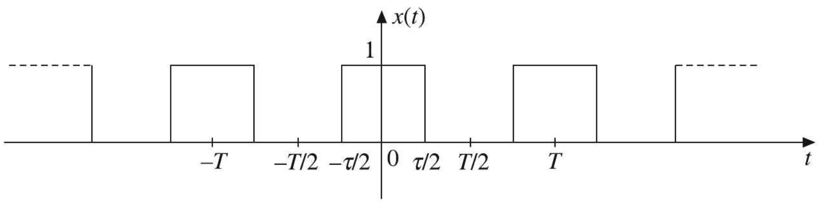


Figure 4.35 Waveform for Example 4.34.

Solution: Differentiating $x(t)$ with respect to time, we get the signal $y(t)$ as shown in Figure 4.36.

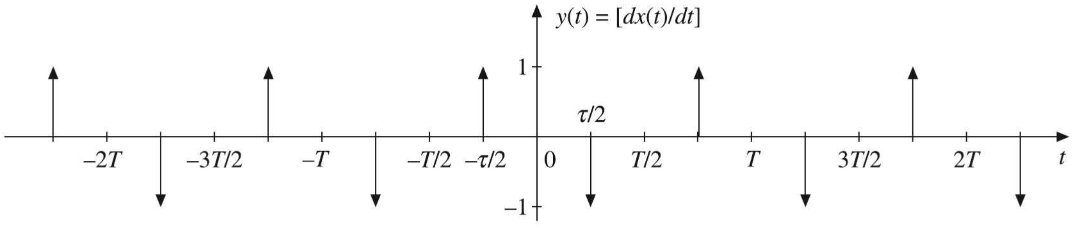


Figure 4.36 Signal $y(t) = [dx(t)/dt]$.

Here $$\omega_0 = \frac{2\pi}{T}$$

Fourier series of $y(t)$ is given by

$$D_n = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left[\delta\left(t+\frac{\tau}{2}\right) - \delta\left(t-\frac{\tau}{2}\right)\right] e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} \delta\left(t+\frac{\tau}{2}\right) e^{-jn\omega_0 t}\, dt - \int_{-T/2}^{T/2} \delta\left(t-\frac{\tau}{2}\right) e^{-jn\omega_0 t}\, dt\right]$$

$$= \frac{1}{T}\left[e^{jn\frac{2\pi}{T}\frac{\tau}{2}} - e^{-jn\frac{2\pi}{T}\frac{\tau}{2}}\right]$$

$$\therefore \qquad D_n = \frac{2j}{T}\left[\sin\left(\frac{n\pi}{T}\tau\right)\right]$$

We have $$y(t) = \frac{d}{dt}x(t)$$

Let C_n be the Fourier series of $x(t)$. Using time differentiation property, we get

$$D_n = jn\omega_0 C_n$$

$$C_n = \frac{1}{jn\omega_0} D_n$$

$$\therefore \qquad = \left(\frac{1T}{jn2\pi}\right)\left[\frac{2j}{T}\sin\left(\frac{n\pi}{T}\tau\right)\right] = \frac{1}{n\pi}\sin\left(\frac{n\pi}{T}\tau\right)$$

$$= \frac{\tau}{T}\,\frac{\sin(n\pi/T)\,\tau}{(n\pi/T)\,\tau} = \frac{\tau}{T}\,\text{sinc}\left(\frac{n\pi}{T}\tau\right)$$

EXAMPLE 4.35 The voltage $v(t)$ having the waveform shown in Figure 4.37 is applied to the circuit shown in Figure 4.38(a). Determine the current $i(t)$ using Fourier series.

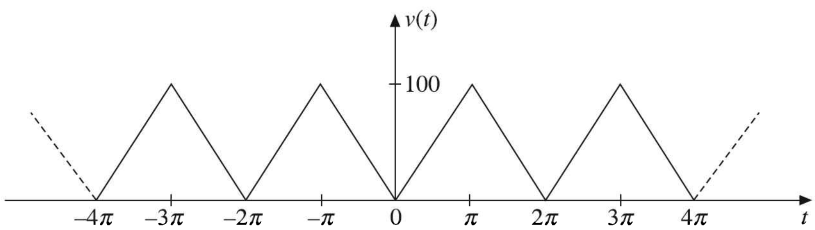


Figure 4.37 Signal for Example 4.35.

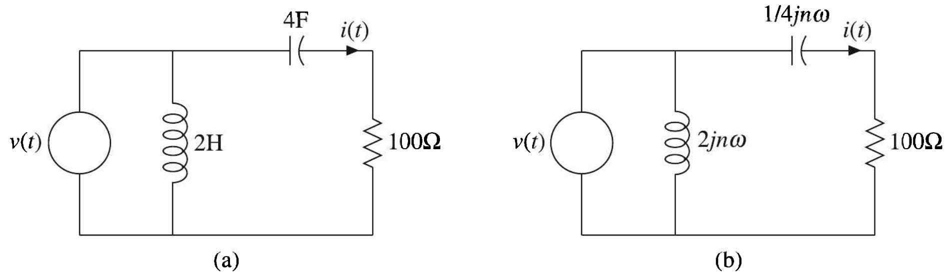


Figure 4.38 (a) Circuit for Example 4.35 (b) Its equivalent.

Solution: The Fourier series of the signal shown in Figure 4.37 can be found as:

[Refer Example 4.4]
$$v(t) = \frac{100}{2} - \frac{4(100)}{\pi^2} \sum_{n=\text{odd}}^{\infty} \frac{\cos nt}{n^2}$$

Since the capacitor in the circuit shown in Figure 4.38(a) does not allow the dc current, the dc term in the input can be neglected and only harmonic components are to be considered. So the given network can be redrawn as shown in Figure 4.38(b).

$$\therefore \qquad v(t) = -\frac{4(100)}{\pi^2} \sum_{n=\text{odd}}^{\infty} \frac{\cos nt}{n^2}$$

$$\therefore \quad i(t) = \frac{v(t)}{100 + \left(\frac{1}{4jn\omega}\right)} = \frac{v(t)}{100 - \frac{j}{4n\omega}} = \frac{4n\omega\, v(t)}{400n\omega - j} = \frac{4n\omega v(t)}{\sqrt{1 + 160000n^2\omega^2}} \underline{\left|- \tan^{-1}\left(\frac{1}{400n\omega}\right)\right.}$$

EXAMPLE 4.36 The waveform shown in Figure 4.39(a) is applied to the circuit shown in Figure 4.39(b). Determine the third harmonic frequency current using Fourier series.

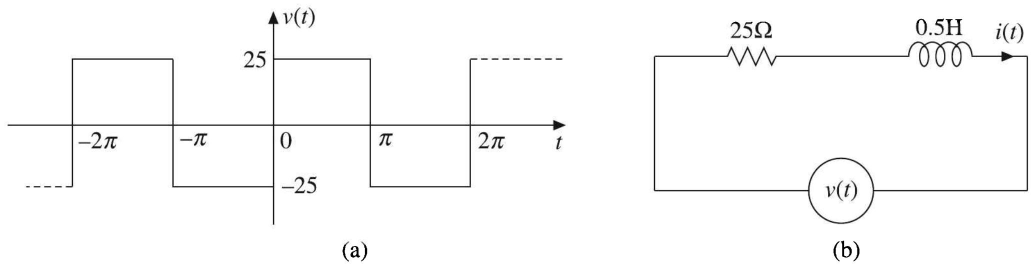


Figure 4.39 (a) Voltage waveform (b) Circuit diagram for Example 4.36.

Solution: The trigonometric Fourier series representation of the voltage waveform shown in Figure 4.39(a) is given by [Refer Example 4.7]

$$x(t) = \sum_{n=\text{odd}}^{\infty} \frac{100}{n\pi} \sin nt = \frac{100}{\pi} \sin t + \frac{100}{3\pi} \sin 3t + \frac{100}{5\pi} \sin 5t$$

The third harmonic component of voltage signal = $(100/3\pi) \sin 3t$.

Suppose we now increase the fundamental period T of the periodic signal $x(t)$. Let us say double the period. Then the amplitude spectrum will appear as shown in Figure 5.1(c). When the period is increased by four times, the spectrum will be as shown in Figure 5.1(d). Observe that the amplitude of the spectrum is decreased but the general shape remains unchanged.

As can be seen from Figure 5.1[(c) and (d)], an increase in fundamental period T results in a spectrum in which the spectral lines become closer and closer. This is to be expected, since $f_0 = 1/T$ and so, as T goes on increasing, f_0 goes on decreasing and the spacing between adjacent spectral lines becomes smaller and smaller.

On the other hand, as the pulses in $x(t)$ are separated by T seconds, as T increases, the interval between adjacent pulses becomes longer and longer. Finally, as T tends to infinity, adjacent pulses become separated by an infinite amount of time, i.e. $x(t)$ becomes a non-periodic function. At the same time, separation between the spectral lines becomes infinitesimally small, or, in other words, a non-periodic signal will be having a continuous spectrum. The envelope of the spectrum depends only upon the pulse shape but not upon the period of repetition.

5.2.1 Derivation of the Fourier Transform of a Non-periodic Signal from the Fourier Series of a Periodic Signal

Let $x(t)$ be a non-periodic function and, $x_T(t)$ be periodic with period T, and let their relation is given by

$$x(t) = \operatorname*{Lt}_{T \to \infty} x_T(t)$$

The Fourier series of a periodic signal $x_T(t)$ is:

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where
$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t)\, e^{-jn\omega_0 t}\, dt \text{ and } \omega_0 = \frac{2\pi}{T}$$

[The term C_n represents the amplitude of the component of frequency $n\omega_0$.]

$\therefore$
$$TC_n = \int_{-T/2}^{T/2} x_T(t)\, e^{-jn\omega_0 t}\, dt$$

Let $n\omega_0 = \omega$ at $T \to \infty$. As $T \to \infty$, we have $\omega_0 = (2\pi/T) \to 0$ and the discrete Fourier spectrum becomes continuous. Further, the summation becomes integral and $x_T(t) \to x(t)$.

Thus, as $T \to \infty$,

$$TC_n = \operatorname*{Lt}_{T \to \infty} \int_{-T/2}^{T/2} x_T(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \left[\operatorname*{Lt}_{T \to \infty} x_T(t) \right] e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X(\omega)$$

$\therefore$
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

[$X(\omega)$ represents the frequency spectrum of $x(t)$ and is called the spectral density function.]

Hence, $X(\omega)$ is called the Fourier transform or the Fourier integral of $x(t)$.

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{T} e^{jn\omega_0 t} \qquad \left[\because \; C_n = \frac{TC_n}{T} = \frac{X(\omega)}{T}\right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi} e^{jn\omega_0 t}\,\omega_0 \qquad \left[n\omega_0 = \omega,\; T = \frac{2\pi}{\omega_0}\right]$$

$$\therefore \qquad x(t) = \underset{T\to\infty}{\text{Lt}}\; x_T(t) = \underset{T\to\infty}{\text{Lt}}\; \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega)\, e^{jn\omega_0 t}\,\omega_0$$

As $T \to \infty$, $\omega_0 = (2\pi/T)$ becomes infinitesimally small and may be represented by $d\omega$. Also the summation becomes integration.

$$\therefore \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

Hence, $x(t)$ is called the *inverse Fourier transform* of $X(\omega)$.
The equations

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

for $X(\omega)$ and $x(t)$ are known as *Fourier transform pair* and can be denoted as:

$$X(\omega) = \text{F}[x(t)]$$

and

$$x(t) = \text{F}^{-1}[X(\omega)]$$

The other notation that can be used to represent the Fourier transform pair is:

$$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

5.3 MAGNITUDE AND PHASE REPRESENTATION OF FOURIER TRANSFORM

The magnitude and phase representation of the Fourier transform is the tool used to analyse the transformed signal.

In general, $X(\omega)$ is a complex valued function of ω. Therefore, $X(\omega)$ can be written as:

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

where $X_R(\omega)$ is the real part of $X(\omega)$ and $X_I(\omega)$ is the imaginary part of $X(\omega)$.

The magnitude of $X(\omega)$ is given by

$$|X(\omega)| = \sqrt{X_R(\omega)^2 + X_I(\omega)^2}$$

and the phase of $X(\omega)$ is given by

$$\underline{|X(\omega)} = \tan^{-1}\frac{X_I(\omega)}{X_R(\omega)}$$

The plot of $|X(\omega)|$ versus ω is known as *amplitude spectrum*, and the plot of $\underline{|X(\omega)}$ versus ω is known as *phase spectrum*. The amplitude spectrum and phase spectrum together is called *frequency spectrum*.

5.4 EXISTENCE OF FOURIER TRANSFORMS

The Fourier transform does not exist for all aperiodic functions. The conditions for a function $x(t)$ to have Fourier transform, called Dirichlet's conditions, are:

1. $x(t)$ is absolutely integrable over the interval $-\infty$ to ∞, that is

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

2. $x(t)$ has a finite number of discontinuities in every finite time interval. Further, each of these discontinuities must be finite.
3. $x(t)$ has a finite number of maxima and minima in every finite time interval.

Almost all the signals that we come across in physical problems satisfy all the above conditions except possibly the absolute integrability condition.

Dirichlet's condition is a sufficient condition but not necessary condition. This means, Fourier transform will definitely exist for functions which satisfy these conditions. On the other hand, in some cases, Fourier transform can be found with the use of impulses even for functions like step function, sinusoidal function, etc. which do not satisfy the convergence condition.

5.5 FOURIER TRANSFORMS OF STANDARD SIGNALS

5.5.1 Impulse Function $\delta(t)$

Given $x(t) = \delta(t)$,

$$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

Then

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} = 1$$

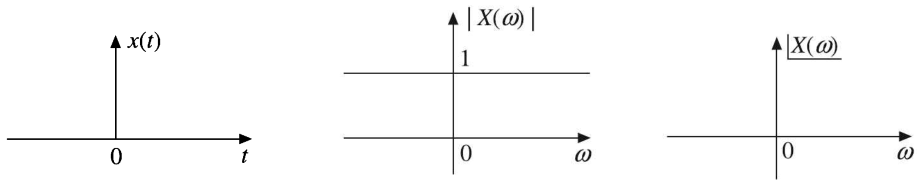
$\therefore$ $$F[\delta(t)] = 1 \quad \text{or} \quad \boxed{\delta(t) \xleftrightarrow{\text{FT}} 1}$$

Hence, the Fourier transform of a unit impulse function is unity.

$$|X(\omega)| = 1 \quad \text{for all } \omega$$

$$\angle X(\omega) = 0 \quad \text{for all } \omega$$

The impulse function with its magnitude and phase spectra are shown in Figure 5.2.



(a) (b) (c)

Figure 5.2 (a) Impulse function, (b) Its magnitude spectrum, (c) Its phase spectrum.

Similarly, $F[\delta(t-t_0)] = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\omega t}\, dt = e^{-j\omega t_0}$ i.e. $\boxed{\delta(t-t_0) \xleftrightarrow{\text{FT}} e^{-j\omega t_0}}$

5.5.2 Single-sided Real Exponential Function $e^{-at}u(t)$

Given $x(t) = e^{-at}\,u(t)$, $$u(t) = \begin{cases} 1 & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Then $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{e^{-\infty} - e^{0}}{-(a+j\omega)}$$

$$= \frac{0-1}{-(a+j\omega)} = \frac{1}{a+j\omega}$$

$\therefore$ $$F[e^{-at}u(t)] = \frac{1}{a+j\omega} \quad \text{or} \quad \boxed{e^{-at}\,u(t) \xleftrightarrow{\text{FT}} \frac{1}{a+j\omega}}$$

Now, $$X(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{(a+j\omega)(a-j\omega)}$$

$$= \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2} = \frac{1}{\sqrt{a^2+\omega^2}} \angle -\tan^{-1}\frac{\omega}{a}$$

$\therefore$ $$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \angle X(\omega) = -\tan^{-1}\frac{\omega}{a} \quad \text{for all } \omega$$

Figure 5.3 shows the single-sided exponential function with its magnitude and phase spectra.

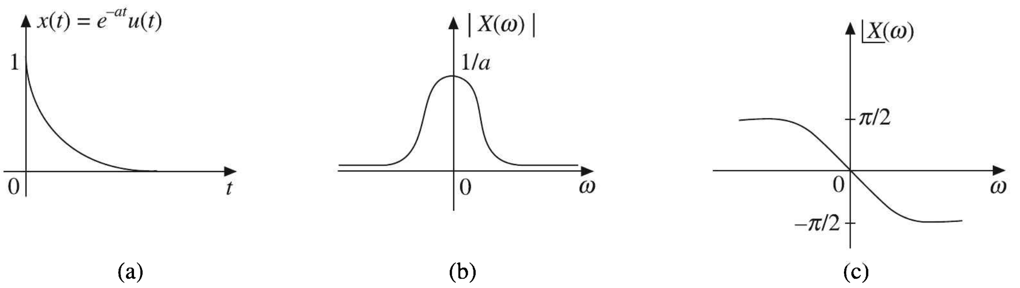


Figure 5.3 (a) One-sided exponential function, (b) Its amplitude spectrum, and (c) Its phase spectrum.

5.5.3 Double-sided Real Exponential Function $e^{-a|t|}$

Given $x(t) = e^{-a|t|}$

$$\therefore \qquad x(t) = e^{-a|t|} = \begin{cases} e^{-a(-t)} = e^{at} & \text{for } t \le 0 \\ e^{-a(t)} = e^{-at} & \text{for } t \ge 0 \end{cases}$$

$$= e^{-a(-t)}\,u(-t) + e^{-at}u(t)$$

$$= e^{at}u(-t) + e^{-at}u(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{at}\,e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \int_{0}^{\infty} e^{-(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt = \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right]_0^{\infty} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

$$= \frac{e^{-\infty} - e^{-0}}{-(a-j\omega)} + \frac{e^{-\infty} - e^{-0}}{-(a+j\omega)} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$\therefore \qquad \mathrm{F}(e^{-a|t|}) = \frac{2a}{a^2+\omega^2} \quad \text{or} \quad \boxed{e^{-a|t|} \xleftrightarrow{\text{FT}} \frac{2a}{a^2+\omega^2}}$$

$$\therefore \qquad |X(\omega)| = \frac{2a}{a^2+\omega^2} \text{ for all } \omega$$

and

$$\angle X(\omega) = 0 \text{ for all } \omega$$

A two-sided exponential function and its amplitude and phase spectra are shown in Figure 5.4.

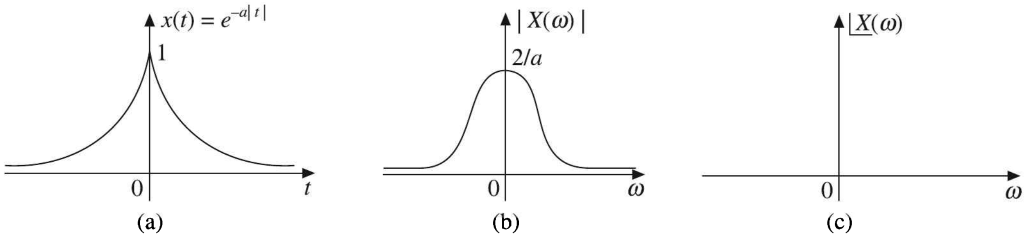


Figure 5.4 (a) Two-sided exponential function, (b) Its magnitude spectrum, and (c) Its phase spectrum.

5.5.4 Complex Exponential Function $e^{j\omega_0 t}$

To find the Fourier transform of complex exponential function $e^{j\omega_0 t}$, consider finding the inverse Fourier transform of $\delta(\omega - \omega_0)$. Let

$$X(\omega) = \delta(\omega - \omega_0)$$

$$x(t) = \mathrm{F}^{-1}[X(\omega)] = \mathrm{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$\therefore$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

$\therefore$
$$\mathrm{F}^{-1}[\delta(\omega - \omega_0)] = \frac{e^{j\omega_0 t}}{2\pi} \quad \text{or} \quad \mathrm{F}^{-1}[2\pi\delta(\omega - \omega_0)] = e^{j\omega_0 t}$$

$\therefore$
$$\mathrm{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0)$$

or
$$\boxed{e^{j\omega_0 t} \xleftrightarrow{\mathrm{FT}} 2\pi\delta(\omega - \omega_0)}$$

5.5.5 Constant Amplitude (1)

Since $x(t) = 1$ is not absolutely integrable, we cannot find its Fourier transform directly. So the Fourier transform of $x(t) = 1$ is determined through inverse Fourier transform of $\delta(\omega)$. We know that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Consider $X(\omega) = \delta(\omega)$
$$\delta(\omega) = \begin{cases} 1 & \text{for } \omega = 0 \\ 0 & \text{for } \omega \neq 0 \end{cases}$$

$$x(t) = \mathrm{F}^{-1}[X(\omega)] = \mathrm{F}^{-1}[\delta(\omega)]$$

$\therefore$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}(1) = \frac{1}{2\pi}$$

$\therefore$ $$\mathrm{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi} \quad \text{or} \quad \mathrm{F}^{-1}[2\pi\delta(\omega)] = 1$$

$\therefore$ $$\mathrm{F}[1] = 2\pi\delta(\omega)$$

or $$\boxed{1 \xleftrightarrow{\text{FT}} 2\pi\delta(\omega)}$$

Similarly, $$\mathrm{F}[A] = 2\pi A\delta(\omega)$$

Alternate method

Let $$x(t) = 1$$

$\therefore$ $$X(\omega) = \mathrm{F}[1] = \int_{-\infty}^{\infty} (1)\, e^{-j\omega t}\, dt$$

We have $$\int_{-\infty}^{\infty} e^{-jxt}\, dx = 2\pi\delta(t)$$

$\therefore$ $$\int_{-\infty}^{\infty} e^{-j\omega t}\, dt = 2\pi\delta(\omega)$$

$\therefore$ $$\mathrm{F}[1] = 2\pi\delta(\omega)$$

Another alternate method

Let $$x(t) = 1 \quad -\infty \le t \le \infty$$

The waveform of a constant function is shown in Figure 5.5. Let us consider a small section of the constant function, say, of duration τ. If we extend the small duration to infinity, we will get back the original function. Therefore,

$$x(t) = \underset{\tau \to \infty}{\mathrm{Lt}} \left[\mathrm{rect}\left(\frac{t}{\tau}\right) \right]$$

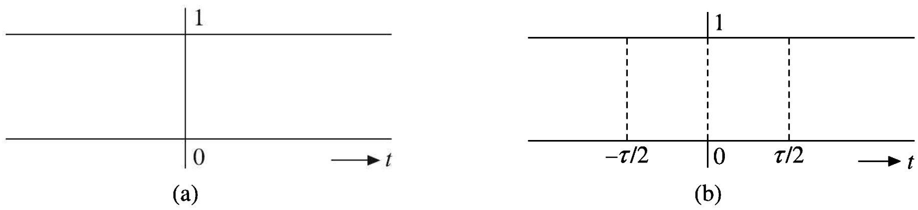


Figure 5.5 (a) Constant function, (b) Small section of the constant function.

where $$\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } -\dfrac{\tau}{2} \le t \le \dfrac{\tau}{2} \\ 0 & \text{elsewhere} \end{cases}$$

By definition, the Fourier transform of $x(t)$ is:

$$X(\omega) = \mathrm{F}[x(t)] = \mathrm{F}\left[\operatorname*{Lt}_{\tau\to\infty} \operatorname{rect}\left(\frac{t}{\tau}\right)\right] = \operatorname*{Lt}_{\tau\to\infty} \mathrm{F}\left[\operatorname{rect}\left(\frac{t}{\tau}\right)\right]$$

$$= \operatorname*{Lt}_{\tau\to\infty} \int_{-\tau/2}^{\tau/2} (1)\, e^{-j\omega t}\, dt = \operatorname*{Lt}_{\tau\to\infty} \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2}$$

$$= \operatorname*{Lt}_{\tau\to\infty} \left[\frac{e^{-j\omega(\tau/2)} - e^{j\omega(\tau/2)}}{-j\omega}\right] = \operatorname*{Lt}_{\tau\to\infty} \left\{\frac{2\sin[\omega(\tau/2)]}{\omega}\right\}$$

$$= \operatorname*{Lt}_{\tau\to\infty} \left\{\tau \frac{\sin[\omega(\tau/2)]}{\omega(\tau/2)}\right\} = \operatorname*{Lt}_{\tau\to\infty} \tau \mathrm{Sa}\left(\omega\frac{\tau}{2}\right) = 2\pi\left[\operatorname*{Lt}_{\tau\to\infty} \frac{\tau/2}{\pi} \mathrm{Sa}\left(\omega\frac{\tau}{2}\right)\right]$$

Using the sampling property of the delta function $\left\{\text{i.e. } \operatorname*{Lt}_{\tau\to\infty} \left[\frac{\tau/2}{\pi} \mathrm{Sa}\left(\omega\frac{\tau}{2}\right)\right] = \delta(\omega)\right\}$, we get

$$X(\omega) = \mathrm{F}\left[\operatorname*{Lt}_{\tau\to\infty} \operatorname{rect}\left(\frac{t}{\tau}\right)\right] = 2\pi\delta(\omega)$$

5.5.6 Signum Function sgn(t)

The signum function is denoted by sgn(t) and is defined by

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

This function is not absolutely integrable. So we cannot directly find its Fourier transform. Therefore, let us consider the function $e^{-a|t|}\operatorname{sgn}(t)$ and substitute the limit $a \to 0$ to obtain the above sgn (t).

Given $\quad x(t) = \operatorname{sgn}(t) = \operatorname*{Lt}_{a\to 0} e^{-a|t|}\operatorname{sgn}(t) = \operatorname*{Lt}_{a\to 0} [e^{-at}u(t) - e^{at}u(-t)]$

$$\therefore \quad X(\omega) = \mathrm{F}[\operatorname{sgn}(t)] = \int_{-\infty}^{\infty} \left[\operatorname*{Lt}_{a\to 0} [e^{-at}u(t) - e^{at}u(-t)]\right] e^{-j\omega t}\, dt$$

$$= \operatorname*{Lt}_{a\to 0}\left[\int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}u(t)\, dt - \int_{-\infty}^{\infty} e^{at}e^{-j\omega t}u(-t)\, dt\right]$$

$$= \operatorname*{Lt}_{a\to 0}\left[\int_{0}^{\infty} e^{-(a+j\omega)t}\, dt - \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt\right] = \operatorname*{Lt}_{a\to 0}\left[\int_{0}^{\infty} e^{-(a+j\omega)t}\, dt - \int_{0}^{\infty} e^{-(a-j\omega)t}\, dt\right]$$

$$= \operatorname*{Lt}_{a\to 0}\left\{\left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} - \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right]_0^{\infty}\right\} = \operatorname*{Lt}_{a\to 0}\left[\frac{e^{-\infty} - e^{0}}{-(a+j\omega)} - \frac{e^{-\infty} - e^{0}}{-(a-j\omega)}\right]$$

$$= \operatorname*{Lt}_{a\to 0}\left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \frac{1}{j\omega} - \frac{1}{-j\omega} = \frac{2}{j\omega}$$

$\therefore$ $$F[\text{sgn}(t)] = \frac{2}{j\omega}$$

or $$\boxed{\text{sgn}(t) \xleftrightarrow{\text{FT}} \frac{2}{j\omega}}$$

$\therefore$ $$|X(\omega)| = \frac{2}{\omega} \quad \text{and} \quad \underline{/X(\omega)} = \frac{\pi}{2} \text{ for } \omega < 0 \quad \text{and} \quad -\frac{\pi}{2} \text{ for } \omega > 0$$

Figure 5.6 shows the signum function and its magnitude and phase spectra.

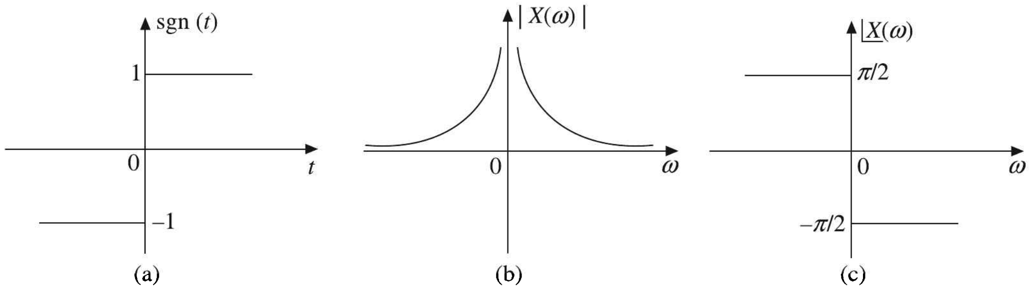


Figure 5.6 (a) Signum function, (b) Its amplitude spectrum, (c) Its phase spectrum.

5.5.7 Unit Step Function $u(t)$

The unit step function is defined by

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Since the unit step function is not absolutely integrable, we cannot directly find its Fourier transform. So express the unit step function in terms of signum function as:

$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

Given $$x(t) = u(t) = \frac{1}{2}[1 + \text{sgn}(t)]$$

$$X(\omega) = F[u(t)] = F\left\{\frac{1}{2}[1 + \text{sgn}(t)]\right\}$$

$$= \frac{1}{2}\{F[1] + F[\text{sgn}(t)]\}$$

We know that

$$F[1] = 2\pi\delta(\omega) \quad \text{and} \quad F[\text{sgn}(t)] = \frac{2}{j\omega}$$

$\therefore$ $$F[u(t)] = \frac{1}{2}\left[2\pi\delta(\omega) + \frac{2}{j\omega}\right] = \pi\delta(\omega) + \frac{1}{j\omega}$$

$\therefore$ $$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

or $$\boxed{u(t) \xleftrightarrow{\text{FT}} \pi\delta(\omega) + \frac{1}{j\omega}}$$

$\therefore$ $|X(\omega)| = \infty$ at $\omega = 0$ and is equal to 0 at $\omega = -\infty$ and $\omega = \infty$.

Figure 5.7 shows the unit step function and its spectrum.

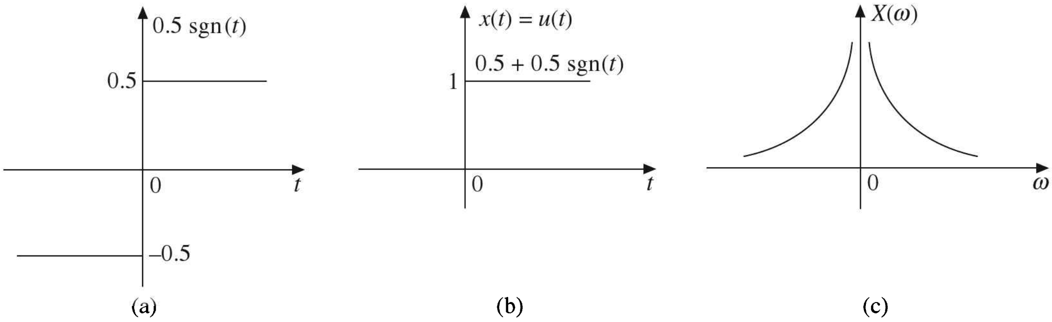


Figure 5.7 (a) 0.5 sgn(t), (b) $u(t)$, (c) Spectrum of $u(t)$.

5.5.8 Rectangular Pulse (Gate pulse) $\Pi\left(\frac{t}{\tau}\right)$ or rect $\left(\frac{t}{\tau}\right)$

Consider a rectangular pulse as shown in Figure 5.8. This is called a unit gate function and is defined as:

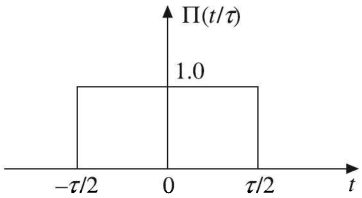


Figure 5.8 Unit gate function.

$$x(t) = \text{rect}\left(\frac{t}{\tau}\right) = \Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$ $$X(\omega) = F\left[\Pi\left(\frac{t}{\tau}\right)\right] = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$$

$$= \int_{-\tau/2}^{\tau/2} (1)\, e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{e^{-j\omega(\tau/2)} - e^{j\omega(\tau/2)}}{-j\omega} = \frac{e^{j\omega(\tau/2)} - e^{-j\omega(\tau/2)}}{j\omega}$$

$$= \frac{\tau}{\omega(\tau/2)}\left[\frac{e^{j\omega(\tau/2)} - e^{-j\omega(\tau/2)}}{2j}\right] = \tau\left[\frac{\sin\omega(\tau/2)}{\omega(\tau/2)}\right]$$

$$= \tau \operatorname{sinc}\frac{\omega\tau}{2}$$

$$= \tau \mathrm{Sa}\frac{\omega\tau}{2}$$

$\therefore \qquad F\left[\Pi\left(\frac{t}{\tau}\right)\right] = \tau \operatorname{sinc}\frac{\omega\tau}{2}$, that is

$$\boxed{\operatorname{rect}\left(\frac{t}{\tau}\right) = \Pi\left(\frac{t}{\tau}\right) \xleftrightarrow{\text{FT}} \tau \operatorname{sinc}\frac{\omega\tau}{2}}$$

Figure 5.9 shows the spectra of the gate function.

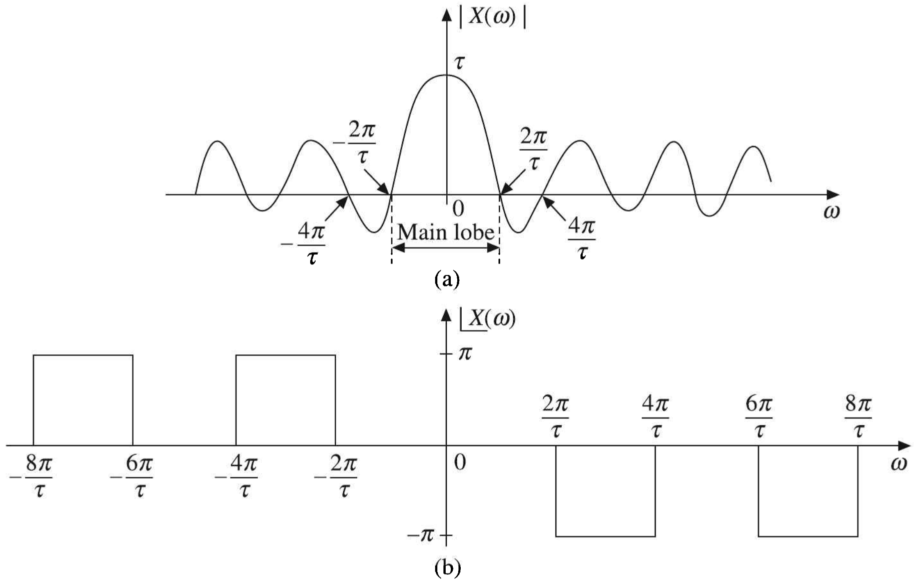


Figure 5.9 (a) Amplitude spectrum, and (b) Phase spectrum of $\Pi(t/\tau)$.

The amplitude spectrum is obtained as follows:

At $\omega = 0$, sinc $(\omega\tau/2) = 1$. Therefore, $|X(\omega)|$ at $\omega = 0$ is equal to τ. At $(\omega\tau/2) = \pm n\pi$, i.e. at

$$\omega = \pm\frac{2n\pi}{\tau},\ n = 1, 2, \ldots,\ \operatorname{sinc}\left(\frac{\omega\tau}{2}\right) = 0$$

The phase spectrum is:

$$\underline{|X(\omega)} = 0 \quad \text{if } \operatorname{sinc}(\omega\tau/2) > 0$$
$$= \pm\pi \quad \text{if } \operatorname{sinc}(\omega\tau/2) < 0$$

The amplitude response between the first two zero crossings is known as main lobe and the portions of the response for $\omega < -(2\pi/\tau)$ and $\omega > (2\pi/\tau)$ are known as side lobes. From the amplitude spectrum, we can find that majority of the energy of the signal is contained in the main lobe. The first zero crossing occurs at $\omega = (2\pi/\tau)$ or at $f = (1/\tau)$ Hz. As the width of the rectangular pulse is made longer, the main lobe becomes narrower. The phase spectrum is odd function of ω. If the amplitude spectrum is positive, then phase is zero, and if the amplitude spectrum is negative, then the phase is $-\pi$ or π.

5.5.9 Triangular Pulse $\Delta\left(\frac{t}{\tau}\right)$

Consider the triangular pulse as shown in Figure 5.10. It is defined as:

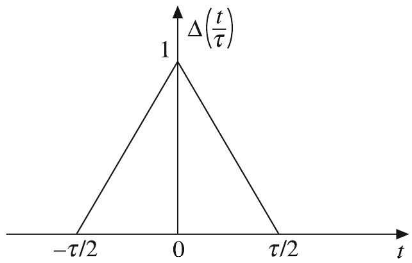


Figure 5.10 Triangular function.

$$x(t) = \Delta\left(\frac{t}{\tau}\right) = \begin{cases} \dfrac{1}{\tau/2}\left(t + \dfrac{\tau}{2}\right) = \left(1 + 2\dfrac{t}{\tau}\right) & \text{for } -\dfrac{\tau}{2} < t < 0 \\ \dfrac{1}{\tau/2}\left(t - \dfrac{\tau}{2}\right) = \left(1 - 2\dfrac{t}{\tau}\right) & \text{for } 0 < t < \dfrac{\tau}{2} \\ 0 & \text{elsewhere} \end{cases}$$

i.e. as
$$x(t) = \Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \dfrac{2|t|}{\tau} & \text{for } |t| < \dfrac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$
$$X(\omega) = \mathrm{F}[x(t)] = \mathrm{F}\left[\Delta\left(\frac{t}{\tau}\right)\right] = \int_{-\infty}^{\infty} \Delta\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$$
$$= \int_{-\tau/2}^{0} \left(1 + \frac{2t}{\tau}\right) e^{-j\omega t}\, dt + \int_{0}^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{-j\omega t}\, dt$$

$$= \int_0^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{j\omega t}\, dt + \int_0^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{-j\omega t}\, dt$$

$$= \int_0^{\tau/2} e^{j\omega t}\, dt - \int_0^{\tau/2} \frac{2t}{\tau} e^{j\omega t}\, dt + \int_0^{\tau/2} e^{-j\omega t}\, dt - \int_0^{\tau/2} \frac{2t}{\tau} e^{-j\omega t}\, dt$$

$$= \int_0^{\tau/2} [e^{j\omega t} + e^{-j\omega t}]\, dt - \frac{2}{\tau} \int_0^{\tau/2} t[e^{j\omega t} + e^{-j\omega t}]\, dt$$

$$= \int_0^{\tau/2} 2 \cos \omega t\, dt - \frac{2}{\tau} \int_0^{\tau/2} 2t \cos \omega t\, dt$$

$$= 2\left[\frac{\sin \omega t}{\omega}\right]_0^{\tau/2} - \frac{4}{\tau}\left[\left[t\frac{\sin \omega t}{\omega}\right]_0^{\tau/2} - \int_0^{\tau/2} \frac{\sin \omega t}{\omega}\, dt\right]$$

$$= 2\left[\frac{\sin \omega t}{\omega}\right]_0^{\tau/2} - \frac{4}{\tau}\left[\left[t\frac{\sin \omega t}{\omega}\right]_0^{\tau/2} + \left[\frac{\cos \omega t}{\omega^2}\right]_0^{\tau/2}\right]$$

$$= \frac{2}{\omega}\left[\sin \omega\frac{\tau}{2}\right] - \frac{4}{\omega\tau}\left[\frac{\tau}{2}\sin\frac{\omega\tau}{2}\right] - \frac{4}{\omega^2\tau}\left[\cos\frac{\omega\tau}{2} - 1\right]$$

$$= \frac{4}{\omega^2\tau}\left[1 - \cos\frac{\omega\tau}{2}\right] = \frac{4}{\omega^2\tau}\left[2\sin^2\frac{\omega\tau}{4}\right]$$

$$= \frac{8}{\omega^2\tau}\left(\frac{\omega\tau}{4}\right)^2 \frac{\sin^2(\omega\tau/4)}{(\omega\tau/4)^2} = \frac{\tau}{2}\,\text{sinc}^2\,\frac{\omega\tau}{4}$$

$\therefore$
$$\text{F}\left[\Delta\left(\frac{t}{\tau}\right)\right] = \frac{\tau}{2}\,\text{sinc}^2\,\frac{\omega\tau}{4}$$

or
$$\boxed{\Delta\left(\frac{t}{\tau}\right) \xleftrightarrow{\text{FT}} \frac{\tau}{2}\,\text{sinc}^2\,\frac{\omega\tau}{4}}$$

Figure 5.11 shows the amplitude spectrum of a triangular pulse.

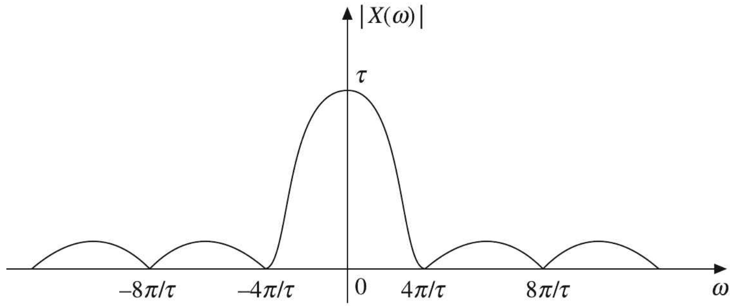


Figure 5.11 Amplitude spectrum of a triangular pulse.

5.5.10 Cosine Wave $\cos \omega_0 t$

Given $$x(t) = \cos \omega_0 t$$

Then $$X(\omega) = \mathrm{F}[x(t)] = \mathrm{F}[\cos \omega_0 t] = \mathrm{F}\left[\frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\right]$$

$$= \frac{1}{2}\left[\mathrm{F}(e^{j\omega_0 t}) + \mathrm{F}(e^{-j\omega_0 t})\right] = \frac{1}{2}\left[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)\right]$$

$$= \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

$\therefore$ $\mathrm{F}[\cos \omega_0 t] = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$ or $\boxed{\cos \omega_0 t \xleftrightarrow{\text{FT}} \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]}$

Figure 5.12 shows the cosine wave and its amplitude and phase spectra.

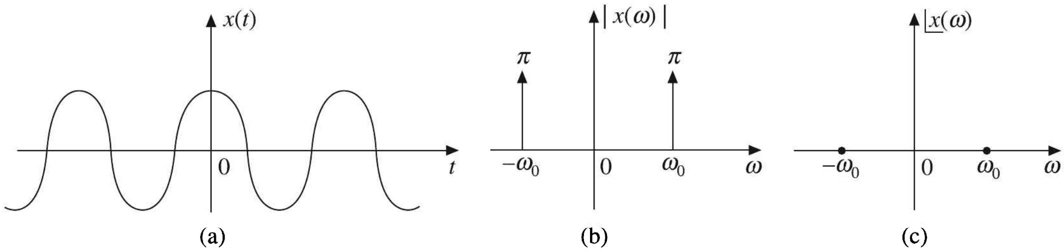


Figure 5.12 (a) Cosine wave, (b) Its magnitude spectrum, (c) Its phase spectrum.

5.5.11 Sine Wave $\sin \omega_0 t$

Given $$x(t) = \sin \omega_0 t$$

Then $$\mathrm{F}[x(t)] = X(\omega) = \mathrm{F}[\sin \omega_0 t] = \mathrm{F}\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]$$

$$= \frac{1}{2j}\left[\mathrm{F}[e^{j\omega_0 t}] - \mathrm{F}[e^{-j\omega_0 t}]\right] = \frac{1}{2j}[2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)]$$

$$= -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

$\therefore$ $\mathrm{F}[\sin \omega_0 t] = -j\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ or $\boxed{\sin \omega_0 t \xleftrightarrow{\text{FT}} -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]}$

Figure 5.13 shows the sine wave signal and its amplitude and phase spectra.

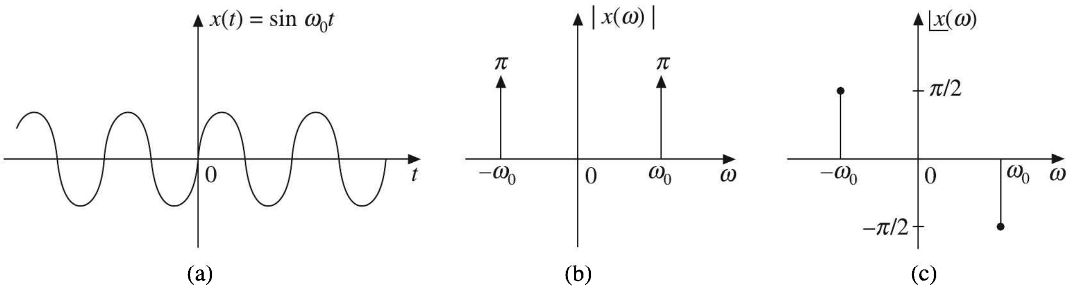


Figure 5.13 (a) Sinusoidal signal, (b) Its amplitude spectrum, (c) Its phase spectrum.

5.6 PROPERTIES OF CONTINUOUS TIME FOURIER TRANSFORM

The Fourier transform has a number of important properties. These properties are useful for deriving Fourier transform pairs as well as for deducing general frequency domain relationships. These also help to find the effect of various time domain operations on the frequency domain. Some of the important properties are discussed as follows:

5.6.1 Linearity Property

The linearity property states that the Fourier transform of a weighted sum of two signals is equal to the weighted sum of their individual Fourier transforms.

i.e. If $$x_1(t) \xleftrightarrow{\text{FT}} X_1(\omega) \quad \text{and} \quad x_2(t) \xleftrightarrow{\text{FT}} X_2(\omega)$$

Then $$ax_1(t) + bx_2(t) \xleftrightarrow{\text{FT}} aX_1(\omega) + bX_2(\omega)$$

where a and b are constants.

Proof: By definition,

$$\text{F}[ax_1(t) + bx_2(t)] = \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} ax_1(t)\, e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} bx_2(t)\, e^{-j\omega t}\, dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt + b\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt$$

$$= aX_1(\omega) + bX_2(\omega)$$

$\therefore$ $$\boxed{ax_1(t) + bx_2(t) \xleftrightarrow{\text{FT}} aX_1(\omega) + bX_2(\omega)}$$

5.6.2 Time Shifting Property

The time shifting property states that if a signal $x(t)$ is shifted by t_0 sec, the spectrum is modified by a linear phase shift of slope $-\omega t_0$, i.e.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$x(t - t_0) \xleftrightarrow{\text{FT}} e^{-j\omega t_0}\, X(\omega)$$

Proof: By definition,

$$\text{F}[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j\omega t}\, dt$$

Let $$t - t_0 = p$$

$\therefore$ $t = p + t_0$ and $dt = dp$

$\therefore$ $$F[x(t - t_0)] = \int_{-\infty}^{\infty} x(p)\, e^{-j\omega(p+t_0)}\, dp$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(p)\, e^{-j\omega p}\, dp$$

$$= e^{-j\omega t_0} X(\omega)$$

$\therefore$ $$\boxed{x(t - t_0) \xleftrightarrow{\text{FT}} e^{-j\omega t_0} X(\omega)}$$

Similarly, $$x(t + t_0) \xleftrightarrow{\text{FT}} e^{j\omega t_0} X(\omega)$$

This property has a very important implication. That is

$$|e^{-j\omega t_0} X(\omega)| = |X(\omega)|$$

and $$\underline{|e^{-j\omega t_0} X(\omega)} = e^{-j\omega t_0} + \underline{|X(\omega)} = \underline{|-\omega t_0} + \underline{|X(\omega)}$$

This shows that shifting a function by t_0 results in multiplying its Fourier transform by $e^{-j\omega t_0}$. Thus, there is no change in magnitude spectrum but the phase spectrum is linearly shifted.

5.6.3 Frequency Shifting Property (Multiplication by an Exponential)

Frequency shifting property states that the multiplication of a time domain signal $x(t)$ by $e^{-j\omega t_0}$ results in the frequency spectrum shifted by ω_0, i.e.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$e^{j\omega_0 t} x(t) \xleftrightarrow{\text{FT}} X(\omega - \omega_0)$$

Proof: By definition,

$$F[e^{j\omega_0 t} x(t)] = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - \omega_0)t}\, dt$$

$$= X(\omega - \omega_0)$$

$\therefore$ $$\boxed{e^{j\omega_0 t} x(t) \xleftrightarrow{\text{FT}} X(\omega - \omega_0)}$$

Similarly, $$e^{-j\omega_0 t} x(t) \xleftrightarrow{\text{FT}} X(\omega + \omega_0)$$

5.6.4 Time Reversal Property

The time reversal property states that

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$x(-t) \xleftrightarrow{\text{FT}} X(-\omega)$$

Proof: By definition,

$$\text{F}[x(-t)] = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt$$

Replacing t by $-t$ in the RHS of the above expression for F[$x(-t)$], we have

$$\text{F}[x(-t)] = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega)t}\, dt = X(-\omega)$$

$\therefore$ $$\boxed{x(-t) \xleftrightarrow{\text{FT}} X(-\omega)}$$

5.6.5 Time Scaling Property

Let $x(at)$ is a compressed version of $x(t)$ when $a > 1$ or expanded version of $x(t)$ when $a < 1$.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$x(at) \xleftrightarrow{\text{FT}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof: By definition,

$$\text{F}[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $$at = p$$

$\therefore$ $$t = \frac{p}{a} \quad \text{and} \quad dt = \frac{dp}{a}$$

$$\text{F}[x(at)] = \int_{-\infty}^{\infty} x(p)\, e^{-j\omega(p/a)}\, \frac{dp}{a}$$

$\therefore$ $$= \frac{1}{a} \int_{-\infty}^{\infty} x(p)\, e^{-j(\omega/a)\,p}\, dp$$

CASE 1 When $a > 0$,

$$\text{F}[x(at)] = \frac{1}{a} \int_{-\infty}^{\infty} x(p)\, e^{-j(\omega/a)\,p}\, dp = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

CASE 2 When $a < 0$,

$$F[at] = \frac{1}{-a}\int_{-\infty}^{\infty} x(p)\, e^{j(\omega/a)\,p}\, dp = \frac{1}{-a}\int_{-\infty}^{\infty} x(p)\, e^{-j[-(\omega/a)]p}\, dp$$

$$= -\frac{1}{a} X\left(-\frac{\omega}{a}\right)$$

$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$\therefore$
$$\boxed{x(at) \xleftrightarrow{\text{FT}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$$

5.6.6 Differentiation in Time Domain Property

The differentiation in time domain property states that the differentiation of a function in time domain is equivalent to the multiplication of its Fourier transform by a factor $j\omega$, i.e.

If
$$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then
$$\frac{d}{dt} x(t) \xleftrightarrow{\text{FT}} j\omega X(\omega)$$

Proof: By definition,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Differentiating both sides w.r.t. t, we have

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}[e^{j\omega t}]\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, j\omega\, e^{j\omega t}\, d\omega$$

$$= j\omega\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\right]$$

$$= j\omega\, F^{-1}[X(\omega)]$$

$\therefore$
$$F\left[\frac{d}{dt}x(t)\right] = j\omega\, X(\omega)$$

$\therefore$
$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\text{FT}} j\omega X(\omega)}$$

In general, $$\frac{d^n x(t)}{dt^n} \xleftrightarrow{\text{FT}} (j\omega)^n X(\omega)$$

5.6.7 Differentiation in Frequency Domain Property

The differentiation in frequency domain property states that the multiplication of a signal $x(t)$ by t is equivalent to differentiation of its Fourier transform in frequency domain, i.e.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$tx(t) \xleftrightarrow{\text{FT}} j\frac{d}{d\omega} X(\omega)$$

Proof: By definition,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Differentiating both sides w.r.t. ω, we have

$$\frac{d}{d\omega}[X(\omega)] = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right] = \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}(e^{-j\omega t})\, dt = \int_{-\infty}^{\infty} x(t)\,(-jt)\, e^{-j\omega t}\, dt$$

$$= -j\int_{-\infty}^{\infty} [tx(t)]\, e^{-j\omega t}\, dt = -j\text{F}[tx(t)]$$

$\therefore$ $$\text{F}[tx(t)] = j\frac{d}{d\omega} X(\omega)$$

$\therefore$ $$\boxed{tx(t) \xleftrightarrow{\text{FT}} j\frac{d}{d\omega} X(\omega)}$$

5.6.8 Time Integration Property

The time integration property states that the integration of a function $x(t)$ in time domain is equivalent to the division of its Fourier transform by $j\omega$, i.e.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\text{FT}} \frac{1}{j\omega} X(j\omega), \text{ if } X(0) = 0$$

Proof: If $X(0) = 0$, this property can be easily proved by using integration by parts as in the case of the differentiation property.

By definition,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Replacing t by a dummy variable τ, we have

$$x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega\tau}\, d\omega$$

Integrating both sides over $-\infty$ to t, we have

$$\int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega\tau}\, d\tau\right] d\omega$$

Interchanging the order of integration, we have

$$\int_{-\infty}^{t} x(\tau)\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(\int_{-\infty}^{t} e^{j\omega\tau}\, d\tau\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\frac{e^{j\omega\tau}}{j\omega}\right]_{-\infty}^{t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{X(\omega)}{j\omega}\right] e^{j\omega t}\, d\omega = \mathrm{F}^{-1}\left[\frac{X(\omega)}{j\omega}\right]$$

$\therefore$
$$\mathrm{F}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega)$$

$\therefore$
$$\boxed{\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\text{FT}} \frac{1}{j\omega} X(\omega)}$$

If $X(0) \neq 0$ then $x(t)$ is not an energy function, and the Fourier transform of $\int_{-\infty}^{t} x(\tau)\, d\tau$ includes an impulse function, that is

$$\mathrm{F}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega) + \pi X(0)\, \delta(\omega)$$

5.6.9 Convolution Property or Theorem

The convolution property or theorem states that the convolution of two signals in time domain is equivalent to the multiplication of their spectra in frequency domain. This is called the *time convolution theorem.*

If $\quad x_1(t) \xleftrightarrow{\text{FT}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\text{FT}} X_2(\omega)$

Then $\quad x_1(t) * x_2(t) \xleftrightarrow{\text{FT}} X_1(\omega)\, X_2(\omega)$

Proof: We know that the convolution of two signals $x_1(t)$ and $x_2(t)$ is given by

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \qquad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

Interchanging the order of integration, we have

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

Substituting $t - \tau = p$ in the second integration, we have

$$t = p + \tau \quad \text{and} \quad dt = dp$$

$$\therefore \qquad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega(p+\tau)}\, dp \right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega p}\, dp \right] e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, X_2(\omega)\, e^{-j\omega\tau}\, d\tau$$

$$= \left[\int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau \right] X_2(\omega) = X_1(\omega)\, X_2(\omega)$$

$$\therefore \qquad \boxed{x_1(t) * x_2(t) \xleftrightarrow{\text{FT}} X_1(\omega)\, X_2(\omega)}$$

5.6.10 Multiplication Property or Theorem

The multiplication property or theorem states that the multiplication of two functions in time domain is equivalent to the convolution of their spectra in the frequency domain. This is called the frequency convolution theorem.

If $\qquad x_1(t) \xleftrightarrow{\text{FT}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\text{FT}} X_2(\omega)$

Then $\qquad x_1(t)\, x_2(t) \xleftrightarrow{\text{FT}} \dfrac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Proof: We know that

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and $$F^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$F[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, e^{j\lambda t}\, d\lambda\right] x_2(t)\, e^{-j\omega t}\, dt$$

Interchanging the order of integration, we get

$$F[x_1(t)\, x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\, e^{j\lambda t}\, e^{-j\omega t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j(\omega-\lambda)t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, X_2(\omega-\lambda)\, d\lambda$$

$$= \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$\therefore$ $$\boxed{x_1(t)\, x_2(t) \xleftrightarrow{\text{FT}} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)}$$

or $$2\pi x_1(t)\, x_2(t) \xleftrightarrow{\text{FT}} X_1(\omega) * X_2(\omega)$$

or $$x_1(t)\, x_2(t) \xleftrightarrow{\text{FT}} X_1(f) * X_2(f)$$

5.6.11 Duality (Symmetry) Property

In spectrum analysis, the duality between the time and the frequency is exhibited. The duality (symmetry) property states that

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$X(t) \xleftrightarrow{\text{FT}} 2\pi x(-\omega)$$

Proof: By definition,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$\therefore$ $$2\pi x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

or $$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Interchanging t and ω, we have

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt = \mathrm{F}[X(t)]$$

$\therefore$ $$\mathrm{F}[X(t)] = 2\pi\, x(-\omega)$$

i.e. $$\boxed{X(t) \xleftrightarrow{\text{FT}} 2\pi x(-\omega)}$$

For even functions,

$$x(-\omega) = x(\omega)$$

$\therefore$ $$X(t) \xleftrightarrow{\text{FT}} 2\pi x(\omega)$$

5.6.12 Modulation Property

The modulation property states that, if a signal $x(t)$ is multiplied by cos $\omega_c t$, its spectrum gets translated up and down in frequency by ω_c, i.e.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$x(t)\cos\omega_c t \xleftrightarrow{\text{FT}} \frac{1}{2}[X(\omega - \omega_c) + X(\omega + \omega_c)]$$

Proof: $$x(t)\cos\omega_c t = x(t)\left[\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right]$$

$\therefore$ $$\mathrm{F}[x(t)\cos\omega_c t] = \mathrm{F}\left[\frac{x(t)}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]\right]$$

$$= \frac{1}{2}\mathrm{F}[x(t)\, e^{j\omega_c t}] + \frac{1}{2}\mathrm{F}[x(t)\, e^{-j\omega_c t}] = \frac{1}{2}X[\omega - \omega_c] + \frac{1}{2}X[\omega + \omega_c]$$

$$= \frac{1}{2}\left[X[\omega - \omega_c] + X[\omega + \omega_c]\right]$$

$\therefore$ $$\boxed{x(t)\cos\omega_c t \xleftrightarrow{\text{FT}} \frac{1}{2}[X(\omega - \omega_c) + X(\omega + \omega_c)]}$$

Similarly, $$x(t)\sin\omega_c t \xleftrightarrow{\text{FT}} \frac{1}{2j}[X(\omega - \omega_c) - X(\omega + \omega_c)]$$

5.6.13 Conjugation Property

The conjugation property states that

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$x^*(t) \xleftrightarrow{\text{FT}} X^*(-\omega)$$

Proof: By definition,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore \quad X^*(\omega) = \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]^* = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

Replacing ω by $-\omega$, we get

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{-j\omega t}\, dt = \text{F}[x^*(t)]$$

$$\therefore \quad \boxed{x^*(t) \xleftrightarrow{\text{FT}} X^*(-\omega)}$$

5.6.14 Autocorrelation Property

The autocorrelation property states that the Fourier transform of the autocorrelation of a time domain signal is equal to the square of the modulus of its spectra.

If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

Then $$R(\tau) \xleftrightarrow{\text{FT}} |X(\omega)|^2$$

Proof: We know that the autocorrelation of a signal $x(t)$ is:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

$$\therefore \quad \text{F}[R(\tau)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt\right] e^{-j\omega\tau}\, d\tau$$

Interchanging the order of integration, we have

$$\text{F}[R(\tau)] = \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} x^*(t-\tau)\, e^{-j\omega\tau}\, d\tau\right] dt$$

Let us substitute $t - \tau = p$ in the second integration, we have $d\tau = dp$.

$$\therefore \qquad F[R(\tau)] = \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} x^*(p)\, e^{-j\omega(t-p)}\, dp \right] dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \int_{-\infty}^{\infty} x^*(p)\, e^{j\omega p}\, dp = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \left[\int_{-\infty}^{\infty} x(p)\, e^{-j\omega p}\, dp \right]^*$$

$$= X(\omega)\, X^*(\omega) = |X(\omega)|^2$$

$$\therefore \qquad \boxed{R(\tau) \xleftrightarrow{\text{FT}} |X(\omega)|^2}$$

5.6.15 Parseval's Relation or Theorem or Property

If $\qquad x_1(t) \xleftrightarrow{\text{FT}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\text{FT}} X_2(\omega)$

then, the Parseval's relation or theorem or property states that

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega$$

for complex $x_1(t)$ and $x_2(t)$.

The Parseval's identity states that the energy content of the signal $x(t)$ is:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Proof:

Parseval's relation

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)\, e^{j\omega t}\, d\omega \right\} x_2^*(t)\, dt$$

Interchanging the order of integration, we have

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) \left\{ \int_{-\infty}^{\infty} x_2^*(t)\, e^{j\omega t}\, dt \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) \left\{ \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt \right\}^* d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) \{X_2(\omega)\}^*\, d\omega = \text{RHS}$$

$$\therefore \quad \boxed{\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega} \quad \text{proved.}$$

Parseval's identity

If $x_1(t) = x_2(t) = x(t)$, then the energy of the signal is given by

$$E = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, X^*(\omega)\, d\omega$$

Since, $|x(t)|^2 = x(t)\, x^*(t)$ and $|X(\omega)|^2 = X(\omega)\, X^*(\omega)$, we get

$$\boxed{E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega} \quad \text{proved.}$$

The Parseval's identity is also called *energy theorem* or *Rayleigh's energy theorem*. The quantity $|X(\omega)|^2$ is called the *energy density spectrum* of the signal $x(t)$. Hence, the Parseval's relation states that, the energy [if $x(t)$ is not periodic] or power [if $x(t)$ is periodic] in the time domain representation of a signal is equal to the energy or power in the frequency domain representation.

5.6.16 Area under the Curve

1. The area under a function $x(t)$ is equal to the value of its Fourier transform $X(\omega)$ at $\omega = 0$, i.e.

   If $$x(t) \xleftrightarrow{\text{FT}} X(\omega)$$

   Then $$\int_{-\infty}^{\infty} x(t)\, dt = \frac{1}{2\pi} X(0)$$

2. The area under the Fourier transform $X(\omega)$ of a function $x(t)$ is equal to the value of the function $x(t)$ at $t = 0$,

   i.e. $$\int_{-\infty}^{\infty} X(\omega)\, d\omega = x(0)$$

5.6.17 Fourier Transform of Complex and Real Functions

If the signal $x(t)$ is complex, it can be represented as:

$$x(t) = x_R(t) + jx_I(t)$$

The Fourier transform of $x(t)$ is given by

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} [x_R(t) + jx_I(t)]\,[\cos \omega t - j \sin \omega t]\, dt$$

$$= \int_{-\infty}^{\infty} [x_R(t) \cos \omega t + x_I(t) \sin \omega t]\, dt + j \int_{-\infty}^{\infty} [x_I(t) \cos \omega t - x_R(t) \sin \omega t]\, dt$$

$$= X_R(\omega) + jX_I(\omega)$$

where
$$X_R(\omega) = \int_{-\infty}^{\infty} [x_R(t) \cos \omega t + x_I(t) \sin \omega t]\, dt$$

and
$$X_I(\omega) = \int_{-\infty}^{\infty} [x_I(t) \cos \omega t - x_R(t) \sin \omega t]\, dt$$

The inverse Fourier transform of $X(\omega)$ is obtained from

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [X_R(\omega) + jX_I(\omega)]\,[\cos \omega t + j \sin \omega t]\, d\omega$$

$$= x_R(t) + jx_I(t)$$

where
$$x_R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [X_R(\omega) \cos \omega t - X_I(\omega) \sin \omega t]\, d\omega$$

and
$$x_I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [X_R(\omega) \sin \omega t + X_I(\omega) \cos \omega t]\, d\omega$$

CASE 1 $x(t)$ is real.

If $x(t)$ is real, then $X_I(t) = 0$ and $X(-\omega) = X^*(\omega)$.

Then
$$X_R(\omega) = \int_{-\infty}^{\infty} x(t) \cos \omega t\, dt$$

$$X_I(\omega) = -\int_{-\infty}^{\infty} x(t) \sin \omega t\, dt$$

CASE 2 $x(t)$ is even and real.

Let
$$x(t) = x_e(t)$$

$$X_R(\omega) = \int_{-\infty}^{\infty} x_e(t) \cos \omega t\, dt = 2 \int_{0}^{\infty} x_e(t) \cos \omega t\, dt$$

$$X_I(\omega) = 0$$

$$\therefore \quad X(\omega) = 2\int_0^{\infty} x_e(t)\cos\omega t\, dt$$

CASE 3 $x(t)$ is odd and real.

Let
$$x(t) = x_o(t)$$

$$X_R(\omega) = 0$$

$$X_I(\omega) = jX(\omega) = -j\int_{-\infty}^{\infty} x_o(t)\sin\omega t = -j2\int_0^{\infty} x_o(t)\sin\omega t$$

For a non-symmetrical function

$$\text{F}[x(t)] = X(\omega) = X_R(\omega) + jX_I(\omega)$$

$$= \int_{-\infty}^{\infty} x_e(t)\cos\omega t\, dt - j\int_{-\infty}^{\infty} x_o(t)\sin\omega t\, dt$$

$$= X_e(\omega) + X_o(\omega)$$

Table B.2 (Appendix B) summarises the properties of Fourier transform. Table B.3 (Appendix B) summarises the commonly used Fourier transform pairs. Table B.4 (Appendix B) summarises the commonly used Fourier transform of signals.

5.7 FOURIER TRANSFORM OF A PERIODIC SIGNAL

Till now we have discussed that the periodic functions can be analysed using Fourier series and that non-periodic functions can be analysed using Fourier transform. But we can find the Fourier transform of a periodic function also. This means that the Fourier transform can be used as a universal mathematical tool in the analysis of both non-periodic and periodic waveforms over the entire interval. Fourier transform of periodic functions may be found using the concept of impulse function.

We know that using Fourier series, any periodic signal can be represented as a sum of complex exponentials. Therefore, we can represent a periodic signal using the Fourier integral. Let us consider a periodic signal $x(t)$ with period T. Then, we can express $x(t)$ in terms of exponential Fourier series as:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

The Fourier transform of $x(t)$ is:

$$X(\omega) = \text{F}[x(t)] = \text{F}\left[\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n \text{F}[e^{jn\omega_0 t}]$$

Using the frequency shifting theorem, we have

$$\mathrm{F}\,[1 e^{jn\omega_0 t}] = \mathrm{F}\,[1]\Big|_{\omega=\omega-n\omega_0} = 2\pi\delta(\omega - n\omega_0)$$

$\therefore$
$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0)$$

where C_ns are the Fourier coefficients associated with $x(t)$ and are given by

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

Thus, the Fourier transform of a periodic function consists of a train of equally spaced impulses. These impulses are located at the harmonic frequencies of the signal and the strength (area) of each impulse is given as $2\pi C_n$.

EXAMPLE 5.1 What are the merits and limitations of Fourier transform?

Solution:

Merits: Some of the merits of the Fourier transform are:

1. The original time function can be uniquely recovered from it.
2. Convolution integrals can be evaluated using the Fourier transform.
3. The Fourier transform is the most useful tool for analysing signals involved in communication systems because here the amplitude and phase characteristics are readily known.

Limitations: The following are the limitations of Fourier transform:

1. It is less powerful than the Laplace transform because in the Fourier transform, the damping factor $\sigma = 0$, but in the Laplace transform, the damping factor σ is finite.
2. There are many functions for which the Laplace transform exists, but the Fourier transform does not exist. In fact, Fourier transform is a special case of Laplace transform with $s = j\omega$, that is Fourier transform is the Laplace transform evaluated along the imaginary axis of the s-plane.

EXAMPLE 5.2 Distinguish between the exponential form of the Fourier series and the Fourier transform. What is the nature of the 'transform pair' in the above two cases?

Solution: The exponential Fourier series representation of a continuous-time periodic signal $x(t)$ is given by

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$$

where
$$\omega_0 = \frac{2\pi}{T} \quad \text{and} \quad C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

Taking Fourier transform on both sides of the above equations for $x(t)$ and $y(t)$, we get

$$X(\omega) = RI(\omega) + L(j\omega)\, I(\omega)$$

and
$$Y(\omega) = L(j\omega)\, I(\omega)$$

The frequency response of the network is:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{L(j\omega)}{R + j\omega L} = \frac{j\omega}{j\omega + (R/L)}$$

The impulse response of the network is given by

$$h(t) = \mathrm{F}^{-1}[H(\omega)] = \mathrm{F}^{-1}\left[\frac{j\omega}{j\omega + (R/L)}\right]$$

$$= \frac{d}{dt}[e^{-(R/L)t}\, u(t)] = -\frac{R}{L} e^{-(R/L)t}\, u(t)$$

5.9 INTRODUCTION TO HILBERT TRANSFORM

When the phase angles of all the positive frequency spectral components of a given signal are shifted by –90° and the phase angles of all the negative frequency spectral components are shifted by +90°, the resulting function of time is called the Hilbert transform of the signal. The amplitude spectrum of the signal is unchanged by Hilbert transform operation. Only phase spectrum of the signal is changed. Hilbert transform differs from other transforms like Fourier transform, Laplace transform, Discrete-time Fourier transform and z-transform in the sense that Hilbert transforming a signal does not bring about a change of domain. The Hilbert transformed signal is also a time domain signal. Consider a real signal $x(t)$ with the Fourier transform $X(\omega)$.

The *Hilbert transform* $\hat{x}(t)$ of a signal $x(t)$ is obtained by convolving $x(t)$ with $1/\pi t$, i.e.

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}$$

$$\therefore \qquad \hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \quad \text{or} \quad \hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(t-\tau)}{\tau} d\tau$$

This clearly shows that Hilbert transform operation of $x(t)$ is a linear operation.

The above definition of a Hilbert transform is applicable to all signals that are Fourier transformable.

The *Inverse Hilbert transform*, by means of which the original signal $x(t)$ is recovered from $\hat{x}(t)$ is defined by

$$x(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau} d\tau$$

The functions $x(t)$ and $\hat{x}(t)$ are said to constitute a *Hilbert transform pair*.

From the definition of Hilbert transform, $\hat{x}(t)$ can be interpreted as the convolution of $x(t)$ with the time function $1/\pi t$.

From the convolution property of Fourier transform, the convolution of two functions in time domain is transformed into the multiplication of their Fourier transform in the frequency domain.

For the time function $1/\pi t$, we have

$$\frac{1}{\pi t} \xleftrightarrow{\text{FT}} -j\,\mathrm{sgn}(\omega)$$

where sgn (ω) is the signum function in the frequency domain given by

$$\mathrm{sgn}(\omega) = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases}$$

The Fourier transform pair $[1/\pi t \xleftrightarrow{\text{FT}} -j\,\mathrm{sgn}(\omega)]$ is obtained by applying duality property of the Fourier transform to

$$\mathrm{sgn}(t) \xleftrightarrow{\text{FT}} \frac{2}{j\omega}$$

The Fourier transform $\hat{X}(\omega)$ of $\hat{x}(t)$ is given by

$$\hat{X}(\omega) = -j\,\mathrm{sgn}(\omega)\,X(\omega)$$

This implies that
$$\hat{X}(\omega) = \begin{cases} -jX(\omega); & \omega > 0 \\ jX(\omega); & \omega < 0 \end{cases}$$

This equation states that given a signal $x(t)$, we may obtain its Hilbert transform $\hat{x}(t)$ by passing $x(t)$ through a linear two port device whose transfer function is $-j\,\mathrm{sgn}(\omega)$.

Since $\hat{X}(\omega)$ is the spectrum of $\hat{x}(t)$ and $X(\omega)$ is the spectrum of $x(t)$, this device may be considered as one that produces a phase shift of –90° for all positive frequencies of the input signal and +90° for all negative frequencies as shown in Figure 5.51.

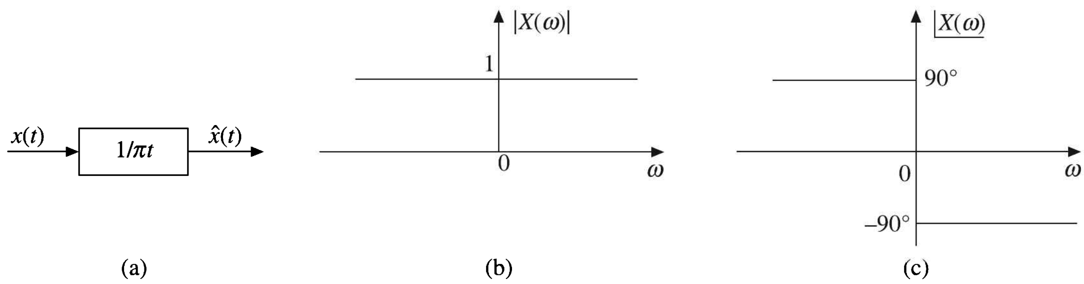


Figure 5.51 (a) Hilbert transformer, (b) Its magnitude response, (c) Its phase response.

The amplitudes of all frequency components in the signal are unaffected by transmission through the device, i.e.

$$|\hat{X}(\omega)| = |X(\omega)|$$

Such an ideal device is called *Hilbert transformer.* The Hilbert transformer may be viewed as an ideal all pass 90° phase-shifter. It is not causal and hence is not physically realizable.

Properties of Hilbert transform

The Hilbert transform differs from the Fourier transform in that it operates exclusively in the time domain.

Property 1: Hilbert transform does not change the domain of a signal.

Property 2: Hilbert transform does not alter the amplitude spectrum of a signal.

A signal $x(t)$ and its Hilbert transform $\hat{x}(t)$ have the same amplitude spectrum, i.e. they have the same energy density spectrum and autocorrelation function.

Property 3: A signal $x(t)$ and its Hilbert transform $\hat{x}(t)$ are orthogonal to each other, i.e.

$$\int_{-\infty}^{\infty} x(t)\,\hat{x}(t)\,dt = 0$$

Property 4: If $\hat{x}(t)$ is the Hilbert transform of $x(t)$, the Hilbert transform of $\hat{x}(t)$ is $-x(t)$.

Applications of Hilbert transform

The Hilbert transform finds extensive applications in the areas of signal processing, analysis and synthesis of signals, design of filters etc.
Some of the important applications are given as follows:

- To realize phase selectivity in the generation of single side band (SSB) modulation systems.
- To represent band pass signals.
- To relate the gain and phase characteristics of linear communication channels and filters of minimum phase type.

The Hilbert transform of the product of a low pass signal and very high pass signal is equal to the product of the low pass signal with the Hilbert transform of the high pass signal, i.e. if $x(t)$ is a low pass signal and $y(t)$ is a very high pass signal and if the spectra of $x(t)$ and $y(t)$ are non overlapping, then

$$\widehat{x(t)\cdot y(t)} = x(t)\cdot \widehat{y(t)}$$

EXAMPLE 5.53 State and prove the properties of Hilbert transform.

Solution: The statements and proofs of the properties of the Hilbert transform are as follows:

Property 1: Hilbert transform does not change the domain of a signal.

Proof: The given signal $x(t)$ is in time domain. The Hilbert transform of $x(t)$, i.e. $\hat{x}(t)$ is obtained by the convolution of $x(t)$ and $1/\pi t$. So $\hat{x}(t)$ is also in domain. It proves that Hilbert transform does not change the domain of a signal.

UNIT-III

Laplace Transforms: Laplace Transforms (L.T) (596-616), Concept of Region of Convergence (ROC) for Laplace Transforms (591-593), Properties of ROC (684 and refer class notes), Properties of L.T (616-627), Inverse Laplace Transform (648-682).

Systems: Classification of Continuous time and discrete time Systems (113-156), impulse response (411), Transfer function (416-417), Response of a linear system (411), Concept of convolution in Time domain and Frequency domain (457-466), Graphical representation of Convolution (466-483).

9.6 ONE-SIDED (UNILATERAL) LAPLACE TRANSFORM OF SOME COMMONLY USED SIGNALS

The unilateral Laplace transform also called single-sided or one-sided Laplace transform is applied to the signals that are causal, i.e. for signals which are zero for time $t < 0$, i.e. only for positive time signals.

The unilateral Laplace transform of the signal $x(t)$ is defined by

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty x(t)\, e^{-(\sigma + j\omega)t}\, dt = \int_0^\infty [x(t)\, e^{-\sigma t}]\, e^{-j\omega t}\, dt$$

Because $|e^{j\omega t}| = 1$, the integral converges if $\int_0^\infty |x(t)\, e^{-\sigma t}\, dt| < \infty$.

9.6.1 Impulse Function [$x(t) = \delta(t)$]

We know that
$$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

$$\therefore \qquad \mathrm{L}[\delta(t)] = X(s) = \int_0^\infty \delta(t) e^{-st}\, dt = e^{-st}\Big|_{t=0} = 1 \qquad \text{(for any } s\text{)}$$

i.e. the ROC is the entire s-plane. The ROC is shown in Figure 9.2.

$$\boxed{\delta(t) \xleftrightarrow{\text{LT}} 1;\ \text{for all } s}$$

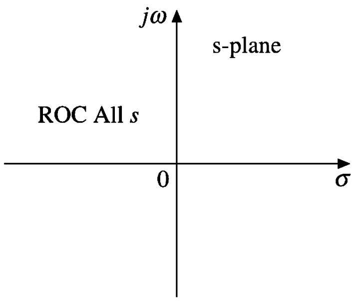


Figure 9.2 ROC of $\delta(t)$.

9.6.2 Step Function [$x(t) = u(t)$]

We know that

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\therefore \qquad L[u(t)] = L[x(t)] = X(s) = \int_0^\infty u(t)\, e^{-st}\, dt$$

$$= \int_0^\infty 1e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}[e^{-\infty} - e^{0}] = \frac{1}{s}$$

This integral converges when $\text{Re}(s) > 0$.

$$\therefore \qquad L[u(t)] = \frac{1}{s}$$

The ROC is $\text{Re}(s) > 0$, i.e. the entire right half of the s-plane as shown in Figure 9.3.

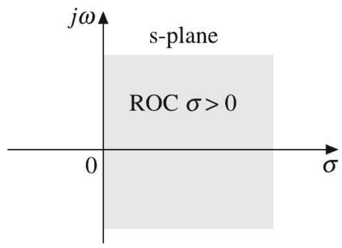


Figure 9.3 ROC of $L[u(t)]$.

Since the one-sided Laplace transform does not take into account the values of $x(t)$ for $t < 0$, the one-sided Laplace transform of $x(t) = 1$ and $x(t) = u(t)$ will be the same.

The Laplace transform of $x(t) = k$ is (k/s) if k is a constant.

$$\boxed{u(t) \xleftrightarrow{\text{LT}} \frac{1}{s};\ \text{ROC};\ \text{Re}(s) > 0}$$

9.6.3 Ramp Function [$x(t) = t\,u(t)$]

$$\therefore \qquad L[tu(t)] = X(s) = \int_0^\infty tu(t)\, e^{-st}\, dt = \int_0^\infty te^{-st}\, dt$$

$$= \left[t\,\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} 1\, dt = 0 - \left[\frac{e^{-st}}{s^2}\right]_0^\infty$$

$$= 0 - \left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

The above integral converges if Re$(s) > 0$, i.e. the ROC is Re$(s) > 0$ as shown in Figure 9.3.

$$\boxed{tu(t) \xleftrightarrow{\text{LT}} \frac{1}{s^2};\ \text{ROC; Re}(s) > 0}$$

9.6.4 Parabolic Function [$x(t) = t^2 u(t)$]

$$\therefore \qquad \text{L}[t^2u(t)] = \int_0^\infty t^2 u(t)\, e^{-st}\, dt = \int_0^\infty t^2 e^{-st}\, dt$$

$$= \left[t^2 \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}(2t)\, dt = 0 + \frac{2}{s}\int_0^\infty te^{-st}\, dt$$

$$= \frac{2}{s}\left\{\left[t\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt\right\}$$

$$= \frac{2}{s}\left(0 + \frac{1}{s}\int_0^\infty e^{-st}\, dt\right) = \frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{2}{s^3}$$

The above integral converges if Re$(s) > 0$, i.e. the ROC is Re$(s) > 0$ as shown in Figure 9.3.

$$\boxed{t^2 u(t) \xleftrightarrow{\text{LT}} \frac{2}{s^3};\ \text{ROC; Re}(s) > 0}$$

9.6.5 Real Exponential Function [$x(t) = e^{at}u(t)$]

$$\therefore \qquad X(s) = \text{L}[e^{at}\, u(t)] = \int_0^\infty e^{at}\, u(t)\, e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{[0-1]}{(s-a)} = \frac{1}{s-a}$$

The above integral converges for Re$(s - a) > 0$, i.e. for Re$(s) > a$.
$\therefore$ ROC is Re$(s) > a$. The ROC is shown in Figure 9.4.

$$\boxed{e^{at}\, u(t) \xleftrightarrow{\text{LT}} \frac{1}{s-a};\ \text{ROC; Re}(s) > a}$$

In a similar way, we can show that

$$\boxed{e^{-at}\, u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+a};\ \text{ROC; Re}(s) > -a}$$

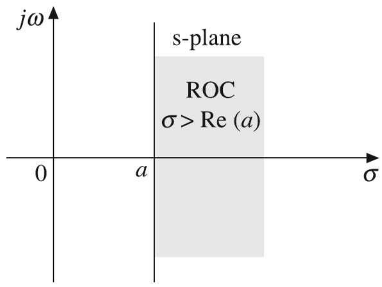


Figure 9.4 ROC of e^{at}.

9.6.6 Complex Exponential Function $[x(t) = e^{j\omega t} u(t)]$

$$\therefore \qquad X(s) = \mathrm{L}[e^{j\omega t} u(t)] = \int_0^\infty e^{j\omega t} u(t)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{j\omega t} e^{-st}\, dt = \int_0^\infty e^{-(s-j\omega)t}\, dt$$

$$= \left[\frac{e^{-(s-j\omega)\,t}}{-(s-j\omega)}\right]_0^\infty = \frac{0-1}{-(s-j\omega)} = \frac{1}{s-j\omega}$$

The ROC is Re(s) > 0 as shown in Figure 9.3.

$$\boxed{e^{j\omega t} u(t) \xleftrightarrow{\text{LT}} \frac{1}{s-j\omega};\ \text{ROC; Re}(s) > 0}$$

In a similar way, we can show that

$$\boxed{e^{-j\omega t} u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+j\omega};\ \text{ROC; Re}(s) > 0}$$

9.6.7 Sine and Cosine Functions $[x(t) = \cos \omega t\ u(t),\ \sin \omega t\ u(t)]$

$$\therefore \qquad X(s) = \mathrm{L}\,[\cos \omega t\ u(t)] = \mathrm{L}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\, u(t)\right]$$

$$= \frac{1}{2}\,\mathrm{L}[e^{j\omega t} u(t)] + \frac{1}{2}\,\mathrm{L}[e^{-j\omega t} u(t)]$$

$$= \frac{1}{2}\left(\frac{1}{s-j\omega}\right) + \frac{1}{2}\left(\frac{1}{s+j\omega}\right) = \frac{1}{2}\left[\frac{s+j\omega+s-j\omega}{(s-j\omega)(s+j\omega)}\right]$$

$$= \frac{s}{s^2+\omega^2} \qquad (\text{for Re}(s) > 0)$$

$$\boxed{\cos \omega t\, u(t) \xleftrightarrow{\text{LT}} \frac{s}{s^2 + \omega^2};\ \text{ROC}; \text{Re}(s) > 0}$$

In a similar way, we can show that

$$\boxed{\sin \omega t\, u(t) \xleftrightarrow{\text{LT}} \frac{\omega}{s^2 + \omega^2};\ \text{ROC}; \text{Re}(s) > 0}$$

9.6.8 Hyperbolic sine and cosine Functions [$x(t)$ = sinh ωt $u(t)$, cosh ωt $u(t)$]

$$\therefore \qquad \text{L}[x(t)] = X(s) = \text{L}[\sinh \omega t\, u(t)]$$

$$= \text{L}\left[\frac{e^{\omega t} - e^{-\omega t}}{2} u(t)\right] = \frac{1}{2}\{\text{L}[e^{\omega t}u(t)] - \text{L}[e^{-\omega t}u(t)]\}$$

$$= \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right)$$

$$= \frac{1}{2}\left[\frac{(s+\omega) - (s-\omega)}{s^2 - \omega^2}\right] = \frac{\omega}{s^2 - \omega^2}$$

$$\boxed{\sinh \omega t\, u(t) \xleftrightarrow{\text{LT}} \frac{\omega}{s^2 - \omega^2};\ \text{ROC}; \text{Re}(s) > 0}$$

In a similar way, we can show that

$$\boxed{\cosh \omega t\, u(t) \xleftrightarrow{\text{LT}} \frac{s}{s^2 - \omega^2};\ \text{ROC}; \text{Re}(s) > 0}$$

9.6.9 Damped sine and cosine Functions [$x(t)$ = e^{-at} sin ωt $u(t)$]

$$\therefore \qquad \text{L}[x(t)] = X(s) = \text{L}\left\{e^{-at}\left[\frac{(e^{j\omega t} - e^{-j\omega t})}{2j}\right] u(t)\right\}$$

$$= \frac{1}{2j}\text{L}[(e^{-(a-j\omega)t} - e^{-(a+j\omega)t})\, u(t)]$$

$$= \frac{1}{2j}\{\text{L}[e^{-(a-j\omega)t}\, u(t)] - \text{L}[e^{-(a+j\omega)t}\, u(t)]\}$$

$$= \frac{1}{2j}\left[\frac{1}{s + (a - j\omega)} - \frac{1}{s + (a + j\omega)}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{(s+a)-j\omega} - \frac{1}{(s+a)+j\omega}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\omega}{(s+a)^2+\omega^2}\right] = \frac{\omega}{(s+a)^2+\omega^2}$$

$$\boxed{e^{-at}\sin\omega t\, u(t) \xleftrightarrow{\text{LT}} \frac{\omega}{(s+a)^2+\omega^2};\ \text{ROC; Re}(s) > -a}$$

In a similar way, we can show that

$$\boxed{e^{-at}\cos\omega t\ u(t) \xleftrightarrow{\text{LT}} \frac{s+a}{(s+a)^2+\omega^2};\ \text{ROC; Re}(s) > -a}$$

9.6.10 Damped Hyperbolic sine and cosine Functions [$x(t) = e^{-at}\sinh \omega t\ u(t)$]

$$\therefore \qquad \text{L}[x(t)] = X(s) = \text{L}[e^{-at}\sinh\omega t\ u(t)]$$

$$= \text{L}\left\{\left[e^{-at}\left(\frac{e^{\omega t}-e^{-\omega t}}{2}\right)\right]u(t)\right\}$$

$$= \frac{1}{2}[\text{L}\,(e^{-at}e^{\omega t} - e^{-at}e^{-\omega t})\,u(t)]$$

$$= \frac{1}{2}\{\text{L}[e^{-(a-\omega)t}\,u(t)] - \text{L}[e^{-(a+\omega)t}u(t)]\}$$

$$= \frac{1}{2}\left(\frac{1}{s+a-\omega} - \frac{1}{s+a+\omega}\right)$$

$$= \frac{\omega}{(s+a)^2-\omega^2}$$

$$\boxed{e^{-at}\sinh\omega t\, u(t) \xleftrightarrow{\text{LT}} \frac{\omega}{(s+a)^2-\omega^2};\ \text{ROC; Re}(s) > -a}$$

Similarly, we can show that

$$\boxed{e^{-at}\cosh\omega t\, u(t) \xleftrightarrow{\text{LT}} \frac{s+a}{(s+a)^2-\omega^2};\ \text{ROC; Re}(s) > -a}$$

EXAMPLE 9.1 Distinguish between unilateral and bilateral Laplace transforms.

Solution: The Laplace transform defined with $-\infty$ as the lower limit for the integral $\left(\text{i.e. } \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt\right)$ is called the bilateral or two-sided Laplace transform and the Laplace transform defined with 0 as the lower limit for the integral $\left(\text{i.e. } \int_{0}^{\infty} x(t)\, e^{-st}\, dt\right)$ is called the unilateral or one-sided Laplace transform. While the two-sided Laplace transform is useful to get a better insight into the systems (with causal and non-causal signals), the one-sided Laplace transform is very useful in practical problems (with causal signals) where the signal is applied to a system at a particular point in time which can be conveniently taken as the time origin or $t = 0$. In the case of unilateral Laplace transform, the ROC is unique and there is a one-to-one correspondence between Laplace transform and inverse Laplace transform, but in the case of bilateral Laplace transform, the ROC is not unique and there is no one-to-one correspondence between the Laplace transform and inverse Laplace transform and the time function obtained from the bilateral inverse Laplace transform depends on the ROC. The ROC of unilateral Laplace transform is always $\text{Re}\,(s) > a$ where a is some constant. The ROC of bilateral Laplace transform may be $\text{Re}(s) > a$ or $\text{Re}(s) < a$ or $a < \text{Re}(s) < b$ where a and b are some constants.

EXAMPLE 9.2 Prove that the signals

(a) $x(t) = e^{-at}\, u(t)$ and (b) $x(t) = -e^{-at}\, u(-t)$

have the same $X(s)$ and differ only in ROC. Also plot their ROCs.

Solution:

(a) Given $$x(t) = e^{-at}\, u(t)$$

$$L[x(t)] = X(s) = L[e^{-at}\, u(t)]$$

$$= \int_0^{\infty} e^{-at}\, u(t)\, e^{-st}\, dt = \int_0^{\infty} e^{-at}\, e^{-st}\, dt$$

$$\therefore \qquad = \int_0^{\infty} e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{t=0}^{t=\infty}$$

$$= \frac{(0-1)}{-(s+a)} = \frac{1}{s+a}$$

This integral converges if $\text{Re}(s + a) > 0$, i.e. $\text{Re}(s)$, $\sigma > -a$. So the ROC is $\sigma > -a$ as shown in Figure 9.5.

$$\therefore \qquad L[e^{-at}\, u(t)] = \frac{1}{s+a};\ \text{ROC; } \sigma > -a$$

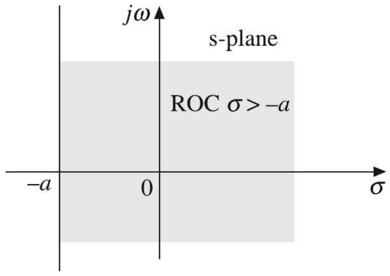


Figure 9.5 ROC of $e^{-at}\,u(t)$.

(b) Given $x(t) = -e^{-at}\,u(-t)$

We know that

$$u(-t) = \begin{cases} 1 & \text{for } t \le 0 \\ 0 & \text{for } t > 0 \end{cases}$$

$$\therefore \quad \mathrm{L}\,[x(t)] = X(s) = \mathrm{L}[-e^{-at}\,u(-t)] = \int_{-\infty}^{\infty} -e^{-at}\,u(-t)\,e^{-st}\,dt$$

$$= \int_{-\infty}^{0} -e^{-at}\,1 e^{-st}\,dt = \int_{-\infty}^{0} -e^{-(s+a)t}\,dt$$

$$= \int_{0}^{\infty} -e^{(s+a)t}\,dt$$

The above integral converges if Re$(s + a) < 0$, i.e. Re$(s) < -a$. So the ROC is Re$(s) < -a$ as shown in Figure 9.6.

$$\therefore \quad X(s) = -\left[\frac{e^{(s+a)t}}{s+a}\right]_{t=0}^{t=\infty} = -\frac{(0-1)}{s+a} = \frac{1}{s+a}$$

$$\therefore \quad \boxed{\mathrm{L}[-e^{-at}\,u(-t)] = \frac{1}{s+a};\ \text{ROC; Re}(s),\ \sigma < -a}$$

From this example, we can observe that the Laplace transforms of the signals $e^{-at}\,u(t)$ and $-e^{-at}\,u(-t)$ are *identical* but their ROCs are *different*. The ROC of $e^{-at}\,u(t)$ is $\sigma > -a$ and that of $-e^{-at}\,u(-t)$ is $\sigma < -a$. The ROCs of these signals are shown in Figures 9.5 and 9.6. Here the complex frequency s is represented graphically.

The shaded region in Figure 9.5 represents the set of points in the s-plane for which the Laplace transform of $e^{-at}\,u(t)$ converges. So it indicates the ROC of $e^{-at}\,u(t)$. The shaded region in Figure 9.6 represents the set of points in the s-plane for which the Laplace transform of $-e^{-at}\,u(-t)$ converges. So it indicates the ROC of $-e^{-at}\,u(-t)$. The signal $e^{-at}\,u(t)$ is a causal signal and the signal $-e^{-at}\,u(-t)$ is a non-causal signal.

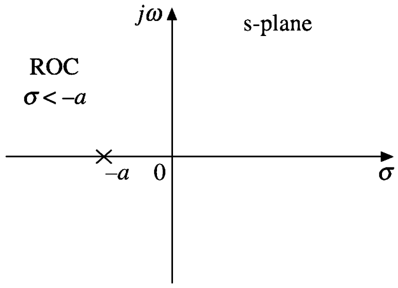


Figure 9.6 ROC of $-e^{-at}\,u(-t)$.

In general, the ROC of a causal signal is $\text{Re}(s) > a$ and the ROC of a non-causal signal is $\text{Re}(s) < a$ where a is some constant.

EXAMPLE 9.3 Find the Laplace transform of the signals:

(a) $x(t) = u(-t)$, (b) $x(t) = e^{-t}\,u(t) + e^{-4t}\,u(t)$ and find their ROCs.

Solution:

(a) Given $$x(t) = u(-t)$$

$$\therefore \qquad L[x(t)] = X(s) = \int_{-\infty}^{\infty} u(-t)\,e^{-st}\,dt = \int_{-\infty}^{0} (1)\,e^{-st}\,dt$$

The above integral converges if $\text{Re}(s) < 0$, i.e. for $s = -\text{ve}$

$$\therefore \qquad X(s) = \left[\frac{e^{-st}}{-s}\right]_{-\infty}^{0} = \frac{e^{0} - e^{-s(-\infty)}}{-s} = -\frac{1}{s};\ \text{ROC; Re}(s) < 0$$

(b) Given $$x(t) = e^{-t}\,u(t) + e^{-4t}\,u(t)$$

$$\therefore \qquad L[x(t)] = X(s) = L[e^{-t}\,u(t) + e^{-4t}\,u(t)]$$

$$= \int_{-\infty}^{\infty} [e^{-t}\,u(t) + e^{-4t}\,u(t)]\,e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-t}\,u(t)\,e^{-st}\,dt + \int_{-\infty}^{\infty} e^{-4t}\,u(t)\,e^{-st}\,dt$$

$$= \underbrace{\underbrace{\int_{0}^{\infty} e^{-(s+1)t}\,dt}_{\substack{\text{Converges if}\\ \text{Re}(s) > -1}} + \underbrace{\int_{0}^{\infty} e^{-(s+4)t}\,dt}_{\substack{\text{Converges if}\\ \text{Re}(s) > -4}}}_{\text{Converges if Re}(s) > -1}$$

$$\therefore \qquad X(s) = \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_{t=0}^{t=\infty} + \left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_{t=0}^{t=\infty} = \frac{(0-1)}{-(s+1)} + \frac{(0-1)}{-(s+4)}$$

$$= \frac{1}{s+1} + \frac{1}{s+4} = \frac{(s+4)+(s+1)}{(s+1)(s+4)}$$

$$= \frac{2s+5}{s^2+5s+4}; \text{ ROC; } \mathrm{Re}(s) > -1$$

The ROC is shown in Figure 9.7. This example shows that the ROC of the sum of two signals is equal to the intersection of the ROCs of the two signals.

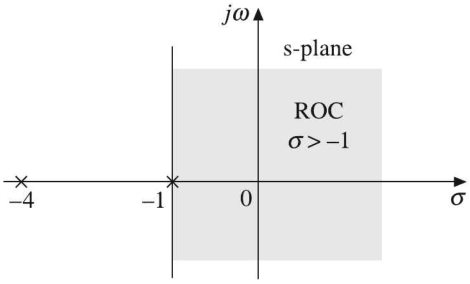


Figure 9.7 ROC of $x(t) = e^{-t} u(t) + e^{-4t} u(t)$.

EXAMPLE 9.4 Find the Laplace transform of the signal

$$x(t) = e^{-2t} u(t) + e^{3t} u(t)$$

Also find the ROC.

Solution: Given $\qquad x(t) = e^{-2t} u(t) + e^{3t} u(t)$

$$\therefore \qquad \mathrm{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} [e^{-2t} u(t) + e^{3t} u(t)]\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-2t} u(t)\, e^{-st}\, dt + \int_{-\infty}^{\infty} e^{3t} u(t)\, e^{-st}\, dt$$

$$= \underbrace{\underbrace{\int_0^{\infty} e^{-(s+2)t}\, dt}_{\substack{\text{Converges if} \\ \mathrm{Re}(s) > -2}} + \underbrace{\int_0^{\infty} e^{-(s-3)t}\, dt}_{\substack{\text{Converges if} \\ \mathrm{Re}(s) > 3}}}_{\text{Converges if } \mathrm{Re}(s) > 3}$$

$$\therefore \qquad X(s) = \left[\frac{e^{-(s+2)t}}{-(s+2)}\right]_{t=0}^{t=\infty} + \left[\frac{e^{-(s-3)t}}{-(s-3)}\right]_{t=0}^{t=\infty}$$

$$= \frac{(0-1)}{-(s+2)} + \frac{(0-1)}{-(s-3)}$$

$$= \frac{1}{s+2} + \frac{1}{s-3} = \frac{s-3+s+2}{(s+2)(s-3)} = \frac{2s-1}{s^2-s-6}$$

ROC; Re(s) > 3 as shown in Figure 9.8.

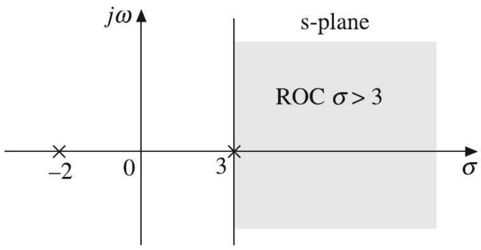


Figure 9.8 ROC of $x(t) = e^{-2t}\, u(t) + e^{3t}\, u(t)$.

EXAMPLE 9.5 Find the Laplace transform of the signal

$$x(t) = e^{-at}\, u(t) - e^{-bt}\, u(-t)$$

and also find its ROC.

Solution: Given $\qquad x(t) = e^{-at}\, u(t) - e^{-bt}\, u(-t)$

$$\therefore \qquad \mathrm{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} [e^{-at}\, u(t) - e^{-bt} u(-t)]\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-st}\, dt + \int_{-\infty}^{\infty} -e^{-bt}\, u(-t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt + \int_{-\infty}^{0} -e^{-bt}\, e^{-st}\, dt$$

$$= \underbrace{\underbrace{\int_{0}^{\infty} e^{-(s+a)t}\, dt}_{\substack{\text{Converges if} \\ \mathrm{Re}(s) > -a}} + \underbrace{\int_{0}^{\infty} -\, e^{(s+b)t}\, dt}_{\substack{\text{Converges if} \\ \mathrm{Re}(s) < -b}}}_{\text{Converges if } -a < \mathrm{Re}(s) < -b}$$

$$\therefore \qquad X(s) = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{t=0}^{t=\infty} + \left[\frac{e^{(s+b)t}}{-(s+b)}\right]_{t=0}^{t=\infty}$$

$$= \frac{(0-1)}{-(s+a)} + \frac{(0-1)}{-(s+b)}$$

$$= \frac{1}{s+a} + \frac{1}{s+b}; \quad \text{ROC;} \ -a < \text{Re}(s) < -b$$

$$= \frac{s+b+s+a}{(s+a)(s+b)} = \frac{2s+(a+b)}{(s+a)(s+b)}; \quad \text{ROC;} \ -a < \text{Re}(s) < -b$$

as shown in Figure 9.9.

Note: The ROC and the Laplace transform exist only if $-a < -b$. The ROC and the Laplace transform do not exist if $-a > -b$.

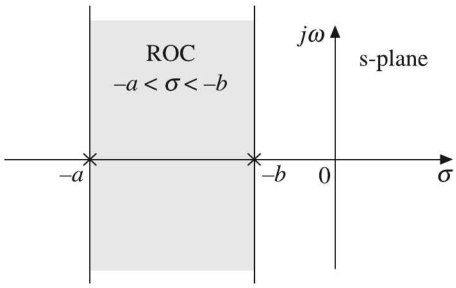


Figure 9.9 ROC of $e^{-at}\,u(t) - e^{-bt}\,u(-t)$.

EXAMPLE 9.6 Find the Laplace transform of the signal $x(t) = e^{-2t}\,u(-t) + e^{-3t}\,u(-t)$ and find its ROC.

Solution: Given $\qquad x(t) = e^{-2t}\,u(-t) + e^{-3t}\,u(-t)$

$$\therefore \qquad \text{L}\,[x(t)] = X(s) = \int_{-\infty}^{\infty} [e^{-2t}\,u(-t) + e^{-3t}\,u(-t)]\,e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-2t}\,u(-t)\,e^{-st}\,dt + \int_{-\infty}^{\infty} e^{-3t}\,u(-t)\,e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{-(s+2)t}\,dt + \int_{-\infty}^{0} e^{-(s+3)t}\,dt$$

$$= \underbrace{\underbrace{\int_{0}^{\infty} e^{(s+2)t}\,dt}_{\substack{\text{Converges if} \\ \text{Re}(s) < -2}} + \underbrace{\int_{0}^{\infty} e^{(s+3)t}\,dt}_{\substack{\text{Converges if} \\ \text{Re}(s) < -3}}}_{\text{Converges if Re}(s) < -3}$$

$$\therefore \qquad X(s) = \left[\frac{e^{(s+2)t}}{s+2}\right]_{t=0}^{t=\infty} + \left[\frac{e^{(s+3)t}}{(s+3)}\right]_{t=0}^{t=\infty}$$

$$= \frac{0-1}{s+2} + \frac{0-1}{s+3} = -\frac{1}{s+2} - \frac{1}{s+3}$$

$$= -\frac{(s+3)+(s+2)}{(s+2)\,(s+3)} = -\frac{(2s+5)}{s^2+5s+6}; \text{ ROC; Re}(s) < -3$$

as shown in Figure 9.10

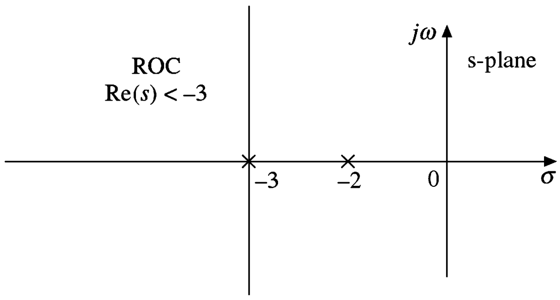


Figure 9.10 ROC of $x(t) = e^{-2t}\,u(-t) + e^{-3t}\,u(-t)$.

EXAMPLE 9.7 Find the Laplace transform of the signal $x(t) = e^{2t}\,u(t) + e^{-2t}\,u(-t)$ and find its ROC.

Solution: Given $\qquad x(t) = e^{2t}\,u(t) + e^{-2t}\,u(-t)$

$$\therefore \qquad \text{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} [e^{2t}\,u(t) + e^{-2t}\,u(-t)]\,e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} e^{2t}\,u(t)\,e^{-st}\,dt + \int_{-\infty}^{\infty} e^{-2t}\,u(-t)\,e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{2t}\,e^{-st}\,dt + \int_{-\infty}^{0} e^{-2t}\,e^{-st}\,dt$$

$$= \underbrace{\underbrace{\int_{0}^{\infty} e^{-(s-2)t}\,dt}_{\substack{\text{Converges if} \\ \text{Re}(s) > 2}} + \underbrace{\int_{-\infty}^{0} e^{-(s+2)t}\,dt}_{\substack{\text{Converges if} \\ \text{Re}(s) < -2}}}_{\text{does not converge for any real value of } s}$$

So there is no ROC and hence the Laplace transform of the given signal does not exist.

EXAMPLE 9.8 Find the Laplace transform of the signal $x(t) = e^{-a|t|}$ and find its ROC.

Solution: Given

$$x(t) = e^{-a|t|}$$

$$\therefore \quad \mathrm{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-a(-t)}\, e^{-st}\, dt + \int_{0}^{\infty} e^{-a(t)}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-st}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s-a)t}\, dt + \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \underbrace{\underbrace{\int_{0}^{\infty} e^{(s-a)t}\, dt}_{\substack{\text{Converges if}\\ \mathrm{Re}(s) < a}} + \underbrace{\int_{0}^{\infty} e^{-(s+a)t}\, dt}_{\substack{\text{Converges if}\\ \mathrm{Re}(s) > -a}}}_{\text{Converges if } -a < \mathrm{Re}(s) < a}$$

$$\therefore \quad X(s) = \left[\frac{e^{(s-a)t}}{s-a}\right]_{t=0}^{t=\infty} + \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{t=0}^{t=\infty}$$

$$= \frac{0-1}{s-a} + \frac{(0-1)}{-(s+a)}$$

$$= -\frac{1}{s-a} + \frac{1}{s+a}$$

$$= \frac{-s-a+s-a}{(s-a)(s+a)} = -\frac{2a}{s^2-a^2}; \ \text{ROC}; -a < \mathrm{Re}(s) < a$$

as shown in Figure 9.11.

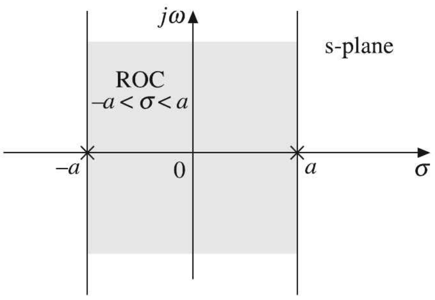


Figure 9.11 ROC of $x(t) = e^{-a|t|}$.

EXAMPLE 9.9 Find the Laplace transform and region of convergence for the following signals:

(a) $x(t) = e^{-5t}\, u(t-1)$ (b) $x(t) = e^{t} \sin 2t$ for $t \le 0$

(c) $x(t) = e^{2t}\, u(-t) + e^{3t}\, u(-t)$ (d) $x(t) = te^{-2|t|}$

Solution:

(a) Given $x(t) = e^{-5t}\, u(t-1)$

$$\therefore \quad L[x(t)] = X(s) = L[e^{-5t}\, u(t-1)]$$

$$= \int_0^{\infty} e^{-5t} u(t-1)\, e^{-st}\, dt$$

$$= \int_1^{\infty} e^{-(s+5)t}\, dt = \left[\frac{e^{-(s+5)t}}{-(s+5)}\right]_1^{\infty}$$

$$= \frac{e^{-(s+5)}}{s+5}$$

The above integral converges if Re(s) > –5. Therefore, ROC is Re(s) > –5.

(b) Given $x(t) = e^{t} \sin 2t$ for $t \le 0$

$$\therefore \quad x(t) = e^{t} \sin 2tu(-t)$$

$$\therefore \quad L[x(t)] = X(s) = \int_{-\infty}^{\infty} e^{t} \sin 2tu(-t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{t} \sin 2te^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{t} \left(\frac{e^{j2t} - e^{-j2t}}{2j}\right) e^{-st}\, dt$$

$$= \frac{1}{2j} \int_{-\infty}^{0} [e^{(1-s+j2)t} - e^{(1-s-j2)t}]\, dt$$

The above integral converges if Re(s) < –1. Therefore, ROC is Re(s) < –1.

$$\therefore \quad X(s) = \frac{1}{2j}\left[\frac{e^{(1-s+j2)t}}{1-s+j2} - \frac{e^{(1-s-j2)t}}{1-s-j2}\right]_{-\infty}^{0} = \frac{1}{2j}\left(\frac{1}{1-s+j2} - \frac{1}{1-s-j2}\right)$$

$$= \frac{1}{2j}\left[\frac{(1-s-j2)-(1-s+j2)}{(1-s)^2+4}\right] = -\frac{2}{(1-s)^2+2^2} = -\frac{2}{(s-1)^2+2^2};$$

(c) Given $$x(t) = e^{2t}\,u(-t) + e^{3t}\,u(-t)$$

Taking Laplace transform on both sides, we have

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} [e^{2t}u(-t) + e^{3t}u(-t)]\,e^{-st}\,dt$$

$$= \int_{-\infty}^{0} [e^{2t} + e^{3t}]\,e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{-(s-2)t}\,dt + \int_{-\infty}^{0} e^{-(s-3)t}\,dt$$

The first integral converges for $\text{Re}(s) < 2$, and the second integral converges if $\text{Re}(s) < 3$.

Therefore, ROC is $\text{Re}(s) < 2$.

$$\therefore \qquad X(s) = \left[\frac{e^{-(s-2)t}}{-(s-2)}\right]_{-\infty}^{0} + \left[\frac{e^{-(s-3)t}}{-(s-3)}\right]_{-\infty}^{0}$$

$$= -\frac{1}{s-2} - \frac{1}{s-3};\ \text{ROC; Re}(s) < 2$$

(d) Given $$x(t) = te^{-2|t|}$$

Taking Laplace transform on both sides, we have

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} te^{-2|t|}\,e^{-st}\,dt = \int_{-\infty}^{0} te^{2t}e^{-st}\,dt + \int_{0}^{\infty} te^{-2t}e^{-st}\,dt$$

$$= \int_{-\infty}^{0} te^{-(s-2)t}\,dt + \int_{0}^{\infty} te^{-(s+2)t}\,dt$$

The first integral converges if $\text{Re}(s) < 2$, and the second integral converges if $\text{Re}(s) > -2$.

Therefore, ROC is $-2 < \text{Re}(s) < 2$.

$$\therefore \quad X(s) = \left\{\left[t\frac{e^{-(s-2)t}}{-(s-2)}\right]_{-\infty}^{0} - \int_{-\infty}^{0} \frac{e^{-(s-2)t}}{-(s-2)}1\,dt\right\} + \left\{\left[t\frac{e^{-(s+2)t}}{-(s+2)}\right]_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-(s+2)t}}{-(s+2)}\,dt\right\}$$

$$= \left\{0 - \left[\frac{e^{-(s-2)t}}{(s-2)^2}\right]_{-\infty}^{0}\right\} + \left\{0 - \left[\frac{e^{-(s+2)t}}{(s+2)^2}\right]_{0}^{\infty}\right\}$$

$$= -\frac{1}{(s-2)^2} + \frac{1}{(s+2)^2};\ \text{ROC; } -2 < \text{Re}(s) < 2$$

EXAMPLE 9.10 Find the Laplace transform of

(a) $\sin^2 3t\ u(t)$ (b) $\cos^2 2t\ u(t)$

(c) $\cos^3 2t\ u(t)$ (d) $[1 + \sin 2t \cos 2t]\, u(t)$

(e) $\cos(\omega t + \theta)\, u(t)$ (f) $\sin(\omega t + \theta)\, u(t)$

Solution:

(a) Given $$x(t) = \sin^2 3t\, u(t) = \left(\frac{1 - \cos 6t}{2}\right) u(t)$$

$\therefore$ $$L[\sin^2 3t\, u(t)] = L\left[\frac{(1 - \cos 6t)}{2}\right] u(t)$$

$$= \frac{1}{2}\{L[u(t)] - L[\cos 6t\, u(t)]\}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 6^2}\right) = \frac{18}{s(s^2 + 36)}$$

(b) Given $$x(t) = \cos^2 2t\, u(t) = \left(\frac{1 + \cos 4t}{2}\right) u(t)$$

$\therefore$ $$L[x(t)] = X(s) = L\left[\frac{(1 + \cos 4t)}{2} u(t)\right]$$

$$= \frac{1}{2}\{L[u(t)] + L[\cos 4t\, u(t)]\}$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + (4)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s^2 + 16}{s(s^2 + 16)}\right]$$

(c) Given $$x(t) = \cos^3 2t\, u(t) = \frac{[\cos 6t + 3 \cos 2t]}{4} u(t)$$

$\therefore$ $$X(s) = L[x(t)] \ = L\left[\left(\frac{\cos 6t + 3 \cos 2t}{4}\right) u(t)\right]$$

$$= \frac{1}{4}\left\{L[\cos 6t\, u(t)] + 3L[\cos 2t\, u(t)]\right\}$$

$$= \frac{1}{4}\left(\frac{s}{s^2 + (6)^2} + 3\frac{s}{s^2 + (2)^2}\right)$$

$$= \frac{1}{4}\left\{\frac{4s^3 + 112s}{[s^2 + (6)^2)(s^2 + (2)^2]}\right\}$$

$$= \frac{s(s^2 + 28)}{(s^2 + 36)(s^2 + 4)}$$

(d) Given $x(t) = [1 + \sin 2t \cos 2t]\, u(t) = \left(1 + \frac{1}{2}\sin 4t\right) u(t)$

$$\therefore \quad \mathrm{L}[x(t)] = X(s) = \mathrm{L}\left[u(t) + \frac{1}{2}\sin 4t\, u(t)\right]$$

$$= \mathrm{L}[u(t)] + \frac{1}{2}\mathrm{L}\,[\sin 4t\, u(t)]$$

$$= \frac{1}{s} + \frac{1}{2}\frac{4}{s^2 + 4^2}$$

$$= \frac{s^2 + 2s + 16}{s(s^2 + 4^2)}$$

(e) Given $x(t) = \cos(\omega t + \theta)\, u(t) = [\cos \omega t \cos \theta - \sin \omega t \sin \theta]\, u(t)$

$$\therefore \quad \mathrm{L}[x(t)] = X(s) = \mathrm{L}[(\cos \omega t \cos \theta - \sin \omega t \sin \theta)\, u(t)]$$

$$= \mathrm{L}[\cos \omega t \cos \theta u(t)] - \mathrm{L}[\sin \omega t \sin \theta u(t)]$$

$$= \cos \theta \mathrm{L}\,[\cos \omega t\, u(t)] - \sin \theta \mathrm{L}\,[\sin \omega t\, u(t)]$$

$$= \cos \theta \left(\frac{s}{s^2 + \omega^2}\right) - \sin \theta \left(\frac{\omega}{s^2 + \omega^2}\right)$$

$$\therefore \quad X(s) = \frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$$

(f) Given $x(t) = \sin(\omega t + \theta)\, u(t) = [\sin \omega t \cos \theta + \cos \omega t \sin \theta]\, u(t)$

$$\therefore \quad \mathrm{L}[x(t)] = X(s) = \mathrm{L}[(\sin \omega t \cos \theta + \cos \omega t \sin \theta)\, u(t)]$$

$$= \mathrm{L}[\sin \omega t \cos \theta\, u(t)] + \mathrm{L}[\cos \omega t \sin \theta u(t)]$$

$$= \cos \theta \mathrm{L}[\sin \omega t\, u(t)] + \sin \theta \mathrm{L}[\cos \omega t\, u(t)]$$

$$= \cos \theta \frac{\omega}{s^2 + \omega^2} + \sin \theta \frac{s}{s^2 + \omega^2}$$

$$\therefore \quad X(s) = \frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2}$$

EXAMPLE 9.11 Is the signal $x(t)$ = [sin(t)] sgn(t), where sgn(t) is the signum function, Laplace transformable or not? State the reason.

Solution: Given signal $x(t) = [\sin(t)]\,\text{sgn}(t)$

We know that

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

$$\therefore \qquad \text{L}[\text{sgn}(t)] = \int_{-\infty}^{\infty} \text{sgn}(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} -e^{-st}\, dt + \int_{0}^{\infty} e^{-st}\, dt$$

The first integral converges only for Re(s) < 0. So ROC R_1 is Re (s) < 0, and the second integral converges for Re (s) > 0. So ROC R_2 is Re(s) > 0.

$$\therefore \qquad \text{ROC of L}[\text{sgn}(t)] = R_1 \cap R_2 = 0$$

Now,
$$\text{L}[\sin(t)] = \frac{1}{s^2+1};\ \text{ROC; Re}(s) > 0$$

$\therefore$ ROC of L{[sin (t)] sgn(t)} does not exist because [0 $\cap$ Re(s) > 0] = 0

$\therefore$ The signal $x(t)$ = [sin(t)] sgn(t) is not Laplace transformable.

EXAMPLE 9.12 Suppose the following facts are given about the signal $x(t)$ with Lapalce transform $X(s)$:

(a) $x(t)$ is real and even.

(b) $X(s)$ has four poles, no zeros in a finite s-plane.

(c) $X(s)$ has a pole at $s = (1/2)e^{j\pi/4}$.

(d) $\displaystyle\int_{-\infty}^{\infty} x(t)\, dt = 4$.

Determine $X(s)$ and ROC.

Solution:

(a) Given $x(t)$ is real and even.

If $X(s)$ has a pole at $s = s_0$, then it will have another pole at $s = -s_0$,

$$X(s) = X(-s)$$

(b) $X(s)$ has 4 poles and no zeros in a finite s-plane.

Let
$$X(s) = \frac{k}{(s-P_1)(s-P_2)(s-P_3)(s-P_4)}$$

(c) Given $X(s)$ has a pole at $s = (1/2)e^{j\pi/4}$.

As $x(t)$ is real and even, $X(s)$ has another pole at $s = (1/2)e^{-j\pi/4}$.

$$\therefore \qquad P_1 = \frac{1+j}{2\sqrt{2}};\ P_2 = \frac{1-j}{2\sqrt{2}}$$

$$\therefore \qquad P_1 + P_2 = \frac{1+j+1-j}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$P_1P_2 = \left(\frac{1+j}{2\sqrt{2}}\right)\left(\frac{1-j}{2\sqrt{2}}\right) = \frac{1+1}{(2\sqrt{2})^2} = \frac{1}{4}$$

$$\therefore \qquad X(s) = \frac{k}{[(s^2 - (s/\sqrt{2}) + (1/4)][s^2 - (P_3 + P_4)s + P_3P_4]}$$

As $x(t)$ is even, $X(s) = X(-s)$, i.e.

$$\frac{k}{[(s^2 - (s/\sqrt{2}) + (1/4)][s^2 - (P_3 + P_4)s + P_3P_4]}$$

$$= \frac{k}{[(s^2 + (s/\sqrt{2}) + (1/4)][s^2 + (P_3 + P_4)s + P_3P_4]}$$

Comparing the coefficients of s, we have

$$-\frac{P_3P_4}{\sqrt{2}} - \frac{(P_3 + P_4)}{4} = \frac{P_3P_4}{\sqrt{2}} + \frac{(P_3 + P_4)}{4}$$

i.e.

$$\frac{2P_3P_4}{\sqrt{2}} = -\frac{2(P_3 + P_4)}{4}$$

$$P_3P_4 = \frac{-\sqrt{2}(P_3 + P_4)}{4} = \frac{-\sqrt{2}(-1/\sqrt{2})}{4}$$

$$\therefore \qquad P_3P_4 = \frac{1}{4}$$

$$\therefore \qquad X(s) = \frac{k}{[(s^2 - (s/\sqrt{2}) + (1/4)][s^2 + (s/\sqrt{2}) + (1/4)]}$$

(d) Given

$$\int_{-\infty}^{\infty} x(t)\, dt = 4$$

$$X(s) = \mathrm{L}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$\therefore \qquad X(0) = \int_{-\infty}^{\infty} x(t)\, e^{0}\, dt = \int_{-\infty}^{\infty} x(t)\, dt = 4$$

Putting $s = 0$ in the equation for $X(s)$, we have

$$X(0) = \frac{k}{(1/4)\,(1/4)}$$

Equating the above two equations for $X(0)$, we have

$$4 = \frac{k}{(1/4)(1/4)} = 16k$$

or

$$k = \frac{1}{4}$$

$$\therefore \qquad X(s) = \frac{1/4}{[(s^2 - (s/\sqrt{2}) + (1/4)]\,[(s^2 + (s/\sqrt{2}) + (1/4)]} = \frac{1/4}{s^4 + (1/16)}$$

$$\therefore \qquad X(s) = \frac{4}{16s^4 + 1};\ \text{ROC; Re}(s) > -\ \frac{1}{16}.$$

9.7 PROPERTIES AND THEOREMS OF LAPLACE TRANSFORM

9.7.1 Linearity Property

The linearity property states that the Laplace transform of a weighted sum of two signals is equal to the weighted sum of individual Laplace transforms. That is,

If $\qquad x_1(t) \xleftrightarrow{\text{LT}} X_1(s)$

and $\qquad x_2(t) \xleftrightarrow{\text{LT}} X_2(s)$

Then $\qquad ax_1(t) + bx_2(t) \xleftrightarrow{\text{LT}} aX_1(s) + bX_2(s)$

Proof: From the definition of Laplace transform, we write

$$\text{L}[ax_1(t) + bx_2(t)] = \int_0^{\infty} [ax_1(t) + bx_2(t)]\, e^{-st}\, dt$$

$$= a\int_0^{\infty} x_1(t) e^{-st}\, dt + b\int_0^{\infty} x_2(t) e^{-st}\, dt$$

$$= aX_1(s) + bX_2(s)$$

$$\boxed{ax_1(t) + bx_2(t) \xleftrightarrow{\text{LT}} aX_1(s) + bX_2(s)}$$

9.7.2 Time Shift Property

The time shift property states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$x(t-t_0) \xleftrightarrow{\text{LT}} e^{-st_0} X(s)$$

Proof: $$\text{L}\,[x(t-t_0)] = \int_0^{\infty} x(t-t_0)\, e^{-st}\, dt$$

Let $$t - t_0 = p$$

$\therefore$ $$dt = dp \text{ and } t = p + t_0$$

$\therefore$ $$\text{L}[x(t-t_0)] = \int_0^{\infty} x(p)\, e^{-s(p+t_0)}\, dp$$

or $$\text{L}[x(t-t_0)] = e^{-st_0} \int_{-t_0}^{\infty} x(p)\, e^{-ps}\, dp$$

$$= e^{-st_0} \int_{-\infty}^{\infty} x(p)\, e^{-ps}\, dp = e^{-st_0} X(s)$$

$$\boxed{x(t-t_0) \xleftrightarrow{\text{LT}} e^{-st_0} X(s)}$$

Similarly,

$$\boxed{x(t+t_0) \xleftrightarrow{\text{LT}} e^{st_0} X(s)}$$

A time shift of t_0 corresponds to multiplication by complex exponential e^{-st}.

9.7.3 Time Scaling Property

The time scaling property states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$x(at) \xleftrightarrow{\text{LT}} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Proof: $$\text{L}\,[x(at)] = \int_0^{\infty} x(at)\, e^{-st}\, dt$$

Let $$at = t_1$$

$\therefore$ $$t = \frac{t_1}{a} \quad \text{and} \quad dt = \frac{dt_1}{a}$$

$$\therefore \qquad \text{L}\,[x(at)] = \int_0^\infty x(t_1)\, e^{-(s/a)t_1} \frac{dt_1}{a} = \frac{1}{a}\int_0^\infty x(t_1)^{-(s/a)t_1}\, dt_1$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right)$$

This is valid for all values of a.

$$\therefore \qquad \text{L}[x(at)] = \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

$$\boxed{x(at) \xleftrightarrow{\text{LT}} \frac{1}{|a|} X\left(\frac{s}{a}\right)}$$

9.7.4 Time Reversal Property

The time reversal property states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$x(-t) \xleftrightarrow{\text{LT}} X(-s)$$

Proof: From the definition of the Laplace transform, we have

$$\text{L}[x(-t)] = \int_{-\infty}^{\infty} x(-t)\, e^{-st}\, dt$$

In RHS, let $$-t = p$$

$$\therefore \qquad dt = -dp$$

$$\therefore \qquad \text{L}[x(-t)] = \int_{-\infty}^{\infty} x(p)\, e^{sp}(-dp)$$

$$= \int_{-\infty}^{\infty} x(p)\, e^{-(-s)p}\, dp = X(-s)$$

$$\boxed{x(-t) \xleftrightarrow{\text{LT}} X(-s)}$$

9.7.5 Transform of Derivatives Property

The transform of derivatives property states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$\frac{d}{dt}x(t) \xleftrightarrow{\text{LT}} sX(s) - x(0^-)$$

Proof: From the definition of Laplace transform, we write

$$\text{L}\left[\frac{d}{dt}x(t)\right] = \int_0^\infty \left[\frac{d}{dt}x(t)\right]e^{-st}\,dt$$

$$= \left[e^{-st}x(t)\right]_0^\infty - \int_0^\infty x(t)\,(-se^{-st})\,dt$$

$$= [0 - x(0^-)] + s\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= -x(0^-) + sX(s)$$

$$\therefore \qquad \text{L}\left[\frac{d}{dt}x(t)\right] = sX(s) - x(0^-)$$

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{\text{LT}} sX(s) - x(0^-)}$$

The Laplace transform of the second derivative, i.e., L[$d^2x(t)/dt^2$] is obtained as follows:

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left[\frac{dx(t)}{dt}\right]$$

$$\therefore \qquad \text{L}\left[\frac{d^2x(t)}{dt^2}\right] = \text{L}\left\{\frac{d}{dt}\left[\frac{dx(t)}{dt}\right]\right\}$$

$$= s\text{L}\left[\frac{dx(t)}{dt}\right] - \frac{dx(0^-)}{dt}$$

$$= s[sX(s) - x(0^-)] - \frac{dx(0^-)}{dt}$$

$$= s^2X(s) - s(x0^-) - \frac{dx(0^-)}{dt}$$

$$\boxed{\frac{d^2x(t)}{dt^2} \xleftrightarrow{\text{LT}} s^2X(s) - sx(0^-) - \frac{dx(0^-)}{dt}}$$

where, $dx(0^-)/dt$ is the derivative of $x(t)$ evaluated at $t = 0$. Similarly,

$$\text{L}\left[\frac{d^nx(t)}{dt^n}\right] = s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\frac{dx(0^-)}{dt} - \cdots - \frac{d^{n-1}x(0^-)}{dt^{n-1}}$$

9.7.6 Transform of Integrals Property

The transform of integrals property states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\text{LT}} \frac{X(s)}{s} + \int_{-\infty}^{0} \frac{x(\tau)\,d\tau}{s}$$

Proof: Let $$f(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$\therefore$ $$\frac{d}{dt} f(t) = x(t)$$

Further, $$f(0^-) = \int_{-\infty}^{0} x(\tau)\,d\tau$$

$\therefore$ $$\text{L}\left[\frac{d}{dt} f(t)\right] = \text{L}[x(t)]$$

i.e. $$sF(s) - f(0^-) = X(s)$$

$\therefore$ $$F(s) = \frac{X(s)}{s} + \frac{f(0^-)}{s}$$

$$= \frac{X(s)}{s} + \int_{-\infty}^{0} \frac{x(\tau)\,d\tau}{s}$$

i.e. $$\text{L}\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{X(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} x(\tau)\,d\tau$$

$$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\text{LT}} \frac{X(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} x(\tau)\,d\tau}$$

9.7.7 Differentiation in *s*-domain Property

The differentiation in s-domain property states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$tx(t) \xleftrightarrow{\text{LT}} -\frac{d}{ds} X(s)$$

Proof:
$$L[x(t)] = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

Differentiating with respect to s on both sides, we get

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left[\int_0^\infty x(t)\, e^{-st}\, dt\right]$$
$$= \left[\int_0^\infty x(t)\, \frac{d}{ds}(e^{-st})\, dt\right]$$
$$= \int_0^\infty x(t)\,(-t)\, e^{-st}\, dt$$
$$= \int_0^\infty [-tx(t)]\, e^{-st}\, dt$$
$$= L[-tx(t)]$$

$\therefore$
$$L\,[-tx(t)] = \frac{d}{ds}\, X(s)$$

$$\boxed{-tx(t) \xleftrightarrow{\text{LT}} \frac{d}{ds}X(s)}$$

In the same way,

$$L[(-t)^2 x(t)] = \frac{d^2X(s)}{ds^2}$$

and
$$L[(-t)^n x(t)] = \frac{d^nX(s)}{ds^n}$$

or
$$L[(tx(t)] = -\frac{d}{ds}X(s)$$

$$L[t^2 x(t)] = (-1)^2\, \frac{d^2X(s)}{ds^2}$$

and
$$L[t^n x(t)] = (-1)^n\, \frac{d^nX(s)}{ds^n}$$

Corollary

If $x(t) = t^n$, then $X(s) = L[t^n] = \frac{n!}{s^{n+1}}$; ROC; $\text{Re}(s) > 0$

$$\boxed{t^n \xleftrightarrow{\text{LT}} \frac{n!}{s^{n+1}}}$$

9.7.8 Frequency Shift Property

The frequency shift property states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$e^{-at}x(t) \xleftrightarrow{\text{LT}} X(s+a)$$

Proof: $$\text{L}\,[e^{-at}x(t)] = \int_0^\infty e^{-at}\,x(t)\,e^{-st}\,dt$$

$$= \int_0^\infty x(t)\,e^{-(s+a)t}\,dt$$

$$= X(s+a)$$

$$\boxed{e^{-at}\,x(t) \xleftrightarrow{\text{LT}} X(s+a)}$$

Multiplication by a complex exponential in time domain introduces a shift in complex frequency s in frequency domain. Similarly,

$$\text{L}\,[e^{at}x(t)] = X(s-a)$$

9.7.9 Time Convolution Property

The time convolution property states that

If $$x_1(t) \xleftrightarrow{\text{LT}} X_1(s)$$

and $$x_2(t) \xleftrightarrow{\text{LT}} X_2(s)$$

Then $$x_1(t) * x_2(t) \xleftrightarrow{\text{LT}} X_1(s)\,X_2(s)$$

Proof: The signals $x_1(t)$ and $x_2(t)$ are causal,

$\therefore$ $$x_1(t) * x_2(t) = \int_0^t x_1(t-\tau)\,x_2(\tau)\,d\tau$$

$$\text{L}[x_1(t) * x_2(t)] = \int_0^\infty \left[\int_0^t x_1(t-\tau)\,x_2(\tau)\,d\tau\right] e^{-st}\,dt$$

$$= \int_0^\infty \left[\int_0^\infty x_1(t-\tau)\,x_2(\tau)\,d\tau\right] e^{-st}\,dt$$

Let $t - \tau = p$, then, $dt = dp$

$$\therefore \qquad \mathrm{L}\,[x_1(t) * x_2(t)] = \int_0^{\infty} x_2(\tau) \int_0^{\infty} x_1(p)\, e^{-s(\tau+p)}\, d\tau\, dp$$

$$= \int_0^{\infty} x_2(\tau)\, e^{-s\tau} d\tau \int_0^{\infty} x_1(p)\, e^{-sp}\, dp$$

$$= X_2(s)\, X_1(s) = X_1(s)\, X_2(s)$$

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{\mathrm{LT}} X_1(s)\, X_2(s)}$$

Thus, the Laplace transform of the convolution of two signals is the product of their respective Laplace transforms.

9.7.10 Multiplication or Modulation or Convolution in *s*-domain Property

The Multiplication/Modulation/Convolution in s-domain property states that

If $\qquad x_1(t) \xleftrightarrow{\mathrm{LT}} X_1(s)$

and $\qquad x_2(t) \xleftrightarrow{\mathrm{LT}} X_2(s)$

Then $\qquad x_1(t)\, x_2(t) \xleftrightarrow{\mathrm{LT}} \dfrac{1}{2\pi j}[X_1(s) * X_2(s)]$

Proof: From the definition of Laplace transform, we have

$$\mathrm{L}[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(u)\, e^{ut}\, du \right] x_2(t)\, e^{-st}\, dt$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(u) \left[\int_{-\infty}^{\infty} x_2(t)\, e^{-(s-u)t}\, dt \right] du$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(u)\, X_2(s-u)\, du$$

$$= \frac{1}{2\pi j}[X_1(s) * X_2(s)]$$

$$\boxed{x_1(t)\,x_2(t) \longleftrightarrow \frac{1}{2\pi j}[X_1(s) * X_2(s)]}$$

9.7.11 Conjugation and Conjugate Symmetry Property

The conjugation property states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$x^*(t) \xleftrightarrow{\text{LT}} X^*(s^*) \qquad \text{[for complex } x(t)\text{]}$$

The conjugate symmetry property states that $X(s) = X^*(s^*)$ for real $x(t)$.

Proof: **Conjugation property**

From the definition of the Laplace transform, we have

$$\text{L}[x^*(t)] = \int_{-\infty}^{\infty} x^*(t) e^{-st}\, dt$$

$$= \left[\int_{-\infty}^{\infty} x(t) e^{-(s^*)t}\, dt\right]^*$$

$$= [X(s^*)]^* = X^*(s^*) \qquad \text{[for complex } x(t)\text{]}$$

$$\boxed{x^*(t) \xleftrightarrow{\text{LT}} X^*(s^*)}$$

Conjugate symmetry property

From the definition of Laplace transform, we have

$$X(s^*) = \int_{-\infty}^{\infty} x(t)\, e^{-(s^*)t}\, dt$$

Taking the conjugate on both sides, we get

$$X^*(s^*) = \left[\int_{-\infty}^{\infty} x(t)\, e^{-(s^*)t}\, dt\right]^*$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(s^*)^*t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = X(s) \qquad \text{[for real } x(t)\text{]}$$

$$\boxed{X(s) = X^*(s^*)}$$

9.7.12 Parseval's Relation or Theorem or Property

The Parseval's relation or theorem or property states that

If $$x_1(t) \xleftrightarrow{\text{LT}} X_1(s)$$

and $$x_2(t) \xleftrightarrow{\text{LT}} X_2(s)$$

Then $$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt \xleftrightarrow{\text{LT}} \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(s) X_2^*(-s^*)\, ds \qquad \text{[for complex } x_1(t) \text{ and } x_2(t)]$$

Proof: We have

$$x_1(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(s)\, e^{st} ds$$

$$\text{LHS} = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(s)\, e^{st}\, ds \right\} x_2^*(t)\, dt$$

Interchanging the order of integration, we have

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(s) \left\{ \int_{-\infty}^{\infty} x_2^*(t)\, e^{st}\, dt \right\} ds$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(s) \left\{ \int_{-\infty}^{\infty} x_2(t)\, e^{-(-s^*)\,t}\, dt \right\}^* ds$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(s) \left\{ \int_{-\infty}^{\infty} X_2(-s^*) \right\}^* ds$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X_1(s)\, \{X_2^*(-s^*)\}\, ds$$

9.7.13 Initial Value Theorem

The initial value theorem enables us to calculate the initial value of a function $x(t)$, i.e. $x(0)$ directly from its transform $X(s)$ without the need for finding the inverse transform of $X(s)$. It states that

If $$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then $$\underset{t\to 0}{\text{Lt}}\; x(t) = x(0) = \underset{s\to\infty}{\text{Lt}}\; sX(s)$$

Proof: Given
$$L[x(t)] = X(s)$$

We have
$$L\left[\frac{dx(t)}{dt}\right] = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt = sX(s) - x(0^-)$$

Taking limit $s \to \infty$ on both sides, we get

$$\underset{s\to\infty}{\text{Lt}}\ L\left[\frac{dx(t)}{dt}\right] = \underset{s\to\infty}{\text{Lt}}\left[\int_0^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt\right] = \underset{s\to\infty}{\text{Lt}}\ [sX(s) - x(0)]$$

i.e.
$$0 = \underset{s\to\infty}{\text{Lt}}\ sX(s) - x(0)$$

$\therefore$
$$x(0) = \underset{s\to\infty}{\text{Lt}}\ sX(s)$$

$$\boxed{x(0) = \underset{t\to 0}{\text{Lt}}\ x(t) = \underset{s\to\infty}{\text{Lt}}\ sX(s)}$$

9.7.14 Final Value Theorem

The final value theorem enables us to determine the final value of a function $x(t)$, i.e. $x(\infty)$ directly from its Laplace transform $X(s)$ without the need for finding the inverse transform of $X(s)$. It states that

If
$$x(t) \xleftrightarrow{\text{LT}} X(s)$$

Then
$$\underset{t\to\infty}{\text{Lt}}\ x(t) = x(\infty) = \underset{s\to 0}{\text{Lt}}\ sX(s)$$

Proof: Given
$$L[x(t)] = X(s)$$

We have
$$L\left[\frac{dx(t)}{dt}\right] = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt = sX(s) - x(0^-)$$

Taking the limit $s \to 0$ on both sides, we obtain

$$\underset{s\to 0}{\text{Lt}}\left[\int_0^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt\right] = \underset{s\to 0}{\text{Lt}}\ [sX(s) - x(0^-)]$$

$\therefore$
$$\int_0^{\infty} \frac{dx(t)}{dt}\, dt = \underset{s\to 0}{\text{Lt}}\ [sX(s) - x(0^-)]$$

$$\left[x(t)\right]_0^{\infty} = \underset{s\to 0}{\text{Lt}}\ [sX(s) - x(0^-)]$$

i.e.
$$x(\infty) - x(0^-) = \underset{s\to 0}{\text{Lt}}\ [sX(s)] - x(0^-)$$

i.e.
$$\underset{t\to\infty}{\text{Lt}}\ x(t) = x(\infty) = \underset{s\to 0}{\text{Lt}}\ sX(s)$$

$$\boxed{x(\infty) = \underset{t\to\infty}{\text{Lt}}\ x(t) = \underset{s\to 0}{\text{Lt}}\ sX(s)}$$

To apply the final value theorem, we must cancel the common factors, if any, in the numerator and denominator of $sX(s)$. If any poles of $sX(s)$ after cancellation of the common factors lie in the right half of the s-plane, then the final value theorem does not hold. A simple pole in $X(s)$ at the origin is permitted. Otherwise all other poles of $X(s)$ must be strictly in the left-half of the s-plane.

9.7.15 Time Periodicity Property (Laplace Transform of Periodic Functions)

The time shift theorem is useful in determining the transform of periodic time functions. Let the function $x(t)$ be a causal periodic waveform which satisfies the condition $x(t) = x(t + nT)$ for all $t > 0$, where T is the period of the function and $n = 0, 1, 2, \ldots$.

Now,
$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

This can be written as:

$$X(s) = \int_0^T x(t)\, e^{-st}\, dt + \int_T^{2T} x(t)\, e^{-st}\, dt + \int_{2T}^{3T} x(t)\, e^{-st}\, dt + \cdots + \int_{nT}^{(n+1)T} x(t) e^{-st}\, dt + \cdots$$

$$= \int_0^T x(t)\, e^{-st}\, dt + e^{-sT} \int_0^T x(t+T)\, e^{-st}\, dt + e^{-2sT} \int_0^T x(t+2T)\, e^{-st}\, dt + \cdots$$

$$+ e^{-nsT} \int_0^T x(t+nT)\, e^{-st}\, dt + \cdots$$

Since, $x(t)$ is periodic, we have

$$x(t) = x(t+T) = x(t+2T) = x(t+3T) = \ldots$$

So the above equation can also be written as:

$$X(s) = \int_0^T x(t)\, e^{-st}\, dt + e^{-sT} \int_0^T x(t)\, e^{-st}\, dt + e^{-2sT} \int_0^T x(t)\, e^{-st} dt + \cdots + e^{-nsT} \int_0^T x(t)\, e^{-st}\, dt + \cdots$$

$$= [1 + e^{-sT} + e^{-2sT} + \cdots + e^{-nsT} + \cdots] \int_0^T x(t)\, e^{-st}\, dt$$

$$= [1 - e^{-sT}]^{-1} \int_0^T x(t)\, e^{-st}\, dt = \frac{1}{1 - e^{-sT}}\, X_1(s)$$

9

Laplace Transforms

9.1 INTRODUCTION

A linear time invariant (LTI) system is described by differential equations. The response of a system for a given input is obtained by solving the differential equations relating its output and input signals. We know that the solution of higher order differential equations is quite tedious and time consuming compared to the solution of algebraic equations. So the Laplace transform is used to solve the differential equations. Laplace transform is a powerful mathematical tool used to convert the differential equations into algebraic equations. It is a simple and systematic method which provides the complete solution in one stroke by taking into account the initial conditions in a natural way at the beginning of the process itself. For solving the differential equations using Laplace transform, we take the Laplace transform of the differential equations (i.e. convert the differential equations in time domain into algebraic equations in frequency domain), insert the initial conditions, solve the resultant algebraic equations (i.e. get the solution in s-domain) and take the inverse Laplace transform of the solution (to get the solution in time domain).

When a complex exponential of any frequency is applied to a system, the output is essentially same as the input except that it is multiplied by a constant. This property of the complex exponential makes the Laplace transform an invaluable tool in the analysis of LTI systems.

The Laplace transform of a time function $x(t)$ is defined as:

$$\mathrm{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

where s is a complex variable and is equal to $s = \sigma + j\omega$. Here the operator L is called the Laplace transform operator which transforms the time domain function $x(t)$ into frequency domain function $X(s)$.

The Laplace transform as defined above with $-\infty$ as the lower limit for the integral is called the 'two-sided' or 'bilateral' Laplace transform. If the lower limit is changed to '0', we get the 'one-sided' or 'unilateral' Laplace transform. Hence the one-sided Laplace transform is defined as

$$\mathrm{L}[x(t)] = X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$$

Clearly the bilateral and unilateral Laplace transforms are equivalent only if $x(t) = 0$ for $t < 0$, i.e. $x(t)$ must be a causal signal. The signal $x(t)$ and its Laplace transform $X(s)$ are said to form a Laplace transform pair denoted by

$$x(t) \xleftrightarrow{\mathrm{LT}} X(s)$$

While the two-sided Laplace transform is useful to get a better insight into the systems (with causal and non-causal signals), the one-sided Laplace transform is very useful in practical problems (with causal signals), where the signal is applied to a system at a particular point in time which can be conveniently taken as the time origin, or $t = 0$. Thus, the one-sided Laplace transform comes in handy in solving differential equations with given initial conditions. Further its region of convergence (ROC) is simple and unique for all causal functions. Hence, it is the one-sided Laplace transform which is used most often for solving practical problems.

9.2 REGION OF CONVERGENCE

For a given value of $x(t)$, the Laplace transform as given by the equation, $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ may not converge for all values of the complex variable s. Since every value of s corresponds to a particular point in the s-plane (with σ along the x-axis and $j\omega$ along the y-axis), we say that the set of points in the s-plane for which the integral on the right hand side of equation for $X(s)$ converges is the region of convergence (ROC) of the Laplace transform of $x(t)$. If there is no value of s, i.e. no point on the s-plane for which the integral converges, we say that $x(t)$ does not have a ROC and hence does not have a Laplace transform, or that it is not Laplace transformable. Most of the functions which are of some use in engineering practice are Laplace transformable. For an $x(t)$ to be Laplace transformable, the usual condition is that it should be piece-wise continuous and must be of exponential order.

If the ROC is not specified, the inverse Laplace transform is not unique. Also in the one-sided Laplace transform, all the time functions are assumed to be positive and hence there is a one-to-one correspondence between the Laplace transform and its inverse Laplace transform. Therefore, no ambiguity will arise even if the ROC is not specified in the one-sided Laplace transform. However, in two-sided Laplace transform, the specification of ROC is essential.

The bilateral Laplace transform of a signal $x(t)$ exists if

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \text{ is finite.}$$

Substituting $s = \sigma + j\omega$ in the above equation, we get

$$\mathrm{L}\,[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt = \int_{-\infty}^{\infty} [x(t)\, e^{-\sigma t}]\, e^{-j\omega t}\, dt = \mathrm{F}[x(t)e^{-\sigma t}]$$

This equation indicates that $X(s)$ is basically the continuous-time Fourier transform of $x(t)e^{-\sigma t}$. So we can say that

The Laplace transform of $x(t)$ is the Fourier transform of $x(t)e^{-\sigma t}$.

9.3 EXISTENCE OF LAPLACE TRANSFORM

The necessary and sufficient conditions for the existence of the Laplace transform are:

1. $x(t)$ should be continuous or piece-wise continuous in the given closed interval.
2. $x(t)e^{-\sigma t}$ must be absolutely integrable.

That is, $X(s)$ exists only if $\int_{-\infty}^{\infty} \left| x(t)\, e^{-\sigma t} \right| dt < \infty$

or only if $\quad \underset{t \to \infty}{\mathrm{Lt}}\; e^{-st}\, x(t) = 0$

The range of σ for which the Laplace transform converges is known as the *region of convergence* (ROC). So the functions which are not Fourier transformable may be Laplace transformable.

Consider a signal $x(t) = e^{2t}\, u(t)$ as shown in Figure 9.1(a). The Fourier transform of $x(t) \left(= \int_{-\infty}^{\infty} e^{2t} u(t)\, e^{-j\omega t}\, dt \right)$ does not exist because it is not absolutely integrable. Now, multiply the signal $x(t)$ with $e^{-\sigma t}$. Then the resultant signal $x(t)e^{-\sigma t}\, u(t) = e^{(2-\sigma)t}\, u(t)$ is absolutely integrable for $\sigma > 2$. Therefore, the Laplace transform of $x(t)$, which is the Fourier transform of $x(t)e^{-\sigma t}$ exists for some values of σ. This concept is illustrated in Figure 9.1(b). Thus, we conclude that *the Laplace transform exists for signals for which the Fourier transform does not exist.*

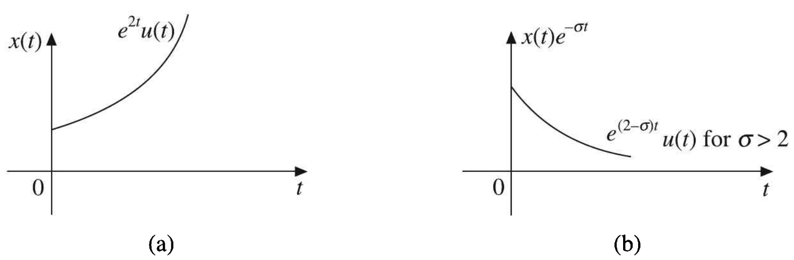


Figure 9.1 Signals (a) $x(t) = e^{2t}\, u(t)$, and (b) $x(t) = e^{(2-\sigma)t}\, u(t)$.

9.10.5 Properties of ROC

The properties of ROC are as follows:

1. The ROC of $X(s)$, the Laplace transform of $x(t)$ is bounded by poles or extends up to infinity.
2. The ROC does not contain any poles.
3. If $x(t)$ is a right-sided signal, the ROC of $X(s)$ extends to the right of the right most pole and no pole is located inside the ROC.
4. If $x(t)$ is a left-sided signal, the ROC of $X(s)$ extends to the left of the left most pole and no pole is located inside the ROC.
5. If $x(t)$ is a two-sided signal, the ROC of $X(s)$ is a strip in the s-plane bounded by poles and no pole is located inside the ROC.
6. Impulse function is the only function for which the ROC is the entire s-plane.
7. The ROC must be a connected region.
8. The ROC of an LTI stable system contains the imaginary axis of s-plane.
9. The ROC of the sum of two or more signals is equal to the intersection of the ROCs of those signals.

EXAMPLE 9.52 Find the Laplace transform and ROC of the right-sided signal

$$x(t) = 4e^{-2t}u(t) + 3e^{-3t}u(t)$$

Solution: The given signal is a right-sided signal. In fact, it is a causal signal.

$$\therefore \qquad X(s) = \frac{4}{s+2} + \frac{3}{s+3}, \text{ ROC; Re}(s) > -2 \text{ and Re}(s) > -3$$

i.e. $$\text{Re}(s) > -2 \cap \text{Re}(s) > -3 = \text{Re}(s) > -2$$

The pole locations and the ROC are shown in Figure 9.20. Observe that the ROC extends to the right of the right most pole and that no pole exists inside the ROC.

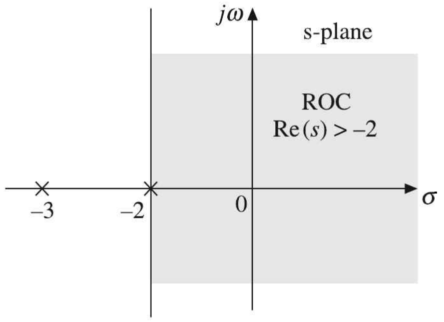


Figure 9.20 ROC and pole location for $x(t) = [4e^{-2t} + 3e^{-3t}]\,u(t)$.

EXAMPLE 9.53 Find the Laplace transform and ROC of the left-sided signal

$$x(t) = [e^{3t} + 5e^{2t}]\,u(-t)$$

We know that, the Fourier transform

$$X(\omega) = [X(s)]_{s=j\omega}$$

$$= \frac{(j\omega)^2 - j\omega + 1}{(j\omega)^2 + j\omega + 1}$$

$$= \frac{(1-\omega^2) - j\omega}{(1-\omega^2) + j\omega}$$

$$\therefore \qquad |X(\omega)| = \frac{\sqrt{(1-\omega^2)^2 + \omega^2}}{\sqrt{(1-\omega^2)^2 + \omega^2}} = 1$$

The magnitude of the Fourier transform of the signal, $|X(\omega)| = 1$.

Some useful Laplace transform pairs are given in Appendix B (Table B.5). The properties of Laplace transform are given in Appendix B (Table B.6).

9.8 INVERSION OF UNILATERAL LAPLACE TRANSFORM

The direct method of finding the inverse Laplace transform of $X(s)$ using the equation

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)\, e^{st}\, ds$$

is quite tedious and time consuming. So the inverse Laplace transform of $X(s)$ is usually obtained by using partial fraction expansion. It uses the one-to-one relationship between a signal and its unilateral Laplace transform. The partial fraction expansion can be used only for proper rational functions, i.e. for functions in which the order of the numerator is smaller than that of the denominator.

Let
$$X(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

If $M = N$, i.e. Deg $N(s)$ = Deg $D(s)$, then divide the numerator by the denominator

$$X(s) = C + \frac{N^1(s)}{D(s)}$$

where C is a constant and Deg $N^1(s)$ < Deg $D(s)$.

Since a constant C has an inverse Laplace transform $C\delta(t)$, Deg $N(s)$ = Deg $D(s)$ implies that $x(t)$, the inverse Laplace transform of $X(s)$ has an impulse.

If Deg $N(s)$ > Deg $D(s)$, divide the numerator by the denominator and write $X(s)$ as

$$X(s) = \text{Terms with powers of } s \text{ greater than or equal to zero} + \frac{N^1(s)}{D(s)}$$

where
$$\text{Deg } N^1(s) < \text{Deg } D(s)$$

In this case, $x(t)$ will include impulses and their derivatives. Usually, we do not come across such functions. So we consider the usual case $X(s) = N(s)/D(s)$ where Deg $N(s) <$ Deg $D(s)$.

Under this condition, we proceed as follows: If the given $X(s)$ is of any of the standard forms, we straight away write down its inverse transform. If the given $X(s)$ is not of any of the standard forms, $D(s)$ is factorized. The roots of the characteristic equation, $D(s) = 0$ are called the poles of $X(s)$.

$X(s)$ can be expressed as:

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{\prod_{k=1}^{N} (s - P_k)}$$

We shall now consider two possible cases that arise. (i) Distinct poles, (ii) Multiple poles.

9.8.1 Distinct Poles

If all the poles P_k are distinct, then we may write $X(s)$ as a sum of single pole terms given by

$$X(s) = \frac{K_1}{s - P_1} + \frac{K_2}{s - P_2} + \cdots + \frac{K_i}{s - P_i} + \cdots + \frac{K_n}{s - P_n}$$

To determine the value of K_i, multiply both sides by $(s - P_i)$ and then put $s = P_i$. We then have

$$(s-P_i)X(s)\Big|_{s=P_i} = \left[(s-P_i)\frac{K_1}{s-P_1} + (s-P_i)\frac{K_2}{s-P_2} + \cdots + (s-P_i)\frac{K_i}{s-P_i} + \cdots + (s-P_i)\frac{K_n}{s-P_n}\right]\Bigg|_{s=P_i}$$

In the RHS of the above equation all the terms except K_i vanish. Hence, we get

$$K_i = (s - P_i)\, X(s)\Big|_{s=P_i}$$

9.8.2 Multiple Poles

In this case, all the poles are not distinct. Some of the poles may repeat. Let the pole P_i repeat l times. Then, if all the other poles are distinct, we write

$$X(s) = \frac{K_1}{s - P_1} + \frac{K_2}{s - P_2} + \cdots + \frac{K_{i1}}{(s - P_i)} + \frac{K_{i2}}{(s - P_i)^2} + \frac{K_{il}}{(s - P_i)^l} + \cdots + \frac{K_n}{(s - P_n)}$$

The coefficient K_{il} is obtained by multiplying both sides of the above equation by $(s - P_i)^l$ and evaluating it at $s = P_i$.

$$\therefore \qquad K_{il} = (s - P_i)^l\, X(s)\Big|_{s=P_i}$$

The coefficients $K_{i(l-b)}$ are evaluated by multiplying both sides of the preceding equation by $(s - P_i)^l$ differentiating $(l - b)$ times and then evaluating the resultant equation at $s = P_i$. Thus,

$$K_{i(l-1)} = \frac{d}{ds}[(s - P_i)^l X(s)]\Big|_{s=P_i}$$

$$\vdots \qquad\qquad \vdots$$

$$K_{i(l-b)} = \frac{1}{b!}\left\{\frac{d^b}{ds^b}[(s - P_i)^l X(s)]\right\}\Bigg|_{s=P_i}$$

9.8.3 Complex Roots

If $X(s)$ has complex poles, then the partial fraction expansion of the same can be expressed as:

$$X(s) = \frac{K_1}{s - P_1} + \frac{K_1^*}{s - P_1^*}$$

where K_1^* is complex conjugate of K_1 and P_1^* is complex conjugate of P_1. In other words, complex conjugate poles result in complex conjugate coefficients.

Alternate method

The above procedure of finding partial fractions involves manipulation of complex numbers. When the denominator has complex roots, the complex pair can be expressed as a single quadratic factor instead of having first order partial fractions. That is, if $s = -a \pm jb$, we can retain the quadratic factor.

$$D(s) = (s + a + jb)(s + a - jb) = (s + a)^2 + b^2$$

EXAMPLE 9.29 Find the inverse Laplace transforms for the following:

(a) $\dfrac{1}{(s+1)^2}$ (b) $\dfrac{1}{(s+1)^2+1}$

(c) $\dfrac{s}{(s+2)^2}$ (d) $\dfrac{s}{(s+2)^2+1}$

(e) $\dfrac{s}{(s-b)^2+a^2}$ (f) $\dfrac{s}{2s^2-8}$

(g) $\dfrac{s}{s^2a^2+b^2}$ (h) $\dfrac{s}{(s^2+a^2)^2}$

(i) $\dfrac{s^2}{(s-1)^4}$ (j) $\dfrac{1}{s(s+2)^3}$

(k) $X(s) = \dfrac{2s+1}{s+2}$; $\text{Re}(s) > -2$

Solution:

(a) Given

$$X(s) = \frac{1}{(s+1)^2}$$

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$= e^{-t}L^{-1}\left(\frac{1}{s^2}\right) = e^{-t}\,tu(t) = te^{-t}\,u(t)$$

(b) Given

$$X(s) = \frac{1}{(s+1)^2+1}$$

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right) = e^{-t}\sin tu(t)$$

(c) Given $X(s) = \dfrac{s}{(s+2)^2}$

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}\right] = L^{-1}\left(\frac{1}{s+2}\right) - 2L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= e^{-2t} - 2e^{-2t}L^{-1}\left(\frac{1}{s^2}\right) = e^{-2t}(1-2t)\,u(t)$$

(d) Given

$$X(s) = \frac{s}{(s+2)^2+1}$$

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = L^{-1}\left[\frac{(s+2)-2}{(s+2)^2+1}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= e^{-2t}L^{-1}\left(\frac{s}{s^2+1}\right) - 2e^{-2t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= [e^{-2t}\cos t - 2e^{-2t}\sin t]\,u(t)$$

(e) Given
$$X(s)=\frac{s}{(s-b)^2+a^2}$$

$$x(t)=\mathrm{L}^{-1}\left[\frac{s}{(s-b)^2+a^2}\right]=\mathrm{L}^{-1}\left[\frac{s-b+b}{(s-b)^2+a^2}\right]$$
$$=\mathrm{L}^{-1}\left[\frac{s-b}{(s-b)^2+a^2}\right]+\mathrm{L}^{-1}\left[\frac{b}{(s-b)^2+a^2}\right]$$
$$=\mathrm{L}^{-1}\left[\frac{s-b}{(s-b)^2+a^2}\right]+\frac{b}{a}\mathrm{L}^{-1}\left[\frac{a}{(s-b)^2+a^2}\right]$$
$$=e^{bt}\mathrm{L}^{-1}\left(\frac{s}{s^2+a^2}\right)+\frac{b}{a}e^{bt}\,\mathrm{L}^{-1}\left(\frac{a}{s^2+a^2}\right)$$
$$=\left(e^{bt}\cos at+\frac{b}{a}e^{bt}\sin at\right)u(t)$$

(f) Given
$$X(s)=\frac{s}{2s^2-8}$$

$\therefore$
$$X(s)=\frac{1}{2}\frac{s}{(s^2-4)}=\frac{1}{2}\left(\frac{s}{s^2-2^2}\right)$$

We know that
$$\cosh \omega t=\frac{s}{s^2-\omega^2}$$

$\therefore$
$$x(t)=\frac{1}{2}\mathrm{L}^{-1}\left(\frac{s}{s^2-2^2}\right)=\frac{1}{2}\cosh 2t$$

(g) Given
$$X(s)=\frac{s}{s^2a^2+b^2}=\frac{1}{a^2}\left[\frac{s}{s^2+(b/a)^2}\right]$$

We know that
$$\mathrm{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right)=\cos \omega t$$

$\therefore$
$$\mathrm{L}^{-1}\left[\frac{s}{s^2+(b/a)^2}\right]=\cos\left(\frac{b}{a}\right)t$$

$\therefore$
$$x(t)=\mathrm{L}^{-1}\left[\frac{1}{a^2}\frac{s}{s^2+(b/a)^2}\right]=\frac{1}{a^2}\cos\left(\frac{b}{a}\right)t$$

(h) Given $$X(s) = \frac{s}{(s^2 + a^2)^2}$$

$$x(t) = \mathrm{L}^{-1}\left[\frac{s}{(s^2 + a^2)^2}\right] = \frac{1}{2a}\left[-\frac{d}{ds}\left\{\frac{a}{s^2 + a^2}\right\}\right]$$

$$= \frac{1}{2a} t\,\mathrm{L}^{-1}\left(\frac{a}{s^2 + a^2}\right) = \frac{1}{2a} t \sin at$$

(i) Given $$X(s) = \frac{s^2}{(s - 1)^4}$$

$$x(t) = \mathrm{L}^{-1}[X(s)] = \mathrm{L}^{-1}\left[\frac{s^2}{(s - 1)^4}\right]$$

s in the numerator corresponds to differential of time domain function

$$\therefore \quad \mathrm{L}^{-1}\left[\frac{s^2}{(s - 1)^4}\right] = \frac{d}{dt}\left\{\mathrm{L}^{-1}\left[\frac{s}{(s - 1)^4}\right]\right\} = \frac{d}{dt}\left\{\frac{d}{dt}\left[\mathrm{L}^{-1}\left(\frac{1}{(s-1)^4}\right)\right]\right\}$$

$$= \frac{d}{dt}\left\{\frac{d}{dt}\left[e^t \mathrm{L}^{-1}\left(\frac{1}{s^4}\right)\right]\right\} = \frac{d}{dt}\left[\frac{d}{dt}\left(\frac{1}{6}t^3 e^t\right)\right]$$

$$= \frac{1}{6}\frac{d}{dt}[t^3 e^t + e^t(3t^2)] = \frac{1}{6}\frac{d}{dt}[t^3 e^t + 3t^2 e^t]$$

$$= \frac{1}{6}[t^3 e^t + 3t^2 e^t + 3t^2 e^t + 6te^t]$$

$$= te^t\left(1 + t + \frac{t^2}{6}\right)$$

(j) Given $$X(s) = \frac{1}{s(s + 2)^3}$$

$$x(t) = \mathrm{L}^{-1}[X(s)] = \mathrm{L}^{-1}\left[\frac{1}{s(s + 2)^3}\right]$$

We know that s in the denominator corresponds to integration

$$\therefore \quad \mathrm{L}^{-1}\left[\frac{1}{s(s+2)^3}\right] = \int_0^t \mathrm{L}^{-1}\left[\frac{1}{(s+2)^3}\right] = \int_0^t e^{-2t}\,\mathrm{L}^{-1}\left(\frac{1}{s^3}\right)$$

$$= \int_0^t e^{-2t}\frac{t^2}{2}\,dt = \frac{1}{2}\left\{\left[\frac{t^2 e^{-2t}}{-2}\right]_0^t - \int_0^t \frac{e^{-2t}}{-2} 2t\,dt\right\}$$

$$= \frac{1}{2}\left\{\frac{t^2 e^{-2t}}{-2} + \left[t\frac{e^{-2t}}{-2}\right]_0^t - \int_0^t \frac{e^{-2t}}{-2}\,dt\right\}$$

$$= \frac{1}{4}[(-t^2 e^{-2t} - te^{-2t})] + \left[\frac{e^{-2t}}{-2}\right]_0^t = \frac{1}{4}\left(-t^2 e^{-2t} - te^{-2t} - \frac{1}{2}e^{-2t} + \frac{1}{2}\right)$$

$$= \frac{1}{8}\left(-2t^2 e^{-2t} - 2te^{-2t} - e^{-2t} + 1\right)$$

(k) Given $$X(s) = \frac{2s+1}{s+2}$$

$$= \frac{2s+4-3}{s+2} = \frac{2(s+2)}{s+2} - \frac{3}{s+2} = 2 - \frac{3}{s+2}$$

Taking inverse Laplace transform on both sides, we have

$$x(t) = 2\delta(t) - 3e^{-2t}\,u(t)$$

EXAMPLE 9.30 Find the inverse Laplace transform of the following:

(a) $X(s) = \dfrac{s+1}{(s+1)^2+4}$; $\text{Re}(s) > -1$

(b) $X(s) = \dfrac{s^3+1}{s(s+1)(s+2)}$

(c) $X(s) = \dfrac{s-1}{(s+1)(s^2+2s+5)}$

(d) $X(s) = \dfrac{3s^2+22s+27}{(s^2+3s+2)(s^2+2s+5)}$

(e) $X(s) = \dfrac{1+e^{-2s}}{3s^2+2s}$

Solution:

(a) Given $$X(s) = \frac{s+1}{(s+1)^2+4};\ \text{Re}(s) > -1$$

$$\text{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] = e^{-t}\text{L}^{-1}\left(\frac{s}{s^2+2^2}\right) = e^{-t}\cos 2t\,u(t)$$

(b) Given $$X(s) = \frac{s^3+1}{s(s+1)(s+2)}$$

$$\text{L}^{-1}\left[\frac{s^3+1}{s(s+1)(s+2)}\right] = \text{L}^{-1}\left(\frac{s^3+1}{s^3+3s^2+2s}\right)$$

Since the order of numerator and denominator are equal, partial fractions cannot be obtained directly.

$\therefore$

$$s^3+3s^2+2s\,)\,s^3+1\,(\,1$$
$$\frac{s^3+3s^2+2s}{-3s^2-2s+1}$$

$\therefore$
$$X(s)=\frac{s^3+1}{s^3+3s^2+2s}=1-\frac{3s^2+2s-1}{s^3+3s^2+2s}$$

$$\frac{3s^2+2s-1}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}=\frac{-1/2}{s}+\frac{7/2}{s+2}$$

where
$$A=\left.\frac{3s^2+2s-1}{(s+1)(s+2)}\right|_{s=0}=-\frac{1}{2}$$

$$B=\left.\frac{3s^2+2s-1}{s(s+2)}\right|_{s=-1}=\frac{3-2-1}{-1(-1+2)}=0$$

$$C=\left.\frac{3s^2+2s-1}{s(s+1)}\right|_{s=-2}=\frac{12-4-1}{2}=\frac{7}{2}$$

$\therefore$
$$X(s)=1-\left(\frac{-1/2}{s}+\frac{7/2}{s+2}\right)=1+\frac{1}{2}\frac{1}{s}-\frac{7}{2}\frac{1}{s+2}$$

$\therefore$
$$x(t)=\delta(t)+\frac{1}{2}u(t)-\frac{7}{2}e^{-2t}\,u(t)$$

(c) Given
$$X(s)=\frac{s-1}{(s+1)(s^2+2s+5)}$$

$$\frac{s-1}{(s+1)(s^2+2s+5)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+2s+5}$$

$\therefore$
$$X(s)=\frac{-1/2}{s+1}+\frac{(1/2)s+(3/2)}{s^2+2s+5}$$
$$=-\frac{1}{2}\frac{1}{s+1}+\frac{(1/2)\,(s+1+2)}{(s+1)^2+2^2}$$
$$=-\frac{1}{2}\frac{1}{s+1}+\frac{1}{2}\frac{s+1}{(s+1)^2+2^2}+\frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

$$x(t)=-\frac{1}{2}e^{-t}\,u(t)+\frac{1}{2}e^{-t}\cos 2t\,u(t)+\frac{1}{2}e^{-t}\sin 2t\,u(t)$$

(d) Given $$X(s) = \frac{3s^2 + 22s + 27}{(s^2 + 3s + 2)(s^2 + 2s + 5)}$$

$$X(s) = \frac{3s^2 + 22s + 27}{(s+1)(s+2)(s^2 + 2s + 5)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs + D}{s^2 + 2s + 5}$$

$\therefore$ $$X(s) = 2\frac{1}{s+1} + \frac{13}{5}\frac{1}{s+2} - \frac{79}{15}\frac{s}{(s+1)^2 + 2^2} - \frac{19}{3}\frac{1}{(s+1)^2 + 2^2}$$

$$= 2\frac{1}{s+1} + \frac{13}{5}\frac{1}{s+2} - \frac{79}{15}\frac{s+1}{(s+1)^2 + 2^2} - \frac{8}{15}\frac{2}{(s+1)^2 + 2^2}$$

$\therefore$ $$x(t) = 2e^{-t}u(t) + \frac{13}{5}e^{-2t}u(t) - \frac{79}{15}e^{-t}\cos 2t\, u(t) - \frac{8}{15}e^{-t}\sin 2t\, u(t)$$

(e) Given $$X(s) = \frac{1 + e^{-2s}}{3s^2 + 2s}$$

$$L^{-1}\left(\frac{1}{3s^2 + 2s}\right) = L^{-1}\left\{\frac{1}{3s\,[s + (2/3)]}\right\} = L^{-1}\left\{-\frac{1}{s} + \frac{1}{[s + (2/3)]}\right\} = e^{-(2/3)t}u(t) - u(t)$$

$$L^{-1}\left(\frac{e^{-2s}}{3s^2 + 2s}\right) = L^{-1}\left(\frac{1}{3s^2 + 2s}\right)_{t=t-2} = e^{-(2/3)(t-2)}u(t-2) - u(t-2)$$

$\therefore$ $$x(t) = e^{-(2/3)t}u(t) - u(t) + e^{-(2/3)(t-2)}u(t-2) - u(t-2)$$

EXAMPLE 9.31 Find the inverse Laplace transform for the following:

(a) $\log\left(\frac{1+s}{s^2}\right)$

(b) $\log\left(1 + \frac{\omega^2}{s^2}\right)$

(c) $\log\left[\frac{s(s+1)}{s^2+1}\right]$

(d) $\log\left(\frac{s^2 + a^2}{s^2 - b^2}\right)$

(e) $\ln\left(\frac{s+a}{s+b}\right)$

Solution:

(a) Given $$X(s) = \log\left(\frac{1+s}{s^2}\right)$$

Let $$L^{-1}[X(s)] = L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right] = x(t)$$

$\therefore \qquad L[x(t)] = \log\left(\dfrac{1+s}{s^2}\right)$

$$L[t\,x(t)] = \frac{-d}{ds}\left[\log\left(\frac{1+s}{s^2}\right)\right]$$

$$= \frac{-d}{ds}[\log(1+s) - \log s^2]$$

$$= \frac{-1}{1+s} + \frac{1}{s^2}2s$$

$$L[tx(t)] = \frac{2}{s} - \frac{1}{s+1}$$

$$tx(t) = L^{-1}\left(\frac{2}{s} - \frac{1}{s+1}\right)$$

$\therefore \qquad = 2L^{-1}\left(\dfrac{1}{s}\right) - L^{-1}\left(\dfrac{1}{s+1}\right) = (2)(1) - e^{-t}$

$\therefore \qquad x(t) = \dfrac{2 - e^{-t}}{t}$

(b) Given $\qquad X(s) = \log\left(1 + \dfrac{\omega^2}{s^2}\right)$

Let $\qquad x(t) = L^{-1}[X(s)] = L^{-1}\left[\log\left(1 + \dfrac{\omega^2}{s^2}\right)\right]$

$\therefore \qquad L[x(t)] = \log\left(1 + \dfrac{\omega^2}{s^2}\right)$

$$= \log\left(\frac{s^2 + \omega^2}{s^2}\right)$$

$$= \log(s^2 + \omega^2) - \log s^2$$

$\therefore \qquad L[t\,x(t)] = \dfrac{-d}{ds}[\log(s^2 + \omega^2) - \log s^2]$

$$= \frac{-1}{s^2 + \omega^2}2s + \frac{1}{s^2}2s$$

$$= \frac{2}{s} - \frac{2s}{s^2 + \omega^2}$$

$$\therefore \qquad t\,x(t) = L^{-1}\left(\frac{2}{s} - \frac{2s}{s^2+\omega^2}\right)$$

$$= L^{-1}\left(\frac{2}{s} - \frac{2s}{s^2+\omega^2}\right)$$

$$= (2)\,(1) - 2\cos\omega t$$

$$= 2\,(1 - \cos\omega t)$$

$$\therefore \qquad x(t) = \frac{2\,(1-\cos\omega t)}{t}$$

(c) Given $\qquad X(s) = \log\left[\frac{s(s+1)}{s^2+1}\right]$

Let $\qquad x(t) = L^{-1}[X(s)]$

$$= L^{-1}\left[\log\frac{s\,(s+1)}{s^2+1}\right]$$

$$\therefore \qquad L[x(t)] = \log\left[\frac{s(s+1)}{s^2+1}\right]$$

$$= \log s\,(s+1) - \log\,(s^2+1)$$

$$= \log s + \log\,(s+1) - \log\,(s^2+1)$$

$$L\,[t\,x(t)] = -\frac{d}{ds}[\log s + \log\,(s+1) - \log(s^2+1)]$$

$$= \frac{-1}{s} - \frac{1}{s+1} + \frac{2s}{s^2+1}$$

$$t\,x(t) = L^{-1}\left(\frac{-1}{s} - \frac{1}{s+1} + \frac{2s}{s^2+1}\right)$$

$$= -L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + 2L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= -1 - e^{-t} + 2\cos t$$

$$\therefore \qquad x(t) = \frac{2\cos t - e^{-t} - 1}{t}$$

(d) Given $$X(s) = \log\left(\frac{s^2 + a^2}{s^2 - b^2}\right)$$

Let $$x(t) = L^{-1}[X(s)] = L^{-1}\left[\log\left(\frac{s^2 + a^2}{s^2 - b^2}\right)\right]$$

$$L[x(t)] = \log\left(\frac{s^2 + a^2}{s^2 - b^2}\right) = \log(s^2 + a^2) - \log(s^2 - b^2)$$

$$L[t\,x(t)] = \frac{-d}{ds}[\log(s^2 + a^2) - \log(s^2 - b^2)]$$

$$= \frac{-1}{s^2 + a^2}2s + \frac{1}{s^2 - b^2}2s$$

$$= \frac{2s}{s^2 - b^2} - \frac{2s}{s^2 + a^2}$$

$\therefore$ $$t\,x(t) = L^{-1}\left(\frac{2s}{s^2 - b^2} - \frac{2s}{s^2 + a^2}\right)$$

$$= 2L^{-1}\left(\frac{s}{s^2 - b^2}\right) - 2L^{-1}\left(\frac{s}{s^2 + a^2}\right)$$

$$= 2\cosh bt - 2\cos at$$

$\therefore$ $$x(t) = \frac{2}{t}(\cosh bt - \cos at)$$

(e) Given $$X(s) = \ln\left(\frac{s + a}{s + b}\right)$$

Let $$x(t) = L^{-1}[X(s)] = L^{-1}\left[\ln\left(\frac{s + a}{s + b}\right)\right]$$

$$L[x(t)] = \ln\left(\frac{s + a}{s + b}\right)$$

$$= \ln(s + a) - \ln(s + b)$$

$$L[tx(t)] = \frac{-d}{ds}[\ln(s + a) - \ln(s + b)]$$

$$= \frac{-1}{s + a} + \frac{1}{s + b} = \frac{1}{s + b} - \frac{1}{s + a}$$

$$\therefore \qquad tx(t) = \text{L}^{-1}\left(\frac{1}{s+b} - \frac{1}{s+a}\right)$$

$$= \text{L}^{-1}\left(\frac{1}{s+b}\right) - \text{L}^{-1}\left(\frac{1}{s+a}\right)$$

$$= e^{-bt} - e^{-at}$$

$$\therefore \qquad x(t) = \frac{e^{-bt} - e^{-at}}{t}$$

EXAMPLE 9.32 Find the inverse Laplace transform of the following functions:

(a) $X(s) = \dfrac{s+4}{s^2+5s+6}$ (b) $X(s) = \dfrac{s^2+3s+4}{s^3+5s^2+7s+3}$

(c) $X(s) = \dfrac{s+1}{s^3+4s^2+6s+4}$ (d) $X(s) = \dfrac{2s-1}{s^2+4s+8}$

(e) $X(s) = \dfrac{1+e^{-2s}}{s^2(s+1)}$ (f) $X(s) = \dfrac{s+2}{s^2(s+1)^2}$

(g) $X(s) = \dfrac{s+3}{s^2+10s+41}$ (h) $X(s) = \dfrac{s^2}{s^4+4a^4}$

Solution:

(a)
$$X(s) = \frac{s+4}{s^2+5s+6} = \frac{s+4}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$\therefore \qquad A = (s+2)\,X(s)\big|_{s=-2} = \left.\frac{s+4}{s+3}\right|_{s=-2} = \frac{-2+4}{-2+3} = 2$$

$$B = (s+3)\,X(s)\big|_{s=-3} = \left.\frac{s+4}{s+2}\right|_{s=-3} = \frac{-3+4}{-3+2} = -1$$

$$\therefore \qquad X(s) = \frac{2}{(s+2)} - \frac{1}{s+3}$$

Taking inverse Laplace transform on both sides, we get

$$x(t) = 2e^{-2t}\,u(t) - e^{-3t}\,u(t)$$

(b)
$$X(s) = \frac{s^2+3s+4}{s^3+5s^2+7s+3} = \frac{s^2+3s+4}{(s+3)(s+1)^2} = \frac{A}{s+3} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

$$\therefore \qquad A = (s+3)\,X(s)\big|_{s=-3} = \left.\frac{s^2+3s+4}{(s+1)^2}\right|_{s=-3} = \frac{9-9+4}{(-3+1)^2} = 1$$

$$B = \frac{1}{1!}\frac{d}{ds}[(s+1)^2\, X(s)]\Big|_{s=-1} = \frac{d}{ds}\left[\frac{s^2+3s+4}{s+3}\right]\Bigg|_{s=-1}$$

$$= \frac{(s+3)\,(2s+3)-(s^2+3s+4)(1)}{(s+3)^2}\Bigg|_{s=-1} = 0$$

$$C = (s+1)^2\, X(s)\Big|_{s=-1} = \frac{s^2+3s+4}{s+3}\Bigg|_{s=-1} = \frac{1-3+4}{-1+3} = 1$$

$\therefore$
$$X(s) = \frac{1}{s+3} + \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform on both sides, we get

$$x(t) = e^{-3t}\,u(t) + te^{-t}\,u(t)$$

(c)
$$X(s) = \frac{s+1}{s^3+4s^2+6s+4} = \frac{s+1}{(s+2)(s^2+2s+2)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+2}$$

$$= \frac{A(s^2+2s+2)+(Bs+C)\,(s+2)}{(s+2)(s^2+2s+2)}$$

$$= \frac{(A+B)\,s^2+(2A+2B+C)\,s+2A+2C}{(s+2)(s^2+2s+2)}$$

Comparing the numerators of LHS and RHS, we get

$$A + B = 0$$

$\therefore$ $$A = -B$$

$$2A + 2B + C = 1$$

$\therefore$ $$C = 1$$

$$2A + 2C = 1$$

$\therefore$ $$2A = 1 - 2 = -1 \quad \text{or} \quad A = -1/2$$

$\therefore$ $$B = 1/2$$

$\therefore$
$$X(s) = \frac{-1/2}{s+2} + \frac{(1/2)\,s+1}{s^2+2s+2} = \frac{-(1/2)}{s+2} + \frac{(1/2)\,(s+1)+(1/2)}{(s+1)^2+(1)^2}$$

$$= \frac{-1/2}{s+2} + \frac{1}{2}\,\frac{(s+1)}{(s+1)^2+(1)^2} + \frac{1}{2}\,\frac{1}{(s+1)^2+(1)^2}$$

Taking inverse Laplace transform on both sides, we get

$$x(t) = -\frac{1}{2}e^{-2t}\,u(t) + \frac{1}{2}e^{-t}\cos t\,u(t) + \frac{1}{2}e^{-t}\sin t\,u(t)$$

(d) $$X(s)=\frac{2s-1}{s^2+4s+8}=\frac{2s-1}{(s+2)^2+(2)^2}$$

$$\frac{2(s+2)-5}{(s+2)^2+(2)^2}=\frac{2(s+2)}{(s+2)^2+(2)^2}-\frac{5}{2}\frac{2}{(s+2)^2+(2)^2}$$

Taking inverse Laplace transform on both sides, we get

$$x(t)=2e^{-2t}\cos 2t\,u(t)-\frac{5}{2}e^{-2t}\sin 2t\,u(t)$$

(e) $$X(s)=\frac{1+e^{-2s}}{s^2(s+1)}=\frac{1}{s^2(s+1)}+\frac{e^{-2s}}{s^2(s+1)}$$

First we will find the inverse Laplace transform of $1/s^2(s+1)$.

Let $$X_1(s)=\frac{1}{s^2(s+1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+1}$$

$$C=(s+1)\,X_1(s)\Big|_{s=-1}=(s+1)\frac{1}{s^2(s+1)}\Bigg|_{s=-1}=\frac{1}{s^2}\Bigg|_{s=-1}=1$$

$$B=s^2X_1(s)\Big|_{s=0}=s^2\frac{1}{s^2(s+1)}\Bigg|_{s=0}=\frac{1}{s+1}\Bigg|_{s=0}=1$$

$$A=\frac{1}{1!}\frac{d}{ds}[s^2X_1(s)]\Big|_{s=0}=\frac{d}{ds}\left[s^2\frac{1}{s^2(s+1)}\right]\Bigg|_{s=0}$$

$$=\frac{(s+1)\,0-1(1)}{(s+1)^2}\Bigg|_{s=0}=-1$$

$\therefore$ $$X_1(s)=-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+1}$$

Taking inverse Laplace transform on both sides, we get

$$x_1(t)=-u(t)+tu(t)+e^{-t}u(t)=[t-1+e^{-t}]u(t)$$

Now, from the time shift theorem, we know that

$$\mathrm{L}^{-1}\left[\frac{e^{-2s}}{s^2(s+1)}\right]=\mathrm{L}^{-1}\left[\frac{1}{s^2(s+1)}\right]\Bigg|_{t\to(t-2)}=x_1(t)\Big|_{t\to t-2}=[t-1+e^{-t}]\,u(t)\Big|_{t\to(t-2)}$$

$$=[(t-2)-1+e^{-(t-2)}]\,u(t-2)$$

$$=[(t-3)+e^{-(t-2)}]\,u(t-2)$$

$\therefore$ $$x(t)=[t-1+e^{-t}]\,u(t)+[(t-3)+e^{-(t-2)}]\,u(t-2)$$

(f) $$X(s)=\frac{s+2}{s^2(s+1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+1}+\frac{D}{(s+1)^2}$$

$$=\frac{As\,(s+1)^2+B(s+1)^2+Cs^2\,(s+1)+Ds^2}{s^2(s+1)^2}$$

$$=\frac{(A+C)s^3+(2A+B+C+D)s^2+(A+2B)s+B}{s^2(s+1)^2}$$

Comparing the numerators of LHS and RHS, we get

$$B=2$$

$$A+2B=1 \quad \text{or} \quad A=1-2B=1-4=-3$$

$$A+C=0$$

$\therefore$ $$A=-C$$

$\therefore$ $$C=3$$

$$2A+B+C+D=0$$

i.e. $$A+B+D=0$$

i.e. $$-3+2+D=0$$

i.e. $$D=1$$

$\therefore$ $$X(s)=\frac{-3}{s}+\frac{2}{s^2}+\frac{3}{s+1}+\frac{1}{(s+1)^2}$$

Taking inverse Laplace transform on both sides, we get

$$x(t)=-3u(t)+2t\,u(t)+3e^{-t}u(t)+te^{-t}u(t)$$

(g) Given $$X(s)=\frac{s+3}{s^2+10s+41}=\frac{s+3}{(s+5)^2+(4)^2}=\frac{s+5-2}{(s+5)^2+(4)^2}$$

$$=\frac{s+5}{(s+5)^2+(4)^2}-\frac{2}{(s+5)^2+(4)^2}=\frac{s+5}{(s+5)^2+(4)^2}-\frac{1}{2}\frac{4}{(s+5)^2+(4)^2}$$

Taking inverse Laplace transform on both sides, we have

$$x(t)=e^{-5t}\cos 4t\,u(t)-\frac{1}{2}e^{-5t}\sin 4t\,u(t)$$

(h) Given $$X(s)=\frac{s^2}{s^4+4a^4}=\frac{s^2}{(s^2-2as+2a^2)(s^2+2as+2a^2)}$$

Taking partial fractions, we have

$$X(s)=\frac{As+B}{s^2-2as+2a^2}+\frac{Cs+D}{s^2+2as+2a^2}$$

$$=\frac{(1/4a)s}{s^2-2as+2a^2}-\frac{(1/4a)s}{s^2+2as+2a^2}$$

$$\therefore \quad x(t) = \mathrm{L}^{-1}[X(s)] = \mathrm{L}^{-1}\left[\frac{(1/4a)s}{s^2 - 2as + 2a^2} - \frac{(1/4a)s}{s^2 + 2as + 2a^2}\right]$$

$$= \frac{1}{4a}\left[\mathrm{L}^{-1}\left(\frac{s}{s^2 - 2as + 2a^2}\right) - \mathrm{L}^{-1}\left(\frac{s}{s^2 + 2as + 2a^2}\right)\right]$$

$$= \frac{1}{4a}\left\{\mathrm{L}^{-1}\left[\frac{(s-a)+a}{(s-a)^2 + a^2}\right] - \mathrm{L}^{-1}\left[\frac{(s+a)-a}{(s+a)^2 + a^2}\right]\right\}$$

$$= \frac{1}{4a}\left\{\begin{array}{l}\mathrm{L}^{-1}\left[\dfrac{s-a}{(s-a)^2 + a^2}\right] + \mathrm{L}^{-1}\left[\dfrac{a}{(s-a)^2 + a^2}\right] - \mathrm{L}^{-1}\left[\dfrac{s+a}{(s+a)^2 + a^2}\right] \\ + \mathrm{L}^{-1}\left[\dfrac{a}{(s+a)^2 + a^2}\right]\end{array}\right\}$$

$$= \frac{1}{4a}\left\{\begin{array}{l}e^{at}\mathrm{L}^{-1}\left[\dfrac{s}{(s^2 + a^2)}\right] + e^{at}\mathrm{L}^{-1}\left[\dfrac{a}{(s^2 + a^2)}\right] - e^{-at}\mathrm{L}^{-1}\left[\dfrac{s}{(s^2 + a^2)}\right] \\ + e^{-at}\mathrm{L}^{-1}\left[\dfrac{a}{(s^2 + a^2)}\right]\end{array}\right\}$$

$$= \frac{1}{4a}[e^{at}\cos at + e^{at}\sin at - e^{-at}\cos at + e^{-at}\sin at]\,u(t)$$

$$= \frac{1}{4a}[\cos at(e^{at} - e^{-at}) + \sin at(e^{at} + e^{-at})]\,u(t)$$

$$= \frac{1}{4a}[2\cos at\sinh at + 2\sin at\cosh at]\,u(t)$$

$$= \frac{1}{2a}[\cos at\sinh at + \sin at\cosh at]\,u(t)$$

EXAMPLE 9.33 Given $X(s) = \dfrac{s^2 + 5s + 1}{s^2 + 3s + 2}$, find $x(t)$ for $t \geq 0$.

Solution: Here the order of numerator and denominator are same. So it is an improper function. Therefore, we cannot directly take the partial fractions. A constant term which corresponds to an impulse function is to be removed by dividing the numerator with the denominator.

$$\begin{array}{r}s^2 + 3s + 2\,)\,s^2 + 5s + 1\,(\,1 \\ \underline{s^2 + 3s + 2\quad\;\;} \\ 2s - 1\quad\;\;\end{array}$$

$$\therefore \qquad X(s) = \frac{s^2+5s+1}{s^2+3s+2} = 1 + \frac{2s-1}{s^2+3s+2} = 1 + \frac{2s-1}{(s+1)(s+2)}$$

Now, $\dfrac{2s-1}{s^2+3s+2}$ is a proper function and so we can take the partial fractions:

$$\frac{2s-1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{-3}{s+1} + \frac{5}{s+2}$$

$$\therefore \qquad X(s) = 1 - 3\frac{1}{s+1} + 5\frac{1}{s+2}$$

Taking inverse Laplace transform on both sides, we have

$$x(t) = \delta(t) - 3e^{-t}u(t) + 5e^{-2t}u(t)$$

EXAMPLE 9.34 Given $X(s) = \dfrac{s^3+5s^2+13s+8}{s^2+4s+8}$, find $x(t)$ for $t \geq 0$.

Solution: Here $X(s)$ is an improper function as the degree of the numerator polynomial is greater than the degree of the denominator polynomial. So we cannot take partial fractions. Hence first divide the numerator by the denominator to obtain a proper function.

$$
\begin{array}{r}
s^2+4s+8 \,)\, s^3+5s^2+13s+8 \,(\, s+1 \\
\underline{s^3+4s^2+8s} \qquad \\
s^2+5s+8 \\
\underline{s^2+4s+8} \\
s \qquad
\end{array}
$$

$$\therefore \qquad X(s) = \frac{s^3+5s^2+13s+8}{s^2+4s+8} = (s+1) + \frac{s}{s^2+4s+8}$$

$$= (s+1) + \frac{(s+2)-2}{(s+2)^2+2^2} = (s+1) + \frac{s+2}{(s+2)^2+2^2} - \frac{2}{(s+2)^2+2^2}$$

Taking inverse Laplace transform, we get

$$x(t) = \delta'(t) + \delta(t) + e^{-2t}\cos 2t\, u(t) - e^{-2t}\sin 2t\, u(t)$$

where $\delta'(t)$ is the first derivative of $\delta(t)$ with respect to time.

EXAMPLE 9.35 Determine the function of time $x(t)$ for each of the following Laplace transforms with their associated regions of convergence:

(a) $X(s) = \dfrac{s^2-s+1}{(s+1)^2}$; $\text{Re}(s) > -1$ (b) $X(s) = \dfrac{(s+1)^2}{s^2-s+1}$; $\text{Re}(s) > \dfrac{1}{2}$

Solution:

(a) Given $$X(s) = \frac{s^2 - s + 1}{(s+1)^2} = \frac{s^2 - s + 1}{s^2 + 2s + 1}; \ \text{Re}(s) > -1$$

Here the order of the numerator and the denominator are same. So the given transfer function is an improper transfer function. First write it as a constant plus a proper transfer function.

$$
\begin{array}{r}
s^2 + 2s + 1 \,)\, s^2 - s + 1 \,(\, 1 \\
\underline{s^2 + 2s + 1} \\
-3s
\end{array}
$$

$$\therefore \quad X(s) = \frac{s^2 - s + 1}{(s+1)^2} = 1 - \frac{3s}{(s+1)^2} = 1 - \frac{3s+3}{(s+1)^2} + \frac{3}{(s+1)^2}$$

Since ROC is Re(s) > –1, the signals must be causal signals. Taking inverse Laplace transform on both sides, we have

$$x(t) = \text{L}^{-1}\left[1 - \frac{3}{s+1} + \frac{3}{(s+1)^2}\right] = \delta(t) - 3e^{-t}u(t) + 3te^{-t}u(t)$$

(b) Given $$X(s) = \frac{(s+1)^2}{s^2 - s + 1} = \frac{s^2 + 2s + 1}{s^2 - s + 1}; \ \text{Re}(s) > \frac{1}{2}$$

In the given $X(s)$, the order of numerator and denominator are same. So it is an improper function. Therefore, first write it as a constant term plus a proper transfer function.

$$
\begin{array}{r}
s^2 - s + 1 \,)\, s^2 + 2s + 1 \,(\, 1 \\
\underline{s^2 - s + 1} \\
3s
\end{array}
$$

$$\therefore \quad X(s) = \frac{(s+1)^2}{s^2 - s + 1} = 1 + 3\frac{s}{s^2 - s + 1} = 1 + 3\frac{s}{[s - (1/2)^2] + (\sqrt{3}/2)^2}$$

$$= 1 + \frac{3\,[s - (1/2)] + (3/2)}{[s - (1/2)^2 + (\sqrt{3}/2)^2}$$

$$= 1 + \frac{3[s - (1/2)]}{[s - (1/2)]^2 + (\sqrt{3}/2)^2} + \sqrt{3}\,\frac{(\sqrt{3}/2)}{[s - (1/2)]^2 + (\sqrt{3}/2)^2}$$

Since ROC is Re(s) > 1/2, the signals must be causal signals. Taking inverse Laplace transform on both sides, we have

$$\therefore \quad x(t) = \delta(t) + 3e^{(1/2)t}\cos\frac{\sqrt{3}}{2}t + \sqrt{3}e^{(1/2)t}\sin\left(\frac{\sqrt{3}}{2}\right)t$$

EXAMPLE 9.36 Find the inverse Laplace transform of the following:

(a) $X(s)=\dfrac{s^2+6s+7}{s^2+3s+2}$; $\text{Re}(s)>-1$ (b) $X(s)=\dfrac{s^3+2s^2+6}{s^2+3s}$; $\text{Re}(s)>0$

Solution:

(a) Given $$X(s)=\frac{s^2+6s+7}{s^2+3s+2};\ \text{Re}(s)>-1$$

The given transfer function is an improper function. So divide the numerator by the denominator, remove a constant term and make it proper.

$$\begin{array}{r} s^2+3s+2\ )\ s^2+6s+7\ (\ 1 \\ \underline{s^2+3s+2} \\ 3s+5 \end{array}$$

$\therefore$ $$X(s)=1+\frac{3s+5}{s^2+3s+2}$$

$$\frac{3s+5}{s^2+3s+2}=\frac{3s+5}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}=\frac{2}{s+1}+\frac{1}{s+2}$$

$\therefore$ $$X(s)=1+\frac{2}{s+1}+\frac{1}{s+2}$$

Since ROC is Re(s) > –1, both the terms must be causal. Taking inverse Laplace transform on both sides, we have

$$x(t)=\delta(t)+2e^{-t}\,u(t)+e^{-2t}\,u(t)$$

(b) Given $$X(s)=\frac{s^3+2s^2+6}{s^2+3s};\ \text{Re}(s)>0$$

The given $X(s)$ is an improper transfer function. So divide the numerator by the denominator, remove some terms and make the remaining transfer function proper.

$$\begin{array}{r} s^2+3s\ )\ s^3+2s^2+6\ (\ s-1 \\ \underline{s^3+3s^2\quad\ } \\ -s^2+6 \\ \underline{-s^2-3s} \\ 3s+6 \end{array}$$

$\therefore$ $$\frac{s^3+2s^2+6}{s^2+3s}=s-1+\frac{3s+6}{s^2+3s}=(s-1)+\frac{3s+6}{s(s+3)}$$

$$\frac{3s+6}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{2}{s} + \frac{1}{s+3}$$

$\therefore$
$$X(s) = s - 1 + \frac{2}{s} + \frac{1}{s+3}$$

Since ROC is Re(s) > 0, both the terms must be causal. So taking inverse Laplace transform, we have

$$x(t) = \delta'(t) - \delta(t) + 2u(t) + e^{-3t}u(t)$$

EXAMPLE 9.37 Find the signal $x(t)$ that corresponds to the Laplace transform

$$X(s) = \frac{3s^2 + 22s + 27}{(s^2 + 3s + 2)(s^2 + 2s + 5)}$$

Solution: Given $X(s) = \dfrac{3s^2 + 22s + 27}{(s^2 + 3s + 2)(s^2 + 2s + 5)} = \dfrac{3s^2 + 22s + 27}{(s+1)(s+2)(s^2 + 2s + 5)}$

Taking partial fractions, we have

$$X(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs + D}{s^2 + 2s + 5} = \frac{2}{s+1} + \frac{1}{s+2} + \frac{-3s+1}{s^2 + 2s + 5}$$

$$= \frac{2}{s+1} + \frac{1}{s+2} - \frac{3(s+1)}{(s+1)^2 + 2^2} + 2\frac{2}{(s+1)^2 + 2^2}$$

Taking inverse Laplace transform on both sides, we have

$$x(t) = 2e^{-t}u(t) + e^{-2t}u(t) - 3e^{-t}\cos 2t\,u(t) + 2e^{-t}\sin 2t\,u(t)$$

EXAMPLE 9.38 Given $x(t) = e^{-t}u(t)$. Find the inverse Laplace transform of $e^{-3s}X(2s)$.

Solution: Given
$$x(t) = e^{-t}u(t)$$

$$X(s) = \text{L}[x(t)] = \text{L}[e^{-t}u(t)] = \frac{1}{s+1}$$

$\therefore$
$$X(2s) = \frac{1}{2s+1} = \frac{1/2}{s + (1/2)}$$

$$\text{L}^{-1}[X(2s)] = \text{L}^{-1}\left[\frac{1/2}{s + (1/2)}\right] = \frac{1}{2}e^{-t/2}u(t)$$

$\therefore$
$$\text{L}^{-1}[e^{-3s}X(2s)] = \text{L}^{-1}[X(2s)]\Big|_{t=t-3} = \frac{1}{2}e^{-t/2}u(t)\Big|_{t=t-3} = \frac{1}{2}e^{-(t-3)/2}u(t-3)$$

$\therefore$
$$\text{L}^{-1}[e^{-3s}X(2s)] = \frac{1}{2}e^{-(t-3)/2}u(t-3) \text{ if } x(t) = e^{-t}u(t)$$

EXAMPLE 9.39 Using the convolution theorem of Laplace transforms, find $y(t) = x_1(t) * x_2(t)$, where $x_1(t)$ and $x_2(t)$ are as follows:

(a) $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = u(t-3)$

(b) $x_1(t) = \sin 3t\, u(t)$ and $x_2(t) = \cos 2t\, u(t)$

(c) $x_1(t) = t u(t)$ and $x_2(t) = u(t-2)$

(d) $x_1(t) = u(t-2)$ and $x_2(t) = u(t-2)$

(e) $x_1(t) = u(t)$ and $x_2(t) = u(t)$

(f) $x_1(t) = tu(t)$ and $x_2(t) = tu(t)$

Solution: Using the convolution property, we have

$$L[x_1(t) * x_2(t)] = X_1(s)\, X_2(s)$$

$$\therefore \quad x_1(t) * x_2(t) = L^{-1}[X_1(s)\, X_2(s)]$$

(a) $x_1(t) = e^{-2t}u(t)$

$$\therefore \quad X_1(s) = \frac{1}{s+2}$$

$x_2(t) = u(t-3)$

$$\therefore \quad X_2(s) = \frac{e^{-3s}}{s}$$

$$\therefore \quad L[x_1(t) * x_2(t)] = X_1(s)\, X_2(s) = \frac{1}{s+2}\,\frac{e^{-3s}}{s} = \frac{e^{-3s}}{s(s+2)}$$

$$\therefore \quad x_1(t) * x_2(t) = L^{-1}\left[\frac{e^{-3s}}{s(s+2)}\right] = L^{-1}\left[\frac{1}{s(s+2)}\right]\Bigg|_{t \to t-3}$$

Let $$Y_1(s) = \frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{1/2}{s} + \frac{-1/2}{s+2}$$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}\left[\frac{1}{s(s+2)}\right] = y_1(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t)$$

$$\therefore \quad x_1(t) * x_2(t) = y_1(t)\big|_{t=t-3} = \frac{1}{2}u(t-3) - \frac{1}{2}e^{-2(t-3)}u(t-3)$$

(b) $$x_1(t) = \sin 3t\, u(t)$$

$$\therefore \quad X_1(s) = \frac{3}{s^2 + (3)^2} = \frac{3}{s^2+9}$$

$$x_2(t) = \cos 2t\, u(t)$$

$\therefore \qquad X_2(s) = \dfrac{s}{s^2+(2)^2} = \dfrac{s}{s^2+4}$

$$L[x_1(t) * x_2(t)] = X_1(s)\, X_2(s) = \frac{3}{s^2+9}\,\frac{s}{s^2+4} = \frac{3s}{(s^2+4)(s^2+9)}$$

Let $\quad X_1(s)X_2(s) = \dfrac{3s}{(s^2+4)(s^2+9)} = \dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{s^2+9} = \dfrac{(3/5)s}{s^2+4} - \dfrac{(3/5)s}{s^2+9}$

$\therefore \qquad X_1(s)\, X_2(s) = \dfrac{3}{5}\,\dfrac{s}{s^2+4} - \dfrac{3}{5}\,\dfrac{s}{s^2+9} = \dfrac{3}{5}\,\dfrac{s}{s^2+(2)^2} - \dfrac{3}{5}\,\dfrac{s}{s^2+(3)^2}$

Taking inverse Laplace transform on both sides, we get

$$y(t) = L^{-1}[X_1(s)\, X_2(s)] = x_1(t) * x_2(t) = \frac{3}{5}\cos 2t\, u(t) - \frac{3}{5}\cos 3t\, u(t)$$

(c) $\qquad x_1(t) = t\, u(t)$

$\therefore \qquad X_1(s) = \dfrac{1}{s^2}$

$\qquad x_2(t) = u(t-2)$

$\therefore \qquad X_2(s) = \dfrac{e^{-2s}}{s}$

$$L[x_1(t) * x_2(t)] = X_1(s)\, X_2(s) = \frac{1}{s^2}\,\frac{e^{-2s}}{s} = \frac{e^{-2s}}{s^3}$$

$\therefore \qquad x_1(t) * x_2(t) = L^{-1}\left(\dfrac{e^{-2s}}{s^3}\right) = L^{-1}\left(\dfrac{1}{s^3}\right)\Bigg|_{t \to t-2} = \dfrac{t^2}{2}\Bigg|_{t \to t-2} = \dfrac{(t-2)^2}{2}$

(d) $\qquad x_1(t) = u(t-2)$

$\therefore \qquad X_1(s) = \dfrac{e^{-2s}}{s}$

$\qquad x_2(t) = u(t-2)$

$\therefore \qquad X_2(s) = \dfrac{e^{-2s}}{s}$

$$L\,[x_1(t) * x_2(t)] = X_1(s)\, X_2(s) = \frac{e^{-2s}}{s}\,\frac{e^{-2s}}{s} = \frac{e^{-4s}}{s^2}$$

$\therefore \qquad x_1(t) * x_2(t) = L^{-1}\left(\dfrac{e^{-4s}}{s^2}\right) = L^{-1}\left(\dfrac{1}{s^2}\right)\Bigg|_{t \to t-4} = t\Big|_{t \to t-4} = t-4$

(e) $$x_1(t) = u(t)$$

$$\therefore \quad X_1(s) = \frac{1}{s}$$

$$x_2(t) = u(t)$$

$$\therefore \quad X_2(s) = \frac{1}{s}$$

$$\mathrm{L}[x_1(t) * x_2(t)] = X_1(s)\, X_2(s) = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}$$

$$\therefore \quad x_1(t) * x_2(t) = \mathrm{L}^{-1}\,[X_1(s)\, X_2(s)] = \mathrm{L}^{-1}\left(\frac{1}{s^2}\right) = tu(t)$$

(f) $$x_1(t) = tu(t)$$

$$\therefore \quad X_1(s) = \frac{1}{s^2}$$

$$x_2(t) = tu(t)$$

$$\therefore \quad X_2(s) = \frac{1}{s^2}$$

$$\mathrm{L}\,[x_1(t) * x_2(t)] = X_1(s)\, X_2(s) = \frac{1}{s^2}\frac{1}{s^2} = \frac{1}{s^4}$$

$$\therefore \quad x_1(t) * x_2(t) = \mathrm{L}^{-1}\,[X_1(s)\, X_2(s)] = \mathrm{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{6}u(t)$$

EXAMPLE 9.40 If $x(t) = e^{-t}\,u(t)$ and $h(t) = e^{-3t}\,u(t)$, determine $y(t) = x(t) * h(t)$ by (a) time domain method and (b) frequency domain method.

Solution:

(a) ***Time domain method***

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_0^t e^{-\tau}\, e^{-3(t-\tau)}\, d\tau$$

$$= e^{-3t}\int_0^t e^{2\tau}\, d\tau = e^{-3t}\left[\frac{e^{2\tau}}{2}\right]_0^t = e^{-3t}\left(\frac{e^{2t}-1}{2}\right) = \frac{e^{-t}-e^{-3t}}{2}u(t)$$

(b) ***Frequency domain method***

Given $$x(t) = e^{-t}\,u(t)$$

$$\therefore \quad X(s) = \frac{1}{s+1}$$

and $$h(t) = e^{-3t}\, u(t)$$

$\therefore$ $$H(s) = \frac{1}{s+3}$$

$$Y(s) = X(s)\, H(s) = \frac{1}{s+1} \times \frac{1}{s+3} = \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+3}\right)$$

As per convolution property of Laplace transform,

$$y(t) = \mathrm{L}^{-1}\,[X(s)\, H(s)] = \mathrm{L}^{-1}\left[\frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+3}\right)\right] = x(t) * h(t)$$

$\therefore$ $$y(t) = x(t) * h(t) = \frac{1}{2}[e^{-t} - e^{-3t}]\, u(t)$$

EXAMPLE 9.41 Find the convolution of the signals $x_1(t) = e^{-2t}\, u(t)$ and $x_2(t) = e^{-4t}\, u(t)$ using the convolution property of Laplace transforms. Verify the result by time domain method.

Solution:

(a) Given $$x_1(t) = e^{-2t}\, u(t)$$

$\therefore$ $$X_1(s) = \frac{1}{s+2}$$

and $$x_2(t) = e^{-4t}\, u(t)$$

$\therefore$ $$X_2(s) = \frac{1}{s+4}$$

As per convolution property of Laplace transforms,

$$\mathrm{L}\,[x_1(t) * x_2(t)] = X_1(s)\, X_2(s)$$

$\therefore$ $$X_1(s)\, X_2(s) = \frac{1}{s+2}\,\frac{1}{s+4} = \frac{1}{2}\left(\frac{1}{s+2} - \frac{1}{s+4}\right) = \mathrm{L}[x_1(t) * x_2(t)]$$

$\therefore$ $$x_1(t) * x_2(t) = \mathrm{L}^{-1}[X_1(s)\, X_2(s)] = \mathrm{L}^{-1}\left[\frac{1}{2}\left(\frac{1}{s+2} - \frac{1}{s+4}\right)\right]$$

$$= \frac{1}{2}\left[\mathrm{L}^{-1}\left(\frac{1}{s+2}\right) - \mathrm{L}^{-1}\left(\frac{1}{s+4}\right)\right] = \frac{1}{2}[e^{-2t} - e^{-4t}]\, u(t)$$

(b) ***Time domain method***

Given $x_1(t) = e^{-2t}\,u(t)$ and $x_2(t) = e^{-4t}\,u(t)$

$$\therefore \quad x_1(t) * x_2(t) = \int_0^t x_1(\tau)\, x_2(t-\tau)\, d\tau = \int_0^t e^{-2\tau}\, e^{-4(t-\tau)}\, d\tau$$

$$= e^{-4t} \int_0^t e^{-2\tau}\, e^{4\tau}\, d\tau = e^{-4t} \int_0^t e^{2\tau}\, d\tau$$

$$= e^{-4t} \left[\frac{e^{2\tau}}{2}\right]_0^t = \frac{1}{2} e^{-4t}\, [e^{2t} - 1] = \frac{1}{2} [e^{-2t} - e^{-4t}]\, u(t)$$

Hence the result is verified.

EXAMPLE 9.42 Find the inverse Laplace transform of the functions:

(a) $X(s) = \dfrac{1}{s(s+1)}$ (b) $X(s) = \dfrac{1}{s^2(s+2)}$

using convolution property of Laplace transforms.

Solution:

(a) Given $X(s) = \dfrac{1}{s(s+1)}$

Let $X(s) = X_1(s)\, X_2(s)$

where $X_1(s) = \dfrac{1}{s}$ and $X_2(s) = \dfrac{1}{s+1}$

$$\therefore \quad x_1(t) = \mathrm{L}^{-1}[X_1(s)] = \mathrm{L}^{-1}\left(\frac{1}{s}\right) = u(t)$$

and
$$x_2(t) = \mathrm{L}^{-1}[X_2(s)] = \mathrm{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}\, u(t)$$

$$\therefore \quad \mathrm{L}^{-1}[X(s)] = \mathrm{L}^{-1}[X_1(s)\, X_2(s)] = \mathrm{L}^{-1}\left[\frac{1}{s(s+1)}\right]$$

$$= x_1(t) * x_2(t) = \int_0^t x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_0^t 1 e^{-(t-\tau)}\, d\tau = e^{-t} \int_0^t e^{\tau}\, d\tau = e^{-t}\, [e^{\tau}]_0^t = e^{-t}\, [e^t - 1] = 1 - e^{-t}$$

$$\therefore \quad \mathrm{L}^{-1}\left[\frac{1}{s(s+1)}\right] = (1 - e^{-t})\, u(t)$$

(b) Given $$X(s) = \frac{1}{s^2(s+2)}$$

Let $$X(s) = X_1(s)\, X_2(s)$$

where $$X_1(s) = \frac{1}{s^2} \quad \text{and} \quad X_2(s) = \frac{1}{s+2}$$

$\therefore$ $$x_1(t) = L^{-1}[X_1(s)] = L^{-1}\left(\frac{1}{s^2}\right) = tu(t)$$

and $$x_2(t) = L^{-1}[X_2(s)] = L^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}u(t)$$

Using convolution property of Laplace transforms, we have

$$L^{-1}[X(s)] = L^{-1}[X_1(s)\, X_2(s)] = L^{-1}\left[\frac{1}{s^2(s+2)}\right] = x_1(t) * x_2(t) = \int_0^t x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_0^t \tau e^{-2(t-\tau)}\, d\tau = e^{-2t}\int_0^t \tau e^{2\tau}\, d\tau = e^{-2t}\left\{\left[\tau \frac{e^{2\tau}}{2}\right]_0^t - \int_0^t \frac{e^{2\tau}}{2}\, d\tau\right\}$$

$$= e^{-2t}\left\{\frac{te^{2t}}{2} - \left[\frac{e^{2\tau}}{4}\right]_0^t\right\} = e^{-2t}\left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4}\right) = \left(\frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2t}\right)$$

$\therefore$ $$L^{-1}\left[\frac{1}{s^2(s+2)}\right] = \left(\frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2t}\right)u(t)$$

EXAMPLE 9.43 Using convolution theorem, find the inverse Laplace transform of

(a) $F(s) = \dfrac{s}{(s^2+\alpha^2)^2}$ (b) $X(s) = \dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Solution:

(a) Given $$F(s) = \frac{s}{(s^2+\alpha^2)^2}$$

Let $$F(s) = F_1(s)\, F_2(s)$$

where $$F_1(s) = \frac{s}{s^2+\alpha^2} \quad \text{and} \quad F_2(s) = \frac{1}{s^2+\alpha^2}$$

$\therefore$ $$f_1(t) = L^{-1}\left(\frac{s}{s^2+\alpha^2}\right) = \cos \alpha t\, u(t)$$

$$f_2(t) = \mathrm{L}^{-1}\left(\frac{1}{s^2+\alpha^2}\right) = \frac{1}{\alpha}\sin\alpha t\,u(t)$$

Using convolution theorem of Laplace transforms, we have

$$\mathrm{L}^{-1}[F(s)] = \mathrm{L}^{-1}[F_1(s)\,F_2(s)] = \mathrm{L}^{-1}\left[\frac{s}{(s^2+\alpha^2)^2}\right]$$

$$= f_1(t) * f_2(t) = \int_0^t f_1(\tau)\,f_2(t-\tau)\,d\tau$$

$$\therefore \quad f(t) = [\cos\alpha t] * \left(\frac{1}{\alpha}\sin\alpha t\right) = \int_0^t \cos\alpha\tau\,\frac{1}{\alpha}\sin\alpha(t-\tau)\,d\tau$$

$$= \frac{1}{\alpha}\int_0^t \left[\frac{\sin\alpha(t-\tau+\tau)+\sin\alpha(t-\tau-\tau)}{2}\right]d\tau$$

$$= \frac{1}{2\alpha}\int_0^t [\sin\alpha t + \sin\alpha(t-2\tau)]\,d\tau = \frac{1}{2\alpha}\left\{\sin\alpha t\left[\tau\right]_0^t + \left[\frac{-\cos\alpha(t-2\tau)}{-2\alpha}\right]_0^t\right\}$$

$$= \frac{1}{2\alpha}\left(t\sin\alpha t + \frac{\cos\alpha t}{2\alpha} - \frac{\cos\alpha t}{2\alpha}\right)$$

$$\therefore \quad x(t) = \frac{1}{2\alpha}\,t\sin\alpha t$$

(b) Given
$$X(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Let
$$X(s) = X_1(s)\,X_2(s) = \left(\frac{s}{s^2+a^2}\right)\left(\frac{s}{s^2+b^2}\right)$$

$$\therefore \quad x_1(t) = \mathrm{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$x_2(t) = \mathrm{L}^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$$

$$\therefore \quad x(t) = \mathrm{L}^{-1}[X(s)] = x_1(t) * x_2(t) = \cos at * \cos bt$$

$$= \int_0^t \cos a\tau\cos b(t-\tau)\,d\tau$$

$$= \int_0^t \left[\frac{\cos(a\tau + bt - b\tau) + \cos(a\tau - bt + b\tau)\, d\tau}{2} \right]$$

$$= \frac{1}{2} \int_0^t \{\cos[(a-b)\,\tau + bt] + \cos[(a+b)\,\tau - bt)]\}\, d\tau$$

$$= \frac{1}{2} \left[\frac{\sin[(a-b)\,\tau + bt]}{a-b} + \frac{\sin[(a+b)\,\tau - bt]}{a+b} \right]_0^t$$

$$= \frac{1}{2} \left\{ \frac{\sin[(a-b)\,t + bt]}{a-b} - \frac{\sin bt}{a-b} + \frac{\sin[(a+b)t - bt]}{a+b} + \frac{\sin bt}{a+b} \right\}$$

$$= \frac{1}{2} \left(\frac{\sin at}{a-b} + \frac{\sin at}{a+b} + \frac{\sin bt}{a+b} - \frac{\sin bt}{a-b} \right)$$

$$= \frac{1}{2} \left(\frac{2a \sin at}{a^2 - b^2} - \frac{2b \sin bt}{a^2 - b^2} \right)$$

$$\therefore \quad x(t) = \frac{a \sin at - b \sin bt}{a^2 - b^2}$$

9.9 INVERSION OF BILATERAL LAPLACE TRANSFORM

So far we discussed about the inversion of unilateral Laplace transform which can be used only for causal signals. To determine the inverse of unilateral Laplace transform, the ROC need not be specified because a given pole always corresponds to a +ve time portion of $x(t)$, i.e. the ROC always lies to the right of the pole, i.e. ROC is always $\text{Re}(s) >$ some constant. The $L^{-1}[1/(s + a)]$ is always equal to $e^{-at}\, u(t)$ for unilateral Laplace transform and the ROC is $\text{Re}(s) > -a$. On the other hand, if $x(t)$ is a non-causal signal, then to find $x(t)$ from $X(s)$, we must know the ROC of $X(s)$. The location of pole along with ROC determines whether a given pole corresponds to a positive or a negative time portion of $x(t)$. If the ROC is $\text{Re}(s) > a$, then the signal is a causal signal. On the other hand, if the ROC is $\text{Re}(s) < a$, then the signal $x(t)$ is a non-causal signal. In other words, if a pole of $X(s)$ lies to the left of the ROC, then this pole gives rise to a causal signal, and if a pole of $X(s)$ lies to the right of the ROC, then this pole gives rise to a non-causal signal, i.e. the $L^{-1}[1/(s + a)]$ may be equal to $e^{-at}\, u(t)$ if ROC is $\text{Re}(s) > -a$ and may be equal to $-e^{-at}\, u(-t)$ if ROC is $\text{Re}(s) < -a$.

EXAMPLE 9.44 Find the inverse Laplace transform of

$$X(s) = \frac{2s + 1}{s + 2}$$

(a) For ROC; $\text{Re}(s) > -2$ (b) For ROC; $\text{Re}(s) < -2$

Solution: Given $X(s) = \dfrac{2s+1}{s+2} = 2\left[\dfrac{s+(1/2)}{s+2}\right] = 2\left[\dfrac{s+2-(3/2)}{s+2}\right]$

$$= 2\left(\frac{s+2}{s+2} - \frac{3/2}{s+2}\right) = 2 - \frac{3}{s+2}$$

(a) Since ROC is Re$(s) > -2$, the pole at $s = -2$ gives rise to a causal signal. Taking inverse Laplace transform on both sides, we have

$$x(t) = \mathrm{L}^{-1}\left(2 - \frac{3}{s+2}\right) = 2\delta(t) - 3e^{-2t}\,u(t)$$

(b) Since ROC is Re$(s) < -2$, the pole at $s = -2$ gives rise to a non-causal signal.

$\therefore$
$$x(t) = \mathrm{L}^{-1}\left(2 - \frac{3}{s+2}\right) = 2\delta(t) + 3e^{-2t}\,u(-t)$$

EXAMPLE 9.45 Find the inverse Laplace transform of

$$X(s) = \frac{3}{(s+1)(s+4)};\ \text{ROC; } -4 < \text{Re}(s) < -1$$

Solution: The given $X(s) = \dfrac{3}{(s+1)(s+4)}$ has two poles, one at $s = -1$ and the other one at $s = -4$. Here the pole at $s = -4$ lies to the left of the ROC, hence this pole gives rise to a causal signal. The pole at $s = -1$ lies to the right of the ROC, hence this pole gives rise to a non-causal signal.

Expanding $X(s)$ into partial fractions, we have

$$X(s) = \frac{3}{(s+1)(s+4)} = \frac{1}{s+1} - \frac{1}{s+4};\ \text{ROC; } -4 < \text{Re}(s) < -1$$

Now, $\quad \mathrm{L}^{-1}\left(\dfrac{1}{s+1}\right) = -e^{-t}u(-t)$ since ROC is Re$(s) < -1$

and $\quad \mathrm{L}^{-1}\left(\dfrac{1}{s+4}\right) = e^{-4t}u(t)$ since ROC is Re$(s) > -4$

$\therefore \quad \mathrm{L}^{-1}[X(s)] = x(t) = -e^{-t}u(-t) - e^{-4t}u(t)$

EXAMPLE 9.46 Find the signal whose bilateral transform is:

$$X(s) = \frac{1}{(s+2)(s+5)}\ \text{ ROC; } -5 < \text{Re}(s) < -2$$

Solution: The given $X(s)$ has two poles, one at $s = -5$ and the other one at $s = -2$. Since ROC is Re$(s) > -5$, the pole at $s = -5$ gives rise to a causal signal. Also since ROC is Re$(s) < -2$, the pole at $s = -2$, gives rise to a non-causal signal.

We have

$$X(s) = \frac{1}{(s+2)(s+5)} = \frac{A}{s+2} + \frac{B}{s+5} = \frac{1}{3}\left(\frac{1}{s+2}\right) - \frac{1}{3}\left(\frac{1}{s+5}\right)$$

Taking inverse of the above bilateral Laplace transform, we obtain

$$x(t) = -\frac{1}{3}e^{-2t}\,u(-t) - \frac{1}{3}e^{-5t}\,u(t)$$

EXAMPLE 9.47 Find the inverse Laplace transform of

$$X(s) = \frac{1}{(s+4)(s-2)}$$

if the region of convergence is

(a) $-4 < \text{Re}(s) < 2$ (b) $\text{Re}(s) > 2$
(c) $\text{Re}(s) < -4$ (d) $2 < \text{Re}(s) < -4$

Solution:
$$X(s) = \frac{1}{(s+4)(s-2)} = \frac{A}{s-2} + \frac{B}{s+4} = \frac{1}{6}\frac{1}{s-2} - \frac{1}{6}\frac{1}{s+4}$$

(a) The $X(s)$ has poles at $s = -4$ and $s = 2$. The strip of ROC for $-4 < \text{Re}(s) < 2$ is shown in Figure 9.15(a). The pole at $s = -4$ which is at the left of the strip of ROC corresponds to the causal signal and the pole at $s = 2$ which is at the right of the strip of ROC corresponds to the anticausal signal. Therefore,

$$x(t) = -\frac{1}{6}e^{2t}u(-t) - \frac{1}{6}e^{-4t}\,u(t)$$

(b) The ROC is $\text{Re}(s) > 2$ as shown in Figure 9.15(b).
Both the poles lie to the left of ROC, so both the poles correspond to causal signal. Therefore,

$$x(t) = \frac{1}{6}e^{2t}\,u(t) - \frac{1}{6}e^{-4t}\,u(t)$$

(c) The ROC is $\text{Re}(s) < -4$ as shown in Figure 9.15(c).

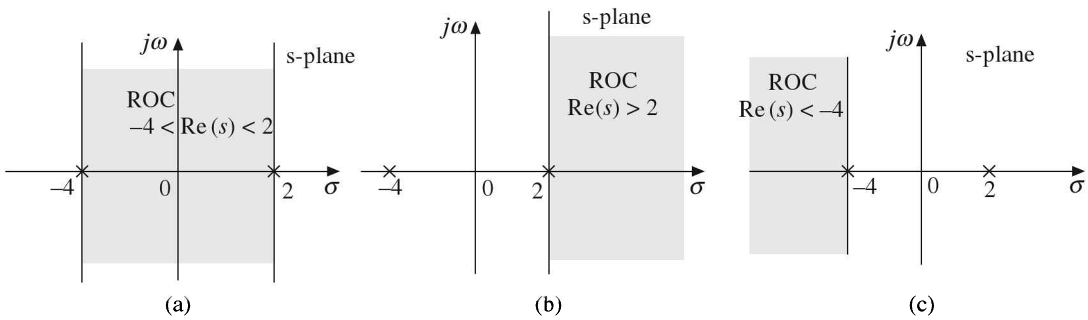


Figure 9.15 All possible ROCs of $X(s) = 1/[(s+4)(s-2)]$.

Both the poles lie to the right of the ROC. So both the poles correspond to anticausal signals. Therefore,

$$x(t) = -\frac{1}{6}e^{2t}u(-t) + \frac{1}{6}e^{-4t}u(-t)$$

(d) Since ROC of the sum of the signals is equal to the intersection of their ROCs, it is not possible to have ROC; $2 < \text{Re}(s) < -4$. So the inverse Laplace transform of the given $X(s)$ for this ROC does not exist.

EXAMPLE 9.48 Find all possible inverse Laplace transforms for $X(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$.

Solution: The given function $X(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$ has three poles, one at $s = -1$, the second one at $s = -2$ and the third one at $s = -3$. Since the inverse Laplace transform depends on the ROC, there are 4 possible ROCs and hence four possible inverse Laplace transforms. They are:

(a) ROC; $\text{Re}(s) > -1$ (b) ROC; $\text{Re}(s) < -3$

(c) ROC; $-3 < \text{Re}(s) < -2$ (d) ROC; $-2 < \text{Re}(s) < -1$

Given $$X(s) = \frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} = \frac{1}{2}\left(\frac{1}{s+1}\right) - \frac{1}{s+2} + \frac{1}{2}\left(\frac{1}{s+3}\right)$$

Taking inverse Laplace transform,

(a) ROC; $\text{Re}(s) > -1$. So all the signals are causal.

$$\therefore \quad L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}u(t),\ L^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}u(t) \quad \text{and} \quad L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}u(t)$$

$\therefore$ For ROC; $\text{Re}(s) > -1$, $x(t) = \dfrac{1}{2}e^{-t}u(t) - e^{-2t}u(t) + \dfrac{1}{2}e^{-3t}u(t)$

(b) ROC; $\text{Re}(s) < -3$. So all the signals are non-causal.

$$\therefore \quad L^{-1}\left(\frac{1}{s+1}\right) = -e^{-t}u(-t),\ L^{-1}\left(\frac{1}{s+2}\right) = -e^{-2t}u(-t)$$

and $$L^{-1}\left(\frac{1}{s+3}\right) = -e^{-3t}u(-t)$$

$\therefore$ For ROC; $\text{Re}(s) < -3$, $x(t) = -\dfrac{1}{2}e^{-t}u(-t) + e^{-2t}u(-t) - \dfrac{1}{2}e^{-3t}u(-t)$

(c) ROC: $-3 < \text{Re}(s) < -2$. So the pole at $s = -3$ gives causal signal and the poles at $s = -1$ and $s = -2$ give non-causal signals.

ROC; $\text{Re}(s) < -1$

$$\therefore \quad L^{-1}\left(\frac{1}{s+1}\right) = -e^{-t}u(-t)$$

ROC; Re$(s) < -2$

$$\therefore \qquad L^{-1}\left(\frac{1}{s+2}\right) = -e^{-2t}u(-t)$$

ROC; Re$(s) > -3$

$$\therefore \qquad L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}u(t)$$

$\therefore$ For ROC; $-3 < \text{Re}(s) < -2$, $x(t) = -\frac{1}{2}e^{-t}u(-t) + e^{-2t}u(-t) + \frac{1}{2}e^{-3t}u(t)$

(d) ROC; $-2 < \text{Re}(s) < -1$. So the pole at $s = -1$ gives a non-causal signal and the poles at $s = -3$ and $s = -2$ give causal signals.

$$\therefore \qquad x(t) = -\frac{1}{2}e^{-t}u(-t) - e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

EXAMPLE 9.49 Find the signal $x(t)$ whose bilateral Laplace transform is:

$$X(s) = \frac{2}{(s^2+4)(s+2)} \quad \text{with ROC; } -2 < \text{Re}(s) < 0$$

Solution: The given function $X(s) = \frac{2}{(s^2+4)(s+2)}$ has one pole at $s = -2$ and one pair of complex poles at $s = \pm j2$. Since ROC; Re$(s) > -2$, the pole at $s = -2$ gives rise to a causal signal and since ROC; Re$(s) < 0$, the complex conjugate pair of poles at $s = \pm j2$ give rise to an anticausal signal.

$$X(s) = \frac{2}{(s^2+4)(s+2)} = \frac{As+B}{s^2+4} + \frac{C}{s+2} = \frac{(-1/4)s + (1/2)}{s^2+4} + \frac{1/4}{s+2}$$

$$\therefore \qquad X(s) = \frac{-(1/4)s + (1/2)}{s^2+4} + \frac{1/4}{s+2} = -\frac{1}{4}\left(\frac{s}{s^2+4}\right) + \frac{1}{4}\left(\frac{2}{s^2+4}\right) + \frac{1}{4}\left(\frac{1}{s+2}\right)$$

Taking inverse Laplace transform on both sides, we have

$$x(t) = \frac{1}{4}\cos 2t\, u(-t) - \frac{1}{4}\sin 2t\, u(-t) + \frac{1}{4}e^{-2t}u(t)$$

EXAMPLE 9.50 Find the inverse Laplace transform of the following:

$$X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}$$

(a) Re$(s) < -5$ (b) Re$(s) > -3$ (c) $-5 < \text{Re}(s) < -3$

Solution: Given

$$X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}$$

Taking partial fractions, we have

$$X(s) = \frac{A}{s+3} + \frac{B}{(s+5)^2} + \frac{C}{s+5} = \frac{2}{s+3} - \frac{10}{(s+5)^2} - \frac{1}{s+5}$$

$$\therefore \qquad X(s) = \frac{2}{s+3} - \frac{10}{(s+5)^2} - \frac{1}{s+5}$$

(a) Since ROC is Re(s) < –5, both the poles give anticausal terms. Taking inverse Laplace transform of $X(s)$ we have

$$x(t) = -2e^{-3t}u(-t) + 10te^{-5t}u(-t) + e^{-5t}u(-t)$$

(b) Since ROC is Re(s) > –3 both the poles give causal terms. Taking inverse Laplace transform of $X(s)$, we have

$$x(t) = 2e^{-3t}u(t) - 10te^{-5t}u(t) - e^{-5t}u(t)$$

(c) Since ROC is –5 < Re(s) < –3 the pole at s = –5 gives causal term and the pole at s = –3 gives non-causal term. Taking inverse Laplace transform of $X(s)$ we have

$$x(t) = -2e^{-3t}u(-t) - 10te^{-5t}u(t) - e^{-5t}u(t)$$

EXAMPLE 9.51 Find all possible inverse Laplace transforms of

$$X(s) = \frac{1}{s(s+1)(s+2)(s+3)}$$

Solution: Given $X(s) = \dfrac{1}{s(s+1)(s+2)(s+3)} = \dfrac{1}{6}\left(\dfrac{1}{s}\right) - \dfrac{1}{2}\left(\dfrac{1}{s+1}\right) + \dfrac{1}{2}\left(\dfrac{1}{s+2}\right) - \dfrac{1}{6}\left(\dfrac{1}{s+3}\right)$

The given $X(s)$ has poles at s = 0, s = –1, s = –2 and s – 3.

Each one of them can give a causal term or a non-causal term depending on the ROC. We know that the ROC of the sum of two or more signals is equal to the intersection of the ROCs of individual signals. So the possible ROCs are:

1. ROC; Re(s) > 0: For this, all the terms must correspond to causal signals.

$$\therefore \qquad x(t) = \frac{1}{6}u(t) - \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{6}e^{-3t}u(t)$$

2. ROC; Re(s) < –3: For this, all the terms must correspond to non-causal signals.

$$\therefore \qquad x(t) = -\frac{1}{6}u(-t) + \frac{1}{2}e^{-t}u(-t) - \frac{1}{2}e^{-2t}u(-t) + \frac{1}{6}e^{-3t}u(-t)$$

3. ROC; –1 < Re(s) < 0: For this, the poles at s = –3, s = –2 and s = –1 correspond to causal signals and the pole at s = 0 correspond to non-causal signal.

$$\therefore \qquad x(t) = -\frac{1}{6}u(-t) - \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{6}e^{-3t}u(t)$$

4. ROC; $-2 < \text{Re}(s) < -1$: For this, the poles at $s = -3$, $s = -2$ correspond to causal signals, and the poles at $s = -1$ and $s = 0$ correspond to non-causal signals.

$$\therefore \qquad x(t) = -\frac{1}{6}u(-t) + \frac{1}{2}e^{-t}u(-t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{6}e^{-3t}u(t)$$

5. ROC; $-3 < \text{Re}(s) < -2$: For this, the pole at $s = -3$ corresponds to causal term and the poles at $s = 0$, $s = -1$ and $s = -2$ correspond to non-causal terms.

$$\therefore \qquad x(t) = -\frac{1}{6}u(-t) + \frac{1}{2}e^{-t}u(-t) - \frac{1}{2}e^{-2t}u(-t) - \frac{1}{6}e^{-3t}u(t)$$

All possible ROCs are plotted in Figure 9.16

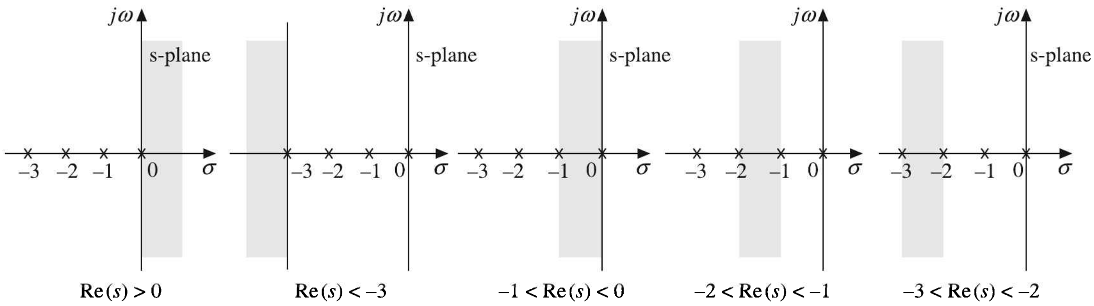


Figure 9.16 ROCs for Example 9.51.

9.10 ROCs FOR VARIOUS CLASSES OF SIGNALS

Earlier we had defined the region of convergence (ROC) as the range of values of Re(s), i.e. σ for which the Laplace transform converges. Also we had determined the Laplace transforms and their ROCs for a number of signals. Now, instead of considering the ROC for each signal, we shall categorise the signals into a few classes and study the nature of the ROC for each of these classes.

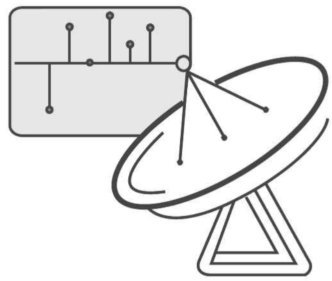
Existence of the Laplace transform

The Lapalce transform of $x(t)$, i.e. $X(s)$ exists only if $\int_{-\infty}^{\infty} |x(t)e^{-st}|\,dt < \infty$. That means $x(t)\,e^{-\sigma t}$ must be absolutely integrable for ROC to exist. The ROC cannot contain any poles, because at a pole, the value of $X(s) = \infty$ and since the ROC is defined as constituting that region of the s-plane in which $X(s)$ converges. The ROC consists of strips of the s-plane parallel to the $j\omega$ axis because the ROC of the Laplace transform of any signal is generally specified in terms of a range of Re(s) – like Re(s) > σ_1 or Re(s) < σ_2 or σ_1 < Re(s) < σ_2.

The signals may be divided into four classes:

(a) Right-sided signals
(b) Left-sided signals
(c) Two-sided signals
(d) Finite duration signals

2



Systems

2.1 INTRODUCTION

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system may also be defined as a set of elements or functional blocks which are connected together and produces an output in response to an input signal. The response or output of the system depends upon the transfer function of the system. It is a cause-and-effect relation between two or more signals. There are various types of systems: electrical systems, mechanical systems, biological systems, opto-electronic systems, electromechanical systems and so on. The actual physical structure of the system determines the exact relation between the input $x(t)$ and the output $y(t)$ and specifies the output for every input. Physical devices such as motor, amplifier, filter, boiler and turbine are examples of systems. In this chapter, we discuss about classification of systems and determination of the type of system from the describing equation. Systems may be single input and single output systems or multi input and multi output systems. In this book, we consider only single input and single output systems.

2.2 CLASSIFICATION OF SYSTEMS

A system is represented by a block diagram as shown in Figure 2.1. An arrow entering the box is the input signal (also called excitation, source or driving function) and an arrow leaving the box is an output signal (also called response). Generally, the input is denoted by $x(t)$ and the output is denoted by $y(t)$.

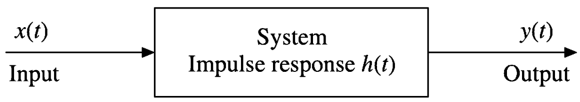


Figure 2.1 A system.

The relation between the input $x(t)$ and the output $y(t)$ of a system has the form

$$y(t) = \text{Operation on } x(t)$$

Mathematically, $$y(t) = T\,[x(t)]$$

which represents that $x(t)$ is transformed to $y(t)$. In other words $y(t)$ is the transformed version of $x(t)$.

Like signals, systems may also be broadly classified as under

(a) Continuous-time systems
(b) Discrete-time systems

Continuous-time system

A continuous-time system is one which transforms continuous-time input signals into continuous-time output signals.

If the input and output of a continuous-time system are $x(t)$ and $y(t)$ then we can say that $x(t)$ is transformed to $y(t)$. That is,

$$y(t) = T[x(t)]$$

Amplifiers, filters, integrators, and differentiators are the examples of continuous-time systems. The block diagram of a continuous-time system is shown in Figure 2.2.

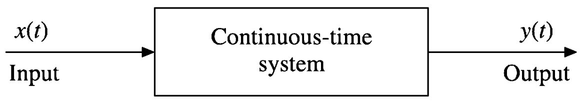


Figure 2.2 Block diagram of continuous-time system.

Discrete-time System

A discrete-time system is one which transforms discrete-time input signals into discrete-time output signals.

If the input and output of a discrete-time system are $x(n)$ and $y(n)$, then we can say that $x(n)$ is transformed to $y(n)$. That is,

$$y(n) = T[x(n)]$$

Microprocessors, semiconductor memories, shift registers, etc. are the examples of discrete-time systems.

The block diagram of a discrete-time system is shown in Figure 2.3.

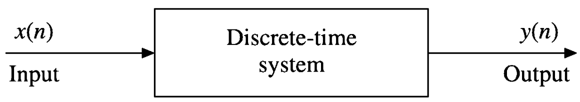


Figure 2.3 Block diagram of discrete-time system.

Both continuous-time and discrete-time systems may be classified as under

1. Lumped parameter and distributed parameter systems
2. Static (memoryless) and dynamic (memory) systems
3. Causal and non-causal systems
4. Linear and non-linear systems
5. Time-invariant and time varying systems

6. Stable and unstable systems.
7. Invertible and non-invertible systems
8. FIR and IIR systems

2.2.1 Lumped Parameter and Distributed Parameter Systems

Lumped parameter systems are the systems in which each component is lumped at one point in space. These systems are described by ordinary differential equations. Distributed parameter systems are the systems in which signals are functions of space as well as time. These systems are described by partial differential equations.

2.2.2 Static and Dynamic Systems

A system is said to be static or memoryless if the response is due to present input alone, i.e. for a static or memoryless system, the output at any instant t (or n) depends only on the input applied at that instant t (or n) but not on the past or future values of input.

For example, the systems defined below are static or memoryless systems.

$$y(t) = x(t)$$
$$y(t) = x^2(t)$$
$$y(n) = x(n)$$
$$y(n) = 2x^2(n)$$

In contrast, a system is said to be dynamic or memory system if the response depends upon past or future inputs.

For example, the systems defined below are dynamic or memory systems.

$$y(t) = x(t-1)$$
$$y(t) = x(t) + x(t+2)$$
$$y(t) = \frac{d^2x(t)}{dt^2} + x(t)$$
$$y(n) = x(2n)$$
$$y(n) = x(n) + x(n-2)$$

Any continuous-time system described by a differential equation or any discrete-time system described by a difference equation is also a dynamic system.

A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a dynamic system.

A summer or accumulator is an example of a discrete-time system with memory. A delay is also a discrete-time system with memory.

EXAMPLE 2.1 Find whether the following systems are dynamic or not:

(a) $y(t) = x(t-3)$ (b) $y(t) = x(2t)$

(c) $y(t) = \frac{d^2x(t)}{dt^2} + 2x(t)$ (d) $y(n) = x(n+2)$

(e) $y(n) = x^2(n)$ (f) $y(n) = x(n-2) + x(n)$

Solution:

(a) Given $$y(t) = x(t-3)$$
The output depends on past value of input. Therefore, the system is dynamic.

(b) Given $$y(t) = x(2t)$$
The output depends on future value of input. Therefore, the system is dynamic.

(c) Given $$y(t) = \frac{d^2x(t)}{dt^2} + 2x(t)$$
The system is described by a differential equation. Therefore, the system is dynamic.

(d) Given $$y(n) = x(n+2)$$
The output depends on the future value of input. Therefore, the system is dynamic.

(e) Given $$y(n) = x^2(n)$$
The output depends on the present value of input alone. Therefore, the system is static.

(f) Given $$y(n) = x(n-2) + x(n)$$
The system is described by a difference equation. Therefore, the system is dynamic.

2.2.3 Causal and Non-causal Systems

A system is said to be causal (or non-anticipative) if the output of the system at any time t depends only on the present and past values of the input but not on future inputs. In other words, we can say that a system is causal if the response or output does not begin before the input function is applied. That is, a causal system is non anticipatory.

Causal systems are real time systems. They are physically realizable.

The impulse response of a causal system is zero for t (or n) < 0, since $\delta(t)$ [or $\delta(n)$] exists only at t (or n) $= 0$.

i.e. $$h(t) = 0 \quad \text{for } t < 0 \quad \text{and} \quad h(n) = 0 \quad \text{for } n < 0$$

Examples for the causal systems are:

$$y(t) = x(t-2) + 2x(t)$$
$$y(t) = tx(t)$$
$$y(n) = nx(n)$$
$$y(n) = x(n-2) + x(n-1) + x(n)$$

A system is said to be non-causal (anticipative) if the output of the system at any time t depends on future inputs. They do not exist in real time. They are not physically realizable. They are anticipatory systems. They produce an output even before the input is given.

Examples for the non-causal systems are:

$$y(t) = x(t+2) + x(t)$$
$$y(t) = x^2(t) + tx(t+1)$$
$$y(n) = x(n) + x(2n)$$
$$y(n) = x^2(n) + 2x(n+2)$$

A resistor is an example of a continuous-time causal system. Image processing systems are examples of non causal systems.

EXAMPLE 2.2 Check whether the following systems are causal or not:

(a) $y(t) = x^2(t) + x(t-4)$ (b) $y(t) = x(2-t) + x(t-4)$

(c) $y(t) = \int_{-\infty}^{3t} x(\tau)d\tau$ (d) $y(t) = x\left(\frac{t}{2}\right)$

(e) $y(t) = x[\sin 2t]$ (f) $y(n) = x(n) + x(n-2)$

(g) $y(n) = x(2n)$ (h) $y(n) = \sin[x(n)]$

(i) $y(n) = x(-n)$

Solution:

(a) Given $y(t) = x^2(t) + x(t-4)$

For $t = -2$ $y(-2) = x^2(-2) + x(-6)$

For $t = 0$ $y(0) = x^2(0) + x(-4)$

For $t = 2$ $y(2) = x^2(2) + x(-2)$

For all values of t, the output depends only on the present and past values of input. Therefore, the system is causal.

(b) Given $y(t) = x(2-t) + x(t-4)$

For $t = -1$ $y(-1) = x(3) + x(-5)$

For $t = 0$ $y(0) = x(2) + x(-4)$

For $t = 1$ $y(1) = x(1) + x(-3)$

For some values of t, the output depends on the future input. Therefore, the system is non-causal.

(c) Given $y(t) = \int_{-\infty}^{3t} x(\tau)\, d\tau$

For $t = 0$ $y(0) = \int_{-\infty}^{0} x(\tau)d\tau = p(0) - p(-\infty)$

For $t = 1$ $y(1) = \int_{-\infty}^{3} x(\tau)\, d\tau = p(3) - p(-\infty)$

where $\int x(\tau)\, d\tau = p(\tau)$

The output $y(1)$ depends on the future value $p(3)$. Therefore, the system is non-causal.

(d) Given $y(t) = x(t/2)$

For $t = -2$ $y(-2) = x(-1)$

For $t = 0$ $y(0) = x(0)$

For $t = 2$ $y(2) = x(1)$

For negative values of t, the output depends on the future input. Therefore, the system is non-causal.

(e) Given $$y(t) = x[\sin 2t]$$
$$y(\pi) = x[\sin 2\pi] = x(0)$$
$$y(-\pi) = x[\sin(-2\pi)] = x(0)$$

As the output at $(-\pi)$ depends on the input that occurs later, the system is non-causal.

(f) Given $$y(n) = x(n) + x(n-2)$$
For $n = -2$ $$y(-2) = x(-2) + x(-4)$$
For $n = 0$ $$y(0) = x(0) + x(-2)$$
For $n = 2$ $$y(2) = x(2) + x(0)$$

For all values of n, the output depends only on the present and past inputs. Therefore, the system is causal.

(g) Given $$y(n) = x(2n)$$
For $n = -2$ $$y(-2) = x(-4)$$
For $n = 0$ $$y(0) = x(0)$$
For $n = 2$ $$y(2) = x(4)$$

For positive values of n, the output depends on the future values of input. Therefore, the system is non-causal.

(h) Given $$y(n) = \sin[x(n)]$$
For $n = -2$ $$y(-2) = \sin[x(-2)]$$
For $n = 0$ $$y(0) = \sin[x(0)]$$
For $n = 2$ $$y(2) = \sin[x(2)]$$

For all values of n, the output depends only on the present value of input. Therefore, the system is causal.

(i) Given $$y(n) = x(-n)$$
For $n = -2$ $$y(-2) = x(2)$$
For $n = 0$ $$y(0) = x(0)$$
For $n = 2$ $$y(2) = x(-2)$$

For negative values of n, the output depends on the future values of input. Therefore, the system is non-causal.

2.2.4 Linear and Non-linear Systems

A system which obeys the principle of superposition and principle of homogeneity is called a linear system, and a system which does not obey the principle of superposition and homogeneity is called a non-linear system.

Homogeneity property means a system which produces an output $y(t)$ for an input $x(t)$ must produce an output $ay(t)$ for an input $ax(t)$.

Superposition property means a system which produces an output $y_1(t)$ for an input $x_1(t)$ and an output $y_2(t)$ for an input $x_2(t)$ must produce an output $y_1(t) + y_2(t)$ for an input $x_1(t) + x_2(t)$.

Combining them we can say that a system is linear if an arbitrary input $x_1(t)$ produces an output $y_1(t)$ and an arbitrary input $x_2(t)$ produces an output $y_2(t)$, then the weighted sum of inputs $ax_1(t) + bx_2(t)$, where a and b are constants, produces an output $ay_1(t) + by_2(t)$ which is the sum of weighted outputs. That is,

$$T[ax_1(t) + bx_2(t)] = aT[x_1(t)] + bT[x_2(t)]$$

For discrete-time linear system,

$$T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

Simply we can say that a system is linear if the output due to weighted sum of inputs is equal to the weighted sum of outputs.

In general, if the describing equation contains square or higher order terms of input and/or output and/or product of input/output and its derivative or a constant, the system will definitely be non-linear.

Few examples of linear systems are filters, communication channels, etc.

EXAMPLE 2.3 Check whether the following systems are linear or not:

(a) $\dfrac{d^2y(t)}{dt^2} + 2t\,y(t) = t^2x(t)$ (b) $2\dfrac{dy(t)}{dt} + 5y(t) = x^2(t)$

(c) $\dfrac{dy(t)}{dt} + y(t) = x(t)\dfrac{dx(t)}{dt}$ (d) $y(t) = x(t^2)$

(e) $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ (f) $y(t) = 2x^2(t)$

(g) $y(t) = e^{x(t)}$ (h) $y(n) = n^2x(n)$

(i) $y(n) = x(n) + \dfrac{1}{2x(n-2)}$ (j) $y(n) = 2x(n) + 4$

(k) $y(n) = x(n)\cos \omega n$

Solution:

(a) Given $$\frac{d^2y(t)}{dt^2} + 2t\,y(t) = t^2x(t)$$

Let an input $x_1(t)$ produce an output $y_1(t)$.

Then $$\frac{d^2y_1(t)}{dt^2} + 2ty_1(t) = t^2x_1(t)$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then
$$\frac{d^2y_2(t)}{dt^2} + 2ty_2(t) = t^2x_2(t)$$

Linear combination of the above equations gives

$$a\frac{d^2y_1(t)}{dt^2} + a2ty_1(t) + b\frac{d^2y_2(t)}{dt^2} + b2ty_2(t) = at^2x_1(t) + bt^2x_2(t)$$

i.e.
$$\frac{d^2}{dt^2}\underbrace{[ay_1(t) + by_2(t)]}_{\text{Weighted sum of outputs}} + 2t\underbrace{[ay_1(t) + by_2(t)]}_{\text{Weighted sum of outputs}} = t^2\underbrace{[ax_1(t) + bx_2(t)]}_{\text{Weighted sum of inputs}}$$

This shows that the weighted sum of inputs to the system produces an output which is equal to the weighted sum of outputs to each of the individual inputs. Therefore, the system is linear.

Note: If RHS is function of weighted sum of inputs and LHS is function of weighted sum of outputs, then the system is linear.

(b) Given
$$2\frac{dy(t)}{dt} + 5y(t) = x^2(t)$$

If an input $x_1(t)$ produces an output $y_1(t)$, then

$$2\frac{dy_1(t)}{dt} + 5y_1(t) = x_1^2(t)$$

Similarly, if an input $x_2(t)$ produces an output $y_2(t)$, then

$$2\frac{dy_2(t)}{dt} + 5y_2(t) = x_2^2(t)$$

The linear combination of the above equations can be written as:

$$a2\frac{dy_1(t)}{dt} + a5y_1(t) + b2\frac{dy_2(t)}{dt} + b5y_2(t) = ax_1^2(t) + bx_2^2(t)$$

i.e.
$$2\frac{d}{dt}\underbrace{[ay_1(t) + by_2(t)]}_{\text{Weighted sum of outputs}} + 5\underbrace{[ay_1(t) + by_2(t)]}_{\text{Weighted sum of outputs}} = \underbrace{ax_1^2(t) + bx_2^2(t)}_{\text{Not a weighted sum of inputs}}$$

The RHS is not a function of weighted sum of inputs. Hence superposition principle is not satisfied. Therefore, the system is non-linear.

(c) Given
$$\frac{dy(t)}{dt} + y(t) = x(t)\frac{dx(t)}{dt}$$

Let an input $x_1(t)$ produce an output $y_1(t)$.

Then
$$\frac{dy_1(t)}{dt} + y_1(t) = x_1(t)\frac{dx_1(t)}{dt}$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$\frac{dy_2(t)}{dt} + y_2(t) = x_2(t)\frac{dx_2(t)}{dt}$$

The linear combination of the above equations can be written as:

$$a\frac{dy_1(t)}{dt} + ay_1(t) + b\frac{dy_2(t)}{dt} + by_2(t) = ax_1(t)\frac{dx_1(t)}{dt} + bx_2(t)\frac{dx_2(t)}{dt}$$

i.e. $$\frac{d}{dt}\underbrace{\left[ay_1(t) + by_2(t)\right]}_{\substack{\text{Weighted sum}\\ \text{of outputs}}} + \underbrace{\left[ay_1(t) + by_2(t)\right]}_{\substack{\text{Weighted sum}\\ \text{of outputs}}} = \underbrace{\left[ax_1(t)\frac{dx_1(t)}{dt} + bx_2(t)\frac{dx_2(t)}{dt}\right]}_{\substack{\text{Not a weighted sum}\\ \text{of inputs}}}$$

Since the RHS is not a weighted sum of inputs, superposition principle is not satisfied. Therefore, the system is non-linear.

(d) Given $$y(t) = x(t^2)$$

Let an input $x_1(t)$ produce an output $y_1(t)$.

Then $$y_1(t) = x_1(t^2)$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$y_2(t) = x_2(t^2)$$

The linear combination of the above equations can be written as:

$$\underbrace{ay_1(t) + by_2(t)}_{\substack{\text{Weighted sum}\\ \text{of outputs}}} = \underbrace{ax_1(t^2) + bx_2(t^2)}_{\substack{\text{Weighted sum}\\ \text{of inputs}}}$$

The LHS is a function of weighted sum of outputs and the RHS is a function of weighted sum of inputs. The superposition principle is satisfied. Therefore, the system is linear.

(e) Given $$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$y(t) = T[x(t)] = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Let an input $x_1(t)$ produce an output $y_1(t)$.

Then $$y_1(t) = T[x_1(t)] = \int_{-\infty}^{t} x_1(\tau)\,d\tau$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$y_2(t) = T[x_2(t)] = \int_{-\infty}^{t} x_2(\tau)\,d\tau$$

The weighted sum of outputs is:

$$ay_1(t) + by_2(t) = a\int_{-\infty}^{t} x_1(\tau)\,d\tau + b\int_{-\infty}^{t} x_2(\tau)\,d\tau = \int_{-\infty}^{t} [ax_1(\tau) + bx_2(\tau)]d\tau$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = \int_{-\infty}^{t} [ax_1(\tau) + bx_2(\tau)]\,d\tau$$

$$y_3(t) = ay_1(t) + by_2(t)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the system is linear.

(f) Given $$y(t) = 2x^2(t)$$

$$y(t) = T[x(t)] = 2x^2(t)$$

For an input $x_1(t)$,

$$y_1(t) = T[x_1(t)] = 2x_1^2(t)$$

For an input $x_2(t)$,

$$y_2(t) = T[x_2(t)] = 2x_2^2(t)$$

The weighted sum of outputs is:

$$ay_1(t) + by_2(t) = a[2x_1^2(t)] + b[2x_2^2(t)] = 2[ax_1^2(t) + bx_2^2(t)]$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = 2[ax_1(t) + bx_2(t)]^2$$

$$y_3(t) \neq ay_1(t) + by_2(t)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the system is non-linear.

(g) Given $$y(t) = e^{x(t)}$$

$$y(t) = T[x(t)] = e^{x(t)}$$

For an input $x_1(t)$,

$$y_1(t) = T[x_1(t)] = e^{x_1(t)}$$

For an input $x_2(t)$,

$$y_2(t) = T[x_2(t)] = e^{x_2(t)}$$

The weighted sum of outputs is:

$$ay_1(t) + by_2(t) = ae^{x_1(t)} + be^{x_2(t)}$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = e^{[ax_1(t)+bx_2(t)]}$$

$$y_3(t) \neq ay_1(t) + by_2(t)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore the system is non-linear.

(h) Given $\quad y(n) = n^2 x(n)$

$$y(n) = T[x(n)] = n^2 x(n)$$

Let an input $x_1(n)$ produce an output $y_1(n)$.

$\therefore \quad y_1(n) = T[x_1(n)] = n^2 x_1(n)$

Let an input $x_2(n)$ produce an output $y_2(n)$.

$\therefore \quad y_2(n) = T[x_2(n)] = n^2 x_2(n)$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[n^2 x_1(n)] + b[n^2 x_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

(i) Given $\quad y(n) = x(n) + \dfrac{1}{2x(n-2)}$

$$y(n) = T[x(n)] = x(n) + \frac{1}{2x(n-2)}$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{2x_1(n-2)}$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{2x_2(n-2)}$$

The weighted sum of outputs is given by

$$ay_1(n) + by_2(n) = a\left[x_1(n) + \frac{1}{2x_1(n-2)}\right] + b\left[x_2(n) + \frac{1}{2x_2(n-2)}\right]$$

$$= [ax_1(n) + bx_2(n)] + \frac{a}{2x_1(n-2)} + \frac{b}{2x_2(n-2)}$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)] + \frac{1}{2[ax_1(n-2) + bx_2(n-2)]}$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(j) Given $$y(n) = 2x(n) + 4$$

$$y(n) = T[x(n)] = 2x(n) + 4$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = 2x_1(n) + 4$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = 2x_2(n) + 4$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[2x_1(n) + 4] + b[2x_2(n) + 4] = 2[ax_1(n) + bx_2(n)] + 4(a + b)$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = 2[ax_1(n) + bx_2(n)] + 4$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(k) Given $$y(n) = x(n) \cos \omega n$$

$$y(n) = T[x(n)] = x(n) \cos \omega n$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = x_1(n) \cos \omega n$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = x_2(n) \cos \omega n$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = ax_1(n) \cos \omega n + bx_2(n) \cos \omega n = [ax_1(n) + bx_2(n)] \cos \omega n$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)] \cos \omega n$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

2.2.5 Time-invariant and Time-varying Systems

Time-invariance is the property of a system which makes the behaviour of the system independent of time. This means that the behaviour of the system does not depend on the time at which the input is applied.

A system is said to be time-invariant (or shift-invariant) if its input/output characteristics do not change with time, i.e. if a time shift in the input results in a corresponding time shift in the output as shown in Figure 2.4, i.e.

If $$x(t) \rightarrow y(t)$$

Then $$x(t - T) \rightarrow y(t - T)$$

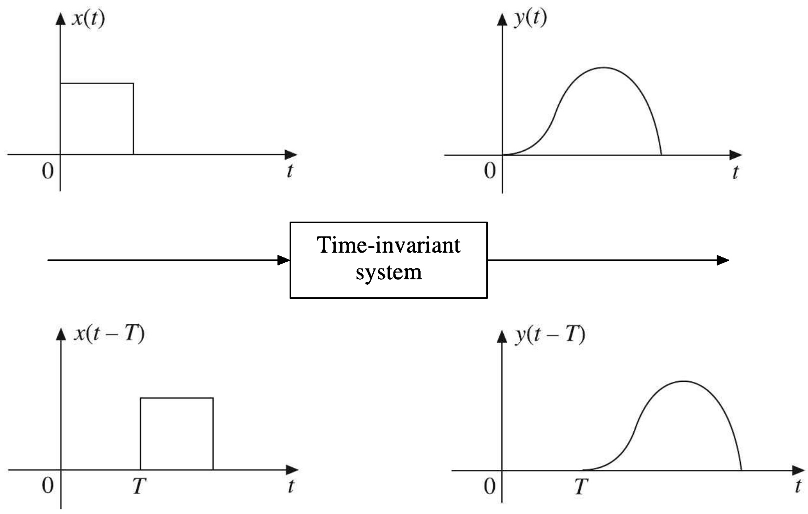


Figure 2.4 Time-invariant system.

A system not satisfying the above requirements is called a time-varying system (or shift varying system)

The time-invariance property of the given continuous-time system can be tested as follows:

Let $x(t)$ be the input and let $x(t - T)$ be the input delayed by T units.

$y(t) = T[x(t)]$ be the output for an input $x(t)$.

$y(t, T) = T[x(t - T)] = y(t)|_{x(t)=x(t-T)}$ be the output for the delayed input $x(t - T)$.

$y(t - T) = y(t)|_{t=t-T}$ be the output delayed by T units.

If

$$y(t, T) = y(t - T)$$

i.e. if the delayed output is equal to the output due to delayed input for all possible values of t, then the system is time-invariant.

On the other hand, if

$$y(t, T) \neq y(t - T)$$

i.e. if the delayed output is not equal to the output due to delayed input, then the system is time-variant.

If the continuous-time system is described by differential equation, the time-invariance can be found by observing the coefficients of the differential equation. If the coefficients of the differential equation are constants, then the system is time-invariant. If the coefficients are functions of time, then the system is time-variant.

The system described by

$$3\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = 5x(t)$$

is time-invariant system because all the coefficients are constants.

The system described by

$$\frac{d^2y(t)}{dt} + 2t\frac{dy(t)}{dt} + 3t^2y(t) = 4x(t)$$

is time varying system because all the coefficients are not constant. (Two are functions of time).

For discrete-time systems, the time invariance property is called shift invariance. The time invariance property of the given discrete-time system can be tested as follows:

Let $x(n)$ be the input and let $x(n - k)$ be the input delayed by k units.

$y(n) = T[x(n)]$ be the output for the input $x(n)$.

$y(n, k) = T[x(n, k)] = y(n)\big|_{x(n)=x(n-k)}$ be the output for the delayed input $x(n - k)$.

$y(n - k) = y(n)\big|_{n=n-k}$ be the output delayed by k units.

If
$$y(n, k) = y(n - k)$$

i.e. if delayed output is equal to the output due to delayed input for all possible values of k, then the system is time-invariant.

On the other hand, if

$$y(n, k) \neq y(n - k)$$

i.e. if the delayed output is not equal to the output due to delayed input, then the system is time-variant.

If the discrete-time system is described by difference equation, the time invariance can be found by observing the coefficients of the difference equation.

If the coefficients of the difference equation are constants, then the system is time-invariant. If the coefficients are functions of time, then the system is time-variant.

The system described by

$$y(n) + 3y(n-1) + 5y(n-2) = 2x(n)$$

is time-invariant system because all the coefficients are constants.

The system described by

$$y(n) - 2ny(n-1) + 3n^2y(n-2) = x(n) + x(n-1)$$

is time-varying system because all the coefficients are not constant (Two are functions of time).

The systems satisfying both linearity and time-invariance properties are popularly known as linear time-invariant or simply LTI systems.

EXAMPLE 2.4 Determine whether the following systems are time-invariant or not:

(a) $y(t) = t^2 x(t)$ (b) $y(t) = x(t)\sin 10\pi t$

(c) $y(t) = x(t^2)$ (d) $y(t) = x(-2t)$

(e) $y(t) = e^{2x(t)}$ (f) $y(n) = x(n/2)$

(g) $y(n) = x(n)$ (h) $y(n) = x^2(n-2)$

(i) $y(n) = x(n) + nx(n-2)$

Solution:

(a) Given $$y(t) = t^2 x(t)$$
$$y(t) = T[x(t)] = t^2 x(t)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = t^2 x(t-T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = (t-T)^2\, x(t-T)$$
$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(b) Given $$y(t) = x(t)\sin 10\pi t$$
$$y(t) = T[x(t)] = x(t)\sin 10\pi t$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = x(t-T)\sin 10\pi t$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = x(t-T)\sin 10\pi(t-T)$$
$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(c) Given $$y(t) = x(t^2)$$
$$y(t) = T[x(t)] = x(t^2)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = x(t^2 - T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = x[(t-T)^2]$$

$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(d) Given $$y(t) = x(-2t)$$

$$y(t) = T[x(t)] = x(-2t)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = x(-2t-T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = x[-2(t-T)] = x(-2t+2T)$$

$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(e) Given $$y(t) = e^{2x(t)}$$

$$y(t) = T[x(t)] = e^{2x(t)}$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = e^{2x(t-T)}$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = e^{2x(t-T)}$$

$$y(t, T) = y(t-T)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(f) Given $$y(n) = x\left(\frac{n}{2}\right)$$

$$y(n) = T[x(n)] = x\left(\frac{n}{2}\right)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x\left(\frac{n}{2} - k\right)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x\left(\frac{n-k}{2}\right)$$

$$y(n, k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(g) Given $$y(n) = x(n)$$

$$y(n) = T[x(n)] = x(n)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x(n-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k)$$

$$y(n, k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(h) Given $$y(n) = x^2(n-2)$$

$$y(n) = T[x(n)] = x^2(n-2)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x^2(n-2-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x^2(n-2-k)$$

$$y(n, k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(i) Given $$y(n) = x(n) + nx(n-2)$$

$$y(n) = T[x(n)] = x(n) + nx(n-2)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x(n-k) + nx(n-2-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k) + (n-k)\,x(n-k-2)$$

$$y(n, k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

EXAMPLE 2.5 Show that the following systems are linear time-invariant systems:

(a) $y(t) = x\left(\dfrac{t}{2}\right)$ (b) $y(t) = \begin{cases} x(t) + x(t-2) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

Solution: To show that a given system is a linear time-invariant system we have to show separately that it is linear and time-invariant.

(a) Given $$y(t) = x\left(\frac{t}{2}\right)$$

For inputs $x_1(t)$ and $x_2(t)$,

$$y_1(t) = x_1\left(\frac{t}{2}\right)$$

$$y_2(t) = x_2\left(\frac{t}{2}\right)$$

The weighted sum of outputs is:

$$ay_1(t) + by_2(t) = ax_1\left(\frac{t}{2}\right) + bx_2\left(\frac{t}{2}\right)$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = ax_1\left(\frac{t}{2}\right) + bx_2\left(\frac{t}{2}\right)$$

$$y_3(t) = ay_1(t) + by_2(t)$$

So the system is linear.

$$y(t, T) = y(t)\big|_{x(t)=x(t-T)} = x\left(\frac{t}{2} - T\right)$$

$$y(t-T) = y(t)\big|_{t=t-T} = x\left(\frac{t-T}{2}\right)$$

$$y(t, T) \neq y(t-T)$$

So the system is time varying.

Hence the given system is linear but time varying. It is not a linear time-invariant system.

(b) Given $$y(t) = \begin{cases} x(t) + x(t-2) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

For inputs $x_1(t)$ and $x_2(t)$,

$$y_1(t) = x_1(t) + x_1(t-2) \quad \text{for } t \geq 0$$

$$y_2(t) = x_2(t) + x_2(t-2) \quad \text{for } t \geq 0$$

The weighted sum of outputs is:

$$ay_1(t) + by_2(t) = a[x_1(t) + x_1(t-2)] + b[x_2(t) + x_2(t-2)]$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = [ax_1(t) + bx_2(t)] + ax_1(t-2) + bx_2(t-2)$$

$$y_3(t) = ay_1(t) + by_2(t)$$

So the system is linear.

$$y(t, T) = y(t)\big|_{x(t)=x(t-T)} = x(t-T) + x(t-2-T)$$

$$y(t-T) = y(t)\big|_{t=t-T} = x(t-T) + x(t-T-2)$$

$$y(t, T) = y(t-T)$$

So the system is time-invariant. Hence the given system is linear time-invariant.

EXAMPLE 2.6 Check whether the following systems are:

1. Static or dynamic
2. Linear or non-linear
3. Causal or non-causal
4. Time-invariant or time-variant

(a) $\dfrac{d^3y(t)}{dt^3} + 2\dfrac{d^2y(t)}{dt^2} + 4\dfrac{dy(t)}{dt} + 3y^2(t) = x(t+1)$

(b) $\dfrac{d^2y(t)}{dt^2} + 2y(t)\dfrac{dy(t)}{dt} + 3t\,y(t) = x(t)$

(c) $y(t) = ev\{x(t)\}$

(d) $y(t) = at^2x(t) + bt\,x(t-4)$

(e) $y(n) = x(n)\,x(n-2)$

(f) $y(n) = \log_{10}|x(n)|$

(g) $y(n) = a^n u(n)$

(h) $y(n) = x^2(n) + \dfrac{1}{x^2(n-1)}$

Solution:

(a) Given $$\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y^2(t) = x(t+1)$$

1. The system is described by a differential equation. Hence the system is dynamic.
2. There is a square term of output [i.e. $y^2(t)$]. So the system is non-linear. This can be proved.

   Let an input $x_1(t)$ produce an output $y_1(t)$. So the differential equation becomes

$$\frac{d^3y_1(t)}{dt^3} + 2\frac{d^2y_1(t)}{dt^2} + 4\frac{dy_1(t)}{dt} + 3y_1^2(t) = x_1(t+1)$$

Let an input $x_2(t)$ produce an output $y_2(t)$. So the differential equation becomes

$$\frac{d^3y_2(t)}{dt^3} + 2\frac{d^2y_2(t)}{dt^2} + 4\frac{dy_2(t)}{dt} + 3y_2^2(t) = x_2(t+1)$$

The linear combination of the above equations becomes

$$a\frac{d^3y_1(t)}{dt^3} + a2\frac{d^2y_1(t)}{dt^2} + a4\frac{dy_1(t)}{dt} + a3y_1^2(t)$$
$$+ b\frac{d^3y_2(t)}{dt^3} + b2\frac{d^2y_2(t)}{dt^2} + b4\frac{dy_2(t)}{dt} + b3y_2^2(t)$$
$$= ax_1(t+1) + bx_2(t+1)$$

i.e. $$\frac{d^3}{dt^3}[ay_1(t) + by_2(t)] + 2\frac{d^2}{dt^2}[ay_1(t) + by_2(t)] + 4\frac{d}{dt}[ay_1(t) + by_2(t)]$$
$$+\underbrace{3[ay_1^2(t) + by_2^2(t)]}_{\substack{\textbf{Not a weighted} \\ \textbf{sum of outputs}}} = ax_1(t+1) + bx_2(t+1)$$

Since one term in LHS is not a weighted sum of outputs, the superposition principle is not valid. Hence the system is non-linear.

3. The output depends on future values of input. Hence the system is non-causal.
4. All the coefficients of the differential equation are constant. Hence the system is time-invariant.

So the given system is dynamic, non-linear, non-causal and time-invariant.

(b) Given $$\frac{d^2y(t)}{dt^2} + 2y(t)\frac{dy(t)}{dt} + 3t\,y(t) = x(t)$$

1. The system is described by a differential equation. Hence the system is dynamic.
2. There is a term with product of output and its derivative {i.e. $y(t)[dy(t)/dt]$}. Hence the system is non-linear. This can be proved.

Let an input $x_1(t)$ produce an output $y_1(t)$

Then $$\frac{d^2y_1(t)}{dt^2} + 2y_1(t)\frac{dy_1(t)}{dt} + 3t\,y_1(t) = x_1(t)$$

Let an input $x_2(t)$ produce an output $y_2(t)$.

Then $$\frac{d^2y_2(t)}{dt^2} + 2y_2(t)\frac{dy_2(t)}{dt} + 3ty_2(t) = x_2(t)$$

The linear combination of the above equations gives

$$a\frac{d^2y_1(t)}{dt^2} + a2y_1(t)\frac{dy_1(t)}{dt} + a3ty_1(t) + \frac{bd^2y_2(t)}{dt^2} + b2y_2(t)\frac{dy_2(t)}{dt^2} + b3t\,y_2(t)$$
$$= ax_1(t) + bx_2(t)$$

i.e. $$\frac{d^2}{dt^2}[ay_1(t)+by_2(t)]+2\underbrace{\left[ay_1(t)\frac{dy_1(t)}{dt}+by_2(t)\frac{dy_2(t)}{dt}\right]}_{\textbf{Not a weighted sum of outputs}}+3t[ay_1(t)+by_2(t)]$$

$$=[ax_1(t)+bx_2(t)]$$

Since one term in LHS is not a weighted sum of outputs, the superposition principle is not valid. Hence the system is non-linear.

3. The output depends on present input only. Hence the system is causal.
4. All the coefficients of the differential equation are not constants. One coefficient is a function of time. Hence the system is time-variant.

So the given system is dynamic, non-linear, causal and time-variant.

(c) Given $$y(t)=ev\{x(t)\}$$

$$y(t)=ev\{x(t)\}=\frac{1}{2}[x(t)+x(-t)]$$

1. For positive values of t, the output depends on past values of input and for negative values of t, the output depends on future values of input. Hence the system is dynamic.

2. $$y(t)=T[x(t)]=\frac{1}{2}[x(t)+x(-t)]$$

For an input $x_1(t)$,

$$y_1(t)=\frac{1}{2}[x_1(t)+x_1(-t)]$$

For an input $x_2(t)$,

$$y_2(t)=\frac{1}{2}[x_2(t)+x_2(-t)]$$

The weighted sum of outputs is:

$$ay_1(t)+by_2(t)=a\frac{1}{2}[x_1(t)+x_1(-t)]+b\frac{1}{2}[x_2(t)+x_2(-t)]$$

$$=\frac{1}{2}\{[ax_1(t)+bx_2(t)]+[ax_1(-t)+bx_2(-t)]\}$$

The output due to weighted sum of inputs is:

$$y_3(t)=T[ax_1(t)+bx_2(t)]=\frac{1}{2}\{[ax_1(t)+bx_2(t)]+[ax_1(-t)+bx_2(-t)]\}$$

$$y_3(t)=ay_1(t)+by_2(t)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. Hence superposition principle is valid and the system is linear.

3. $$y(-2) = \frac{1}{2}[x(-2) + x(2)]$$

   i.e. for negative values of t, the output depends on future values of input. Hence the system is non-causal.

4. Given $$y(t) = \frac{1}{2}[x(t) + x(-t)]$$

   The output due to input delayed by T units is:

   $$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = \frac{1}{2}[x(t-T) + x(-t-T)]$$

   The output delayed by T units is:

   $$y(t-T) = y(t)\big|_{t=t-T} = \frac{1}{2}[x(t-T) + x(-t+T)]$$

   $$y(t, T) \neq y(t-T)$$

   So the system is time-variant.

   So the given system is dynamic, linear, non-causal and time-variant.

(d) Given $$y(t) = at^2 x(t) + bt\, x(t-4)$$

1. The output depends on past inputs. So it requires memory. Hence it is a dynamic system.

2. Given $$y(t) = at^2 x(t) + bt\, x(t-4)$$

   For an input $x_1(t)$,

   $$y_1(t) = at^2 x_1(t) + btx_1(t-4)$$

   For an input $x_2(t)$,

   $$y_2(t) = at^2 x_2(t) + btx_2(t-4)$$

   The weighted sum of outputs is:

   $$py_1(t) + qy_2(t) = pat^2 x_1(t) + pbt\, x_1(t-4) + qat^2 x_2(t) + qbt\, x_2(t-4)]$$

   $$= at^2[px_1(t) + qx_2(t)] + bt[px_1(t-4) + qx_2(t-4)]$$

   The output due to weighted sum of inputs is:

   $$y_3(t) = T[px_1(t) + qx_2(t)] = at^2[px_1(t) + qx_2(t)] + bt[px_1(t-4) + qx_2(t-4)]$$

   $$y_3(t) = ay_1(t) + by_2(t)$$

   Superposition principle is satisfied. Hence the system is linear.

3. The output depends only on the present and past inputs. It does not depend on future inputs. Hence the system is causal.

4. Given $$y(t) = T[x(t)] = at^2 x(t) + bt\, x(t-4)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = at^2 x(t-T) + bt\, x(t-4-T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = a(t-T)^2 x(t-T) + b(t-T)x(t-T-4)$$

$$y(t, T) \neq y(t-T)$$

Hence the system is time-variant.

So the given system is dynamic, linear, causal and time-variant.

(e) Given $$y(n) = x(n)\, x(n-2)$$

1. The output depends on past values of input. So it requires memory. Hence the system is dynamic.
2. The only term contains the product of input and delayed input. So the system is non-linear. This can be proved.

   Let an input $x_1(n)$ produce an output $y_1(n)$.

   Then $$y_1(n) = x_1(n)\, x_1(n-2)$$

   Let an input $x_2(n)$ produce an output $y_2(n)$.

   Then $$y_2(n) = x_2(n)\, x_2(n-2)$$

   The weighted sum of outputs is:

   $$ay_1(n) + by_2(n) = ax_1(n)\, x_1(n-2) + bx_2(n)\, x_2(n-2)$$

   The output due to weighted sum of inputs is:

   $$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]\,[ax_1(n-2) + bx_2(n-2)]$$

   $$y_3(n) \neq ay_1(n) + by_2(n)$$

   Hence the system is non-linear.
3. The output depends only on the present and past values of input. It does not depend on future values of input. So the system is causal.
4. Given $$y(n) = x(n)\, x(n-2)$$

   The output due to input delayed by k units is:

   $$y(n, k) = y(n)\big|_{x(n)=x(n-k)} = x(n-k)\, x(n-2-k)$$

   The output delayed by k units is:

   $$y(n-k) = y(n)\big|_{n=n-k} = x(n-k)\; x(n-k-2)$$

   $$y(n, k) = y(n-k)$$

   Hence the system is time-invariant.

   So the given system is dynamic, non-linear, causal and time-invariant.

(f) Given $$y(n) = \log_{10}|x(n)|$$

1. The output depends on present value of input only. Hence the system is static.
2. Given $$y(n) = \log_{10}|x(n)|$$

   Let an input $x_1(n)$ produce an output $y_1(n)$.

   Then $$y_1(n) = \log_{10}|x_1(n)|$$

   Let an input $x_2(n)$ produce an output $y_2(n)$.

   Then $$y_2(n) = \log_{10}|x_2(n)|$$

   The weighted sum of outputs is:

   $$ay_1(n) + by_2(n) = a\log_{10}|x_1(n)| + b\log_{10}|x_2(n)|$$

   The output due to weighted sum of inputs is:

   $$y_3(n) = T[ax_1(n) + bx_2(n)] = \log_{10}|ax_1(n) + bx_2(n)|$$

   $$y_3(n) \neq ay_1(n) + by_2(n)$$

   Hence the system is non-linear.
3. The output does not depend upon future inputs. Hence the system is causal.
4. $$y(n) = T[x(n)] = \log_{10}|x(n)|$$

   The output due to input delayed by k units is:

   $$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = \log_{10}|x(n-k)|$$

   The output delayed by k units is:

   $$y(n-k) = y(n)\big|_{n=n-k} = \log_{10}|x(n-k)|$$

   $$y(n, k) = y(n-k)$$

   Hence the system is time-invariant.

   So the given system is static, non-linear, causal and time-invariant.

(g) Given $$y(n) = a^n x(n)$$

1. The output at any instant depends only on the present values of input. Hence the system is static.
2. Given $$y(n) = a^n x(n)$$

   For an input $x_1(n)$,

   $$y_1(n) = a^n x_1(n)$$

   For an input $x_2(n)$,

   $$y_2(n) = a^n x_2(n)$$

   The weighted sum of outputs is:

   $$py_1(n) + qy_2(n) = pa^n x_1(n) + qa^n x_2(n) = a^n[px_1(n) + qx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[px_1(n) + qx_2(n)] = a^n[px_1(n) + qx_2(n)]$$

$$y_3(n) = py_1(n) + qy_2(n)$$

Hence the system is linear.

3. The output depends only on the present input. It does not depend on future inputs. Hence the system is causal.

4. Given $$y(n) = T[x(n)] = a^n x(n)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = a^n x(n-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = a^{n-k} x(n-k)$$

$$y(n, k) \neq y(n-k)$$

Hence the system is time-variant.

So the given system is static, linear, causal and time-variant.

(h) Given $$y(n) = x^2(n) + \frac{1}{x^2(n-1)}$$

1. The output at any instant depends upon past input. So memory is required. Hence the system is dynamic.

2. Given $$y(n) = x^2(n) + \frac{1}{x^2(n-1)}$$

There are square terms of input. So the system is non-linear. This can be proved.

For an input $x_1(n)$,

$$y_1(n) = x_1^2(n) + \frac{1}{x_1^2(n-1)}$$

For an input $x_2(n)$,

$$y_2(n) = x_2^2(n) + \frac{1}{x_2^2(n-1)}$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = ax_1^2(n) + \frac{a}{x_1^2(n-1)} + bx_2^2(n) + \frac{b}{x_2^2(n-1)}$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]^2 + \frac{1}{[ax_1(n-1) + bx_2(n-1)]^2}$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

Hence the system is non-linear.

3. The output does not depend on future values of input. Hence the system is causal.

4. Given $$y(n) = T[x(n)] = x^2(n) + \frac{1}{x^2(n-1)}$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x^2(n-k) + \frac{1}{x^2(n-1-k)}$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x^2(n-k) + \frac{1}{x^2(n-k-1)}$$

$$y(n, k) = y(n-k)$$

Hence the system is time-invariant.

So the given system is dynamic, non-linear, causal and time-invariant.

2.2.6 Stable and Unstable Systems

A bounded signal is a signal whose magnitude is always a finite value. For example, a sinewave is a bounded signal. A system is said to be bounded-input, bounded-output (BIBO) stable, if and only if every bounded input produces a bounded output. The output of a stable system does not diverge or does not grow unreasonably large.

Let the input signal $x(t)$ be bounded (finite), i.e.

$$|x(t)| \leq M_x < \infty \text{ for all } t$$

where M_x is a positive real number.

If $$|y(t)| \leq M_y < \infty$$

i.e. if $y(t)$ is also bounded, then the system is BIBO stable. Otherwise, the system is unstable. That is, we say that a system is unstable even if one bounded input produces an unbounded output.

It is very important to know about the stability of the system. Stability indicates the usefulness of the system. The stability can be found from the impulse response of the system which is nothing but the output of the system for a unit impulse input. If the impulse response is absolutely integrable for a continuous-time system or absolutely summable for a discrete-time system, then the system is stable.

BIBO stability criterion

The necessary and sufficient condition for a system to be BIBO stable is given by the expression

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

where $h(t)$ is the impulse response of the system. This is called BIBO stability criterion.

Proof: Consider a linear time-invariant system with $x(t)$ as input and $y(t)$ as output. The input and output of the system are related by the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Taking absolute values on both sides, we have

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right|$$

Using the fact that the absolute value of the integral of the product of two terms is always less than or equal to the integral of the product of their absolute values, we have

$$\left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right| \le \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|\, d\tau$$

If the input $x(\tau)$ is bounded, i.e. there exists a finite number M_x such that,

$$|x(\tau)| \le M_x < \infty$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(t-\tau)|\, d\tau$$

Changing the variables by $m = t - \tau$, the output is bounded if

$$\int_{-\infty}^{\infty} |h(m)|\, dm < \infty$$

Replacing m by t, we have $$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

which is the necessary and sufficient condition for a system to be BIBO stable.

Figure 2.5 shows bounded and unbounded signals.

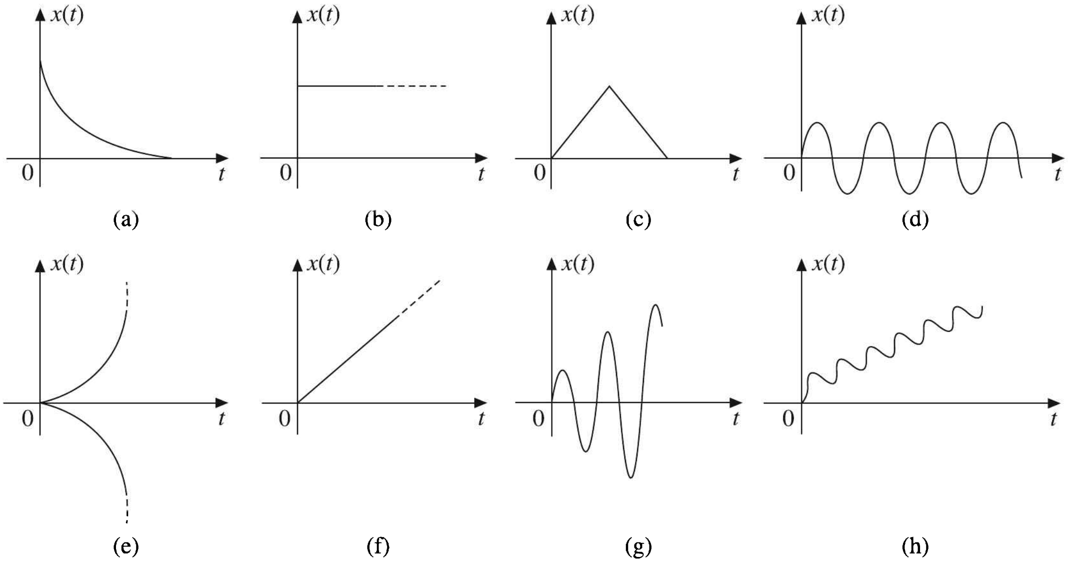


Figure 2.5 (a), (b), (c), (d) Bounded signals, (e), (f), (g), (h) Unbounded signals.

Figure 2.6 shows stable and unstable systems.

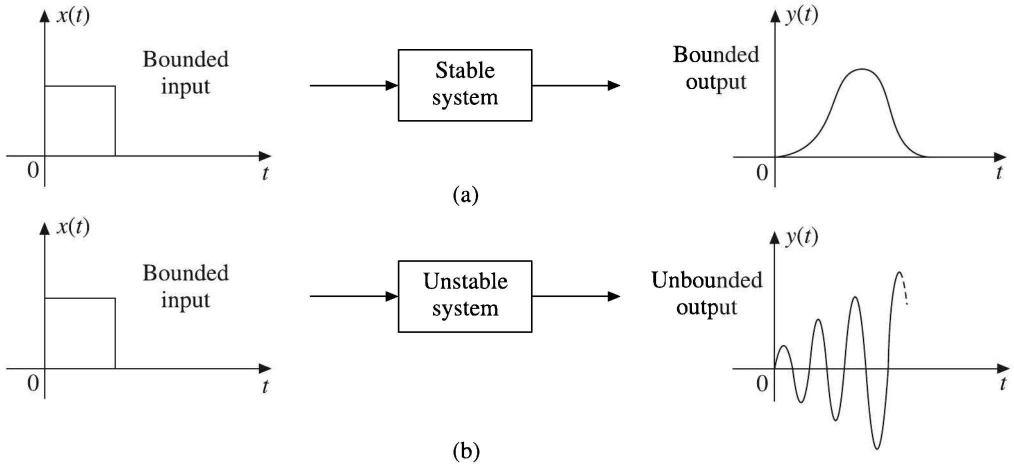


Figure 2.6 (a) Stable system, (b) Unstable system.

EXAMPLE 2.7 Find whether the following systems are stable or not:

(a) $y(t) = e^{x(t)};\ |x(t)| \le 8$ (b) $y(t) = (t + 5)\, u(t)$

(c) $h(t) = (2 + e^{-3t})\, u(t)$ (d) $h(t) = e^{2t}\, u(t)$

(e) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ (f) $h(t) = \dfrac{1}{RC} e^{-t/RC}\, u(t)$

(g) $h(t) = \omega_0 \left|\sin \omega_0 t\, u(t)\right|$

Solution:

(a) Given $$y(t) = e^{x(t)};\ |x(t)| \le 8$$

Here the input is bounded, $|x(t)| \le 8$

Therefore for stability, the output must be bounded.
The output $y(t)$ becomes

$$e^{-8} \le y(t) \le e^{8}$$

Hence $y(t)$ is also bounded. Therefore, the system is stable.

(b) Given $$y(t) = (t + 5)\, u(t)$$

$\therefore$ $$y(t) = t + 5 \quad \text{for } t \ge 0$$

So, as $t \to \infty$, $y(t) \to \infty$
Hence the output grows without any bound and hence the given system is unstable.

(c) Given $$h(t) = (2 + e^{-3t})\, u(t)$$

For a system to be stable,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Here $$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} (2 + e^{-3t})\, u(t)\, dt = \int_{0}^{\infty} (2 + e^{-3t})\, dt = \left[2t + \frac{e^{-3t}}{-3} \right]_0^{\infty} = \infty$$

Since the impulse response is not absolutely integrable, i.e. since $\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$, the system is unstable.

(d) Given $$h(t) = e^{2t} u(t)$$

$\therefore$ $$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left| e^{2t} u(t) \right| dt = \int_{0}^{\infty} e^{2t} dt = \left[\frac{e^{2t}}{2} \right]_0^{\infty} = \infty$$

Since the impulse response is not absolutely integrable, i.e. since $\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$, the system is unstable.

(e) Given $$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

For an input $\delta(t)$,

$$y(t) = h(t)$$

Therefore, $$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

For stability,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

In this case,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} u(t)\, dt = \infty$$

Therefore, the impulse response is unbounded and the system is unstable.

(f) Given $$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

For stability,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

In this case,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left| \frac{1}{RC} e^{-t/RC} u(t) \right| dt = \int_{0}^{\infty} \left| \frac{1}{RC} e^{-t/RC} \right| dt = 1$$

Therefore, the system is stable.

(g) Given $$h(t) = \omega_0 \left| \sin \omega_0 t\, u(t) \right|$$

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \omega_0 \int_{0}^{\infty} |\sin \omega_0 \tau|\, d\tau$$

This integral does not converge and so output is not bounded. Hence the system is unstable.

EXAMPLE 2.8 Check the stability of the system defined by

(a) $y(n) = ax(n-7)$

(b) $y(n) = x(n) + \frac{1}{2} x(n-1) + \frac{1}{4} x(n-2)$

(c) $h(n) = a^n$ for $0 < n < 11$

(d) $h(n) = 2^n u(n)$

(e) $h(n) = u(n)$

Solution:

(a) Given $$y(n) = ax(n-7)$$

Let $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

$\therefore$ $$h(n) = a\delta(n-7)$$

A system is stable if its impulse response $h(n)$ is absolutely summable.

i.e. $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} a\delta(n-7) = a \quad \text{for } n = 7$$

Hence the given system is stable if the value of a is finite.

(b) Given $$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$$

Let $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

$\therefore$ $$h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)$$

A discrete-time system is stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

The given $h(n)$ has a value only at $n = 0$, $n = 1$, and $n = 2$. For all other values of n from $-\infty$ to ∞, $h(n) = 0$.

At $n = 0$, $h(0) = \delta(0) + \frac{1}{2}\delta(0-1) + \frac{1}{4}\delta(0-2) = \delta(0) + \frac{1}{2}\delta(-1) + \frac{1}{4}\delta(-2) = 1$

At $n = 1$, $h(1) = \delta(1) + \frac{1}{2}\delta(1-1) + \frac{1}{4}\delta(1-2) = \delta(1) + \frac{1}{2}\delta(0) + \frac{1}{4}\delta(-1) = \frac{1}{2}$

At $n = 2$, $h(2) = \delta(2) + \frac{1}{2}\delta(2-1) + \frac{1}{4}\delta(2-2) = \delta(2) + \frac{1}{2}\delta(1) + \frac{1}{4}\delta(0) = \frac{1}{4}$

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} < \infty \quad \text{(a finite value)}$$

Hence the system is stable.

(c) Given $$h(n) = a^n \quad \text{for } 0 < n < 11$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |a^n| = \sum_{n=0}^{11} a^n = \frac{1-a^{12}}{1-a}$$

This value is finite for finite value of a. Hence the system is stable if a is finite.

(d) Given $$h(n) = 2^n u(n)$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |2^n u(n)| = \sum_{n=0}^{\infty} 2^n = \infty$$

The impulse response is not absolutely summable. Hence this system is unstable.

(e) Given $$h(n) = u(n)$$

For stability,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty$$

So the output is not bounded and the system is unstable.

EXAMPLE 2.9 Check whether the following digital systems are BIBO stable or not:

(a) $y(n) = ax(n+1) + bx(n-1)$

(b) $y(n) = \text{maximum of } [x(n), x(n-1), x(n-2)]$

(c) $y(n) = ax(n) + b$

(d) $y(n) = e^{-x(n)}$

(e) $y(n) = ax(n) + bx^2(n-1)$

Solution:

(a) Given $$y(n) = ax(n+1) + bx(n-1)$$

If $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

Hence the impulse response is:

$$h(n) = a\delta(n+1) + b\delta(n-1)$$

When $n = 0, h(0) = a\delta(1) + b\delta(-1) = 0$

When $n = 1, h(1) = a\delta(2) + b\delta(0) = b$

When $n = 2, h(2) = a\delta(3) + b\delta(1) = 0$

In general, $$h(n) = \begin{cases} b & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| = b$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

So the system is BIBO stable if $|b| < \infty$.

(b) Given $y(n) = \text{maximum of } [x(n), x(n-1), x(n-2)]$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

$\therefore$ $h(n) = \text{maximum of } [\delta(n), \delta(n-1), \delta(n-2)]$

$h(0) = \text{maximum of } [\delta(0), \delta(-1), \delta(-2)] = 1$

$h(1) = \text{maximum of } [\delta(1), \delta(0), \delta(-1)] = 1$

$h(2) = \text{maximum of } [\delta(2), \delta(1), \delta(0)] = 1$

$h(3) = \text{maximum of } [\delta(3), \delta(2), \delta(1)] = 0$

Similarly $h(4) = 0 = h(5) = h(6) \ldots$

$\therefore$
$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots$$
$$= 1 + 1 + 1 + 0 + 0 + \cdots = 3$$

So the given system is BIBO stable.

(c) Given $y(n) = ax(n) + b$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is:
$$h(n) = a\delta(n) + b$$

When $n = 0$, $h(0) = a\delta(0) + b = a + b$

When $n = 1$, $h(1) = a\delta(1) + b = b$

Here, $h(1) = h(2) = \ldots = h(n) = b$

Therefore,
$$h(n) = \begin{cases} a+b & \text{when } n = 0 \\ b & \text{when } n \neq 0 \end{cases}$$

The necessary and sufficient condition for BIBO stability is:
$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore,
$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots + \cdots$$
$$= |a+b| + |b| + |b| + \cdots + |b| + \cdots$$

This series never converges since the ratio between the successive terms is one. Hence the given system is BIBO unstable.

(d) Given $y(n) = e^{-x(n)}$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is:

$$h(n) = e^{-\delta(n)}$$

When $n = 0$, $h(0) = e^{-\delta(0)} = e^{-1}$

When $n = 1$, $h(1) = e^{-\delta(1)} = e^{0} = 1$

In general,

$$h(n) = \begin{cases} e^{-1} & \text{when } n = 0 \\ 1 & \text{when } n \neq 0 \end{cases}$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore, $$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots$$

$$= e^{-1} + 1 + 1 + 1 + \cdots + 1 + \cdots$$

Since the given sequence never converges, it is BIBO unstable.

(e) Given $$y(n) = ax(n) + bx^2(n-1)$$

If $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

Hence the impulse response is:

$$h(n) = a\delta(n) + b\delta^2(n-1)$$

When $n = 0$, $h(0) = a\delta(0) + b\delta^2(-1) = a$

When $n = 1$, $h(1) = a\delta(1) + b\delta^2(0) = b$

When $n = 2$, $h(2) = a\delta(2) + b\delta^2(1) = 0$

Hence, $$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots$$

$$= |a| + |b| + 0 + 0 + \cdots$$

Hence, the given system is BIBO stable if $|a| + |b| < \infty$.

EXAMPLE 2.10 Determine whether each of the system with impulse response/output listed below is (i) causal (ii) stable.

(a) $h(n) = 3^n u(-n)$

(b) $h(n) = \cos \dfrac{n\pi}{2}$

(c) $h(n) = \delta(n) + \cos n\pi$

(d) $h(n) = e^{3n} u(n-2)$

(e) $y(n) = \cos x(n)$

(f) $y(n) = \sum_{k=-\infty}^{n+5} x(k)$

(g) $y(n) = \log|x(n)|$ (h) $h(n) = [u(n) - u(n-15)]\, 2^n$

(i) $h(n) = 4^n u(2-n)$ (j) $h(n) = e^{-5|n|}$

Solution:

(a) Given $$h(n) = 3^n u(-n)$$

$u(-n)$ exists for $-\infty < n \le 0$. Hence $h(n) \ne 0$ for $n < 0$. So the system is non-causal.

For stability $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} 3^n u(-n) = \sum_{n=-\infty}^{0} 3^n = \sum_{n=0}^{\infty} 3^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$$

$$= \left(1 - \frac{1}{3}\right)^{-1} = \frac{1}{1-(1/3)} = \frac{3}{2} < \infty$$

So the system is stable.

(b) Given $$h(n) = \cos\frac{n\pi}{2}$$

$\cos(n\pi/2)$ exists for $-\infty < n < \infty$. So $h(n) \ne 0$ for $n < 0$. Therefore, the system is non-causal.

For stability $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} \left|\cos\frac{n\pi}{2}\right| = \infty$$

because for odd values of n, $\left|\cos\frac{n\pi}{2}\right| = 0$ and for even values of n, $\left|\cos\frac{n\pi}{2}\right| = 1$.

So the system is unstable.

(c) Given $$h(n) = \delta(n) + \cos n\pi$$

$\delta(n) = 1$ for $n = 0$ and $\delta(n) = 0$ for $n \ne 0$. $|\cos n\pi| = 1$ for all values of n.

For stability $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(-\infty)| + \cdots + |h(-1)| + |h(0)| + |h(1)| + \cdots + |h(\infty)|$$

$$= 1 + \cdots + 1 + 2 + 1 + \cdots + 1 = \infty$$

Therefore, the system is unstable.

(d) Given $$h(n) = e^{3n}u(n-2)$$

$u(n-2)$ exists only for $n \geq 2$. So $h(n) = 0$ for $n < 0$. Hence the system is causal.

For stability $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |e^{3n}u(n-2)| = \sum_{n=2}^{\infty} e^{3n}$$
$$= e^6 + e^9 + e^{12} + \cdots$$
$$= \infty$$

Therefore, the system is unstable.

(e) Given $$y(n) = \cos x(n)$$

For the system to be stable, it has to satisfy the following condition:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

If $x(n) = \delta(n)$, then the impulse response is:

$$h(n) = \cos \delta(n)$$

For $n = 0, h(0) = \cos \delta(0) = \cos 1 = 0.54$

For $n = 1, h(1) = \cos \delta(1) = \cos 0 = 1$

For $n = 2, h(2) = \cos \delta(2) = \cos 0 = 1$

For $n = -1, h(-1) = \cos \delta(-1) = \cos 0 = 1$

For $n = -2, h(-2) = \cos \delta(-2) = \cos 0 = 1$

Then $$\sum_{n=-\infty}^{\infty} |h(n)| = |h(-\infty)| + \cdots + |h(-2)| + |h(-1)| + |h(0)| + |h(1)| + |h(2)| + \cdots + |h(\infty)|$$
$$= 1 + 1 + \cdots + 1 + 0.54 + 1 + 1 + \cdots + 1$$
$$= \infty$$

The system is unstable.

(f) Given $$y(n) = \sum_{k=-\infty}^{n+5} x(k)$$

For the system to be stable,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Let $x(n) = \delta(n)$, then $y(n) = h(n)$. So for the given system.

$$h(n) = \sum_{k=-\infty}^{n+5} \delta(k)$$

For $n = -6$ $$h(-6) = \sum_{k=-\infty}^{-1} \delta(k) = 0$$

For $n = -5$ $$h(-5) = \sum_{k=-\infty}^{0} \delta(k) = 1$$

For $n = 1$ $$h(1) = \sum_{k=-\infty}^{6} \delta(k) = 1$$

For $n = -\infty$ to $n = -6$, $h(n) = 0$ and for $n = -5$ to $n = \infty$, $h(n) = 1$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} |h(n)| = 0 + 0 + \cdots + 1 + 1 + 1 + \cdots = \infty$$

So the given system is unstable.

(g) Given $$y(n) = \log_{10} |x(n)|$$

The output depends only on the present input. Hence the system is causal. The impulse response is:

$$h(n) = \log_{10} |\delta(n)|$$

$$h(0) = \log_{10} |\delta(0)| = \log_{10} 1 = 0$$

$$h(1) = \log_{10} |\delta(1)| = \log_{10} 0 = 0$$

$$h(2) = \log_{10} |\delta(2)| = \log_{10} 0 = 0$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} |h(n)| = 0 + 0 + \cdots = 0$$

(h) Given $$h(n) = [u(n) - u(n-15)]\, 2^n$$

$h(n) = 0$ for $n < 0$. So the system is causal.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} [u(n) - u(n-15)] 2^n$$

$$= \sum_{n=0}^{14} (1) 2^n = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{14} < \infty$$

Therefore, the system is stable.

(i) Given $$h(n) = 4^n u(2-n)$$

$h(n) \neq 0$ for $n < 0$. So the system is non-causal.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} 4^n u(2-n) = \sum_{n=-\infty}^{2} 4^n = \sum_{n=-\infty}^{0} 4^n + \sum_{n=1}^{2} 4^n$$

$$= \sum_{n=0}^{\infty} 4^{-n} + \sum_{n=1}^{2} 4^n = \left(1 - \frac{1}{4}\right)^{-1} + 4 + 4^2 = \left\{\frac{1}{[1-(1/4)]} + 20\right\} < \infty$$

Therefore, the system is stable.

(j) Given $$h(n) = e^{-5|n|}$$

The system is non-causal since $h(n) \neq 0$ for $n < 0$.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} e^{-5n} = \sum_{n=-\infty}^{-1} e^{5n} + \sum_{n=0}^{\infty} e^{-5n} = \sum_{n=1}^{\infty} e^{-5n} + \sum_{n=0}^{\infty} e^{-5n}$$

$$= \frac{e^{-5}}{1-e^{-5}} + \frac{1}{1-e^{-5}} = \frac{1+e^{-5}}{1-e^{-5}} < \infty$$

Therefore, the system is stable.

EXAMPLE 2.11 Comment about the linearity, stability, time invariance and causality for the following filter:

$$y(n) = 2x(n+1) + [x(n-1)]^2$$

Solution: Given $$y(n) = 2x(n+1) + [x(n-1)]^2$$

1. There is a square term of delayed input [i.e., $x(n-1)^2$] in the difference equation. So the system is nonlinear.
2. The output depends on the future value of input [i.e., $2x(n+1)$]. So the system is non-causal.
3. For $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \quad h(n) = 2\delta(n+1) + \{\delta(n-1)\}^2$$

$$h(0) = 2\delta(1) + \{\delta(-1)\}^2 = 0 + 0 = 0$$
$$h(1) = 2\delta(2) + \{\delta(0)\}^2 = 0 + 1 = 1$$
$$h(-1) = 2\delta(0) + \{\delta(-2)\}^2 = 2 + 0 = 2$$
$$h(-2) = 2\delta(-1) + \{\delta(-3)\}^2 = 0 + 0 = 0$$
$$h(n) = 0, \text{ for any other } n$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 0 + 1 + 2 + 0 + 0 + \cdots = 3 < \infty$$

Impulse response is absolutely summable. So the system is stable. Also we can say that since the output depends only on the delayed and advanced inputs, if the input is bounded, the output is bounded. So the system is BIBO stable.

4. The output due to delayed input is given by

$$y(n, k) = 2x(n+1-k) + \{x(n-1-k)\}^2$$

The delayed output is

$$y(n-k) = 2x(n+1-k) + \{x(n-k-1)\}^2$$

$$\therefore \quad y(n, k) = y(n-k)$$

Therefore, the system is time-invariant.

Also, we can say that since the system is described by constant coefficient difference equation, the system is time-invariant. So the given system is non-linear, stable, time-invariant and non-causal.

EXAMPLE 2.12 State whether the following system is linear, causal, time-invariant and stable:

$$y(n) + y(n-1) = x(n) + x(n-2)$$

Solution: Given $\quad y(n) = -y(n-1) + x(n) + x(n-2)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Therefore, the weighted sum of outputs is:

$$ay_1(n) + by_2(n) = -[ay_1(n-1) + by_2(n-1)] + [ax_1(n) + bx_2(n)] + [ax_1(n-2) + bx_2(n-2)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = -\{ay_1(n-1) + by_2(n-1)\} + \{ax_1(n) + bx_2(n)\} + \{ax_1(n-2) + bx_2(n-2)\}$$

$$y_3(n) = ay_1(n) + by_2(n)$$

So the system is linear.

2. The output depends only on the present and past inputs and past outputs. So the system is causal.
3. All the coefficients of the differential equation are constants. So the system is time-invariant.
4. For $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \qquad h(n) = -h(n-1) + \delta(n) + \delta(n-2)$$

$$h(0) = -h(-1) + \delta(0) + \delta(-2) = 1$$

$$h(1) = -h(0) + \delta(1) + \delta(-1) = -1$$

$$h(2) = -h(1) + \delta(2) + \delta(0) = 1 + 0 + 1 = 2$$

$$h(3) = -h(2) + \delta(3) + \delta(1) = -2 + 0 + 0 = -2$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + 2 + 2 + \cdots = \infty$$

i.e., the impulse response is not absolutely summable. So the system is unstable. Therefore, the given system is non-linear, causal, time-variant and unstable.

EXAMPLE 2.13 Determine whether the following system is linear, stable, causal and time-invariant using appropriate tests:

$$y(n) = nx(n) + x(n+2) + y(n-2)$$

Solution: Given $\quad y(n) = nx(n) + x(n+2) + y(n-2)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$ay_1(n) + by_2(n) = n[ax_1(n) + bx_2(n)] + [ax_1(n+2) + bx_2(n+2)] + [ay_1(n-2) + by_2(n-2)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = n\{ax_1(n) + bx_2(n)\} + \{ax_1(n+2) + bx_2(n+2)\} + \{ay_1(n-2) + by_2(n-2)\}$$

$$y_3(n) = ay_1(n) + by_2(n)$$

So the system is linear.

2. For $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \qquad h(n) = n\delta(n) + \delta(n+2) + h(n-2)$$

$$h(-2) = -2\delta(-2) + \delta(0) + h(-4) = 1$$

$$h(0) = 0\delta(0) + \delta(2) + h(-2) = 0 + 0 + 1 = 1$$

$$h(1) = 1\delta(1) + \delta(3) + h(-1) = 0$$

$$h(2) = 2\delta(2) + \delta(4) + h(0) = 1$$

$$h(3) = 3\delta(3) + \delta(5) + h(1) = 0$$

$$h(4) = 4\delta(4) + \delta(6) + h(2) = 1$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 0 + 1 + 0 + \cdots = \infty$$

So the system is unstable.

3. $y(2) = 2x(2) + x(4) + y(0)$

The output depends on future inputs. So the system is non-causal.

4. The coefficient of $x(n)$ is a function of time. So it is a time-varying system. Therefore, the given system is linear, unstable, non-causal and time-varying.

EXAMPLE 2.14 Find the linearity, invariance, causality of the following systems:

(a) $y(n) = -ax(n-1) + x(n)$

(b) $y(n) = x(n^2) + x(-n)$

Solution:

(a) Given $\qquad y(n) = -ax(n-1) + x(n)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$py_1(n) + qy_2(n) = -a[px_1(n-1) + qx_2(n-1)] + [px_1(n) + qx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = -a[px_1(n-1) + qx_2(n-1)] + [px_1(n) + qx_2(n)]$$

$$y_3(n) = py_1(n) + qy_2(n)$$

So the system is linear.

2. The output depends only on the present and past inputs. So the system is causal.

3. The output due to delayed input is:

$$y(n, k) = -ax(n-1-k) + x(n-k)$$

The delayed output is:

$$y(n-k) = -ax(n-1-k) + x(n-k)$$

$$y(n, k) = y(n - k)$$

So the system is time-invariant. Therefore, the given system is linear, causal and time invariant.

(b) Given $$y(n) = x(n^2) + x(-n)$$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$ay_1(n) + by_2(n) = [ax_1(n^2) + bx_2(n^2)] + [ax_1(-n) + bx_2(-n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = \{ax_1(n^2) + bx_2(n^2)\} + \{ax_1(-n) + bx_2(-n)\}$$
$$y_3(n) = ay_1(n) + by_2(n)\}$$

So the system is linear.

2. $y(-2) = x(4) + x(2)$

$y(2) = x(4) + x(-2)$

The output depends upon future inputs. So the system is non-causal.

3. The output due to delayed input is:

$$y(n, k) = x(n^2 - k) + x(-n - k)$$

The delayed output is:

$$y(n - k) = x\{(n - k)^2\} + x\{-(n - k)\}$$
$$y(n, k) \neq y(n - k)$$

So the system is time-variant. Therefore, the system is linear, non-causal and time variant.

EXAMPLE 2.15 Test the causality and stability of the following system:

$$y(n) = x(n) - x(-n - 1) + x(n - 1)$$

Solution: Given $$y(n) = x(n) - x(-n - 1) + x(n - 1)$$

1. $y(-2) = x(-2) - x(1) + x(-3)$
   For negative values of n, the output depends on future values of input. So the system is non-causal.
2. For $x(n) = \delta(n)$, $y(n) = h(n)$

$\therefore$
$$h(n) = \delta(n) - \delta(-n - 1) + \delta(n - 1)$$
$$h(0) = \delta(0) - \delta(-1) + \delta(-1) = 1 - 0 + 0 = 1$$
$$h(1) = \delta(1) - \delta(-2) + \delta(0) = 0 - 0 + 1 = 1$$
$$h(-1) = \delta(-1) - \delta(0) + \delta(-2) = 0 - 1 + 0 = -1$$
$$h(n) = 0 \text{ for any other value of } n$$
$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + 1 + 0 + 0 + \cdots = 3 < \infty$$

i.e., the impulse response is absolutely summable. So the system is stable.

EXAMPLE 2.16 If a system is represented by the following difference equation:

$$y(n) = 3y^2(n-1) - nx(n) + 4x(n-1) - x(n+1),\ n \geq 0$$

(a) Is the system linear? Explain.
(b) Is the system shift-invariant? Explain.
(c) Is the system causal? Why or why not?

Solution: Given $y(n) = 3y^2(n-1) - nx(n) + 4x(n-1) - x(n+1),\ n \geq 0$

(a) No, the system is non-linear because there is a square term of delayed output in the difference equation.
(b) No, the system is shift variant because the coefficient of $x(n)$ is not a constant. It is a function of time.
(c) No, the system is non-causal because the output depends on future inputs.

EXAMPLE 2.17 Test the following systems for linearity, time invariance, stability and causality:

(a) $y(n) = a^{\{x(n)\}}$ (b) $y(n) = \sin\left\{\dfrac{2\pi bfn}{F}\right\} x(n)$

Solution:

(a) Given $y(n) = a^{\{x(n)\}}$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$py_1(n) + qy_2(n) = pa^{\{x_1(n)\}} + qa^{\{x_2(n)\}}$$

The output due to weighted sum of inputs is:

$$y_3(n) = a^{\{px_1(n) + qx_2(n)\}}$$

$$y_3(n) \neq py_1(n) + qy_2(n)$$

So the system is non-linear.

2. The output due to delayed input is:

$$y(n, k) = a^{\{x(n-k)\}}$$

The delayed output is:

$$y(n-k) = a^{\{x(n-k)\}}$$

$$y(n, k) = y(n-k)$$

So the system is shift-invariant.

3. When input $x(n) = \delta(n)$, $y(n) = h(n)$

$$h(n) = a^{\{\delta(n)\}}$$

$$h(0) = a^{\{\delta(0)\}} = a$$

$$h(n) = a^0 = 1 \quad \text{(for any other } n\text{)}$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + \cdots + a + 1 + 1 + \cdots + = \infty$$

The impulse response is not absolutely summable. So the system is unstable.

4. The output depends only on the present input. So the system is causal. Therefore, the given system is nonlinear, shift invariant, unstable and causal.

(b) Given $$y(n) = \sin\left\{\frac{2\pi bfn}{F}\right\} x(n) = \sin\frac{n\omega b}{F} x(n)$$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$py_1(n) + qy_2(n) = p\sin\left(\frac{n\omega b}{F}\right)x_1(n) + q\sin\left(\frac{n\omega b}{F}\right)x_2(n)$$

$$= \sin\left(\frac{n\omega b}{F}\right)[px_1(n) + qx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = \sin\left(\frac{n\omega b}{F}\right)[px_1(n) + qx_2(n)]$$

$\therefore$ $$y_3(n) = py_1(n) + qy_2(n)$$

So the system is linear.

2. The output due to delayed input is:

$$y(n, k) = \sin\left(\frac{n\omega b}{F}\right)x(n-k)$$

The delayed output is:

$$y(n-k) = \sin\left[\frac{(n-k)\,\omega b}{F}\right]x(n-k)$$

$$y(n, k) \neq y(n-k)$$

So the system is shift invariant.

3. When $x(n) = \delta(n)$, $y(n) = h(n)$

$\therefore$ $$h(n) = \sin\left(\frac{n\omega b}{F}\right)\delta(n)$$

$$h(0) = \sin(0)\,\delta(0) = 0$$

$$h(1) = \sin\left(\frac{\omega b}{F}\right)\delta(1) = 0$$

$$h(-1) = \sin\left(\frac{-\omega b}{F}\right)\delta(-1) = 0$$

$$h(n) = 0 \quad \text{for all other } n$$

$\sum_{n=-\infty}^{\infty} |h(n)| = 0$. So the system is stable.

4. The output depends upon present input only. So the system is causal. Therefore, the given system is linear, shift invariant, stable and causal.

2.2.7 Invertible and Non-invertible Systems

If a system has a unique relationship between its input $x(t)$ [or $x(n)$] and output $y(t)$ [or $y(n)$], the system is known as invertible. Therefore, for an invertible system if $y(t)$ [or $y(n)$] is known, $x(t)$ [or $x(n)$] can be found out unambiguously and uniquely. On the other hand, if the system does not have a unique relationship between its input and output, the system is said to be non-invertible.

In other words a system is known as invertible only if an inverse system exists which when cascaded with the original system produces an output equal to the input of the first system.

For $y(t) = 3x(t)$ [or $y(n) = 3x(n)$], the system is said to be invertible, whereas for $y(t) = 2x^2(t)$ [or $y(n) = 2x^2(n)$], the system is said to be non-invertible. Mathematically, a system is to be invertible if

$$x(t) = T^{-1}\{T[x(t)]\}$$

or

$$x(n) = T^{-1}\{T[x(n)]\}$$

The block diagram representation of both an invertible and non-invertible system is shown in Figure 2.7.

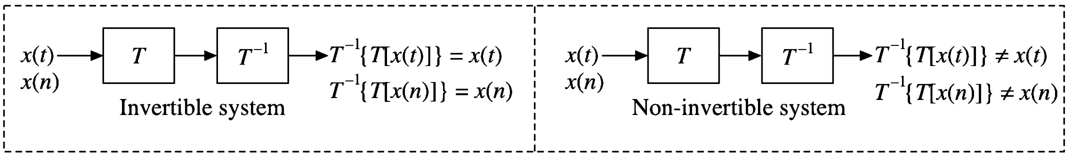


Figure 2.7 Invertibility property in continuous domain.

2.2.8 FIR and IIR Systems

Linear time invariant systems can be classified according to the type of impulse response. If the impulse response sequence is of finite duration, the system is called a finite impulse response(FIR) system and if the impulse response sequence is of infinite duration, the system is called an infinite impulse response(IIR) system.

An example of FIR system is described by

$$h(n) = \begin{cases} -2 & n = 2, 4 \\ 2 & n = 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

An example of IIR system is described by

$$h(n) = 2^n\, u(n)$$

Impulse response

Impulse response is the output of the system for a unit impulse input. For understanding the system behaviour, unit impulse response is very important.

If Input $x(t) = \delta(t)$

then Output $y(t) = h(t)$

$$L[\delta(t)] = 1 \quad \text{and} \quad F[\delta(t)] = 1$$

Once the transfer function $H(\omega)$ of an LTI system is known in frequency domain, the impulse response of the system can be found by finding the inverse Fourier transform of $H(\omega)$.

i.e. $$h(t) = F^{-1}[H(\omega)]$$

Similarly, once the transfer function $H(s)$ of an LTI system is known in s-domain, the impulse response of the system can be found by finding the inverse Laplace transform of $H(s)$.

i.e. $$h(t) = L^{-1}[H(s)]$$

Response of a linear system

Once the impulse response $h(t)$ is known, the response of the linear system $y(t)$ for any given input $x(t)$ can be obtained by convolving the input with the impulse response of the system.

$$y(t) = h(t) * x(t) = x(t) * h(t)$$

If a noncausal signal is applied to a noncausal system, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

If a causal signal is applied to a non-causal system, then

$$y(t) = \int_{-\infty}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

If a noncausal signal is applied to a causal system, then

$$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

If a causal signal is applied to a causal system, then

$$y(t) = \int_{0}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

Step response

The step response of an LTI continuous-time system can be obtained by using convolution integral. If $u(t)$ is an input to a system with impulse response $h(t)$, then the step response is given by

$$h(t) = \frac{ds(t)}{dt}$$

The unit impulse response is the first derivative of the unit step response.

Therefore the unit step response can also be used to characterize an LTI system because a unit impulse response can be characterized from the unit step response.

6.4 TRANSFER FUNCTION OF AN LTI SYSTEM

A continuous-time system is shown in Figure 6.4.

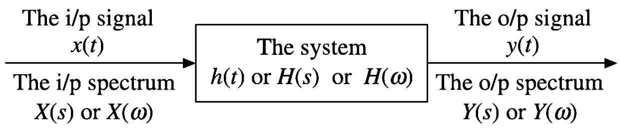


Figure 6.4 A system.

The transfer function of a continuous-time LTI system may be defined using Fourier transform or Laplace transform. The transfer function is defined only under zero initial conditions.

The transfer function of a LTI system $H(\omega)$ is defined as the ratio of the Fourier transform of the output signal to the Fourier transform of the input signal when the initial conditions are zero. Or simply we can say that it is the ratio of output to input in frequency domain when the initial conditions are neglected. Or simply we can say that the transfer function $H(\omega)$ of a LTI system is the Fourier transform of its impulse response.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$H(\omega)$ is a complex quantity having magnitude and phase.

$$H(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

The transfer function in frequency domain $H(\omega)$ is also called the frequency response of the system. The frequency response is amplitude response plus phase response.

$|H(\omega)|$ = Amplitude response of the system.

$\theta(\omega) = \angle H(\omega)$ = Phase response of the system.

We can say that $H(\omega)$ is a frequency domain representation of a system.

Since $$Y(\omega) = H(\omega)\, X(\omega)$$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

and $$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

$H(\omega)$ has conjugate symmetry property.

$$H(-\omega) = H^*(\omega)$$

i.e.
$$|H(-\omega)| = |H(\omega)|$$

and
$$\underline{/H(-\omega)} = -\underline{/H(\omega)}$$

The impulse response $h(t)$ of a system is the inverse Fourier transform of its transfer function (frequency response) $H(\omega)$.

$$H(\omega) = \mathrm{F}[h(t)]$$

or
$$h(t) = \mathrm{F}^{-1}[H(\omega)]$$

The transfer function of a system in s-domain (Laplace domain) is defined as the ratio of the Laplace transform of the output of the system to the Laplace transform of the input of the system when the initial conditions are neglected. Or simply we can say that it is the ratio of the output to input in s-domain when the initial conditions are zero. The transfer function of a system can also be defined as the Laplace transform of the impulse response of the system.

$$H(s) = \frac{Y(s)}{X(s)}$$

or
$$H(s) = \mathrm{L}\,[h(t)]$$

The impulse response is nothing but the inverse Laplace transform of the transfer function $H(s)$

$$h(t) = \mathrm{L}^{-1}\,[H(s)]$$

Once the transfer function in s-domain $H(s)$ is known, the transfer function in frequency domain $H(\omega)$ can be found by replacing s in $H(s)$ by $j\omega$.

i.e.
$$H(\omega) = H(s)\big|_{s = j\omega}$$

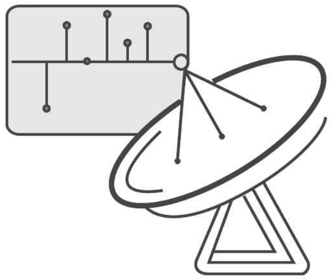
6.5 FILTER CHARACTERISTICS OF LINEAR SYSTEMS

For a given system an input signal $x(t)$ gives rise to a response signal $y(t)$. The system, therefore, processes the signal $x(t)$ in a way that is characteristic of the system. The spectral density function of the input signal $x(t)$ is given by $X(s)$ or $X(\omega)$, and the spectral density function of the response signal $y(t)$ is given by $Y(s)$ or $Y(\omega)$. $Y(s) = H(s)\ X(s)$ or $Y(\omega) = H(\omega)\ X(\omega)$ where $H(s)$ or $H(\omega)$ is the transfer function or system function of the system (shown in Figure 6.1).

The system, therefore, modifies the spectral density function of the input. The system acts as a kind of filter for various frequency components. Some frequency components are boosted in strength, i.e. they are amplified. Some frequency components are weakened in strength, i.e. they are attenuated and some may remain unaffected. Similarly, each frequency component suffers a different amount of phase shift in the process of transmission. The

7

Convolution and Correlation of Signals



7.1 INTRODUCTION

Convolution is a mathematical way of combining two signals to form a third signal. Convolution is important because it relates the input signal and impulse response of the system to the output of the system. Correlation is again a mathematical operation that is similar to convolution. Correlation also uses two signals to form a third signal. It is very widely used in practice, particularly in communication engineering. Basically, it compares two signals in order to determine the degree of similarity between them. Radar, sonar, and digital communications use correlation of signals very extensively. Correlation may be cross correlation or autocorrelation. When one signal is correlated with another signal to form a third signal, it is called cross correlation. When a signal is correlated with itself to form another signal, it is called autocorrelation.

In this chapter, the properties of convolution, convolution theorems associated with Fourier transforms and graphical convolution of two signals are discussed. Also correlation, energy density spectra, power density spectra, Rayleigh's theorem and Parseval's theorem are discussed.

7.2 CONCEPT OF CONVOLUTION

Convolution is a mathematical operation which is used to express the input–output relationship of an LTI system. It is a most important operation in LTI continuous-time systems. It relates input and impulse response of the system to output.

An arbitrary driving function $x(t)$ can be expressed as a continuous sum of impulse functions. The response $y(t)$ is then given by the continuous sum of responses to various impulse components. In fact, the convolution integral precisely expresses the response as a continuous sum of responses to individual impulse components.

Consider an LTI system which is initially relaxed at $t = 0$. If the input to the system is an impulse, then the output of the system is denoted by $h(t)$ and is called the impulse response of the system.

The impulse response is denoted as:

$$h(t) = T[\delta(t)]$$

We know that any arbitrary signal $x(t)$ can be represented as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

The system output is given by

$$y(t) = T[x(t)]$$

$\therefore$

$$y(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$$

For a linear system,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,T[\delta(t-\tau)]\,d\tau$$

If the response of the system due to impulse $\delta(t)$ is $h(t)$, then the response of the system due to delayed impulse is:

$$h(t, \tau) = T[\delta(t - \tau)]$$

Substituting this value of $T[\delta(t - \tau)]$ in the expression for $y(t)$, we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t, \tau)\,d\tau$$

For a time invariant system, the output due to input delayed by τ sec is equal to the output delayed by τ sec. That is,

$$h(t, \tau) = h(t - \tau)$$

Substituting this value of $h(t, \tau)$ in the expression for $y(t)$, we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

This is called convolution integral, or simply convolution. The convolution of two signals $x(t)$ and $h(t)$ can be represented as:

$$y(t) = x(t) * h(t)$$

In general, the lower and upper limits of integration in the convolution integral depend on whether the signal $x(t)$ and the impulse response $h(t)$ are causal or not. If $h(t)$ is causal, then $h(t - \tau) = 0$ for $\tau > t$. Therefore, the upper limit of integration is t for a causal $h(t)$. If $x(t)$ is causal, then $x(t) = 0$ for $t < 0$. Therefore, the lower limit of integration is 0 for a causal $x(t)$. Thus,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{if both } x(t) \text{ and } h(t) \text{ are non-causal}$$

$$= \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau \quad \text{if } x(t) \text{ is non-causal and } h(t) \text{ is causal}$$

$$= \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{if } x(t) \text{ is causal and } h(t) \text{ is non-causal}$$

$$= \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau \quad \text{if both } x(t) \text{ and } h(t) \text{ are causal}$$

7.3 PROPERTIES OF CONVOLUTION

Let us consider two signals $x_1(t)$ and $x_2(t)$. The convolution of two signals $x_1(t)$ and $x_2(t)$ is given by

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

The properties of convolution are as follows:

Commutative property The commutative property of convolution states that

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Distributive property The distributive property of convolution states that

$$x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

Associative property The associative property of convolution states that

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

Shift property The shift property of convolution states that if

$$x_1(t) * x_2(t) = z(t)$$

Then $\quad x_1(t) * x_2(t-T) = z(t-T)$

Similarly, $\quad x_1(t-T) * x_2(t) = z(t-T)$

and $\quad x_1(t-T_1) * x_2(t-T_2) = z(t-T_1-T_2)$

Convolution with an impulse Convolution of a signal $x(t)$ with a unit impulse is the signal itself. That is,

$$x(t) * \delta(t) = x(t)$$

Width property Let the duration of $x_1(t)$ and $x_2(t)$ be T_1 and T_2 respectively. Then the duration of the signal obtained by convolving $x_1(t)$ and $x_2(t)$ is $T_1 + T_2$.

EXAMPLE 7.1 Find the convolution of the following signals:

(i) $x_1(t) = e^{-2t}\, u(t)$; $x_2(t) = e^{-4t}\, u(t)$

(ii) $x_1(t) = t\, u(t)$; $x_2(t) = t\, u(t)$

(iii) $x_1(t) = \cos t\, u(t)$; $x_2(t) = u(t)$

(iv) $x_1(t) = e^{-3t}\, u(t)$; $x_2(t) = u(t + 3)$

(v) $x_1(t) = r(t)$; $x_2(t) = e^{-2t}\, u(t)$

Solution:

(i) Given $x_1(t) = e^{-2t}\, u(t)$; $x_2(t) = e^{-4t}\, u(t)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} e^{-2\tau}\, u(\tau)\, e^{-4(t-\tau)}\, u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t - \tau) = 1$ for $(t - \tau) \ge 0$ or for $\tau < t$.
Hence $u(\tau)\, u(t - \tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\, u(t - \tau) = 0$.

$$\therefore \quad x_1(t) * x_2(t) = \int_0^t e^{-2\tau}\, e^{-4(t-\tau)}\, d\tau$$

$$= e^{-4t} \int_0^t e^{2\tau}\, d\tau = e^{-4t} \left[\frac{e^{2\tau}}{2} \right]_0^t$$

$$= e^{-4t} \left(\frac{e^{2t} - 1}{2} \right) = \frac{e^{-2t} - e^{-4t}}{2} \qquad (\text{for } t \ge 0)$$

$$= \frac{e^{-2t} - e^{-4t}}{2}\, u(t)$$

(ii) Given $x_1(t) = t\, u(t)$; $x_2(t) = t\, u(t)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \tau u(\tau)\, (t-\tau)\, u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t - \tau) = 1$ for $(t - \tau) \ge 0$ or for $\tau < t$.
Hence $u(\tau)\, u(t - \tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\, u(t - \tau) = 0$.

$$\therefore \qquad x_1(t) * x_2(t) = \int_0^t \tau(t-\tau)\, d\tau$$

$$= \int_0^t t\tau\, d\tau - \int_0^t \tau^2 d\tau = t\left[\frac{\tau^2}{2}\right]_0^t - \left[\frac{\tau^3}{3}\right]_0^t$$

$$= t\left(\frac{t^2}{2} - 0\right) - \left(\frac{t^3}{3} - 0\right) = \frac{t^3}{2} - \frac{t^3}{3}$$

$$= \frac{t^3}{6} \qquad (\text{for } t \geq 0)$$

$$\therefore \qquad x_1(t) * x_2(t) = \frac{t^3}{6} u(t)$$

(iii) Given $\qquad x_1(t) = \cos t\, u(t);\ x_2(t) = u(t)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \qquad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \cos\tau\, u(\tau)\, u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t - \tau) = 1$ for $(t - \tau) \geq 0$ or for $\tau < t$.
Hence $u(\tau)\, u(t - \tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\, u(t - \tau) = 0$.

$$\therefore \qquad x_1(t) * x_2(t) = \int_0^t \cos\tau\, d\tau$$

$$= [\sin\tau]_0^t$$

$$= \sin t \qquad \text{for } t \geq 0$$

$$\therefore \qquad x_1(t) * x_2(t) = \sin t\, u(t)$$

(iv) Given $\qquad x_1(t) = e^{-3t} u(t);\ x_2(t) = u(t + 3)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \qquad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} e^{-3\tau}\, u(\tau)\, u(t+3-\tau)$$

In this case, $u(\tau) = 0$ for $\tau < 0$ and $u(t + 3 - \tau) = 0$ for $\tau > t + 3$.
$u(\tau)\ u(t + 3 - \tau) = 1$ only for $0 < \tau < t + 3$. For all other values of τ, $u(\tau)\ u(t + 3 - \tau) = 0$.

$$\therefore \qquad x_1(t) * x_2(t) = \int_0^{t+3} e^{-3\tau}\, d\tau$$

$$= \left[\frac{e^{-3\tau}}{-3}\right]_0^{t+3} = \frac{e^{-3(t+3)} - 1}{-3}$$

$$= \frac{1 - e^{-3(t+3)}}{3}$$

$$\therefore \qquad y(t) = 0 \qquad \text{(for } t < -3\text{)}$$

$$= \frac{1 - e^{-3(t+3)}}{3} \qquad \text{(for } t > -3\text{)}$$

(v) Given $\qquad x_1(t) = r(t) = tu(t);\ x_2(t) = e^{-2t}\, u(t)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t - \tau)\, d\tau$$

$$\therefore \qquad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \tau u(\tau)\, e^{-2(t-\tau)}\, u(t - \tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t - \tau) = 1$ for $(t - \tau) \geq 0$ or for $\tau < t$.
Hence $u(\tau)\ u(t - \tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\ u(t - \tau) = 0$.

$$\therefore \qquad x_1(t) * x_2(t) = \int_0^{t} \tau e^{-2(t-\tau)}\, d\tau$$

$$= e^{-2t} \int_0^{t} \tau e^{2\tau}\, d\tau = e^{-2t} \left\{\left[\frac{\tau e^{2\tau}}{2}\right]_0^{t} - \int_0^{t} \frac{e^{2\tau}}{2} d\tau\right\}$$

$$= e^{-2t} \left\{\left[\frac{te^{2t}}{2} - \left[\frac{e^{2\tau}}{4}\right]_0^{t}\right]\right\} = e^{-2t} \left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4}\right)$$

$$= \frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4} \qquad \text{(for } t \geq 0\text{)}$$

$$\therefore \qquad x_1(t) * x_2(t) = \left(\frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4}\right) u(t)$$

7.4 CONVOLUTION THEOREMS

Convolution of signals may be done either in time domain or in frequency domain. So there are following two theorems of convolution associated with Fourier transforms:

1. Time convolution theorem
2. Frequency convolution theorem

7.4.1 Time Convolution Theorem

The time convolution theorem states that convolution in time domain is equivalent to multiplication of their spectra in frequency domain. Mathematically, if

$$x_1(t) \longleftrightarrow X_1(\omega)$$

and
$$x_2(t) \longleftrightarrow X_2(\omega)$$

Then
$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)\, X_2(\omega)$$

Proof:
$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t}\, dt$$

We have
$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$\therefore$
$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} [x_1(\tau)\, x_2(t-\tau)\, d\tau] \right\} e^{-j\omega t} dt$$

Interchanging the order of integration, we have

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau) e^{-j\omega t}\, dt \right] d\tau$$

Letting $t - \tau = p$, in the second integration, we have

$$t = p + \tau \text{ and } dt = dp$$

$\therefore$
$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p) e^{-j\omega(p+\tau)} dp \right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p) e^{-j\omega p} dp \right] e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, X_2(\omega)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau\, X_2(\omega)$$

$$= X_1(\omega)\, X_2(\omega)$$

$\therefore \qquad x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)\, X_2(\omega)$

This is time convolution theorem.

7.4.2 Frequency Convolution Theorem

The frequency convolution theorem states that the multiplication of two functions in time domain is equivalent to convolution of their spectra in frequency domain. Mathematically, if

$$x_1(t) \longleftrightarrow X_1(\omega)$$

and
$$x_2(t) \longleftrightarrow X_2(\omega)$$

Then
$$x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

Proof:
$$\mathrm{F}[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} [x_1(t)\, x_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda) e^{j\lambda t}\, d\lambda\right] x_2(t)\, e^{-j\omega t}\, dt$$

Interchanging the order of integration, we get

$$\mathrm{F}[x_1(t)\, x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t) e^{-j\omega t}\, e^{j\lambda t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t) e^{-j(\omega-\lambda)t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega-\lambda)\, d\lambda$$

$$= \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

$\therefore$
$$x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

or
$$2\pi x_1(t)\, x_2(t) \longleftrightarrow X_1(\omega) * X_2(\omega)$$

This is frequency convolution theorem in radian frequency.
In terms of frequency, we get

$$\mathrm{F}[x_1(t)\, x_2(t)] = X_1(f) * X_2(f)$$

EXAMPLE 7.2 Find the convolution of the signals $x_1(t) = e^{-at}\,u(t)$; $x_2(t) = e^{-bt}\,u(t)$ using Fourier transform.

Solution: Given $$x_1(t) = e^{-at}\,u(t)$$

$$\therefore \qquad X_1(\omega) = \frac{1}{a + j\omega}$$

$$x_2(t) = e^{-bt}\,u(t)$$

$$\therefore \qquad X_2(\omega) = \frac{1}{b + j\omega}$$

We know that $$\mathrm{F}[x_1(t) * x_2(t)] = X_1(\omega)\,X_2(\omega)$$

$$\therefore \qquad x_1(t) * x_2(t) = \mathrm{F}^{-1}[X_1(\omega)\,X_2(\omega)]$$

$$\therefore \qquad x_1(t) * x_2(t) = \mathrm{F}^{-1}\left[\frac{1}{(a + j\omega)\,(b + j\omega)}\right] = \mathrm{F}^{-1}\left[\frac{1}{(b-a)}\left(\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right)\right]$$

$$= \frac{1}{b-a}\left[\mathrm{F}^{-1}\left(\frac{1}{a+j\omega}\right) - \mathrm{F}^{-1}\left(\frac{1}{b+j\omega}\right)\right]$$

$$= \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right]$$

EXAMPLE 7.3 Find the convolution of the signals $x_1(t) = 2e^{-2t}\,u(t)$ and $x_2(t) = u(t)$ using Fourier transform.

Solution: Given $$x_1(t) = 2e^{-2t}u(t)$$

$$\therefore \qquad X_1(\omega) = \frac{2}{j\omega + 2}$$

$$x_2(t) = u(t)$$

$$\therefore \qquad X_2(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\therefore \qquad X_1(\omega)\,X_2(\omega) = \frac{2}{j\omega+2}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \frac{2}{j\omega(j\omega+2)} + \frac{2\pi\delta(\omega)}{j\omega+2}$$

Since $x_1(t) * x_2(t) = \mathrm{F}^{-1}[X_1(\omega)\,X_2(\omega)]$, we have

$$x_1(t) * x_2(t) = \mathrm{F}^{-1}\left[\frac{2}{j\omega(j\omega+2)} + \frac{2\pi\delta(\omega)}{j\omega+2}\right] = \mathrm{F}^{-1}\left[\frac{1}{j\omega} - \frac{1}{j\omega+2} + \frac{2\pi\delta(\omega)}{j\omega+2}\right]$$

Since $\delta(\omega) = 1$ for $\omega = 0$ and $\delta(\omega) = 0$ for $\omega \neq 0$, we have $\dfrac{2\pi\delta(\omega)}{j\omega+2} = \pi\delta(\omega)$.

$$\therefore \quad x_1(t) * x_2(t) = \mathrm{F}^{-1}\left[\frac{1}{j\omega} + \pi\delta(\omega) - \frac{1}{j\omega+2}\right] = \mathrm{F}^{-1}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \mathrm{F}^{-1}\left(\frac{1}{j\omega+2}\right)$$

$$= u(t) - e^{-2t}\,u(t) = (1 - e^{-2t})\,u(t)$$

EXAMPLE 7.4 Find the convolution of signals using Fourier transform.

$$x_1(t) = e^{-t}\,u(t) \quad \text{and} \quad x_2(t) = e^{-t}\,u(t)$$

Solution: Given

$$x_1(t) = e^{-t}\,u(t)$$

$$\therefore \quad X_1(\omega) = \frac{1}{j\omega+1}$$

$$x_2(t) = e^{-t}\,u(t)$$

$$\therefore \quad X_2(\omega) = \frac{1}{j\omega+1}$$

$$\therefore \quad X_1(\omega)\,X_2(\omega) = \frac{1}{(j\omega+1)}\,\frac{1}{(j\omega+1)} = \frac{1}{(j\omega+1)^2}$$

Since $x_1(t) * x_2(t) = \mathrm{F}^{-1}[X_1(\omega)\,X_2(\omega)]$, we have

$$x_1(t) * x_2(t) = \mathrm{F}^{-1}\left[\frac{1}{(j\omega+1)^2}\right] = te^{-t}\,u(t)$$

7.5 GRAPHICAL PROCEDURE TO PERFORM CONVOLUTION

The convolution of two signals can be performed using graphical method. The procedure is:

1. For the given signals $x(t)$ and $h(t)$, replace the independent variable t by a dummy variable τ and plot the graph for $x(\tau)$ and $h(\tau)$.
2. Keep the function $x(\tau)$ fixed. Visualise the function $h(\tau)$ as a rigid wire frame and rotate (or invert) this frame about the vertical axis ($\tau = 0$) to obtain $h(-\tau)$.
3. Shift the inverted frame along the τ-axis by t sec. The shifted frame now represents $h(t - \tau)$.
4. Plot the graph for $x(\tau)$ and $h(t - \tau)$ on the same axis beginning with very large negative time shift t.
5. For a particular value of $t = a$, integration of the product $x(\tau)\,h(a - \tau)$ represents the area under the product curve (common area). This common area represents the convolution of $x(t)$ and $h(t)$ for a shift of $t = a$.

   i.e. $$\int_{-\infty}^{\infty} x(\tau)\,h(a-\tau)\,d\tau = [x(t) * h(t)]_{t=a}$$

6. Increase the time shift t and take the new interval whenever the function either $x(\tau)$ or $h(t - \tau)$ changes. The value of t at which the change occurs defines the end of the current interval and the beginning of a new interval. Calculate $y(t)$ using Step 5.
7. The value of convolution obtained at different values of t (both positive and negative values) may be plotted on a graph to get the combined convolution.

EXAMPLE 7.5 Find the convolution of the following signals by graphical method:

$$x(t) = e^{-3t}\,u(t);\; h(t) = u(t + 3)$$

Solution: Given $\quad x(t) = e^{-3t}\,u(t)$ and $h(t) = u(t + 3)$

The output
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\,d\tau$$

The two functions $x(\tau)$ and $h(\tau)$ will be

$$x(\tau) = e^{-3\tau}\,u(\tau) = e^{-3\tau} \quad \text{for } \tau \geq 0$$

and
$$h(\tau) = u(\tau + 3) = 1 \quad \text{for } \tau \geq -3$$

$\therefore$
$$h(-\tau) = u(-\tau + 3)$$

$h(-\tau)$ can be obtained by folding $h(\tau)$ about $\tau = 0$. Figure 7.1 shows the plots of $x(\tau)$ and $h(\tau)$.

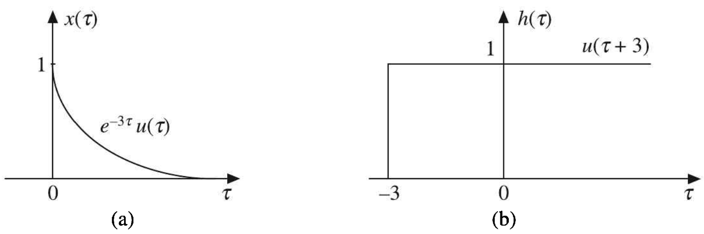


Figure 7.1 Plots of (a) $x(\tau)$, and (b) $h(\tau)$.

Figure 7.2 shows the plots of $x(\tau)$ and $h(-\tau)$.

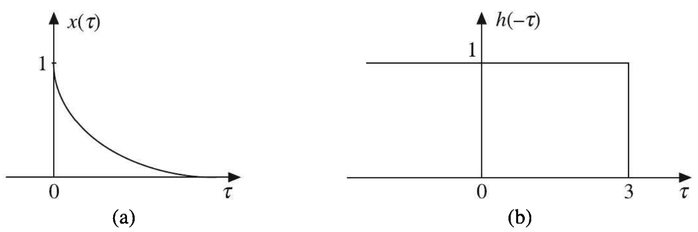
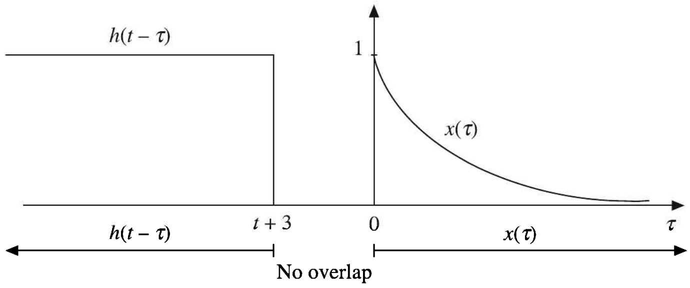


Figure 7.2 Plots of (a) $x(\tau)$, and (b) $h(-\tau)$.

Figure 7.3 shows the plots of $x(\tau)$ and $h(t - \tau)$ together on the same time axis. Here the signal $h(t - \tau)$ is sketched for $t < -3$. $x(\tau)$ and $h(t - \tau)$ do not overlap. Therefore, the product $x(\tau)$ $h(t - \tau)$ is equal to zero.

$$\therefore \qquad y(t) = 0 \quad (\text{for } t < -3)$$



$$\begin{aligned} \text{For} \quad t+3 &< 0 \quad \text{or } t < -3 \\ h(t-\tau) &= 0 \quad \text{for } \tau > t+3 \\ x(\tau) &= 0 \quad \text{for } \tau < 0 \end{aligned}$$

Figure 7.3 Plots of $x(\tau)$, and $h(t - \tau)$ when there is no overlap.

Now, increase the time shift t until the signal $h(t - \tau)$ intersects $x(\tau)$. Figure 7.4 shows the situation for $t > -3$. Here $x(\tau)$ and $h(t - \tau)$ overlapped [This overlapping continues for all values of $t > -3$ upto $t = \infty$ because $x(\tau)$ exists for all values of $\tau > 0$]. But $x(\tau) = 0$ for $\tau < 0$ and $h(t - \tau) = 0$ for $\tau > t + 3$. Therefore, the integration interval is from $\tau = 0$ to $\tau = t + 3$.

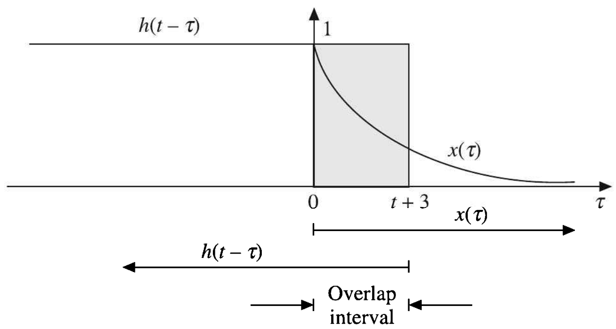


Figure 7.4 Plots of $x(\tau)$, and $h(t - \tau)$ with overlap.

$$y(t) = \int_0^{t+3} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^{t+3} e^{-3\tau}\, d\tau = \left[\frac{e^{-3\tau}}{-3}\right]_0^{t+3}$$

$$= \frac{e^{-3(t+3)} - 1}{-3}$$

$$= \frac{1 - e^{-3(t+3)}}{3}$$

$\therefore \qquad y(t) = 0 \qquad \text{for } t < -3$

$$= \frac{1 - e^{-3(t+3)}}{3} \qquad \text{for } t > -3$$

EXAMPLE 7.6 The input and the impulse response to the system are given by

$$x(t) = u(t+2)$$

$$h(t) = u(t-3)$$

Determine the output of the system graphically.

Solution: Given $x(t) = u(t + 2)$ and $h(t) = u(t - 3)$. The output $y(t)$ is the convolution of $x(t)$ and $h(t)$, i.e.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau$$

The two functions $x(\tau)$ and $h(\tau)$ will be

$$x(\tau) = u(\tau + 2) = 1 \quad (\text{for } \tau \geq -2)$$

and $\qquad h(\tau) = u(\tau - 3) = 1 \quad (\text{for } \tau \geq 3)$

The functions $x(\tau)$, $h(\tau)$ and $h(-\tau)$ are plotted as shown in Figure 7.5.

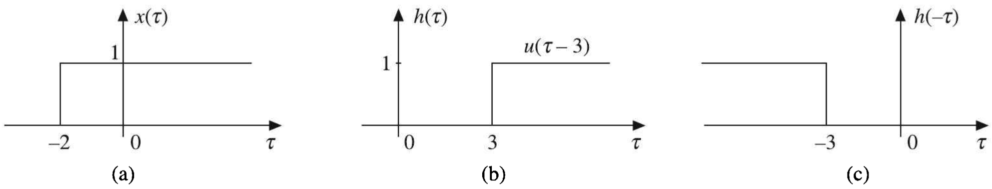


Figure 7.5 Plots of (a) $x(\tau)$, (b) $h(\tau)$, and (c) $h(-\tau)$.

Figure 7.6 shows the plots of the functions $x(\tau)$ and $h(t - \tau)$ together on the same axis. $h(t - \tau)$ is sketched for $t - 3 < -2$, i.e. for $t < 1$.

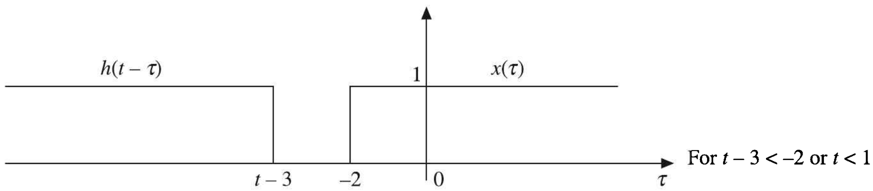


Figure 7.6 Plots of $x(\tau)$, and $h(t - \tau)$ for $t < 1$.

For $t < 1$, $x(\tau)$ and $h(t - \tau)$ do not overlap because

$$x(\tau) = 0 \quad (\text{for } \tau < -2)$$

and $\qquad h(t - \tau) = 0 \quad (\text{for } t - 3 < -2)$

$\therefore \qquad y(t) = 0$

Figure 7.7 shows the plots of $x(\tau)$ and $h(t-\tau)$ when $t-3 < -2$ or $t > 1$. Now, there is an overlap between signals $x(\tau)$ and $h(t-\tau)$ in the interval $-2 < \tau < t-3$.

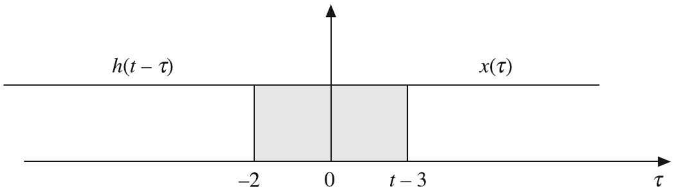


Figure 7.7 Plots of $x(\tau)$, and $h(t-\tau)$ when there is overlap.

From Figure 7.7, we have three separate regions as follows:

For $-\infty < \tau < -2$, $x(\tau)\ h(t-\tau) = 0$, since there is no overlap.

For $-2 < \tau < t-3$, $x(\tau)\ h(t-\tau) \neq 0$, since there is an overlap.

For $t-3 \leq \tau \leq \infty$, $x(\tau)\ h(t-\tau) = 0$, since there is no overlap.

Based on the above, we can write the convolution integral as:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{-2} (0\times 1)\,d\tau + \int_{-2}^{t-3} (1\times 1)\,d\tau + \int_{t-3}^{\infty} (1\times 0)\,d\tau$$

$$= \int_{-2}^{t-3} d\tau = [\tau]_{-2}^{t-3}$$

$$= t-3+2 = t-1$$

$$\therefore \qquad y(t) = 0 \qquad (\text{for } t < 1)$$

$$= t-1 \qquad (\text{for } t > 1)$$

This function can also be written as:

$$y(t) = (t-1)\,u(t-1)$$

This is unit ramp delayed by 1. Its plot is shown in Figure 7.8.

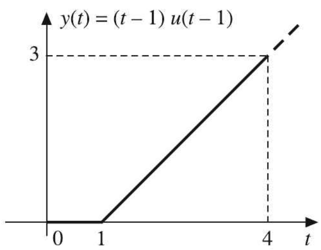


Figure 7.8 Plot of $y(t) = u(t+2) * u(t-3)$.

EXAMPLE 7.7 The impulse response of the circuit is given as $h(t) = e^{-2t}\,u(t)$. This circuit is excited by an input of $x(t) = e^{-4t}\,[u(t) - u(t - 2)]$. Determine the output of the circuit.

Solution: Here the impulse response and input are:

$$h(t) = e^{-2t}\,u(t) = e^{-2t} \qquad (\text{for } t \ge 0)$$

and
$$x(t) = e^{-4t}[u(t) - u(t-2)] = e^{-4t} \qquad (\text{for } 0 < t < 2)$$

The output of the circuit $y(t)$ can be obtained by convolution of $x(t)$ and $h(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Writing $x(t)$ and $h(t)$ in terms of τ, we have

$$x(\tau) = e^{-4\tau} \quad (\text{for } 0 \le \tau \le 2)$$

and
$$h(\tau) = e^{-2\tau} \quad (\text{for } \tau \ge 0)$$

Figure 7.9 shows the plots of $x(\tau)$, $h(\tau)$ and $h(-\tau)$ w.r.t. τ.

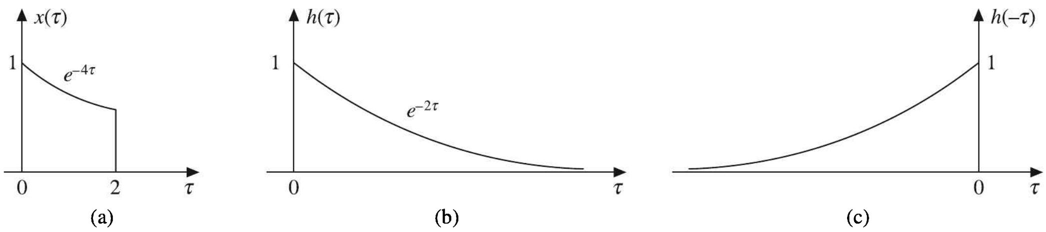


Figure 7.9 Plots of (a) $x(\tau)$, (b) $h(\tau)$ and (c) $h(-\tau)$.

For the convolution of $x(t)$ and $h(t)$, we require $h(t - \tau)$.

$\therefore$
$$h(t-\tau) = e^{-2(t-\tau)}\,u(t-\tau) = e^{-2(t-\tau)} \quad (\text{for } t - \tau > 0 \text{ or } \tau < t)$$

The plots of $x(\tau)$ and $h(t - \tau)$ drawn on the same time axis are shown in Figure 7.10 for $t < 0$. The plots do not overlap.

$\therefore$
$$y(t) = 0 \quad (\text{for } t < 0)$$

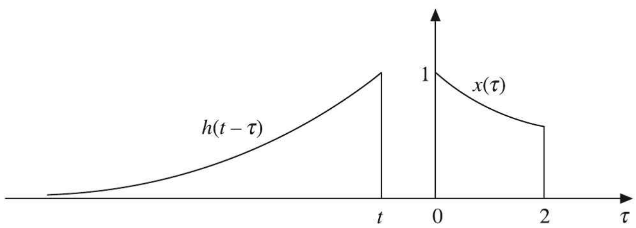


Figure 7.10 Plots of $x(\tau)$, and $h(t - \tau)$ for $t < 0$.

For $0 \le t \le 2$

Figure 7.11 shows the plots of $x(\tau)$ and $h(t - \tau)$ for $0 \le t \le 2$ drawn on the same time axis. Observe that there is an overlap between $x(\tau)$ and $h(t - \tau)$ as shown by the shaded area only for 0 to t.

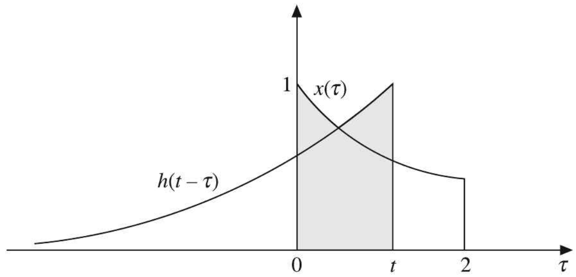


Figure 7.11 Plots of $x(\tau)$ and $h(t - \tau)$ when there is an overlap.

For $0 \le t \le 2$ we can write the convolution as:

$$y(t) = \int_{-\infty}^{0} (0) \times h(t-\tau)\,d\tau + \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau + \int_{t}^{2} x(\tau) \times (0)\,d\tau$$

$$= \int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{0}^{t} (e^{-4\tau})\,(e^{-2(t-\tau)})\,d\tau$$

$$= e^{-2t} \int_{0}^{t} e^{-2\tau}\,d\tau$$

$$= e^{-2t}\left[\frac{e^{-2\tau}}{-2}\right]_0^t = e^{-2t}\left(\frac{e^{-2t}-1}{-2}\right) = \frac{1}{2}e^{-2t}(1-e^{-2t})$$

$$\therefore \qquad y(t) = \frac{1}{2}e^{-2t}(1-e^{-2t}) \qquad (\text{for } 0 \le t \le 2)$$

For $t > 2$

Now, consider the case $t > 2$. For $t > 2$, the plots of $x(\tau)$ and $h(t - \tau)$ drawn on the same time axis are shown in Figure 7.12. In this figure, observe that $x(\tau)$ and $h(t - \tau)$ overlap only for $0 \le t \le 2$ as shown by the shaded area.

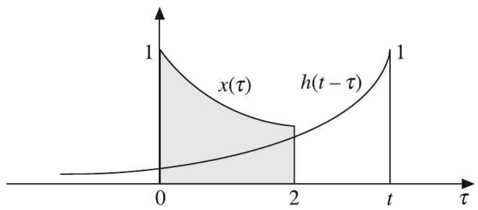


Figure 7.12 Plots of $x(\tau)$, and $h(t - \tau)$ for $t > 2$.

Hence we can write the convolution equation as:

$$y(t) = \int_{-\infty}^{0} (0) \times h(t-\tau)\, d\tau + \int_{0}^{2} x(\tau)\, h(t-\tau)\, d\tau + \int_{2}^{t} (0) \times h(t-\tau)\, d\tau$$

$$= \int_{0}^{2} x(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{2} e^{-4\tau} e^{-2(t-\tau)}\, d\tau = e^{-2t} \int_{0}^{2} e^{-2\tau}\, d\tau$$

$$= e^{-2t}\left[\frac{e^{-2\tau}}{-2}\right]_0^2 = e^{-2t}\left(\frac{e^{-4}-1}{-2}\right)$$

$$= \frac{1}{2} e^{-2t}(1-e^{-4}) \qquad \text{for } t > 2$$

Thus, we obtained the convolution as follows:

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ \dfrac{1}{2}(1-e^{-2t})e^{-2t} & \text{for } 0 < t < 2 \\ \dfrac{1}{2}(1-e^{-4})e^{-2t} & \text{for } t > 2 \end{cases}$$

This function is plotted in Figure 7.13.

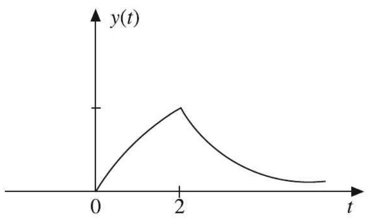


Figure 7.13 Plot of $y(t)$.

At t = 2, the value of $y(t) = \frac{1}{2}(1-e^{-4})\, e^{-4} = 0.009$

In Figure 7.13, observe that $y(t)$ increases from t = 0 to t = 2. It has the maximum value at t = 2. Then $y(t)$ decays exponentially.

EXAMPLE 7.8 Obtain the convolution of the following two functions:

$$x(t) = \begin{cases} 1 & \text{for } -3 \le t \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$h(t) = \begin{cases} 2 & \text{for } 0 \le t \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Solution: The function $x(t)$ is a pulse of amplitude 1 from –3 to 3. The function $h(t)$ is a pulse of amplitude 2 from 0 to 3. Changing the variable t to τ, $x(\tau)$, $h(\tau)$ and $h(-\tau)$ are plotted as shown in Figure 7.14.

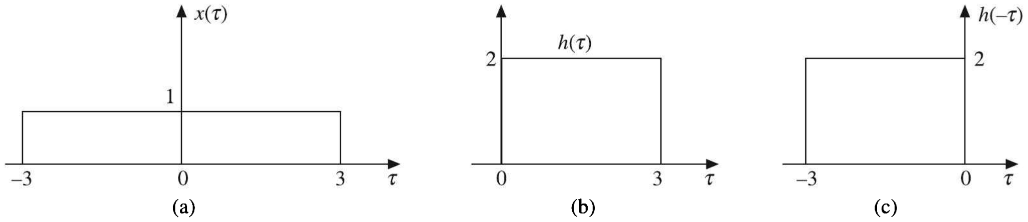


Figure 7.14 Plots of (a) $x(\tau)$, (b) $h(\tau)$, and (c) $h(-\tau)$.

For $t < -3$

The plots of $x(\tau)$ and $h(t - \tau)$ for $t < -3$ drawn on the same time axis are shown in Figure 7.15. It can be observed that there is no overlap between $x(\tau)$ and $h(t - \tau)$ for $t < -3$. Therefore, the product $x(\tau)\, h(t - \tau)$ will be always zero.

$$\therefore \qquad y(t) = 0 \quad (\text{for } t < -3)$$

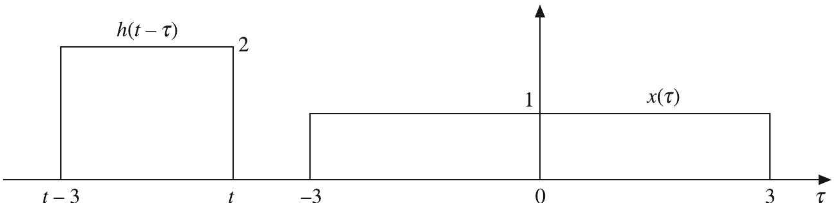


Figure 7.15 Plots of $x(\tau)$, and $h(t - \tau)$ for $t < -3$.

For $-3 < t < 0$

Figure 7.16 shows the plots of $x(\tau)$ and $h(t - \tau)$ for $-3 \le t \le 0$. Observe that there is a partial overlap between $x(\tau)$ and $h(t - \tau)$ as shown by the shaded area. The signals overlap in the interval –3 to t.

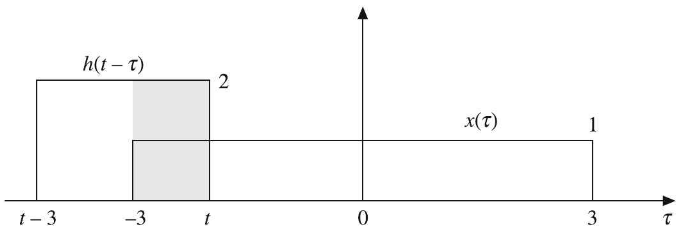


Figure 7.16 Plots of $x(\tau)$, and $h(t - \tau)$ for $-3 < t < 0$.

$$y(t) = \int_{-3}^{t} x(\tau)\, h(t - \tau)\, d\tau$$

$$= \int_{-3}^{t} (1)(2)\, d\tau = 2\int_{-3}^{t} d\tau = 2[\tau]_{-3}^{t}$$

$$= 2[t - (-3)] = 2(t + 3) \qquad (\text{for } -3 < t < 0)$$

For $0 \le t \le 3$

Figure 7.17 shows the waveforms of $x(\tau)$ and $h(t - \tau)$ for $0 \le t \le 3$. For this range of t, both the pulses fully overlap each other. The shaded area shows the overlap.

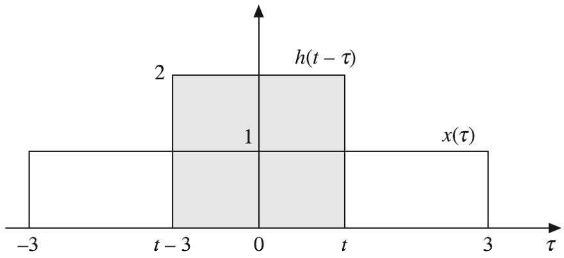


Figure 7.17 Plots of $x(\tau)$, and $h(t - \tau)$ for $0 \le t \le 3$.

The convolution for this range is

$$y(t) = \int_{t-3}^{t} x(\tau)\,h(t-\tau)\,d\tau = \int_{t-3}^{t} (1)(2)\,d\tau$$

$$= 2\int_{t-3}^{t} d\tau = 2[\tau]_{t-3}^{t} = 2[t-(t-3)] = 6$$

$\therefore$ $\qquad y(t) = 6 \quad (\text{for } 0 \le t \le 3)$

For $3 < t < 6$

The plots of $x(\tau)$ and $h(t - \tau)$ are shown in Figure 7.18 for $3 \le t \le 6$. For this range, $x(\tau)$ and $h(t - \tau)$ partially overlap each other. The shaded area in the figure shows the overlap. The overlap is in the interval $t - 3 \le t \le 3$.

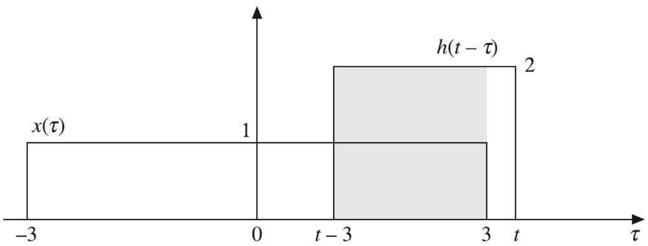


Figure 7.18 Plots of $x(\tau)$, and $h(t - \tau)$ for $3 \le t \le 6$.

For this range, we can write the convolution as:

$$y(t) = \int_{t-3}^{3} x(\tau)\,h(t-\tau)\,d\tau = \int_{t-3}^{3} (1)\,(2)\,d\tau$$

$$= 2\int_{t-3}^{3} d\tau = 2[\tau]_{t-3}^{3} = 2\,[3-(t-3)]$$

$$= 2(6-t) \qquad (\text{for } 3 \le t \le 6)$$

For $t > 6$ there is no overlap between $x(\tau)$ and $h(t - \tau)$ as shown in Figure 7.19(a). Thus, $x(\tau)\, h(t - \tau) = 0$, and so the convolution is zero, i.e.

$$y(t) = 0 \quad (\text{for } t > 6)$$

The result of convolution obtained by combining all the above is given as:

$$\therefore \qquad y(t) = \begin{cases} 0 & \text{for } t < -3 \\ 2(t+3) & \text{for } -3 \le t < 0 \\ 6 & \text{for } 0 < t < 3 \\ 2(6-t) & \text{for } 3 \le t \le 6 \\ 0 & \text{for } t > 6 \end{cases}$$

The plot of $y(t)$ is shown in Figure 7.19(b).

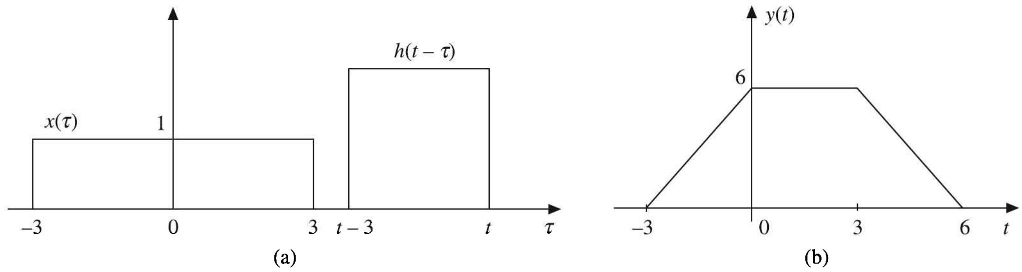


Figure 7.19 (a) Plots of $x(\tau)$ and $h(t - \tau)$ for $t > 6$, and (b) Plot of $y(t)$.

EXAMPLE 7.9 Find the convolution of the signals

$$x(t) = \Pi\left(\frac{t-2}{4}\right);\ h(t) = u(t-8)$$

Solution: Changing the variable t to τ, we have

$$x(\tau) = \Pi\left(\frac{\tau-2}{4}\right) \quad \text{and} \quad h(\tau) = u(\tau - 8)$$

$x(\tau)$, $h(\tau)$, and $h(-\tau)$ are plotted as shown in Figure 7.20.

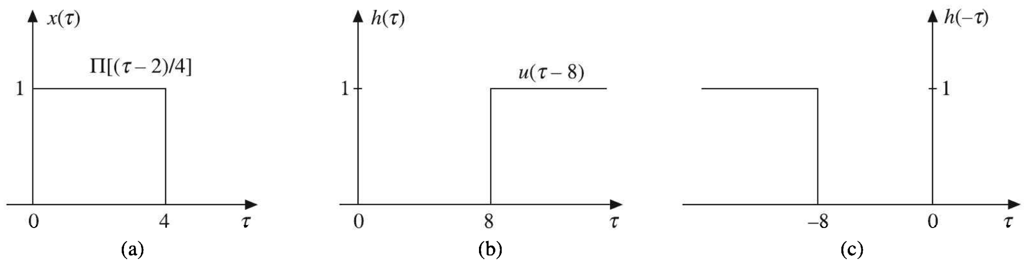


Figure 7.20 Plots of (a) $x(\tau)$ (b) $h(\tau)$ and (c) $h(-\tau)$.

For $0 < t < 8$

Figure 7.21 shows $x(\tau)$ and $h(t - \tau)$ plotted on the same time axis for $t - 8 < 0$, i.e. for $t < 8$. In this case, there is no overlap of signals $x(\tau)$ and $h(t - \tau)$. Therefore, $x(\tau)\, h(t - \tau) = 0$.

$$\therefore \qquad y(t) = 0 \quad \text{for } t < 8$$

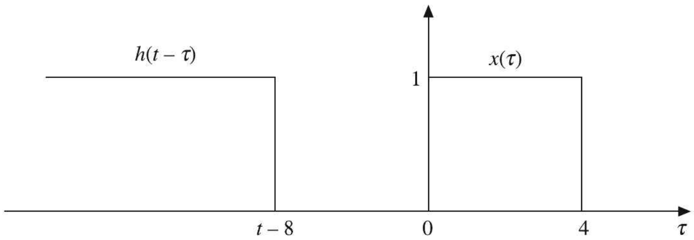


Figure 7.21 Plots of $x(\tau)$, and $h(t - \tau)$ for $t < 8$.

For $8 < t < 12$

Figure 7.22 shows $x(\tau)$ and $h(t - \tau)$ plotted on the same time axis for $t > 8$. In this case, there is a partial overlap of $x(\tau)$ and $h(t - \tau)$. This overlap is shown by the shaded area.

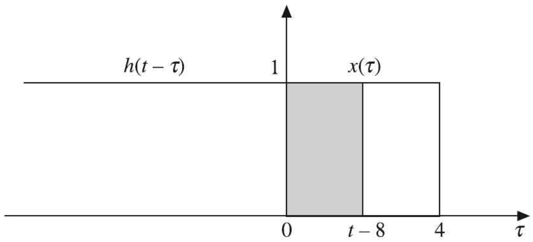


Figure 7.22 Plots of $x(\tau)$, and $h(t - \tau)$ for $t > 8$.

From Figure 7.22, we have

$$x(\tau)\, h(t-\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ 0 & \text{for } \tau \ge t-8 \\ 1 & \text{for } 0 < \tau < t-8 \end{cases}$$

Therefore, for $8 \le t \le 12$,

$$y(t) = \int_0^{t-8} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^{t-8} d\tau = [\tau]_0^{t-8} = t-8$$

Figure 7.23(a) shows the plots of $x(\tau)$ and $h(t - \tau)$ for $t > 12$. Here again there is a partial overlap of $x(\tau)$ and $h(t - \tau)$ between 0 and 4.

In this case, $$x(\tau)\, h(t-\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ 0 & \text{for } \tau > 4 \\ 1 & \text{for } 0 < \tau < 4 \end{cases}$$

$\therefore$
$$y(t) = \int_0^4 x(\tau)\, h(t-\tau)\, d\tau = \int_0^4 d\tau = [\tau]_0^4 = 4$$

Combining all, we get

$$y(t) = \begin{cases} 0 & \text{for } t \le 8 \\ t-8 & \text{for } 8 \le t \le 12 \\ 4 & \text{for } t \ge 12 \end{cases}$$

The plot of $y(t)$ is shown in Figure 7.23(b).

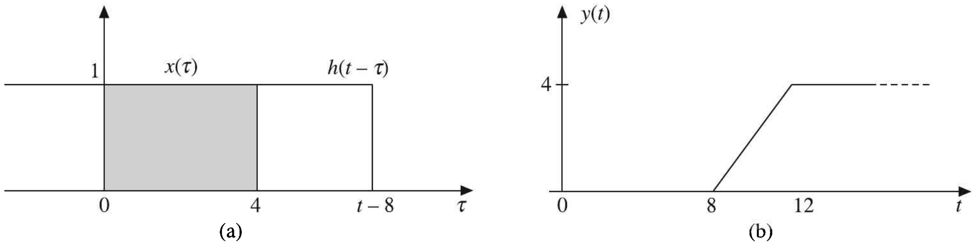


Figure 7.23 (a) Plots of $x(\tau)$, and $h(t-\tau)$ for $t > 12$, and (b) $y(t)$.

EXAMPLE 7.10 Find the convolution of the signals $x(t)$ and $h(t)$ shown in Figure 7.24.

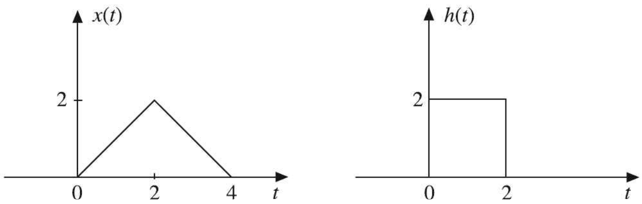


Figure 7.24 Waveforms $x(t)$ and $h(t)$.

Solution: The given signals $x(t)$ and $h(t)$ can be mathematically expressed as:

$$x(t) = \begin{cases} t & \text{for } 0 \le t \le 2 \\ 4-t & \text{for } 2 \le t \le 4 \end{cases}$$

and
$$h(t) = 2 \qquad \text{for } 0 \le t \le 2$$

Changing the variable t to τ, the plots of $x(\tau)$, $h(\tau)$ and $h(-\tau)$ are shown in Figure 7.25.

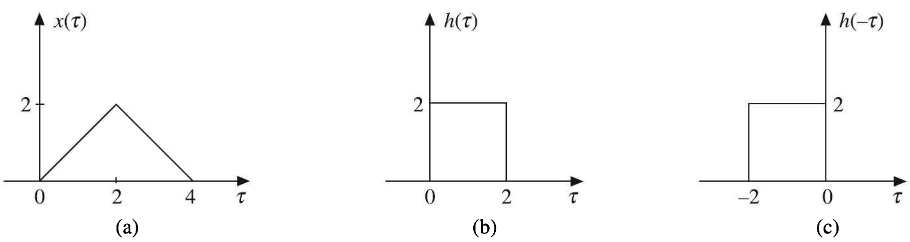


Figure 7.25 Plots of (a) $x(\tau)$, (b) $h(\tau)$, and (c) $h(-\tau)$.

For $t < 0$

Figure 7.26 shows the plots of $x(\tau)$ and $h(t - \tau)$ drawn on the same time axis for $t < 0$. There is no overlap between $x(\tau)$ and $h(t - \tau)$. Therefore, $x(\tau)\, h(t - \tau) = 0$ for $t < 0$.

$$\therefore \qquad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = 0$$

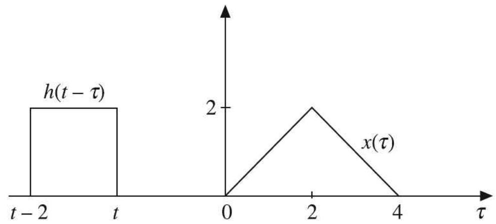


Figure 7.26 Plots of $x(\tau)$, and $h(t - \tau)$ for $t < 0$.

For $0 < t < 2$

Figure 7.27 shows the plots of $x(\tau)$ and $h(t - \tau)$ drawn on the same time axis for $0 < t < 2$. There is a partial overlap between $x(\tau)$ and $h(t - \tau)$. The shaded area in the figure indicates the overlap.

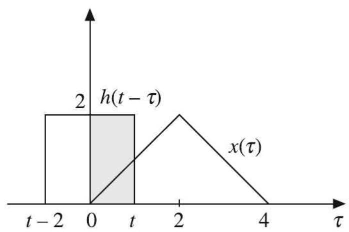


Figure 7.27 Plots of $x(\tau)$, and $h(t - \tau)$ for $0 \le t \le 2$.

Now, $\qquad x(\tau) = \tau$

and $\qquad h(t - \tau) = 2 \qquad$ (for $0 < t < 2$)

$$\therefore \qquad y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau = \int_0^t (\tau)(2)\, d\tau$$

$$= 2\left[\frac{\tau^2}{2}\right]_0^t = t^2$$

For $2 \le t \le 4$

Figure 7.28 shows the plots of $x(\tau)$ and $h(t - \tau)$ drawn on the same time axis for $2 \le t \le 4$. Now, again there is a partial overlap between $x(\tau)$ and $h(t - \tau)$. The shaded area in the figure indicates the overlap. The interval of integration is $t - 2$ to t.

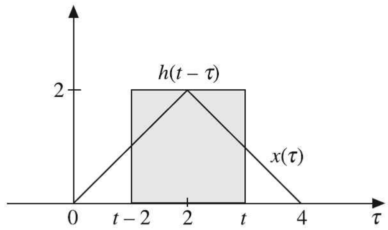


Figure 7.28 Plots of $x(\tau)$, and $h(t-\tau)$ for $2 \le t \le 4$.

$$\therefore \qquad y(t) = \int_{t-2}^{2} x(\tau)\, h(t-\tau)\, d\tau + \int_{2}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-2}^{2} \tau(2)\, d\tau + \int_{2}^{t} (4-\tau)(2)\, d\tau = 2\left[\frac{\tau^2}{2}\right]_{t-2}^{2} + \int_{2}^{t} (8-2\tau)\, d\tau$$

$$= \frac{2}{2}\left[4-(t-2)^2\right] + \left[8\tau - \frac{2\tau^2}{2}\right]_2^t = -2t^2 + 12t - 12$$

For $4 \le t \le 6$

Figure 7.29 shows the plots of $x(\tau)$ and $h(t-\tau)$ drawn on the same time axis for $4 \le t \le 6$. There is a partial overlap of $x(\tau)$ and $h(t-\tau)$. The shaded area in the figure represents the overlap. The interval of integration is $t-2$ to 4.

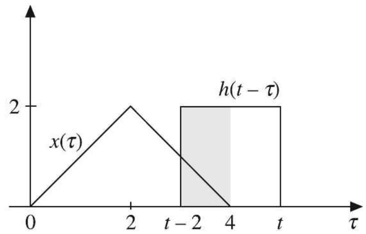


Figure 7.29 Plots of $x(\tau)$, and $h(t-\tau)$ for $4 < t < 6$.

$$\therefore \qquad y(t) = \int_{t-2}^{4} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-2}^{4} (4-\tau)\,(2)\, d\tau = \int_{t-2}^{4} (8d\tau - 2\tau\, d\tau)$$

$$= [8\tau - \tau^2]_{t-2}^{4} = 8[4-(t-2)] - [4^2 - (t-2)^2]$$

$$= 8(6-t) - [16 - (t^2 - 4t + 4)] = 48 - 8t - (16 - t^2 + 4t - 4)$$

$$= t^2 - 12t + 36$$

For $t \geq 6$

The plots of $x(\tau)$ and $h(t - \tau)$ for $t > 6$ are shown in Figure 7.30.

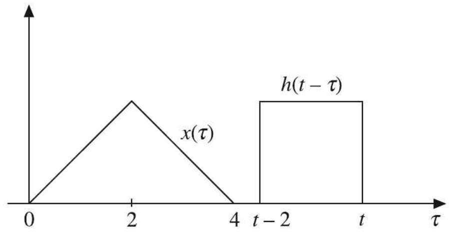


Figure 7.30 Plots of $x(\tau)$, and $h(t - \tau)$ for $t > 6$.

There is no overlap between $x(\tau)$ and $h(t - \tau)$.

$$\therefore \qquad y(t) = 0$$

Combining all, we have

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ t^2 & \text{for } 0 \leq t \leq 2 \\ -2t^2 + 12t - 12 & \text{for } 2 \leq t \leq 4 \\ t^2 - 12t + 36 & \text{for } 4 \leq t \leq 6 \\ 0 & \text{for } t \geq 6 \end{cases}$$

EXAMPLE 7.11 Find $x_1(t) * x_2(t)$ for the functions $x_1(t)$ and $x_2(t)$ shown in Figure 7.31[(a) and (b)].

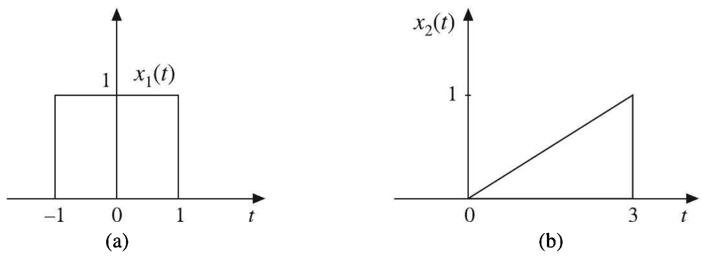


Figure 7.31 Waveforms for Example 7.11.

Solution: Here $x_1(t)$ has a simpler mathematical description than that of $x_2(t)$, so it is preferable to invert $x_1(t)$. Hence we shall determine $x_2(t) * x_1(t)$ rather than $x_1(t) * x_2(t)$.

Thus,
$$y(t) = x_2(t) * x_1(t) = \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t - \tau)\, d\tau$$

The expressions for $x_1(t)$ and $x_2(t)$ are:

$$x_1(t) = 1 \quad (\text{for } -1 \leq t \leq 1)$$

and $$x_2(t) = \frac{t}{3} \quad (\text{for } 0 \le t \le 3)$$

$\therefore$ $$x_1(-\tau) = 1 \quad \text{for } -1 \le t \le 1 \text{ and } x_2(\tau) = \frac{\tau}{3}$$

Figure 7.32 shows the plots of $x_2(\tau)$, $x_1(\tau)$, and $x_1(-\tau)$.

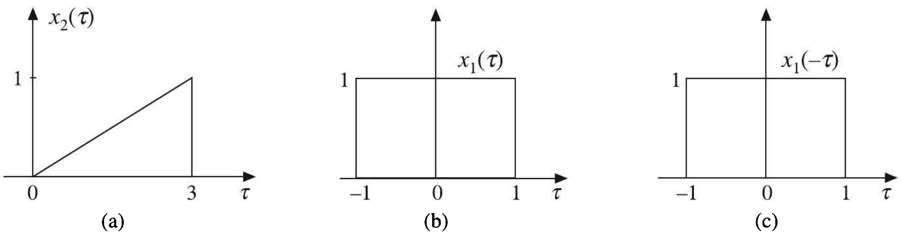


Figure 7.32 Plots of (a) $x_2(\tau)$, (b) $x_1(\tau)$, and (c) $x_1(-\tau)$.

Figure 7.33(a) shows $x_2(\tau)$ and $x_1(t - \tau)$ which is $x_1(-\tau)$ shifted by t for $t < -1$. Because the edges of $x_1(-\tau)$ are at $\tau = -1$ and 1, the edges of $x_1(t - \tau)$ are at $-1 + t$ and $1 + t$. The two functions do not overlap for $t < -1$.

$\therefore$ $$y(t) = 0 \quad \text{for } t < -1$$

Figure 7.33(b) shows $x_2(\tau)$ and $x_1(t - \tau)$ plotted on the same time axis for $-1 < t < 2$.

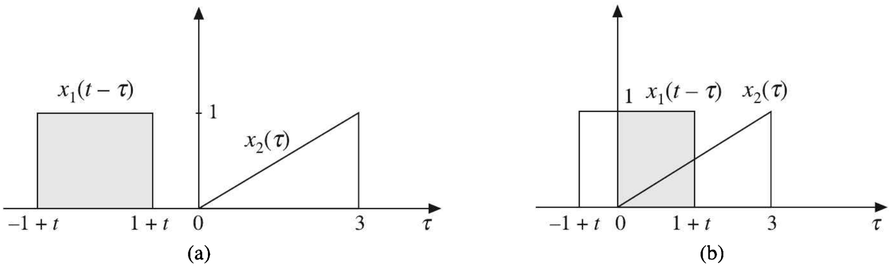


Figure 7.33 Plots of $x_1(t - \tau)$, and $x_2(\tau)$ for (a) $t < -1$, and (b) $-1 < t < 1$.

The two functions overlap over the interval $(0, 1 + t)$ so that

$$y(t) = \int_0^{1+t} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

$$= \int_0^{1+t} \frac{1}{3}\tau\, d\tau = \frac{1}{3}\left[\frac{\tau^2}{2}\right]_0^{1+t}$$

$$= \frac{1}{6}(t+1)^2 \quad (\text{for } -1 \le t \le 1)$$

This situation is depicted in Figure 7.33(b) which is valid only for $-1 \le t \le 1$.

For $t > 1$ but <2, the situation is illustrated in Figure 7.34(a). The two functions overlap only over the range $-1 + t$ to $1 + t$. Note that the expressions for $x_2(\tau)$ and $x_1(t - \tau)$ do not change, only their range of integration changes. Therefore,

$$y(t) = \int_{-1+t}^{1+t} \frac{1}{3}\tau \, d\tau = \frac{2}{3}t \qquad (\text{for } 1 \le t \le 2)$$

Note that the above two expressions for $y(t)$ apply at $t = 1$, the transition point between their respective ranges. We can verify that both expressions really yield a value of 2/3 at $t = 1$, so that $y(1) = 2/3$.

For $t \ge 2$ but < 4, the situation is as shown in Figure 7.34(b).

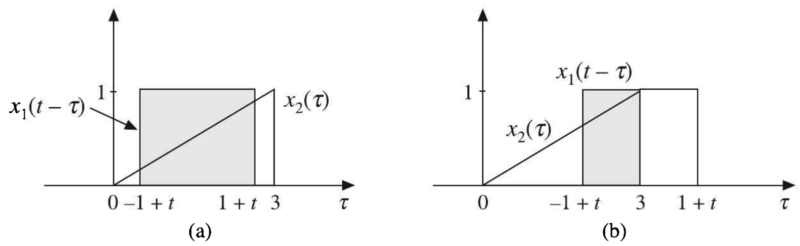


Figure 7.34 Plots of $x_1(t - \tau)$ and $x_2(\tau)$ for (a) $1 < t < 2$, and (b) $2 < t < 4$.

The functions $x_2(\tau)$ and $x_1(t - \tau)$ overlap over the interval from $-1 + t$ to 3 so that

$$y(t) = \int_{-1+t}^{3} \frac{1}{3}\tau \, d\tau = -\frac{1}{6}(t^2 - 2t - 8)$$

Again both the equations $y(t) = (2/3)t$ and $y(t) = -(1/6)(t^2 - 2t - 8)$ apply at the transition point at $t = 2$. We can verify that $y(t) = 4/3$ at that point.

For $t \ge 4$, $x_1(t - \tau)$ has been shifted so far to the right that it no longer overlaps with $x_2(\tau)$ as shown in Figure 7.35(a). Consequently $y(t) = 0$ for $t \ge 4$.

The complete plot of $y(t)$ is shown in Figure 7.35(b).

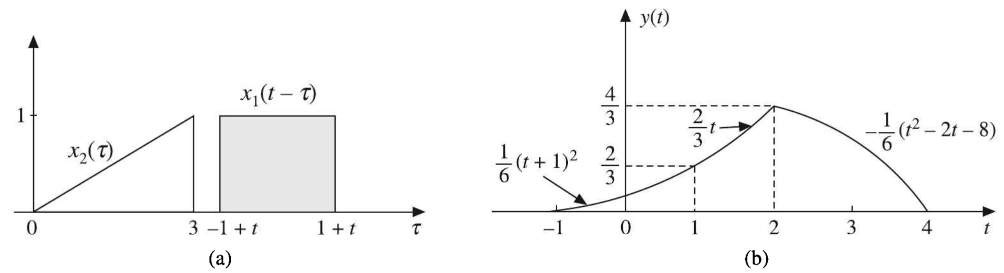


Figure 7.35 (a) Plots of $x_1(t - \tau)$, and $x_2(\tau)$ for $t \ge 4$ and (b) $y(t)$.

7.6 SIGNAL COMPARISON: CORRELATION OF FUNCTIONS

Concept of correlation

The signals may be compared on the basis of similarity of waveforms. Quantitatively, a comparison may be based upon the amount of the component of one waveform contained in the other waveform. If $x_1(t)$ and $x_2(t)$ are two waveforms, then the waveform $x_1(t)$ contains an amount $C_{12}x_2(t)$ of that particular waveform $x_2(t)$ in the interval (t_1, t_2), where

UNIT-IV

Signal Transmission through Linear Systems: Filter characteristic of Linear System (417-418), Distortion less transmission through a system (418-420), Signal bandwidth, System Bandwidth (420-421), Ideal LPF, HPF, and BPF characteristics (421-423), Causality and Paley-Wiener criterion for physical realization (424).
Relationship between Bandwidth and Rise Time (424-427)

Correlation: Cross Correlation and Auto Correlation of Functions, Properties of Correlation Functions (483-491), Energy Density Spectrum (491-493), Parsevals Theorem (491-492), Power Density Spectrum (493-497), Relation between Autocorrelation Function and Energy/Power Spectral Density Function (497-498), Relation between Convolution and Correlation (498-516), Detection of Periodic Signals in the presence of Noise by Correlation (516-518), Extraction of Signal from Noiseby Filtering (518-520).

$$H(-\omega) = H^*(\omega)$$

i.e.

$$|H(-\omega)| = |H(\omega)|$$

and

$$\underline{/H(-\omega)} = -\underline{/H(\omega)}$$

The impulse response $h(t)$ of a system is the inverse Fourier transform of its transfer function (frequency response) $H(\omega)$.

$$H(\omega) = \mathrm{F}[h(t)]$$

or

$$h(t) = \mathrm{F}^{-1}[H(\omega)]$$

The transfer function of a system in s-domain (Laplace domain) is defined as the ratio of the Laplace transform of the output of the system to the Laplace transform of the input of the system when the initial conditions are neglected. Or simply we can say that it is the ratio of the output to input in s-domain when the initial conditions are zero. The transfer function of a system can also be defined as the Laplace transform of the impulse response of the system.

$$H(s) = \frac{Y(s)}{X(s)}$$

or

$$H(s) = \mathrm{L}\,[h(t)]$$

The impulse response is nothing but the inverse Laplace transform of the transfer function $H(s)$

$$h(t) = \mathrm{L}^{-1}\,[H(s)]$$

Once the transfer function in s-domain $H(s)$ is known, the transfer function in frequency domain $H(\omega)$ can be found by replacing s in $H(s)$ by $j\omega$.

i.e.

$$H(\omega) = H(s)\big|_{s = j\omega}$$

6.5 FILTER CHARACTERISTICS OF LINEAR SYSTEMS

For a given system an input signal $x(t)$ gives rise to a response signal $y(t)$. The system, therefore, processes the signal $x(t)$ in a way that is characteristic of the system. The spectral density function of the input signal $x(t)$ is given by $X(s)$ or $X(\omega)$, and the spectral density function of the response signal $y(t)$ is given by $Y(s)$ or $Y(\omega)$. $Y(s) = H(s)\,X(s)$ or $Y(\omega) = H(\omega)\,X(\omega)$ where $H(s)$ or $H(\omega)$ is the transfer function or system function of the system (shown in Figure 6.1).

The system, therefore, modifies the spectral density function of the input. The system acts as a kind of filter for various frequency components. Some frequency components are boosted in strength, i.e. they are amplified. Some frequency components are weakened in strength, i.e. they are attenuated and some may remain unaffected. Similarly, each frequency component suffers a different amount of phase shift in the process of transmission. The system, therefore, modifies the spectral density function of the input according to its filter characteristics. The modification is carried out according to the transfer function $H(s)$ or $H(\omega)$, which represents the response of the system to various frequency components. $H(\omega)$ acts as a weighting function or spectral shaping function to the different frequency components in the input signal. An LTI system, therefore, acts as a filter. A filter is basically a frequency selective network.

Some LTI systems allow the transmission of only low frequency components and stop all high frequency components. They are called low-pass filters (LPFs).

Some LTI systems allow the transmission of only high frequency components and stop all low frequency components. They are called high-pass filters (HPFs).

Some LTI systems allow transmission of only a particular band of frequencies and stop all other frequency components. They are called band-pass filters (BPFs).

Some LTI systems reject only a particular band of frequencies and allow all other frequency components. They are called band-rejection filters (BRFs).

The band of frequency that is allowed by the filter is called pass-band.

The band of frequency that is severely attenuated and not allowed to pass through the filter is called stop-band or rejection-band.

An LTI system may be characterised by its pass-band, stop-band and half power bandwidth.

6.6 DISTORTIONLESS TRANSMISSION THROUGH A SYSTEM

The change of shape of the signal when it is transmitted through a system is called distortion. Transmission of a signal through a system is said to be distortionless if the output is an exact replica of the input signal. This replica may have different magnitude and also it may have different time delay. A constant change in magnitude and a constant time delay are not considered as distortion. Only the shape of the signal is important. Mathematically, we can say that a signal $x(t)$ is transmitted without distortion if the output

$$y(t) = kx(t - t_d)$$

where k is a constant representing the change in amplitude (amplification or attenuation) and t_d is delay time. A distortionless system and typical input and output waveforms are shown in Figure 6.5.

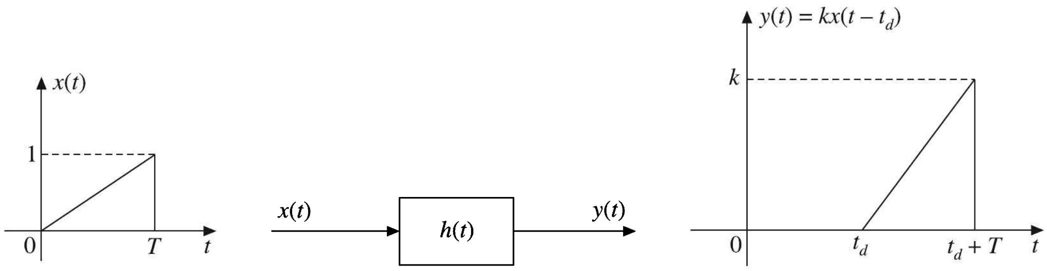


Figure 6.5 Distortionless system.

Taking Fourier transform on both sides of the equation for $y(t)$ and using the shifting property, we have

$$Y(\omega) = ke^{-j\omega t_d} X(\omega)$$

Therefore, for distortionless transmission, the transfer function of the system must be of the form

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = ke^{-j\omega t_d}$$

Taking inverse Fourier transform, the corresponding impulse response must be

$$h(t) = k\delta(t - t_d)$$

It is evident that the magnitude of the transfer function

$$|H(\omega)| = k$$

and that it is constant for all values of ω.

The phase shift

$$\theta(\omega) = \underline{|H}(\omega) = -\,\omega t_d$$

and it varies linearly with frequency.

In general, $\qquad \theta(\omega) = n\pi - \omega t_d \quad (n \text{ integral})$

So for distortionless transmission of a signal through a system, the magnitude $|H(\omega)|$ should be a constant, i.e. all the frequency components of the input signal must undergo the same amount of amplification or attenuation, i.e. the system bandwidth is infinite and the phase spectrum should be proportional to frequency. But, in practice, no system can have infinite bandwidth and hence distortionless conditions are never met exactly.

The magnitude and phase characteristics of a distortionless transmission system is shown in Figure 6.6.

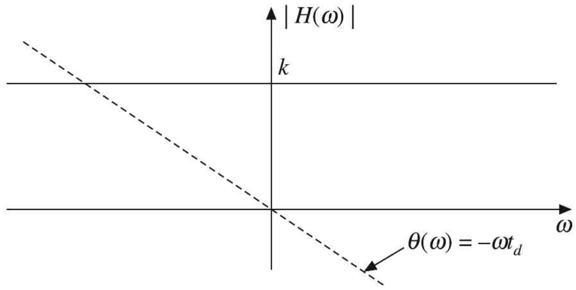


Figure 6.6 The magnitude and phase characteristics of a distortionless transmission system.

Linear phase systems

For distortionless transmission, there should not be any phase distortion. No phase distortion means the phase should be linear. Therefore, for distortionless transmission, the system must be of linear phase type. For linear phase systems, the impulse response is symmetrical about $t = t_d$. This can be proved as follows:

For linear phase system,

$$H(\omega) = |H(\omega)|\, e^{-j\omega t_d}$$

The impulse response of such a system is obtained by finding the inverse Fourier transform.

$$\begin{aligned}
h(t) &= \mathrm{F}^{-1}[H(\omega)] \\
&= \mathrm{F}^{-1}[|H(\omega)|\, e^{-j\omega t_d}] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \{|H(\omega)|\, e^{-j\omega t_d}\}\, e^{j\omega t}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|\, e^{j\omega(t-t_d)}\, d\omega \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} |H(\omega)|\, e^{j\omega(t-t_d)}\, d\omega + \int_{0}^{\infty} |H(\omega)|\, e^{j\omega(t-t_d)}\, d\omega\right] \\
&= \frac{1}{2\pi}\left[\int_{0}^{\infty} |H(\omega)|\, e^{-j\omega(t-t_d)}\, d\omega + \int_{0}^{\infty} |H(\omega)|\, e^{j\omega(t-t_d)}\, d\omega\right] \\
&= \frac{1}{2\pi}\left[\int_{0}^{\infty} |H(\omega)|\, [e^{j\omega(t-t_d)} + e^{-j\omega(t-t_d)}]\, d\omega\right] \\
&= \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)| \cos \omega(t - t_d)\, d\omega
\end{aligned}$$

$$h(t_d + t) = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)| \cos \omega t\, d\omega$$

$$h(t_d - t) = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)| \cos \omega t\, d\omega$$

$$\therefore \qquad h(t_d + t) = h(t_d - t)$$

This shows that for a linear phase system, the impulse response $h(t)$ is symmetrical about t_d, and it is non-causal (non-zero for $t < 0$).

The maximum value of $h(t)$ is at $t = t_d$ and is given by

$$h_{\max} = h(t_d) = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)|\, d\omega$$

6.7 SIGNAL BANDWIDTH

The spectral components of a signal extend from $-\infty$ to ∞. Any practical signal has finite energy. As a result, the spectral components approach zero as ω tends to ∞. Therefore, we neglect the spectral components which have negligible energy and select only a band of frequency components which have most of the signal energy. This band of frequencies that contain most of the signal energy is known as the bandwidth of the signal. Normally, the band is selected such that it contains around 95% of total energy depending on the precision.

6.8 SYSTEM BANDWIDTH

For distortionless transmission, we need a system with infinite bandwidth. Due to physical limitations, it is impossible to construct a system with infinite bandwidth. Actually, a satisfactory distortionless transmission can be achieved by systems with finite, but fairly large bandwidths, if the magnitude $|H(\omega)|$ is constant over this band.

The constancy of the magnitude $|H(\omega)|$ in a system is usually specified by its bandwidth. The bandwidth of a system is defined as the range of frequencies over which the magnitude $|H(\omega)|$ remains within $1/\sqrt{2}$ times (within 3 dB) of its value at midband. The bandwidth of a system whose $|H(\omega)|$ plot is shown in Figure 6.7 is $(\omega_2 - \omega_1)$ where ω_2 is called the upper cutoff frequency or upper 3 dB frequency or upper half power frequency and ω_1 is called the lower cutoff frequency or lower 3 dB frequency or lower half power frequency.

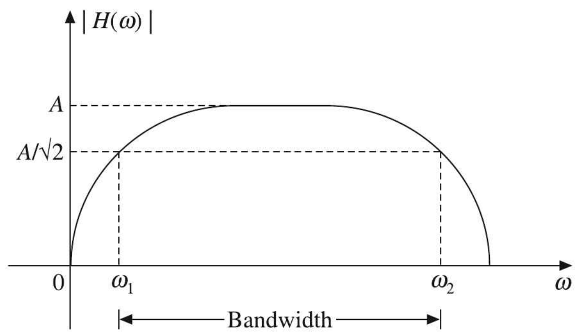


Figure 6.7 Bandwidth.

The band limited signals can be transmitted without distortion, if the system bandwidth is atleast equal to the signal bandwidth.

6.9 IDEAL FILTER CHARACTERISTICS

A filter is a frequency selective network. It allows transmission of signals of certain frequencies with no attenuation or with very little attenuation, and it rejects or heavily attenuates signals of all other frequencies.

An ideal filter has very sharp cutoff characteristics, and it passes signals of certain specified band of frequencies exactly and totally rejects signals of frequencies outside this band. Its phase spectrum is linear.

Filters are usually classified according to their frequency response characteristics as low-pass filter (LPF), high-pass filter (HPF), band-pass filter (BPF) and band-elimination or band-stop or band-reject filter (BEF, BSF, BRF). Ideal versions of these filters are described below and their magnitude responses are shown in Figure 6.8.

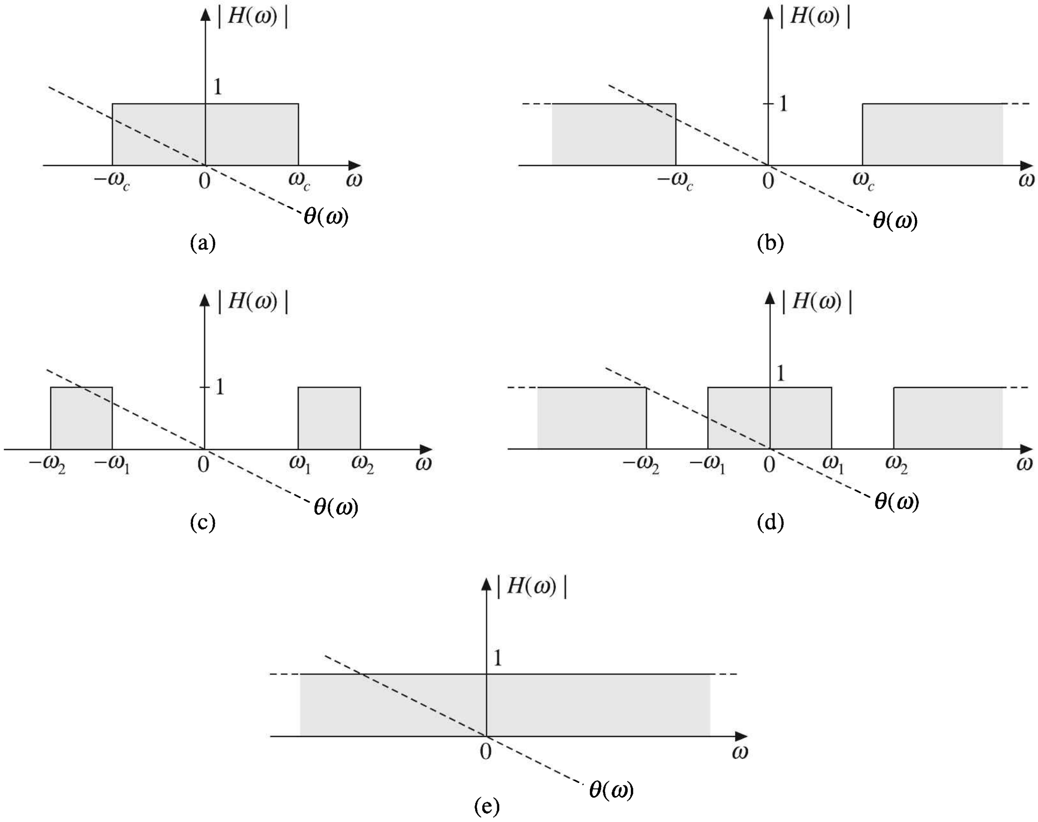


Figure 6.8 Ideal filters (a) LPF, (b) HPF, (c) BPF, (d) BRF, (e) All pass filter.

Ideal LPF

An ideal low-pass filter transmits, without any distortion, all of the signals of frequencies below a certain frequency ω_c radians per second. The signals of frequencies above ω_c radians/second are completely attenuated. ω_c is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

The transfer function of an ideal LPF is given by

$$|H(\omega)| = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

The frequency response characteristics of an ideal LPF are shown in Figure 6.8(a). It is a gate function.

Ideal HPF

An ideal high-pass filter transmits, without any distortion, all of the signals of frequencies above a certain frequency ω_c radians/second and attenuates completely the signals of frequencies below ω_c radians/second, where ω_c is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal HPF is specified by

$$|H(\omega)| = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$

The frequency response characteristics of an ideal HPF are shown in Figure 6.8(b).

Ideal BPF

An ideal band-pass filter transmits, without any distortion, all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ radians/second and attenuates completely the signals of frequencies outside this band. $(\omega_2 - \omega_1)$ is the bandwidth of the band-pass filter. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal BPF is specified by

$$|H(\omega)| = \begin{cases} 1, & |\omega_1| < \omega < |\omega_2| \\ 0, & \omega < |\omega_1| \text{ and } \omega > |\omega_2| \end{cases}$$

The frequency response characteristics of an ideal BPF are shown in Figure 6.8(c).

Ideal BRF

An ideal band rejection filter rejects totally all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ radians/second and transmits without any distortion all signals of frequencies outside this band. $(\omega_2 - \omega_1)$ is the rejection band. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal BRF is specified by

$$|H(\omega)| = \begin{cases} 0, & |\omega_1| < \omega < |\omega_2| \\ 1, & \omega < |\omega_1| \text{ and } \omega > |\omega_2| \end{cases}$$

The frequency response characteristics of an ideal BRF are shown in Figure 6.8(d).

In addition to these filters, there is one more filter called an all pass filter.

An all pass filter transmits signals of all frequencies without any distortion, that is, its bandwidth is ∞ as shown in Figure 6.8(e).

An ideal all pass filter is specified by

$$|H(\omega)| = 1 \qquad \text{(for all frequencies)}$$

The corresponding phase function for distortionless transmission is $-\omega t_d$.
All ideal filters are non-causal systems. Hence none of them is physically realizable.

6.10 CAUSALITY AND PALEY-WIENER CRITERION FOR PHYSICAL REALIZATION

A system is said to be causal if it does not produce an output before the input is applied. For an LTI system to be causal, the condition to be satisfied is its impulse response must be zero for t less than zero, i.e.

$$h(t) = 0 \quad \text{for } t < 0$$

Physical realizability implies that it is physically possible to construct that system in real time. A physically realizable system cannot have a response before the input is applied. This is known as causality condition. It means the unit impulse response $h(t)$ of a physically realizable system must be causal. This is the time domain criterion of physical realizability. In the frequency domain, this criterion implies that a necessary and sufficient condition for a magnitude function $H(\omega)$ to be physically realizable is:

$$\int_{-\infty}^{\infty} \frac{\ln |H(\omega)|}{1+\omega^2} d\omega < \infty$$

The magnitude function $|H(\omega)|$ must, however, be square-integrable before the Paley-Wiener criterion is valid, that is,

$$\int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega < \infty$$

A system whose magnitude function violates the Paley-Wiener criterion has a non-causal impulse response, that is, the response exists prior to the application of the driving function.

The following conclusions can be drawn from the Paley-Wiener criterion:

1. The magnitude function $|H(\omega)|$ may be zero at some discrete frequencies, but it cannot be zero over a finite band of frequencies since this will cause the integral in the equation of Paley-Wiener criterion to become infinity. That means ideal filters are not physically realizable.
2. The magnitude function $|H(\omega)|$ cannot fall off to zero faster than a function of exponential order. It implies, a realizable magnitude characteristic cannot have too great a total attenuation.

6.11 RELATIONSHIP BETWEEN BANDWIDTH AND RISE TIME

We know that the transfer function of an ideal LPF is given by

$$H(\omega) = |H(\omega)| \, e^{-j\omega t_d}$$

where
$$H(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

ω_c is called the cutoff frequency.

$$\therefore \qquad H(\omega) = e^{-j\omega t_d} \qquad -\omega_c \le \omega \le \omega_c, \text{ i.e. } |\omega| \le \omega_c$$

$$= 0 \qquad \omega > |\omega_c|$$

The impulse response $h(t)$ of the LPF is obtained by taking the inverse Fourier transform of the transfer function $H(\omega)$.

$$h(t) = \mathrm{F}^{-1}[H(\omega)] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega t_d} e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_d)} \, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(t-t_d)}}{j(t-t_d)} \right]_{-\omega_c}^{\omega_c}$$

$$\therefore \qquad = \frac{1}{2\pi} \left[\frac{e^{j\omega_c(t-t_d)} - e^{-j\omega_c(t-t_d)}}{j(t-t_d)} \right]$$

$$= \frac{1}{\pi(t-t_d)} [\sin \omega_c(t-t_d)]$$

$$= \frac{\omega_c}{\pi} \left[\frac{\sin \omega_c(t-t_d)}{\omega_c(t-t_d)} \right]$$

The impulse response of the ideal LPF is shown in Figure 6.9. The impulse response has a peak value at $t = t_d$. This value ω_c/π is proportional to cutoff frequency ω_c. The width of the main lobe is $2\pi/\omega_c$. As $\omega_c \to \infty$, the LPF becomes an all pass filter. As $t_d \to 0$, the output response peak $\to \infty$, that is, the output response approaches input.

Further, the impulse response $h(t)$ is non-zero for $t < 0$, even though the input $\delta(t)$ is applied at $t = 0$. That is, the impulse response begins before the input is applied. In real life, no system exhibits such type of characteristics. Hence we can conclude that ideal LPF is physically not realizable.

If the impulse response is known, the step response can be obtained by convolution.

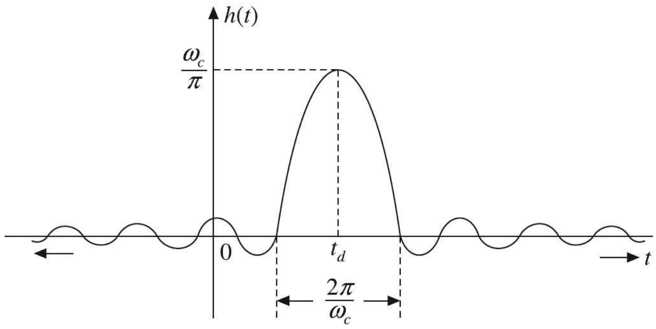


Figure 6.9 Impulse response of the ideal LPF.

The step response $$y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

We have $$h(t) = \frac{\omega_c}{\pi}\,\frac{\sin \omega_c(t - t_d)}{\omega_c(t - t_d)}$$

∴ $$y(t) = \int_{-\infty}^{t} \frac{\omega_c}{\pi}\,\frac{\sin \omega_c(\tau - t_d)}{\omega_c(\tau - t_d)}\,d\tau$$

Let $$x = \omega_c(\tau - t_d)$$

∴ $$dx = \omega_c d\tau \quad \text{or} \quad d\tau = \frac{dx}{\omega_c}$$

∴ $$y(t) = \int_{-\infty}^{\omega_c(t-t_d)} \frac{\omega_c}{\pi}\,\frac{\sin x}{x}\,\frac{dx}{\omega_c} = \frac{1}{\pi}\int_{-\infty}^{\omega_c(t-t_d)} \frac{\sin x}{x}\,dx$$

$$= \frac{1}{\pi}\left[\text{Si}(x)\right]_{-\infty}^{\omega_c(t-t_d)}$$

where Si is the sine integral function.

The properties of sine integral functions are:

1. $\text{Si}(x)$ is an odd function, that is $\text{Si}(-x) = -\text{Si}(x)$
2. $\text{Si}(0) = 0$
3. $\text{Si}(\infty) = \pi/2$ and $\text{Si}(-\infty) = -(\pi/2)$

A sketch of $\text{Si}(x)$ is shown in Figure 6.10(a).

The step response can be expressed as:

$$y(t) = \frac{1}{\pi}\{\text{Si}[\omega_c(t - t_d)] - \text{Si}(-\infty)\}$$

$$= \frac{1}{\pi}\left\{\text{Si}[\omega_c(t - t_d)] + \frac{\pi}{2}\right\}$$

$$= \frac{1}{2} + \frac{1}{\pi}\,\text{Si}[\omega_c(t - t_d)]$$

If $\omega_c \to \infty$, then the response is:

$$y(t) = \frac{1}{2} + \frac{1}{\pi}\,\text{Si}(\infty) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2}\right) = 1$$

If $\omega_c \to -\infty$, then the response is:

$$y(t) = \frac{1}{2} + \frac{1}{\pi}\,\text{Si}(-\infty) = \frac{1}{2} + \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = 0$$

The step response of LPF is shown in Figure 6.10(b).

From Figure 6.10(b), we can observe that $y(t)$ approaches a delayed unit step $u(t - t_d)$. But the abrupt rise of input corresponds to more gradual rise of the output.

The rise time t_r is defined as the time required for the response to reach from 0% to 100% of the final value. To find it, draw a tangent at $t = t_d$ with the line $y(t) = 0$ and $y(t) = 1$. From Figure 6.10(b), we have

$$\left.\frac{dy(t)}{dt}\right|_{t=t_d} = \frac{1}{t_r} = \frac{\omega_c}{\pi}\left.\frac{\sin\omega_c(t-t_d)}{\omega_c(t-t_d)}\right|_{t=t_d} = \frac{\omega_c}{\pi}$$

$$\therefore \qquad t_r = \frac{\pi}{\omega_c}$$

For a low-pass filter

$$\text{Cut off frequency} = \text{Bandwidth}$$

So the rise time is inversely proportional to the bandwidth.

$$\text{Bandwidth} \times \text{Rise time} = \text{Constant}$$

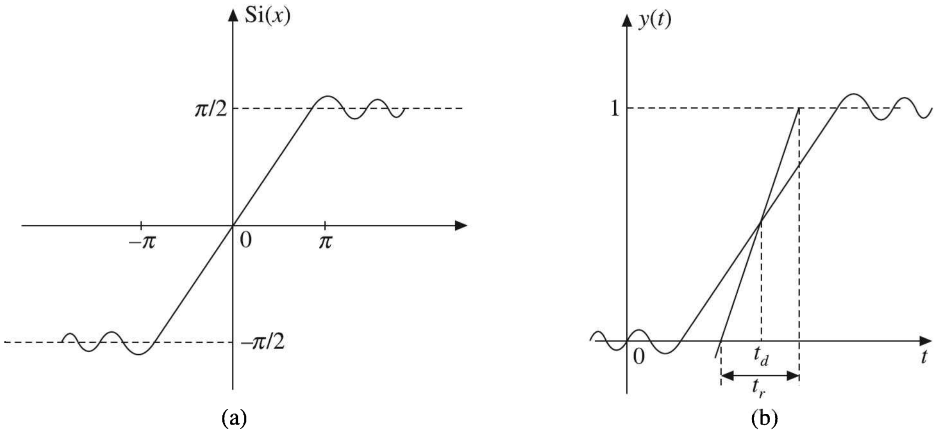


Figure 6.10 (a) Si function, (b) Step response of an ideal LPF.

EXAMPLE 6.1 Let the system function of an LTI system be $1/(j\omega + 2)$. What is the output of the system for an input $(0.8)^t\, u(t)$?

Solution: Given that the transfer function of the system

$$H(\omega) = \frac{1}{j\omega + 2}$$

The input $\qquad x(t) = (0.8)^t\, u(t)$

The impulse response $\quad h(t) = \mathrm{F}^{-1}[H(\omega)] = \mathrm{F}^{-1}\left(\frac{1}{j\omega + 2}\right) = e^{-2t}\, u(t)$

We know that the output $y(t)$ is the convolution of input $x(t)$ and impulse response $h(t)$.

$$y(t) = \int_{-1+t}^{1+t} \frac{1}{3}\tau\, d\tau = \frac{2}{3}t \qquad (\text{for } 1 \le t \le 2)$$

Note that the above two expressions for $y(t)$ apply at $t = 1$, the transition point between their respective ranges. We can verify that both expressions really yield a value of 2/3 at $t = 1$, so that $y(1) = 2/3$.

For $t \ge 2$ but < 4, the situation is as shown in Figure 7.34(b).

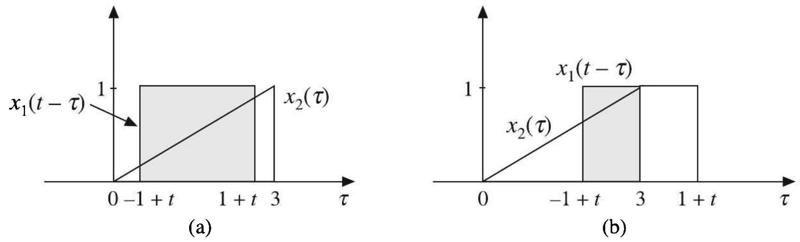


Figure 7.34 Plots of $x_1(t - \tau)$ and $x_2(\tau)$ for (a) $1 < t < 2$, and (b) $2 < t < 4$.

The functions $x_2(\tau)$ and $x_1(t - \tau)$ overlap over the interval from $-1 + t$ to 3 so that

$$y(t) = \int_{-1+t}^{3} \frac{1}{3}\tau\, d\tau = -\frac{1}{6}(t^2 - 2t - 8)$$

Again both the equations $y(t) = (2/3)t$ and $y(t) = -(1/6)(t^2 - 2t - 8)$ apply at the transition point at $t = 2$. We can verify that $y(t) = 4/3$ at that point.

For $t \ge 4$, $x_1(t - \tau)$ has been shifted so far to the right that it no longer overlaps with $x_2(\tau)$ as shown in Figure 7.35(a). Consequently $y(t) = 0$ for $t \ge 4$.

The complete plot of $y(t)$ is shown in Figure 7.35(b).

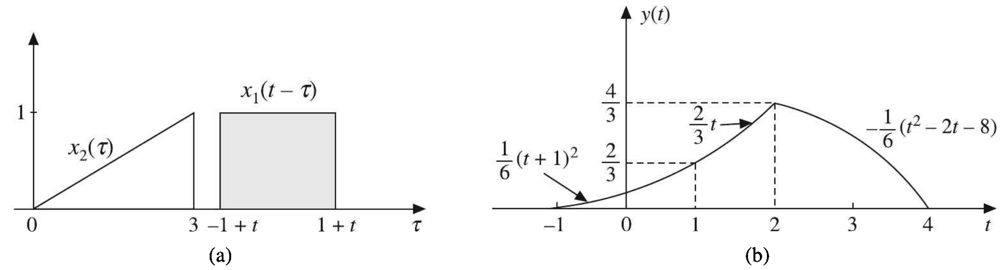


Figure 7.35 (a) Plots of $x_1(t - \tau)$, and $x_2(\tau)$ for $t \ge 4$ and (b) $y(t)$.

7.6 SIGNAL COMPARISON: CORRELATION OF FUNCTIONS

Concept of correlation

The signals may be compared on the basis of similarity of waveforms. Quantitatively, a comparison may be based upon the amount of the component of one waveform contained in the other waveform. If $x_1(t)$ and $x_2(t)$ are two waveforms, then the waveform $x_1(t)$ contains an amount $C_{12}x_2(t)$ of that particular waveform $x_2(t)$ in the interval (t_1, t_2), where

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)x_2(t)\,dt}{\int_{t_1}^{t_2} x_2^2(t)\,dt}$$

The magnitude of the integral in the numerator might be taken as an indication of similarity. If this integral vanishes, i.e.

$$\int_{t_1}^{t_2} x_1(t)\,x_2(t)\,dt = 0$$

then the two signals have no similarity over the interval (t_1, t_2). Such signals are said to be orthogonal over the specified interval.

The integral $\int_{t_1}^{t_2} x_1(t)\,x_2(t)\,dt$ forms the basis of comparison of the two signals $x_1(t)$ and $x_2(t)$ over the interval (t_1, t_2).

In general we are interested in comparing the two signals over the interval $(-\infty, \infty)$. So the test integral becomes

$$\int_{-\infty}^{\infty} x_1(t)\,x_2(t)\,dt$$

However, there is a difficulty with this test integral which can be illustrated with the example of radar pulse. Figure 7.36 shows a transmitted pulse and a received pulse which is delayed w.r.t. transmitted pulse by T s. Obviously, the two waveforms are identical except that one is delayed w.r.t. the other. Yet the test integral $\int_{-\infty}^{\infty} x_1(t)\,x_2(t)\,dt$ yields zero because the product $x_1(t)\,x_2(t)$ is zero everywhere. This indicates that the two waveforms have no measure of similarity which is obviously a wrong conclusion. Hence in order to search for a

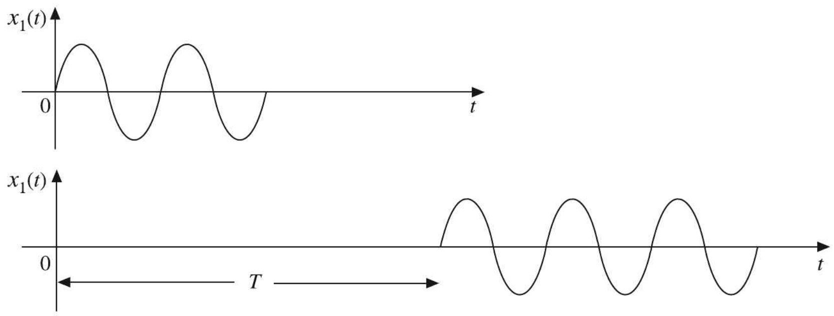


Figure 7.36 Signal comparison.

similarity between the two waveforms, we must shift one waveform w.r.t. the other by various amounts and see whether a similarity exists for some amount of shift of one function w.r.t. the other.

Therefore, the test integral is modified as

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

where τ is the searching or scanning parameter. This integral is a function of τ. This integral is known as the cross correlation function between $x_1(t)$ and $x_2(t)$ and is denoted by $R_{12}(\tau)$.

It is immaterial whether we shift the function $x_1(t)$ by an amount of τ in the negative direction or shift the function $x_2(t)$ by the same amount in the positive direction. Thus

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\, dt$$

Thus the correlation of two functions or signals or waveforms is a measure of similarity between those signals. The correlation is of two types: cross correlation and autocorrelation. The autocorrelation and cross correlation are defined separately for energy (or aperiodic) signals and power (or periodic) signals.

7.6.1 Cross Correlation

The cross correlation between two different waveforms or signals is a measure of similarity or match or relatedness or coherence between one signal and the time delayed version of another signal. That means the cross correlation between two signals indicates how much one signal is related to the time delayed version of another signal.

Cross correlation of energy signals

Consider two general complex signals $x_1(t)$ and $x_2(t)$ of finite energy. The cross correlation of these two energy signals denoted by $R_{12}(\tau)$ is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2{}^*(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t+\tau) x_2{}^*(t)\, dt$$

If the two signals $x_1(t)$ and $x_2(t)$ are real, then

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\, dt$$

If $x_1(t)$ and $x_2(t)$ have some similarity, then the cross correlation $R_{12}(\tau)$ will have some finite value over the range of τ. Also if

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = 0$$

i.e. if $\qquad R_{12}(0) = 0$

then the two signals $x_1(t)$ and $x_2(t)$ are called *orthogonal signals*. That is the cross correlation for orthogonal signals is zero.

Another form of cross correlation between $x_2(t)$ and $x_1(t)$ is defined as:

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$$

In the above equations, the cross correlation function $R_{12}(\tau)$ is a function of the variable τ. The variable τ is called the *delay parameter* or the *scanning parameter* or the *searching parameter*. It is time delay or time shift of one of the two signals. The delay parameter τ determines the correlation between two signals. Two signals with no cross correlation at $\tau = 0$ can have significant cross correlation by adjusting the parameter τ. Two signals for which the cross correlation is zero for all values of τ are called *uncorrelated* or *incoherent signals*.

Properties of cross correlation function for energy signals

Following are the properties of cross correlation for energy signals:

1. The cross correlation functions exhibit conjugate symmetry, i.e.

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

   That is unlike convolution, cross correlation is not in general commutative, i.e.

$$R_{12}(\tau) \neq R_{21}(-\tau)$$

2. If $\qquad R_{12}(0) = 0$

   i.e. if $\qquad \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = 0$

   then the two signals are said to be orthogonal over the entire time interval.

3. The cross correlation of two energy signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of Fourier transform of second signal.

   i.e. $\qquad R_{12}(\tau) \longleftrightarrow X_1(\omega) X_2^*(\omega)$

   This is known as *correlation theorem*.

Cross correlation of power (periodic) signals

The cross correlation function $R_{12}(\tau)$ for two periodic signals $x_1(t)$ and $x_2(t)$ may be defined with the help of average form of correlation. If the two periodic signals $x_1(t)$ and $x_2(t)$ have the same time period T, then cross correlation is defined as:

$$R_{12}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_1(t) x_2^*(t-\tau)\, dt$$

The cross correlation of two periodic functions is defined in another form as:

$$R_{21}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x_2(t)x_1^*(t-\tau)\,dt$$

Properties of cross correlation function for power (periodic) signals

Following are the properties of cross correlation for power signals:

1. The Fourier transform of the cross correlation of two signals is equal to the multiplication of Fourier transform of one signal and complex conjugate of Fourier transform of other signal.

$$R_{12}(\tau) \longleftrightarrow X_1(\omega)X_2^*(\omega)$$

2. If
$$R_{12}(0) = 0$$
i.e. if
$$\underset{T\to\infty}{\text{Lt}}\ \frac{1}{T}\int_{-T/2}^{T/2} x_1(t)\,x_2^*(t)\,dt = 0$$
then the signals are said to be orthogonal over the entire time interval.

3. The cross correlation exhibits conjugate symmetry, i.e.
$$R_{12}(\tau) = R_{21}^*(-\tau)$$

4. Unlike convolution, the cross correlation is not commutative, i.e.
$$R_{12}(\tau) \neq R_{21}(\tau)$$

EXAMPLE 7.12 Prove that $R_{12}(\tau) = R_{21}^*(-\tau)$ i.e. the cross correlation exhibits conjugate symmetry.

Solution: The cross correlation of two signals $x_1(t)$ and $x_2(t)$ is given as:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\,x_2^*(t-\tau)\,dt$$

Let $t - \tau = n$ in the above equation for $R_{12}(\tau)$,

$$\therefore \qquad R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(n+\tau)x_2^*(n)\,dn$$

Also we know that

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)x_1^*(t-\tau)\,dt$$

Let $t = n$ in the above equation for $R_{21}(\tau)$.

$$\therefore \qquad R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(n)\,x_1^*(n-\tau)\,dn$$

$\therefore$
$$R_{21}^*(\tau) = \int_{-\infty}^{\infty} x_2^*(n)\, x_1(n-\tau)\, dn$$

$\therefore$
$$R_{21}^*(-\tau) = \int_{-\infty}^{\infty} x_2^*(n)\, x_1(n+\tau)\, dn$$

Comparing the above two equations for $R_{12}(\tau)$ and $R_{21}^*(-\tau)$, we can write

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

7.6.2 Autocorrelation

The autocorrelation function gives the measure of match or similarity or relatedness or coherence between a signal and its time delayed version. This means that the autocorrelation function is a special form of cross correlation function. It is defined as the correlation of a signal with itself.

The autocorrelation is defined separately for energy signals and power signals.

Autocorrelation for energy signals

The autocorrelation of an energy signal $x(t)$ is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

where τ is called the delay parameter and the signal $x(t)$ is shifted by τ in positive direction.

If $x(t)$ is shifted by τ in negative direction, then

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau) x^*(t)\, dt$$

Properties of autocorrelation function of energy signals

Following are the properties of autocorrelation for energy signals:

1. The autocorrelation function exhibits conjugate symmetry, i.e.

$$R(\tau) = R^*(-\tau)$$

Thus, it states that the real part of $R(\tau)$ is an even function of τ and the imaginary part of $R(\tau)$ is an odd function of τ.

Proof: The autocorrelation of an energy signal $x(t)$ is given by

$$R(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t-\tau)\, dt$$

Taking the complex conjugate, we have

$$R^*(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t-\tau)\, dt$$

$$\therefore \qquad R^*(-\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt = R(\tau)$$

$$\therefore \qquad R(\tau) = R^*(-\tau)$$

2. The value of autocorrelation function of an energy signal at origin (i.e. at $\tau = 0$) is equal to the total energy of that signal, i.e.

$$R(0) = E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

Proof: We have

$$R(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t-\tau)\, dt$$

Putting $\tau = 0$ gives

$$R(0) = \int_{-\infty}^{\infty} x(t) x^*(t)\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt = E$$

3. If τ is increased in either direction, the autocorrelation $R(\tau)$ reduces. As τ reduces autocorrelation, $R(\tau)$ increases and it is maximum at $\tau = 0$, i.e. at the origin. Therefore,

$$|R(\tau)| \le R(0) \quad \text{(for all } \tau)$$

Proof: Consider the functions $x(t)$ and $x(t + \tau)$. $[x(t) \pm x(t + \tau)]^2$ is always greater than or equal to zero since it is squared, i.e.

$$x^2(t) + x^2(t+\tau) \pm 2x(t)\, x(t+\tau) \ge 0$$

or

$$x^2(t) + x^2(t+\tau) \ge \pm 2x(t)\, x(t+\tau)$$

Integrating both the sides, we get

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2\, dt \ge 2 \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

$$\therefore \qquad E + E \ge 2R(\tau) \qquad \text{[If } x(t) \text{ is real valued function]}$$

$$\therefore \qquad E \ge R(\tau)$$

or

$$R(0) \ge |R(\tau)| \qquad \text{(Since } R(0) = E)$$

4. The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(\omega)$ of energy signal form a Fourier transform pair.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

Autocorrelation theorem

The autocorrelation theorem states that the Fourier transform of autocorrelation function $R(\tau)$ yields the energy density function of signal $x(t)$, i.e.

$$\mathrm{F}[R(\tau)] = |X(\omega)|^2 = \psi(\omega)$$

Proof: The Fourier transform of autocorrelation function $R(\tau)$ is:

$$F[R(\tau)] = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\,x(t-\tau)\,e^{-j\omega\tau}\,dt\,d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \int_{-\infty}^{\infty} x(t-\tau)e^{j\omega(t-\tau)}\,d\tau$$

$$= X(\omega)\int_{-\infty}^{\infty} x(t-\tau)\,e^{j\omega(t-\tau)}\,d\tau$$

Letting $t - \tau = n$ in the second integral, we have

$$F[R(\tau)] = X(\omega)\int_{-\infty}^{\infty} x(n)e^{j\omega n}\,dn$$

$$= X(\omega)\,X(-\omega) = |X(\omega)|^2$$

$$= \psi(\omega)$$

For a given function $x(t)$, there is a unique autocorrelation function but the reverse is not true. A given autocorrelation function may correspond to an infinite variety of waveforms.

Autocorrelation function for power (periodic) signals

The autocorrelation function of a periodic signal with any period T is given by

$$R(\tau) = \underset{T\to\infty}{\text{Lt}}\ \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x^*(t-\tau)\,dt$$

Properties of autocorrelation function for power signals

Following are the properties of autocorrelation function for power signals:

1. The autocorrelation function exhibits conjugate symmetry, i.e.

$$R(\tau) = R^*(-\tau)$$

Proof: We have

$$R(\tau) = \underset{T\to\infty}{\text{Lt}}\ \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x^*(t-\tau)\,dt$$

$$\therefore \quad R^*(\tau) = \underset{T\to\infty}{\text{Lt}}\ \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\,x(t-\tau)\,dt$$

$$\therefore \quad R^*(-\tau) = \underset{T\to\infty}{\text{Lt}}\ \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\,x(t+\tau)\,dt = R(\tau)$$

$$\therefore \quad R(\tau) = R^*(-\tau)$$

2. The autocorrelation function at origin is equal to the average power of that signal, i.e.
$$R(0) = P$$
3. The autocorrelation function $R(\tau)$ has maximum value at the origin, i.e.
$$|R(\tau)| \le R(0)$$
The value of autocorrelation reduces as τ increases from origin.
4. The autocorrelation function $R(\tau)$ and power spectral density $S(\omega)$ form a Fourier transform pair, i.e.
$$R(\tau) \longleftrightarrow S(\omega)$$
5. The autocorrelation function is periodic with the same period as the periodic signal itself, i.e.
$$R(\tau) = R(\tau \pm nT), \quad n = 1, 2, 3, \ldots$$

7.7 ENERGY DENSITY SPECTRUM

Spectral density It is the distribution of energy or power of a signal per unit bandwidth as a function of frequency.

Energy signals Signals with finite energy and zero average power, i.e. $0 < E < \infty$ and $P = 0$ are called energy signals, e.g. aperiodic signals like pulse.

Normalized energy The normalized energy, or simply energy of a signal $x(t)$ is defined as the energy dissipated by a voltage signal applied across 1-Ω resistor (or by a current signal flowing through 1-Ω resistor). Mathematically,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

The energy of a signal exists only if E is finite, i.e. only if $0 < E < \infty$.

Parseval's theorem for energy signals (Rayleigh's energy theorem) Parseval's theorem defines the energy of a signal in terms of its Fourier transform. Using Parseval's theorem, the energy of a signal $x(t)$ can be evaluated directly from its frequency spectrum $X(\omega)$ without the knowledge of its time domain version, i.e. $x(t)$.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

or
$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Proof: Consider a function $x(t)$ such that

$$x(t) \longleftrightarrow X(\omega)$$

Let $x^*(t)$ be the conjugate of $x(t)$ such that

$$x^*(t) \longleftrightarrow X^*(-\omega)$$

The energy of a signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} x^*(t)\, x(t)\, dt$$

Replacing $x(t)$ by its inverse Fourier transform, we have

$$E = \int_{-\infty}^{\infty} x^*(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \right] dt$$

Interchanging the order of integration,

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) X^*(-\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Let $\omega = 2\pi f$

$\therefore$ $d\omega = 2\pi df$

Normally $X(2\pi f)$ is written as $X(f)$, then we have

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

This is called Parseval's theorem for energy signals (also called Rayleigh's energy theorem).

Energy spectral density

The ESD function gives the distribution of energy of a signal in the frequency domain. For an energy signal, the total area under the spectral density curve plotted as a function of frequency is equal to the total energy of the signal. It is also called energy density spectrum (ESD or ED). It is designated by $\psi(\omega)$ and given by

$$\psi(\omega) = |X(\omega)|^2$$

Let $x(t)$ and $y(t)$ be the input and output respectively of a linear system.
Let $x(t) \longleftrightarrow X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ and $H(\omega)$ be the system transfer function.
Then, we have

$$Y(\omega) = H(\omega)\, X(\omega)$$

The ESD of the input $x(t)$ is:

$$\psi_x(\omega) = |X(\omega)|^2$$

The ESD of the output $y(t)$ is:

$$\psi_y(\omega) = |Y(\omega)|^2$$

$$\therefore \quad \psi_y(\omega) = |Y(\omega)|^2 = |H(\omega)\, X(\omega)|^2$$
$$= |H(\omega)|^2 |X(\omega)|^2 = |H(\omega)|^2 \psi_x(\omega)$$

$$\therefore \quad \psi_y(\omega) = |H(\omega)|^2 \psi_x(\omega)$$

Thus, the ESD of the output (response) of a linear system is the product of ESD of input (excitation) and square of the magnitude of the transfer function.

Energy of the output signal $E_y = \int_{-\infty}^{\infty} \psi_y(f)\, df = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \psi_y(\omega)\, d\omega = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega$

$$= \frac{1}{2\pi} 2\int_{0}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega$$

If the LTI system is an ideal LPF with lower and upper cutoff frequencies f_L and f_H respectively, then $|H(\omega)| = 1$ for $f_L < f < f_H$.

$$\therefore \quad E_y = \frac{1}{\pi}\int_{f_L}^{f_H} \psi_x(\omega)\, d\omega$$

or
$$E_y = \frac{1}{\pi}\int_{f_L}^{f_H} \psi_x(2\pi f)\, 2\pi df = 2\int_{f_L}^{f_H} \psi_x(f)\, df$$

Properties of ESD: The following are the properties of ESD.

1. The total area under the energy density spectrum is equal to the total energy of the signal.

   i.e.
   $$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$$

2. If $x(t)$ is the input to an LTI system with impulse response $h(t)$, then the input and output ESD functions are related as:
   $$\psi_y(\omega) = |H(\omega)|^2 \psi_x(\omega)$$
   or
   $$\psi_y(f) = |H(f)|^2 \psi_x(f)$$

3. The autocorrelation function $R(\tau)$ and ESD $\psi(\omega)$ form a Fourier transform pair, i.e.
   $$R(\tau) \longleftrightarrow \psi(\omega)$$
   or
   $$R(\tau) \longleftrightarrow \psi(f)$$

7.8 POWER DENSITY SPECTRUM

Power signals Signals with finite average power and infinite energy, i.e. $0 < P < \infty$ and $E = \infty$ are called power signals, e.g. periodic signals.

Average power It is defined as the average power dissipated by a voltage $x(t)$ applied across 1-Ω resistor (or by a current signal flowing through 1-Ω resistor). Mathematically,

$$P = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

The power P defined above is actually the mean square value or the time average of the squared signal. Thus, we may write

$$P = \overline{x^2(t)} = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Parseval's power theorem Parseval's power theorem defines the power of a signal in terms of its Fourier series coefficients, i.e. in terms of the harmonic components present in the signal. Mathematically, it is given by

$$P = \sum_{n=-\infty}^{\infty} |C(n)|^2$$

Proof: Consider a function $x(t)$. We know that

$$|x(t)|^2 = x(t)\, x^*(t)$$

where $x^*(t)$ is the conjugate of $x(t)$.

The average power of $x(t)$ for one cycle is:

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t) \, dt$$

But, we have the exponential Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$\therefore \qquad P = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t} x^*(t) \, dt$$

Interchanging the order of summation and integration, we get

$$P = \sum_{n=-\infty}^{\infty} C_n \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} |C_n|^2$$

$$\therefore \qquad P = \sum_{n=-\infty}^{\infty} |C_n|^2$$

This is called Parseval's power theorem. It states that the power of a signal is equal to the sum of square of the magnitudes of various harmonics present in the discrete spectrum.

Power spectral density (PSD)

The distribution of average power of the signal in the frequency domain is called power spectral density or power density or power density spectrum (PSD or PD).

To derive the PSD, assume the power signal as a limiting case of the energy signal. Consider a power signal $x(t)$, extending to infinity as shown in Figure 7.37.

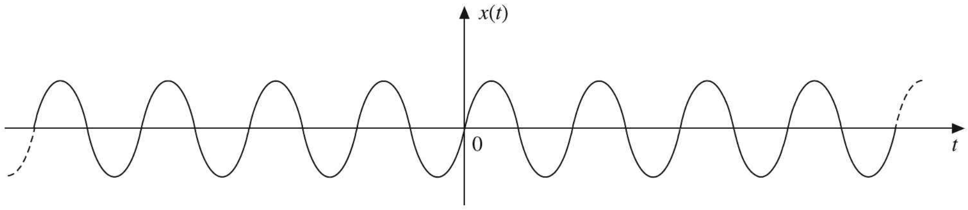


Figure 7.37 Power signal.

Let us truncate this signal so that it is zero outside the interval $|\tau/2|$ as shown in Figure 7.38. Let this truncated signal be $x_\tau(t)$.

$$\therefore \qquad x_\tau(t) = \begin{cases} x(t), & |t| < \dfrac{\tau}{2} \\ 0, & \text{elsewhere} \end{cases}$$

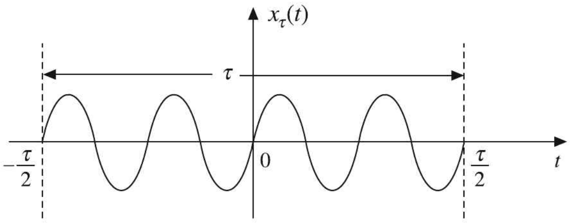


Figure 7.38 Truncated power signal.

The signal $x_\tau(t)$ is of finite duration τ and hence it is an energy signal with energy E given by

$$E = \int_{-\infty}^{\infty} |x_\tau(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_\tau(\omega)|^2\, d\omega$$

where
$$x_\tau(t) \longleftrightarrow X_\tau(\omega)$$

As $x(t)$ over the interval, $\left(-\dfrac{\tau}{2} \text{ to } \dfrac{\tau}{2}\right)$ is same as $x_\tau(t)$ over the interval $-\infty$ to ∞, we have

$$\int_{-\infty}^{\infty} |x_\tau(t)|^2\, dt = \int_{-\tau/2}^{\tau/2} |x(t)|^2\, dt$$

$$\therefore \qquad \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2\, dt = \frac{1}{2\pi} \frac{1}{\tau} \int_{-\infty}^{\infty} |X_\tau(\omega)|^2\, d\omega$$

If $\tau \to \infty$, the left hand side of above equation represents the average power P of the function $x(t)$.

$\therefore$
$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\tau\to\infty}{\text{Lt}} \frac{|X_\tau(\omega)|^2}{\tau}\, d\omega$$

If $\tau \to \infty$, $|X_\tau(\omega)|^2/\tau$ approaches a finite value.

Let this finite value is denoted by $S(\omega)$, i.e.

$$S(\omega) = \underset{\tau\to\infty}{\text{Lt}} \frac{|X_\tau(\omega)|^2}{\tau}$$

The average power P of the function $x(t)$ is given by

$$P = \overline{x^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df$$

where $\overline{x^2(t)}$ is the mean square value of $x(t)$.

The average power is, therefore, given by

$$P = 2\frac{1}{2\pi}\int_{0}^{\infty} S(\omega)\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\, d\omega = 2\int_{0}^{\infty} S(f)\, df$$

The PSD of a periodic function is given by

$$S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(\omega - n\omega_0)$$

or alternately
$$S(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0)$$

The input and output relation of a linear system in terms of PSD is given by

$$S_y(\omega) = |H(\omega)|^2\, S_x(\omega)$$

or
$$S_y(f) = |H(f)|^2\, S_x(f)$$

Properties of PSD The following are the properties of PSD:

1. The area under the PSD function is equal to the average power of that signal, i.e.

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df$$

2. The input and output PSDs of an LTI system are related as:

$$S_y(\omega) = |H(\omega)|^2\, S_x(\omega)$$

3. The autocorrelation function $R(\tau)$ and PSD $S(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow S(\omega)$$

The comparison of ESD and PSD is given in Table 7.1

Table 7.1 Comparision of ESD and PSD

S.No.	ESD	PSD
1.	It gives the distribution of energy of a signal in frequency domain.	It gives the distribution of power of a signal in frequency domain.
2.	It is given by $\psi(\omega) = \lvert X(\omega)\rvert^2$	It is given by $S(\omega) = \underset{\tau\to\infty}{\text{Lt}} \dfrac{\lvert X(\omega)\rvert^2}{\tau}$
3.	The total energy is given by $E = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$	The total power is given by $P = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df$
4.	The autocorrelation for an energy signal and its ESD form a Fourier transform pair. $R(\tau) \longleftrightarrow \psi(\omega)$ or $R(\tau) \longleftrightarrow \psi(f)$	The autocorrelation for a power signal and its PSD form a Fourier transform pair $R(\tau) \longleftrightarrow S(\omega)$ or $R(\tau) \longleftrightarrow S(f)$

7.9 RELATION BETWEEN AUTOCORRELATION FUNCTION AND ENERGY/POWER SPECTRAL DENSITY FUNCTION

7.9.1 Relation between ESD and Autocorrelation Function $R(\tau)$

The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

Proof: The autocorrelation of a function $x(t)$ is given as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Replacing $x^*(t-\tau)$ by its inverse transform, we have

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(t-\tau)}\, d\omega\right]^* dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega(t-\tau)}\, d\omega\right] dt$$

Interchanging the order of integration, we have

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt\right] e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega) e^{j\omega\tau}\, d\omega \qquad [\text{since } |X(\omega)|^2 = \psi(\omega)]$$

$$= \mathrm{F}^{-1}[\psi(\omega)]$$

$$\therefore \qquad \psi(\omega) = \mathrm{F}[R(\tau)]$$

This proves that $R(\tau)$ and $\psi(\omega)$ form a Fourier transform pair.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

7.9.2 Relation between Autocorrelation Function $R(\tau)$ and Power Spectral Density (PSD)

The autocorrelation function $R(\tau)$ and the power spectral density (PSD), $S(\omega)$ of a power signal form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow S(\omega)$$

Proof: The autocorrelation function of a power (periodic) signal $x(t)$ in terms of Fourier series coefficients is given as:

$$R(\tau) = \sum_{n=-\infty}^{\infty} C_n C_{-n} e^{jn\omega_0 \tau}$$

where C_n and C_{-n} are the exponential Fourier series coefficients.

$$\therefore \qquad R(\tau) = \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0 \tau}$$

Taking Fourier transform on both sides, we have

$$\mathrm{F}[R(\tau)] = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0 \tau} \right) e^{-j\omega\tau}\, d\tau$$

Interchanging the order of integration and summation, we get

$$\mathrm{F}[R(\tau)] = \sum_{n=-\infty}^{\infty} |C_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)}\, d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of the periodic function $x(t)$.

$$\therefore \qquad \mathrm{F}[R(\tau)] = S(\omega) \quad [\text{or } S(f)]$$

and $$\mathrm{F}^{-1}[S(\omega)] \quad \{\text{or } \mathrm{F}^{-1}[S(f)]\} = R(\tau)$$

i.e. $$R(\tau) \longleftrightarrow S(\omega) \quad [\text{or } S(f)]$$

7.10 RELATION BETWEEN CONVOLUTION AND CORRELATION

There is a striking resemblance between the operation of convolution and correlation. Indeed the two integrals are closely related. To obtain the cross correlation of $x_1(t)$ and $x_2(t)$ according to the equation $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$, we multiply $x_1(t)$ with function $x_2(t)$

displaced by τ sec. The area under the product curve is the cross correlation between $x_1(t)$ and $x_2(t)$ at τ. On the other hand, the convolution of $x_1(t)$ and $x_2(t)$ at $t = \tau$ is obtained by folding $x_2(t)$ backward about the vertical axis at the origin and taking the area under the product curve of $x_1(t)$ and the folded function $x_2(-t)$ displaced by τ. It, therefore, follows that the cross correlation of $x_1(t)$ and $x_2(t)$ is the same as the convolution of $x_1(t)$ and $x_2(-t)$.

The same conclusion can be arrived at analytically as follows:

The convolution of $x_1(t)$ and $x_2(-t)$ is given by

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(\tau - t)\, d\tau$$

Replacing the dummy variable τ in the above integral by another variable n, we have

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n - t)\, dn$$

Changing the variable from t to τ, we get

$$x_1(\tau) * x_2(-\tau) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n - \tau)\, dn = R_{12}(\tau)$$

Hence $\qquad R_{12}(\tau) = x_1(t) * x_2(-t)\big|_{t=\tau}$

Similarly, $\qquad R_{21}(\tau) = x_2(t) * x_1(-t)\big|_{t=\tau}$

All of the techniques used to evaluate the convolution of two functions can be directly applied in order to find the correlation of two functions. Similarly, all of the results derived for convolution also apply to correlation.

If one of the function is an even function of t, let us say $x_2(t)$ is an even function of t, i.e.

$$x_2(t) = x_2(-t)$$

then the cross correlation and convolution are equivalent.

EXAMPLE 7.13 A signal $x(t) = e^{-2t}\, u(t)$ is passed through an ideal low pass filter with cut off frequency of 1 rad/sec.

(i) Test whether the input is an energy signal.

(ii) Find the input and output energy.

Solution:

(i) Given input signal $\qquad x(t) = e^{-2t}\, u(t)$

Energy of the input signal $E_i = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} \left|e^{-2t}u(t)\right|^2 dt = \int_0^{\infty} e^{-4t}\, dt$

$$= \left[\frac{e^{-4t}}{-4}\right]_0^{\infty} = \frac{1}{4} = 0.25 \text{ joule}$$

Now, Input power $P_i = \underset{T\to\infty}{\text{Lt}} \dfrac{1}{2T} E_i = \underset{T\to\infty}{\text{Lt}} \dfrac{1}{2T}\left(\dfrac{1}{4}\right) = 0$

$\therefore$ Input power = 0 and energy is finite, so it is an energy signal.

(ii) Input energy is calculated as E_i = 0.25 joule. Now, ESD of the output $y(t)$ is given by

$$\psi_y(\omega) = |H(\omega)|^2\, \psi_x(\omega)$$

But
$$\psi_x(\omega) = |X(\omega)|^2$$

Now,
$$X(\omega) = \text{F}[x(t)] = \text{F}[e^{-2t}\, u(t)] = \frac{1}{2 + j\omega}$$

$\therefore$
$$\psi_x(\omega) = |X(\omega)|^2 = \frac{1}{2^2 + \omega^2} = \frac{1}{\omega^2 + 4}$$

For a LPF, the square of the transfer function is given by

$$|H(\omega)|^2 = \begin{cases} 1, & |\omega| < 1 \\ 0, & \text{otherwise} \end{cases}$$

$\therefore$
$$\psi_y(\omega) = |H(\omega)|^2\, \psi_x(\omega) = \begin{cases} 1/(\omega^2 + 4), & |\omega| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the total energy of the output signal is given by

$$E_o = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_y(\omega)\, d\omega = \frac{1}{\pi}\int_0^{\infty} \psi_y(\omega)\, d\omega$$

$$= \frac{1}{\pi}\int_0^1 \frac{1}{\omega^2 + 4}\, d\omega$$

$$= \frac{1}{2\pi}\tan^{-1}\left(\frac{1}{2}\right)$$

Therefore,
$$E_o = \frac{0.46}{2\pi} = 0.049 \text{ joule}$$

EXAMPLE 7.14 A function $x(t)$ has a PSD of $S_x(\omega)$. Find the PSD of (a) integral of $x(t)$ and (b) time derivative of $x(t)$.

Solution: Given function is $x(t)$ with a PSD of $S_x(\omega)$.

(a) Let
$$x(t) \longleftrightarrow X(\omega)$$

Then for the integral of $x(t)$, we have

$$\int_{-\infty}^{\infty} x(\tau)\, d\tau \longleftrightarrow \frac{1}{j\omega} X(\omega)$$

Therefore, if $x(t)$ is the input with Fourier transform $X(\omega)$, then the Fourier transform of the output is:

$$Y(\omega) = \frac{1}{j\omega} X(\omega)$$

Therefore, Transfer function $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{(1/j\omega)X(\omega)}{X(\omega)} = \dfrac{1}{j\omega}$

Now, $$|H(\omega)|^2 = \frac{1}{\omega^2}$$

Now, the output PSD is given by

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

$\therefore$ $$S_y(\omega) = \frac{1}{\omega^2} S_x(\omega)$$

(b) Let $$x(t) \longleftrightarrow X(\omega)$$

Then for derivative of $x(t)$, we have

$$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega)$$

Now, Transfer function $H(\omega) = \dfrac{j\omega X(\omega)}{X(\omega)} = j\omega$

$\therefore$ $$|H(\omega)|^2 = \omega^2$$

Output PSD is given by

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

$\therefore$ $$S_y(\omega) = \omega^2 S_x(\omega)$$

EXAMPLE 7.15 Determine the energy spectral density (ESD) of a gate function of width τ and amplitude A as shown in Figure 7.39.

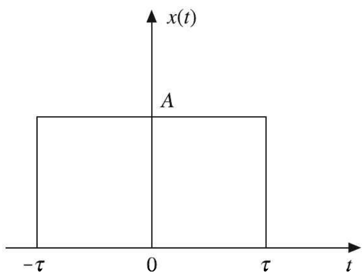


Figure 7.39 Signal for Example 7.15.

Solution: From Figure 7.39, the gate function of width τ and amplitude A is given by

$$x(t) = \begin{cases} A, & |t| < \tau \\ 0, & \text{elsewhere} \end{cases}$$

Therefore,
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\tau}^{\tau} A e^{-j\omega t}\, dt = A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau}^{\tau} = A\left(\frac{e^{-j\omega\tau} - e^{j\omega\tau}}{-j\omega}\right)$$

$$= \frac{2A}{\omega}\left(\frac{e^{j\omega\tau} - e^{-j\omega\tau}}{j2}\right) = \frac{2A}{\omega}\sin\omega\tau = 2A\tau\,\frac{\sin\omega\tau}{\omega\tau}$$

$$= 2A\tau \operatorname{sinc}\omega\tau$$

$\therefore$
$$X(\omega) = 2A\tau \operatorname{sinc}\omega\tau$$

Therefore, the Fourier transform of a gating function is a sinc function.

We know that ESD
$$\psi(\omega) = |X(\omega)|^2$$

$\therefore$ ESD
$$\psi(\omega) = |X(\omega)|^2 = [2A\tau \operatorname{sinc}\omega\tau]^2 = 4A^2\tau^2 \operatorname{sinc}^2 \omega\tau$$

EXAMPLE 7.16 The autocorrelation function of an aperiodic power signal is:

$$R(\tau) = e^{-\tau^2/2\sigma^2}$$

Determine the PSD and the normalized average power content of the signal.

Solution: Given
$$R(\tau) = e^{-\tau^2/2\sigma^2}$$

We have
$$R(\tau) \longleftrightarrow S(\omega)$$

i.e.
$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-(\tau^2/2\sigma^2)}\, e^{-j\omega\tau}\, d\tau$$

$$= \sqrt{2\pi\sigma^2}\; e^{-(\omega\sigma)^2/2}$$

$\therefore$ Normalized average power $P = R(0) = e^0 = 1$ watt

EXAMPLE 7.17 A power signal $g(t)$ has a PSD $S_g(\omega) = N/A^2$, $-2\pi B < \omega < 2\pi B$, shown in Figure 7.40 where A and N are constants.

Determine the PSD and the mean square value of its derivative $dg(t)/dt$.

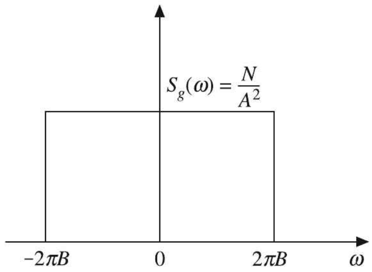


Figure 7.40 Signal for Example 7.17.

Solution: Given

$$\text{Input signal} = g(t)$$

$$\text{Input PSD, } S_g(\omega) = \frac{N}{A^2}$$

Let

$$g(t) \longleftrightarrow G(\omega)$$

$\therefore$

$$\frac{d}{dt} g(t) \longleftrightarrow j\omega G(\omega)$$

$$\text{Transfer function } H(\omega) = \frac{\text{Output}}{\text{Input}} = \frac{j\omega G(\omega)}{G(\omega)} = j\omega$$

$\therefore$

$$H(\omega) = j\omega$$

$\therefore$

$$|H(\omega)|^2 = \omega^2$$

Therefore,

$$\text{Output PSD, } S_y(\omega) = |H(\omega)|^2 S_g(\omega)$$

$\therefore$

$$S_y(\omega) = \frac{\omega^2 N}{A^2}$$

Mean square value of output = Area under output PSD

$$= \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} \frac{\omega^2 N}{A^2} d\omega$$

$$= \frac{N}{2\pi A^2} \left[\frac{\omega^3}{3} \right]_{-2\pi B}^{2\pi B}$$

$$= \frac{N}{6\pi A^2} [8\pi^3 B^3 - (-8\pi^3 B^3)]$$

$$= \frac{N}{6\pi A^2} (16\pi^3 B^3)$$

$$= \frac{8\pi^2 N B^3}{3A^2}$$

EXAMPLE 7.18 Energies of signals $x_1(t)$ and $x_2(t)$ are E_1 and E_2 respectively.

(i) Show that, in general, the energy of signal $x_1(t) + x_2(t)$ is not $E_1 + E_2$.

(ii) Under what condition is the energy of $x_1(t) + x_2(t)$ is equal to $E_1 + E_2$?

(iii) Can the energy of the signal $x_1(t) + x_2(t)$ be zero? If so, under what conditions?

Solution: Given: The energy of $x_1(t)$ is E_1 and energy of $x_2(t)$ is E_2. In general, energy of a signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$\therefore$
$$E_1 = \int_{-\infty}^{\infty} |x_1(t)|^2\, dt$$

$$E_2 = \int_{-\infty}^{\infty} |x_2(t)|^2\, dt$$

(i) Energy of signal $x_1(t) + x_2(t)$ is:

$$E = \int_{-\infty}^{\infty} |x_1(t) + x_2(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} |x_1(t)|^2\, dt + \int_{-\infty}^{\infty} |x_2(t)|^2\, dt + 2\int_{-\infty}^{\infty} |x_1(t)||x_2(t)|dt$$

Therefore,
$$E = E_1 + E_2 + 2\int_{-\infty}^{\infty} |x_1(t)||x_2(t)|dt$$

Therefore,
$$E \neq E_1 + E_2$$

(ii) When $x_1(t)$ and $x_2(t)$ are orthogonal, i.e.

$$\int_{-\infty}^{\infty} |x_1(t)||x_2(t)|\, dt = 0$$

Then
$$E = E_1 + E_2$$

(iii) When $x_1(t) = x_2(t) = 0$ then the energy $E = 0$.

EXAMPLE 7.19 A signal $x(t)$ has energy E. Calculate the energy of the signal $x(3t)$.

Solution: The energy of a signal $x(t)$ is:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

Now, for signal $x(3t)$, we have the energy

$$E' = \int_{-\infty}^{\infty} |x(3t)|^2\, dt$$

Let $3t = p$,

$$\therefore \qquad dt = \frac{dp}{3}$$

$$\therefore \qquad E' = \int_{-\infty}^{\infty} |x(p)|^2 \frac{dp}{3} = \frac{1}{3} \int_{-\infty}^{\infty} |x(p)|^2 \, dp = \frac{1}{3} E$$

EXAMPLE 7.20 Verify Parseval's theorem for the energy signal $x(t) = e^{-4t} u(t)$.

Solution: The energy of a signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| e^{-4t} u(t) \right|^2 dt$$

$$= \int_0^{\infty} \left| e^{-4t} \right|^2 dt = \int_0^{\infty} e^{-8t} \, dt = \left[\frac{e^{-8t}}{-8} \right]_0^{\infty} = \frac{1}{8} \text{ joule}$$

Now, according to Parseval's theorem, we have

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

$$X(\omega) = \text{F}[x(t)] = \int_{-\infty}^{\infty} e^{-4t} u(t) e^{-j\omega t} \, dt$$

$$= \int_0^{\infty} e^{-(4+j\omega)t} \, dt = \left[\frac{e^{-(4+j\omega)t}}{-(4+j\omega)} \right]_0^{\infty} = \frac{1}{4 + j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{4^2 + \omega^2}}$$

$$\therefore \qquad E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4^2 + \omega^2} \, d\omega = \frac{1}{8\pi} \tan^{-1} \left[\frac{\omega}{4} \right]_{-\infty}^{\infty}$$

$$= \frac{1}{8\pi} \left[\frac{\pi}{2} - \left(\frac{-\pi}{2} \right) \right] = \frac{1}{8} \text{ joule}$$

Thus, from the above equations, we see that energy is same in both the cases. Hence Parseval's theorem is verified.

EXAMPLE 7.21 A filter has an input $x(t) = e^{-t} u(t)$ and its impulse response $h(t) = e^{-3t} u(t)$. Find the energy spectral density of the output.

Solution: Given $\qquad x(t) = e^{-t} u(t)$

$$\therefore \qquad X(\omega) = \text{F}[e^{-t} u(t)] = \frac{1}{1 + j\omega}$$

$\therefore$ ESD of input $\psi_x(\omega) = |X(\omega)|^2 = \dfrac{1}{1+\omega^2}$

Given $h(t) = e^{-3t}\, u(t)$

$\therefore$ $H(\omega) = \text{F}[e^{-3t}\, u(t)] = \dfrac{1}{j\omega + 3}$

$\therefore$ $|H(\omega)|^2 = \dfrac{1}{9+\omega^2}$

$\therefore$ ESD of output $\psi_y(\omega) = |H(\omega)|^2\, \psi_x(\omega)$

$$= \left(\frac{1}{9+\omega^2}\right)\left(\frac{1}{1+\omega^2}\right)$$

EXAMPLE 7.22 Figure 7.41 shows the PSD of the signal $x(t)$. Find out its average power.

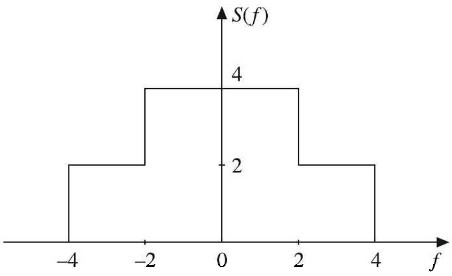


Figure 7.41 PSD for Example 7.22.

Solution: We know that the total average power is equal to the area under the PSD curve, i.e.

$$P = \int_{-\infty}^{\infty} S(f)\,df = \int_{-4}^{-2} 2df + \int_{-2}^{2} 4\,df + \int_{2}^{4} 2df$$

$$= 2[f]_{-4}^{-2} + 4[f]_{-2}^{2} + 2[f]_{2}^{4}$$

$$= 2\,(-2+4) + 4(2+2) + 2(4-2)$$

$$= 24 \text{ watt}$$

EXAMPLE 7.23 For the signal $x(t) = e^{-at}\, u(t)$, find out the total energy contained in the frequency band $|f| \le f_1$, where $f_1 = a/2\pi$.

Solution: Given $x(t) = e^{-at}\, u(t)$

$$X(\omega) = \frac{1}{a+j\omega} = \frac{1}{a+j2\pi f}$$

$\therefore$ $\text{ESD} = |X(\omega)|^2 = \dfrac{1}{a^2 + (2\pi f)^2} = \psi(f)$

We know that the total energy of $x(t)$ is given by area under the ESD curve. Energy contained in the band $|a| \le f_1$ is equal to area under the ESD curve from $-f_1$ to $+f_1$, i.e.

$$\text{Energy in } (-f_1, f_1) = \int_{-f_1}^{f_1} \psi(f)\, df = \int_{-f_1}^{f_1} \frac{1}{a^2 + (2\pi f)^2}\, df$$

$$= \frac{1}{2\pi a}\left[\tan^{-1}\left(\frac{2\pi f}{a}\right)_{-f_1}^{f_1}\right]$$

$$= \frac{1}{4a}$$

EXAMPLE 7.24 Determine the autocorrelation function and energy spectral density of $x(t) = e^{-at}\, u(t)$.

Solution:

(i) **To obtain autocorrelation function:**

The given function is:

$$x(t) = e^{-at}\, u(t)$$

This is an energy signal. Autocorrelation function for energy signals is given as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

If $x(t)$ is real, then the above equation becomes

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$$

Here $x(t) = e^{-at}\, u(t)$ is a real signal. Substituting this value of $x(t)$ in the equation for $R(\tau)$, we get

$$R(\tau) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-a(t-\tau)}\, u(t-\tau)\, dt$$

We know that $u(t) = 1$ for $t \ge 0$ and $u(t - \tau) = 1$ for $t \ge \tau$. Hence the limits of integration will be from τ to ∞, i.e.

$$R(\tau) = \int_{\tau}^{\infty} e^{-at}\, e^{-a(t-\tau)}\, dt = \int_{\tau}^{\infty} e^{-at}\, e^{-at}\, e^{a\tau}\, dt$$

$$= e^{a\tau}\int_{\tau}^{\infty} e^{-2at}\, dt = e^{a\tau}\left[\frac{e^{-2at}}{-2a}\right]_{\tau}^{\infty} = e^{a\tau}\left(\frac{e^{-\infty}}{-2a} + \frac{e^{-2a\tau}}{2a}\right)$$

$$= \frac{e^{-a\tau}}{2a}$$

This is an autocorrelation function for positive value of τ. This is because we have considered $u(t - \tau) = 1$ for 'τ' positive. We know that autocorrelation has conjugate symmetry,

$$R(\tau) = R^*(-\tau)$$

In this example, we obtained $R(\tau)$, which is real. Hence the above equation becomes

$$R(\tau) = R(-\tau)$$

Thus, $R(\tau)$ is an even function of τ. Hence the autocorrelation function becomes

$$R(\tau) = \frac{e^{-a|\tau|}}{2a}$$

Figure 7.42 shows the sketch of the above autocorrelation function.

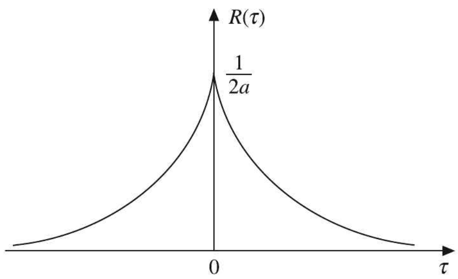


Figure 7.42 Autocorrelation function.

(ii) **To obtain energy spectral density:**

We know that the autocorrelation function and energy spectral density form a Fourier transform pair.

$$\therefore \qquad \psi(\omega) = \text{F}\{R(\tau)\}$$

$$= \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} \frac{e^{-a|\tau|}}{2a} e^{-j\omega\tau}\, d\tau$$

$$= \frac{1}{2a} \int_{-\infty}^{\infty} e^{-a|\tau|} e^{-j\omega\tau}\, d\tau = \frac{1}{2a} \frac{2a}{a^2 + \omega^2}$$

$$= \frac{1}{a^2 + \omega^2}$$

This is an expression for the energy spectral density.

EXAMPLE 7.25 Consider a pulse function given by

$$x(t) = \begin{cases} \cos\left(\dfrac{\pi t}{T}\right), & \text{for } -\dfrac{T}{2} < t < \dfrac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Find:

(i) Autocorrelation function

(ii) Energy spectral density

Solution:

(i) **To obtain autocorrelation function:**

The given signal is defined only for $\left(-\frac{T}{2}, \frac{T}{2}\right)$. Hence we will use an equation for energy signals, i.e.

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Here $$x(t) = \cos\left(\frac{\pi t}{T}\right) \quad \text{for } -\frac{T}{2} < t < \frac{T}{2}$$

This is a real function. Hence the equation for $R(\tau)$ becomes

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$$

$$= \int_{-T/2}^{T/2} \cos\frac{\pi t}{T} \cos\frac{\pi}{T}(t-\tau)\, dt$$

$$= \int_{-T/2}^{T/2} \cos\frac{\pi t}{T} \cos\left(\frac{\pi t}{T} - \frac{\pi \tau}{T}\right) dt$$

Using the relation,

$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

we have $$R(\tau) = \int_{-T/2}^{T/2} \frac{1}{2}\left\{\cos\left(\frac{\pi\tau}{T}\right) + \cos\left(\frac{2\pi t}{T} - \frac{\pi\tau}{T}\right)\right\} dt$$

$$= \frac{1}{2}\int_{-T/2}^{T/2} \cos\left(\frac{\pi\tau}{T}\right) dt + \frac{1}{2}\int_{-T/2}^{T/2} \cos\left(\frac{2\pi t}{T} - \frac{\pi\tau}{T}\right) dt$$

$$= \frac{1}{2}\cos\left(\frac{\pi\tau}{T}\right)[t]_{-T/2}^{T/2} + \frac{1}{2}\frac{1}{2\pi/T}\left[\sin\left(\frac{2\pi t}{T} - \frac{\pi\tau}{T}\right)\right]_{-T/2}^{T/2}$$

On simplifying the above equation, we get

$$R(\tau) = \frac{T}{2}\sin\left(\frac{\pi t}{T}\right)$$

This is the autocorrelation function of a given signal.

(ii) **To obtain energy spectral density:**

We know that

$$R(\tau) \longleftrightarrow \psi(\omega)$$

i.e.
$$\psi(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} \frac{T}{2} \sin\left(\frac{\pi\tau}{T}\right) e^{-j\omega\tau}\, d\tau$$

$$= \frac{T}{2} \int_{-\infty}^{\infty} \sin\left(\frac{\pi\tau}{T}\right) e^{-j\omega\tau}\, d\tau$$

We know that $\sin\theta = (e^{j\theta} - e^{-j\theta})/2j$. Hence the above equation for $\psi(\omega)$ becomes

$$\psi(\omega) = \frac{T}{2} \int_{-\infty}^{\infty} \frac{e^{j(\pi\tau/T)} - e^{-j(\pi\tau/T)}}{2j} e^{-j\omega\tau}\, d\tau$$

$$= \frac{T}{4j} \int_{-\infty}^{\infty} \left[e^{-j[\omega - (\pi/T)]\tau} - e^{-j[\omega + (\pi/T)]\tau} \right] d\tau$$

$$= \frac{T}{4j} \left\{ \int_{-\infty}^{\infty} e^{-j[\omega - (\pi/T)]\tau}\, d\tau - \int_{-\infty}^{\infty} e^{-j[\omega + (\pi/T)]\tau}\, d\tau \right\}$$

Here, $\int_{-\infty}^{\infty} e^{-j\omega\tau}\, d\tau = \delta(\omega)$. Hence above equation becomes

$$\psi(\omega) = \frac{T}{4j} \left[\delta\left(\omega - \frac{\pi}{T}\right) - \delta\left(\omega + \frac{\pi}{T}\right) \right]$$

This is the required expression for ESD.

EXAMPLE 7.26 Find the zero-lag value of the autocorrelation function of the signal

$$x(t) = 5\cos\left(\frac{\pi t}{T}\right), \quad -\frac{T}{2} \le t \le \frac{T}{2}$$

Solution: The given signal $x(t) = 5\cos\left(\frac{\pi t}{T}\right)$ is a cosine signal with period $T_0 = \frac{(2\pi)}{\pi/T} = 2T$.

Hence $5\cos\left(\frac{\pi t}{T}\right), \ -\frac{T}{2} \le t \le \frac{T}{2}$ is a cosine pulse as shown in Figure 7.43.

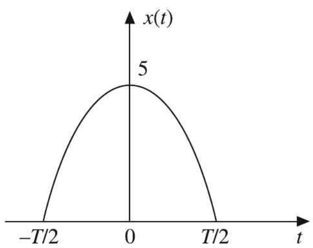


Figure 7.43 Cosine pulse.

We know that $R(0)$ = Energy of the signal $x(t)$

and
$$R(0) = \int_{-\infty}^{\infty} 5\cos\left(\frac{\pi t}{T}\right) 5\cos\left(\frac{\pi t}{T}\right) dt = 25 \int_{-T/2}^{T/2} \cos^2\left(\frac{\pi t}{T}\right) dt$$

$$= \frac{25}{2} \int_{-T/2}^{T/2} \left(1 + \cos\frac{2\pi t}{T}\right) dt = 12.5\left[t + \frac{\sin(2\pi t/T)}{2\pi/T}\right]_{-T/2}^{T/2}$$

$$= 12.5\left[\frac{T}{2} - \left(-\frac{T}{2}\right) + 0 - 0\right] = 12.5T$$

EXAMPLE 7.27 Find the autocorrelation of the signal

$$x(t) = A\sin(\omega_0 t + \theta)$$

Solution: Given:
$$x(t) = A\sin(\omega_0 t + \theta)$$

$$R(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A\sin(\omega_0 t + \theta)\, A\sin[\omega_0(t - \tau) + \theta]\, dt$$

Let
$$\phi = \omega_0 t + \theta$$

$\therefore$
$$dt = \frac{d\phi}{\omega_0}$$

For $t = -T_0/2$, $\phi = -\pi + \theta$ and for $t = T_0/2$, $\phi = \pi + \theta$

$$R(\tau) = \frac{A^2}{T_0} \int_{-T_0/2}^{T_0/2} \sin\phi \sin(\phi - \omega_0\tau) \frac{d\phi}{\omega_0}$$

$$= \frac{A^2}{T_0\omega_0} \int_{-\pi+\theta}^{\pi+\theta} \sin\phi(\sin\phi\cos\omega_0\tau - \cos\phi\sin\omega_0\tau)\, d\phi$$

$$= \frac{A^2}{T_0\omega_0} \int_{-\pi+\theta}^{\pi+\theta} \left[\cos\omega_0\tau\left(\frac{1 - \cos 2\phi}{2}\right) - \sin\omega_0\tau\left(\frac{\sin 2\phi}{2}\right)\right] d\phi$$

$$= \frac{A^2}{4\pi}\left\{\cos\omega_0\tau\,[\phi]_{-\pi+\theta}^{\pi+\theta}\right\} - \frac{A^2}{4\pi}\cos\omega_0\tau \int_{-\pi+\theta}^{\pi+\theta} \cos 2\phi\, d\phi$$

$$- \frac{A^2}{4\pi}\sin\omega_0\tau \int_{-\pi+\theta}^{\pi+\theta} \sin 2\phi\, d\phi$$

$\therefore$
$$= \frac{A^2}{4\pi}\cos\omega_0\tau\,[\pi + \theta - (-\pi + \theta)] = \frac{A^2}{2}\cos\omega_0\tau$$

$\therefore$ Autocorrelation of $x(t) = A\sin(\omega_0 t + \theta)$ is $R(\tau) = A^2/2 \cos\omega_0\tau$.

EXAMPLE 7.28 For the periodic function shown in Figure 7.44 determine and plot the autocorrelation.

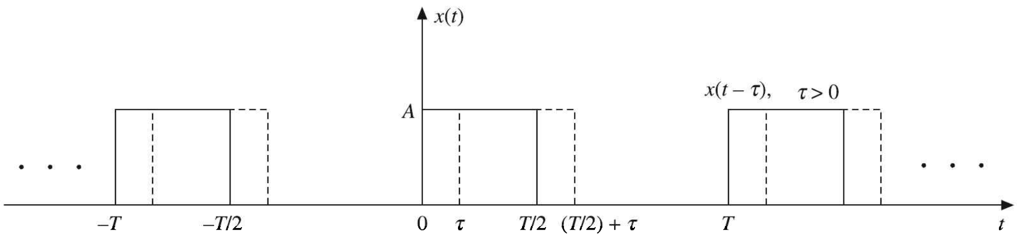


Figure 7.44 Figure for Example 7.28.

Solution: We know that for a periodic signal with period T,

$$R(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x(t-\tau)\, dt$$

$\therefore$
$$R(\tau) = \frac{1}{T}\int_{0}^{T/2} x(t)\, x(t-\tau)\, dt$$

When $x(t)$ is shifted to the right (i.e. $\tau > 0$), $x(t - \tau)$ will be as shown by the dotted-line waveform.

$\therefore$
$$R(\tau) = \frac{1}{T}\int_{\tau}^{T/2} A^2 dt = \frac{A^2}{T}\left(\frac{T}{2} - \tau\right); \quad \tau > 0$$

Since τ takes positive as well as negative values and $R(\tau)$ must have even symmetry [since $x(t)$ is real valued], we have

$$R(\tau) = \frac{A^2}{T}\left(\frac{T}{2} - |\tau|\right)$$

Hence $R(\tau)$ is a periodic triangular wave with period T, as shown in Figure 7.45.

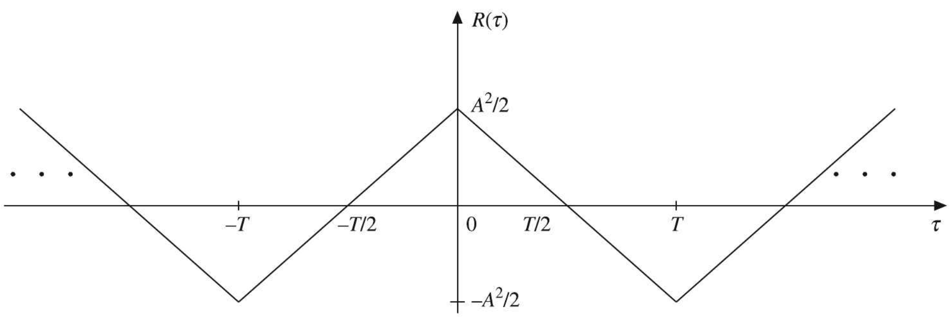


Figure 7.45 Periodic triangular wave.

EXAMPLE 7.29 The signal $x(t) = \cos \omega_0 t + 2 \sin 3\omega_0 t + 0.5 \sin 4\omega_0 t$ is filtered by an RC low pass filter with a 3-dB frequency $f_c = 2f_0$. Find the output power S_0.

Solution: Given $x(t) = \cos \omega_0 t + 2 \sin 3\omega_0 t + 0.5 \sin 4\omega_0 t$

When the above signal $x(t)$ is passed through RC low pass filter with $f_c = 2f_0$, $\sin 3\omega_0 t$ and $0.5 \sin 4\omega_0 t$ are filtered out and cannot reach the output.

The output of low pass filter will be

$$x_0(t) = \cos \omega_0 t$$

$$\text{Output average power } S_0 = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x_0(t)|^2 \, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} (\cos \omega_0 t)^2 \, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} \left(\frac{1 + \cos 2\omega_0 t}{2}\right) dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{4T} \left[t + \frac{\sin 2\omega_0 t}{2\omega_0} \right]_{-T}^{T}$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{4T} \left\{ [T - (-T)] + \left[\frac{\sin 2\omega_0 T + \sin 2\omega_0 T}{2\omega_0} \right] \right\}$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{4T} (2T + 0) = \frac{1}{2} \text{W}$$

EXAMPLE 7.30 If $x(t) = \sin \omega_0 t$, find

(i) $R(\tau)$ (ii) ESD

Solution: Given $x(t) = \sin \omega_0 t$

(i)
$$R(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \sin \omega_0 t \sin [\omega_0 (t - \tau)] \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \sin \omega_0 t \, [\sin \omega_0 t \cos \omega_0 \tau - \cos \omega_0 t \sin \omega_0 \tau] \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \cos \omega_0 \tau \left(\frac{1 - \cos 2\omega_0 t}{2}\right) - \sin \omega_0 \tau \left(\frac{\sin 2\omega_0 t}{2}\right) dt$$

$$= \frac{1}{2T} \left\{ \cos \omega_0 \tau \, [t]_{-T/2}^{T/2} \right\} - 0 - 0 = \frac{1}{2} \cos \omega_0 \tau$$

(ii) ESD $\psi(\omega) = \mathrm{F}[R(\tau)] = \mathrm{F}\left(\frac{1}{2}\cos\omega_0\tau\right)$

$$= \frac{\pi}{2}[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$

EXAMPLE 7.31 A signal $x(t)$ is given by $x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$. Find the autocorrelation and PSD of $x(t)$.

Solution: Given $x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$

(i) **Autocorrelation:**
Autocorrelation of function $x(t)$ is given by

$$R(\tau) = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt$$

$$R(\tau) = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{-T/2}^{T/2} \left[C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)\right]\left[C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + n\omega_0\tau + \theta_n)\right]$$

$$= \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{-T/2}^{T/2} \left[C_0^2 + C_0C_n\sum_{n=1}^{\infty} \cos(n\omega_0 t + n\omega_0\tau + \theta_n) + \sum_{n=1}^{\infty} C_0C_n\cos(n\omega_0 t + \theta_n)\right.$$

$$\left. + C_n^2\sum_{n=1}^{\infty}\cos(n\omega_0 t + \theta_n)\cos(n\omega_0 t + n\omega_0\tau + \theta_n)\, dt\right]$$

$$= \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{-T/2}^{T/2} C_0^2 dt + \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{-T/2}^{T/2} C_0C_n\sum_{n=1}^{\infty}\cos(n\omega_0 t + n\omega_0\tau + \theta_n)\, dt$$

$$+ \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=1}^{\infty} C_0C_n\cos(n\omega_0 t + \theta_n)\, dt$$

$$+ \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=1}^{\infty} C_n^2\cos(n\omega_0 t + \theta_n)\cos(n\omega_0 t + n\omega_0\tau + \theta_n)\, dt$$

$$= C_0^2 + 0 + 0 + \sum_{n=1}^{\infty} \underset{T\to\infty}{\mathrm{Lt}} \frac{C_n^2}{2T}\int_{-T/2}^{T/2} 2\cos(n\omega_0 t + \theta_n)\cos(n\omega_0 t + n\omega_0\tau + \theta_n)\, dt$$

$$= C_0^2 + \sum_{n=1}^{\infty} \underset{T\to\infty}{\mathrm{Lt}} \frac{C_n^2}{2T}\int_{-T/2}^{T/2} [\cos(2n\omega_0 t + n\omega_0\tau + 2\theta_n) + \cos(n\omega_0 t)]\, dt$$

$$= C_0^2 + \sum_{n=1}^{\infty} \underset{T\to\infty}{\text{Lt}} \frac{C_n^2}{2T} \int_{-T/2}^{T/2} \cos n\omega_0\tau \, dt = C_0^2 + \sum_{n=1}^{\infty} \underset{T\to\infty}{\text{Lt}} \frac{C_n^2}{2T} \cos n\omega_0\tau \left(\frac{T}{2}+\frac{T}{2}\right)$$

$$= C_0^2 + \sum_{n=1}^{\infty} \underset{T\to\infty}{\text{Lt}} \frac{C_n^2}{2T} \cos n\omega_0\tau T$$

$$\therefore \qquad R(\tau) = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 \cos n\omega_0\tau$$

Power spectral density (PSD) and autocorrelation form a Fourier transform pair.

$$\therefore \qquad \text{PSD} = \text{F}[R(\tau)]$$

$$\text{PSD} = \text{F}\left[C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 \cos n\omega_0\tau\right]$$

$$= C_0^2\,[2\pi\delta(\omega)] + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 \pi\,[\delta(\omega - n\omega_0) + \delta(\omega + n\omega_0)]$$

EXAMPLE 7.32 Find the cross correlation between triangular and gate functions shown in Figure 7.46.

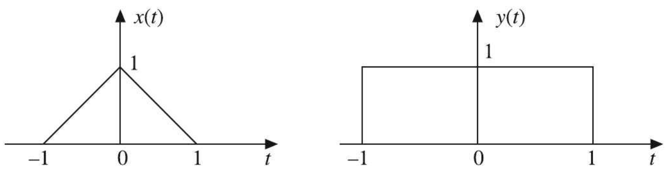


Figure 7.46 Waveforms for Example 7.32.

Solution: By definition

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t+\tau)\, dt$$

Based on Table 7.2 which gives graphical evaluation of correlation, we can draw the correlation function $R_{12}(\tau)$ as shown in Figure 7.47. We can observe that correlation of triangular and gate function leads to triangular function, but with double time period.

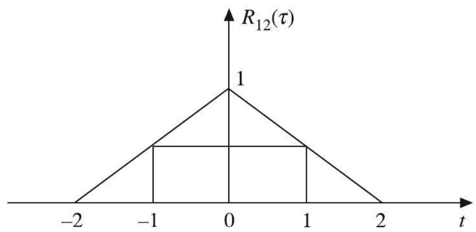
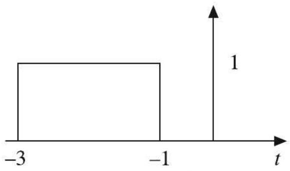
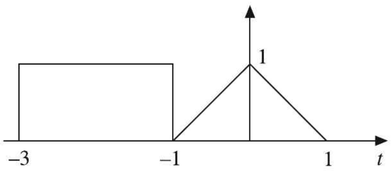
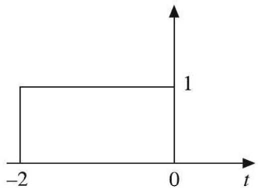
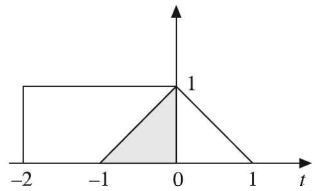
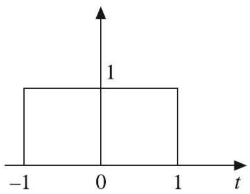
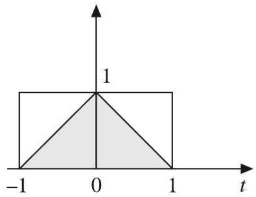
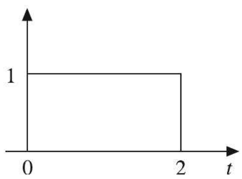
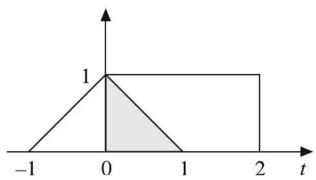
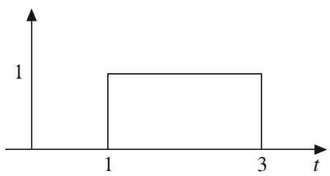
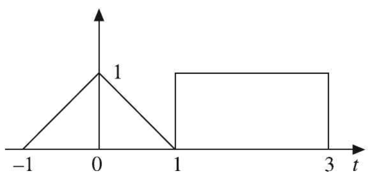


Figure 7.47 Correlation function.

Table 7.2 Graphical Evaluation of Correlation

τ	$y(t + \tau)$	$x(t)\ y(t + \tau)$	*Integration*
+2			Common area = 0
+1			Common area = 0.5 between interval –1 to 0
0			Common area = 1 between interval –1 to +1
–1			Common area = 0.5 between interval 0 to 1
–2			Common area = 0

7.11 DETECTION OF PERIODIC SIGNALS IN THE PRESENCE OF NOISE BY CORRELATION

Detection of periodic signals masked by random noise is of great importance. It finds applications in the detection of radar and sonar signals, the detection of periodic components in brain waves, the detection of the cyclical components in ocean wave analysis and in many areas of geophysics including metrology. Correlation techniques provide a powerful tool in the solution of the above problems. Both autocorrelation and cross correlation can be used in the detection of a periodic signal masked by noise.

The noise signal encountered in practice is a signal with random amplitude variation. Such a signal is uncorrelated with any periodic signal.

If $s(t)$ is a periodic signal and $n(t)$ represents the noise signal, then the cross correlation function of $s(t)$ and $n(t)$ is:

$$R_{sn}(\tau) = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} s(t)\, n(t-\tau)\, dt = 0 \quad \text{(for all } \tau)$$

7.11.1 Detection by Autocorrelation

Let $s(t)$ be a periodic signal mixed with a noise signal $n(t)$. Then the received signal which is also periodic is:

$$y(t) = s(t) + n(t)$$

Let $R_{yy}(\tau)$, $R_{ss}(\tau)$, and $R_{nn}(\tau)$ denote the autocorrelation functions of $y(t)$, $s(t)$ and $n(t)$, respectively. Then

$$R_{yy}(\tau) = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, y(t-\tau)\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} [s(t) + n(t)]\,[s(t-\tau) + n(t-\tau)]\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} [s(t)\, s(t-\tau) + n(t)\, n(t-\tau) + s(t)\, n(t-\tau) + n(t)\, s(t-\tau)]\, dt$$

$$= R_{ss}(\tau) + R_{nn}(\tau) + R_{sn}(\tau) + R_{ns}(\tau)$$

Since the periodic signal $s(t)$ and noise signal $n(t)$ are uncorrected,

$$R_{sn}(\tau) = R_{ns}(\tau) = 0$$

Therefore,

$$R_{yy}(\tau) = R_{ss}(\tau) + R_{nn}(\tau)$$

Thus, $R_{yy}(\tau)$ has two components: $R_{ss}(\tau)$ and $R_{nn}(\tau)$.

We know that the autocorrelation function of a periodic signal is also a periodic function of the same frequency and the autocorrelation function of a non-periodic function tends to zero for large values of τ.

Since $s(t)$ is a periodic signal and $n(t)$ is a non-periodic signal, $R_{ss}(\tau)$ is a periodic function, whereas $R_{nn}(\tau)$ becomes arbitrarily small for large values of τ. Therefore, for large values of τ, $R_{yy}(\tau)$ is essentially equal to $R_{ss}(\tau)$. Therefore, $R_{yy}(\tau)$ will exhibit a periodic nature at sufficiently large values of τ.

The autocorrelation function of the given signal can be calculated by the numerical techniques used for convolution.

7.11.2 Detection by Cross Correlation

The detection of a periodic signal can be carried out by cross correlating the received signal with another periodic signal of the same frequency. Detection by cross correlation is much more effective than that by the autocorrelation. The disadvantage, however is, that it is necessary to know before hand the frequency of the signal to be detected. In many cases, the frequency is known before hand. If the frequency is not known before hand, it may be determined by the autocorrelation technique.

Let the received signal be

$$y(t) = s(t) + n(t)$$

Let $l(t)$ be the locally generated signal of same frequency as that of $s(t)$.

$$\therefore \qquad R_{yl}(\tau) = \underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} [s(t) + n(t)]\, l(t-\tau)\, dt$$

$$= R_{sl}(\tau) + R_{nl}(\tau)$$

The term $l(t)$ is a periodic function and is uncorrelated with the random noise signal. Hence $R_{nl}(\tau) = 0$.

$$\therefore \qquad R_{yl}(\tau) = R_{sl}(\tau)$$

Since $s(t)$ and $l(t)$ are signals of the same frequency, $R_{sl}(\tau)$ is also a periodic function of the same frequency. So if the cross correlation of the contaminated signal $y(t)$ with $l(t)$ yields a periodic signal, $y(t)$ must contain a periodic component of the same frequency as that of $l(t)$.

7.12 EXTRACTION OF A SIGNAL FROM NOISE BY FILTERING

A signal masked by noise can be detected either by correlation techniques or by filtering. Actually the two techniques are equivalent. The correlation technique is a means of extraction of a given signal in the time domain, whereas filtering achieves exactly the same results in the frequency domain. Correlation in the time domain corresponds to filtering in the frequency domain.

Relationship between correlation and filtering

Let $x(t)$ and $y(t)$ be two periodic signals, and let $R_{xy}(\tau)$ be their cross correlation function. We know that

If $$x(t) \longleftrightarrow X(\omega)$$

and $$y(t) \longleftrightarrow Y(\omega)$$

then $$R_{xy}(\tau) \longleftrightarrow X(\omega)\, Y(-\omega)$$

It is obvious that the operation of cross correlation of $x(t)$ and $y(t)$ in the time domain is equivalent to multiplication of the spectra $X(\omega)$ and $Y(-\omega)$ in the frequency domain. So the cross correlation of $x(t)$ and $y(t)$ can be achieved either in time domain as shown in Figure 7.48(a) by applying $x(t)$ and $y(t)$ to a cross correlator or in frequency domain as shown in Figure 7.48(b) by applying $x(t)$ to a linear system with transfer function $Y(-\omega)$ [or impulse response $y(-t)$]. This second operation essentially represents filtering.

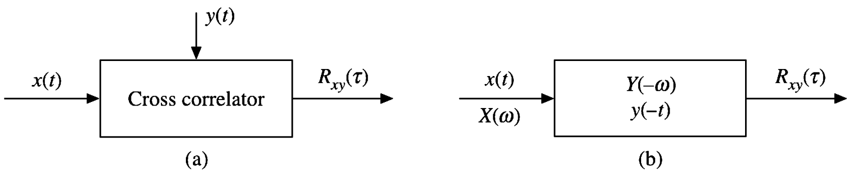


Figure 7.48 Correlation in (a) time domain, (b) frequency domain.

So we can conclude that: *the cross correlation function of signals* $x(t)$ *and* $y(t)$ *is the response of a system with transfer function* $Y(-\omega)$ [*or the impulse response* $y(-t)$] *when the driving function is* $x(t)$.

Detection by filtering

Let $s(t)$ and $n(t)$ be the desired periodic signal component and the random noise component respectively, then the received signal $y(t)$ is:

$$y(t) = s(t) + n(t)$$

The periodic component $s(t)$ present in $y(t)$ can be detected by cross correlating $y(t)$ with another periodic signal $l(t)$ of the same period as that of $s(t)$. To perform this cross correlation, we need a system which has a unit impulse response $l(-t)$ or which has a transfer function $L(-\omega)$ where

$$l(t) \longleftrightarrow L(\omega)$$

$$l(-t) \longleftrightarrow L(-\omega)$$

Since $l(t)$ is a periodic signal of period T_0, $L(\omega)$, the Fourier transform of $l(t)$ will consist of impulses located at $\omega = 0, \pm\omega_0, \pm 2\omega_0, \ldots, \pm n\omega_0$, where $\omega_0 = 2\pi/T_0$. Obviously $L(-\omega)$ also consists of impulses located at these frequencies. These impulses have magnitudes equal to 2π times the corresponding coefficients of the exponential Fourier series for $l(-t)$.

Thus, if $l(t)$ is expanded by the Fourier series,

$$l(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T_0}\right)$$

Then the Fourier transform $L(\omega)$ of $l(t)$ is given by

$$L(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0)$$

Since

$$L(-\omega) = L^*(\omega)$$

$$L(-\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n^* \delta(\omega - n\omega_0)$$

Thus, $L(-\omega)$ consists of impulses located at $\omega = 0, \pm\omega_0, \pm 2\omega_0, \ldots$, etc. It is obvious that the transfer function $L(-\omega)$ represents a system which attenuates all of the frequencies except $\omega = 0, \pm\omega_0, \pm 2\omega_0, \ldots, \pm n\omega_0, \ldots$.

These frequency components also go through relative attenuation given by the corresponding coefficient C_r^* for the rth harmonic. The output signal, therefore, consists of a signal with frequency components $\omega = 0, \pm\omega_0, \pm 2\omega_0, \ldots$, etc. It is evident that the operation of cross correlation is equivalent to filtering in the frequency domain which allows to pass through only the frequency components of the fundamental frequency of $s(t)$ and its harmonics. In essence, we are merely filtering out all of the noise signal and extracting the desired periodic signal $s(t)$ by a filter which allows only the frequency components present in $s(t)$ to pass through. The various harmonics of $s(t)$, however, go through a different relative attenuation (proportional to C_r^* for rth harmonic) and hence the output of this filter has the same fundamental frequency (ω_0) as that of $s(t)$ but, in general, its waveform is different from $s(t)$.

If the nature of the filter were such that the relative attenuation of all of the frequency components were uniform, then the output would be an exact replica of $s(t)$.

For this to happen we have to cross correlate $s(t)$ with a uniform impulse train. The perfect filtering is equivalent to cross correlating over an infinite time interval. In practice, it is impossible to design filters which allow only discrete frequency components and attenuate completely all of the other frequencies.

We can conclude that the cross correlation of $y(t)$ [$= s(t) + n(t)$] with the periodic impulse function $\delta_{T_0}(t)$ of the same frequency as that of $s(t)$ yields the actual waveform of the desired periodic component.

MATLAB PROGRAMS

Program 7.1

```
% Finding circular convolution of two discrete-time sequences
clc;clear all;close all;
x1=input('enter sequence1');
x2=input('enter sequence2');
Nx1=length(x1);
Nx2=length(x2);
N=max(Nx1,Nx2)
newx1=[x1 zeros(1,N-Nx1)];
newx2=[x2 zeros(1,N-Nx2)];
disp('circular convolution is')
y=cconv(newx1,newx2,N)
subplot(3,1,1);stem(x1);
title('x1(n)');
subplot(3,1,2);stem(x2);
title('x2(n)');
subplot(3,1,3);stem(y);
title('y(n)');
```

Output:

```
enter sequence1 [1 2 3 4 5]
enter sequence2 [1 2 1 1]
N =
 5
circular convolution is
y =
 18 13 13 13 18
```

UNIT-V

Sampling Theorem: Impulse Sampling (547-548)- Graphical and analytical proof for sampling of Band Limited Signals (541-545), Reconstruction of signal from its samples (552-574), Effect of under sampling – Aliasing (546), Natural and Flat top Sampling (548-552), Discrete time processing of continuous time signals, Introduction to Band Pass Sampling (575-578).

Z–Transforms: Concepts of Z- Transform of a Discrete Sequence (754-783), ROC (755) and it's properties (783-788), Properties of z-transforms (788-814). Inverse z-transform – Power series method, Residue Theorem method, Convolution method and Partial fraction expansion method (814-855).

most part of the energy is retained. This LPF used for band-limiting a signal before sampling, as shown in Figure 8.4, is generally referred to as an *anti-aliasing filter* since it is used primarily for preventing aliasing.

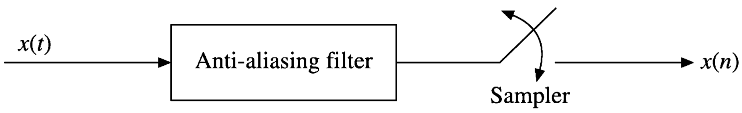


Figure 8.4 Anti-aliasing filter.

8.7 SAMPLING TECHNIQUES

Sampling of a signal is done in several ways. Basically there are three types of sampling techniques:

1. Instantaneous sampling or impulse sampling
2. Natural sampling
3. Flat top sampling

Out of these three methods, instantaneous or impulse sampling is also called ideal sampling, whereas the natural sampling and Flat top sampling are called practical sampling methods. These sampling techniques are discussed in detail in the subsequent sections.

8.7.1 Ideal or Impulse Sampling

Ideally, sampling should be done instantaneously so that the kth element of the sequence obtained by sampling represents the value of $x(t)$ at $t = kT$. The operation is shown in Figure 8.5(a). Assume that the fictitious sampler closes almost for zero time once in every T sec. It is equivalent to transmitting the input signal to the output for a very very short time (almost zero time) once in every T sec. The mechanical switch can be replaced by an electronic switch which is basically a Pulse Amplitude Modulator as shown in Figure 8.5(b).

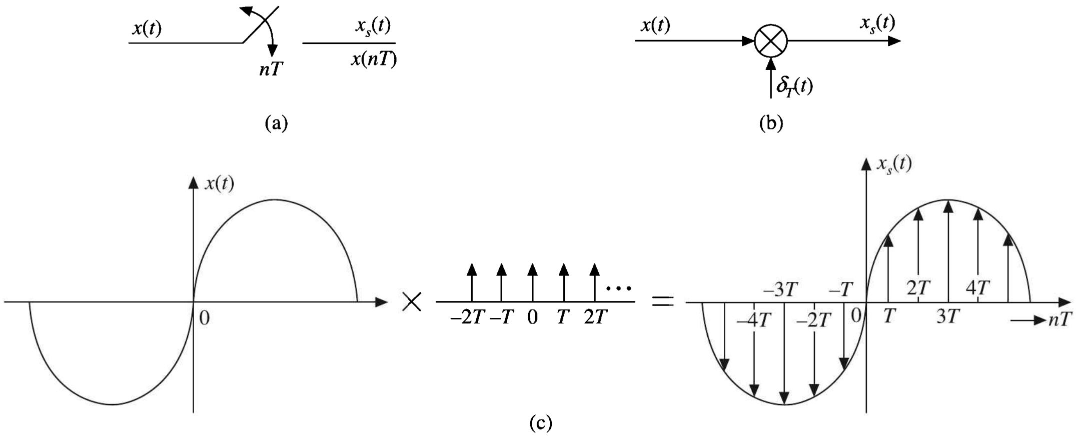


Figure 8.5 Ideal sampling.

Now, the operation is equivalent to multiplying the input signal $x(t)$ by an impulse train $\delta_T(t)$ as shown in Figure 8.5(c). So the output of the sampler is a train of impulses of height equal to the instantaneous value of the input signal at the sampling instant.

The impulse train, also called the sampling function is represented as:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The sampled signal is given by

$$x_s(t) = x(t)\,\delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

$\therefore$

$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

or

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

This equation gives the spectrum of ideally sampled signal. It shows that the spectrum $X_s(\omega)$ is an infinite sum of shifted replicas of $X(\omega)$ spaced $n\omega_s$ apart, where $n = \pm 1, \pm 2$, etc. and scaled by a factor $1/T$. However, it may be noted that ideal or instantaneous sampling is possible only in theory because it is impossible to have a pulse with pulse width approaching zero. Practically, the flat top sampling or natural sampling is used.

8.7.2 Natural Sampling

Natural sampling, also called sampling, using a sequence of pulses is the most practical way of accomplishing sampling of a band-limited signal. This is achieved by multiplying the signal $x(t)$ with a pulse train $p_T(t)$ as shown in Figure 8.6. Each pulse of $p_T(t)$ is of short duration τ and occurs at a sampling period of T sec. The output of the sampler is same as the input during that short duration τ. Hence it is termed as *natural sampling*.

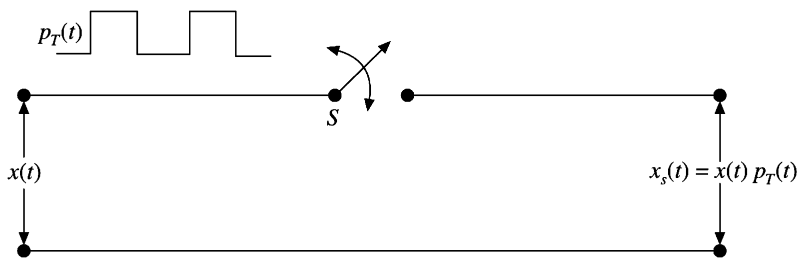
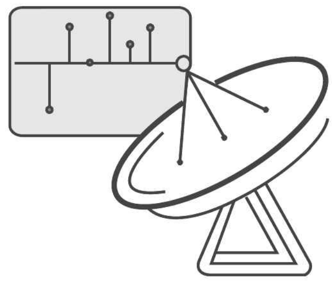


Figure 8.6 Natural sampler.

Figure 8.7 explains the process of natural sampling. Figure 8.7(a) is the signal $x(t)$ to be sampled, and Figure 8.7(b) is its spectrum $X(f)$. Figure 8.7(c) is the pulse train $p_T(t)$, and Figure 8.7(d) is its spectrum $P(f)$. Figure 8.7(e) is the output of the sampler $x_s(t)$, and

8

Sampling



8.1 INTRODUCTION

Earlier we had defined a continuous-time signal as one which is defined for all values of time and a discrete-time signal as one which is defined only over a discrete set of points in time. Most of the signals that we encounter in practice are continuous-time (analog) signals. Analog signal processing, representation, transmission and recovery fall under the category of analog communications which have certain drawbacks. In digital communications, which is more advantageous, it is required to transform an analog signal into a discrete-time signal.

The process of converting a continuous-time signal into a discrete-time signal is called *sampling*. After sampling, the signal is defined at discrete instants of time and the time interval between two successive sampling instants is called *sampling period* or *sampling interval*. In the process of sampling, one of the important factors that we have to consider is—the sampling rate must be kept sufficiently high so that the original signal can be reconstructed from its samples.

8.2 SAMPLING

The sampling operation can be represented by a fictitious switch shown in Figure 8.1. The switch is closed for a very short interval of time τ (ideally, $\tau = 0$), once every T sec during which the signal is available at the output. Therefore, if the input is $x(t)$, then the output $x_s(t)$ is $x(nT)$, $n = 0, \pm 1, \pm 2, \dots$ and $x(nT)$ is called the sampled sequence of $x(t)$, where T is called the sampling period or sampling interval. It is the time interval between successive samples and the sampling frequency is given by $f_s = (1/T)$ Hz. Although a mechanical switch is shown in Figure 8.1, in actual practice, an electronic switch may be used.

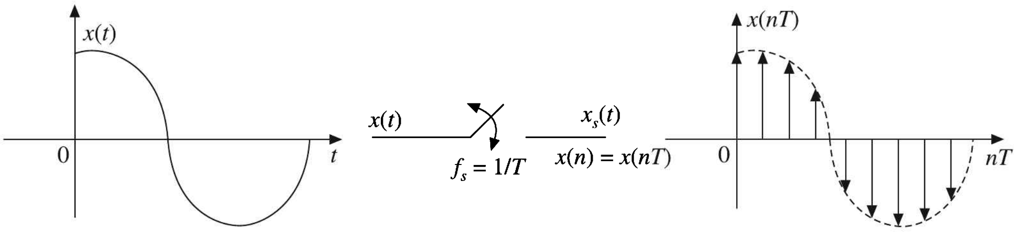


Figure 8.1 Sampling operation.

8.3 SAMPLING THEOREM

The sampling theorem is one of the most useful theorems since it applies to digital communication systems. The sampling theorem states that *A band limited signal $x(t)$ with $X(\omega) = 0$ for $|\omega| \geq \omega_m$ [i.e. $X(f) = 0$ for $f \geq f_m$] can be represented into and uniquely determined from its samples $x(nT)$ if the sampling frequency $f_s \geq 2f_m$, where f_m is the highest frequency component present in it.* That is, for signal recovery, the sampling frequency must be atleast twice the highest frequency present in the signal.

This theorem is known as *uniform sampling theorem* since it pertains to the specification of a given signal by its samples at uniform intervals of $1/2f_m$ sec.

It is also called *low pass sampling theorem* because it applies to low pass signals, i.e. signals for which $X(f) = 0$ for all frequencies such that $|f| \geq f_m$, where f_m is some finite frequency.

Proof: The sampling operation can be represented as shown in Figure 8.2. $x(t)$ is a continuous-time band limited signal to be sampled which has no spectral components above f_m cycles per sec. That means $X(\omega)$, the Fourier transform of $x(t)$ is 0 for $\omega > \omega_m \cdot \delta_T(t)$ is an impulse train which samples at a rate of f_s Hz and $x_s(t)$ is the sampled signal. T is the sampling period and $f_s = (1/T)$ is the sampling frequency.

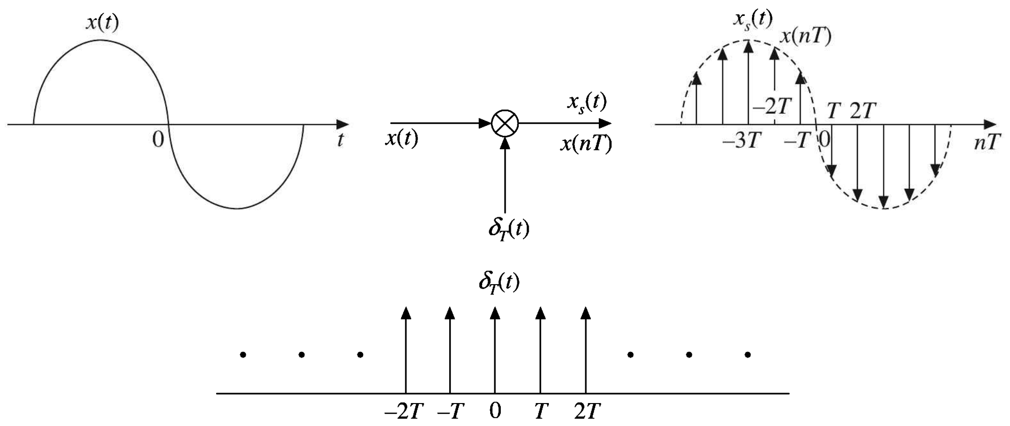


Figure 8.2 Sampling operation.

$x_s(t)$ is the product of signal $x(t)$ and impulse train $\delta_T(t)$. It is a sequence of impulses located at regular intervals of T sec and having strength equal to the values of $x(t)$ at the corresponding instants.

$$\therefore \qquad x_s(t) = x(t)\ \delta_T(t) \text{ where } \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The exponential form of Fourier series of $\delta_T(t)$ is:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t} \text{ where } C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_s t} = \frac{1}{T}$$

$$\therefore \qquad \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_s t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$x_s(t) = x(t)\ \delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_s t}$$

Taking Fourier transform on both sides, we have

$$\text{F}[x_s(t)] = \text{F}\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_s t}\right] = \frac{1}{T}\sum_{n=-\infty}^{\infty} \text{F}[x(t) e^{jn\omega_s t}]$$

i.e.

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}n\right)$$

or

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

where $X(\omega)$ or $X(f)$ is the spectrum of input signal and $X_s(\omega)$ or $X_s(f)$ is the spectrum of the sampled signal.

Thus, the Fourier transform of the sampled signal is given by an infinite sum of shifted replicas of the Fourier transform of the original signal.

The signal $x(t)$ is band limited to f_m. The term $X[\omega - (2\pi/T)n]$ is the shifting of $X(\omega)$ from $\omega = 0$ to $\omega = (2\pi/T)n$. Hence $X_s(\omega)$ is the sum of shifted replicas of $(1/T)\ X(\omega)$ centering at $(2\pi/T)\,n$, $n = 0, \pm1, \pm2, \ldots$. Figure 8.3 shows the plot of $X(\omega)$ and $X_s(\omega)$ for various values of π/T. It shows that [Figure 8.3(b) and (c)] if $(\pi/T) \geq \omega_m$, the replicas will not overlap and as a result, the frequency spectrum of $TX_s(\omega)$ in the frequency range $[-(\pi/T), \pi/T]$ is identical to $X(\omega)$. $X(\omega)$ can be recovered from $X_s(\omega)$ by passing it through a low pass filter which has sharp cutoff at $\omega = \pi/T$ (or with bandwidth B, where $\omega_m \leq B \leq \omega_s - \omega_m$). If $(\pi/T) < \omega_m$ [Figure 8.3(d)], the successive frequency spectra will overlap and the original signal cannot be recovered from the sampled signal. Therefore, we can say that for signal recovery,

$$\omega_s - \omega_m \geq \omega_m, \text{ i.e. } \omega_s \geq 2\omega_m$$

or $$f_s - f_m \geq f_m, \text{ i.e. } f_s \geq 2f_m$$

or $$\frac{\pi}{T} \geq \omega_m, \text{ i.e. } \pi f_s \geq 2\pi f_m, \text{ i.e. } f_s \geq 2f_m$$

or $$\frac{1}{f_s} \leq \frac{1}{2f_m}, \text{ i.e. } T \leq \frac{1}{2f_m}$$

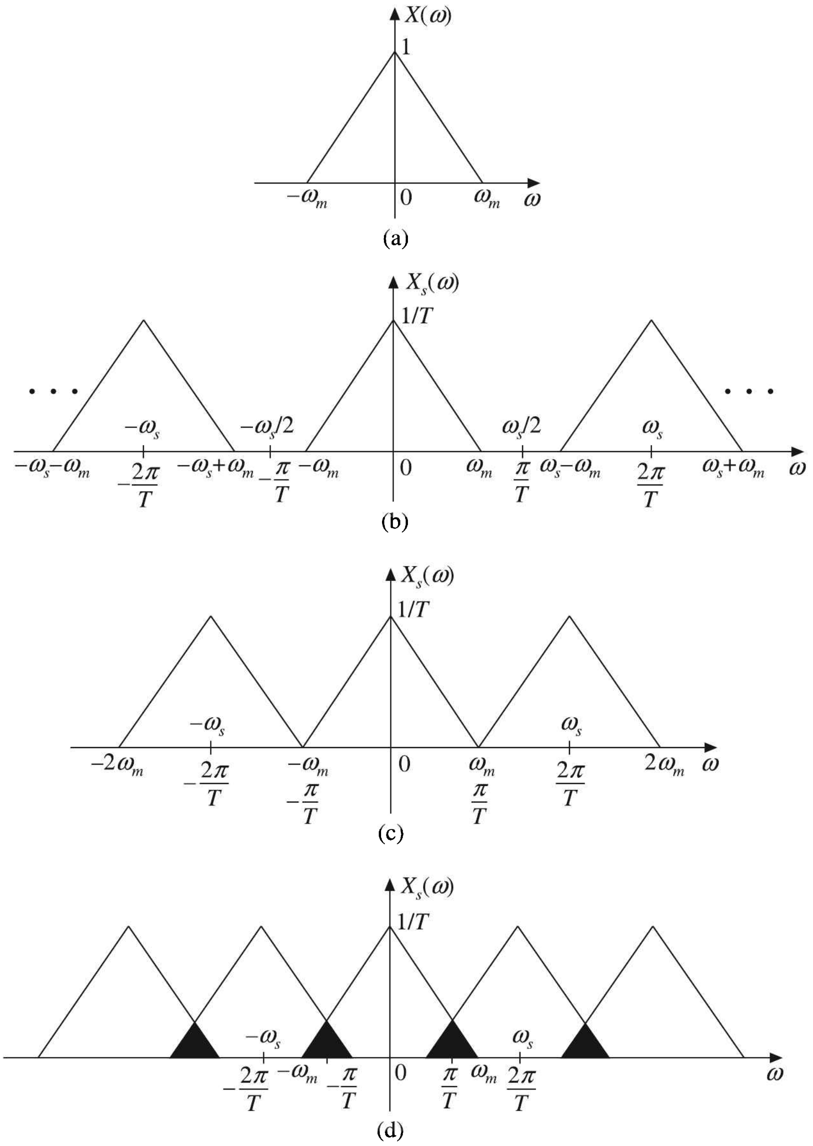


Figure 8.3 (a) Frequency spectrum of continuous-time signal $x(t)$, (b), (c) and (d) Frequency spectrum of sampled signal $x_s(t)$ for $(\pi/T) > \omega_m$, $(\pi/T) = \omega_m$ and $(\pi/T) < \omega_m$ respectively.

So we can conclude that if the sampling interval T is small $[<(1/2f_m)]$, $X(\omega)$ can be recovered from $X_s(\omega)$, but if T becomes larger than $1/2f_m$, then there is an overlap between successive cycles and $X(\omega)$ cannot be recovered from $X_s(\omega)$. This proves the sampling theorem.

From the previous discussion, we can observe that when the spectra overlap, it is impossible to retrieve $x(t)$ from $x_s(t)$.

Thus, we find that in general, there are two basic conditions to be satisfied if $x(t)$ is to be recovered from its samples.

1. $x(t)$ should be band-limited to some frequency ω_m.
2. The sampling frequency ω_s should be atleast twice the band-limiting frequency ω_m [i.e. $\omega_s \geq 2\omega_m$].

From Figure 8.3, we can observe that:

1. $X_s(\omega)$ is a repetitive version of $X(\omega)$ with $X(\omega)$ repeating itself at regular intervals of ω_s, the sampling frequency.
2. When $\omega_s > 2\omega_m$ [Figure 8.3(b)], the spectral replicates have a larger separation between them, known as *guard band*, which makes the process of filtering much easier and effective. Even a non-ideal filter which does not have a sharp cutoff can also be used.
3. When $\omega_s = 2\omega_m$ [Figure 8.3(c)], there is no separation between the replicates, so no *guard band* exists, and $X(\omega)$ can be obtained from $X_s(\omega)$ by using only an ideal low pass filter (LPF) with sharp cutoff.
4. When $\omega_s < 2\omega_m$ [Figure 8.3(d)], the low frequency components in $X_s(\omega)$ overlap on the high frequency components of $X(\omega)$, there is distortion and $X(\omega)$ cannot be recovered from $X_s(\omega)$ by using any filter. This type of distortion is called *aliasing*. Aliasing can be avoided if $f_s \geq 2f_m$ or $T \leq (1/2f_m)$.

Since it is impossible to build filters having an infinite sharpness of cutoff, a guard band between f_m and $f_s - f_m$ is preferred.

The impulse train at the output of the sampler is processed through an ideal LPF with gain T and cutoff frequency greater than ω_m and less than $\omega_s - \omega_m$. The resulting output signal will exactly equal $x(t)$.

8.4 NYQUIST RATE OF SAMPLING

Nyquist rate of sampling is the theoretical minimum sampling rate at which a signal can be sampled and still be reconstructed from its samples without any distortion. It is 'the theoretical minimum' because when the Nyquist rate of sampling is used, only an ideal LPF can be used to extract $X(\omega)$ from $X_s(\omega)$, i.e. to recover $x(t)$ from $x_s(t)$. It is always equal to $2f_m$ where f_m is the maximum frequency component present in the signal.

A signal sampled at greater than Nyquist rate is said to be *over sampled* and a signal sampled at less than its Nyquist rate is said to be *under sampled.*

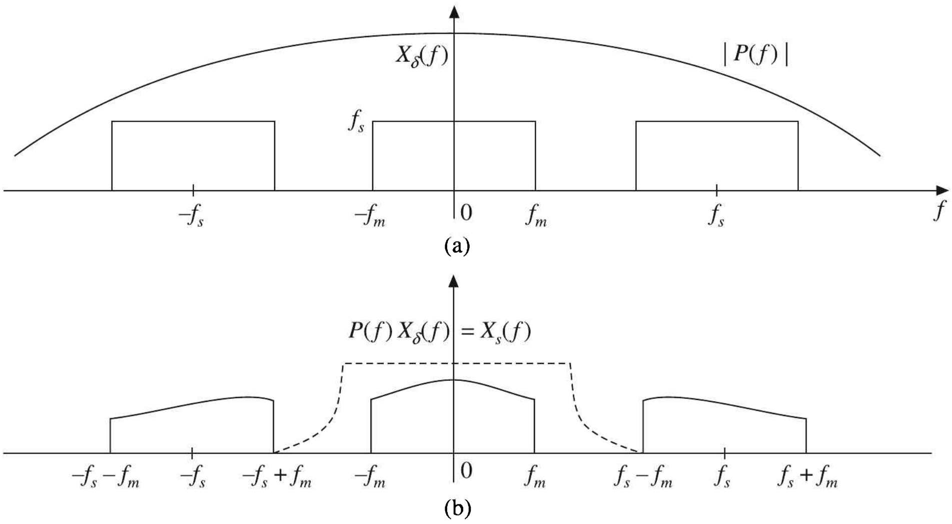


Figure 8.10 (a) Plot of $P(f)$ and $X_\delta(f)$, (b) Plot of $X_s(f) = P(f)\,X_\delta(f)$.

Observe that the magnitudes of the high frequency components in $X_s(f)$ are relatively reduced as compared to the magnitudes of the low frequency components because of the multiplication of $X_\delta(f)$ by $P(f)$. So we can only get a distorted version of $x(t)$, but not exact $x(t)$, by passing $x_s(t)$ through an LPF.

This distortion, wherein the amplitudes of the high frequency components are reduced relative to the amplitudes of the low frequency components, in the reconstructed signal $x(t)$ obtained from the flat top sampled version of the signal is referred to as the *aperture effect.* This aperture effect can be reduced by using an equalizer with transfer function $H_e(f)$ in cascade with the reconstruction filter and adjusting $H_e(f)$ so that

$$H_e(f) = \frac{1}{P(f)}; \quad |f| \le f_m$$

8.8 DATA RECONSTRUCTION

The process of obtaining the analog signal $x(t)$ from the sampled signal $x_s(t)$ is called data reconstruction or interpolation. We know that

$$x_s(t) = x(t)\,\delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

or

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

since $\delta(t - nT)$ is zero except at the sampling instants $t = nT$. The reconstruction filter, which is assumed to be linear and time invariant, has unit impulse response $h(t)$. The reconstruction filter output, $y(t)$ is given by the convolution,

$$y(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\, \delta(\lambda - nT)\, h(t-\lambda)\, d\lambda$$

or, upon changing the order of summation and integration,

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(\lambda - nT)\, h(t-\lambda)\, d\lambda$$

i.e.

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT)$$

8.8.1 Ideal Reconstruction Filter

If $x(t)$ is sampled at a frequency exceeding the Nyquist rate and if the sampled signal $x_s(t)$ is passed through an ideal LPF, with bandwidth greater than f_m but less than $f_s - f_m$ and a pass band amplitude response of T, the filter output is $x(t)$. We choose the bandwidth of the ideal reconstruction filter to be $0.5f_s$. The transfer function of this ideal reconstruction filter is, therefore,

$$H(f) = \begin{cases} T, & |f| < 0.5 f_s \\ 0, & \text{otherwise} \end{cases}$$

as shown in Figure 8.11.

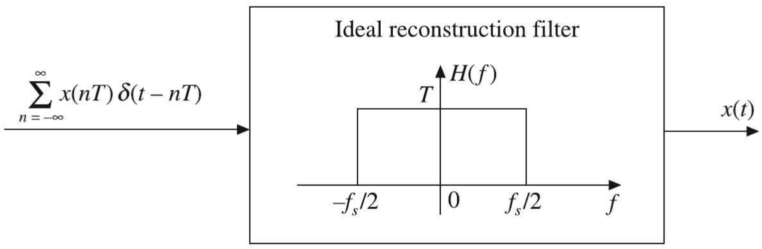


Figure 8.11 Reconstruction filtering.

The impulse response of the ideal reconstruction filter is given by

$$h(t) = \int_{-f_s/2}^{f_s/2} T e^{j2\pi f t}\, df$$

which is

$$h(t) = T\left[\frac{e^{j2\pi f t}}{j2\pi t}\right]_{-f_s/2}^{f_s/2} = \frac{T}{j2\pi t}\left[e^{j\pi f_s t} - e^{-j\pi f_s t}\right]$$

$$= \frac{1}{\pi f_s t}\left(\frac{e^{j\pi f_s t} - e^{-j\pi f_s t}}{2j}\right) = \frac{\sin \pi f_s t}{\pi f_s t}$$

or
$$h(t) = \text{sinc } f_s t$$

Substituting this value of $h(t)$ in the expression for output $y(t)$, we get

$$y(t) = x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc } f_s(t - nT)$$

A more convenient form for this expression, which is often referred to as an interpolation formula is:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}\left(\frac{t}{T} - n\right)$$

This shows that the original data signal can be reconstructed by weighing each sample by a sinc function centred at the sample time and summing.

The reconstruction filter discussed above is non-causal and the impulse response is not limited. So it cannot be used for real time applications.

In practice, several other methods are used to reconstruct the signal. Some of the important ones among them are:

- Zero order hold
- First order hold
- Linear interpolator

Most commonly the signal is reconstructed using zero order hold.

8.8.2 Zero Order Hold

One of the most widely used interpolator is the zero order hold (ZOH). The ZOH reconstructs the continuous-time signal from its samples by holding the given sample for an interval until the next sample is received as shown in Figure 8.12. So the ZOH generates step approximations.

Mathematically,

$$\tilde{x}_a(t) = x(n) \quad \text{for } nT \le n \le (n+1)T$$

In particular,

$$\begin{aligned} \tilde{x}_a(t) &= x(0) && \text{for } 0 \le t \le T \\ &= x(T) && \text{for } T \le t \le 2T \\ &= x(2T) && \text{for } 2T \le T \le 3T \\ &\ \ \vdots \end{aligned}$$

The impulse response of a zero order hold is given by

$$\begin{aligned} h(t) &= 1 && 0 \le t \le T \\ &= 0 && \text{otherwise} \end{aligned}$$

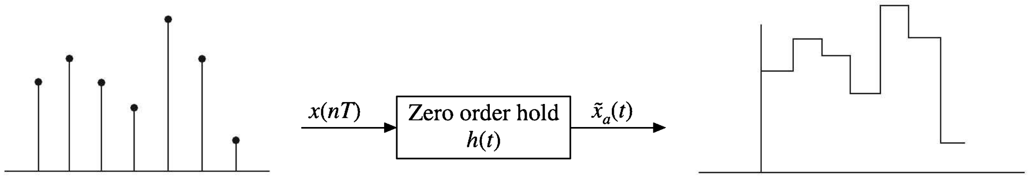


Figure 8.12 Zero order hold.

8.8.3 Transfer Function of a Zero Order Hold

The output $\tilde{x}_a(t)$ of a zero order hold is the convolution of its input $x(nT)$ and its impulse response $h(t)$, i.e.

$$\tilde{x}_a(t) = x(nT) * h(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT)$$

For zero order hold,

$$h(t) = u(t) - u(t - T)$$

$$\therefore \qquad h(t - nT) = u(t - nT) - u[t - (n+1)T]$$

$$\therefore \qquad \tilde{x}_a(t) = \sum_{n=-\infty}^{\infty} x(nT)\,[u(t - nT) - u[t - (n+1)T]$$

Taking Laplace transform on both sides, we have

$$\text{L}\,[\tilde{x}_a(t)] = \tilde{X}_a(s) = \text{L}\left[\sum_{n=-\infty}^{\infty} x(nT)\,\{u(t - nT) - u(t - (n+1)T\}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\left(\frac{e^{-nTs}}{s} - \frac{e^{-(n+1)Ts}}{s}\right)$$

$$= \frac{1 - e^{-Ts}}{s} \sum_{n=-\infty}^{\infty} x(nT)\, e^{-nTs} = \left(\frac{1 - e^{-Ts}}{s}\right) X^*(s)$$

$$\therefore \qquad \text{Transfer function of zero order hold} = \frac{\tilde{X}_a(s)}{X^*(s)} = \frac{1 - e^{-Ts}}{s}$$

Since the output of the ZOH consists of steps, it consists of higher order harmonics. To remove these harmonics, the output of ZOH is applied to an LPF. This filter tends to smooth the corners on the step approximations generated by the ZOH. Hence this filter is often called a *smoothing filter*.

EXAMPLE 8.1 Determine the Nyquist rate corresponding to each of the following signals:

(a) $x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$ (b) $x(t) = \dfrac{\sin(4000\pi t)}{\pi t}$

(c) $x(t) = \left[\dfrac{\sin(4000\pi t)}{\pi t}\right]^2$

Solution:

(a) Given

$$x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$$

Highest frequency component in '1' is zero
Highest frequency component in $\cos 2000\pi t = \cos \omega_{m1} t$ is $\omega_{m1} = 2000\pi$.
Highest frequency component in $\sin 4000\pi t = \sin \omega_{m2} t$ is $\omega_{m2} = 4000\pi$.
So the maximum frequency component in $x(t)$ is $\omega_m = 4000\pi$ [highest of 0, 2000π, 4000π).

$\therefore$ $$2\pi f_m = 4000\pi$$

or $$f_m = \frac{4000\pi}{2\pi} = 2000 \text{ Hz}$$

$\therefore$ $$\text{Nyquist rate } f_N = 2f_m = 2 \times 2000 = 4000 \text{ Hz}$$

and $$\text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m} = \frac{1}{4000} \text{ sec} = 0.25 \text{ ms}$$

Alternative method

Given $$x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$$

Taking Fourier transform on both sides, we get

$$X(\omega) = 2\pi\delta(\omega) + \pi[\delta(\omega + 2000\pi) + \delta(\omega - 2000\pi)] + j\pi[\delta(\omega + 4000\pi) - \delta(\omega - 4000\pi)]$$

The frequency spectrum is shown in Figure 8.13.

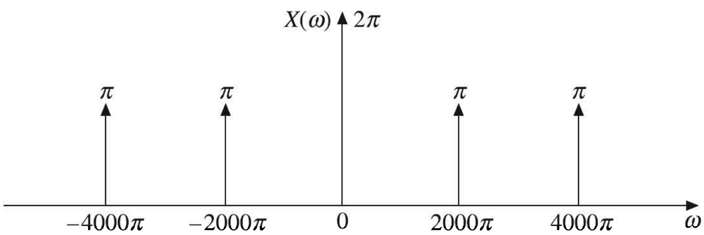


Figure 8.13 Spectrum of Example 8.1(a).

The highest frequency component in Figure 8.13 is $\omega_m = 4000\pi$.

$$\therefore \qquad f_m = \frac{\omega_m}{2\pi}$$

$$= \frac{4000\pi}{2\pi} = 2000 \text{ Hz}$$

$$\therefore \qquad \text{Nyquist rate } f_N = 2f_m$$

$$= 2 \times 2000 = 4000 \text{ Hz}$$

and

$$\text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m}$$

$$= \frac{1}{4000} = 0.25 \text{ ms}$$

(b) Given $$x(t) = \frac{\sin(4000\pi t)}{\pi t}$$

The highest frequency component in sin $4000\pi t$ = sin $\omega_m t$ is $\omega_m = 4000\pi$. So the maximum frequency component in $x(t) = 4000\pi$.

$$\therefore \qquad 2\pi f_m = 4000\pi$$

or

$$f_m = \frac{4000\pi}{2\pi} = 2000 \text{ Hz}$$

$$\therefore \qquad \text{Nyquist rate } f_N = 2f_m$$

$$= 2 \times 2000 = 4000 \text{ Hz}$$

$$\therefore \qquad \text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m}$$

$$= \frac{1}{4000} = 0.25 \text{ ms}$$

Alternative method

We know that the Fourier transform of a rectangular pulse is a sinc function as represented in Figure 8.14(a). Using duality property, the Fourier transform of a sinc function is a rectangular pulse as shown in Figure 8.14(b).

Given $$x(t) = \frac{\sin 4000\pi t}{\pi t} = 4000\left(\frac{\sin 4000\pi t}{4000\pi t}\right) = 4000 \text{ sinc}(4000\pi t)$$

It is a sinc function of the form τ sinc$[t(\tau/2)]$.

$$\therefore \qquad \frac{t\tau}{2} = 4000\pi t$$

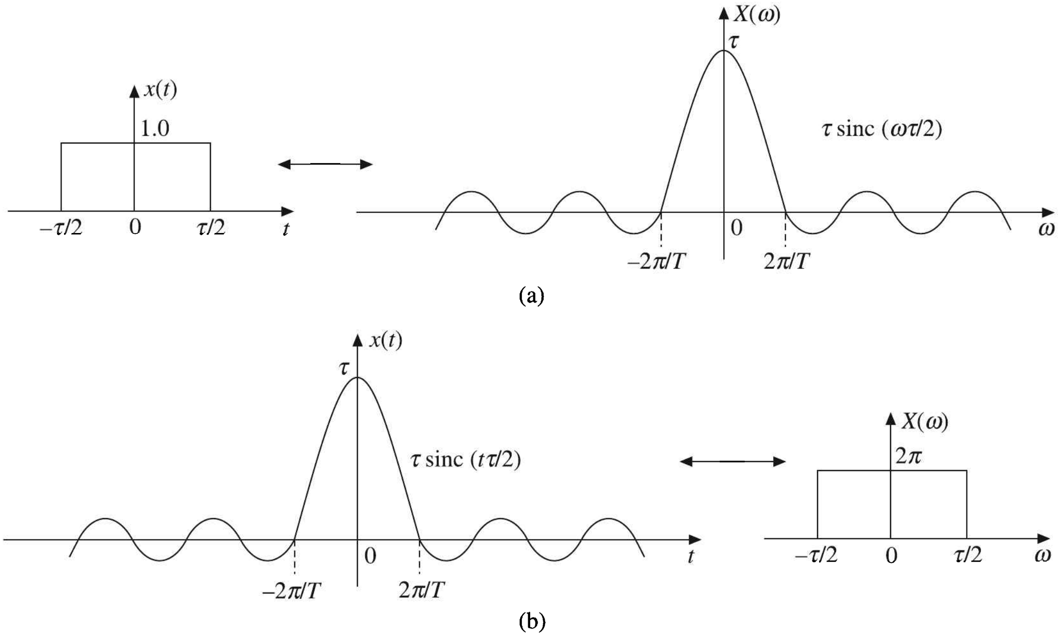


Figure 8.14 Fourier transform pairs of (a) rectangular function, (b) sinc function.

or $$\frac{\tau}{2} = 4000\pi$$

or $$\omega_m = \frac{\tau}{2} = 4000\pi$$

or $$f_m = \frac{\omega_m}{2\pi} = \frac{4000\pi}{2\pi} = 2000 \text{ Hz}$$

$\therefore$ $$\text{Nyquist rate } f_N = 2f_m = 2 \times 2000 = 4000 \text{ Hz}$$

and $$\text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m} = \frac{1}{4000} = 0.25 \text{ ms}$$

(c) Given $$x(t) = \left(\frac{\sin(4000\pi t)}{\pi t}\right)^2 = \left(\frac{\sin(4000\pi t)}{\pi t}\right)\left(\frac{\sin 4000\pi t}{\pi t}\right)$$

Highest frequency component in $\dfrac{\sin(4000\pi t)}{\pi t}$ is $\omega_m = 4000\pi$.

$\therefore$ Highest frequency component in

$$\left[\frac{\sin(4000\pi t)}{\pi t}\right]\left[\frac{\sin(4000\pi t)}{\pi t}\right]=\left[\frac{1-\cos 8000\pi t}{(\pi t)^2}\right] \text{ is:}$$

$\omega_m = 4000\pi + 4000\pi = 8000\pi$ [sum of the highest frequency components]

or $$f_m = \frac{\omega_m}{2\pi} = \frac{8000\pi}{2\pi} = 4000 \text{ Hz}$$

$\therefore$ $$\text{Nyquist rate } f_N = 2f_m = 2 \times 4000 = 8000 \text{ Hz}$$

and $$\text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m} = \frac{1}{8000} = 0.125 \text{ ms}$$

Alternative method

We know that the Fourier transform of a triangular pulse is a sinc square function as represented in Figure 8.15(a). Using duality property, the Fourier transform of a sinc square function is a triangular pulse as shown in Figure 8.15(b).

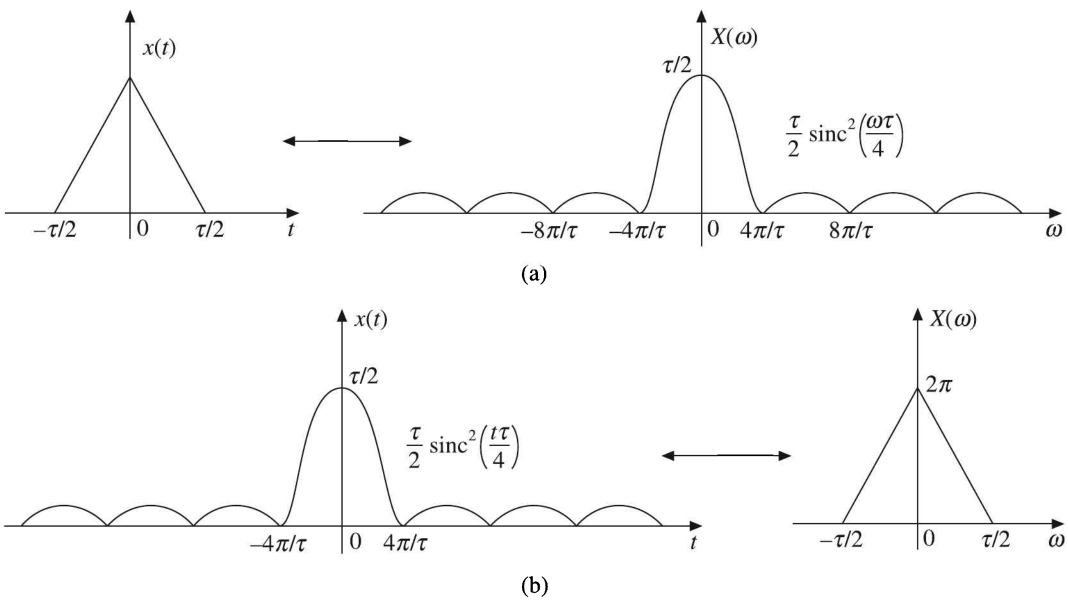


Figure 8.15 Fourier transform pairs of (a) triangular function, (b) sinc square function.

Given $$x(t) = \left(\frac{\sin 4000\pi t}{\pi t}\right)^2 = (4000)^2\left(\frac{\sin 4000\pi t}{4000\pi t}\right)^2$$

$$= (4000)^2 \text{ sinc}^2(4000\pi t)$$

It is a sinc square function of the form $\frac{\tau}{2}\text{sinc}^2\left(\frac{t\tau}{4}\right)$.

$\therefore$ $$\frac{t\tau}{4} = 4000\pi t$$

or $$\omega_m = \frac{\tau}{2} = 8000\pi$$

or $$f_m = \frac{\omega_m}{2\pi} = \frac{8000\pi}{2\pi} = 4000 \text{ Hz}$$

$\therefore$ $$\text{Nyquist rate } f_N = 2f_m$$

$$= 2 \times 4000 = 8000 \text{ Hz}$$

and $$\text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m}$$

$$= \frac{1}{8000} = 0.125 \text{ ms.}$$

EXAMPLE 8.2 Find the Nyquist rate and the Nyquist interval for the following signals:

(a) rect (300*t*) (b) –10 sin $40\pi t$ cos $300\pi t$

Solution:

(a) Given $$x(t) = \text{rect }(300t) = \Pi\left(\frac{t}{\tau}\right)$$

$\therefore$ $$\frac{t}{\tau} = 300t \quad \text{or} \quad \frac{1}{\tau} = 300 \quad \text{or} \quad \tau = \frac{1}{300}$$

We know that

$$\text{F}\left[\Pi\left(\frac{t}{\tau}\right)\right] = \tau \text{ sinc}\left(\frac{\omega\tau}{2}\right)$$

$\therefore$ $$\text{F}[x(t)] = X(\omega) = \tau \text{ sinc}\left(\frac{\omega\tau}{2}\right) = \left(\frac{1}{300}\right)\text{sinc}\left(\frac{2\pi f}{2}\frac{1}{300}\right)$$

$$= \frac{1}{300}\text{sinc}\left(\frac{\pi f}{300}\right)$$

The Fourier transform of a rectangular pulse is a sinc function and it stays upto infinity. Hence, the Nyquist rate is also infinite and the Nyquist interval is zero.

(b) Given $x(t) = -10 \sin(40\pi t) \cos(300\pi t)$

Let $x(t) = (-10)\, x_1(t)\, x_2(t)$

$x_1(t) = \sin(40\pi t)$ has highest frequency $\omega_{m1} = 40\pi$

and $x_2(t) = \cos(300\pi t)$ has highest frequency $\omega_{m2} = 300\pi$

Now, the product of two time domain functions results in the convolution of their spectra.

So if the individual highest frequency components are ω_{m1} and ω_{m2}, then the convoluted spectra will have highest frequency component $\omega_m = \omega_{m1} + \omega_{m2} = 40\pi + 300\pi = 340\pi$.

$$\therefore \quad f_m = \frac{\omega_m}{2\pi} = \frac{340\pi}{2\pi} = 170 \text{ Hz}$$

$$\therefore \quad \text{Nyquist rate } f_N = 2f_m$$

$$= 2 \times 170 = 340 \text{ Hz}$$

$$\text{and} \quad \text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m}$$

$$= \frac{1}{340} \text{ sec}$$

Alternative method

Given $x(t) = -10 \sin(40\pi t) \cos(300\pi t)$

$$= -5\,[2 \sin(40\pi t) \cos(300\pi t)]$$

$$= -5\,[\sin(300\pi t + 40\pi t) + \sin(40\pi t - 300\pi t)]$$

$$= -5 \sin 340\pi t + 5 \sin 260\pi t$$

The highest frequency component in $\sin 340\pi t = \sin \omega_{m1} t$ is $\omega_{m1} = 340\pi$.
The highest frequency component in $\sin 260\pi t = \sin \omega_{m2} t$ is $\omega_{m2} = 260\pi$.
Therefore, the highest frequency component in $x(t)$ is:

$$\omega_m = 340\pi \qquad [\text{higher of } \omega_{m1} = 340\pi \text{ and } \omega_{m2} = 260\pi]$$

$$\therefore \quad f_m = \frac{\omega_m}{2\pi} = \frac{340\pi}{2\pi} = 170 \text{ Hz}$$

$$\therefore \quad \text{Nyquist rate } f_N = 2f_m$$

$$= 2 \times 170 = 340 \text{ Hz}$$

$$\text{and} \quad \text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m}$$

$$= \frac{1}{340} \text{ sec}$$

EXAMPLE 8.3 Determine the Nyquist sampling rate and Nyquist sampling interval for

(a) $x(t) = 2\,\text{sinc}(100\pi t)$ (b) $x(t) = \dfrac{1}{2}\text{sinc}(100\pi t) + \dfrac{1}{3}\text{sinc}(50\pi t)$

(c) $x(t) = \text{sinc}(100\pi t) + 3\,\text{sinc}^2(60\pi t)$ (d) $x(t) = \text{sinc}(80\pi t)\,\text{sinc}(120\pi t)$

Solution:

(a) Given $$x(t) = 2\,\text{sinc}(100\pi t) = 2\frac{\sin(100\pi t)}{100\pi t} = 2\frac{\sin(\omega_m t)}{\omega_m t}$$

$\therefore$ $$\omega_m = 100\pi$$

or $$f_m = \frac{\omega_m}{2\pi} = \frac{100\pi}{2\pi} = 50 \text{ Hz}$$

$\therefore$ Nyquist sampling rate $f_N = 2f_m$

$$= 2 \times 50 = 100 \text{ Hz}$$

and Nyquist sampling interval $= \dfrac{1}{f_N} = \dfrac{1}{2f_m}$

$$= \frac{1}{100} = 10 \text{ ms}$$

(b) Given $$x(t) = \frac{1}{2}\text{sinc}(100\pi t) + \frac{1}{3}\text{sinc}(50\pi t)$$

i.e. $$x(t) = \frac{1}{2}\frac{\sin(100\pi t)}{100\pi t} + \frac{1}{3}\frac{\sin(50\pi t)}{50\pi t}$$

$$= \frac{1}{2}\frac{\sin(\omega_{m1} t)}{\omega_{m1} t} + \frac{1}{3}\frac{\sin(\omega_{m2} t)}{\omega_{m2} t}$$

$\therefore$ $$\omega_{m1} = 100\pi \quad \text{and} \quad \omega_{m2} = 50\pi$$

$$\omega_m = \omega_{m1} = 100\pi \qquad [\text{larger of } \omega_{m1} \text{ and } \omega_{m2}]$$

Highest frequency component $f_m = \dfrac{\omega_m}{2\pi} = \dfrac{100\pi}{2\pi} = 50$ Hz

$\therefore$ Nyquist sampling rate $f_N = 2f_m$

$$= 2 \times 50 = 100 \text{ Hz}$$

and Nyquist sampling interval $= \dfrac{1}{f_N} = \dfrac{1}{2f_m}$

$$= \frac{1}{100} = 10 \text{ ms}$$

(c) Given $x(t) = \text{sinc}(100\pi t) + 3\,\text{sinc}^2(60\pi t) = x_1(t) + x_2(t)$

$$x_1(t) = \text{sinc}\,(100\pi t) = \frac{\sin(100\pi t)}{100\pi t} = \frac{\sin\,\omega_{m1}t}{\omega_{m1}t}$$

$\therefore$ $\omega_{m1} = 100\pi$ rad/sec

$$x_2(t) = 3\,\sin c^2(60\pi t) = 3\left[\frac{\sin(60\pi t)}{60\pi t}\right]^2 = 3\left[\frac{\sin(60\pi t)}{60\pi t}\right]\left[\frac{\sin(60\pi t)}{60\pi t}\right]$$

$\therefore$ $\omega_{m2} = 60\pi + 60\pi = 120\pi$ rad/sec

$\therefore$ $\omega_m = \omega_{m2} = 120\pi$ [The larger of ω_{m1} and ω_{m2}]

$\therefore$ $$f_m = \frac{\omega_m}{2\pi} = \frac{120\pi}{2\pi} = 60 \text{ Hz}$$

$\therefore$ Nyquist sampling rate $f_N = 2f_m$

$= 2 \times 60 = 120$ Hz

and Nyquist interval $= \dfrac{1}{f_N} = \dfrac{1}{2f_m}$

$= \dfrac{1}{120}$ sec

(d) Given $x(t) = \text{sinc}(80\pi t)\,\text{sinc}(120\pi t) = x_1(t)\,x_2(t)$

$$x_1(t) = \text{sinc}(80\pi t) = \frac{\sin(80\pi t)}{80\pi t} = \frac{\sin\,\omega_{m1}t}{\omega_{m1}t}$$

$\therefore$ $\omega_{m1} = 80\pi$ rad/sec

$$x_2(t) = \text{sinc}(120\pi t) = \frac{\sin(120\pi t)}{120\pi t} = \frac{\sin\,\omega_{m2}t}{\omega_{m2}t}$$

$\therefore$ $\omega_{m2} = 120\pi$ rad/sec

$\therefore$ $\omega_m = \omega_{m1} + \omega_{m2} = 80\pi + 120\pi = 200\pi$ rad/sec

$\therefore$ $$f_m = \frac{\omega_m}{2\pi} = \frac{200\pi}{2\pi} = 100 \text{ Hz}$$

$\therefore$ Nyquist sampling rate $f_N = 2f_m = 2 \times 100 = 200$ Hz

and Nyquist interval $= \dfrac{1}{f_N} = \dfrac{1}{2f_m}$

$= \dfrac{1}{200} = 5$ ms

EXAMPLE 8.4 Consider the signal $x(t) = (\sin 50\pi t/\pi t)^2$ which is to be sampled with a sampling frequency of $\omega_s = 150\pi$ to obtain a signal $g(t)$ with Fourier transform $G(\omega)$. Determine the maximum value of ω_0 for which it is guaranteed that $G(\omega) = 75\ X(\omega)$ for $|\omega| \le \omega_0$, where $X(\omega)$ is the Fourier transform of $x(t)$.

Solution: Given $x(t) = \left(\dfrac{\sin 50\pi t}{\pi t}\right)^2 = \left(\dfrac{\sin 50\pi t}{\pi t}\right)\left(\dfrac{\sin 50\pi t}{\pi t}\right) = \left(\dfrac{\sin \omega_{m1} t}{\pi t}\right)^2$

$\therefore$ $$\omega_{m1} = 50\pi$$

and $$\omega_m = 50\pi + 50\pi = 100\pi \text{ rad/sec}$$

or $$f_m = \frac{\omega_m}{2\pi} = \frac{100\pi}{2\pi} = 50 \text{ Hz}$$

Given $$\omega_s = 150\pi \text{ rad/sec}$$

$\therefore$ $$f_s = \frac{150\pi}{2\pi} = 75 \text{ Hz}$$

$\therefore$ $$\omega_s < 2\omega_m, \text{ i.e. } f_s < 2f_m$$

So the sampling theorem is not satisfied and the original signal cannot be recovered from its samples. The resultant signal will be distorted due to aliasing effect.

$G(\omega) = 75\ X(\omega)$ is satisfied for lesser values of ω (i.e. for $\omega < 100\pi$).

Thus, the maximum value of ω_0 will be obtained at which the signal $G(\omega)$ is perfectly extracted from $X(\omega)$.

where $$G(\omega) = 75X(\omega) = \frac{1}{T_s} X(\omega)$$

$$f_s = \frac{1}{T_s} = 75 \text{ Hz}$$

So at $\omega_0 = 75\pi$, the bandwidth of the signal $X(\omega)$ is 75π. To recover $x(t)$, the sampling frequency must be $f_s = 75$ Hz. Thus, the signal $x(t)$ is perfectly recovered if $\omega_0 = 75\pi$ rad/sec.

EXAMPLE 8.5 A signal $x(t) = 2 \cos 400\pi t + 6 \cos 640\pi t$ is ideally sampled at $f_s = 500$ Hz. If the sampled signal is passed through an ideal low pass filter with a cutoff frequency of 400 Hz, what frequency components will appear in the output? Sketch the output spectrum. Also find the output signal.

Solution: Given

$$x(t) = 2 \cos 400\pi t + 6 \cos 640\pi t$$

i.e. $$x(t) = 2 \cos [2\pi(200)t] + 6 \cos [2\pi(320)t]$$

Taking Fourier transform on both sides, we get

$$X(f) = [\delta(f + 200) + \delta(f - 200)] + 3[\delta(f + 320) + \delta(f - 320)]$$

The spectrum of the given signal is shown in Figure 8.16.

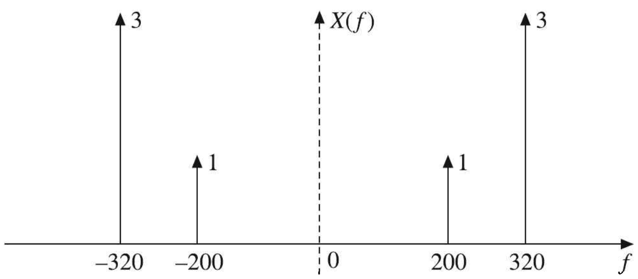


Figure 8.16 Spectrum of $x(t)$.

The spectrum of the sampled signal is:

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

as shown in Figure 8.17.

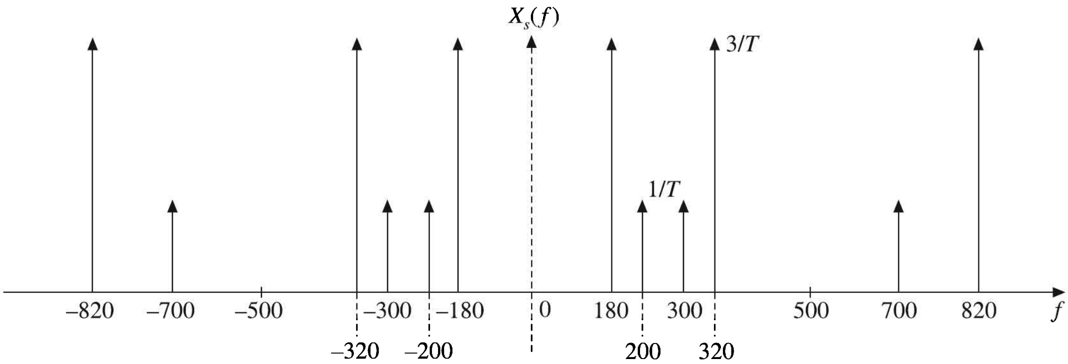


Figure 8.17 Spectrum of $x_s(t)$.

It is a periodic repetition of $X(f)$ at regular intervals of $\pm 500n$.

If the sampled signal is passed through an ideal low pass filter with a cutoff frequency of 400 Hz [shown in Figure 8.18(a)], the frequency spectrum of the output signal will be as shown in Figure 8.18(b).

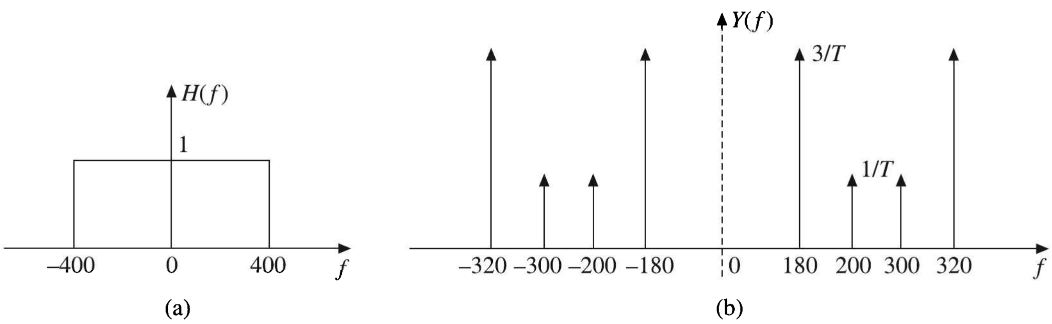


Figure 8.18 (a) Spectrum of ideal LPF, (b) Spectrum of $y(t)$.

So the frequency components that will appear in the output are:

$$-320 \text{ Hz}, -300 \text{ Hz}, -200 \text{ Hz}, -180 \text{ Hz}, 180 \text{ Hz}, 200 \text{ Hz}, 300 \text{ Hz}, 320 \text{ Hz}$$

The spectrum of the output signal is:

$$Y(f) = \frac{1}{T}[\delta(f+200) + \delta(f-200)] + [\delta(f+300) + \delta(f-300)]$$
$$+ 3[\delta(f+180) + \delta(f-180)] + 3[\delta(f+320) + \delta(f-320)]$$

Taking inverse Fourier transform on both sides, the output signal is:

$$y(t) = \frac{1}{T}\{2\cos[2\pi(200)t] + 2\cos[2\pi(300)t] + 6\cos[2\pi(180)t] + 6\cos[2\pi(320)t]\}$$
$$= \frac{1}{T}\{2\cos 400\pi t + 2\cos 600\pi t + 6\cos 360\pi t + 6\cos 640\pi t\}$$

EXAMPLE 8.6 A signal $x(t) = \cos 100\pi t + \cos 250\pi t$ is sampled at $f_s = 150$ Hz and the sampled signal $x_s(t)$ is passed through an ideal low pass filter with bandwidth $B = 75$ Hz. Assuming ideal sampling, sketch the frequency spectrum of the output $y(t)$ of the LPF. Also find $y(t)$. If the original signal is to be recovered from the sampled signal, what must be the sampling frequency and what must be the bandwidth of the low pass filter?

Solution: Given $x(t) = \cos 100\pi t + \cos 250\pi t = \cos 2\pi(50)t + \cos 2\pi(125)t$

Therefore, the frequency components are $f_1 = 50$ Hz and $f_2 = 125$ Hz.

or $$f_m = f_2 = 125 \text{ Hz}$$

Given sampling frequency $f_s = 150$ Hz and bandwidth $B = 75$ Hz.

$f_s = 150 \text{ Hz} < 2f_m = 250$ Hz. So aliasing effect will be there, and it is not possible to extract the original signal by a filter of any bandwidth. The spectrum of the sampled signal will have the following frequency components:

$$f_1, f_s \pm f_1, 2f_s \pm f_1 = 50, (150 \pm 50), (300 \pm 50), \ldots$$
$$= 50, 100, 200, 250, 350, \ldots$$
$$f_2, f_s \pm f_2, 2f_s \pm f_2 = 125, (150 \pm 125), (300 \pm 125), \ldots$$
$$= 125, 25, 275, 175, 425, \ldots$$

The replicas of these components will be present at negative frequencies, i.e. at $-f_1$, $-f_s \pm f_1$, $-2f_s \pm f_1$, ... and $-f_2$, $-f_s \pm f_2$, $-2f_s \pm f_2$, ...

As the bandwidth $B = 75$ Hz, the LPF passes only 25 Hz and 50 Hz frequency components to the output. The spectra $X_s(f)$ of the sampled signal $x_s(t)$ is shown in Figure 8.19. The output of the low pass filter $Y(f)$ is also shown in figure.

$$Y(f) = \frac{1}{2}[\delta(f+25) + \delta(f-25)] + \frac{1}{2}[\delta(f+50) + \delta(f-50)]$$

$$\therefore \quad y(t) = \cos 2\pi(25)t + \cos 2\pi(50)t = \cos 50\pi t + \cos 100\pi t$$

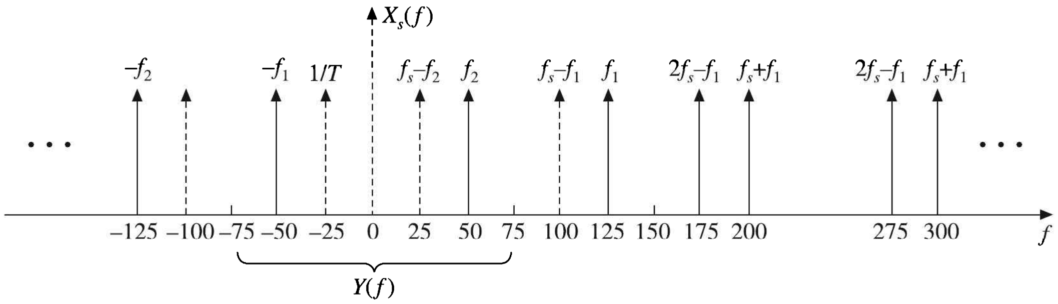


Figure 8.19 Spectrum of $X_s(f)$.

If the original signal is to be recovered, the sampling frequency $f_s \geq 250$ Hz and the bandwidth of the filter used must be 125 Hz < BW < 175 Hz.

EXAMPLE 8.7 A low pass signal $x(t)$ has a spectrum $X(f)$ given by

$$X(f) = 1 - \frac{|f|}{200}, \quad |f| < 200$$

Assume that $x(t)$ is ideally sampled at $f_s = 300$ Hz. Sketch the spectrum of $x_s(t)$.

Solution: The spectrum $X(f) = 1 - (|f|/200)$, $|f| < 200$ is a triangular pulse as shown in Figure 8.20(a).

From the spectrum, we observe that $f_m = 200$ Hz.

$\therefore$ Nyquist rate $f_N = 2f_m$

$$= 2 \times 200 = 400 \text{ Hz}$$

and Sampling frequency $f_s = 300$ Hz $< 2f_m = 400$ Hz

So aliasing problem is there, and there will be overlapping of spectra.

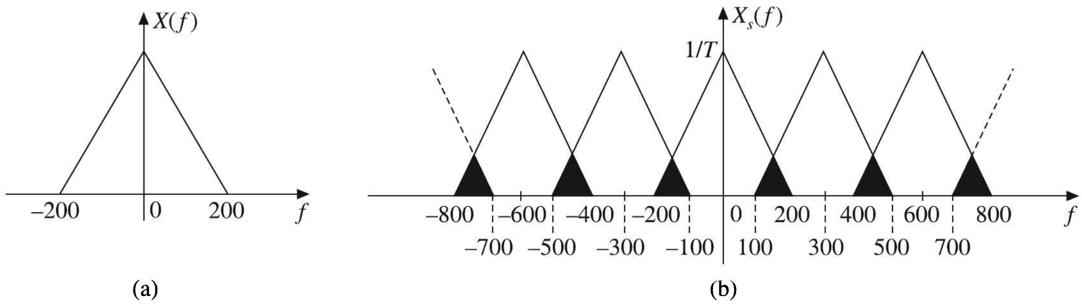


Figure 8.20 (a) Frequency spectrum of $x(t)$, (b) Frequency spectrum of $x_s(t)$.

We know that

$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

or
$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

The frequency spectrum of sampled signal is shown in Figure 8.20(b). Observe that $X(f)$ gets repeated at $f_s, -f_s, 2f_s, -2f_s, 3f_s, -3f_s, \ldots$

EXAMPLE 8.8 The signal $y(t)$ is generated by convolving a band-limited signal $x_1(t)$ with another band-limited signal $x_2(t)$, that is

$$y(t) = x_1(t) * x_2(t)$$

where $\quad X_1(\omega) = 0 \quad \text{for } |\omega| > 1000\pi,\ X_2(\omega) = 0 \quad \text{for } |\omega| > 2000\pi$

Impulse train sampling is performed on $y(t)$ to obtain $y_p(t) = \sum_{n=-\infty}^{\infty} y(nT)\, \delta(t - nT)$

Specify the range of values for sampling period T which ensures that $y(t)$ is recoverable from $y_p(t)$.

Solution: Given
$$y(t) = x_1(t) * x_2(t)$$

$\therefore$
$$Y(\omega) = X_1(\omega)\, X_2(\omega)$$

Since
$$X_1(\omega) = 0 \quad \text{for } |\omega| \geq 1000\pi$$

and
$$X_2(\omega) = 0 \quad \text{for } |\omega| \geq 2000\pi$$

$\therefore$
$$Y(\omega) = X_1(\omega)\, X_2(\omega) = 0 \quad \text{for } |\omega| \geq 1000\pi$$

The highest frequency component present in $y(t)$ is $\omega_m = 1000\pi$.

$\therefore$
$$f_m = \frac{\omega_m}{2\pi} = \frac{1000\pi}{2\pi} = 500 \text{ Hz}$$

$\therefore$
$$\text{Nyquist rate } f_N = 2f_m$$
$$= 2 \times 500 = 1000 \text{ Hz}$$

For $y(t)$ to be recoverable,
$$f_s \geq 2f_m$$

i.e.
$$\frac{1}{f_s} \leq \frac{1}{2f_m}$$

i.e.
$$T \leq \frac{1}{2f_m} = \frac{1}{1000} = 1 \text{ ms}$$

i.e.
$$T \leq 1 \text{ ms}$$

So the range of T for recovery of $y(t)$ from $y_p(t)$ is:
$$0 \leq T \leq 1 \text{ ms}$$

EXAMPLE 8.9 The signal $x(t)$ with Fourier transform $X(\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$ can undergo impulse sampling without aliasing provided that the sampling period $T < (\pi/\omega_0)$. Justify.

Solution: Given $X(\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$

The frequency spectrum is shown in Figure 8.21. ω_0 is the highest frequency component present in $x(t)$.

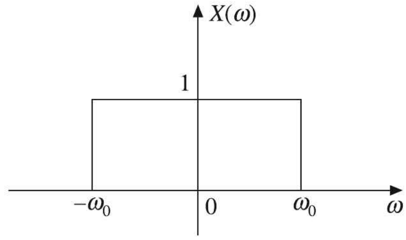


Figure 8.21 Frequency spectrum of $x(t)$.

We know that for no aliasing,

$$f_s \geq 2f_m$$

i.e.

$$\frac{1}{T} \geq 2\frac{\omega_0}{2\pi}$$

i.e.

$$\frac{1}{T} \geq \frac{\omega_0}{\pi}$$

or

$$T \leq \frac{\pi}{\omega_0}$$

This justifies that the given signal $x(t)$ can undergo impulse sampling without aliasing provided the sampling period $T < (\pi/\omega_0)$.

EXAMPLE 8.10 The signal $x(t)$ with Fourier transform $X(\omega) = u(\omega) - u(\omega - \omega_0)$ can undergo impulse train sampling without aliasing, provided that the sampling period $T < (2\pi/\omega_0)$. Justify.

Solution: Given that $X(\omega) = u(\omega) - u(\omega - \omega_0)$, the spectrum $X(\omega)$ is shown in Figure 8.22.

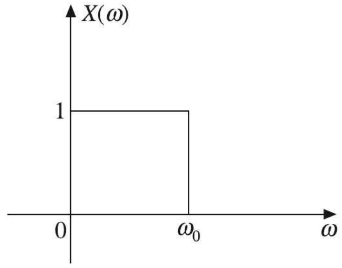


Figure 8.22 Spectrum for Example 8.10.

The given sampling period is:

$$T < \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi f_0} = \frac{1}{f_0}, \quad \text{i.e. } T < \frac{1}{f_0}$$

Since $$T = \frac{1}{f_s}$$

Then $$\frac{1}{f_s} < \frac{1}{f_0}$$

or $$f_s > f_0$$

This does not satisfy the Nyquist condition $f_s > 2f_0$.

The signal $x(t)$ with $X(\omega) = u(\omega) - u(\omega - \omega_m)$ cannot undergo impulse sampling without aliasing. Therefore, the above statement is false.

EXAMPLE 8.11 Consider the following sampling and reconstruction block shown in Figure 8.23(a):

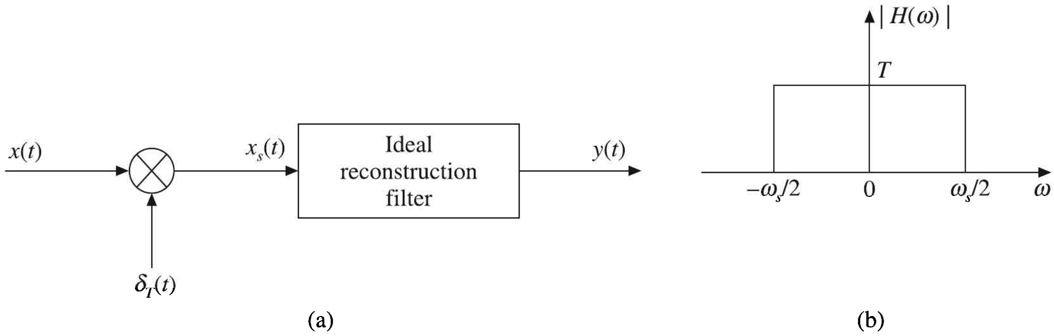


Figure 8.23 (a) Sampling and reconstruction block, (b) Spectrum of LPF.

The output of the ideal reconstruction filter can be found by sending the signal $x_s(t)$ through an ideal low pass filter having characteristics as shown in Figure 8.23(b). For the following signals, draw the spectrum $|X_s(\omega)|$ and find expressions for $x(nT)$ and $y(t)$:

(a) $x(t) = 2 + \cos(30\pi t) + 2\cos(60\pi t)$ for $T = 0.02$ sec

(b) If $X(\omega) = 1/(1 + j\omega)$ and $T = 1$ sec, draw $|X_s(\omega)|$. Test for aliasing.

Solution: (a) Given $x(t) = 2 + \cos(30\pi t) + 2\cos(60\pi t)$ for $T = 0.02$ sec

$$x(nT) = 2 + \cos[30\pi(n \times 0.02)] + 2\cos[60\pi(n \times 0.02)]$$
$$= 2 + \cos(0.6n\pi) + 2\cos(1.2n\pi)$$

$T = 0.02$ sec

$$\therefore \quad f_s = \frac{1}{T} = \frac{1}{0.02} = 50 \text{ Hz}, \ \omega_s = 2\pi f_s = 100\pi \text{ rad/sec.}$$

Highest frequency in $x(t)$ is ω_m = 60π rad/sec.

$$\therefore \qquad f_m = \frac{\omega_m}{2\pi} = \frac{60\pi}{2\pi} = 30 \text{ Hz}$$

and $\qquad$ Nyquist rate $f_N = 2f_m = 2 \times 30 = 60$ Hz

The sampling frequency $f_s < 2f_m$. Therefore, the successive spectrum of $X(\omega)$ in $X_s(\omega)$ will overlap and $x(t)$ cannot be recovered from $x(nT)$.

$$X(\omega) = 4\pi\delta(\omega) + \pi[\delta(\omega + 30\pi) + \delta(\omega - 30\pi)] + 2\pi[\delta(\omega + 60\pi) + \delta(\omega - 60\pi)]$$

$$\omega_s = 100\pi \text{ rad/sec} < 2\omega_m = 120\pi \text{ rad/sec}$$

Therefore, the spectrum $X_s(\omega)$ can be obtained by periodically repeating $X(\omega)$ for every 100π rad/sec (50 Hz). The spectrum $X(\omega)$ is shown in Figure 8.24 and the spectrum $X_s(\omega)$ is shown in Figure 8.25.

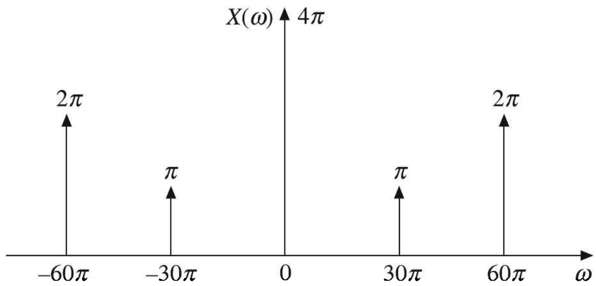


Figure 8.24 Spectrum $X(\omega)$.

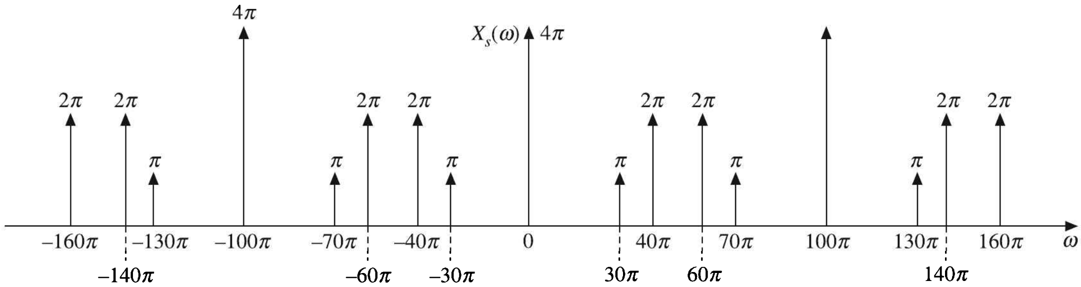


Figure 8.25 Spectrum $X_s(\omega)$.

The frequency spectrum of the low pass filter is shown in Figure 8.26(a). The output spectrum of the low pass filter $Y(\omega)$ is shown in Figure 8.26(b).

From Figure 8.26(b), we have

$$Y(\omega) = 4\pi\delta(\omega) + \pi[\delta(\omega + 30\pi) + \delta(\omega - 30\pi)] + 2\pi[\delta(\omega + 40\pi) + \delta(\omega - 40\pi)]$$

Taking inverse Fourier transform on both sides, we get

$$y(t) = 2 + \cos(30\pi t) + 2\cos(40\pi t)$$

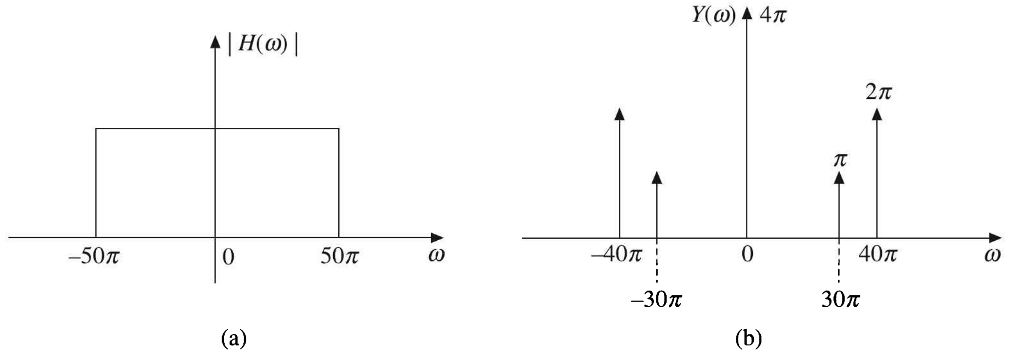


Figure 8.26 Frequency spectrum of (a) LPF, (b) Output of LPF.

(b) Given $$T = 1 \text{ sec}$$

$\therefore$ $$f_s = \frac{1}{T} = 1\text{ Hz}$$

$$\omega_s = 2\pi f_s = 2\pi \text{ rad/sec}$$

$$X(\omega) = \frac{1}{1+j\omega},\ |X(\omega)| = \frac{1}{\sqrt{1+\omega^2}}.$$

The spectrum of $|X(\omega)|$ is plotted as shown in Figure 8.27(a). We can observe that the spectrum of $|X(\omega)|$ is not band-limited. Therefore, aliasing of successive frequency components occurs even if ω_s is large. The spectra $|X_s(\omega)|$ is shown in Figure 8.27(b). It is a periodic repetition of $X(\omega)$ at $\omega = \pm(2\pi/T)n = \pm 2\pi n$.

$$X(\omega) = \frac{1}{1+j\omega}$$

$\therefore$ $$x(t) = e^{-t},\ x(nT) = e^{-nT} = e^{-n}$$

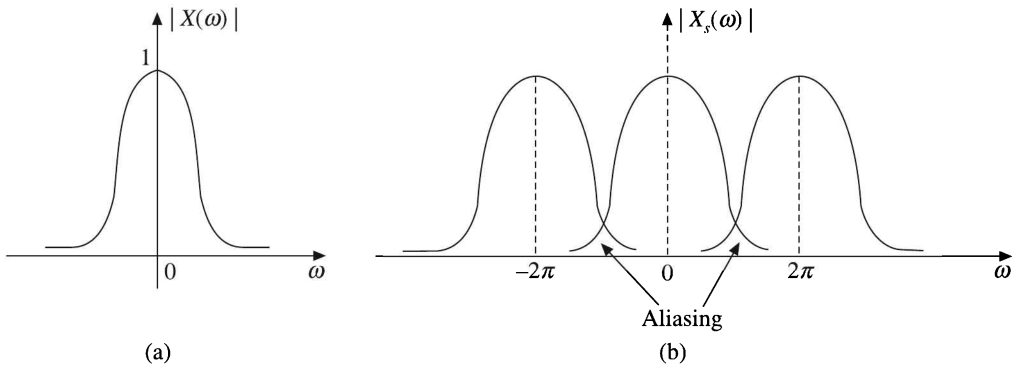


Figure 8.27 (a) Spectrum $|X(\omega)|$, (b) Spectrum $|X_s(\omega)|$.

EXAMPLE 8.12 The signal $x(t) = u(t + T_0) - u(t - T_0)$ can undergo impulse sampling without aliasing provided that the sampling period $T < 2T_0$. Justify.

Solution: Given signal $x(t) = u(t + T_0) - u(t - T_0)$ is shown in Figure 8.28.

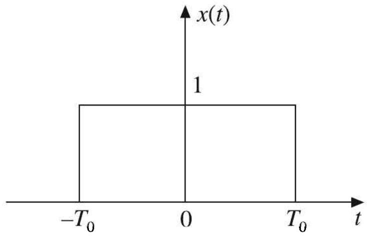


Figure 8.28 Signal $x(t) = u(t + T_0) - u(t - T_0)$.

The sampling is free from aliasing if the sampling frequency $f_s \geq 2f_m$, where f_m is the highest frequency component in $x(t)$.

Given that $$T \leq 2T_0, \text{ i.e. } T < \frac{1}{2f_0}$$

But the Nyquist rate, $$f_s = \frac{1}{T}$$

Then $$\frac{1}{f_s} < \frac{1}{2f_0}, \text{ i.e. } f_s > 2f_0$$

So the sampling theorem is satisfied. Hence the above statement is true.

EXAMPLE 8.13 A signal with Fourier transform $X(\omega)$ undergoes impulse train sampling to generate $x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$ where $T = 10^{-4}$. For each of the following sets of constraints on $x(t)$ and/or $X(\omega)$, does the sampling theorem guarantee that $x(t)$ can be recovered exactly from $x_p(t)$?

(a) $X(\omega) = 0$ for $|\omega| > 5000\pi$

(b) $x(t)$ real and $X(\omega) = 0$ for $|\omega| > 5000\pi$

Solution: The sampling theorem states that a band-limited signal $x(t)$ with $X(\omega) = 0$ for $|\omega| > \omega_m$ can be easily recovered from its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$ if the sampling frequency ω_s is greater than or equal to twice the maximum frequency component present in the signal, i.e.

$$\omega_s \geq 2\omega_m$$

Here the sampling frequency $\omega_s = \dfrac{2\pi}{T} = \dfrac{2\pi}{10^{-4}} = 20000\pi$ rad/sec.

(a) Given $X(\omega) = 0 \quad \text{for } |\omega_m| > 5000\pi$

The highest frequency component $\omega_m = 5000\pi$ rad/sec

Since $\omega_s > 2\omega_m = 10000\pi$ rad/sec, the sampling theorem is satisfied and $x(t)$ can be recovered from $x_p(t)$.

(b) $x(t)$ real and $X(\omega) = 0 \quad \text{for } |\omega| > \omega_m$

As $x(t)$ is real, we have

$$X(\omega) = \overline{X(-\omega)}$$

$$\therefore \qquad X(\omega) = \overline{X(-\omega)} = 0 \quad \text{for } |\omega| > 5000\pi$$

$$\therefore \qquad X(\omega) = 0 \quad \text{for } \omega < -5000\pi \quad \text{and} \quad \text{for } \omega > 5000\pi$$

i.e. $X(\omega) = 0 \quad \text{for } |\omega| > 5000\pi$

So obviously $\omega_s > 2\omega_m$

Therefore, the signal $x(t)$ can be recovered from $x_p(t)$.

EXAMPLE 8.14 Given a continuous-time signal $x(t)$ with Nyquist rate ω_N. Determine the Nyquist rate for the following continuous-time signals:

(a) $y(t) = x^2(t)$

(b) $y(t) = x(t) \cos \omega_0 t$

Solution: Given $x(t)$ has a Nyquist rate of ω_N.

Therefore, $X(\omega) = 0 \quad \text{for } |\omega| > \dfrac{\omega_N}{2}$

(a) Given $y(t) = x^2(t)$

Therefore, $$Y(\omega) = \frac{1}{2\pi}[X(\omega) * X(\omega)]$$

It is clear that $Y(\omega) = 0 \quad \text{for } \omega > \dfrac{\omega_N}{2} + \dfrac{\omega_N}{2} = \omega_N$

Therefore, the Nyquist rate for $y(t)$ is $2\omega_N$.

(b) Given $y(t) = x(t) \cos \omega_0 t$

Therefore, $$Y(\omega) = \frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)]$$

It is clear that $Y(\omega) = 0 \quad \text{for } \omega > \omega_0 + \dfrac{\omega_N}{2}$.

Therefore, the Nyquist rate for $y(t) = 2\left(\omega_0 + \dfrac{\omega_N}{2}\right) = 2\omega_0 + \omega_N$.

Nyquist interval

Nyquist interval is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

$$\text{Nyquist rate } f_N = 2f_m \text{ Hz}$$

$$\text{Nyquist interval } = \frac{1}{f_N} = \frac{1}{2f_m} \text{ sec.}$$

8.5 EFFECTS OF UNDER SAMPLING—ALIASING

When $\omega_s < 2\omega_m$, i.e. when the signal is under sampled, $X(\omega)$, the spectrum of $x(t)$ is no longer replicated in $X_s(\omega)$, and thus is no longer recoverable by low pass filtering. This effect in which the individual terms in equation $X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$ overlap is referred to as aliasing. This process of spectral overlap is also called *frequency folding effect.* In fact, aliasing is defined as the phenomenon in which a high frequency component in the frequency spectrum of signal takes identity of a lower frequency component in the spectrum of the sampled signal.

Aliasing can occur if either of the following conditions exists:

1. The signal is not band-limited to a finite range.
2. The sampling rate is too low.

Theoretically if the signal is not band-limited, there is no way of avoiding the aliasing problem with the basic sampling scheme employed. However, the spectra of most real life signals are such that they may be assumed to be band-limited. Further, a common practice employed in many sampled data systems is to filter the continuous-time signals before sampling to ensure that it does meet the band-limited criterion closely enough for all practical purposes.

To avoid aliasing, it should be ensured that:

1. $x(t)$ is strictly band-limited (this can be ensured by using anti-aliasing filter before the sampler).
2. f_s is greater than $2f_m$.

8.6 ANTI-ALIASING FILTER

Sampling theorem states that a signal can be perfectly reconstructed from its samples only if it is band-limited. In practice no signal is strictly band-limited, i.e. in general signals have frequency spectra consisting of low frequency components as well as high frequency noise components. When a signal is sampled, with sampling frequency f_s, all signals with frequency range higher than $\omega_s/2$ appear as signal frequencies between 0 and $\omega_s/2$ creating aliasing. Therefore, to avoid aliasing errors caused by the undesired high frequency signals, it is necessary to first band-limit $x(t)$ to some appropriate frequency f_m by using an LPF such that

Now, the operation is equivalent to multiplying the input signal $x(t)$ by an impulse train $\delta_T(t)$ as shown in Figure 8.5(c). So the output of the sampler is a train of impulses of height equal to the instantaneous value of the input signal at the sampling instant.

The impulse train, also called the sampling function is represented as:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The sampled signal is given by

$$x_s(t) = x(t)\,\delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

$\therefore$
$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

or
$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

This equation gives the spectrum of ideally sampled signal. It shows that the spectrum $X_s(\omega)$ is an infinite sum of shifted replicas of $X(\omega)$ spaced $n\omega_s$ apart, where $n = \pm 1, \pm 2$, etc. and scaled by a factor $1/T$. However, it may be noted that ideal or instantaneous sampling is possible only in theory because it is impossible to have a pulse with pulse width approaching zero. Practically, the flat top sampling or natural sampling is used.

8.7.2 Natural Sampling

Natural sampling, also called sampling, using a sequence of pulses is the most practical way of accomplishing sampling of a band-limited signal. This is achieved by multiplying the signal $x(t)$ with a pulse train $p_T(t)$ as shown in Figure 8.6. Each pulse of $p_T(t)$ is of short duration τ and occurs at a sampling period of T sec. The output of the sampler is same as the input during that short duration τ. Hence it is termed as *natural sampling*.

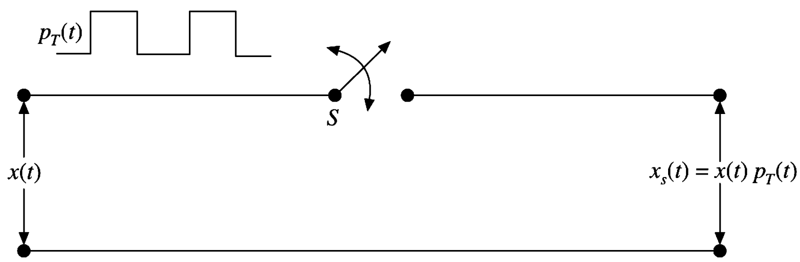


Figure 8.6 Natural sampler.

Figure 8.7 explains the process of natural sampling. Figure 8.7(a) is the signal $x(t)$ to be sampled, and Figure 8.7(b) is its spectrum $X(f)$. Figure 8.7(c) is the pulse train $p_T(t)$, and Figure 8.7(d) is its spectrum $P(f)$. Figure 8.7(e) is the output of the sampler $x_s(t)$, and

Figure 8.7(f) is the output spectrum $X_s(f)$. From Figure 8.7(f), it is clear that $X(f)$ can be recovered from $X_s(f)$, i.e. $x(t)$ can be recovered from $x_s(t)$, if $f_s > 2f_m$ by using an LPF whose gain is constant atleast upto $f = f_m$ and whose cutoff frequency B is such that $f_m < B < f_s - f_m$.

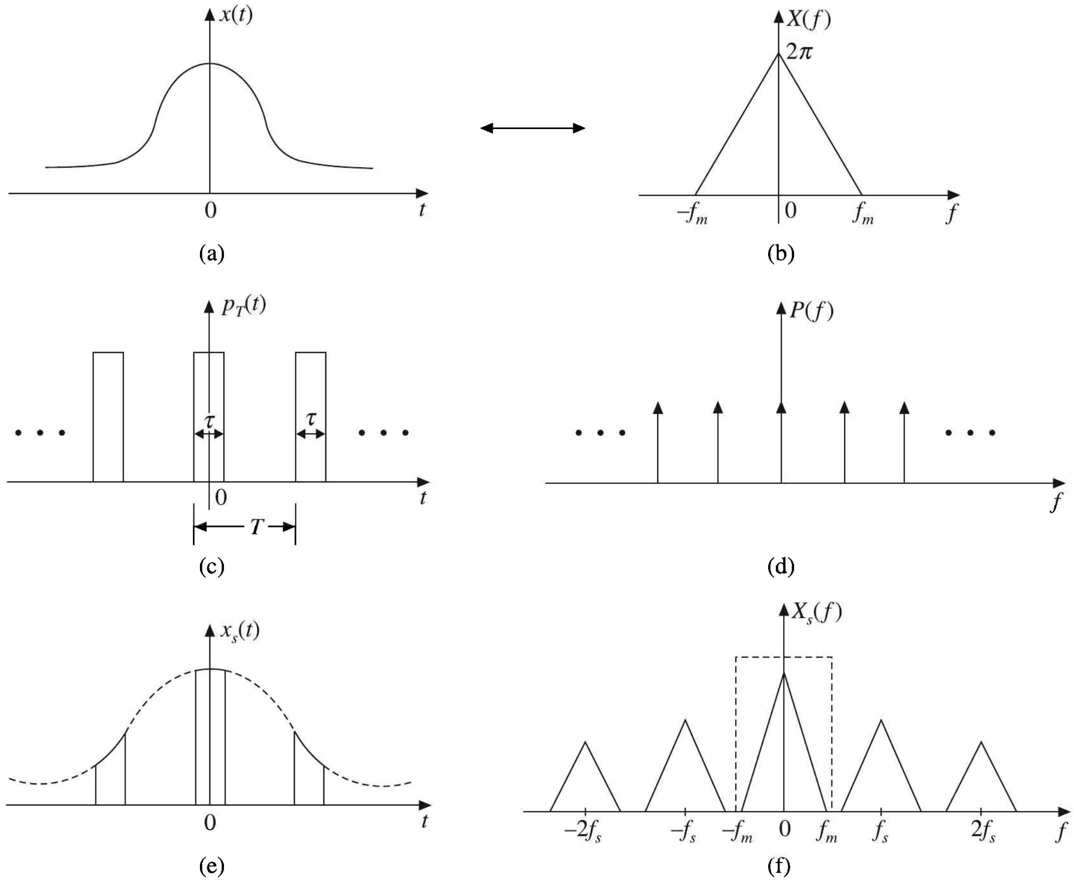


Figure 8.7 Natural sampling.

The output of the sampler is:

$$x_s(t) = x(t)\, p_T(t)$$

where
$$p_T(t) = \sum_{n=-\infty}^{\infty} p(t - nT)$$

As $p_T(t)$ is a periodic pulse train, let us write its Fourier series expansion

$$p_T(t) = \sum_{n=-\infty}^{\infty} p(t - nT) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_s t}$$

where $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} p_T(t)\, e^{-j2\pi n f_s t}\, dt$$

Since τ the width of $p(t)$, single pulse in $p_T(t)$ is very much less than T and $p(t) = 0$ for $|t| \geq \tau/2$, we may write

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} p_T(t)\, e^{-j2\pi n f_s t}\, dt = \frac{1}{T}\int_{-\infty}^{\infty} p(t)\, e^{-j2\pi n f_s t}\, dt$$

$$C_n = f_s P(n f_s)$$

where $$P(nf_s) = \mathrm{F}\,[p(t)]\big|_{f = nf_s}$$

$\therefore$ $$p_T(t) = f_s \sum_{n=-\infty}^{\infty} P\,(nf_s)\, e^{j2\pi n f_s t}$$

and $$X_s(f) = \mathrm{F}\,[x_s(t)] = \mathrm{F}\left[f_s \sum_{n=-\infty}^{\infty} P(nf_s)\, x(t)\, e^{j2\pi n f_s t} \right]$$

$$= f_s \sum_{n=-\infty}^{\infty} P(nf_s)\, X(f)\, \delta(f - nf_s)$$

Since $$\mathrm{F}\,[e^{j2\pi n f_s t}] = \delta(f - nf_s)$$

Hence $$X_s(f) = f_s \sum_{n=-\infty}^{\infty} P(nf_s)\, X(f - nf_s)$$

If $x(t)$ has a spectrum $X(f)$, as shown in Figure 8.7(b), then $X_s(f)$, the spectrum of the sampled version of $x(t)$ will appear as shown in Figure 8.7(f).

8.7.3 Flat Top Sampling

This is the simplest and most popular sampling method that uses the sample and hold (S/H) circuit with flat top samples. This is also called practical sampling. Here the top of the samples remain constant which is equal to the instantaneous value of the base band signal $x(t)$ at the beginning of sampling. The duration or width of each sample is τ and the sampling rate, $f_s = 1/T$.

The schematic of a 'sample and hold' (S/H) circuit is shown in Figure 8.8(a), and a typical output waveform from an S/H circuit is shown in Figure 8.8(b).

The S/H circuit essentially consists of two switches S_1 and S_2 and a capacitor C connected as shown in Figure 8.8(a). With S_2 open, S_1 is closed for a very brief period at each sampling instant. The capacitor C then gets charged to a voltage equal to the value of the input signal $x(t)$ at the sampling instant and holds it for a period τ at the end of which S_2 is closed to allow the capacitor to discharge. This sequence of operations is repeated at the next and all subsequent sampling instants. The switches S_1 and S_2 are generally FET switches and are operated by giving appropriate pulses to their gates. An actual S/H circuit uses one or two op-amps also. The voltage across C appears as $x_s(t)$ and is sketched in Figure 8.8(b).

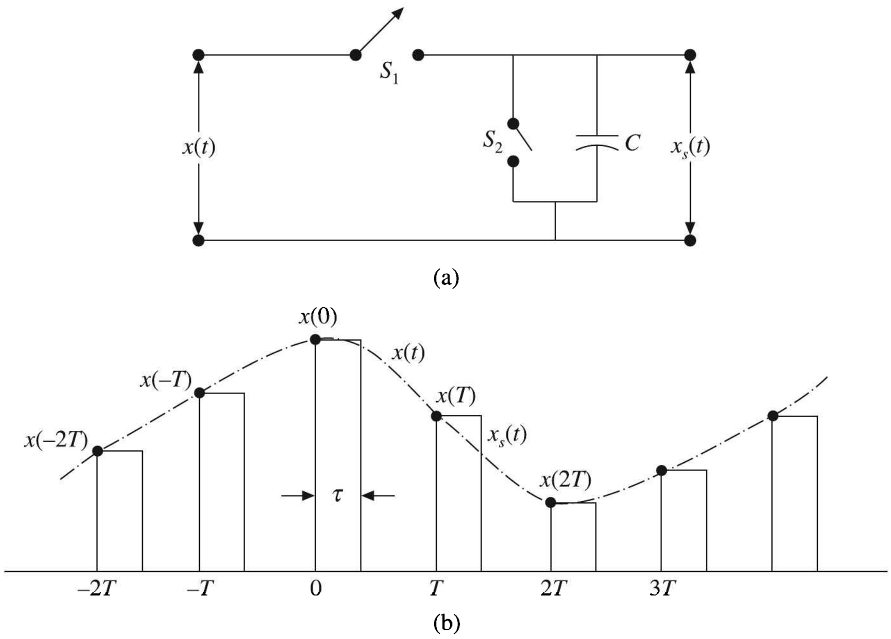


Figure 8.8 (a) Schematic of an S/H circuit, (b) Signal $x(t)$ and output of S/H circuit.

From the Figure, it is obvious that the sampled version, $x_s(t)$ consists of a sequence of rectangular pulses, the leading edge of the kth pulse being at $t = kT$ and the amplitude of the pulse being the value of $x(t)$ at $t = kT$, i.e. $x(kT)$.

The sampled signal $x_s(t)$ is the convolution of rectangular pulses $p(t)$ and the ideally sampled version of $x(t)$, i.e. of $x_\delta(t)$

$$\therefore \qquad X_s(f) = P(f)\, X_\delta(f)$$

Assume that $x(t)$ has a spectrum as shown in Figure 8.9.

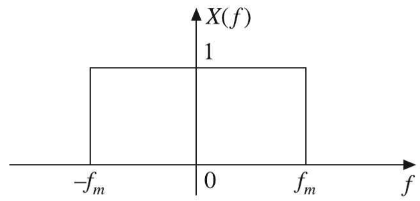


Figure 8.9 Assumed shape of $X(f)$.

Since $p(t)$ is a rectangular pulse of width τ, its Fourier transform, $P(f)$ which is a sinc function will have a shape as shown in Figure 8.10 and will have its first zero values only at $f = \pm(1/\tau)$. Since $\tau < T$, these zero values of $|P(f)|$ which occur at $\pm(1/\tau)$, will be far away from f_s and $-f_s$. Since $X_s(f) = P(f)\, X_\delta(f)$, its plot will be as shown in Figure 8.10(b).

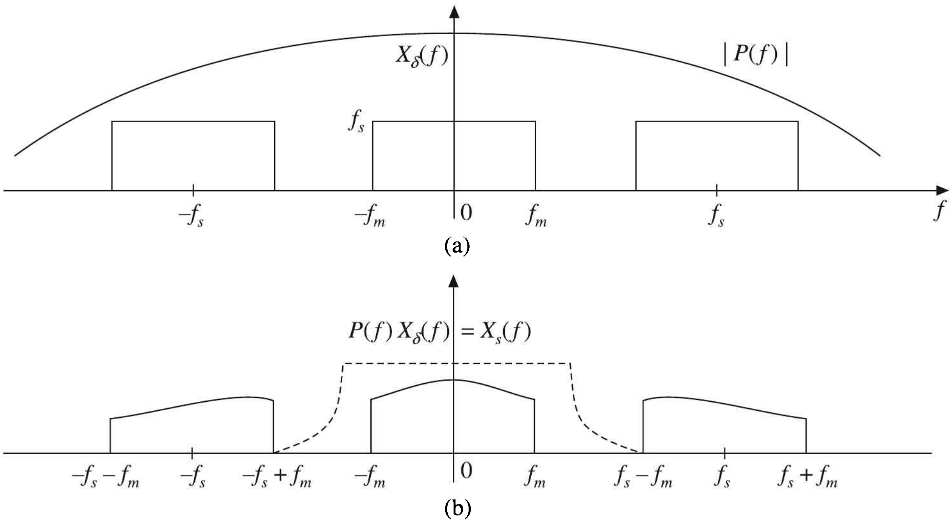


Figure 8.10 (a) Plot of $P(f)$ and $X_\delta(f)$, (b) Plot of $X_s(f) = P(f)\,X_\delta(f)$.

Observe that the magnitudes of the high frequency components in $X_s(f)$ are relatively reduced as compared to the magnitudes of the low frequency components because of the multiplication of $X_\delta(f)$ by $P(f)$. So we can only get a distorted version of $x(t)$, but not exact $x(t)$, by passing $x_s(t)$ through an LPF.

This distortion, wherein the amplitudes of the high frequency components are reduced relative to the amplitudes of the low frequency components, in the reconstructed signal $x(t)$ obtained from the flat top sampled version of the signal is referred to as the *aperture effect.* This aperture effect can be reduced by using an equalizer with transfer function $H_e(f)$ in cascade with the reconstruction filter and adjusting $H_e(f)$ so that

$$H_e(f) = \frac{1}{P(f)}; \; |f| \le f_m$$

8.8 DATA RECONSTRUCTION

The process of obtaining the analog signal $x(t)$ from the sampled signal $x_s(t)$ is called data reconstruction or interpolation. We know that

$$x_s(t) = x(t)\,\delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

or

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

since $\delta(t - nT)$ is zero except at the sampling instants $t = nT$. The reconstruction filter, which is assumed to be linear and time invariant, has unit impulse response $h(t)$. The reconstruction filter output, $y(t)$ is given by the convolution,

8.9 SAMPLING OF BAND PASS SIGNALS

The sampling theorem, which we have discussed earlier, is called low pass sampling theorem because it applies only to low pass signals, i.e. signals for which $X(f) = 0$ for $f \geq f_m$, where f_m has a finite value. Here the signal which is band-limited to f_m Hz has to be sampled at least at $2f_m$ samples per second if the analog signal is to be reconstructed. In the case of band pass signals, that is signals for which $X(f) = 0$ for all frequencies outside the range $f_1 \leq f \leq f_2$, where $f_1 > 0$, this rule does not apply. For this class of signals, while there will be no aliasing if $f_s > 2f_2$, there might be no aliasing even if $f_s < 2f_2$ provided f_s satisfies certain conditions.

Sampling of a band pass signal with $f_s > 2f_2$ to prevent aliasing has two disadvantages.

1. The spectrum of the sampled signal will have spectral gaps.
2. If f_2 is large, the sampling rate is also very large, and therefore, have practical limitations.

To overcome this, the band pass sampling theorem is defined as follows:

Let the bandwidth of the band pass signal shown in Figure 8.29(a) be $B = f_2 - f_1 = 2f_m$. Then the band pass sampling theorem states that $x(t)$ can be recovered without any error what so ever from its samples $x(nT)$ taken at regular intervals of T, if the sampling rate f_s is such that

$$f_s = \frac{1}{T} = \frac{2f_2}{m} \qquad \text{(which is smaller than the Nyquist rate } 2f_2\text{)}$$

where m is the largest integer not exceeding f_2/B.

If we assume that the highest frequency component present in the band pass signal is multiple of bandwidth, i.e. $f_2 = KB = K_2 f_m$, then the band pass sampling theorem states that

The band pass signal $x(t)$ whose maximum bandwidth is $2f_m$ can be completely represented into and recovered from its samples if it is sampled at the minimum rate of twice the bandwidth.

Hence for band pass signals of bandwidth $2f_m$, the minimum sampling rate is equal to twice that of bandwidth, i.e. $f_s = 2 \times \text{BW} = 4f_m$ samples per second, or $T = 1/4f_m$ sec.

If the spectrum of the band pass signal is $X(\omega)$, then the spectrum of the sampled band pass signal is:

$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - 2nB)$$

where $X_s(\omega)$ is the sum of the original Fourier transform $X(\omega)$ and shifted replicas of $X(\omega)$ and then scaled by $1/T$.

Figure 8.29(b) shows the spectrum of the original signal and sampled signal for $K = 3$. From this figure, we can find that the original signal can be recovered by passing $x(t)$ through an ideal band pass filter with a pass band given by $f_1 \leq |f| \leq f_2$ with a gain of T.

It may be noted that sampling frequencies higher than what is given by the equation $f_s = 2f_2/m$ may not always permit recovery of $x(t)$ without distortion (i.e. they may not be able to avoid aliasing unless $f_s > 2f_2$. The required sampling rate for a band pass filter depends on m, i.e. on (f_2/B).

If $f_s > 2f_2$, there will not be any aliasing and perfect reconstruction is possible. Also if $f_2 = KB$ where K is an integer, a sampling rate $f_s = 2B$ would suffice and will not produce any aliasing.

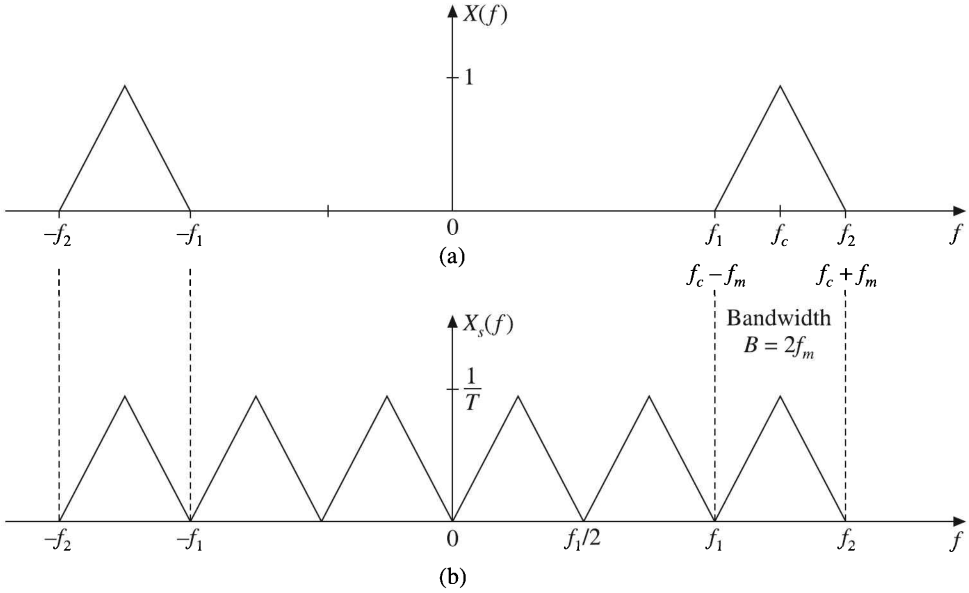


Figure 8.29 (a) Spectrum of band pass filter (BPF), (b) Spectrum of sampled BPF.

EXAMPLE 8.15 The spectral range of a function extends from 5.6 MHz to 6.8 MHz. Find the minimum sampling rate and maximum sampling time.

Solution: Given frequency range = 5.6 MHz to 6.8 MHz

$\therefore$ Bandwidth $B = 2f_m = (6.8 - 5.6)$ MHz $= 1.2$ MHz

$\therefore$ Minimum sampling rate $f_s = 2B = 4f_m = 2 \times 1.2$ MHz $= 2.4$ MHz

$\therefore$ Maximum sampling time $T_s = \dfrac{1}{f_s} = \dfrac{1}{2.4 \times 10^6} = 0.417\ \mu\text{s}.$

EXAMPLE 8.16 A band pass signal with a spectrum shown in Figure 8.30 is ideally sampled. Sketch the spectrum of the sampled signal when f_s = 20 Hz, 30 Hz, and 40 Hz. Indicate if and how the signal can be recovered.

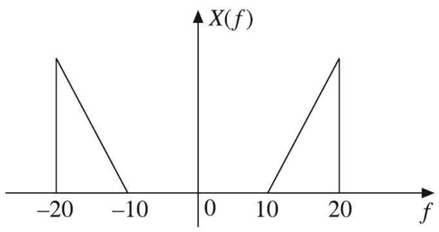


Figure 8.30 Spectrum for Example 8.16.

Solution:

(a) f_s = 20 Hz

When f_s = 20 Hz, the corresponding spectrum of the sampled signal $X_s(f)$ is shown in Figure 8.31. It is a periodic repetition of $X(f)$ at regular intervals of $\pm nf_s$, i.e. $\pm 20n$, n = 1, 2, ...

The base band signal can be recovered by passing the sampled signal through a band pass filter with a pass band from 10 Hz to 20 Hz.

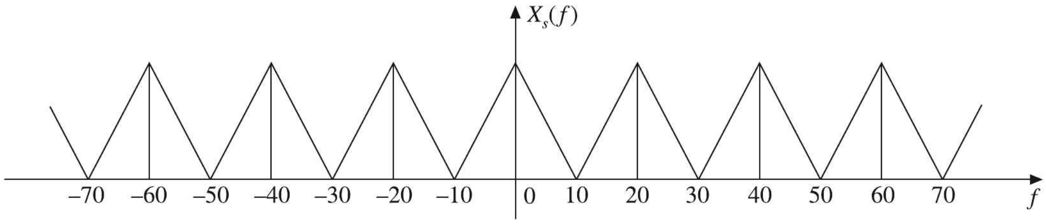


Figure 8.31 Spectrum when f_s = 20 Hz.

(b) f_s = 30 Hz.

When f_s = 30 Hz, the corresponding spectrum of the sampled signal $X_s(f)$ is shown in Figure 8.32. It is a periodic repetition of $X(f)$ at regular intervals of $\pm 30n$. Due to aliasing effect, the signal cannot be recovered.

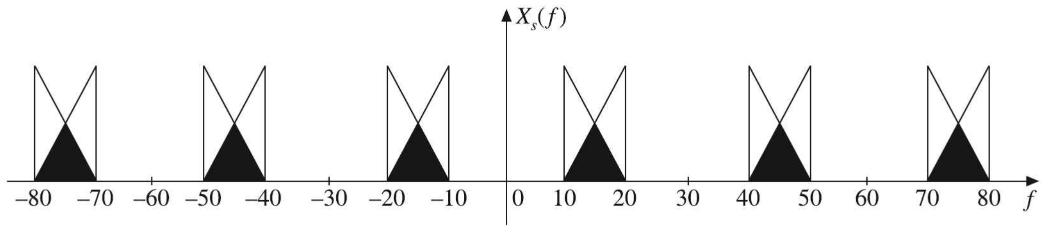


Figure 8.32 Spectrum when f_s = 30 Hz.

(c) f_s = 40 Hz

When f_s = 40 Hz, the corresponding spectrum of the sampled signal $X_s(f)$ is shown in Figure 8.33. The base band signal can be recovered by passing the sampled signal through a band pass filter with a pass band from 10 Hz to 20 Hz.

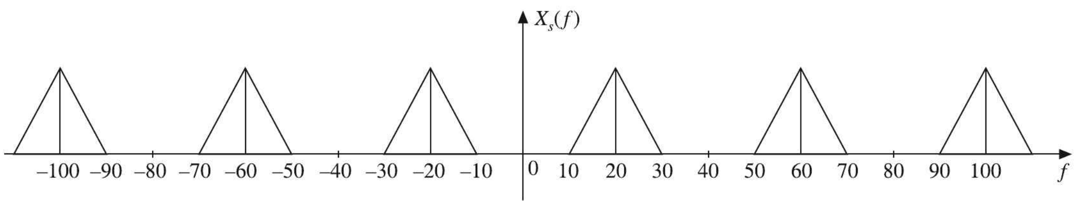


Figure 8.33 Spectrum when f_s = 40 Hz.

EXAMPLE 8.17 A band pass signal has a spectrum as depicted in Figure 8.34. What is the minimum sampling frequency that can be used? Show that no aliasing takes place when this sampling frequency is used.

Solution: Here, f_2, the highest frequency component is 30 kHz. If we use the low pass sampling theorem, a minimum sampling frequency of 60 kHz would be needed.

However, it is not necessary to use such a high sampling frequency. Since it is a band pass signal, a sampling frequency equal to f_s could suffice where

$$f_s = \frac{2f_2}{m}$$

where m is the largest integer less than $\frac{f_2}{B} = \frac{30}{8} = 3.75$

Thus, $$m = 3$$

Therefore, the minimum sampling frequency that can be used is:

$$f_s = 2 \times \frac{30}{3} = 20 \text{ kHz}$$

Figure 8.35 shows that no aliasing occurs with this sampling frequency.

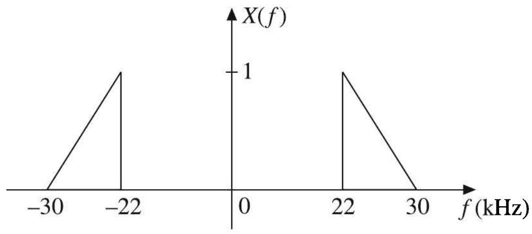


Figure 8.34 Spectrum of band pass signal

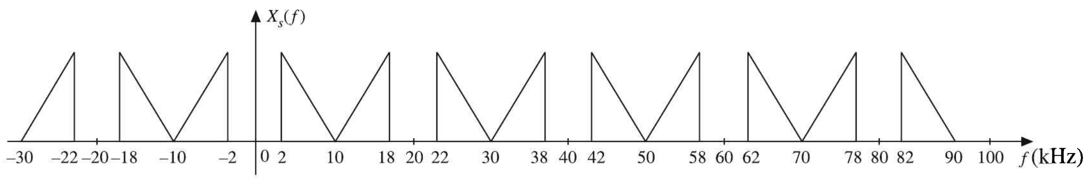


Figure 8.35 Spectrum of sampled version of $x(t)$ with f_s kHz.

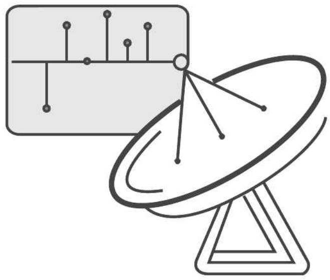
MATLAB PROGRAMS

Program 8.1

```
% Verification of the Sampling theorem
t=-10:0.001:10;
% continuous time signal x(t)
fm=0.1;
xt=cos(2*pi*fm*t);
subplot(2,2,1);plot(t,xt);
title('continuous time signal x(t)');
```

10

Z-Transforms



10.1 INTRODUCTION

We know that a linear time invariant (LTI) continuous-time system is represented by differential equations and a linear time invariant discrete-time system is represented by difference equations. The direct solution of higher order differential equations as well as higher order difference equations is quite tedious and time consuming. So usually they are solved by indirect methods. To solve the differential equations which are in time domain, they are first converted into algebraic equations in frequency domain using Laplace transforms, the algebraic equations are manipulated in s-domain and the result obtained in frequency domain is converted back into time domain using inverse Laplace transform. The Laplace transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself. The Z-transform plays the same role for discrete-time systems as that played by Laplace transform for continuous-time systems. The Z-transform is the discrete-time counterpart of the Laplace transform. It is the Laplace transform of the discretized version of the continuous-time signal $x(t)$. Just as the Laplace transform is a powerful mathematical tool to convert the differential equations into algebraic equations, the Z-transform is a powerful mathematical tool used to convert the difference equations into algebraic equations. To solve the difference equations which are in time domain, they are converted first into algebraic equations in z-domain using Z-transform, the algebraic equations are manipulated in z-domain and the result obtained is converted back into time domain using inverse Z-transform. Like the Laplace transform, the Z-transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and the initial conditions can be introduced in the beginning of the process itself. Also just as the Laplace transform is a very useful tool in the analysis of linear time invariant (LTI) systems, the Z-transform is a very useful tool in the analysis of linear shift invariant (LSI) systems. Like the Laplace transform, the Z-transform may be one-sided (unilateral) or two-sided (bilateral). As in the case of Laplace transform, it is the one-sided or unilateral Z-transform that is more useful because we mostly deal with causal sequences. Further, it is eminently suited for solving difference equations with initial conditions. Just as the range of values of s for which $X(s)$ converges is called the ROC of $X(s)$ the range of values of z for which $X(z)$ converges is called the ROC of $X(z)$.

The *bilateral* or *two-sided* Z-transform of a discrete-time signal or a sequence $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable.

The *one-sided* or *unilateral* Z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

Region of convergence (ROC)

For any given sequence, the Z-transform may or may not converge.

The set of values of z or equivalently the set of points in z-plane, for which $X(z)$ converges is called the region of convergence (ROC) of $X(z)$.

If there is no value of z (i.e. no point in the z-plane) for which $X(z)$ converges, then the sequence $x(n)$ is said to be having no Z-transform.

Advantages of Z-transform

1. The Z-transform converts the difference equations of a discrete-time system into linear algebraic equations so that the analysis become easy and simple.
2. Convolution in time domain is converted into multiplication in z-domain.
3. Z-transform exists for most of the signals for which discrete-time Fourier transform (DTFT) does not exist.

Limitation

Frequency domain response cannot be achieved and cannot be plotted.

10.2 RELATION BETWEEN DISCRETE TIME FOURIER TRANSFORM (DTFT) AND Z-TRANSFORM

The discrete-time Fourier transform (DTFT) of a sequence $x(n)$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

For the existence of DTFT, the above summation should converge, i.e. $x(n)$ must be absolutely summable. The Z-transform of the sequence $x(n)$ is given by

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable and is given by

$$z = re^{j\omega}$$

where r is the radius of a circle.

$$\therefore \qquad X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\omega n}$$

For the existence of Z-transform, the above summation should converge, i.e. $x(n)\, r^{-n}$ must be absolutely summable, i.e. $\sum_{n=-\infty}^{\infty} |x(n)\, r^{-n}| < \infty$.

The above equation represents the discrete-time Fourier transform of a signal $x(n)\, r^{-n}$. Hence, we can say that the Z-transform of $x(n)$ is same as the discrete-time Fourier transform of $x(n)\, r^{-n}$.

For the DTFT to exist, the discrete sequence $x(n)$ must be absolutely summable, i.e. $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$.

So for many sequences the DTFT may not exist but the Z-transform may exist. When $r = 1$, the DTFT is same as the Z-transform, i.e. the DTFT is nothing but the Z-transform evaluated along the unit circle centred at the origin of the z-plane.

The set of values of z for which $X(z)$ converges is called the ROC of $X(z)$.

10.3 Z-TRANSFORMS OF SOME COMMON SEQUENCES

10.3.1 The Unit-sample Sequence (The Unit-impulse Sequence) [$x(n) = \delta(n)$]

We know that

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{for } n \neq 0$$

$$\therefore \qquad X(z) = \mathrm{Z}[x(n)] = \mathrm{Z}[\delta(n)] = \sum_{n=0}^{\infty} \delta(n)\, z^{-n} = 1 \quad \text{(for all } z\text{)}$$

i.e. the ROC is the entire z-plane.

$$\boxed{\delta(n) \xleftrightarrow{\text{ZT}} 1 \quad \text{for all } z}$$

10.3.2 The Unit-step Sequence [$x(n) = u(n)$]

We know that

$$u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

$$\therefore \qquad X(z) = \text{Z}\,[x(n)] = Z\,[u(n)] = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 1z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

The above series converges if $|z^{-1}| < 1$, i.e. ROC is $|z| > 1$. So, the ROC is the exterior of the unit circle in the z-plane.

$$\boxed{u(n) \xleftrightarrow{\text{ZT}} \frac{z}{z-1} = \frac{1}{1-z^{-1}};\ \text{ROC};\ |z|>1}$$

10.3.3 The Unit-ramp Sequence [$x(n) = r(n) = nu(n)$]

We know that

$$r(n) = n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

$$\therefore \qquad X(z) = \text{Z}[x(n)] = \text{Z}[r(n)] = \sum_{n=0}^{\infty} r(n)\, z^{-n} = \sum_{n=0}^{\infty} nz^{-n}$$

$$= 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + \cdots = z^{-1}(1 + 2z^{-1} + 3z^{-2} + \cdots)$$

$$= z^{-1}(1 - z^{-1})^{-2} = z^{-1}\frac{1}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$$

The above series converges if $|z^{-1}| < 1$, i.e ROC is $|z| > 1$. So the ROC is the exterior of the unit circle in the z-plane.

$$\boxed{nu(n) \xleftrightarrow{\text{ZT}} \frac{z^{-1}}{[1-z^{-1}]^2} = \frac{z}{(z-1)^2};\ \text{ROC};\ |z|>1}$$

10.3.4 The Exponential Sequence [$x(n) = e^{-j\omega n}u(n)$]

$$x(n) = e^{-j\omega n}u(n) = e^{-j\omega n} \quad \text{for } n \ge 0$$
$$= 0 \quad \text{for } n < 0$$

$$\therefore \quad X(z) = Z[e^{-j\omega n}u(n)] = \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} = \sum_{n=0}^{\infty} (z^{-1}e^{-j\omega})^n$$
$$= 1 + (z^{-1}e^{-j\omega}) + (z^{-1}e^{-j\omega})^2 + (z^{-1}e^{-j\omega})^3 + \cdots$$
$$= [1 - (z^{-1}e^{-j\omega})]^{-1} = \frac{1}{1 - z^{-1}e^{-j\omega}} = \frac{z}{z - e^{-j\omega}}$$

The above series converges if $|z^{-1}| < 1$, i.e. ROC $|z| > 1$.

$$\boxed{e^{-j\omega n}u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1 - z^{-1}e^{-j\omega}} = \frac{z}{z - e^{-j\omega}};\ \text{ROC; } |z| > 1}$$

On similar lines,

$$\boxed{e^{j\omega n}u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1 - z^{-1}e^{j\omega}} = \frac{z}{z - e^{j\omega}};\ \text{ROC; } |z| > |1|}$$

10.3.5 The Sinusoidal Sequence [$x(n) = \sin \omega n\, u(n)$]

$$x(n) = \sin \omega n\, u(n) = \sin \omega n \quad \text{for } n \ge 0$$
$$= 0 \quad \text{for } n < 0$$

$$\therefore \quad X(z) = Z[\sin \omega n\, u(n)] = \sum_{n=0}^{\infty} \sin \omega n\, z^{-n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) z^{-n} = \frac{1}{2j}\sum_{n=0}^{\infty} (e^{j\omega n}z^{-n} - e^{-j\omega n}z^{-n})$$
$$= \sum_{n=0}^{\infty} \frac{(z^{-1}e^{j\omega})^n - (z^{-1}e^{-j\omega})^n}{2j}$$
$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} (z^{-1}e^{j\omega})^n - \sum_{n=0}^{\infty} (z^{-1}e^{-j\omega})^n\right]$$

The above series converges for $|z^{-1}| < 1$, i.e $|z| > 1$.

$$\therefore \quad X(z) = \frac{1}{2j}\left(\frac{1}{1-z^{-1}e^{j\omega}} - \frac{1}{1-z^{-1}e^{-j\omega}}\right) = \frac{1}{2j}\left(\frac{z}{z-e^{j\omega}} - \frac{z}{z-e^{-j\omega}}\right)$$

$$= \frac{1}{2j}\left[\frac{z(z-e^{-j\omega}) - z(z-e^{j\omega})}{(z-e^{j\omega})(z-e^{-j\omega})}\right] = \frac{1}{2j}\left[\frac{z(e^{j\omega}-e^{-j\omega})}{z^2 - z(e^{j\omega}+e^{-j\omega})+1}\right]$$

$$= \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$$

$$\boxed{\sin\omega n u(n) \xleftrightarrow{\text{ZT}} \frac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega + z^{-2}} = \frac{z\sin\omega}{z^2-2z\cos\omega+1};\ \text{ROC};\ |z|>1}$$

10.3.6 The Cosinusoidal Sequence [$x(n) = \cos\omega n u(n)$]

$$\cos\omega n u(n) = \cos\omega n \quad \text{for } n \ge 0$$
$$= 0 \quad \text{for } n < 0$$

$$\text{Z}[\cos\omega n u(n)] = \text{Z}\left[\frac{e^{j\omega n}+e^{-j\omega n}}{2}u(n)\right] = \frac{1}{2}\{\text{Z}\,[e^{j\omega n}u(n)] + \text{Z}[e^{-j\omega n}u(n)]\}$$

$$= \frac{1}{2}\left(\frac{z}{z-e^{j\omega}} + \frac{z}{z-e^{-j\omega}}\right) = \frac{1}{2}\left[\frac{z(z-e^{-j\omega}) + z(z-e^{j\omega})}{(z-e^{j\omega})(z-e^{-j\omega})}\right];\ |z|>1$$

$$= \frac{1}{2}\left\{\frac{z[2z-(e^{j\omega}+e^{-j\omega})]}{z^2 - z(e^{j\omega}+e^{-j\omega})+1}\right\} = \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1};\ \text{ROC};\ |z|>1$$

$$\boxed{\cos\omega n u(n) \xleftrightarrow{\text{ZT}} \frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega + z^{-2}} = \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1};\ \text{ROC};\ |z|>1}$$

EXAMPLE 10.1 Distinguish between one-sided and two-sided Z-transforms. What are their applications?

Solution: The one-sided or unilateral Z-transform is defined only for causal signals and is given by

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

The two-sided or bilateral Z-transform is defined for bidirectional signals and is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

If the signal is causal, i.e. $x(n) = 0$ for $n < 0$, then the one-sided and two-sided Z-transforms are identical.

Several properties, viz., superposition, scale change, integration, and differentiation are identical in both the Z-transforms; but the initial value theorem is fundamentally a unilateral Z-transform property.

The time reversal property of the bilateral Z-transform has no meaning in the unilateral Z-transform.

The inverse Z-transform of unilateral Z-transform is unique because its ROC is unique. Its ROC is always $|z| > a$. The inverse Z-transform of bilateral Z-transform is not unique because its ROC is not unique. Its ROC may be $|z| > a$ or $|z| < a$ or $a < |z| < b$ where a and b are some constants. The one-sided Z-transform is more practical because most of the signals are causal signals.

Applications

Unilateral Z-transform with time delay can be used to solve linear constant coefficient differential equations with zero initial conditions.

Z-transform can be used to analyse linear systems.

EXAMPLE 10.2 Compare Laplace transform and Z-transform.

Solution: The Laplace and Z-transforms are compared as follows:

Laplace transform	*Z-transform*
1. It is used to analyse LTI continuous-time systems.	1. It is used to analyse LTI discrete-time systems.
2. It converts differential equations which are in time domain into algebraic equations in s-domain.	2. It converts difference equations which are in time domain into algebraic equations in z-domain.
3. It is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself.	3. It is also a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself.
4. Laplace transform may be one-sided (unilateral) or two-sided (bilateral).	4. Z-transform also may be one-sided (unilateral) or two-sided (bilateral).
5. The range of values of s for which $X(s)$ converges is called ROC of $X(s)$.	5. The range of values of z for which $X(z)$ converges is called ROC of $X(z)$
6. The ROC of $X(s)$ consists of strips parallel to $j\omega$ axis in s-plane.	6. The ROC of $X(z)$ consists of a ring or disc in z-plane centred at the origin.
7. If the real part of s, i.e. $\sigma = 0$, then the Laplace transform becomes the continuous-time Fourier transform.	7. If the magnitude of z, i.e. $|z| = 1$, then the Z-transform becomes DTFT.
8. Convolution in time domain is equal to multiplication in s-domain.	8. Convolution in time domain is equal to multiplication in z-domain.

EXAMPLE 10.3 Prove that, for causal sequences, the ROC is the exterior of a circle of radius r.

Solution: Causal sequences are the sequences defined for only positive integer values of n and do not exist for negative times, i.e.,

$$x(n) = 0 \quad \text{for } n < 0$$

Consider a causal sequence,

$$x(n) = r^n u(n)$$

From the definition of Z-transform of $x(n)$, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} r^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (rz^{-1})^n$$

The above summation converges for

$$|rz^{-1}| < 1, \text{ i.e. for } |z| > r$$

Hence, for the causal sequences, the ROC is the exterior of a circle of radius r.

EXAMPLE 10.4 Derive the relation between Laplace and Z-transforms.

Solution: Let $x(t)$ be a continuous signal. The discrete version of the signal $x(t)$ is $x^*(t)$. $x^*(t)$ is obtained by sampling $x(t)$ with a sampling period of T sec, i.e. $x^*(t)$ is obtained by multiplying $x(t)$ with a sequence of impulses T sec apart.

$$\therefore \qquad x^*(t) = \sum_{n=0}^{\infty} x(nT)\, \delta(t - nT)$$

The Laplace transform of $x^*(t)$ is given by

$$\mathrm{L}[x^*(t)] = X^*(s) = \mathrm{L}\left[\sum_{n=0}^{\infty} x(nT)\, \delta(t - nT)\right] = \sum_{n=0}^{\infty} x(nT)\, \mathrm{L}[\delta(t - nT)] = \sum_{n=0}^{\infty} x(nT)\, e^{-nTs}$$

The Z-transform of signal $x(nT)$ is given by

$$\mathrm{Z}[x^*(t)] = \mathrm{Z}[x(nT)] = \sum_{n=0}^{\infty} x(nT)\, z^{-n}$$

So the relation between Laplace transform and Z-transform is:

$$L[x^*(t)] = Z[x^*(t)] \text{ with } z = e^{Ts}$$

EXAMPLE 10.5 Find the Z-transform of

(a) $n\delta(n)$ (b) $\delta(n-4)$
(c) $\delta(n+3)$ (d) $n\delta(n-2)$
(e) $y(n) = x(n-1)\,u(n)$ (f) $y(n) = x(n+1)\,u(n)$

Solution:

(a) Given $x(n) = n\delta(n)$

We know that
$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{for } n \neq 0$$

From the definition of Z-transform, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n} = \sum_{n=-\infty}^{\infty} n\delta(n)\,z^{-n}$$
$$= -\infty\delta(-\infty)\,z^{\infty} - \cdots - 1\delta(-1)\,z^{1} + 0\delta(0)\,z^{0} + 1\delta(1)\,z^{-1} + \cdots + \infty\delta(\infty)\,z^{-\infty}$$
$$= -\infty \times 0 \times z^{\infty} - \cdots - 1 \times 0 \times z^{1} + 0 \times 1 \times z^{0} + 1 \times 0 \times z^{-1} + \cdots + \infty \times 0 \times z^{-\infty}$$
$$= 0$$

Hence $Z[n\delta(n)] = 0$

(b) Given $x(n) = \delta(n-4)$

We know that
$$\delta(n-4) = 1 \quad \text{for } n = 4$$
$$= 0 \quad \text{otherwise}$$

$$\therefore \quad X(z) = Z[\delta(n-4)] = \sum_{n=-\infty}^{\infty} \delta(n-4)\,z^{-n} = z^{-4}$$

Hence $Z[\delta(n-4)] = z^{-4}$; ROC; Entire z-plane except at $z = 0$

(c) Given $x(n) = \delta(n+3)$

We know that
$$\delta(n+3) = 1 \quad \text{for } n = -3$$
$$= 0 \quad \text{otherwise}$$

$$\therefore \quad X(z) = Z[\delta(n+3)] = \sum_{n=-\infty}^{\infty} \delta(n+3)\,z^{-n} = z^{3}$$

Hence $Z[\delta(n+3)] = z^3$; ROC; Entire z-plane except at $z = \infty$

(d) Given $$x(n) = n\delta(n-2)$$

We know that
$$\delta(n-2) = 1 \quad \text{for } n = 2$$
$$= 0 \quad \text{otherwise}$$

$$\therefore \qquad X(z) = Z[n\delta(n-2)] = \sum_{n=-\infty}^{\infty} n\delta(n-2)\, z^{-n} = 2z^{-2}$$

Hence $Z[n\delta(n-2)] = 2z^{-2}$; ROC; Entire z-plane except at $z = 0$

(e) Given $$y(n) = x(n-1)\, u(n)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n-1)\, u(n)\, z^{-n} = \sum_{n=0}^{\infty} x(n-1)\, z^{-n}$$

Let $$n - 1 = k$$

$$\therefore \qquad n = k + 1$$

$$\therefore \qquad Y(z) = \sum_{k=-1}^{\infty} x(k)\, z^{-(k+1)} = z^{-1} \sum_{k=-1}^{\infty} x(k)\, z^{-k}$$
$$= z^{-1}\left[\sum_{k=0}^{\infty} x(k)\, z^{-k} + x(-1)\, z\right]$$
$$= z^{-1} \sum_{k=0}^{\infty} x(k)\, z^{-k} + x(-1)$$
$$= z^{-1} X(z) + x(-1)$$

$$\therefore \qquad Z[x(n-1)\, u(n)] = z^{-1} X(z) + x(-1)$$

(f) Given $$y(n) = x(n+1)\, u(n)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n+1)\, u(n)\, z^{-n} = \sum_{n=0}^{\infty} x(n+1)\, z^{-n}$$

Let $$n + 1 = k$$

$$\therefore \qquad n = k - 1$$

$$\therefore \qquad Y(z) = \sum_{k=1}^{\infty} x(k)\, z^{-(k-1)} = z \sum_{k=1}^{\infty} x(k)\, z^{-k}$$
$$= z\left[\sum_{k=0}^{\infty} x(k)\, z^{-k} - x(0) z^{-0}\right] = z \sum_{k=0}^{\infty} x(k)\, z^{-k} - zx(0)$$
$$= zX(z) - zx(0)$$

$$\therefore \qquad Z[x(n+1)\, u(n)] = zX(z) - zx(0)$$

EXAMPLE 10.6 Prove that the sequences

(a) $x(n) = a^n u(n)$ and (b) $x(n) = -a^n u(-n-1)$

have the same $X(z)$ and differ only in ROC. Also plot their ROCs.

Solution:

(a) The given sequence $a^n u(n)$ is a causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases} \quad \text{because } u(n) = \begin{cases} 1 & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$\therefore \quad Z[x(n)] = Z[a^n u(n)] = \sum_{n=0}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} [az^{-1}]^n = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

This is a geometric series of infinite length, and converges if $|az^{-1}| < 1$, i.e. if $|z| > |a|$.

$$\therefore \quad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}; \text{ROC; } |z| > |a|$$

which implies that the ROC is exterior to the circle of radius a as shown in Figure 10.1(a)

$$\boxed{a^n u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1 - az^{-1}} = \frac{z}{z-a}; \text{ ROC; } |z| > a}$$

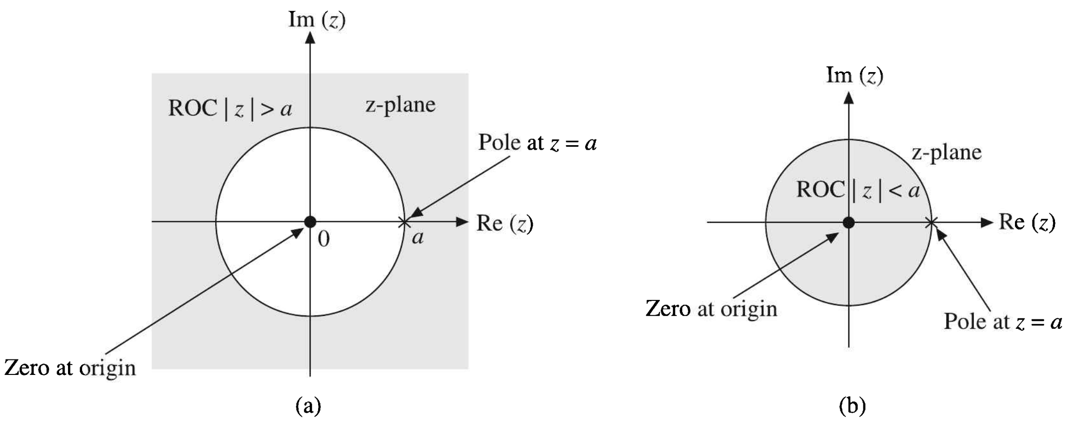


Figure 10.1 (a) ROC of $a^n u(n)$, (b) ROC of $-a^n u(-n-1)$.

(b) The given signal $x(n) = -a^n u(-n - 1)$ is a non-causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} -a^n, & n \le -1 \\ 0, & n \ge 0 \end{cases} \quad \text{because } u(-n-1) = \begin{cases} 1 & \text{for } n \le -1 \\ 0 & \text{for } n \ge 0 \end{cases}$$

$$\therefore \quad X(z) = Z[-a^n u(-n-1)] = \sum_{n=-\infty}^{\infty} -a^n u(-n-1)\, z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} -a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1} z)^n$$

This is a geometric series of infinite length and converges if $|a^{-1}z| < 1$ or $|z| < |a|$. Hence

$$X(z) = -\left[\sum_{n=0}^{\infty} (a^{-1}z)^n - 1\right] = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1 - a^{-1}z} = -\frac{a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}; \text{ ROC; } |z| < |a|$$

That is, the ROC is the interior of the circle of radius a as shown in Figure 10.1(b).

From this example, we can observe that the Z-transform of the sequences $a^n u(n)$ and $-a^n u(-n - 1)$ are same, even though the sequences are different. Only ROC differentiates them. Therefore, to find the correct inverse Z-transform, it is essential to know the ROC. The ROCs are shown in Figure 10.1[(a) and (b)].

In general, the ROC of a causal signal is $|z| > a$ and the ROC of a non-causal signal is $|z| < a$, where a is some constant.

EXAMPLE 10.7 Find the Z-transform of the following sequences:

(a) $y(n) = x(n-2)\, u(n)$ (b) $y(n) = x(n+2)\, u(n)$

Solution:

(a) Given $y(n) = x(n-2)\, u(n)$

$$\therefore \quad Y(z) = Z[y(n)] = Z[x(n-2)\, u(n)] = \sum_{n=0}^{\infty} x(n-2)\, u(n)\, z^{-n} = \sum_{n=0}^{\infty} x(n-2)\, z^{-n}$$

Let $n - 2 = p$, i.e. $n = p + 2$

$$\therefore \quad Y(z) = \sum_{p=-2}^{\infty} x(p)\, z^{-(p+2)} = z^{-2}\left[\sum_{p=-2}^{\infty} x(p)\, z^{-p}\right]$$

$$= z^{-2}\left[x(-2)\, z^2 + x(-1)\, z^1 + \sum_{p=0}^{\infty} x(p)\, z^{-p}\right]$$

$$= z^{-2} X(z) + z^{-1} x(-1) + x(-2)$$

(b) Given $$y(n) = x(n+2)\,u(n)$$

$$\therefore \quad Y(z) = \text{Z}[y(n)] = \text{Z}\big[x(n+2)\,u(n)\big] = \sum_{n=0}^{\infty} x(n+2)\,u(n)\,z^{-n} = \sum_{n=0}^{\infty} x(n+2)\,z^{-n}$$

Let $n + 2 = p$, i.e. $n = p - 2$

$$\therefore \quad Y(z) = \sum_{p=2}^{\infty} x(p)\,z^{-(p-2)} = z^2\left[\sum_{p=2}^{\infty} x(p)\,z^{-p}\right]$$

$$= z^2\left[\sum_{p=0}^{\infty} x(p)\,z^{-p} - x(0) - x(1)\,z^{-1}\right]$$

$$= z^2 X(z) - z^2 x(0) - zx(1)$$

EXAMPLE 10.8 Find the Z-transform and ROC of $x(n) = a^n\,u(-n)$.

Solution: The given sequence is a non-causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} a^n, & n \le 0 \\ 0, & n > 0 \end{cases}$$

$$\therefore \quad \text{Z}[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} a^n\,u(-n)\,z^{-n} = \sum_{n=-\infty}^{0} a^n z^{-n} = \sum_{n=0}^{\infty} (a^{-1}z)^n$$

The above series converges if $|a^{-1}z| < 1$ or $|z| < a$.

$$\therefore \quad X(z) = \frac{1}{1 - a^{-1}z} = \frac{a}{a - z}; \text{ ROC; } |z| < a$$

EXAMPLE 10.9 Determine the Z-transform and ROC of

$$x(n) = a^n u(n) - b^n u(-n-1)$$

Solution: The given sequence is a two-sided infinite duration sequence.

$$\therefore \quad \text{Z}[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(-n-1)]\,z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} [az^{-1}]^n - \sum_{n=1}^{\infty} [b^{-1}z]^n$$

The first series converges if $|az^{-1}| < 1$ or $|z| > |a|$ and the second series converges if $|b^{-1}z| < 1$ or $|z| < |b|$. If $|b| < |a|$, the two ROCs do not overlap as shown in Figure 10.2(a) and the conditions $|z| > |a|$ and $|z| < |b|$ cannot be satisfied simultaneously, so the Z-transform $X(z)$ does not exist.

If $|b| > |a|$, the two ROCs overlap as shown in Figure 10.2(b) and the conditions $|z| > |a|$ and $|z| < |b|$ can be satisfied simultaneously, so $X(z)$ exists. Therefore, the ROC of $X(z)$ is $|a| < |z| < |b|$. This implies that *for an infinite duration two-sided signal, the ROC is a ring in the* z*-plane.*

$$\therefore \qquad X(z) = \frac{z}{z-a} + \frac{z}{z-b}; \text{ ROC; } |a| < |z| < |b|$$

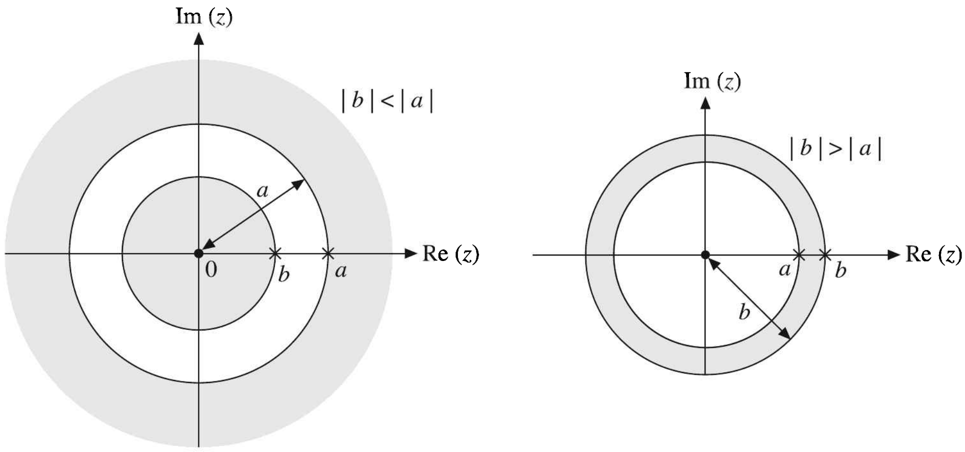


Figure 10.2 ROC of two-sided sequence for (a) $|b| < |a|$, and (b) $|b| > |a|$.

From the above discussion, the following conclusions can be drawn:

1. The ROC of a causal sequence is $|z| > |a|$, i.e. it is the exterior of a circle of radius a, where $z = a$ is the largest pole in $X(z)$.
2. The ROC of a non-causal sequence is $|z| < |a|$, i.e. it is the interior of a circle of radius a, where $z = a$ is the smallest pole in $X(z)$.
3. The ROC of a two-sided sequence is a ring in z-plane or the Z-transform does not exist at all.
4. The ROC of the sum of two or more sequences is equal to the intersection of the ROCs of those sequences.

EXAMPLE 10.10 Find the Z-transform and ROC of $X(z)$ for

$$x(n) = 3\left(\frac{5}{7}\right)^n u(n) + 2\left(-\frac{1}{3}\right)^n u(n)$$

Also find the pole-zero location.

Solution: Given $$x(n) = 3\left(\frac{5}{7}\right)^n u(n) + 2\left(-\frac{1}{3}\right)^n u(n)$$

We have $$X(z)=\sum_{n=-\infty}^{\infty} x(n)\,z^{-n}=\sum_{n=-\infty}^{\infty}\left[3\left(\frac{5}{7}\right)^n u(n)+2\left(-\frac{1}{3}\right)^n u(n)\right]z^{-n}$$

$$=\sum_{n=0}^{\infty}\left[3\left(\frac{5}{7}\right)^n z^{-n}+2\left(-\frac{1}{3}\right)^n z^{-n}\right]=3\sum_{n=0}^{\infty}\left(\frac{5}{7}\right)^n z^{-n}+2\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n}$$

$$=3\sum_{n=0}^{\infty}\left(\frac{5}{7}z^{-1}\right)^n+2\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n$$

The first summation converges for $\left|\frac{5}{7}z^{-1}\right|<1$ or $|z|>\frac{5}{7}$ and the second summation converges for $\left|-\frac{1}{3}z^{-1}\right|<1$ or $|z|>\frac{1}{3}$.

Therefore, $X(z)$ converges for $|z| > 5/7$.

Now, $$X(z)=3\frac{1}{1-(5/7)z^{-1}}+2\frac{1}{1+(1/3)z^{-1}}=3\frac{z}{z-(5/7)}+2\frac{z}{z+(1/3)}$$

$$=\frac{3z[z+(1/3)]+2z[z-(5/7)]}{[z-(5/7)][z+(1/3)]}=\frac{5z^2-(3/7)z}{[z-(5/7)][z+(1/3)]}$$

$$=\frac{5z^2-(3/7)z}{[z-(5/7)][z+(1/3)])}=\frac{5z[z-(3/35)]}{[z-(5/7)][z+(1/3)]}$$

The poles of $X(z)$ are at $z = (5/7)$ and $z = -(1/3)$. So the ROC is $|z| > (5/7)$, i.e. the ROC is the exterior of the circle with radius 5/7. The zeros are at $z = 0$ and $z = (3/35)$. The pole-zero location is shown in Figure 10.3.

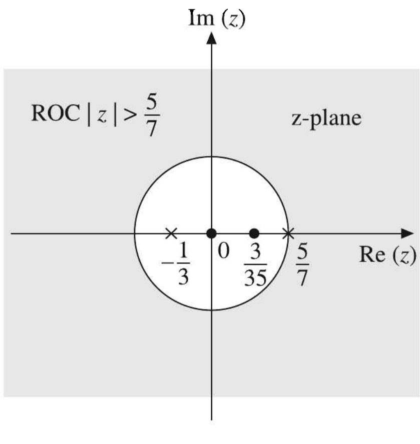


Figure 10.3 Pole-zero location and ROC for Example 10.10.

EXAMPLE 10.11 Determine the Z-transform and ROC of

$$x(n) = \left(\frac{1}{2}\right)^n u(-n) - 2^n u(-n-1)$$

Also indicate the pole-zero locations.

Solution: Given
$$x(n) = \left(\frac{1}{2}\right)^n u(-n) - 2^n u(-n-1)$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n u(-n) - 2^n u(-n-1)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} -(2)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{-n} z^n - \sum_{n=1}^{\infty} 2^{-n} z^n$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{-1} z\right]^n - \sum_{n=1}^{\infty} (2^{-1} z)^n$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{-1} z\right]^n - \left[-1 + \sum_{n=0}^{\infty} (2^{-1} z)^n\right]$$

The first series converges if $\left|\left(\frac{1}{2}\right)^{-1} z\right| < 1$ or $|z| < \frac{1}{2}$ and the second series converges if $|2^{-1}z| < 1$ or $|z| < 2$. So $X(z)$ converges for $|z| < 1/2$ and the ROC is the interior of the circle of radius 1/2. The ROC is $|z| < 1/2$.

$$\therefore \qquad X(z) = \frac{1}{1-(1/2)^{-1}z} + 1 - \frac{1}{1-2^{-1}z}$$

$$= \frac{1}{1-(1/2)^{-1}z} - \frac{2^{-1}z}{1-2^{-1}z}$$

$$= -\frac{0.5}{z-0.5} + \frac{z}{z-2} = \frac{z^2 - z + 1}{(z-0.5)(z-2)}$$

$$= \frac{(z-0.5-j\sqrt{0.75})(z-0.5+j\sqrt{0.75})}{(z-0.5)(z-2)}$$

The ROC and the pole-zero locations are shown in Figure 10.4.

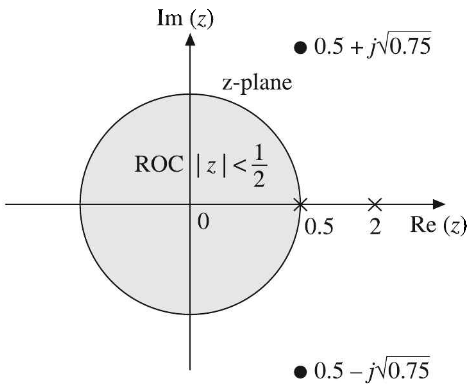


Figure 10.4 Pole-zero location and ROC for Example 10.11.

EXAMPLE 10.12 Find the Z-transform of the sequence

$$x(n) = \left(\frac{1}{4}\right)^n \cos\left(\frac{\pi}{3}n\right) u(n)$$

Also sketch the ROC and pole-zero location.

Solution: Given $x(n) = \left(\frac{1}{4}\right)^n \cos\left(\frac{\pi}{3}n\right) u(n) = \left(\frac{1}{4}\right)^n \left[\frac{e^{j(\pi/3)n} + e^{-j(\pi/3)n}}{2}\right] u(n)$

$$= \frac{1}{2}\left[\left(\frac{1}{4}e^{j(\pi/3)}\right)^n + \left(\frac{1}{4}e^{-j(\pi/3)}\right)^n\right] u(n)$$

We have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[\left(\frac{1}{4}e^{j(\pi/3)}\right)^n + \left(\frac{1}{4}e^{-j(\pi/3)}\right)^n\right] u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{4}e^{j(\pi/3)}\right)^n z^{-n} + \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{4}e^{-j(\pi/3)}\right)^n z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{1}{4}e^{j(\pi/3)} z^{-1}\right]^n + \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{1}{4}e^{-j(\pi/3)} z^{-1}\right]^n$$

$$= \frac{1}{2}\frac{1}{1-(1/4)e^{j(\pi/3)}\, z^{-1}} + \frac{1}{2}\frac{1}{1-(1/4)\, e^{-j(\pi/3)}\, z^{-1}}$$

The first series converges if $\left|\frac{1}{4}z^{-1}e^{j(\pi/3)}\right| < 1$, i.e. $|z| > \frac{1}{4}$ and the second series converges if $\left|\frac{1}{4}z^{-1}e^{-j(\pi/3)}\right| < 1$, i.e. $|z| > \frac{1}{4}$. So the ROC is $|z| > \frac{1}{4}$.

$$\therefore \quad X(z) = \frac{1}{2}\frac{z}{z-(1/4)\,e^{j(\pi/3)}} + \frac{1}{2}\frac{z}{z-(1/4)\,e^{-j(\pi/3)}}$$

$$= \frac{z[z-(1/4)\cos(\pi/3)]}{[z-(1/4)\,e^{j(\pi/3)}]\,[z-(1/4)\,e^{-j(\pi/3)}]} = \frac{z[z-(1/8)]}{[z-(1/8)-j(\sqrt{3}/8)]\,[z-(1/8)+j(\sqrt{3}/8)]}$$

The poles are at $z = (1/4)\,e^{j(\pi/3)}$ and $z = (1/4)\,e^{-j(\pi/3)}$ and the zeros are at $z = 0$ and $z = (1/8)$. The pole-zero location and the ROC are sketched in Figure 10.5.

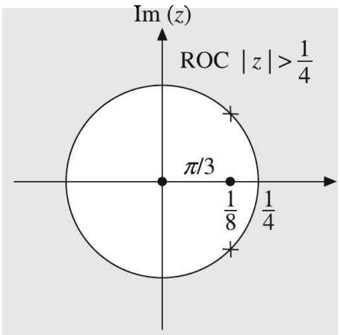


Figure 10.5 Pole-zero location and ROC for Example 10.12.

EXAMPLE 10.13 Find the Z-transform and ROC of

$$x(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{2}\right)^{2n} u(n)$$

Sketch the ROC and pole-zero location.

Solution: Given $x(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{2}\right)^{2n} u(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{4}\right)^n u(n)$

We have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n} = \sum_{n=-\infty}^{\infty}\left[2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{4}\right)^n u(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2\left(\frac{5}{6}\right)^n u(-n-1)\,z^{-n} + \sum_{n=-\infty}^{\infty} 3\left(\frac{1}{4}\right)^n u(n)\,z^{-n} = \sum_{n=-\infty}^{-1} 2\left(\frac{5}{6}\right)^n z^{-n} + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n z^{-n}$$

$$= \sum_{n=1}^{\infty} 2\left[\left(\frac{5}{6}\right)^{-1} z\right]^n + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}z^{-1}\right)^n$$

The first series converges if $(5/6)^{-1}\, z < 1$ or $z < (5/6)$ and the second series converges if $(1/4)\, z^{-1} < 1$ or $z > (1/4)$.

So the region of convergence for $X(z)$ is $(1/4) < z < (5/6)$, i.e. it is a ring with $(1/4) < z < (5/6)$.

$$\therefore \qquad X(z) = 2\left\{-1 + \sum_{n=0}^{\infty}\left[\left(\frac{5}{6}\right)^{-1} z\right]^n\right\} + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}z^{-1}\right)^n$$

$$= \left\{2\left[-1 + \frac{1}{1-(5/6)^{-1}z}\right] + 3\frac{1}{1-(1/4)z^{-1}}\right\} = -2\frac{z}{z-(5/6)} + 3\frac{z}{z-(1/4)}$$

$$= \frac{-2z^2 + (z/2) + 3z^2 - (5z/2)}{[z-(5/6)]\,[z-(1/4)]} = \frac{z(z-2)}{[z-(5/6)]\,[z-(1/4)]}$$

The ROC and the pole-zero plot are shown in Figure 10.6.

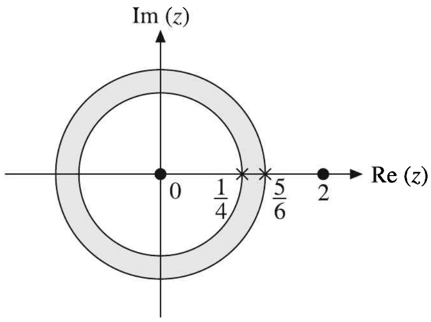


Figure 10.6 Pole-zero plot and ROC for Example 10.13.

EXAMPLE 10.14 Find the Z-transform and ROC of

$$x(n) = a^{|n|};\ |a| < 1$$

Solution: Given $\quad x(n) = a^{|n|};\ |a| < 1$

We have

$$X(z) = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} a^n z^n + \sum_{n=0}^{\infty} a^n z^{-n} = -1 + \sum_{n=0}^{\infty} (az)^n + \sum_{n=0}^{\infty} (az^{-1})^n$$

The first series $(az)^n$ converges for $az < 1$, i.e. if $|z| < |1/a|$ and the second series converges if $az^{-1} < 1$, i.e. if $|z| > |a|$. So $X(z)$ converges if $|a| < |z| < |1/a|$.

$$\therefore \qquad X(z) = -1 + \frac{1}{1-az} + \frac{1}{1-az^{-1}} = \frac{az}{1-az} + \frac{z}{z-a}$$

$$= -\frac{z}{z-(1/a)} + \frac{z}{z-a} = \frac{z[a-(1/a)]}{(z-a)[z-(1/a)]}$$

$$\therefore \qquad X(z) = \frac{z[a-(1/a)]}{[z-(1/a)](z-a)}; \text{ ROC; } |a| < |z| < \left|\frac{1}{a}\right|$$

Alternatively

$$a^{|n|} = a^n\, u(n) + a^{-n}\, u(-n-1) = a^n\, u(n) + \left(\frac{1}{a}\right)^n u(-n-1)$$

$$\therefore \qquad \text{Z}\,[a^{|n|}] = \text{Z}[a^n\, u(n)] + \text{Z}\left[\left(\frac{1}{a}\right)^n u(-n-1)\right]$$

$$= \frac{z}{z-a} - \frac{z}{z-(1/a)} = \frac{z\,[a-(1/a)]}{(z-a)\,[z-z-(1/a)]}; \text{ ROC; } |a| < |z| < \left|\frac{1}{a}\right|$$

EXAMPLE 10.15 Find the Z-transform of the following sequences:

(a) $x(n) = (a^n + a^{-n})\, u(n)$; where a is real

(b) $x(n) = (-1)^n\, 3^{-n}\, u(n)$

(c) $x(n) = (1 + 2^n + 3^n)\, u(n)$

(d) $x(n) = 2(n-1)(1/2)^n\, u(n-1)$

(e) $x(n) = \begin{cases} 2^n, & n \le 0 \\ 0, & n > 0 \end{cases}$

Solution:

(a) The given sequence is:

$$x(n) = (a^n + a^{-n})\, u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} (a^n + a^{-n})\, u(n)\, z^{-n} = \sum_{n=0}^{\infty} (a^n + a^{-n})\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} a^{-n} z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=0}^{\infty} (a^{-1}z^{-1})^n$$

The first series converges if $|az^{-1}| < 1$, i.e. if $|z| > |a|$ and the second series converges if $|a^{-1}z^{-1}| < 1$, i.e. if $|z| > |1/a|$.

$$\therefore \qquad X(z) \text{ converges if } |z| > \text{Max}\left[|a|, \left|\frac{1}{a}\right|\right]$$

$$\therefore \qquad X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-a^{-1}z^{-1}} = \frac{z}{z-a} + \frac{z}{z-(1/a)}$$

$$= \frac{z\{2z-[a+(1/a)]\}}{(z-a)[z-(1/a)]}; \text{ ROC; } |z| > \text{Max}\left(|a|, \frac{1}{|a|}\right)$$

(b) Given $\qquad x(n) = (-1)^n\, 3^{-n}\, u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} (-1)^n\, 3^{-n}\, u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} [-1(3)^{-1}]^n\, z^{-n} = \sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n$$

The above series converges if

$$\left|-\frac{1}{3}z^{-1}\right| < 1, \text{ i.e. if } \left|\frac{1}{3}z^{-1}\right| < 1, \text{ i.e. if } |z| > \frac{1}{3}$$

$$\therefore \qquad X(z) = \frac{1}{1-[-(1/3)\, z^{-1}]} = \frac{1}{1+(1/3)\, z^{-1}} = \frac{z}{z+(1/3)}; \text{ ROC; } |z| > \frac{1}{3}$$

(c) Given $\qquad x(n) = (1 + 2^n + 3^n)\, u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} (1+2^n+3^n)\, u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n} + \sum_{n=0}^{\infty} 3^n z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n + \sum_{n=0}^{\infty} (2z^{-1})^n + \sum_{n=0}^{\infty} (3z^{-1})^n$$

The first series converges if $|z^{-1}| < 1$, i.e. if $|z| > 1$.

The second series converges if $|2z^{-1}| < 1$, i.e. if $|z| > 2$.

The third series converges if $|3z^{-1}| < 1$, i.e. if $|z| > 3$.

So $X(z)$ converges if $|z| > 3$, i.e. the ROC is $|z| > 3$.

$$\therefore \qquad X(z) = \frac{1}{1-z^{-1}} + \frac{1}{1-2z^{-1}} + \frac{1}{1-3z^{-1}}$$

$$= \frac{z}{z-1} + \frac{z}{z-2} + \frac{z}{z-3}; \text{ ROC; } |z| > 3$$

(d) The given sequence is:

$$x(n) = 2(n-1)\left(\frac{1}{2}\right)^n u(n-1)$$

$$\therefore \quad X(z) = \sum_{n=1}^{\infty} 2(n-1)\left(\frac{1}{2}\right)^n u(n-1)\, z^{-n} = \sum_{n=1}^{\infty} 2(n-1)\,(0.5\, z^{-1})^n$$

$$= 2\,[0 + (0.5z^{-1})^2 + 2(0.5z^{-1})^3 + 3(0.5z^{-1})^4 + 4(0.5z^{-1})^5 + \cdots]$$

$$= 2(0.5z^{-1})^2\,[1 + 2(0.5z^{-1}) + 3(0.5z^{-1})^2 + 4(0.5z^{-1})^3 + \cdots]$$

$$= 0.5z^{-2}\,[(1 - 0.5z^{-1})^{-2}] = \frac{0.5z^{-2}}{(1 - 0.5z^{-1})^2}$$

$$\therefore \quad X(z) = \frac{0.5}{(z - 0.5)^2}; \text{ ROC; } |z| > 0.5$$

(e) The given sequence is:

$$x(n) = \begin{cases} 2^n, & n \le 0 \\ 0, & n > 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{0} 2^n z^{-n} = \sum_{n=0}^{\infty} 2^{-n} z^n$$

$$= \sum_{n=0}^{\infty} (2^{-1}z)^n = \frac{1}{1 - 2^{-1}z} = \frac{2}{2 - z} \text{ ROC; } |z| < 2$$

(because for the above series to converge $|2^{-1} z| < 1$, i.e. $|z| < 2$.)

EXAMPLE 10.16 Determine the Z-transform and ROC of the signal

$$x(n) = \left(\frac{1}{3}\right)^n [u(n) - u(n-6)]$$

Sketch the ROC, poles and zeros in the z-plane.

Solution: Given

$$x(n) = \left(\frac{1}{3}\right)^n [u(n) - u(n-6)]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n [u(n) - u(n-6)]\, z^{-n}$$

$$= \sum_{n=0}^{5} \left(\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{5} \left(\frac{1}{3} z^{-1}\right)^n$$

$$= \frac{1 - [(1/3)\, z^{-1}]^6}{1 - (1/3)z^{-1}} = \frac{1}{z^5}\, \frac{z^6 - (1/3)^6}{z - (1/3)} \qquad \left[\begin{aligned} \sum_{n=0}^{N-1} \alpha^n &= \frac{1 - \alpha^N}{1 - \alpha} && \text{if } \alpha \ne 1 \\ &= N && \text{if } \alpha = 1 \end{aligned}\right]$$

The given sequence is a finite duration right-sided sequence. So ROC is all z except z = 0. The ROC and pole-zero locations are shown in Figure 10.7.

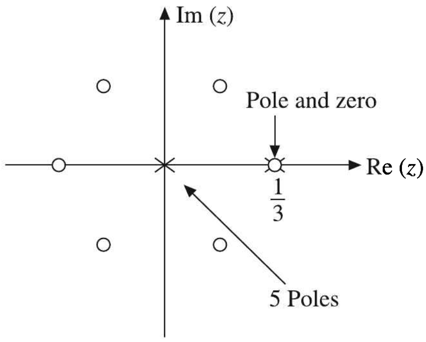


Figure 10.7 Pole-zero location and ROC for Example 10.16.

EXAMPLE 10.17 Find the Z-transform and ROC of

$$x(n) = \left(\frac{1}{2}\right)^n u(n-2)$$

Sketch the pole-zero location and ROC.

Solution: Given $$x(n) = \left(\frac{1}{2}\right)^n u(n-2)$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n-2)\, z^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n$$

The above series converges if $\left|\frac{1}{2}z^{-1}\right| < 1$, i.e. if $|z| > \frac{1}{2}$

So, the ROC of $X(z)$ is $|z| > \frac{1}{2}$.

$$\therefore \qquad X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n - \sum_{n=0}^{1} \left(\frac{1}{2}z^{-1}\right)^n = \frac{1}{1-(1/2)\,z^{-1}} - \left(1+\frac{1}{2}z^{-1}\right)$$

$$= \frac{1-[1-(1/4)\,z^{-2}]}{1-(1/2)z^{-1}} = \frac{[(1/2)z^{-1}]^2}{1-(1/2)z^{-1}}$$

$$= \frac{1}{4z[z-(1/2)]}; \text{ ROC; } |z| > \frac{1}{2}$$

The ROC and pole-zero locations are shown in Figure 10.8.

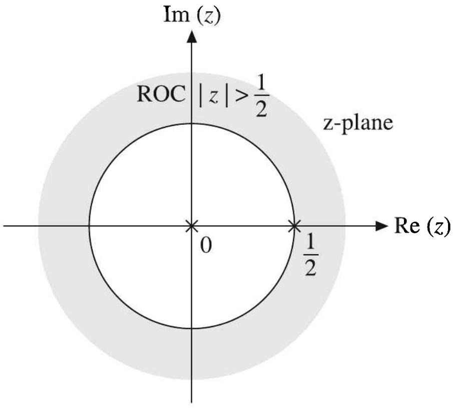


Figure 10.8 Pole-zero location and ROC for Example 10.17.

EXAMPLE 10.18 Find the Z-transform of the following:

(a) $x(n) = \left(\frac{1}{4}\right)^n u(n) - \cos\left(\frac{n\pi}{4}\right) u(n)$

(b) $x(n) = a^n \sin(n\omega)\, u(n)$

(c) $x(n) = a^n \cos\frac{n\pi}{2}$

(d) $x(n) = a^{-n} u(-n-1)$

(e) $x(n) = u(-n)$

(f) $x(n) = \frac{b^n}{n!}$ for $n \ge 0$

(g) $x(n) = \left\{\begin{matrix} 2, 1, 3, 5, 0, 7 \\ \uparrow \end{matrix}\right\}$

(h) $x(n) = u(-n+1)$

(i) $x(n) = u(-n-2)$

(j) $x(n) = 2^n u(n-2)$

Solution:

(a) Given $x(n) = \left(\frac{1}{4}\right)^n u(n) - \cos\left(\frac{n\pi}{4}\right) u(n)$

$$Z[x(n)] = X(z) = Z\left[\left(\frac{1}{4}\right)^n u(n) - \cos\left(\frac{n\pi}{4}\right) u(n)\right]$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{4}\right)^n u(n) - \cos\left(\frac{n\pi}{4}\right) u(n)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} - \sum_{n=0}^{\infty} \left[\frac{e^{j(n\pi/4)} + e^{-j(n\pi/4)}}{2}\right] z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{4}z^{-1}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}[e^{(j\pi/4)}z^{-1}]^n - \frac{1}{2}\sum_{n=0}^{\infty}[e^{-(j\pi/4)}z^{-1}]^n$$

$$= \frac{1}{1-(1/4)z^{-1}} - \frac{1}{2}\frac{1}{1-e^{(j\pi/4)}z^{-1}} - \frac{1}{2}\frac{1}{1-e^{-(j\pi/4)}z^{-1}}$$

$$= \frac{z}{z-(1/4)} - \frac{1}{2}\left[\frac{z}{z-e^{(j\pi/4)}} + \frac{z}{z-e^{(-j\pi/4)}}\right]$$

$$= \frac{z}{z-(1/4)} - \frac{1}{2}\frac{z[2z-(e^{(j\pi/4)}+e^{-(j\pi/4)})]}{z^2-z(e^{(j\pi/4)}+e^{(-j\pi/4)})+1}$$

$$= \frac{z}{z-(1/4)} - \frac{z[z-\cos(\pi/4)]}{z^2-2z\cos(\pi/4)+1}$$

(b) Given $$x(n) = a^n \sin(n\omega)\, u(n)$$

$$\mathrm{Z}[x(n)] = X(z) = \mathrm{Z}\left[a^n\left(\frac{e^{j\omega n}-e^{-j\omega n}}{2j}\right)\right]$$

$$= \sum_{n=0}^{\infty} a^n\left(\frac{e^{j\omega n}-e^{-j\omega n}}{2j}\right)z^{-n} = \frac{1}{2j}\left[\sum_{n=0}^{\infty}(ae^{j\omega}z^{-1})^n - \sum_{n=0}^{\infty}(ae^{-j\omega}z^{-1})^n\right]$$

$$= \frac{1}{2j}\left(\frac{1}{1-ae^{j\omega}z^{-1}} - \frac{1}{1-ae^{-j\omega}z^{-1}}\right) = \frac{1}{2j}\left(\frac{z}{z-ae^{j\omega}} - \frac{z}{z-ae^{-j\omega}}\right)$$

$$= \frac{1}{2j}\left[\frac{z(z-ae^{-j\omega}-z+ae^{j\omega})}{(z-ae^{j\omega})(z-ae^{-j\omega})}\right] = \frac{1}{2j}\left[\frac{za(e^{j\omega}-e^{-j\omega})}{z^2-2za\cos\omega+a^2}\right]$$

$$= \frac{za\sin\omega}{z^2-2za\cos\omega+a^2}$$

(c) Given $$x(n) = a^n\cos\frac{n\pi}{2} = a^n\,\frac{e^{j(n\pi/2)}+e^{-j(n\pi/2)}}{2}\,u(n)$$

$$\therefore \quad \mathrm{Z}[x(n)] = \mathrm{Z}\left(a^n\cos\frac{n\pi}{2}\right) = \mathrm{Z}\left[a^n\left[\frac{e^{j(n\pi/2)}+e^{-j(n\pi/2)}}{2}\right]u(n)\right]$$

$$= \sum_{n=0}^{\infty} a^n\left[\frac{e^{j(n\pi/2)}+e^{-j(n\pi/2)}}{2}\right]z^{-n}$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{\infty}[ae^{j(\pi/2)}z^{-1}]^n + \sum_{n=0}^{\infty}[ae^{-j(\pi/2)}z^{-1}]^n\right\}$$

$$= \frac{1}{2}\left[\frac{1}{1 - ae^{(j\pi/2)}\,z^{-1}} + \frac{1}{1 - ae^{(-j\pi/2)}\,z^{-1}}\right] = \frac{1}{2}\left(\frac{z}{z - ae^{j\pi/2}} + \frac{z}{z - ae^{-j\pi/2}}\right)$$

$$= \frac{1}{2}\,\frac{z(z - ae^{-j\pi/2} + z - ae^{j\pi/2})}{z^2 - za\,(e^{j\pi/2} + e^{-j\pi/2}) + a^2} = \frac{1}{2}\,\frac{z\{2z - 2a\,[(e^{j\pi/2} + e^{-j\pi/2})/2]\}}{z^2 - 2za\,[(e^{j\pi/2} + e^{-j\pi/2})/2] + a^2}$$

$$= \frac{z[z - a\cos(\pi/2)]}{z^2 - 2az\cos(\pi/2) + a^2} = \frac{z^2}{z^2 + a^2}$$

(d) Given $$x(n) = a^{-n}\,u(-n-1)$$

$$Z[x(n)] = X(z) = Z\,[a^{-n}\,u(-n-1)] = \sum_{n=-\infty}^{\infty} a^{-n}\,u(-n-1)\,z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}z^{-n} = \sum_{n=-\infty}^{-1} (az)^{-n} = \sum_{n=1}^{\infty} (az)^{n}$$

$$= \sum_{n=0}^{\infty} (az)^{n} - 1 = \frac{1}{1 - az} - 1 = \frac{az}{1 - az} = \frac{z}{(1/a) - z} = -\frac{z}{z - (1/a)}$$

i.e. $$Z[a^{-n}\,u(-n-1)] = -\frac{z}{z - (1/a)}$$

(e) Given $$x(n) = u(-n)$$

$$Z\,[x(n)] = X(z) = Z\,[u(-n)] = \sum_{n=-\infty}^{\infty} u(-n)\,z^{-n}$$

$$= \sum_{n=-\infty}^{0} 1z^{-n} = \sum_{n=0}^{\infty} z^{n} = \frac{1}{1 - z} \quad \text{or} \quad -\frac{1}{z - 1}$$

i.e. $$Z[u(-n)] = \frac{1}{1 - z}$$

(f) Given $$x(n) = \frac{b^n}{n!} \quad \text{for } n \geq 0$$

From the definition of Z-transform of $x(n)$, we can write

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n} = \sum_{n=0}^{\infty} \frac{b^n}{n!}\,z^{-n} = \sum_{n=0}^{\infty} \frac{(bz^{-1})^n}{n!}$$

We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\therefore \qquad X(z) = e^{bz^{-1}}$$

i.e. $$Z\left(\frac{b^n}{n!}\right) \text{ for } n \geq 0 \text{ is } e^{bz^{-1}}$$

(g) Given $$x(n) = \left\{\begin{matrix} 2, 1, 3, 5, 0, 7 \\ \uparrow \end{matrix}\right\}$$

We have unilateral Z-transform

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

The given sequence $x(n)$ exists for both positive and negative values of n. But unilateral Z-transform is defined only for non-negative values of n.

Unilateral Z-transform is:

$$X(z) = 3 + 5z^{-1} + 7z^{-3}$$

We have bilateral Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Bilateral Z-transform is:

$$X(z) = 2z^2 + z + 3 + 5z^{-1} + 7z^{-3}$$

(h) Given $$x(n) = u(-n+1)$$

From the definition of Z-transform, we have

$$X(z) = \text{Z}\,[u(-n+1)] = \sum_{n=-\infty}^{\infty} u(-n+1)\, z^{-n} = \sum_{n=-\infty}^{1} z^{-n} = \sum_{n=-1}^{\infty} z^{n} = z^{n}\Big|_{n=-1} + \sum_{n=0}^{\infty} z^{n}$$

$$= z^{-1} + \frac{1}{1-z} = \frac{z^{-1}}{1-z} = -\frac{1}{z(z-1)}; \text{ ROC; } 0 < |z| < 1$$

(i) Given $$x(n) = u(-n-2)$$

From the definition of Z-transform, we have

$$\text{Z}[u(-n-2)] = \sum_{n=-\infty}^{\infty} u(-n-2)\, z^{-n} = \sum_{n=-\infty}^{-2} z^{-n} = \sum_{n=2}^{\infty} z^{n}$$

$$= \sum_{n=0}^{\infty} z^{n} - \sum_{n=0}^{1} z^{n} = \frac{1}{1-z} - 1 - z = \frac{z^2}{1-z}; \text{ ROC; } |z| < 1$$

(j) Given $$x(n) = 2^n\, u(n-2)$$

$$X(z) = Z[x(n)] = \sum_{n=0}^{\infty} 2^n\, u(n-2)\, z^{-n} = \sum_{n=2}^{\infty} (2z^{-1})^n = \sum_{n=0}^{\infty} (2z^{-1})^n - \sum_{n=0}^{1} (2z^{-1})^n$$

$$= \frac{1}{1-2z^{-1}} - (2z^{-1})^0 - (2z^{-1})^1 = \frac{z}{z-2} - 1 - \frac{2}{z} = \frac{4}{z(z-2)}$$

EXAMPLE 10.19 Consider the sequence

$$x(n) = \begin{cases} a^n & 0 \le n \le N-1,\ a < 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $X(z)$.

Solution: $$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1-(a/z)^N}{1-(a/z)} = \frac{(z^N - a^N)/z^N}{(z-a)/z} = \frac{z^N - a^N}{z^{N-1}(z-a)}$$

$\therefore$ $$X(z) = \frac{1}{z^{N-1}}\, \frac{z^N - a^N}{z-a}$$

EXAMPLE 10.20 For the given signal as under:

(a) Determine the parameter values for which Z-transform will exist.

(b) Find the Z-transform.

(c) Plot the ROC.

$$x(n) = -b^n\, u(-n-1) + (0.5)^n\, u(n)$$

Solution: Given $$x(n) = -b^n\, u(-n-1) + (0.5)^n\, u(n)$$

$$Z\,[x(n)] = X(z) = Z\,[-b^n\, u(-n-1) + (0.5)^n\, u(n)]$$

$$= \sum_{n=-\infty}^{\infty} [-b^n\, u(-n-1) + (0.5)^n\, u(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -b^n u(-n-1)\, z^{-n} + \sum_{n=-\infty}^{\infty} (0.5)^n\, u(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -b^n z^{-n} + \sum_{n=0}^{\infty} (0.5)^n\, z^{-n}$$

$$= -\sum_{n=1}^{\infty} b^{-n} z^n + \sum_{n=0}^{\infty} (0.5z^{-1})^n = -\sum_{n=1}^{\infty} (b^{-1}z)^n + \sum_{n=0}^{\infty} (0.5z^{-1})^n$$

The first summation converges for $b^{-1}z < 1$, i.e. for $|z| < b$ and the second summation converges for $0.5z^{-1} < 1$, i.e. for $|z| > 0.5$.

$$\therefore \qquad X(z) = -\left[\sum_{n=0}^{\infty}(b^{-1}z)^n - 1\right] + \sum_{n=0}^{\infty}(0.5z^{-1})^n$$

$$= 1 - \frac{1}{1-b^{-1}z} + \frac{1}{1-0.5z^{-1}}$$

$$= -\frac{b^{-1}z}{1-b^{-1}z} + \frac{1}{1-0.5z^{-1}}$$

$$= -\frac{1}{(1/b^{-1}z)-1} + \frac{1}{1-0.5z^{-1}}$$

$$= -\frac{z}{b-z} + \frac{1}{1-0.5z^{-1}}$$

$$= \frac{z}{z-b} + \frac{z}{z-0.5}$$

So the ROC or the parameter values for which the Z-transform will exist is $0.5 < |z| < b$. The ROC is plotted in Figure 10.9.

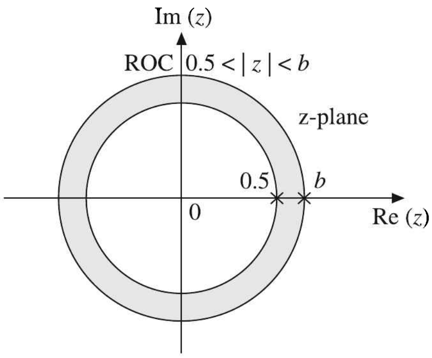


Figure 10.9 Pole-zero plot and ROC for Example 10.20.

EXAMPLE 10.21 For the sequence

$$x(n) = 2^n, \; n < 0$$

$$= \left(\frac{1}{2}\right)^n, \; n = 0, 2, 4, \ldots$$

$$= \left(\frac{1}{3}\right)^n, \; n = 1, 3, 5, \ldots$$

Prove that the absolute convergence region is given by $(1/2) < z < 2$.

Solution: For the given $x(n)$,

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 2^n z^{-n} + \sum_{n=0,2,4,\ldots}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=1}^{\infty} (2^{-1}z)^n + \sum_{n=0,2,4,\ldots}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{1}{3}z^{-1}\right)^n$$

In the above expression,

The first series converges for $\left|2^{-1}z\right| < 1$ i.e. for $|z| < 2$

The second series converges for $\left|\frac{1}{2}z^{-1}\right| < 1$, i.e. for $|z| > \frac{1}{2}$

The third series converges for $\left|\frac{1}{3}z^{-1}\right| < 1$ i.e. for $|z| > \frac{1}{3}$

The absolute convergence region for $X(z)$ is the intersection of the above three convergence regions.

So the ROC is given by $|1/2| < |z| < 2$

$$\therefore \quad X(z) = \sum_{n=1}^{\infty} (2^{-1}z)^n + \sum_{n=0,2,4}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=1,3,5}^{\infty} (\frac{1}{3}z^{-1})^n$$

$$= \sum_{n=0}^{\infty} (2^{-1}z)^n - 1 + \sum_{n=0,1,2,\ldots}^{\infty} \left[\left(\frac{1}{2}z^{-1}\right)^2\right]^n + \left(\frac{1}{3}z^{-1}\right) \sum_{n=0,1,2,\ldots}^{\infty} \left[\left(\frac{1}{3}z^{-1}\right)^2\right]^n$$

$$= \frac{1}{1-2^{-1}z} - 1 + \sum_{n=0,1,2,\ldots}^{\infty} \left[\left(\frac{1}{2}z^{-1}\right)^2\right]^n + \left(\frac{1}{3}z^{-1}\right) \sum_{n=0,1,2,\ldots}^{\infty} \left[\left(\frac{1}{3}z^{-1}\right)^2\right]^n$$

$$= -\frac{z}{z-2} + \frac{1}{1-[(1/2)z^{-1}]^2} + \frac{1}{3}z^{-1} \frac{1}{1-[(1/3)z^{-1}]^2}$$

$$= -\frac{z}{z-2} + \frac{z^2}{z^2-(1/4)} + \frac{1}{3}\frac{z}{z^2-(1/3)}$$

10.4 Z-TRANSFORM AND ROC OF FINITE DURATION SEQUENCES

Finite duration sequences are sequences having a finite number of samples. Finite duration sequences may be right hand sequences or left hand sequences or two-sided sequences.

Right hand sequence

A right hand sequence is one for which $x(n) = 0$ for $n < n_0$, where n_0 is positive or negative but finite. If $n_0 \geq 0$, the resulting sequence is a causal or a positive time sequence. For a causal or a positive time sequence, the ROC is entire z-plane except at $z = 0$.

EXAMPLE 10.22 Find the ROC and Z-transform of the causal sequence

$$x(n) = \{\underset{\uparrow}{1}, 0, -2, 3, 5, 4\}$$

Solution: The given sequence values are:

$$x(0) = 1,\ x(1) = 0,\ x(2) = -2,\ x(3) = 3,\ x(4) = 5 \text{ and } x(5) = 4.$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(0) + x(1)z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + x(5)z^{-5}$$

$$\therefore \qquad Z\,[x(n)] = X(z) = 1 - 2z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

The $X(z)$ converges for all values of z except at $z = 0$.

EXAMPLE 10.23 A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, -2, 0, 4, -3\}$. Find $X(z)$ and its ROC.

Solution: Given $\qquad x(n) = \{5, 3, -2, 0, 4, -3\}$

$$\therefore \qquad x(n) = 5\delta(n) + 3\delta(n-1) - 2\delta(n-2) + 4\delta(n-4) - 3\delta(n-5)$$

The given sequence is a right-sided sequence. So the ROC is entire z-plane except at $z = 0$. Taking Z-transform on both sides of the above equation, we have

$$\therefore \qquad X(z) = 5 + 3z^{-1} - 2z^{-2} + 4z^{-4} - 3z^{-5}$$

ROC: Entire z-plane except at $z = 0$.

Left hand sequence

A left hand sequence is one for which $x(n) = 0$ for $n \geq n_0$ where n_0 is positive or negative but finite. If $n_0 \leq 0$, the resulting sequence is anticausal sequence. For such type of sequence, the ROC is entire z-plane except at $z = \infty$.

EXAMPLE 10.24 Find the Z-transform and ROC of the anticausal sequence.

$$x(n) = \{4, 2, 3, -1, -2, \underset{\uparrow}{1}\}$$

Solution: The given sequence values are:

$$x(-5) = 4,\ x(-4) = 2,\ x(-3) = 3,\ x(-2) = -1,\ x(-1) = -2,\ x(0) = 1$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values, $X(z)$ is:

$$X(z) = x(-5)\, z^5 + x(-4)\, z^4 + x(-3)\, z^3 + x(-2)z^2 + x(-1)\, z + x(0)$$

$$\therefore \qquad Z[x(n)] = X(z) = 4z^5 + 2z^4 + 3z^3 - z^2 - 2z + 1$$

The $X(z)$ converges for all values of z except at $z = \infty$.

Two-sided sequence

A sequence that has finite duration on both the left and right sides is known as a two-sided sequence. For a two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.

EXAMPLE 10.25 Find the Z-transform and ROC of the sequence

$$x(n) = \{2, 1, -3, 0, \underset{\uparrow}{4}, 3, 2, 1, 5\}$$

Solution: The given sequence values are:

$$x(-4) = 2,\ x(-3) = 1,\ x(-2) = -3,\ x(-1) = 0,\ x(0) = 4,\ x(1) = 3,\ x(2) = 2,\ x(3) = 1,\ x(4) = 5$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 2z^4 + z^3 - 3z^2 + 4 + 3z^{-1} + 2z^{-2} + z^{-3} + 5z^{-4}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

EXAMPLE 10.26 Find the Z-transform of the following sequences:

(a) $u(n) - u(n-4)$ (b) $u(-n) - u(-n-3)$ (c) $u(2-n) - u(-2-n)$

Solution:

(a) The given sequence is:

$$x(n) = u(n) - u(n-4)$$

From Figure 10.10, we notice that the sequence values are:

$$\begin{aligned} x(n) &= 1, \quad \text{for } 0 \le n \le 3 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

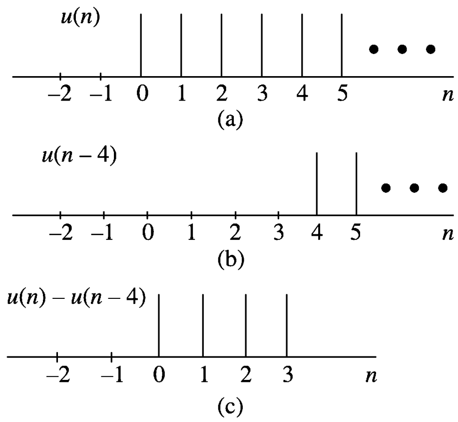


Figure 10.10 Sequences (a) $u(n)$, (b) $u(n-4)$ and (c) $u(n) - u(n-4)$.

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$$

The ROC is entire z-plane except at $z = 0$.

(b) The given sequence is:

$$x(n) = u(-n) - u(-n-3)$$

From Figure 10.11, we notice that the sequence values are:

$$\begin{aligned} x(n) &= 1, \quad \text{for } -2 \le n \le 0 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

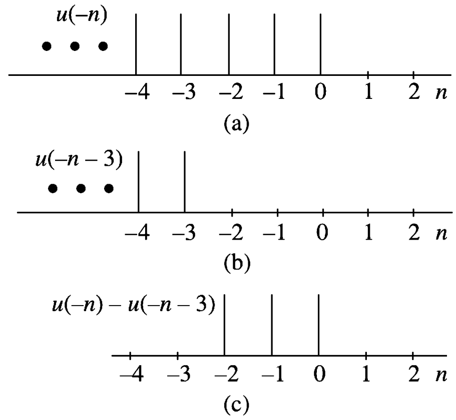


Figure 10.11 Sequences (a) $u(-n)$, (b) $u(-n-3)$, and (c) $u(-n) - u(-n-3)$.

We know that $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z + z^2$$

The ROC is entire z-plane except at $z = \infty$.

(c) The given sequence is:

$$x(n) = u(2-n) - u(-2-n)$$

From Figure 10.12, we notice that the sequence values are:

$$x(n) = 1, \quad \text{for } -1 \le n \le 2$$
$$= 0, \quad \text{otherwise}$$

Substituting the sequence values, we get

$$X(z) = z + 1 + z^{-1} + z^{-2}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

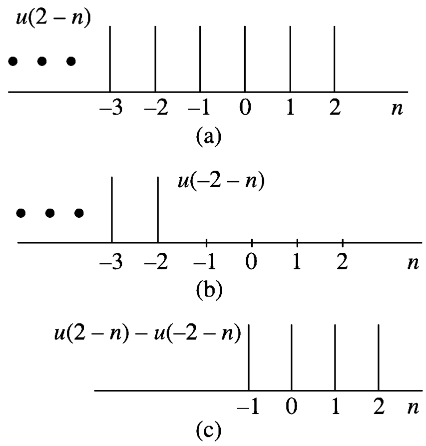


Figure 10.12 Sequences (a) $u(2-n)$, (b) $u(-2-n)$, and (c) $u(2-n) - u(-2-n)$.

10.5 PROPERTIES OF ROC

1. The ROC is a ring or disk in the z-plane centred at the origin.
2. The ROC cannot contain any poles.
3. If $x(n)$ is an infinite duration causal sequence, the ROC is $|z| > \alpha$, i.e. it is the exterior of a circle of radius α.
   If $x(n)$ is a finite duration causal sequence (right-sided sequence), the ROC is entire z-plane except at $z = 0$.
4. If $x(n)$ is an infinite duration anticausal sequence, the ROC is $|z| < \beta$, i.e. it is the interior of a circle of radius β.
   If $x(n)$ is a finite duration anticausal sequence (left-sided sequence), the ROC is entire z-plane except at $z = \infty$.

5. If $x(n)$ is a finite duration two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.
6. If $x(n)$ is an infinite duration, two-sided sequence, the ROC consists of a ring in the z-plane (ROC; $\alpha < |z| < \beta$) bounded on the interior and exterior by a pole, not containing any poles.
7. The ROC of an LTI stable system contains the unit circle.
8. The ROC must be a connected region. If $X(z)$ is rational, then its ROC is bounded by poles or extends up to infinity.
9. $x(n) = \delta(n)$ is the only signal whose ROC is entire z-plane.

10.6 PROPERTIES OF Z-TRANSFORM

The Z-transform has several properties that can be used in the study of discrete-time signals and systems. They can be used to find the closed form expression for the Z-transform of a given sequence. Many of the properties are analogous to those of the DTFT. They make the Z-transform a powerful tool for the analysis and design of discrete-time LTI systems. In general, both one-sided and two-sided Z-transforms have almost same properties.

10.6.1 Linearity Property

The linearity property of Z-transform states that, the Z-transform of a weighted sum of two signals is equal to the weighted sum of individual Z-transforms. That is, the linearity property states that

If $$x_1(n) \xleftrightarrow{\text{ZT}} X_1(z), \text{ with ROC} = R_1$$

and $$x_2(n) \xleftrightarrow{\text{ZT}} X_2(z), \text{ with ROC} = R_2$$

Then $$ax_1(n) + bx_2(n) \xleftrightarrow{\text{ZT}} aX_1(z) + bX_2(z), \text{ with ROC} = R_1 \cap R_2$$

Proof: We know that $$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$\therefore$ $$Z[ax_1(n) + bx_2(n)] = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} ax_1(n)\, z^{-n} + \sum_{n=-\infty}^{\infty} bx_2(n)\, z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n)\, z^{-n}$$

$$= aX_1(z) + bX_2(z);\ \text{ROC};\ R_1 \cap R_2$$

$$\boxed{ax_1(n) + bx_2(n) \xleftrightarrow{\text{ZT}} aX_1(z) + bX_2(z)}$$

The ROC for the Z-transform of a sum of sequences is equal to the intersection of the ROCs of the individual transforms.

10.6.2 Time Shifting Property

The time shifting property of Z-transform states that

If $\quad x(n) \xleftrightarrow{\text{ZT}} X(z)$, with zero initial conditions with ROC = R

Then
$$x(n-m) \xleftrightarrow{\text{ZT}} z^{-m} X(z)$$

with ROC = R except for the possible addition or deletion of the origin or infinity.

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore \quad Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

Let $n - m = p$ in the summation, then $n = m + p$.

$$Z[x(n-m)] = \sum_{p=-\infty}^{\infty} x(p) z^{-(m+p)}$$

$$\therefore \quad = z^{-m} \sum_{p=-\infty}^{\infty} x(p) z^{-p}$$

$$= z^{-m} X(z)$$

$$\boxed{x(n-m) \xleftrightarrow{\text{ZT}} z^{-m} X(z)}$$

Similarly
$$\boxed{x(n+m) \xleftrightarrow{\text{ZT}} z^{m} X(z)}$$

If the initial conditions are not neglected, we have

(a) Time delay $\quad Z[x(n-m)] = z^{-m} X(z) + z^{-m} \sum_{k=1}^{m} x(-k) z^{k}$

(b) Time advance $\quad Z[x(n+m)] = z^{m} X(z) - z^{m} \sum_{k=0}^{m-1} x(k) z^{-k}$

Proof:

(a)
$$Z[x(n-m)] = \sum_{n=0}^{\infty} x(n-m) z^{-n} = \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)} z^{-m}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

Let $n - m = p$ in the summation, then

$$Z[x(n-m)] = z^{-m} \sum_{p=-m}^{\infty} x(p)\, z^{-p}$$

$$= z^{-m} \left[\sum_{p=0}^{\infty} x(p)\, z^{-p} + \sum_{p=-m}^{-1} x(p)\, z^{-p} \right]$$

Substituting $p = -k$ in the second summation, we obtain

$$Z[x(n-m)] = z^{-m} X(z) + z^{-m} \sum_{k=1}^{m} x(-k)\, z^{k}$$

$$\therefore \quad Z[x(n-m)] = z^{-m} X(z) + z^{-(m-1)} x(-1) + z^{-(m-2)} x(-2) + z^{-(m-3)} x(-3) + \cdots$$

(b)

$$Z[x(n+m)] = \sum_{n=0}^{\infty} x(n+m) z^{-n} = \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)}\, z^{m}$$

$$= z^{m} \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)}$$

Let $p = n + m$, then

$$Z[x(n+m)] = z^{m} \sum_{p=m}^{\infty} x(p)\, z^{-p}$$

$$= \left[z^{m} \sum_{p=0}^{\infty} x(p)\, z^{-p} - \sum_{p=0}^{m-1} x(p)\, z^{-p} \right]$$

$$= z^{m} X(z) - z^{m} \sum_{p=0}^{m-1} x(p)\, z^{-p}$$

i.e. $$Z[x(n+m)] = z^{m} X(z) - z^{m} x(0) - z^{m-1} x(1) - z^{m-2} x(2)$$

This time shifting property is very useful in finding the output $y(n)$ of a system described in difference equation for an input $x(n)$.

$$Z[x(n+1)] = zX(z) - z x(0)$$

$$Z[x(n+2)] = z^{2} X(z) - z^{2} x(0) - z x(1)$$

$$Z[x(n+3)] = z^{3} X(z) - z^{3} x(0) - z^{2} x(1) - z x(2)$$

$$Z[x(n-1)] = z^{-1} X(z) + x(-1)$$

$$Z[x(n-2)] = z^{-2} X(z) + z^{-1} x(-1) + x(-2)$$

$$Z[x(n-3)] = z^{-3} X(z) + z^{-2} x(-1) + z^{-1} x(-2) + x(-3)$$

10.6.3 Multiplication by an Exponential Sequence Property

The multiplication by an exponential sequence property of Z-transform states that

If $$x(n) \xleftrightarrow{\text{ZT}} X(z) \text{ with ROC } = R$$

Then $$a^n x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{a}\right) \text{ with ROC } = |a|R$$

where a is a complex number.

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \quad Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

$$\boxed{a^n x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{a}\right)}$$

Note: $$e^{j\omega n} x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{e^{j\omega}}\right) = X(e^{-j\omega} z)$$

$$e^{-j\omega n} x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{e^{-j\omega}}\right) = X(e^{j\omega} z)$$

10.6.4 Time Reversal Property

The time reversal property of Z-transform states that

If $$x(n) \xleftrightarrow{\text{ZT}} X(z), \text{ with ROC } = R$$

Then $$x(-n) \xleftrightarrow{\text{ZT}} X\left(\frac{1}{z}\right), \text{ with ROC } = \frac{1}{R}$$

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \qquad Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

Let $p = -n$ in the above summation, then

$$Z\,[x(-n)] = \sum_{p=-\infty}^{\infty} x(p)\, z^{p}$$

$$Z[x(-n)] = \sum_{p=-\infty}^{\infty} x(p)\, (z^{-1})^{-p}$$

$$= X(z^{-1}) = X\left(\frac{1}{z}\right)$$

$$\boxed{x(-n) \xleftrightarrow{\text{ZT}} X(z^{-1})}$$

10.6.5 Time Expansion Property

The time expansion property of Z-transform states that

If $\qquad x(n) \xleftrightarrow{\text{ZT}} X(z)$, with ROC $= R$

Then $\qquad x_k(n) \xleftrightarrow{\text{ZT}} X(z^k)$, with ROC $= R^{1/k}$

where $\qquad x_k(n) = x\left(\dfrac{n}{k}\right)$, if n is an integer multiple of k

$\qquad = 0,$ otherwise

$x_k(n)$ has $k - 1$ zeros inserted between successive values of the original signal.

Proof: We know that

$$Z\,[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore \qquad Z[x_k(n)] = \sum_{n=-\infty}^{\infty} x_k(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{k}\right) z^{-n}$$

Let $\qquad \dfrac{n}{k} = p$

$$Z\,[x_k(n)] = \sum_{p=-\infty}^{\infty} x(p)\, z^{-pk}$$

$$\therefore \qquad = \sum_{p=-\infty}^{\infty} x(p)\, (z^{k})^{-p}$$

$$= X(z^k)$$

$$\boxed{x\left(\frac{n}{k}\right) \xleftrightarrow{\text{ZT}} X(z^k)}$$

10.6.6 Multiplication by *n* or Differentiation in *z*-domain Property

The multiplication by n or differentiation in z-domain property of Z-transform states that

If $\qquad x(n) \xleftrightarrow{\text{ZT}} X(z)$, with ROC $= R$

Then $\qquad nx(n) \xleftrightarrow{\text{ZT}} -z\dfrac{d}{dz}X(z)$, with ROC $= R$

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Differentiating both sides with respect to z, we get

$$\frac{d}{dz}X(z) = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-n)\, z^{-n-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty} [nx(n)]z^{-n}$$

$$= -z^{-1}Z[nx(n)]$$

$$\therefore \qquad Z[nx(n)] = -z\frac{d}{dz}X(z)$$

$$\boxed{nx(n) \xleftrightarrow{\text{ZT}} -z\frac{d}{dz}X(z)}$$

In the same way, $\qquad Z[n^k x(n)] = (-1)^k z^k \dfrac{d^k X(z)}{dz^k}$

10.6.7 Conjugation Property

The conjugation property of Z-transform states that

If $\qquad x(n) \xleftrightarrow{\text{ZT}} X(z)$, with ROC $= R$

Then $\qquad x^*(n) \xleftrightarrow{\text{ZT}} X^*(z^*)$, with ROC $= R$

Proof: We have

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$\therefore$

$$Z[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n)\, z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}\right]^*$$

$$= [X(z^*)]^* = X^*(z^*)$$

$$\boxed{x^*(n) \xleftrightarrow{\text{ZT}} X^*(z^*)}$$

10.6.8 Convolution Property

The convolution property of Z-transform states that the Z-transform of the convolution of two signals is equal to the multiplication of their Z-transforms, i.e.

If $\quad x_1(n) \xleftrightarrow{\text{ZT}} X_1(z),$ with ROC $= R_1$

and $\quad x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$ with ROC $= R_2$

Then $\quad x_1(n) * x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z),$ with ROC $= R_1 \cap R_2$

Proof: We know that

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Let $\quad x(n) = x_1(n) * x_2(n)$

We have

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$\therefore$

$$Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\, z^{-(n-k)}\, z^{-k}$$

Interchanging the order of summations,

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-(n-k)}$$

Replacing $(n - k)$ by p in the second summation, we get

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{p=-\infty}^{\infty} x_2(p)\, z^{-p}$$

$$= X_1(z)\, X_2(z)$$

$$\boxed{x_1(n) * x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z);\ \text{ROC};\ R_1 \cap R_2}$$

10.6.9 The Multiplication Property or Complex Convolution Property

The multiplication property or complex convolution property of Z-transform states that

If $\qquad x_1(n) \xleftrightarrow{\text{ZT}} X_1(z)$ and $x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$

Then $\qquad x_1(n)\, x_2(n) \xleftrightarrow{\text{ZT}} \dfrac{1}{2\pi j} \oint_c X_1(v)\, X_2\left(\dfrac{z}{v}\right) v^{-1}\, dv$

Proof: From the definition of Z-transform, we have

$$\text{Z}\,[x_1(n)\, x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n)\, x_2(n)]\, z^{-n}$$

But from the inverse Z-transform, we have

$$x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$$

Changing the complex variable z to v and substituting back, we get

$$\text{Z}\,[x_1(n)\, x_2(n)] = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint X_1(v)\, v^{n-1}\, dv\right] x_2(n)\, z^{-n}$$

$$= \frac{1}{2\pi j} \oint X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n)\, (v^{-1}z)^{-n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j} \oint X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

$$\boxed{x_1(n)\, x_2(n) \xleftrightarrow{\text{ZT}} \frac{1}{2\pi j} \oint_c X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv}$$

10.6.10 Correlation Property

The correlation property of Z-transform states that

If $\qquad x_1(n) \xleftrightarrow{\text{ZT}} X_1(z) \quad \text{and} \quad x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$

Then $\qquad R_{12}(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z^{-1})$

Proof: From the definition of the Z-transform, we have

$$Z\,[x_1(n) \otimes x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) \otimes x_2(n)]\, z^{-n}$$

But from the correlation sum, we have

$$x_1(n) \otimes x_2(n) = \sum_{l=-\infty}^{\infty} x_1(l)\, x_2(l-n) \quad \text{or} \quad x_1(n) \otimes x_2(n) = \sum_{l=-\infty}^{\infty} x_1(l-n)\, x_2(l)$$

Substituting back, we get

$$Z\,[x_1(n) \otimes x_2(n)] = \sum_{n=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} x_1(l)\, x_2(l-n)\right] z^{-n}$$

Interchanging the order of summation, we have

$$Z\,[x_1(n) \otimes x_2(n)] = \sum_{l=-\infty}^{\infty} x_1(l) \left[\sum_{n=-\infty}^{\infty} x_2(l-n)\, z^{-n}\right]$$

Letting $l - n = m$ in the second summation, we have $n = l - m$.

$$\therefore \qquad Z\,[x_1(n) \otimes x_2(n)] = \sum_{l=-\infty}^{\infty} x_1(l) \left[\sum_{m=\infty}^{-\infty} x_2(m)\, z^{-(l-m)}\right]$$

$$= \left[\sum_{l=-\infty}^{\infty} x_1(l)\, z^{-l}\right] \left[\sum_{m=-\infty}^{\infty} x_2(m)\, z^{m}\right]$$

$$= \left[\sum_{l=-\infty}^{\infty} x_1(l)\, z^{-l}\right] \left[\sum_{m=-\infty}^{\infty} x_2(m)\, (z^{-1})^{-m}\right]$$

$$= X_1(z)\, X_2(z^{-1})$$

$$\boxed{R_{12}(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z^{-1})}$$

10.6.11 Accumulation Property

The accumulation property of Z-transform states that

If $$x(n) \xleftrightarrow{\text{ZT}} X(z)$$

Then $$\sum_{k=-\infty}^{n} x(k) \xleftrightarrow{\text{ZT}} \frac{1}{1-z^{-1}} X(z)$$

Proof: From the definition of Z-transform, we have

$$Z\left[\sum_{k=-\infty}^{n} x(k)\right] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{n} x(k)\, z^{-n}$$

Substituting $n - k = r$ (or $n = k + r$ or $k = n - r$) at RHS, we get

$$Z\left[\sum_{k=-\infty}^{n} x(k)\right] = \sum_{k=-\infty-r}^{k=\infty-r} \sum_{r=n-(-\infty)}^{r=n-n} x(k)\, z^{-(k+r)}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{r=\infty}^{0} x(k)\, z^{-k} z^{-r} = \left(\sum_{r=0}^{\infty} z^{-r}\right) \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

$$= \left(\sum_{r=0}^{\infty} z^{-r}\right) X(z) = \frac{1}{1-z^{-1}} X(z)$$

$$\boxed{\sum_{k=-\infty}^{n} x(k) \xleftrightarrow{\text{ZT}} \frac{1}{1-z^{-1}} X(z)}$$

10.6.12 Parseval's Theorem or Relation or Property

The Parseval's relation or theorem or property of Z-transform states that

If $$x_1(n) \xleftrightarrow{\text{ZT}} X_1(z) \quad \text{and} \quad x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$$

Then $$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint X_1(z)\, X_2^*\, [(z^*)^{-1}]\, z^{-1}\, dz \quad \text{for complex } x_1(n) \text{ and } x_2(n)$$

Proof: Substituting the relation of the inverse Z-transform at LHS, we get

$$\text{LHS} = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint X_1(z)\, z^{n-1}\, dz\right] x_2^*(n)$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left[\sum_{n=-\infty}^{\infty} x_2^*(n)\, (z^{-1})^{-n}\right] z^{-1} dz$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left\{ \sum_{n=-\infty}^{\infty} x_2(n) [(z^*)^{-1}]^{-n} \right\}^* z^{-1}\, dz$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left\{ X_2 [(z^*)^{-1}] \right\}^* z^{-1}\, dz$$

$$= \frac{1}{2\pi j} \oint X_1(z)\, X_2^* [(z^*)^{-1}]\, z^{-1}\, dz = \text{RHS}$$

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint X_1(z)\, X_2^* [(z^*)^{-1}]\, z^{-1}\, dz}$$

10.6.13 Initial Value Theorem

The initial value theorem of Z-transform states that, for a causal signal $x(n)$

If $$x(n) \xleftrightarrow{\text{ZT}} X(z)$$

Then $$\underset{n \to 0}{\text{Lt}}\, x(n) = x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

Proof: We know that for a causal signal

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots$$

Taking the limit $z \to \infty$ on both sides, we have

$$\underset{z \to \infty}{\text{Lt}}\, X(z) = \underset{z \to \infty}{\text{Lt}} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \cdots \right] = x(0) + 0 + 0 + \cdots = x(0)$$

$\therefore$ $$\underset{n \to 0}{\text{Lt}}\, x(n) = x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

$$\boxed{x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)}$$

This theorem helps us to find the initial value of $x(n)$ from $X(z)$ without taking its inverse Z-transform.

10.6.14 Final Value Theorem

The final value theorem of Z-transform states that, for a causal signal

If $$x(n) \xleftrightarrow{\text{ZT}} X(z)$$

and if $X(z)$ has no poles outside the unit circle, and it has no double or higher order poles on the unit circle centred at the origin of the z-plane, then

$$\underset{n\to\infty}{\text{Lt}}\; x(n) = x(\infty) = \underset{z\to 1}{\text{Lt}}\; (z-1)\, X(z)$$

Proof: We know that for a causal signal

$$Z\,[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$Z\,[x(n+1)] = zX(z) - zx(0) = \sum_{n=0}^{\infty} x(n+1)\, z^{-n}$$

$$\therefore \quad Z\,[x(n+1)] - Z\,[x(n)] = zX(z) - zx(0) - X(z) = \sum_{n=0}^{\infty} x(n+1)\, z^{-n} - \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

i.e.

$$(z-1)\, X(z) - zx(0) = \sum_{n=0}^{\infty} [x(n+1) - x(n)]\, z^{-n}$$

i.e. $(z-1)\, X(z) - zx(0) = [x(1) - x(0)]\, z^{-0} + [x(2) - x(1)]\, z^{-1} + [x(3) - x(2)]\, z^{-2} + \cdots$

Taking limit $z \to 1$ on both sides, we have

$$\underset{z\to 1}{\text{Lt}}\; (z-1)\, X(z) - x(0) = [x(1) - x(0) + x(2) - x(1) + x(3) - x(2) + \cdots + x(\infty) - x(\infty - 1)$$

$$= x(\infty) - x(0)$$

$$\therefore \quad x(\infty) = \underset{z\to 1}{\text{Lt}}\; (z-1)\, X(z)$$

or

$$x(\infty) = \underset{z\to 1}{\text{Lt}}\; (1 - z^{-1})\, X(z)$$

$$\boxed{x(\infty) = \underset{z\to 1}{\text{Lt}}\; (z-1)\, X(z)}$$

This theorem enables us to find the steady-state value of $x(n)$, i.e. $x(\infty)$ without taking the inverse Z-transform of $X(z)$.

Some common Z-transform pairs are given in Table B.7 (Appendix B). The properties of Z-transform are given in Table B.8 (Appendix B).

EXAMPLE 10.27 Using properties of Z-transform, find the Z-transform of the following signals:

(a) $x(n) = u(-n)$ (b) $x(n) = u(-n+1)$

(c) $x(n) = u(-n-2)$ (d) $x(n) = 2^n\, u(n-2)$

Solution:

(a) Given $\quad x(n) = u(-n)$

We know that $\quad Z\,[u(n)] = \dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$; ROC; $|z| > 1$

Using the time reversal property,

$$Z[u(-n)] = \left.\frac{z}{z-1}\right|_{z=(1/z)} = \frac{1/z}{(1/z)-1} = \frac{1}{1-z} = -\frac{1}{z-1};\ \text{ROC; } |z|<1$$

(b) Given $\qquad x(n) = u(-n+1) = u[-(n-1)]$

$\therefore \qquad Z[x(n)] = X(z) = Z[u(-n+1)] = Z\{u[-(n-1)]\}$

$$= z^{-1}Z[u(-n)] = z^{-1}\frac{1}{1-z} = -\frac{1}{z(z-1)}$$

(c) Given $\qquad x(n) = u(-n-2) = u[-(n+2)]$

$\therefore \qquad Z[x(n)] = X(z) = Z[u(-n-2)] = Z\{u[-(n+2)]\}$

$$= z^2 Z[u(-n)] = \frac{z^2}{1-z} = -\frac{z^2}{z-1}$$

(d) Given $\qquad x(n) = 2^n u(n-2)$

$$Z[u(n-2)] = z^{-2}Z[u(n)] = z^{-2}\frac{z}{z-1} = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}$$

$$Z[2^n u(n-2)] = Z[u(n-2)]\Big|_{z=(z/2)} = \left.\frac{1}{z(z-1)}\right|_{z=(z/2)} = \frac{1}{(z/2)[(z/2)-1]} = \frac{4}{z(z-2)}$$

EXAMPLE 10.28 Using properties of Z-transform, find the Z-transform of the sequence

(a) $x(n) = \alpha^{n-2}u(n-2)$ $\qquad$ (b) $x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

Solution:

(a) The Z-transform of the sequence $x(n) = \alpha^n u(n)$ is given by

$$X(z) = \frac{z}{z-\alpha};\ \text{ROC; } |z|>|\alpha|$$

Using the time shifting property of Z-transform, we have

$$Z[x(n-m)] = z^{-m}X(z)$$

In the same way,

$$Z[\alpha^{n-2}u(n-2)] = z^{-2}Z[\alpha^n u(n] = z^{-2}\frac{z}{z-\alpha} = \frac{1}{z(z-\alpha)};\ \text{ROC; } |z|>|\alpha|$$

(b) Given
$$x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

implies that $x(n) = u(n) - u(n-N)$.

We know that
$$Z[u(n)] = \frac{z}{z-1}$$

Using the time shifting property, we have
$$Z[u(n-N)] = z^{-N} Z[u(n)] = z^{-N} \frac{z}{z-1}$$

Using the linearity property, we have
$$Z[u(n) - u(n-N)] = Z[u(n)] - Z[u(n-N)] = \frac{z}{z-1} - z^{-N}\frac{z}{z-1} = \frac{z}{z-1}[1 - z^{-N}]$$

EXAMPLE 10.29 Using appropriate properties, find the Z-transform of the signal
$$x(n) = 2(3)^n u(-n)$$

Solution: Given
$$x(n) = 2(3)^n u(-n)$$

We know that
$$Z[u(n)] = \frac{1}{1-z^{-1}}; \text{ ROC; } |z| > 1$$

Using the time reversal property, we have
$$Z[u(-n)] = Z[u(n)]\big|_{z\to 1/z} = \frac{1}{1-z}; \text{ ROC; } |z| > |\alpha|$$

Using scaling in z-domain property, we have
$$Z[3^n u(-n)] = Z[u(-n)]\big|_{z\to z/3} = \frac{1}{1-z}\bigg|_{z\to z/3} = \frac{1}{1-(z/3)}; \text{ ROC; } |z| < 3$$

Using the linearity property, we have
$$Z[2(3)^n u(-n)] = 2Z[3^n u(-n)] = 2\frac{1}{1-(z/3)} = \frac{2}{1-(1/3)z}; \text{ ROC; } |z| < 3$$

EXAMPLE 10.30 Using appropriate properties, find the Z-transform of
$$x(n) = n^2 \left(\frac{1}{3}\right)^n u(n-2)$$

Solution: Given $x(n) = n^2 \left(\frac{1}{3}\right)^n u(n-2)$
$$= n^2 \left(\frac{1}{3}\right)^2 \left(\frac{1}{3}\right)^{n-2} u(n-2) = \frac{1}{9} n^2 \left(\frac{1}{3}\right)^{n-2} u(n-2)$$

We know that

$$Z[u(n)] = \frac{z}{z-1};\ \text{ROC};\ |z| > 1$$

From the property of multiplication by an exponential, we have

$$Z\left[\left(\frac{1}{3}\right)^n u(n)\right] = Z[u(n)]\Big|_{z \to [z/(1/3)] = 3z} = \frac{3z}{3z-1} = \frac{z}{z-(1/3)},\ |z| > \frac{1}{3}$$

Using the time shifting property, we have

$$Z\left[\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = z^{-2}\, Z\left[\left(\frac{1}{3}\right)^n u(n)\right] = z^{-2}\frac{z}{z-(1/3)} = \frac{1}{z[z-(1/3)]} = \frac{1}{z^2-(1/3)z}$$

Using differentiation in z-domain property, we have

$$Z\left[n\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = -z\frac{d}{dz}\left\{Z\left[\left(\frac{1}{3}\right)^{n-2} u(n-2)\right]\right\}$$

$$= -z\frac{d}{dz}\left[\frac{1}{z^2-(1/3)z}\right] = -z\left\{\frac{-1\,[2z-(1/3)]}{[z^2-(1/3)z]^2}\right\}$$

$$= \frac{2z^2-(1/3)z}{[z^2-(1/3)z]^2}$$

Again using differentiation in z-domain property, we have

$$Z\left[n^2\left(\frac{1}{3}\right)^{n-2} u(n-2)\right]$$

$$= -z\frac{d}{dz}\left\{Z\left[n\left(\frac{1}{3}\right)^{n-2} u(n-2)\right]\right\} = -z\frac{d}{dz}\left\{\frac{2z^2-(1/3)z}{[z^2-(1/3)z]^2}\right\}$$

$$= -z\left\{\frac{[z^2-(1/3)z]^2\,[4z-(1/3)] - [2z^2-(1/3)z]\,2[z^2-(1/3)z]\,[2z-(1/3)]}{[z^2-(1/3)z]^4}\right\}$$

$$= -z\left[\frac{-4z^3+z^2-(1/9)z}{[z^2-(1/3)z]^3}\right] = \left\{\frac{z^2\,[4z^2-z+(1/9)]}{z^3\,[z-(1/3)]^3}\right\} = \left\{\frac{4z^2-z+(1/9)}{z[z-(1/3)]^3}\right\}$$

Using the linearity property, we get

$$Z\left[\frac{1}{9}n^2\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = \frac{1}{9}\left\{\frac{4z^2-z+(1/9)}{z[z-(1/3)]^3}\right\}$$

EXAMPLE 10.31 Using appropriate properties of Z-transform, find the Z-transform of the signal

$$x(n) = n2^n \sin\left(\frac{\pi}{2}n\right)u(n)$$

Solution: Given

$$x(n) = n2^n \sin\left(\frac{\pi}{2}n\right)u(n)$$

We know that

$$\mathrm{Z}\left[\sin\left(\frac{\pi}{2}n\right)u(n)\right] = \frac{z\sin(\pi/2)}{z^2 - 2z\cos(\pi/2) + 1} = \frac{z}{z^2+1}$$

Using the multiplication by an exponential property, we have

$$\mathrm{Z}\left[2^n \sin\left(\frac{\pi}{2}n\right)u(n)\right] = \mathrm{Z}\left[\sin\left(\frac{\pi}{2}n\right)u(n)\right]\Bigg|_{z\to(z/2)}$$

$$= \frac{z}{z^2+1}\Bigg|_{z\to(z/2)} = \frac{z/2}{(z/2)^2+1}$$

$$= \frac{2z}{z^2+4}$$

Using differentiation in z-domain property, we have

$$\mathrm{Z}\left[n2^n \sin\left(\frac{\pi}{2}n\right)u(n)\right] = -z\frac{d}{dz}\left\{\mathrm{Z}\left[2^n \sin\left(\frac{\pi}{2}n\right)u(n)\right]\right\}$$

$$= -z\frac{d}{dz}\left(\frac{2z}{z^2+4}\right) = -z\left[\frac{(z^2+4)(2) - 2z(2z)}{(z^2+4)^2}\right]$$

$$= -z\left[\frac{-2z^2+8}{(z^2+4)^2}\right] = \frac{2z(z^2-4)}{(z^2+4)^2}$$

EXAMPLE 10.32 Determine the Z-transform of the following signals and sketch the ROC:

(a) $x_1(n) = \begin{cases} (1/2)^n, & n \ge 0 \\ (1/4)^{-n}, & n < 0 \end{cases}$

(b) Using properties of Z-transform, determine $x_2(n) = x_1(n + 2)$.

Solution:

(a) Given

$$x_1(n) = \begin{cases} (1/2)^n, & n \ge 0 \\ (1/4)^{-n}, & n < 0 \end{cases}$$

$$\therefore \qquad X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{4}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}$$

$$= \sum_{n=-1}^{-\infty} \left(\frac{1}{4}z\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^{n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4}z\right)^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^{n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}z\right)^{n} - 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^{n}$$

The first sequence converges if $|(1/4)z| < 1$, i.e. $|z| < 4$ and the second sequence converges if $|(1/2)z^{-1}| < 1$, i.e. if $|z| > 1/2$.

So $X_1(z)$ converges if $1/2 < |z| < 4$.

$$\therefore \qquad X_1(z) = \frac{(1/4)z}{1-(1/4)z} + \frac{1}{1-(1/2)z^{-1}}; \text{ ROC; } \frac{1}{2} < |z| < 4$$

$$= -\frac{z}{z-4} + \frac{z}{z-(1/2)} = \frac{-(7/2)z}{(z-4)[z-(1/2)]}$$

(b) $$x_2(n) = x_1(n+2)$$

Using the time shifting property

$$X_2(z) = \text{Z}[x_1(n+2)] = z^2 \text{Z}[x_1(n)]$$

$$= z^2 X_1(z) = \frac{-(7/2)\, z^3}{(z-4)[z-(1/2)]}; \text{ ROC; } \frac{1}{2} < |z| < 4$$

The pole-zero location and ROC are shown in Figure 10.13.

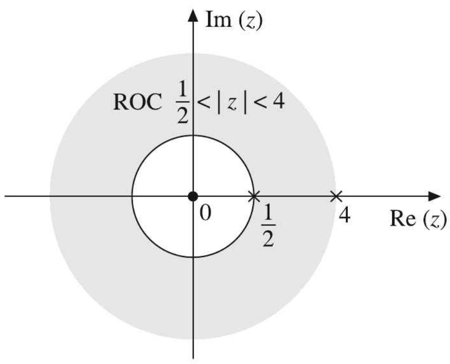


Figure 10.13 Pole-zero location and ROC for Example 10.32.

EXAMPLE 10.33 Determine the Z-transform of the following signal:

$$x(n) = \frac{1}{3}(n^2 + n)\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

Solution: Given

$$x(n) = \frac{1}{3}(n^2 + n)\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$= \frac{1}{3}n^2\left(\frac{1}{2}\right)^{n-1} u(n-1) + \frac{1}{3}n\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

We know that

$$Z\left[\left(\frac{1}{2}\right)^{n} u(n)\right] = \frac{z}{z-(1/2)}; \text{ ROC; } |z| > \frac{1}{2}$$

Using the time shifting property, we have

$$Z\left[\left(\frac{1}{2}\right)^{n-1} u(n-1)\right] = z^{-1}Z\left[\left(\frac{1}{2}\right)^{n} u(n)\right] = z^{-1}\frac{z}{z-(1/2)} = \frac{1}{z-(1/2)}$$

Using the multiplication by n property, we have

$$Z\left[n\left(\frac{1}{2}\right)^{n-1} u(n-1)\right] = -z\frac{d}{dz}\left\{Z\left[\left(\frac{1}{2}\right)^{n-1} u(n-1)\right]\right\} = -z\frac{d}{dz}\left[\frac{1}{z-(1/2)}\right]$$

$$= -z\left[\frac{-1}{[z-(1/2)^2}\right] = \frac{z}{[z-(1/2)]^2}$$

Using the multiplication by n property, we have

$$Z\left[n^2\left(\frac{1}{2}\right)^{n-1} u(n-1)\right] = -z\frac{d}{dz}\left\{Z\left[n\left(\frac{1}{2}\right)^{n-1} u(n-1)\right]\right\} = -z\frac{d}{dz}\left\{\frac{z}{[z-(1/2)]^2}\right\}$$

$$= -z\left\{\frac{[z-(1/2)]^2 - z\,2\,[z-(1/2)]}{[z-(1/2)]^4}\right\} = -z\frac{[z-(1/2)-2z]}{[z-(1/2)]^3}$$

$$= \frac{-z[-z-(1/2)]}{[z-(1/2)]^3} = \frac{z\,[z+(1/2)]}{[z-(1/2)]^3}$$

Using the linearity property, we have

$$X(z) = \frac{1}{3}\left\{Z\left[n^2\left(\frac{1}{2}\right)^{n-1} u(n-1)\right] + Z\left[n\left(\frac{1}{2}\right)^{n-1} u(n-1)\right]\right\}$$

$$= \frac{1}{3}\left\{\frac{z\,[z+(1/2)]}{[z-(1/2)]^3} + \frac{z}{[z-(1/2)]^2}\right\} = \frac{(2/3)\,z^2}{[z-(1/2)]^3}$$

EXAMPLE 10.34 Express the Z-transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

in terms of $X(z)$.

Solution: Given

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$\therefore$
$$y(n-1) = \sum_{k=-\infty}^{n-1} x(k)$$

$\therefore$
$$y(n) - y(n-1) = \sum_{k=-\infty}^{n} x(k) - \sum_{k=-\infty}^{n-1} x(k) = x(n)$$

Taking Z-transform on both sides, we have

$$Y(z) - z^{-1}Y(z) = X(z)$$

i.e.
$$Y(z)(1 - z^{-1}) = X(z)$$

$\therefore$
$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

EXAMPLE 10.35 Find the Z-transform of $x(n) = n^2 u(n)$.

Solution: Given
$$x(n) = n^2 u(n)$$

We have

$$\text{Z}[u(n)] = \frac{z}{z-1}$$

Using the multiplication by n property, we have

$$\text{Z}[nu(n)] = -z\frac{d}{dz}[\text{Z}\{u(n)\}] = -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z\left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

Again using the multiplication by n property, we have

$$\text{Z}[n^2u(n)] = -z\frac{d}{dz}\{\text{Z}[nu(n)]\} = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2 - z2(z-1)}{(z-1)^4}\right] = -z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= \frac{-z(-z-1)}{(z-1)^3} = \frac{z(z+1)}{(z-1)^3}$$

EXAMPLE 10.36 Find the Z-transform of the signal $x(n) = n\left(\frac{1}{2}\right)^{n+2} u(n+2)$.

Solution: The given sequence is:

$$x(n) = n\left(\frac{1}{2}\right)^{n+2} u(n+2) = (n+2)\left(\frac{1}{2}\right)^{n+2} u(n+2) - 2\left(\frac{1}{2}\right)^{n+2} u(n+2)$$

Using the time shifting property, the Z-transform is:

$$X(z) = \mathrm{Z}\,[x(n)] = \mathrm{Z}\left[(n+2)\left(\frac{1}{2}\right)^{n+2} u(n+2)\right] - \mathrm{Z}\left[2\left(\frac{1}{2}\right)^{n+2} u(n+2)\right]$$

$$= z^2\mathrm{Z}\left[n\left(\frac{1}{2}\right)^{n} u(n)\right] - 2z^2\mathrm{Z}\left[\left(\frac{1}{2}\right)^{n} u(n)\right]$$

$$= z^2\left\{-z\frac{d}{dz}\left[\frac{z}{z-(1/2)}\right]\right\} - 2z^2\left[\frac{z}{z-(1/2)}\right]$$

$$= z^2\left\{-z\left[\frac{[z-(1/2)]-z}{[z-(1/2)]^2}\right]\right\} - 2z^2\left[\frac{z}{z-(1/2)}\right]$$

$$= \frac{(1/2)z^3(3-4z)}{[z-(1/2)]^2}$$

Alternate method

Given $$x(n) = n\left(\frac{1}{2}\right)^{n+2} u\,(n+2)$$

$$X(z) = -z\frac{d}{dz}\left\{\mathrm{Z}\left[\left(\frac{1}{2}\right)^{n+2} u(n+2)\right]\right\} = -z\frac{d}{dz}\left\{z^2\mathrm{Z}\left[\left(\frac{1}{2}\right)^{n} u(n)\right]\right\}$$

$$= -z\frac{d}{dz}\left\{z^2\left[\frac{z}{z-(1/2)}\right]\right\} = -z\frac{d}{dz}\left[\frac{z^3}{z-(1/2)}\right] = -z\left\{\frac{[z-(1/2)]\,3z^2 - z^3}{[z-(1/2)]^2}\right\}$$

$$= \frac{(1/2)\,z^3(3-4z)}{[z-(1/2)]^2}$$

EXAMPLE 10.37 Find the Z-transform of the following signal:

$$x(n) = n\left(\frac{1}{2}\right)^{n} u(n) * \left[\delta(n) - \frac{1}{2}\delta(n-1)\right]$$

Solution: The given sequence is:

$$x(n) = n\left(\frac{1}{2}\right)^n u(n) * \left[\delta(n) - \frac{1}{2}\delta(n-1)\right]$$

Let $\quad x(n) = x_1(n) * x_2(n)$

where $\quad x_1(n) = n\left(\frac{1}{2}\right)^n u(n)$ and $x_2(n) = \delta(n) - \frac{1}{2}\delta(n-1)$

$$\therefore \quad X_1(z) = \mathrm{Z}\left[n\left(\frac{1}{2}\right)^n u(n)\right] = -z\frac{d}{dz}\left[\frac{z}{z-(1/2)}\right]$$

$$= -\frac{z\{[z-(1/2)]-z\}}{[z-(1/2)]^2} = \frac{(1/2)z}{[z-(1/2)]^2}$$

$$X_2(z) = \mathrm{Z}\left[\delta(n) - \frac{1}{2}\delta(n-1)\right] = 1 - \frac{1}{2}z^{-1} = \frac{z-(1/2)}{z}$$

Using the convolution property of Z-transforms, we get

$$X(z) = X_1(z)\,X_2(z) = \frac{(1/2)z}{[z-(1/2)]^2}\,\frac{z-(1/2)}{z} = \frac{(1/2)}{z-(1/2)};\ \text{ROC; } |z| > \frac{1}{2}$$

Alternate method

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty}\left[k\left(\frac{1}{2}\right)^k u(k)\right]\left[\delta(n-k) - \frac{1}{2}\delta(n-k-1)\right]$$

$$= \sum_{k=-\infty}^{\infty} k\left(\frac{1}{2}\right)^k u(k)\,\delta(n-k) - \frac{1}{2}\sum_{k=-\infty}^{\infty} k\left(\frac{1}{2}\right)^k u(k)\,\delta(n-k-1)$$

$$= n\left(\frac{1}{2}\right)^n u(n) - \frac{1}{2}(n-1)\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

Taking Z-transform on both sides, we have

$$X(z) = \frac{(1/2)z}{[z-(1/2)]^2} - \frac{1}{2}z^{-1}\left[\frac{(1/2)z}{[z-(1/2)]^2}\right] = \frac{(1/2)}{z-(1/2)};\ \text{ROC; } |z| > \frac{1}{2}$$

EXAMPLE 10.38 Find the Z-transform of the following signal using convolution property of Z-transforms.

$$x(n) = \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{4}\right)^n u(n)$$

Solution: Let

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$\therefore$

$$X_1(z) = \frac{z}{z-(1/2)}; \text{ ROC; } |z| > \frac{1}{2}$$

and

$$x_2(n) = \left(\frac{1}{4}\right)^n u(n)$$

$\therefore$

$$X_2(z) = \frac{z}{z-(1/4)}; \text{ ROC; } |z| > \frac{1}{4}$$

Now,

$$x(n) = x_1(n) * x_2(n)$$

$\therefore$

$$\text{Z}[x(n)] = X(z) = \text{Z}[x_1(n) * x_2(n)] = X_1(z)\, X_2(z); \text{ ROC; } |z| > \frac{1}{2}$$

$$= \frac{z}{z-(1/2)} \frac{z}{z-(1/4)}; \text{ ROC; } |z| > \frac{1}{2}$$

EXAMPLE 10.39 Find the Z-transform of the signal

$$x(n) = n\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]$$

Solution: Let

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$\therefore$

$$X_1(z) = \frac{1}{1-(1/2)z^{-1}} = \frac{z}{z-(1/2)}; \text{ ROC; } |z| > \frac{1}{2}$$

and

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

$\therefore$

$$X_2(z) = \frac{1}{1-(1/3)z^{-1}} = \frac{z}{z-(1/3)}; \text{ ROC; } |z| > \frac{1}{3}$$

Using convolution in the time domain property, we have

$$\text{Z}[x_1(n) * x_2(n)] = X_1(z)\, X_2(z)$$

$$\therefore \qquad Z\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right] = \frac{z}{z-(1/2)}\,\frac{z}{z-(1/3)}$$

Using differentiation in z-domain property, we have

$$Z\left\{n\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]\right\} = -z\frac{d}{dz}\left\{Z\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]\right\}$$

$$= -z\frac{d}{dz}\left\{\frac{z^2}{[z-(1/2)]\,[z-(1/3)]}\right\}$$

$$= -z\left[\frac{[z^2-(5/6)z+(1/6)]\,2z - z^2\,[2z-(5/6)]}{[z^2-(5/6)z+(1/6)]^2}\right]$$

$$= -z\left[\frac{2z^3-(5/3)z^2+(1/3)z-2z^3+(5/6)z^2}{[z^2-(5/6)z+(1/6)]^2}\right]$$

$$= \frac{(5/6)\,z^2\,[z-(2/5)]}{[z-(1/2)]^2\,[z-(1/3)]^2}$$

EXAMPLE 10.40 Using Z-transform, find the convolution of the sequences

$$x_1(n) = \{2, 1, 0, -1, 3\};\; x_2(n) = \{1, -3, 2\}$$

Solution: From the convolution property of Z-transforms, we have

$$Z\{x_1(n) * x_2(n)\} = X_1(z)\,X_2(z) \text{ which implies that}$$

$$x_1(n) * x_2(n) = Z^{-1}[X_1(z)\,X_2(z)]$$

Given $\qquad x_1(n) = \{2, 1, 0, -1, 3\}$

$\therefore \qquad X_1(z) = 2 + z^{-1} - z^{-3} + 3z^{-4}$

and $\qquad x_2(n) = \{1, -3, 2\}$

$\therefore \qquad X_2(z) = 1 - 3z^{-1} + 2z^{-2}$

$$\therefore \qquad X_1(z)\,X_2(z) = (2 + z^{-1} - z^{-3} + 3z^{-4})\,(1 - 3z^{-1} + 2z^{-2})$$

$$= 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}$$

Taking inverse Z-transform on both sides,

$$x(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

EXAMPLE 10.41 Find the convolution of the sequences

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$$

using (a) Convolution property of Z-transforms and (b) Time domain method

Solution:

(a) **Convolution property of Z-transforms**

Given $\quad x_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$

$$\therefore \quad X_1(z) = \mathrm{Z}\left[\left(\frac{1}{2}\right)^n u(n)\right] = \frac{1}{1-(1/2)\,z^{-1}} = \frac{z}{z-(1/2)};\ \text{ROC; } |z| > \frac{1}{2}$$

and

$$X_2(z) = \mathrm{Z}\left[\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = z^{-2}\,\mathrm{Z}\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$= z^{-2}\frac{1}{1-(1/3)z^{-1}} = \frac{z^{-1}}{z-(1/3)};\ \text{ROC; } |z| > \frac{1}{3}$$

We know that

$$x(n) = x_1(n) * x_2(n)$$

$$\therefore \quad \mathrm{Z}[x(n)] = X(z) = \mathrm{Z}\,[x_1(n) * x_2(n)] = X_1(z)\,X_2(z)$$

$$\therefore \quad \mathrm{Z}\,[x_1(n) * x_2(n)] = \frac{z}{z-(1/2)}\,\frac{z^{-1}}{z-(1/3)} = \frac{1}{[z-(1/2)]\,[z-(1/3)]}$$

$$\therefore \quad x(n) = \mathrm{Z}^{-1}\left\{\frac{1}{[z-(1/2)]\,[z-(1/3)]}\right\} = \mathrm{Z}^{-1}\left[\frac{1}{z-(1/2)} - \frac{1}{z-(1/3)}\right]6$$

$$= 6\left[\left(\frac{1}{2}\right)^{n-1} u(n-1) - \left(\frac{1}{3}\right)^{n-1} u(n-1)\right]$$

(b) **Time domain method**

$$x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\,x_2(n-k)$$

$$= \sum_{k=0}^{n-2}\left(\frac{1}{2}\right)^k u(k)\left(\frac{1}{3}\right)^{n-2-k} u(n-2-k)$$

$$= \sum_{k=0}^{n-2}\left(\frac{1}{2}\right)^k\left(\frac{1}{3}\right)^{n-2-k} = \sum_{k=0}^{n-2}\left(\frac{1}{2}\right)^k\left(\frac{1}{3}\right)^n\left(\frac{1}{3}\right)^{-2}\left(\frac{1}{3}\right)^{-k}$$

$$= 9\left(\frac{1}{3}\right)^n \sum_{k=0}^{n-2}\left(\frac{3}{2}\right)^k = 9\left(\frac{1}{3}\right)^n \left[\frac{1-(3/2)^{n-1}}{1-(3/2)}\right] = -18\left(\frac{1}{3}\right)^n\left[1-\left(\frac{3}{2}\right)^{n-1}\right]$$

$$= -6\left[\left(\frac{1}{3}\right)^{n-1} u(n-1) - \left(\frac{1}{3}\right)^{n-1}\left(\frac{3}{2}\right)^{n-1} u(n-1)\right]$$

$$= -6\left[\left(\frac{1}{3}\right)^{n-1} u(n-1) - \left(\frac{1}{2}\right)^{n-1} u(n-1)\right]$$

EXAMPLE 10.42 Find the convolution of the sequences $x_1(n) = (1/3)^n\, u(n)$ and $x_2(n) = (1/5)^n\, u(n)$ using (a) Convolution property of Z-transforms and (b) Time domain method.

Solution:

(a) **Convolution property of Z-transforms**

Given $$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{5}\right)^n u(n)$$

$$\therefore \quad X_1(z) = \mathrm{Z}\left[\left(\frac{1}{3}\right)^n u(n)\right] = \frac{1}{1-(1/3)\,z^{-1}} = \frac{z}{z-(1/3)}; \text{ ROC; } |z| > \frac{1}{3}$$

and $$X_2(z) = \mathrm{Z}\left[\left(\frac{1}{5}\right)^n u(n)\right] = \frac{1}{1-(1/5)\,z^{-1}} = \frac{z}{z-(1/5)}; \text{ ROC; } |z| > \frac{1}{5}$$

We know that

$$x(n) = x_1(n) * x_2(n)$$

$$\therefore \quad \mathrm{Z}\,[x(n)] = X(z) = \mathrm{Z}\,[x_1(n) * x_2(n)] = X_1(z)\, X_2(z)$$

$$\therefore \quad \mathrm{Z}\,[x_1(n) * x_2(n)] = \frac{z}{z-(1/3)}\,\frac{z}{z-(1/5)}$$

$$\therefore \quad x(n) = \mathrm{Z}^{-1}\left[\frac{z}{z-(1/3)}\,\frac{z}{z-(1/5)}\right] = \mathrm{Z}^{-1}\left[\frac{5}{2}\,\frac{z}{z-(1/3)} - \frac{3}{2}\,\frac{z}{z-(1/5)}\right]$$

$$= \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n)$$

(b) **Time domain method**

Given $$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{5}\right)^n u(n)$$

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k u(k) \left(\frac{1}{5}\right)^{n-k} u(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-k} = \left(\frac{1}{5}\right)^n \sum_{k=0}^{n} \left(\frac{1}{3} \times \frac{5}{1}\right)^k = \left(\frac{1}{5}\right)^n \sum_{k=0}^{n} \left(\frac{5}{3}\right)^k$$

$$= \left(\frac{1}{5}\right)^n \left[\frac{1-(5/3)^{n+1}}{1-(5/3)}\right] = -\frac{3}{2}\left\{\left(\frac{1}{5}\right)^n \left[1 - \left(\frac{5}{3}\right)^n \frac{5}{3}\right]\right\}$$

$$= -\frac{3}{2}\left(\frac{1}{5}\right)^n + \frac{3}{2}\left(\frac{1}{5}\right)^n \left(\frac{5}{3}\right)^n \frac{5}{3} = \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n)$$

EXAMPLE 10.43 Using final value theorem, find $x(\infty)$, if $X(z)$ is given by

(a) $\dfrac{z+1}{(z-0.6)^2}$ (b) $\dfrac{z+2}{4(z-1)(z+0.7)}$ (c) $\dfrac{2z+3}{(z+1)(z+3)(z-1)}$

Solution:

(a) Given $$X(z) = \frac{z+1}{(z-0.6)^2}$$

Looking at $X(z)$, we notice that the ROC of $X(z)$ is $|z| > 0.6$ and $(z-1)\,X(z)$ has no poles on or outside the unit circle. Therefore,

$$x(\infty) = \underset{z\to 1}{\text{Lt}}\, (z-1)\, X(z) = \underset{z\to 1}{\text{Lt}}\, (z-1)\frac{z+1}{(z-0.6)^2} = 0$$

(b) Given $$X(z) = \frac{z+2}{4(z-1)(z+0.7)}$$

$$(z-1)\, X(z) = \frac{z+2}{4(z+0.7)}$$

$(z-1)\, X(z)$ has no poles on or outside the unit circle.

$$\therefore \quad x(\infty) = \underset{z\to 1}{\text{Lt}}\, (z-1)\, X(z) = \underset{z\to 1}{\text{Lt}}\left[\frac{z+2}{4(z+0.7)}\right] = \frac{3}{6.8} = 0.44$$

(c) Given $$X(z) = \frac{2z+3}{(z+1)(z+3)(z-1)}$$

$$(z-1)\, X(z) = \frac{(z-1)(2z+3)}{(z+1)(z+3)(z-1)} = \frac{2z+3}{(z+1)(z+3)}$$

$(z-1)\, X(z)$ has one pole on the unit circle and one pole outside the unit circle. So $x(\infty)$ tends to infinity as $n \to \infty$.

EXAMPLE 10.44 Find $x(0)$ if $X(z)$ is given by

(a) $\dfrac{z^2+2z+2}{(z+1)(z+0.5)}$ (b) $\dfrac{z+3}{(z+1)(z+2)}$

Solution:

(a) Given $$X(z) = \frac{z^2+2z+2}{(z+1)(z+0.5)} = \frac{1+(2/z)+(2/z^2)}{[1+(1/z)][1+(0.5/z)]}$$

$$x(0) = \underset{z\to\infty}{\text{Lt}}\ X(z) = \underset{z\to\infty}{\text{Lt}}\ \frac{[1+(2/z)+(2/z^2)]}{[1+(1/z)][1+(0.5/z)]} = 1$$

(b) Given $$X(z) = \frac{z+3}{(z+1)(z+2)} = \frac{z[1+(3/z)]}{z^2[1+(1/z)][1+(2/z)]} = \frac{1}{z}\frac{1+(3/z)}{[1+(1/z)][1+(2/z)]}$$

$$x(0) = \underset{z\to\infty}{\text{Lt}}\ X(z) = \underset{z\to\infty}{\text{Lt}}\ \frac{1}{z}\frac{1+(3/z)}{[1+(1/z)][1+(2/z)]} = 0$$

EXAMPLE 10.45 Prove that the final value of $x(n)$ for $X(z) = z^2/(z-1)(z-0.2)$ is 1.25 and its initial value is unity.

Solution: Given $$X(z) = \frac{z^2}{(z-1)(z-0.2)}$$

The final value theorem states that

$$\underset{n\to\infty}{\text{Lt}}\ x(n) = x(\infty) = \underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z)$$

$$\therefore \qquad x(\infty) = \underset{z\to 1}{\text{Lt}}\ (z-1)\frac{z^2}{(z-1)(z-0.2)} = \underset{z\to 1}{\text{Lt}}\ \frac{z^2}{z-0.2} = \frac{1}{1-0.2} = 1.25$$

The initial value theorem states that

$$\underset{n\to 0}{\text{Lt}}\ x(n) = x(0) = \underset{z\to\infty}{\text{Lt}}\ X(z)$$

$$\therefore \qquad x(0) = \underset{z\to\infty}{\text{Lt}}\ \frac{1}{[1-(1/z)][1-(0.2/z)]} = 1$$

10.7 INVERSE Z-TRANSFORM

In the last few sections, we have defined the Z-transform, found out the Z-transform of some simple signals (sequences) and then studied some Z-transform properties and theorems. These properties and theorems enable us to determine the Z-transforms of more complicated signals in terms of the Z-transforms of simple signals.

We shall now discuss about the inverse Z-transform. The process of finding the time domain signal $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform which is denoted as:

$$x(n) = Z^{-1}[X(z)]$$

We have

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\omega n}$$

This is the DTFT of the signal $x(n)\, r^{-n}$. Hence the inverse discrete-time Fourier transform (IDTFT) of $X(re^{j\omega})$ must be $x(n)\, r^{-n}$. Therefore, we can write

$$x(n) r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega}) e^{j\omega n}\, d\omega$$

i.e.
$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega}) (re^{j\omega})^n\, d\omega$$

We have

$$z = re^{j\omega}$$

$\therefore$
$$\frac{dz}{d\omega} = jre^{j\omega}, \text{ i.e. } d\omega = \frac{dz}{jre^{j\omega}}$$

$\therefore$
$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where the symbol $\oint_c$ denotes integration around the circle of radius $|z| = r$ in a counter clockwise direction.

This is the direct method of finding the inverse Z-transform of $X(z)$. It is quite tedious. So inverse Z-transform is normally found using indirect methods. The Z-transform $X(z)$ is a ratio of two polynomials in z given by

$$X(z) = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$

The roots of the numerator polynomial are those values of z for which $X(z) = 0$ and are referred to as the zeros of $X(z)$. The roots of the denominator polynomial are those values of z for which $X(z) = \infty$ and are referred to as poles of $X(z)$. In z-plane, zero locations are denoted by • (a small circle) symbol and the pole locations with × (cross) symbol.

Basically there are four methods that are often used to find the inverse Z-transform. They are:

(a) Power series method or long division method
(b) Partial fraction expansion method
(c) Complex inversion integral method (also known as the residue method)
(d) Convolution integral method

The long division method is simple, but does not give a closed form expression for the time signal. Further, it can be used only if the ROC of the given $X(z)$ is either of the form $|z| > \alpha$ or of the form $|z| < \alpha$, i.e. it is useful only if the sequence $x(n)$ is either purely right-sided or purely left-sided. The partial fraction expansion method enables us to determine the time signal $x(n)$ making use of our knowledge of some basic Z-transform pairs and Z-transform theorems. The inversion integral method requires a knowledge of the theory of complex variables, but is quite powerful and useful. The convolution integral method uses convolution property of Z-transforms and can be used when given $X(z)$ can be written as the product of two functions.

10.7.1 Long Division Method

The Z-transform of a two-sided sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The $X(z)$ has both positive powers of z as well as negative powers of z. We cannot obtain a two-sided sequence by long division. If the sequence $x(n)$ is causal, then

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

has only negative powers of z, with ROC; $|z| > \alpha$.

If the sequence $x(n)$ is anticausal, then

$$X(z) = \sum_{n=-\infty}^{0} x(n)\, z^{-n} = \cdots + x(-3)\, z^{3} + x(-2)\, z^{2} + x(-1)\, z^{1} + x(0)\, z^{0}$$

has only positive powers of z, with ROC; $|z| < \alpha$.

Since determination of the inverse Z-transform of $X(z)$ is only the determination of $x(n)$, i.e. $x(0)$, $x(1)$, $x(2)$, ... if it is a causal signal or $x(0)$, $x(-1)$, $x(-2)$, ... if it is an anticausal signal. To determine the inverse Z-transform, if $X(z)$ is a ratio of the polynomials

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

we can generate a series in z by dividing the numerator of $X(z)$ by its denominator.
If $X(z)$ converges for $|z| > \alpha$, we obtain the series

$$X(z) = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

We can identify the coefficients of z^{-n} as $x(n)$ of a causal sequence.
If $X(z)$ converges for $|z| < \alpha$, we obtain the series

$$X(z) = x(0) + x(-1)\, z^{1} + x(-2)\, z^{2} + \cdots$$

we can identify the coefficients of z^{-n} as $x(n)$ of a non-causal sequence.

For getting a causal sequence, first put $N(z)$ and $D(z)$ either in descending powers of z or in ascending powers of z^{-1} before long division.

For getting a non-causal sequence, first put $N(z)$ and $D(z)$ either in ascending powers of z or in descending powers of z^{-1} before long division. This method is best illustrated by the following examples.

EXAMPLE 10.46 Find the inverse Z-transform of

$$X(z) = z^3 + 2z^2 + z + 1 - 2z^{-1} - 3z^{-2} + 4z^{-3}$$

Solution: We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \ldots x(-3)\, z^3 + x(-2)\, z^2 + x(-1)\, z^1 + x(0) + x(1)\, z^{-1}$$

$$+ x(2)\, z^{-2} + x(3)\, z^{-3} + \cdots$$

Comparing this $X(z)$ with the given $X(z)$, we have

$$x(n) = \{1, 2, 1, \underset{\uparrow}{1}, -2, -3, 4\}$$

Alternatively, taking inverse Z-transform of $X(z)$, we have

$$x(n) = \delta(n+3) + 2\delta(n+2) + \delta(n+1) + \delta(n) - 2\delta(n-1) - 3\delta(n-2) + 4\delta(n-3)$$

EXAMPLE 10.47 Determine the inverse Z-transform of

(a) $X(z) = \dfrac{1}{z-a}$; ROC; $|z| > a$ (b) $X(z) = \dfrac{1}{1-az^{-1}}$; ROC; $|z| > a$

(c) $X(z) = \dfrac{1}{1-z^{-4}}$; ROC; $|z| > 1$

Solution:

(a) Given $X(z) = \dfrac{1}{z-a}$; ROC; $|z| > a$

$$= \frac{1}{z(1-az^{-1})} = z^{-1}(1-az^{-1})^{-1} = z^{-1}(1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots)$$

$$= z^{-1} + az^{-2} + a^2z^{-3} + \cdots = \sum_{n=1}^{\infty} a^{n-1} z^{-n} = \sum_{n=1}^{\infty} a^{n-1} u(n-1)\, z^{-n}$$

$\therefore$ $$x(n) = a^{n-1} u(n-1)$$

(b) Given $$X(z) = \frac{1}{1-az^{-1}}; \text{ ROC; } |z| > a$$

By Taylor's series expansion, we have

$$X(z) = \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n\, u(n)\, z^{-n}$$

Therefore, $$x(n) = a^n\, u(n)$$

(c) From infinite sum formula, we have

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}; \ |\alpha| < 1$$

Given $\quad X(z) = \dfrac{1}{1-z^{-4}} = \sum_{k=0}^{\infty} (z^{-4})^k = \sum_{k=0}^{\infty} z^{-4k} \quad [|z^{-4}| < 1, \text{ i.e.} |z| > 1]$

Taking inverse Z-transform on both sides, we get

$$x(n) = \sum_{k=0}^{\infty} \delta(n-4k)$$

$\therefore \quad x(n) = 1,$ when $n = 4k$, i.e. when n is an integer multiple of 4

$\qquad\quad = 0,$ otherwise

EXAMPLE 10.48 Find the inverse Z-transform of

$$X(z) = \sum_{k=4}^{8} \left(\frac{1}{2k}\right) z^{-k}; \text{ ROC}; |z| > 0$$

Solution: Given

$$X(z) = \sum_{k=4}^{8} \left(\frac{1}{2k}\right) z^{-k}$$

$$= \frac{1}{8} z^{-4} + \frac{1}{10} z^{-5} + \frac{1}{12} z^{-6} + \frac{1}{14} z^{-7} + \frac{1}{16} z^{-8}$$

Taking the inverse Z-transform, we get

$$x(n) = \frac{1}{8}\delta(n-4) + \frac{1}{10}\delta(n-5) + \frac{1}{12}\delta(n-6) + \frac{1}{14}\delta(n-7) + \frac{1}{16}\delta(n-8)$$

$$= \sum_{k=4}^{8} \left(\frac{1}{2k}\right) \delta(n-k)$$

EXAMPLE 10.49 Determine the inverse Z-transform of the sequences:

(a) $X(z) = \cos(3z)$; ROC; $|z| < \infty$ (b) $X(z) = \sin(z)$; ROC; $|z| < \infty$

Solution:

(a) Given $\quad X(z) = \cos(3z)$

The corresponding $x(n)$ must be a left-sided sequence because ROC is $|z| < \infty$. From the trigonometric series or Taylor series, we have

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}$$

$$\therefore \quad X(z) = \cos(3z) = \sum_{k=0}^{\infty} (-1)^k \frac{(3z)^{2k}}{(2k)!} = 1 - \frac{(3z)^2}{2!} + \frac{(3z)^4}{4!} - \frac{(3z)^6}{6!} + \cdots$$

$$= \cdots - \frac{81}{80}z^6 + \frac{27}{8}z^4 - \frac{9}{2}z^2 + 1$$

Therefore, the inverse Z-transform is:

$$x(n) = \left[\begin{array}{c} \ldots, -\frac{81}{80}, 0, \frac{27}{8}, 0, -\frac{9}{2}, 0, \underset{\uparrow}{1} \end{array}\right]$$

(b) Given $X(z) = \sin(2z)$. The corresponding $x(n)$ must be a left-sided sequence because ROC is $|z| < \infty$. From the trigonometric series or Taylor series, we have

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}$$

$$\therefore \quad X(z) = \sin(2z) = \sum_{k=0}^{\infty} (-1)^k \frac{(2z)^{2k+1}}{(2k+1)!} = (2z) - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!} + \cdots$$

$$= \cdots - \frac{8}{315}z^7 + \frac{4}{15}z^5 - \frac{4}{3}z^3 + 2z$$

Therefore, the inverse Z-transform is:

$$x(n) = \left[\begin{array}{c} \ldots, -\frac{8}{315}, 0, \frac{4}{15}, 0, -\frac{4}{3}, 0, 2, \underset{\uparrow}{0} \end{array}\right]$$

EXAMPLE 10.50 Determine the inverse Z-transform of the following transformed signals:

(a) $X(z) = e^z + e^{1/z}$ (b) $X(z) = \log_{10}(1 + az^{-1})$; ROC; $|z| > a$

Solution:

(a) Given $X(z) = e^z + e^{1/z}$

From the exponential series or Taylor series, we have

$$X(z) = e^z + e^{1/z} = e^z + e^{z^{-1}}$$

$$= \left\{1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right\} + \left\{1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \cdots\right\}$$

$$= \cdots + \frac{1}{24}z^4 + \frac{1}{6}z^3 + \frac{1}{2}z^2 + z + 2 + z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{6}z^{-3} + \frac{1}{24}z^{-4} + \cdots$$

Therefore, the inverse Z-transform is:

$$x(n) = \left\{ \dots, \frac{1}{24}, \frac{1}{6}, \frac{1}{2}, 1, \underset{\uparrow}{2}, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24} \dots \right\}$$

(b) Given $$X(z) = \log_{10}(1 + az^{-1})$$

Since the given signal is of base 10, manipulating, we get

$$X(z) = \log_{10}(1 + az^{-1}) = \frac{\log_e(1 + az^{-1})}{\log_e(10)}$$

$$= \frac{1}{\log_e(10)}\left[az^{-1} - \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} + \frac{(az^{-1})^5}{5} - \cdots \right]$$

$$= -\sum_{n=1}^{\infty} \frac{(-az^{-1})^n}{n \log_e(10)} = -\sum_{n=1}^{\infty} \frac{(-a)^n}{n \log_e(10)} z^{-n}$$

Therefore, the inverse Z-transform is:

$$x(n) = \left[\underset{\uparrow}{0}, \frac{a}{\log_e(10)}, -\frac{a^2}{2\log_e(10)}, \frac{a^3}{3\log_e(10)}, -\frac{a^4}{4\log_e(10)}, \cdots \right]$$

$$= -\frac{(-a)^n}{n \log_e(10)} u(n-1)$$

EXAMPLE 10.51 Using power series expansion method, determine the inverse Z-transform of

$$X(z) = \ln(1 + z^{-1});\ \text{ROC};\ |z| > 0$$

Solution: Given $$X(z) = \ln(1 + z^{-1})$$

The corresponding $x(n)$ must be right-sided sequence because ROC is $|z| > 0$. We know that

$$\ln(1 + \theta) = \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \frac{\theta^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\theta^k}{k};\ \text{if } |\theta| < 1$$

$$\therefore \quad X(z) = \ln(1 + z^{-1}) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(z^{-1})^k}{k} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^{-k}}{k}$$

Taking inverse Z-transform on both sides, we get

$$x(n) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\delta(n-k)}{k}$$

EXAMPLE 10.52 Using power series expansion method, determine the inverse Z-transform of

$$X(z) = \ln(1 - 2z);\ \text{ROC};\ |z| < \frac{1}{2}$$

Solution: Given

$$X(z) = \ln(1 - 2z);\ \text{ROC};\ |z| < \frac{1}{2}$$

Since ROC is $|z| = (1/2)$, the corresponding sequence $x(n)$ must be a left-sided sequence. We know that

$$\ln(1-\theta) = -\theta - \frac{\theta^2}{2} - \frac{\theta^3}{3} - \frac{\theta^4}{4} - \cdots = -\sum_{k=1}^{\infty} \frac{\theta^k}{k}$$

$$\therefore \qquad X(z) = \ln(1-2z) = -\sum_{k=1}^{\infty} \frac{(2z)^k}{k} = -\sum_{k=1}^{\infty} \frac{2^k}{k} z^k$$

Taking inverse Z-transform on both sides, we get

$$x(n) = -\sum_{k=1}^{\infty} \frac{2^k}{k} \delta(n+k)$$

EXAMPLE 10.53 Determine the inverse Z-transform of

$$X(z) = \ln\left(\frac{\alpha}{\alpha - z^{-1}}\right);\ \text{ROC};\ |z| > \frac{1}{\alpha}$$

Solution: Given $X(z) = \ln\left(\dfrac{\alpha}{\alpha - z^{-1}}\right) = \ln\left[\dfrac{1}{1-(\alpha z)^{-1}}\right] = -\ln[1-(\alpha z)^{-1}]$

We know that

$$\ln(1-\theta) = -\theta - \frac{\theta^2}{2} - \frac{\theta^3}{3} - \frac{\theta^4}{4} - \cdots$$

$$\therefore \quad X(z) = -\ln[1-(\alpha z)^{-1}] = (\alpha z)^{-1} + \frac{(\alpha z)^{-2}}{2} + \frac{(\alpha z)^{-3}}{3} + \frac{(\alpha z)^{-4}}{4} + \cdots = \sum_{k=1}^{\infty} \frac{[(\alpha z)^{-1}]^k}{k}$$

The above series converges for $(\alpha z)^{-1} < 1$, i.e. for $|z| > (1/|\alpha|)$.

$$\therefore \qquad X(z) = \sum_{k=1}^{\infty} \frac{\alpha^{-k}}{k} z^{-k}$$

Taking inverse Z-transform, we get

$$x(n) = \sum_{k=1}^{\infty} \frac{\alpha^{-k}}{k} \delta(n-k)$$

$$x(n) = \frac{\alpha^{-n}}{n} \quad \text{for } n > 0, \text{ i.e. } x(n) = \left(\frac{\alpha^{-n}}{n}\right) u(n-1)$$

EXAMPLE 10.54 Determine the inverse Z-transform of

(a) $X(z) = \log_e \left(\dfrac{1}{1 - a^{-1}z}\right)$; ROC; $|z| < |a|$ (b) $X(z) = \log_e \left(\dfrac{1}{1 - az^{-1}}\right)$; ROC; $|z| > |a|$

Solution:

(a) Given $$X(z) = \log_e \left(\frac{1}{1 - a^{-1}z}\right); \text{ ROC; } |z| < |a|$$

$$X(z) = \log_e \left(\frac{1}{1 - a^{-1}z}\right) = -\log_e (1 - a^{-1}z)$$

$$= -\left[-a^{-1}z - \frac{(a^{-1}z)^2}{2} - \frac{(a^{-1}z)^3}{3} - \frac{(a^{-1}z)^4}{4} - \cdots\right]$$

$\therefore$ $$X(z) = \cdots + \frac{1}{4a^4}z^4 + \frac{1}{3a^3}z^3 + \frac{1}{2a^2}z^2 + \frac{1}{a}z = \sum_{n=-\infty}^{-1} \left(\frac{a^n}{-n}\right) z^{-n}$$

Hence $$x(n) = -\frac{a^n}{n} \text{ for } n < 0$$

that is, $$x(n) = \left(-\frac{a^n}{n}\right) u(-n-1)$$

(b) Given $$X(z) = \log_e \left(\frac{1}{1 - az^{-1}}\right); \text{ ROC; } |z| > |a|$$

$$X(z) = \log_e \left(\frac{1}{1 - az^{-1}}\right) = -\log_e (1 - az^{-1})$$

$$= -\left[-az^{-1} - \frac{(az^{-1})^2}{2} - \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} - \cdots\right]$$

$$\therefore \; X(z) = az^{-1} + \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} + \frac{(az^{-1})^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(az^{-1})^k}{k} = \sum_{k=1}^{\infty} \frac{a^k}{k} z^{-k}$$

Taking inverse Z-transform, we have

$$x(n) = \sum_{k=1}^{\infty} \frac{a^k}{k} \delta(n-k) = \sum_{n=1}^{\infty} \frac{a^n}{n} = \frac{a^n}{n} u(n-1)$$

EXAMPLE 10.55 Discuss the methods by which inverse Z-transformation can be found out?

Solution: The process of finding $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform and is denoted as:

$$x(n) = Z^{-1}[X(z)]$$

There are four methods often used to find the inverse Z-transform.

1. Power series method or long division method
2. Partial fraction method
3. Complex inversion integral method or residue method
4. Convolution integral method

The long division method is simple and the advantage of this method is: it is more general and can be applied to any problem, but the disadvantage is: it is difficult to get the solution in closed form. Further it can be used only if the ROC of the given $X(z)$ is either of the form $|z| > \alpha$ or of the form $|z| < \alpha$, i.e. it is useful only if the sequence $x(n)$ is either purely right-sided or purely left-sided. It cannot be used for bidirectional sequences.

If $X(z)$ is a ratio of two polynomials, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

We can generate a series in z by dividing the numerator by the denominator. If ROC is $|z| > a$, it yields a causal sequence.

$$X(z) = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

So $X(z)$ is to be expressed in negative powers of z.

If ROC is $|z| < a$, it yields an anticausal sequence.

$$X(z) = x(0) + x(-1)\, z^{1} + x(-2)\, z^{2} + \cdots$$

So $X(z)$ is to be expressed in positive powers of z.

In long division method to realise a causal sequence (i.e. if ROC is $|z| > a$) both numerator and denominator are expressed either in descending powers of z or in ascending powers of z^{-1}, and the numerator is divided by the denominator continuously. To realise an anticausal sequence (i.e. ROC is $|z| < a$) both numerator and denominator are expressed either in ascending powers of z or in descending power of z^{-1} and the numerator is divided by the denominator continuously.

For partial fraction expansion method, $X(z)/z$ must be proper and the denominator should be in factored form. If it is not proper, it is to be written as the sum of a polynomial and a proper transfer function. The proper function $X(z)/z$ is written in terms of partial fractions and inverse Z-transform of each partial fraction is found by using the table of standard Z-transform pairs and all of them are added.

In the residue method, the inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where c is a circle in the z-plane in the ROC of $X(z)$. The above equation can also be written as

$$x(n) = \sum \text{Residues of } X(z)\, z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z - z_i)\, X(z)\, z^{n-1}\Big|_{z = z_i}$$

If $X(z)$ has repeated poles of order k, the residue at that pole is given by

$$\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(z-z_i)^k X(z)\, z^{n-1}\Big|_{z=z_i}$$

The convolution integral method uses convolution property of Z-transforms to determine the inverse Z-transform and can be used when the given $X(z)$ can be written as the product of two functions. In this method, the given $X(z)$ is written as the product of two functions $X_1(z)$ and $X_2(z)$ and the inverse Z-transform of those two functions, i.e. $x_1(n)$ and $x_2(n)$ are determined separately. The inverse Z-transform of $X(z)$ is then obtained by convolving $x_1(n)$ and $x_2(n)$ in time domain.

EXAMPLE 10.56 Determine the inverse Z-transform of

$$X(z) = \log_e(1 + az^{-1});\ \text{ROC};\ |z| > a$$

Solution: Given $$X(z) = \log_e(1 + az^{-1})$$

Differentiating the given Z-transform with respect to z, we get

$$\frac{dX(z)}{dz} = \frac{1}{1+az^{-1}}(-az^{-2}) = \frac{-az^{-2}}{1+az^{-1}}$$

or $$-z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}} = az^{-1}\left[\frac{1}{1-(-a)z^{-1}}\right]$$

But from the property of differentiation in z-domain, we have

$$nx(n) \xleftrightarrow{\text{ZT}} -z\frac{dX(z)}{dz}$$

Hence $$nx(n) = Z^{-1}\left[az^{-1}\left(\frac{1}{1-(-a)z^{-1}}\right)\right] = -Z^{-1}\left[\frac{(-a)z^{-1}}{1-(-a)z^{-1}}\right]$$

$$= -(-a)\,Z^{-1}\left[\frac{1}{z-(-a)}\right] = -(-a)\,[(-a)^{n-1}u(n-1)] = -(-a)^n u(n-1)$$

$\therefore$ $$x(n) = -\frac{(-a)^n}{n}u(n-1)$$

Alternately

$$X(z) = \log_e(1+az^{-1}) = az^{-1} - \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} + \cdots$$

$$= -\sum_{n=1}^{\infty}\frac{(-az^{-1})^n}{n} = -\sum_{n=1}^{\infty}\frac{(-a)^n}{n}z^{-n} = -\frac{(-a)^n}{n}u(n-1)$$

$\therefore$ $$x(n) = -\frac{(-a)^n}{n}u(n-1)$$

EXAMPLE 10.57 Using long division, determine the inverse Z-transform of

$$X(z) = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1}; \text{ ROC; } |z| > 1$$

Solution: Since ROC is $|z| > 1$, $x(n)$ must be a causal sequence. For getting a causal sequence, the $N(z)$ and $D(z)$ of $X(z)$ must be put either in descending powers of z or in ascending powers of z^{-1} before performing long division.

In the given $X(z)$ both $N(z)$ and $D(z)$ are already in descending powers of z.

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\ \hline
z^3 - 3z^2 + 4z + 1 & z^2 + 2z \\
 & \underline{z^2 - 3z + 4 + z^{-1}} \\
 & \quad 5z - 4 - z^{-1} \\
 & \quad \underline{5z - 15 + 20z^{-1} + 5z^{-2}} \\
 & \qquad 11 - 21z^{-1} - 5z^{-2} \\
 & \qquad \underline{11 - 33z^{-1} + 44z^{-2} + 11z^{-3}} \\
 & \qquad\quad 12z^{-1} - 49z^{-2} - 11z^{-3} \\
 & \qquad\quad \underline{12z^{-1} - 36z^{-2} + 48z^{-3} + 12z^{-4}} \\
 & \qquad\qquad -13z^{-2} - 59z^{-3} - 12z^{-4}
\end{array}$$

$$\therefore \quad X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \cdots$$

$$\therefore \quad x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Writing $N(z)$ and $D(z)$ of $X(z)$ in ascending powers of z^{-1}, we have

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1} = \frac{z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 4z^{-2} + z^{-3}}$$

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\ \hline
1 - 3z^{-1} + 4z^{-2} + z^{-3} & z^{-1} + 2z^{-2} \\
 & \underline{z^{-1} - 3z^{-2} + 4z^{-3} + z^{-4}} \\
 & \quad 5z^{-2} - 4^{-3} - z^{-4} \\
 & \quad \underline{5z^{-2} - 15z^{-3} + 20z^{-4} + 5z^{-5}} \\
 & \qquad 11z^{-3} - 21z^{-4} - 5z^{-5} \\
 & \qquad \underline{11z^{-3} - 33z^{-4} + 44z^{-5} + 11z^{-6}} \\
 & \qquad\quad 12z^{-4} - 49z^{-5} - 11z^{-6} \\
 & \qquad\quad \underline{12z^{-4} - 36z^{-5} + 48z^{-6} + 12z^{-7}} \\
 & \qquad\qquad -13z^{-5} - 59z^{-6} - 12z^{-7}
\end{array}$$

$$\therefore \quad X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \ldots$$

$$\therefore \quad x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Observe that both the methods give the same sequence $x(n)$.

EXAMPLE 10.58 Using long division, determine the inverse Z-transform of

$$X(z) = \frac{z^2 + z + 2}{z^3 - 2z^2 + 3z + 4}; \text{ ROC; } |z| < 1$$

Solution: Since ROC is $|z| < 1$, $x(n)$ must be a non-causal sequence. For getting a non-causal sequence, the $N(z)$ and $D(z)$ must be put either in ascending powers of z or in descending powers of z^{-1} before performing long division.

$$X(z) = \frac{z^2 + z + 2}{z^3 - 2z^2 + 3z + 4} = \frac{2 + z + z^2}{4 + 3z - 2z^2 + z^3}$$

$$\frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4$$

$$4 + 3z - 2z^2 + z^3 \;\Big)\; 2 + z + z^2$$

$$2 + \frac{3}{2}z - z^2 + \frac{1}{2}z^3$$

$$-\frac{1}{2}z + 2z^2 - \frac{1}{2}z^3$$

$$-\frac{1}{2}z - \frac{3}{8}z^2 + \frac{1}{4}z^3 - \frac{1}{8}z^4$$

$$\frac{19}{8}z^2 - \frac{3}{4}z^3 + \frac{1}{8}z^4$$

$$\frac{19}{8}z^2 + \frac{57}{32}z^3 - \frac{19}{16}z^4 + \frac{19}{32}z^5$$

$$-\frac{81}{32}z^3 + \frac{21}{16}z^4 - \frac{19}{32}z^5$$

$$-\frac{81}{32}z^3 - \frac{243}{128}z^4 + \frac{81}{64}z^5 - \frac{81}{128}z^6$$

$$\frac{411}{128}z^4 - 129z^5 + \frac{81}{128}z^6$$

$$\therefore \qquad X(z) = \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 \ldots$$

$$\therefore \qquad x(n) = \left\{ \ldots, \frac{411}{512}, -\frac{81}{128}, \frac{19}{32}, -\frac{1}{8}, \underset{\uparrow}{\frac{1}{2}} \right\}$$

Also
$$X(z) = \frac{2 + z + z^2}{4 + 3z - 2z^2 + z^3} = \frac{2z^{-3} + z^{-2} + z^{-1}}{4z^{-3} + 3z^{-2} - 2z^{-1} + 1}$$

$$
\begin{array}{r|l}
 & \frac{1}{2}-\frac{1}{8}z+\frac{19}{32}z^2-\frac{81}{128}z^3+\frac{411}{512}z^4+\cdots \\
\hline
4z^{-3}+3z^{-2}-2z^{-1}+1 & 2z^{-3}+z^{-2}+z^{-1} \\
 & 2z^{-3}+\frac{3}{2}z^{-2}-z^{-1}+\frac{1}{2} \\
\cline{2-2}
 & -\frac{1}{2}z^{-2}+2z^{-1}-\frac{1}{2} \\
 & -\frac{1}{2}z^{-2}-\frac{3}{8}z^{-1}+\frac{1}{4}-\frac{1}{8}z \\
\cline{2-2}
 & \frac{19}{8}z^{-1}-\frac{3}{4}+\frac{1}{8}z \\
 & \frac{19}{8}z^{-1}+\frac{57}{32}-\frac{19}{16}z+\frac{19}{32}z^2 \\
\cline{2-2}
 & -\frac{81}{32}+\frac{21}{16}z-\frac{19}{32}z^2 \\
 & -\frac{18}{32}-\frac{243}{128}z+\frac{81}{64}z^2-\frac{81}{128}z^3 \\
\cline{2-2}
 & \frac{411}{128}z-\frac{119}{69}z^2+\frac{81}{128}z^3
\end{array}
$$

$$\therefore \qquad X(z)=\frac{1}{2}-\frac{1}{8}z+\frac{19}{32}z^2-\frac{81}{128}z^3+\frac{411}{512}z^4+\cdots$$

$$\therefore \qquad x(n)=\left\{\ldots,\frac{411}{512},-\frac{81}{128},\frac{19}{32},-\frac{1}{8},\underset{\uparrow}{\frac{1}{2}}\right\}$$

Observe that both the methods give the same sequence $x(n)$.

We can say from the above examples that, this method does not give $x(n)$ in a closed form expression in terms of n and hence is useful only if one is interested in determining the first few terms of the sequence $x(n)$.

EXAMPLE 10.59 Using the power series expansion technique, find the inverse Z-transform of the following $X(z)$:

(a) $X(z)=\dfrac{z}{2z^2-3z+1}$; ROC; $|z|<\dfrac{1}{2}$ (b) $X(z)=\dfrac{z}{2z^2-3z+1}$; ROC; $|z|>1$

Solution:

(a) Given $$X(z)=\frac{z}{2z^2-3z+1};\ \text{ROC; } |z|<\frac{1}{2}$$

Since ROC is $|z| < 1/2$, the given signal is a non-causal signal. So express both numerator and denominator in ascending powers of z or in descending powers of z^{-1} and make long division.

$$
\begin{array}{r|l}
 & z + 3z^2 + 7z^3 + 15z^4 + 31z^5 + 63z^6 + \cdots \\
\hline
1 - 3z + 2z^2 & z \\
 & \underline{z - 3z^2 + 2z^3} \\
 & \quad 3z^2 - 2z^3 \\
 & \quad \underline{3z^2 - 9z^3 + 6z^4} \\
 & \qquad 7z^3 - 6z^4 \\
 & \qquad \underline{7z^3 - 21z^4 + 14z^5} \\
 & \qquad\quad 15z^4 - 14z^5 \\
 & \qquad\quad \underline{15z^4 - 45z^5 + 30z^6} \\
 & \qquad\qquad 31z^5 - 30z^6 \\
 & \qquad\qquad \underline{31z^5 - 93z^6 + 62z^7} \\
 & \qquad\qquad\quad 63z^6 - 62z^7
\end{array}
$$

$$\therefore \quad X(z) = \frac{z}{2z^2 - 3z + 1}; \text{ ROC; } |z| < \frac{1}{2} = z + 3z^2 + 7z^3 + 15z^4 + 31z^5 + 63z^6 + \cdots$$

$$\therefore \quad x(n) = Z^{-1}[z/(2z^2 - 3z + 1)]$$

$$= \delta(n+1) + 3\delta(n+2) + 7\delta(n+3) + 15\delta(n+4) + 31\delta(n+5) + 63\delta(n+6) + \cdots$$

$$\therefore \qquad x(n) = \{\ldots, 63, 31, 15, 7, 3, 1, 0\}$$

(b) Given $\quad X(z) = z/(2z^2 - 3z + 1)$; ROC; $|z| > 1$

Since ROC is $|z| > 1$, the given signal is a causal signal. So express both numerator and denominator in descending powers of z or in ascending powers of z^{-1} before long division.

$$
\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \frac{31}{32}z^{-5} \\
\hline
2z^2 - 3z + 1 & z \\
 & \underline{z - \frac{3}{2} + \frac{1}{2}z^{-1}} \\
 & \quad \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & \quad \underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}} \\
 & \qquad \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\
 & \qquad \underline{\frac{7}{4}z^{-1} - \frac{21}{8}z^{-2} + \frac{7}{8}z^{-3}} \\
 & \qquad\quad \frac{15}{8}z^{-2} - \frac{7}{8}z^{-3} \\
 & \qquad\quad \underline{\frac{15}{8}z^{-2} - \frac{45}{16}z^{-3} + \frac{15}{16}z^{-4}} \\
 & \qquad\qquad \frac{31}{16}z^{-3} - \frac{15}{16}z^{-4}
\end{array}
$$

$$\therefore \quad X(z) = \frac{z}{2z^2 - 3z + 1};\ \text{ROC; } |z| > 1 = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \frac{31}{32}z^{-5} + \cdots$$

$$\therefore \quad x(n) = Z^{-1}\left(\frac{z}{2z^2 - 3z + 1}\right)$$

$$= \frac{1}{2}\delta(n-1) + \frac{3}{4}\delta(n-2) + \frac{7}{8}\delta(n-3) + \frac{15}{16}\delta(n-4) + \frac{31}{32}\delta(n-5) + \cdots$$

$$\therefore \qquad x(n) = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \cdots\right\}$$

EXAMPLE 10.60 Determine the inverse Z-transform of $X(z) = \dfrac{1}{2 - 4z^{-1} + 2z^{-2}}$ by long division method when

(a) ROC; $|z| > 1$ (b) ROC; $|z| < 1/2$

Solution: Given $\quad X(z) = \dfrac{1}{2 - 4z^{-1} + 2z^{-2}} = \dfrac{z^2}{2z^2 - 4z + 2}$

(a) ROC; $|z| > 1$

When ROC is $|z| > 1$, all the terms must be causal. So express both numerator and denominator of $X(z)$ in descending powers of z before long division.

$$\begin{array}{r|l}
 & \frac{1}{2} + z^{-1} + \frac{3}{2}z^{-2} + 2z^{-3} + \frac{5}{2}z^{-4} + 3z^{-5} \\
\hline
2z^2 - 4z + 2 & z^2 \\
 & \underline{z^2 - 2z + 1} \\
 & \quad 2z - 1 \\
 & \quad \underline{2z - 4 + 2z^{-1}} \\
 & \qquad 3 - 2z^{-1} \\
 & \qquad \underline{3 - 6z^{-1} + 3z^{-2}} \\
 & \qquad\quad 4z^{-1} - 3z^{-2} \\
 & \qquad\quad \underline{4z^{-1} - 8z^{-2} + 4z^{-3}} \\
 & \qquad\qquad 5z^{-2} - 4z^{-3} \\
 & \qquad\qquad \underline{5z^{-2} - 10z^{-3} + 5z^{4}} \\
 & \qquad\qquad\quad 6z^{-3} - 5z^{-4}
\end{array}$$

$$\therefore \qquad X(z) = \frac{1}{2 - 4z^{-1} + 2z^{-2}} = \frac{1}{2} + z^{-1} + \frac{3}{2}z^{-2} + 2z^{-3} + \frac{5}{2}z^{-4} + 3z^{-5} + \cdots$$

$$\therefore \quad x(n) = Z^{-1}\left(\frac{1}{2 - 4z^{-1} + 2z^{-2}}\right)$$

$$= \frac{1}{2}\delta(n) + \delta(n-1) + \frac{3}{2}\delta(n-2) + 2\delta(n-3) + \frac{5}{2}\delta(n-4) + 3\delta(n-5) + \cdots$$

$\therefore \qquad x(n)=\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\right\}$

(b) ROC; $|z| < 1/2$

When ROC is $|z| < 1/2$ all the terms must be anticausal. So express both numerator and denominator in ascending powers of z or in descending powers of z^{-1} before making long division.

$$\therefore \qquad X(z)=\frac{z^2}{2-4z+2z^2}$$

$$\begin{array}{r|l}
 & \frac{1}{2}z^2 + z^3 + \frac{3}{2}z^4 + 2z^5 + \frac{5}{2}z^6 + \cdots \\
\hline
2 - 4z + 2z^2 & z^2 \\
 & z^2 - 2z^3 + z^4 \\
\hline
 & 2z^3 - z^4 \\
 & 2z^3 - 4z^4 + 2z^5 \\
\hline
 & 3z^4 - 2z^5 \\
 & 3z^4 - 6z^5 + 3z^6 \\
\hline
 & 4z^5 - 3z^6 \\
 & 4z^5 - 8z^6 + 4z^7 \\
\hline
 & 5z^6 - 4z^7
\end{array}$$

$$\therefore \qquad X(z)=\frac{1}{2-4z^{-1}+2z^{-2}}=\frac{1}{2}z^2+z^3+\frac{3}{2}z^4+2z^5+\frac{5}{2}z^6+\cdots$$

$$\therefore \qquad x(n)=Z^{-1}[X(z)]=Z^{-1}\left(\frac{1}{2-4z^{-1}+2z^{-2}}\right) \quad \text{for ROC; } |z|<\frac{1}{2}$$

$$x(n)=\frac{1}{2}\delta(n+2)+\delta(n+3)+\frac{3}{2}\delta(n+4)+2\delta(n+5)+\frac{5}{2}\delta(n+6)+\cdots$$

$$\therefore \qquad x(n)=\left\{\cdots \frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}, 0, 0\right\}$$

EXAMPLE 10.61 Find the inverse Z-transform of $X(z)$ using long division method.

$$X(z)=\frac{2+3z^{-1}}{(1+z^{-1})[1+(1/4)z^{-1}-(1/8)z^{-2}]}$$

Solution: Given $X(z)=\frac{2+3z^{-1}}{(1+z^{-1})[1+(1/4)z^{-1}-(1/8)z^{-2}]}=\frac{2+3z^{-1}}{1+(5/4)z^{-1}+(1/8)z^{-2}-(1/8)z^{-3}}$

Let us assume that a causal sequence is to be obtained.

Long division

$$
\begin{array}{r|l}
 & 2+\frac{1}{2}z^{-1}-\frac{7}{8}z^{-2}+\frac{41}{32}z^{-3}-\frac{183}{128}z^{-4}+\cdots \\
\hline
1+\frac{5}{4}z^{-1}+\frac{1}{8}z^{-2}-\frac{1}{8}z^{-3} & 2+3z^{-1} \\
 & 2+\frac{5}{2}z^{-1}+\frac{1}{4}z^{-2}-\frac{1}{4}z^{-3} \\
\cline{2-2}
 & \frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}+\frac{1}{4}z^{-3} \\
 & \frac{1}{2}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{16}z^{-3}-\frac{1}{16}z^{-4} \\
\cline{2-2}
 & -\frac{7}{8}z^{-2}+\frac{3}{16}z^{-3}+\frac{1}{16}z^{-4} \\
 & -\frac{7}{8}z^{-2}-\frac{35}{32}z^{-3}-\frac{7}{64}z^{-4}+\frac{7}{64}z^{-5} \\
\cline{2-2}
 & \frac{41}{32}z^{-3}+\frac{11}{64}z^{-4}-\frac{7}{64}z^{-5} \\
 & \frac{41}{32}z^{-3}+\frac{205}{128}z^{-4}+\frac{41}{256}z^{-5}-\frac{41}{256}z^{-6} \\
\cline{2-2}
 & -\frac{183}{128}z^{-4}
\end{array}
$$

$\therefore$
$$X(z)=2+\frac{1}{2}z^{-1}-\frac{7}{8}z^{-2}+\frac{41}{32}z^{-3}-\frac{183}{128}z^{-4}+\cdots$$

Taking inverse Z-transform on both sides, we have

$$x(n)=2\delta(n)+\frac{1}{2}\delta(n-1)-\frac{7}{8}\delta(n-2)+\frac{41}{32}\delta(n-3)-\frac{183}{128}\delta(n-4)+\cdots$$

or
$$x(n)=\left\{2,\frac{1}{2},\frac{-7}{8},\frac{41}{32},\frac{-183}{128},\ldots\right\}$$

EXAMPLE 10.62 Given

(a) $X(z)=\dfrac{1}{1-az^{-1}}$; ROC; $|z|>a$ (b) $X(z)=\dfrac{z^2}{z^2-1.5z+0.5}$; $|z|<0.5$

Find $x(n)$ using long division.

Solution:

(a) Given $X(z)=\dfrac{1}{1-az^{-1}}=\dfrac{z}{z-a}$; ROC; $|z|>a$

Since ROC is $|z|>a$, the signal must be a causal signal. Hence, the numerator and denominator must be in descending powers of z for long division.

$$
\begin{array}{r|l}
 & 1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+\frac{a^4}{z^4}+\cdots \\ \hline
z-a & z \\
 & z-a \\ \hline
 & a \\
 & a-\frac{a^2}{z} \\ \hline
 & \frac{a^2}{z} \\
 & \frac{a^2}{z}-\frac{a^3}{z^2} \\ \hline
 & \frac{a^3}{z^2} \\
 & \frac{a^3}{z^2}-\frac{a^4}{z^3} \\ \hline
 & \frac{a^4}{z^3}
\end{array}
$$

$\therefore$ $$X(z)=\frac{z}{z-a}=1+az^{-1}+a^2z^{-2}+a^3z^{-3}+a^4z^{-4}+\cdots$$

Taking inverse Z-transform on both sides, we have

$$x(n)=1+a\delta(n-1)+a^2\delta(n-2)+a^3\delta(n-3)+a^4\delta(n-4)+\cdots$$

$\therefore$ $$x(n)=\{1, a, a^2, a^3, a^4, \ldots\}$$

(b) Given $$X(z)=\frac{z^2}{z^2-1.5z+0.5}; \ |z|<0.5$$

Since ROC is $|z|<0.5$ the signal must be an anticausal signal. Hence the numerator and denominator must be in ascending powers of z for long division.

$$
\begin{array}{r|l}
 & 2z^2+6z^3+14z^4+30z^5+\cdots \\ \hline
0.5-1.5z+z^2 & z^2 \\
 & z^2-3z^3+2z^4 \\ \hline
 & 3z^3-2z^4 \\
 & 3z^3-9z^4+6z^5 \\ \hline
 & 7z^4-6z^5 \\
 & 7z^4-21z^5+14z^6 \\ \hline
 & 15z^5-14z^6
\end{array}
$$

$$\therefore \quad X(z) = \frac{z}{z^2 - 1.5z + 0.5} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + \cdots$$

$$\therefore \quad x(n) = 2\delta(n+2) + 6\delta(n+3) + 14\delta(n+4) + 30\delta(n+5) + \cdots$$
$$= \{\ldots, 30, 14, 6, 2, 0, 0\}$$

10.7.2 Partial Fraction Expansion Method

To find the inverse Z-transform of $X(z)$ using partial fraction expansion method, its denominator must be in factored form. It is similar to the partial fraction expansion method used earlier for the inversion of Laplace transforms. However, in this case, we try to obtain the partial fraction expansion of $X(z)/z$ instead of $X(z)$. This is because, the Z-transform of time domain signals have z in their numerators. This method can be applied only if $X(z)/z$ is a proper rational function (i.e. the order of its denominator is greater than the order of its numerator). If $X(z)/z$ is not proper, then it should be written as the sum of a polynomial and a proper function before applying this method. The disadvantage of this method is that, the denominator must be factored. Using known Z-transform pairs and the properties of Z-transform, the inverse Z-transform of each partial fraction can be found.

Consider a rational function $X(z)/z$ given by

$$\frac{X(z)}{z} = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$

When $M < N$, it is a proper function.
When $M \geq N$, it is not a proper function, so write it as:

$$\frac{X(z)}{z} = \underbrace{c_0 z^{N-M} + c_1 z^{N-M-1} + \cdots + c_{N-M}}_{\text{polynomial}} + \underbrace{\frac{N_1(z)}{D(z)}}_{\text{Proper rational function}}$$

There are two cases for the proper rational function $X(z)/z$.

Case 1 $X(z)/z$ has all distinct poles.

When all the poles of $X(z)/z$ are distinct, then $X(z)/z$ can be expanded in the form

$$\frac{X(z)}{z} = \frac{C_1}{z - P_1} + \frac{C_2}{z - P_2} + \cdots + \frac{C_N}{z - P_N}$$

The coefficients C_1, C_2, ..., C_N can be determined using the formula

$$C_k = (z - P_k)\left.\frac{X(z)}{z}\right|_{z=P_k}, \quad k = 1, 2, \ldots, N$$

Case 2 $X(z)/z$ has l-repeated poles and the remaining N-l poles are simple. Let us say the kth pole is repeated l times. Then, $X(z)/z$ can be written as:

$$\frac{X(z)}{z} = \underbrace{\frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \cdots}_{(N\text{-}l)\text{ terms}} + \frac{C_{k1}}{z-P_k} + \frac{C_{k2}}{(z-P_k)^2} + \cdots + \frac{C_{kl}}{(z-P_k)^l}$$

where
$$C_{kl} = (z-P_k)^l \left.\frac{X(z)}{z}\right|_{z=P_k}$$

In general,
$$C_{ki} = \frac{1}{(l-i)!}\frac{d^{l-i}}{dz^{l-i}}\left[(z-P_k)^l \frac{X(z)}{z}\right]\Bigg|_{z=P_k}$$

If $X(z)$ has a complex pole, then the partial fraction can be expressed as:

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_1^*}{z-P_1^*}$$

where C_1^* is complex conjugate of C_1 and P_1^* is complex conjugate of P_1.

In other words, complex conjugate poles result in complex conjugate coefficients in the partial fraction expansion.

EXAMPLE 10.63 Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}};\ \text{ROC};\ |z|>1$$

Solution: Given
$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}} = \frac{z}{3z^2-4z+1}$$

$$= \frac{z}{3[z^2-(4z/3)+(1/3)]} = \frac{1}{3}\frac{z}{(z-1)[z-(1/3)]}$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{1}{3}\frac{1}{(z-1)[z-(1/3)]} = \frac{A}{z-1} + \frac{B}{z-(1/3)}$$

where A and B can be evaluated as follows:

$$A = (z-1)\left.\frac{X(z)}{z}\right|_{z=1} = (z-1)\frac{1}{3}\frac{1}{(z-1)[z-(1/3)]}\Bigg|_{z=1} = \frac{1}{3}\frac{1}{1-(1/3)} = \frac{1}{2}$$

$$B = \left(z-\frac{1}{3}\right)\left.\frac{X(z)}{z}\right|_{z=1/3} = \left(z-\frac{1}{3}\right)\frac{1}{3}\frac{1}{(z-1)[z-(1/3)]}\Bigg|_{z=1/3} = \frac{1}{3}\frac{1}{(1/3)-1} = -\frac{1}{2}$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z-(1/3)}$$

or $$X(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-(1/3)}\right]; \text{ ROC; } |z| > 1$$

Since ROC is $|z| > 1$, both the sequences must be causal. Therefore, taking inverse Z-transform, we have

$$x(n) = \frac{1}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right]; \text{ ROC; } |z| > 1$$

EXAMPLE 10.64 Find the inverse Z-transform of the following:

$$X(z) = \frac{(1/6)\,z^{-1}}{[1-(1/2)\,z^{-1}]\,[1-(1/3)\,z^{-1}]}; \text{ ROC; } |z| > \frac{1}{2}$$

Solution: Given $$X(z) = \frac{(1/6)\,z^{-1}}{[1-(1/2)\,z^{-1}]\,[1-(1/3)\,z^{-1}]}; \text{ ROC; } |z| > \frac{1}{2}$$

Multiplying the numerator and denominator of $X(z)$ by z^2, we have

$$X(z) = \frac{(1/6)\,z}{[z-(1/2)]\,[z-(1/3)]}$$

The above equation can be expressed in partial fraction form as:

$$\frac{X(z)}{z} = \frac{1/6}{[z-(1/2)]\,[z-(1/3)]} = \frac{C_1}{z-(1/2)} + \frac{C_2}{z-(1/3)}$$

where C_1 and C_2 can be evaluated as follows:

$$C_1 = \left(z - \frac{1}{2}\right)\frac{X(z)}{z}\bigg|_{z=1/2} = \frac{1}{6}\,\frac{1}{z-(1/3)}\bigg|_{z=1/2} = \frac{1}{6}\,\frac{1}{(1/2)-(1/3)} = 1$$

$$C_2 = \left(z - \frac{1}{3}\right)\frac{X(z)}{z}\bigg|_{z=1/3} = \frac{1}{6}\,\frac{1}{z-(1/2)}\bigg|_{z=1/3} = \frac{1}{6}\,\frac{1/6}{(1/3)-(1/2)} = -1$$

$\therefore$ $$\frac{X(z)}{z} = \frac{1}{z-(1/2)} - \frac{1}{z-(1/3)}$$

or $$X(z) = \frac{z}{z-(1/2)} - \frac{z}{z-(1/3)}; \text{ ROC; } |z| > \frac{1}{2}$$

Since ROC is $|z| > 1/2$, both the sequences must be causal. Taking inverse Z-transform, we have

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE 10.65 Find the inverse Z-transform of

$$X(z)=\frac{z(z-1)}{(z+1)^3\,(z+2)};\ \text{ROC};\ |z|>2$$

Solution: Given
$$X(z)=\frac{z(z-1)}{(z+1)^3\,(z+2)};\ \text{ROC};\ |z|>2$$

$$\therefore\qquad \frac{X(z)}{z}=\frac{z-1}{(z+1)^3\,(z+2)}=\frac{C_1}{z+1}+\frac{C_2}{(z+1)^2}+\frac{C_3}{(z+1)^3}+\frac{C_4}{z+2}$$

where the constants C_1, C_2, C_3 and C_4 can be obtained as follows:

$$C_4=(z+2)\left.\frac{X(z)}{z}\right|_{z=-2}=\left.\frac{z-1}{(z+1)^3}\right|_{z=-2}=\frac{-2-1}{(-2+1)^3}=3$$

$$C_3=(z+1)^3\left.\frac{X(z)}{z}\right|_{z=-1}=\left.\frac{z-1}{(z+2)}\right|_{z=-1}=\frac{-1-1}{-1+2}=-2$$

$$C_2=\frac{1}{1!}\frac{d}{dz}\left.\left[(z+1)^3\frac{X(z)}{z}\right]\right|_{z=-1}=\frac{d}{dz}\left.\left(\frac{z-1}{z+2}\right)\right|_{z=-1}=\left.\frac{(z+2)\,(1)-(z-1)\,(1)}{(z+2)^2}\right|_{z=-1}=3$$

$$C_1=\frac{1}{2!}\frac{d^2}{dz^2}\left.\left[(z+1)^3\frac{X(z)}{z}\right]\right|_{z=-1}=\frac{1}{2!}\frac{d^2}{dz^2}\left.\left(\frac{z-1}{z+2}\right)\right|_{z=-1}$$

$$=\frac{1}{2!}\frac{d}{dz}\left.\left[\frac{3}{(z+2)^2}\right]\right|_{z=-1}=\frac{1}{2}\left.\frac{-3\times 2(z+2)}{(z+2)^4}\right|_{z=-1}=\frac{-3(-1+2)}{(-1+2)^3}=-3$$

$$\therefore\qquad \frac{X(z)}{z}=\frac{-3}{z+1}+\frac{3}{(z+1)^2}-\frac{2}{(z+1)^3}+\frac{3}{z+2}$$

$$\therefore\qquad X(z)=\frac{-3z}{z+1}+\frac{3z}{(z+1)^2}-\frac{2z}{(z+1)^3}+\frac{3z}{z+2};\ \text{ROC};\ |z|>2$$

Since ROC is $|z|>2$, all the above sequences must be causal. Taking inverse Z-transform on both sides, we have

$$x(n)=-3(-1)^n\,u(n)+3n(-1)^n\,u(n)-2(n)\,(n-1)\,(-1)^n\,u(n)+3(-2)^n u(n)$$
$$=[-3+3n-2n(n-1)]\,(-1)^n\,u(n)+3(-2)^n\,u(n)$$

EXAMPLE 10.66 Find the inverse Z-transform of

$$X(z)=\frac{2z-7}{z^2-5z+6};\ \text{ROC};\ |z|<2$$

Solution: Given $$X(z)=\frac{2z-7}{z^2-5z+6};\ \text{ROC};\ |z|<2$$

$$\frac{X(z)}{z}=\frac{2z-7}{z(z^2-5z+6)}=\frac{2z-7}{z(z-2)(z-3)}=\frac{C_1}{z}+\frac{C_2}{z-2}+\frac{C_3}{z-3}$$

where C_1, C_2 and C_3 are determined as follows:

$$C_1=z\frac{X(z)}{z}\bigg|_{z=0}=\frac{2z-7}{(z-2)(z-3)}\bigg|_{z=0}=-\frac{7}{6}$$

$$C_2=(z-2)\frac{X(z)}{z}\bigg|_{z=2}=\frac{2z-7}{z(z-3)}\bigg|_{z=2}=\frac{3}{2}$$

$$C_3=(z-3)\frac{X(z)}{z}\bigg|_{z=3}=\frac{2z-7}{z(z-2)}\bigg|_{z=3}=-\frac{1}{3}$$

$$\therefore \qquad \frac{X(z)}{z}=-\frac{7}{6}\left(\frac{1}{z}\right)+\frac{3}{2}\left(\frac{1}{z-2}\right)-\frac{1}{3}\left(\frac{1}{z-3}\right)$$

$$\therefore \qquad X(z)=-\frac{7}{6}+\frac{3}{2}\frac{z}{z-2}-\frac{1}{3}\frac{z}{z-3};\ \text{ROC};\ |z|<2$$

Since ROC is $|z|<2$, all the sequences must be anticausal. Hence taking inverse Z-transform of $X(z)$, we have

$$x(n)=-\frac{7}{6}\delta(n)-\frac{3}{2}(2)^n\,u(-n-1)+\frac{1}{3}(3)^n\,u(-n-1)$$

EXAMPLE 10.67 Find the inverse Z-transform of

$$X(z)=\frac{3z^{-1}}{(1-z^{-1})(1-2z^{-1})}$$

(a) If ROC; $|z|>2$ (b) If ROC; $|z|<1$ (c) If ROC; $1<|z|<2$

Solution: Given $$X(z)=\frac{3z^{-1}}{(1-z^{-1})(1-2z^{-1})}$$

Multiplying the numerator and denominator by z^2, we have

$$X(z)=\frac{3z}{(z-1)(z-2)}$$

$$\therefore \qquad \frac{X(z)}{z}=\frac{3}{(z-1)(z-2)}=\frac{C_1}{z-1}+\frac{C_2}{z-2}$$

where C_1 and C_2 are determined as follows:

$$C_1 = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{3}{z-2}\bigg|_{z=1} = -3$$

$$C_2 = (z-2)\frac{X(z)}{z}\bigg|_{z=2} = \frac{3}{z-1}\bigg|_{z=2} = 3$$

$\therefore$
$$\frac{X(z)}{z} = -\frac{3}{z-1} + \frac{3}{z-2}$$

or
$$X(z) = -3\frac{z}{z-1} + 3\frac{z}{z-2}$$

The pole-zero plot is shown in Figure 10.14.

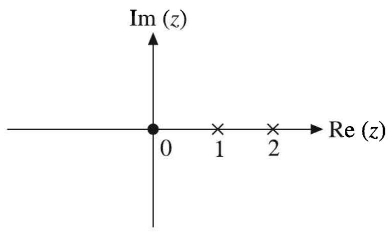


Figure 10.14 Pole-zero plot for Example 10.67.

We know that the ROC of the sum of two signals is equal to the intersection of the ROCs of the two signals. Therefore,

(a) If ROC is $|z| > 2$, both the signals must be causal.

Hence $$x(n) = -3u(n) + 3(2)^n u(n)$$

The ROC is sketched as shown in Figure 10.15(a).

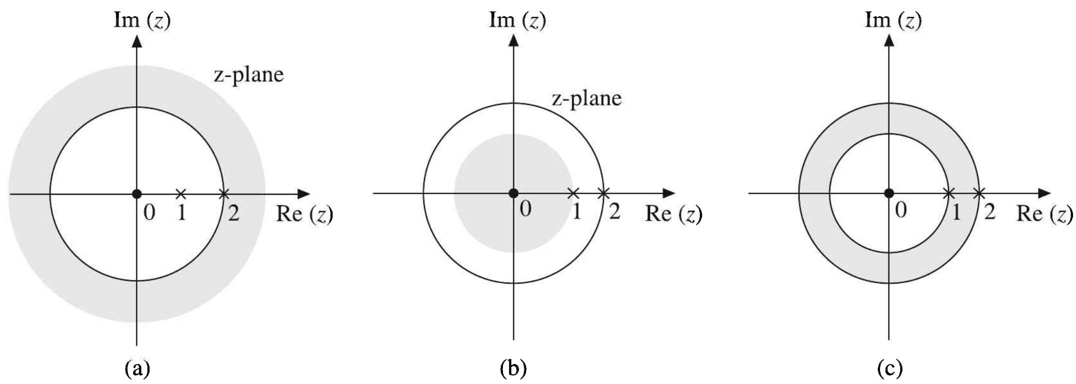


Figure 10.15 (a) ROC $|z| > 2$, (b) ROC $|z| < 1$, (c) ROC $1 < |z| < 2$.

(b) If ROC is $|z| < 1$, both the signals must be anticausal.

Hence $$x(n) = 3u(-n-1) - 3(2)^n u(-n-1)$$

The ROC is sketched as shown in Figure 10.15(b).

(c) If ROC is $1 < |z| < 2$, one signal (with pole at $z = 1$) must be causal and the second one (with pole at $z = 2$) must be anticausal.

Hence $$x(n) = 3u(n) - 3(2)^n u(-n-1)$$

The ROC is sketched as shown in Figure 10.15(c).

EXAMPLE 10.68 Determine all possible signals $x(n)$ associated with Z-transform.

$$X(z) = \frac{(1/4)z^{-1}}{[1-(1/2)z^{-1}][1-(1/4)z^{-1}]}$$

Solution: Given $$X(z) = \frac{(1/4)z^{-1}}{[1-(1/2)z^{-1}][1-(1/4)z^{-1}]}$$

Multiplying the numerator and denominator with z^2, we obtain

$$X(z) = \frac{(1/4)z}{[z-(1/2)][z-(1/4)]}$$

Now, $X(z)$ has two poles, one at $z = (1/2)$ and the other at $z = 1/4$ as shown in Figure 10.16. The possible ROCs are:

(a) ROC; $|z| > \frac{1}{2}$ (b) ROC; $|z| < \frac{1}{4}$ (c) ROC; $\frac{1}{4} < |z| < \frac{1}{2}$

Hence there are three possible signals $x(n)$ corresponding to these ROCs.

Now, $$\frac{X(z)}{z} = \frac{1/4}{[z-(1/2)][z-(1/4)]} = \frac{C_1}{z-(1/2)} + \frac{C_2}{z-(1/4)} = \frac{1}{z-(1/2)} - \frac{1}{z-(1/4)}$$

or $$X(z) = \frac{z}{z-(1/2)} - \frac{z}{z-(1/4)}$$

(a) ROC; $|z| > \frac{1}{2}$

Here both the poles, i.e. $z = (1/2)$ and $z = (1/4)$ correspond to causal terms.

$\therefore$ $$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

(b) ROC; $|z| < \frac{1}{4}$

Here both the poles must correspond to anticausal terms.

$\therefore$ $$x(n) = -\left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{4}\right)^n u(-n-1)$$

(c) ROC; $\frac{1}{4} < |z| < \frac{1}{2}$

Here the pole at $z = (1/4)$ must correspond to causal term and the pole at $z = (1/2)$ must correspond to anticausal term.

$$\therefore \qquad x(n) = -\left(\frac{1}{2}\right)^n u(-n-1) - \left(\frac{1}{4}\right)^n u(n)$$

The ROCs are shown in Figure 10.16.

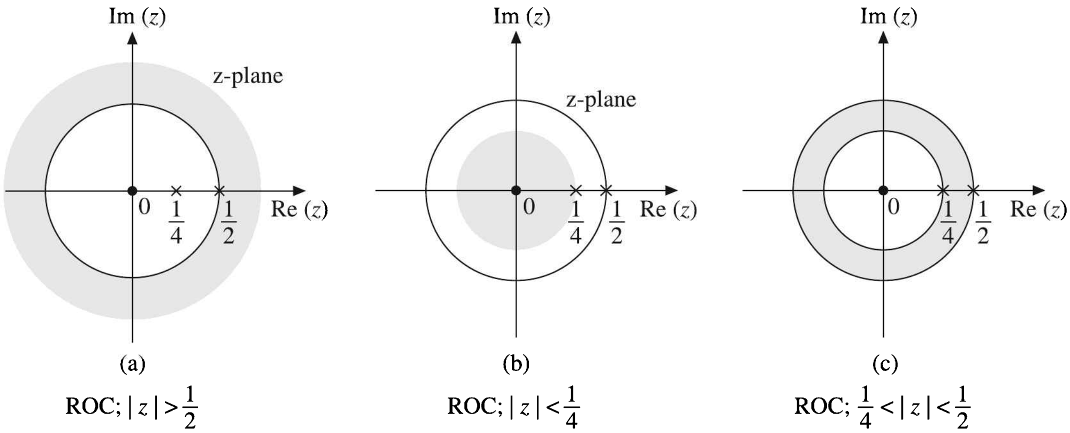


Figure 10.16 ROCs for Example 10.68.

EXAMPLE 10.69 Determine all possible values of $x(n)$ for

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - (3/2)z^{-1} + (1/2)z^{-2}}$$

Solution: Given
$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - (3/2)z^{-1} + (1/2)z^{-2}}$$

Multiplying the numerator and denominator by z^2, we have

$$X(z) = \frac{z^2 + 2z + 1}{z^2 - (3/2)z + (1/2)} = \frac{(z+1)^2}{(z-1)[z-(1/2)]}$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{(z+1)^2}{z(z-1)[z-(1/2)]} = \frac{C_1}{z} + \frac{C_2}{z-1} + \frac{C_3}{z-(1/2)} = \frac{2}{z} + \frac{8}{z-1} - \frac{9}{z-(1/2)}$$

$$\therefore \qquad X(z) = 2 + \frac{8z}{z-1} - \frac{9z}{z-(1/2)}$$

$X(z)$ has two poles one at $z = 1$ and the other at $z = (1/2)$. The possible ROCs are:

(a) $|z|>1$ (b) $|z|<\frac{1}{2}$ (c) $\frac{1}{2}<|z|<1$

So there are three possible values of $x(n)$.

(a) When ROC; $|z|>1$, both terms must be causal.

$$\therefore \qquad x(n)=2\delta(n)+8u(n)-9\left(\frac{1}{2}\right)^n u(n)$$

(b) When ROC; $|z|<\frac{1}{2}$, both terms must be anticausal.

$$\therefore \qquad x(n)=2\delta(n)-8u(-n-1)+9\left(\frac{1}{2}\right)^n u(-n-1)$$

(c) When ROC; $\frac{1}{2}<|z|<1$, the term with pole at $z = 1/2$ must be causal and the term with pole at $z = 1$ must be anticausal.

$$\therefore \qquad x(n)=2\delta(n)-8u(-n-1)-9\left(\frac{1}{2}\right)^n u(n)$$

EXAMPLE 10.70 Determine all possible values of $x(n)$ for

$$X(z)=\frac{z^3+z^2+(3/2)z+(1/2)}{z^3+(3/2)z^2+(1/2)z}$$

Solution: Given $X(z)=\dfrac{z^3+z^2+(3/2)z+(1/2)}{z^3+(3/2)z^2+(1/2)z}=\dfrac{z^3+z^2+(3/2)z+(1/2)}{z(z+1)[z+(1/2)]}$

$$\therefore \qquad \frac{X(z)}{z}=\frac{z^3+z^2+(3/2)z+(1/2)}{z^2(z+1)[z+(1/2)]}$$

Taking partial fractions, we have

$$\frac{X(z)}{z}=\frac{z^3+z^2+(3/2)z+(1/2)}{z^2(z+1)[z+(1/2)]}=\frac{A}{z^2}+\frac{B}{z}+\frac{C}{z+1}+\frac{D}{z+(1/2)}=\frac{1}{z^2}+\frac{2}{z+1}-\frac{1}{z+(1/2)}$$

$$\therefore \qquad X(z)=\frac{1}{z}+\frac{2z}{z+1}-\frac{z}{z+(1/2)}=z^{-1}+2\frac{z}{z+1}-\frac{z}{z+(1/2)}$$

(a) When ROC is $|z| > 1$, both the terms must be causal.

$$\therefore \qquad x(n)=\delta(n-1)+2(-1)^n u(n)-\left(-\frac{1}{2}\right)^n u(n)$$

(b) When ROC is $|z| > 1/2$, both the terms must be anticausal.

$$\therefore \qquad x(n) = \delta(n-1) - 2(-1)^n u(-n-1) + \left(-\frac{1}{2}\right)^n u(-n-1)$$

(c) When ROC is $1/2 < |z| < 1$, the term with pole at $z = 1/2$ must be causal and the term with pole at $z = 1$ must be anticausal.

$$\therefore \qquad x(n) = \delta(n-1) + 2(-1)^n u(n) + \left(-\frac{1}{2}\right)^n u(-n-1)$$

EXAMPLE 10.71 Determine all possible inverse Z-transforms of the following $X(z)$:

$$X(z) = \frac{4 - 3z^{-1} + 3z^{-2}}{(z+2)(z-3)^2}$$

using partial fraction expansion method.

Solution: Given $\qquad X(z) = \dfrac{4 - 3z^{-1} + 3z^{-2}}{(z+2)(z-3)^2} = \dfrac{z^{-2}(4z^2 - 3z + 3)}{(z+2)(z-3)^2}$

The given $X(z)$ has 3 poles, one at $z = -2$ and a double pole at $z = 3$. So it can have three ROCs.

(a) ROC; $|z| > 3$ $\quad$ (b) ROC; $|z| < 2$ $\quad$ (c) ROC; $2 < |z| < 3$

So it can have three inverse Z-transforms.

Now, $$\frac{X(z)}{z} = \frac{z^{-2}(4z^2 - 3z + 3)}{z(z+2)(z-3)^2} = z^{-3}\left[\frac{4z^2 - 3z + 3}{(z+2)(z-3)^2}\right]$$

Let $$X_1(z) = \frac{(4z^2 - 3z + 3)}{(z+2)(z-3)^2} = \frac{C_1}{z+2} + \frac{C_2}{(z-3)^2} + \frac{C_3}{z-3} = \left[\frac{1}{z+2} + \frac{6}{(z-3)^2} + \frac{3}{z-3}\right]$$

$$\therefore \qquad X(z) = z^{-3}\left[\frac{z}{z+2} + \frac{6z}{(z-3)^2} + \frac{3z}{z-3}\right]$$

Now, $$Z^{-1}\left(\frac{z}{z+2}\right) = (-2)^n u(n);\ Z^{-1}\left[\frac{z}{(z-3)^2}\right] = n(3)^n u(n);\ Z^{-1}\left(\frac{z}{z-3}\right) = (3)^n u(n)$$

(a) When ROC is $|z| > 3$, all the terms in $x(n)$ will be causal.

$$\therefore \qquad x(n) = (-2)^{n-3} u(n-3) + 6(n-3)3^{n-3} u(n-3) + 3(3)^{n-3} u(n-3)$$

(b) When ROC is $|z| < 2$, all the terms in $x(n)$ will be anticausal.

$$\therefore \quad x(n) = -(-2)^{n-3} u(-n-4) - 6(n-3)(3)^{n-3} u(-n-4) - 3(3)^{n-3} u(-n-4)$$

(c) When ROC is $2 < |z| < 3$ the term with pole at $z = 2$ will be causal and the terms with pole at $z = 3$ will be anticausal.

$$\therefore \quad x(n) = (-2)^{n-3} u(n-3) - 6(n-3)(3)^{n-3} u(-n-4) - 3(3)^{n-3} u(-n-4)$$

EXAMPLE 10.72 Find all possible inverse Z-transforms of the following function

$$X(z) = \frac{z^2}{z-1} + \frac{z^2}{z-3}$$

Solution: Given $\quad X(z) = \dfrac{z^2}{z-1} + \dfrac{z^2}{z-3} = z\left(\dfrac{z}{z-1}\right) + z\left(\dfrac{z}{z-3}\right)$

The poles are at $z = 1$ and $z = 3$. So the possible ROCs are:

(a) $|z| > 3$ (b) $|z| < 1$ (c) $1 < |z| < 3$

Therefore, using time shifting property, the inverse Z-transforms are:

(a) $x(n) = u(n+1) + 3^{n+1} u(n+1)$; ROC; $|z| > 3$

(b) $x(n) = -u(-n) - 3^{n+1} u(-n)$; ROC; $|z| < 1$

(c) $x(n) = u(n+1) - 3^{n+1} u(-n)$; ROC; $1 < |z| < 3$

EXAMPLE 10.73 Find all possible inverse Z-transforms of the following function:

$$X(z) = \frac{z(z^2 - 4z + 5)}{z^3 - 6z^2 + 11z - 6}$$

Solution: Given $\quad X(z) = \dfrac{z(z^2 - 4z + 5)}{z^3 - 6z^2 + 11z - 6} = \dfrac{z(z^2 - 4z + 5)}{(z-1)(z-2)(z-3)}$

The poles of $X(z)$ are at $z = 1$, $z = 2$ and $z = 3$. So the possible ROCs are:

(a) $|z| > 3$ (b) $|z| < 1$ (c) $2 < |z| < 3$ and (d) $1 < |z| < 2$

Using partial fraction method, we have

$$\frac{X(z)}{z} = \frac{z^2 - 4z + 5}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{z-3}$$

or

$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3}$$

Therefore, the possible inverse Z-transforms are:

(a) $x(n) = u(n) - 2^n u(n) + 3^n u(n)$; ROC; $|z| > 3$

(b) $x(n) = -u(-n-1) + 2^n u(-n-1) - 3^n u(-n-1)$; ROC; $|z| < 1$

(c) $x(n) = u(n) - 2^n u(n) - 3^n u(-n-1)$; ROC; $2 < |z| < 3$

(d) $x(n) = u(n) + 2^n u(-n-1) - 3^n u(-n-1)$; ROC; $1 < |z| < 2$

EXAMPLE 10.74 Find the inverse Z-transform of the following function:

$$X(z) = \frac{1}{1 - 1.8z^{-1} + 0.8z^{-2}}$$

for the following ROC:

(a) $|z| > 1$ (b) $|z| < 0.8$ (c) $0.8 < |z| < 1$

Solution: Given $X(z) = \dfrac{1}{1 - 1.8z^{-1} + 0.8z^{-2}} = \dfrac{z^2}{z^2 - 1.8z + 0.8} = \dfrac{z^2}{(z-1)(z-0.8)}$

Taking partial fractions of $X(z)/z$, we have

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.8)} = \frac{A}{z-1} + \frac{B}{z-0.8} = \frac{5}{z-1} - \frac{4}{z-0.8}$$

$$\therefore \qquad X(z) = 5\left(\frac{z}{z-1}\right) - 4\left(\frac{z}{z-0.8}\right)$$

The given function has two poles one at $z = 1$ and the other one at $z = 0.8$. So it has three possible ROCs:

(a) ROC; $|z| > 1$ (b) ROC; $|z| < 0.8$ (c) ROC; $0.8 < |z| < 1$

(a) For ROC; $|z| > 1$, both the terms must be causal

$$\therefore \qquad x(n) = 5u(n) - 4(0.8)^n\, u(n)$$

(b) For ROC; $|z| < 0.8$, both the terms must be non-causal

$$\therefore \qquad x(n) = -5u(-n-1) + 4(0.8)^n u(-n-1)$$

(c) For ROC; $0.8 < |z| < 1$, the term with pole at $z = 0.8$ must be causal and the term with pole at $z = 1$ must be non-causal.

$$\therefore \qquad x(n) = -5u(-n-1) - 4(0.8)^n\, u(n)$$

EXAMPLE 10.75 (a) Find the inverse Z-transform of the following function:

$$X(z) = \frac{2z^3 - 5z^2 + z + 4}{(z-1)(z-2)};\ \text{ROC; } |z| < 1$$

(b) Find the inverse Z-transform of

$$X(z) = \frac{3}{z-2};\ \text{ROC; } |z| > 2$$

Solution: (a) Given $X(z) = \dfrac{2z^3 - 5z^2 + z + 4}{(z-1)(z-2)} = \dfrac{2z^3 - 5z^2 + z + 4}{z^2 - 3z + 2}$

$$\therefore \qquad \frac{X(z)}{z} = \frac{2z^3 - 5z^2 + z + 4}{z^3 - 3z^2 + 2z}$$

Since the order of the numerator is equal to that of the denominator, the function $X(z)/z$ is an improper function. So divide the numerator by the denominator and remove a term such that the remaining function is a proper function.

$$\begin{array}{r|l} & \quad 2 \\ \cline{2-2} z^3 - 3z^2 + 2z & 2z^3 - 5z^2 + z + 4 \\ & \underline{2z^3 - 6z^2 + 4z} \\ & \qquad\quad z^2 - 3z - 4 \end{array}$$

$$\therefore \qquad \frac{X(z)}{z} = 2 + \frac{z^2 - 3z + 4}{z^3 - 3z^2 + 2z} = 2 + \frac{z^2 - 3z + 4}{z(z-1)(z-2)}$$

$$\frac{z^2 - 3z + 4}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2} = \frac{2}{z} - \frac{2}{z-1} + \frac{1}{z-2}$$

$$\therefore \qquad \frac{X(z)}{z} = 2 + \frac{2}{z} - \frac{2}{z-1} + \frac{1}{z-2}$$

or

$$X(z) = 2z + 2 - 2\frac{z}{z-1} + \frac{z}{z-2}$$

Taking inverse Z-transform on both sides, we have

$$x(n) = 2\delta(n+1) + 2\delta(n) - 2u(n) + (2)^n u(n)$$

(b) Given

$$X(z) = \frac{3}{z-2} \quad |z| > 2$$

We have

$$X(z) = \frac{3}{z-2} = 3z^{-1}\frac{z}{z-2}$$

We know that

$$\mathrm{Z}^{-1}\left(\frac{z}{z-2}\right) = (2)^n u(n)$$

$$\therefore \qquad \mathrm{Z}^{-1}\left(z^{-1}\frac{z}{z-2}\right) = \mathrm{Z}^{-1}\left(\frac{z}{z-2}\right)\Bigg|_{n \to n-1}$$

$$= (2)^n u(n)\big|_{n=n-1} = (2)^{n-1} u(n-1)$$

$$\therefore \qquad \mathrm{Z}^{-1}\left(\frac{3}{z-2}\right) = \mathrm{Z}^{-1}\left(3z^{-1}\frac{z}{z-2}\right) = 3(2)^{n-1} u(n-1)$$

EXAMPLE 10.76 Find the inverse Z-transform of

$$X(z)=\frac{2+z^3+3z^{-4}}{z^2+4z+3};\ \text{ROC};\ |z|>0$$

Solution: Given $$X(z)=\frac{2+z^3+3z^{-4}}{z^2+4z+3}=(2+z^3+3z^{-4})\frac{1}{(z^2+4z+3)}$$

Let $$X_1(z)=\frac{1}{z^2+4z+3}=\frac{1}{(z+1)(z+3)}$$

Taking partial fractions of $X_1(z)/z$ we have

$$\frac{X_1(z)}{z}=\frac{1}{z(z+1)(z+3)}=\frac{A}{z}+\frac{B}{z+1}+\frac{C}{z+3}=\frac{1}{3}\left(\frac{1}{z}\right)-\frac{1}{2}\left(\frac{1}{z+1}\right)+\frac{1}{6}\left(\frac{1}{z+3}\right)$$

$\therefore$ $$X_1(z)=\frac{1}{3}-\frac{1}{2}\left(\frac{z}{z+1}\right)+\frac{1}{6}\left(\frac{z}{z+3}\right)$$

Taking inverse Z-transform, we have

$$Z^{-1}[X_1(z)]=x_1(n)=\frac{1}{3}\delta(n)-\frac{1}{2}(-1)^n u(n)+\frac{1}{6}(-3)^n u(n)$$

Using linearity and shifting properties, we have

$$Z^{-1}\left(\frac{2}{z^2+4z+3}\right)=2Z^{-1}\left(\frac{1}{z^2+4z+3}\right)=2\left[\frac{1}{3}\delta(n)-\frac{1}{2}(-1)^n u(n)+\frac{1}{6}(-3)^n u(n)\right]$$

$$Z^{-1}\left(\frac{z^3}{z^2+4z+3}\right)=Z^{-1}\left(\frac{1}{z^2+4z+3}\right)\bigg|_{n=n+3}$$

$$=\left[\frac{1}{3}\delta(n+3)-\frac{1}{2}(-1)^{n+3}u(n+3)+\frac{1}{6}(-3)^{n+3}u(n+3)\right]$$

$$Z^{-1}\left(\frac{z^{-4}}{z^2+4z+3}\right)=Z^{-1}\left(\frac{1}{z^2+4z+3}\right)\bigg|_{n=n-4}$$

$$=\left[\frac{1}{3}\delta(n-4)-\frac{1}{2}(-1)^{n-4}u(n-4)+\frac{1}{6}(-3)^{n-4}u(n-4)\right]$$

$\therefore$ $$x(n)=2\left[\frac{1}{3}\delta(n)-\frac{1}{2}(-1)^n u(n)+\frac{1}{6}(-3)^n u(n)\right]$$

$$+\left[\frac{1}{3}\delta(n+3)-\frac{1}{2}(-1)^{n+3}u(n+3)+\frac{1}{6}(-3)^{n+3}u(n+3)\right]$$

$$+\left[\frac{1}{3}\delta(n-4)-\frac{1}{2}(-1)^{n-4}u(n-4)+\frac{1}{6}(-3)^{n-4}u(n-4)\right]$$

10.7.3 Residue Method

The inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where c is a circle in the z-plane in the ROC of $X(z)$. The above equation can be evaluated by finding the sum of all residues of the poles that are inside the circle c. Therefore,

$$x(n) = \sum \text{Residues of } X(z) z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z - z_i)\, X(z)\, z^{n-1} \Big|_{z = z_i}$$

If $X(z)\, z^{n-1}$ has no poles inside the contour c for one or more values of n, then $x(n) = 0$ for these values.

EXAMPLE 10.77 Explain in detail, the contour integration method of finding inverse Z-transform.

Solution: The Z-transform of a discrete-time sequence $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable. In polar form, $z = re^{j\omega}$ where r is the radius of a circle.

$$\therefore \qquad X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n} = \sum_{n-\infty}^{\infty} [x(n) r^{-n}]\, e^{-j\omega n}$$

The above equation represents the Fourier transform of a signal $x(n)\, r^{-n}$. Hence, the inverse DTFT of $X(re^{j\omega})$ must be $x(n)\, r^{-n}$. Therefore, we can write

$$x(n)\, r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})\, e^{j\omega n}\, d\omega$$

$$z = re^{j\omega}$$

$$dz = re^{j\omega}\, j\, d\omega$$

$$d\omega = \frac{dz}{re^{j\omega} j} = \frac{dz}{jz}$$

$$\therefore \qquad x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})\, (re^{j\omega})^n\, d\omega$$

Substituting $z = re^{j\omega}$ and $d\omega = dz/jz$ in the above equation, we have

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where the symbol $\oint_c$ denotes integration around the circle of radius $|z| = r$ in a counter clockwise direction.

The above equation can be evaluated by finding the sum of all residues of the poles that are inside the circle c. Therefore, the above equation can be written as:

$$x(n) = \sum \text{(Residues of } X(z)\,z^{n-1} \text{ at the poles inside } c)$$

$$= \sum_i (z - z_i)\, X(z)\, z^{n-1}\Big|_{z = z_i}$$

If $X(z)\, z^{n-1}$ has multiple poles of order k, the residue at the pole $z = z_i$ is given by

$$\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(z - z_i)^k\, X(z)\, z^{n-1}\Big|_{z = z_i}$$

EXAMPLE 10.78 Using residue method, find the inverse Z-transform of

$$X(z) = \frac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}};\ \text{ROC};\ |z| > 3$$

Solution: Given $X(z) = \dfrac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}} = \dfrac{z(z+2)}{z^2+4z+3} = \dfrac{z(z+2)}{(z+1)(z+3)}$

$$\therefore \quad x(n) = \sum \text{Residues of } X(z)\,z^{n-1} \text{ at the poles of } X(z)\,z^{n-1} \text{ within } c$$

$$= \sum \text{Residues of } \frac{z(z+2)\,z^{n-1}}{(z+1)\,(z+3)} = \frac{z^n(z+2)}{(z+1)(z+3)} \text{ at the poles of same within } c$$

$$\therefore \quad x(n) = \sum \text{Residues of } \frac{z^n(z+2)}{(z+1)\,(z+3)} \text{ at poles } z = -1 \text{ and } z = -3$$

$$= (z+1)\frac{z^n(z+2)}{(z+1)\,(z+3)}\Bigg|_{z=-1} + \frac{(z+3)\,z^n(z+2)}{(z+1)\,(z+3)}\Bigg|_{z=-3}$$

$$= \frac{1}{2}(-1)^n\, u(n) + \frac{1}{2}(-3)^n\, u(n)$$

EXAMPLE 10.79 Determine the inverse Z-transform using the complex integral

$$X(z) = \frac{3z^{-1}}{[1-(1/2)\,z^{-1}]^2};\ \text{ROC};\ |z| > \frac{1}{4}$$

Solution: We know that the inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz$$

at the poles inside c where c is a circle in the z-plane in the ROC of $X(z)$.

This can be evaluated by finding the sum of all residues of the poles that are inside the circle c. Therefore, the above equation can be written as:

$$x(n) = \sum \text{Residues of } X(z)z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z - z_i)\, X(z)\, z^{n-1}\Big|_{z=z_i}$$

If there is a pole of multiplicity k, then the residue at that pole is:

$$= \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}[(z-z_i)^k\; X(z)\,z^{n-1}] \text{ at the pole } z = z_i$$

Given $$X(z) = \frac{3z^{-1}}{[1-(1/2)z^{-1}]^2} = \frac{3z}{[z-(1/2)]^2}$$

The given $X(z)$ has a pole of order 2 at $z = 1/2$.

$$x(n) = \sum \text{Residues of } X(z)\,z^{n-1} \text{ at its poles}$$

$$= \sum \text{Residue of } 3z^n/[z-(1/2)]^2 \text{ at the pole } z = (1/2) \text{ of multiplicity 2}$$

$\therefore$ $$x(n) = \frac{1}{1!}\frac{d}{dz}\left[\left(z-\frac{1}{2}\right)^2 \frac{3z^n}{[z-(1/2)]^2}\right]\Bigg|_{z=1/2} = 3nz^{n-1}\Big|_{z=1/2} = 3n\left(\frac{1}{2}\right)^{n-1} u(n)$$

EXAMPLE 10.80 Given $X(z) = z/(z-1)^3$, find $x(n)$ using contour integration method.

Solution: Given $$X(z) = \frac{z}{(z-1)^3}$$

The given $X(z)$ has a pole of order 3 at $z = 1$, i.e. $k = 3$.

$\therefore$ $$x(n) = \sum \text{Residues of } X(z)\,z^{n-1} \text{ at its poles}$$

$$= \sum \text{Residues of } \frac{z^n}{(z-1)^3} \text{ at the pole } z = 1 \text{ of multiplicity 3}$$

$\therefore$ $$x(n) = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-1)^3 \frac{z}{(z-1)^3} z^{n-1}\right]\Bigg|_{z=1}$$

$$= \frac{1}{2}\frac{d^2}{dz^2}[z^n]\Big|_{z=1} = \frac{1}{2}\frac{d}{dz}[nz^{n-1}]\Big|_{z=1} = \frac{1}{2}n(n-1)\,z^{n-2}\Big|_{z=1}$$

$$= \frac{1}{2}n(n-1) \quad \text{for } n > 0$$

$$= 0 \quad \text{for } n < 0$$

$\therefore$ $$x(n) = \frac{n(n-1)}{2}u(n)$$

EXAMPLE 10.81 Determine the inverse Z-transform using residue method.

$$X(z) = \frac{3z^{-1}}{(1-z^{-1})(1-4z^{-1})}; \text{ ROC; } 1 < |z| < 4$$

Solution: Given $X(z) = \dfrac{3z^{-1}}{(1-z^{-1})(1-4z^{-1})} = \dfrac{3z}{(z-1)(z-4)}$; ROC; $1 < |z| < 4$

$X(z)$ has two poles, one at $z = 1$ and the second one at $z = 4$.

Residue of $X(z)\, z^{n-1}$ at pole $z = 1$ is:

$$(z-1)\frac{3z\,z^{n-1}}{(z-1)(z-4)}\bigg|_{z=1} = \frac{3z^n}{z-4}\bigg|_{z=1} = -1$$

Residue of $X(z)\, z^{n-1}$ at pole $z = 4$ is:

$$(z-4)\frac{3z\,z^{n-1}}{(z-1)(z-4)}\bigg|_{z=4} = \frac{3z^n}{z-1}\bigg|_{z=4} = (4)^n$$

The contour of integration c lies in the annual region of ROC.

$$x(n) = \begin{cases} -\text{ Residue of } X(z)\,z^{n-1} \text{ at pole } z = 4 \text{ for } n < 0 \\ +\text{ Residue of } X(z)\,z^{n-1} \text{ at pole } z = 1 \text{ for } n \ge 0 \end{cases}$$

$\therefore$ For $n < 0$, $x(n) = (4)^n$

and For $n \ge 0$, $x(n) = -1$

Therefore, $$x(n) = -u(n) - (4)^n\, u(-n-1)$$

EXAMPLE 10.82 Find all possible inverse Z-transforms of $X(z) = \dfrac{z(z+1)}{z^2 - 3z + 2}$ using contour integration (residue theorem) method.

Solution: Given $$X(z) = \frac{z(z+1)}{z^2-3z+2} = \frac{z(z+1)}{(z-1)(z-2)}$$

$X(z)$ has two poles, one at $z = 1$ and the second one at $z = 2$. So there are three possible inverse Z-transforms.

(a) With ROC; $|z| > 2$

(b) With ROC; $|z| < 1$

(c) With ROC; $1 < |z| < 2$

Residue of $X(z)\, z^{n-1}$ at pole $z = 1$ is:

$$\frac{(z-1)\,z^n(z+1)}{(z-1)(z-2)}\bigg|_{z=1} = (1)^n(-2) = -2$$

Residue of $X(z)\, z^{n-1}$ at pole $z = 2$ is:

$$\left.\frac{(z-2)\, z^n (z+1)}{(z-1)(z-2)}\right|_{z=2} = 3(2)^n$$

So the possible inverse Z-transforms are

(a) ROC; $|z| > 2$, both the poles are interior to c. So the residues are positive.

$\therefore$ $$x(n) = -2u(n) + 3(2)^n\, u(n)$$

(b) ROC; $|z| < 1$, both the poles are exterior to c. So the residues are negative.

$\therefore$ $$x(n) = 2u(-n-1) - 3(2)^n\, u(-n-1)$$

(c) ROC; $1 < |z| < 2$, the pole at $z = 1$ is interior to c and the pole at $z = 2$ is exterior to c. So the residue at $z = 1$ is positive and the residue at $z = 2$ is negative.

$\therefore$ $$x(n) = -2u(n) - 3(2)^n\, u(-n-1)$$

EXAMPLE 10.83 Find all possible inverse Z-transforms of $X(z) = \dfrac{z}{(z+1)^2 (z+2)^3}$ using contour integration (residue) method.

Solution: Given $$X(z) = \frac{z}{(z+1)^2 (z+2)^3}$$

$X(z)$ has two poles, one of order 2 at $z = -1$ and the second one of order 3 at $z = -2$. So there are three possible inverse Z-transforms:

(a) with ROC; $|z| > 2$

(b) with ROC; $|z| < 1$

(c) with ROC; $1 < |z| < 2$

We know that $x(n)$ is given by the sum of the residues of $X(z)\, z^{n-1}$ at the poles of $X(z)$.

Residue of $X(z) z^{n-1} = \dfrac{z^n}{(z+1)^2 (z+2)^3}$ at the pole $z = -1$ of order 2 is:

$$\frac{1}{1!}\frac{d}{dz}\left[(z+1)^2 \frac{z^n}{(z+1)^2 (z+2)^3}\right]\Bigg|_{z=-1} = \frac{d}{dz}\left[\frac{z^n}{(z+2)^3}\right]\Bigg|_{z=-1}$$

$$= \left.\frac{(z+2)^3 (nz^{n-1}) - z^n 2(z+2)^2}{(z+2)^6}\right|_{z=-1}$$

$$= -(n+3)(-1)^n$$

Residue of $X(z) z^{n-1} = \dfrac{z^n}{(z+1)^2 (z+2)^3}$ at the pole $z = -2$ of order 3 is:

$$\frac{1}{2!}\frac{d^2}{dz^2}\left[(z+2)^3\frac{z^n}{(z+1)^2(z+2)^3}\right]\Bigg|_{z=-2} = \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z^n}{(z+1)^2}\right]\Bigg|_{z=-2}$$

$$= (-2)^n[0.125n^2 - 1.125n + 3]$$

(a) When ROC; $|z| > 2$, both the residues must be positive.

$$\therefore \qquad x(n) = -(n+3)(-1)^n u(n) + (0.125n^2 - 1.125n + 3)(-2)^n u(n)$$

(b) When ROC; $|z| < 1$, both the residues must be negative.

$$\therefore \qquad x(n) = (n+3)(-1)^n u(-n-1) - (0.125n^2 - 1.125n + 3)(-2)^n u(-n-1)$$

(c) When ROC; $1 \leq |z| \leq 2$, residue at pole $z = -1$ is positive and the residue at pole at $z = -2$ is negative.

$$\therefore \qquad x(n) = -(n+3)(-1)^n u(n) - (0.125n^2 - 1.125n + 3)(-2)^n u(-n-1)$$

EXAMPLE 10.84 Find the inverse Z-transform of $X(z) = \dfrac{1}{z^3(z-1)}$ using (a) Contour integration method (b) Verify by any other method.

Solution: (a) ***Contour integration method***

Given $$X(z) = \frac{1}{z^3(z-1)}$$

$X(z)z^{n-1} = \dfrac{z^{n-1}}{z^3(z-1)}$ has a pole of 3rd order at $z = 0$ and a simple pole at $z = 1$.

$\therefore$ $x(n)$ = Sum of the residues of the pole of 3rd order at $z = 0$ and the pole at $z = 1$.

Residue of pole of 3rd order at $z = 0$ is:

$$\frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z^3(z^{n-1})}{z^3(z-1)}\right]\Bigg|_{z=0} = \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z^{n-1}}{z-1}\right]\Bigg|_{z=0}$$

$$= \frac{1}{2}\frac{d}{dz}\left[\frac{(z-1)(n-1)z^{n-2} - z^{n-1}(1)}{(z-1)^2}\right]\Bigg|_{z=0}$$

$$= \frac{1}{2}\frac{(z-1)^2\{(n-1)[(n-1)z^{n-2} - (n-2)z^{(n-3)}]\,z^{n-2}\}}{(z-1)^4}$$

$$- [(z-1)(n-1)\,z^{n-2} - z^{n-1}]\,2(z-1)\big|_{z=0} = 0$$

Residue of pole at $z = 0$ is:

$$(z-1)\frac{z^{n-1}}{z^3(z-1)}\Bigg|_{z=0} = \frac{z^{n-1}}{z^3}\Bigg|_{z=1} = z^{n-4}\Big|_{z=1} = (1)^{n-4}$$

$$\therefore \qquad x(n) = 0\,u(n) + (1)^{n-4}\,u(n-4) = u(n-4)$$

(b) ***Second method***

Given $$X(z)=\frac{1}{z^3(z-1)}=z^{-4}\left(\frac{z}{z-1}\right)$$

We know that $$Z^{-1}\left(\frac{z}{z-1}\right)=u(n)$$

$\therefore$ $$Z^{-1}\left(z^{-4}\frac{z}{z-1}\right)=Z^{-1}\left[\frac{z}{z-1}\right]_{n=n-4}=u(n-4)$$

10.7.4 Convolution Method

The inverse Z-transform can also be determined using convolution method. In this method, the given $X(z)$ is splitted into $X_1(z)$ and $X_2(z)$ such that $X(z) = X_1(z)\,X_2(z)$. Then, $x_1(n)$ and $x_2(n)$ are obtained by taking the inverse Z-transform of $X_1(z)$ and $X_2(z)$ respectively. Then, $x(n)$ is obtained by performing convolution of $x_1(n)$ and $x_2(n)$ in time domain.

$$Z\,[x_1(n) * x_2(n)] = X_1(z)\,X_2(z) = X(z)$$

$\therefore$ $$x(n)=Z^{-1}[X(z)]=Z^{-1}[Z\{x_1(n)*x_2(n)\}]=x_1(n)*x_2(n)=\sum_{k=0}^{n}x_1(k)\,x_2(n-k)$$

EXAMPLE 10.85 Explain how the analysis of discrete-time invariant system can be obtained using convolution properties of Z-transform.

Solution: One of the most important properties of Z-transform used in the analysis of discrete-time systems is convolution property. According to this property, the Z-transform of the convolution of two signals is equal to the multiplication of their Z-transforms. The analysis of a discrete-time system means the input to the system $x(n)$ and its impulse response $h(n)$ are known and we have to determine the output $y(n)$ of the system. If the input sequence to the LTI system is $x(n)$ and the impulse response is $h(n)$, then first determine the Z-transforms of $x(n)$ and $h(n)$.

Let $Z\,[x(n)] = X(z)$ and $Z\,[h(n)] = H(z)$, then obtain the product of these Z-transforms, i.e. $Y(z) = H(z)\,X(z)$. We know that the output $y(n) = x(n) * h(n)$.

As per the linear convolution property,

$$Y(z)=Z[x(n)*h(n)]=Z[x(n)]\,Z[h(n)]=X(z)\,H(z)$$

So having obtained $Y(z) = X(z)\,H(z)$, take the inverse Z-transform of $X(z)\,H(z)$. This gives $y(n)$ which is the response of the system for the input $x(n)$.

EXAMPLE 10.86 Find the inverse Z-transform of $X(z)=\dfrac{z^2}{(z-2)(z-3)}$ using convolution property of Z-transforms.

Solution: Given $$X(z)=\frac{z^2}{(z-2)(z-3)}$$

Let $$X(z) = X_1(z)X_2(z) = \frac{z}{z-2}\frac{z}{z-3}$$

$\therefore$ $$x_1(n) = Z^{-1}[X_1(z)] = Z^{-1}\left(\frac{z}{z-2}\right) = 2^n u(n)$$

$$x_2(n) = Z^{-1}[X_2(z)] = Z^{-1}\left(\frac{z}{z-3}\right) = 3^n u(n)$$

$\therefore$ $$x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{n} 2^k u(k)\, 3^{n-k} u(n-k)$$

$$= 3^n \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k = 3^n \left[\frac{1-(2/3)^{n+1}}{1-(2/3)}\right]$$

$$= 3^{n+1}\left[1-\left(\frac{2}{3}\right)^{n+1}\right] = 3^{n+1} u(n) - 2^{n+1} u(n)$$

EXAMPLE 10.87 Find the inverse Z-transform of $X(z) = \dfrac{z}{(z-1)[z-(1/2)]}$ using convolution property of Z-transforms.

Solution: Given $$X(z) = \frac{z}{(z-1)[z-(1/2)]}$$

Let $$X(z) = X_1(z)\, X_2(z) = \frac{z}{(z-1)}\frac{1}{[z-(1/2)]}$$

$\therefore$ $$x_1(n) = Z^{-1}[X_1(z)] = Z^{-1}\left(\frac{z}{z-1}\right) = u(n)$$

$$x_2(n) = Z^{-1}[X_2(z)] = Z^{-1}\left[\frac{1}{z-(1/2)}\right] = \left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$\therefore$ $$x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k) x_2(n-k)$$

$$= \sum_{k=0}^{n-1} u(k)\left(\frac{1}{2}\right)^{n-1-k} u(n-1-k)$$

$$= \left(\frac{1}{2}\right)^{n-1}\left[\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^{-k}\right] = \left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{n-1}\left[\left(\frac{1}{2}\right)^{-1}\right]^k$$

$$= \left(\frac{1}{2}\right)^{n-1} \left\{\frac{1-[(1/2)^{-1}]^n}{1-(1/2)^{-1}}\right\} = \left(\frac{1}{2}\right)^{n-1} \left[\frac{1-(1/2)^{-n}}{-1}\right]$$

$$= \left(\frac{1}{2}\right)^{-1} - \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{-1} = 2u(n) - 2\left(\frac{1}{2}\right)^{n} u(n)$$

10.8 TRANSFORM ANALYSIS OF LTI SYSTEMS

The Z-transform plays an important role in the analysis and design of discrete-time LTI systems.

10.8.1 System Function and Impulse Response

Consider a discrete-time LTI system having an impulse response $h(n)$ as shown in Figure 10.17.

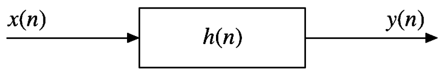


Figure 10.17 Discrete-time LTI system.

Let us say it gives an output $y(n)$ for an input $x(n)$. Then, we have

$$y(n) = x(n) * h(n)$$

Taking Z-transform on both sides, we get

$$Y(z) = X(z)\, H(z)$$

where

$Y(z)$ = Z-transform of the output $y(n)$

$X(z)$ = Z-transform of the input $x(n)$

$H(z)$ = Z-transform of the impulse response $h(n)$

$$\therefore \qquad H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$ is called *the system function* or the *transfer function* of the LTI discrete system and is defined as:

The ratio of the Z-transform of the output sequence $y(n)$ to the Z-transform of the input sequence $x(n)$ when the initial conditions are neglected.

If the input $x(n)$ is an impulse sequence, then $X(z) = 1$. So $Y(z) = H(z)$. So the transfer function is also defined as the Z-transform of the impulse response of the system.

The poles and zeros of the system function offer an insight into the system characteristics. The poles of the system are defined as the values of z for which the system function $H(z)$ is infinity and the zeros of the system are the values of z for which the system function $H(z)$ is zero.