22-6-21

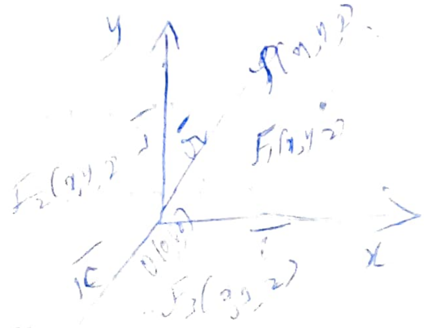
UNIT-5

VECTOR DIFFERENTIAL CALCULUS

Scalar point function (scalar field): If to each point 'P', whose position vector '$\bar{r}$', of a region 'R' in space there corresponds a definite scalar denoted by $\phi(\bar{r})$, then $\phi(\bar{r})$ is called a scalar point func^n in R. The region so defined is known as a scalar field.

Vector point function (vector field): Let 'P' be a point of region 'R' in space. Let its position vector be $\bar{r}$. Then to each point 'P' of the region 'R', if we associate a vector denoted by $\bar{F}(\bar{r})$, it is called the vector point function. The region 'R' is called a vector field.

Vector differential operator: The operator '∇' is known as a vector differential operator, which is defined as
$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$



$\bar{r} = \overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

★ **Gradient of a scalar function:** Suppose $\phi(x,y,z)$ be a scalar function, then gradient of the scalar function 'ϕ' is denoted by '$\nabla\phi$' (or) 'grad ϕ' and is defined as

$$\text{grad}\,\phi = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = \Sigma\bar{i}\frac{\partial\phi}{\partial x}$$

NOTE: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ be the position vector

$\therefore r = |\bar{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$ —(1)

Diff (1) partially w.r.t 'x'

$2r\frac{\partial r}{\partial x} = 2x+0+0$, $\therefore \boxed{\frac{dr}{dx} = \frac{x}{r}}$

Similarly $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$, $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

Problems:

1) Find grad ϕ, if $\phi = 2xz - y^2$ at the point $(1,3,2)$

Sol:

$$\nabla\phi = \text{grad}\,\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = \Sigma\bar{i}\frac{\partial\phi}{\partial x} \quad \text{—(1)}$$

Now,

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= 2z \\ \frac{\partial\phi}{\partial y} &= -2y \\ \frac{\partial\phi}{\partial z} &= 2x \end{aligned}\right\}$$

(1) $\Rightarrow \nabla\phi = \text{grad}\,\phi$

$= \bar{i}(2z) + \bar{j}(-2y) + \bar{k}(2x)$

$$\nabla\phi_{(1,3,2)} = 4\bar{i} - 6\bar{j} + 2\bar{k}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ x & y & z \end{matrix}$$

2) Show that $grad(r^n) = \nabla(r^n) = nr^{n-2}\bar{r}$

Sol:

$$L.H.S = \nabla(r^n) = \bar{i}\frac{\partial}{\partial x}(r^n) + \bar{j}\frac{\partial}{\partial y}(r^n) + \bar{k}\frac{\partial}{\partial z}(r^n)$$

$$= \sum \bar{i}\frac{\partial}{\partial x}(r^n) \quad -(1)$$

$$\frac{\partial}{\partial x}(r^n) = nr^{n-1}\frac{\partial r}{\partial x} \quad \left(\because r(x,y,z) \Rightarrow \frac{\partial}{\partial x}(r^n) = nr^{n-1}\frac{\partial r}{\partial x}\right)$$

$$= nr^{n-1}\left(\frac{x}{r}\right) = nr^{n-2}x$$

$$\therefore (1) \Rightarrow \sum \bar{i}(nr^{n-2}x) = nr^{n-2}\sum x\bar{i}$$

$$= nr^{n-2}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= nr^{n-2}\bar{r}$$

$\therefore$ LHS = RHS

3) Show that $grad\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$

Sol: put $n = -1$ in problem (2), we get

$$grad(r^{-1}) = grad\left(\frac{1}{r}\right) = -r^{-1-2}\bar{r} = \frac{-\bar{r}}{r^3}$$

4) Show that $grad(r) = \frac{\bar{r}}{r}$

Sol: put $n=1$ in problem(2), we get

$$grad(r) = r^{1-2}\bar{r} = \frac{\bar{r}}{r}$$

5) Find grad $(\log_e r)$ (or) $\nabla(\log_e r)$

Sol:

$$\nabla(\log_e r) = \Sigma \bar{i} \frac{\partial}{\partial x}(\log r)$$

$$= \Sigma \bar{i}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right) \quad \left(\because \frac{d}{\partial x}(\log x) = \frac{1}{x}\right)$$

$$= \Sigma \bar{i}\left(\frac{1}{r}\left(\frac{x}{r}\right)\right)$$

$$= \frac{1}{r^2}\Sigma x\bar{i} = \frac{1}{r^2}[x\bar{i}+y\bar{j}+z\bar{k}]$$

$$= \frac{\bar{r}}{r^2}$$

6) Show that grad $(f(r)) = \frac{\bar{r}\, f'(r)}{r}$

Sol:

$$\text{grad}(f(r)) = \Sigma \bar{i}\frac{\partial}{\partial x}(f(r))$$

$$= \Sigma \bar{i}\, f'(r)\frac{\partial r}{\partial x}$$

$$= \Sigma \bar{i}\, f'(r)\cdot\frac{x}{r}$$

$$= \frac{f'(r)}{r}\Sigma x\bar{i} = \frac{f'(r)}{r}(x\bar{i}+y\bar{j}+z\bar{k})$$

$$= \frac{f'(r)\cdot\bar{r}}{r}$$

7) If '$\bar{a}$' be a constant vector, show that grad $(\bar{a}\,\bar{r}) = \bar{a}$.

Sol: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ where a_1, a_2, a_3 are constants

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{a}\cdot\bar{r} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})\cdot(x\bar{i} + y\bar{j} + z\bar{k}) = a_1x + a_2y + a_3z$

$\text{grad}(\bar{a}\cdot\bar{r}) = \sum \bar{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z)$

$= \sum \bar{i}(a_1 + 0 + 0)$

$= \sum a_1\bar{i}$

$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

$\therefore \text{grad}(\bar{a}\cdot\bar{r}) = \bar{a}$

8) Evaluate $\nabla\phi$ where $\phi = x^3 + y^3 + z^3 - 3xyz$ at $(1,-1,2)$

9) Prove that $\nabla(r) = \frac{\bar{r}}{r}$, $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

10) If $\phi = xy + yz + zx$, find grad ϕ at $(1,1,1)$

Sol:

8) $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 3x^2 - 3yz$

$\frac{\partial\phi}{\partial y} = 3y^2 - 3xz$

$\frac{\partial\phi}{\partial z} = 3z^2 - 3xy$

$\nabla\phi = (3x^2 - 3yz)\bar{i} + (3y^2 - 3xz)\bar{j} + (3z^2 - 3xy)\bar{k}$

$\nabla\phi_{(1,-1,2)} = 9\bar{i} - 3\bar{j} + 15\bar{k}$

$$\nabla(r) = \sum \bar{i} \frac{\partial}{\partial x}(r)$$

$$= \sum \bar{i}\left(\frac{x}{r}\right)$$

$$= \frac{1}{r} \sum x\bar{i}$$

$$= \frac{\bar{r}}{r}$$

$$\nabla(\phi) = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = y+z$$

$$\frac{\partial \phi}{\partial y} = x+z$$

$$\frac{\partial \phi}{\partial z} = y+x$$

$$\nabla\phi = (y+z)\bar{i} + (x+z)\bar{j} + (y+x)\bar{k}$$

$$\nabla\phi_{(1,1,1)} = 2\bar{i} + 2\bar{j} + 2\bar{k}$$

Directional Derivative: 23-6-21

The directional derivative of a scalar point function $\phi(x,y,z)$ at the point $P(x,y,z)$ in the direction of a unit vector $\bar{e}$ is def... / equal to $\bar{e}$ grad ϕ (or) $\bar{e}\cdot\nabla\phi$

NOTE:

1). $\nabla\phi$ is a vector normal to the level surface $\phi(x,y,z)$

2) The greatest value of the directional derivative (D.D) of a scalar point function 'ϕ' is given by $|\nabla\phi|$

Problems:

1) Find the directional derivative of the function $\phi = xy+yz+zx$ in the direction of vector $\bar{a} = \bar{i}+2\bar{j}+2\bar{k}$ at the point $(1,2,0)$

Sol: Given $\phi = xy+yz+zx$

let $\bar{a} = \bar{i}+2\bar{j}+2\bar{k}$, $\bar{e} = \frac{\bar{a}}{|a|} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1+2^2+2^2}}$

$\therefore \bar{e} = \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})$

$$\text{grad}(\phi) = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(y+z)+\bar{j}(x+z)+\bar{k}(y+x)$$

$$\nabla\phi_{(1,2,0)} = \bar{i}(2)+\bar{j}(1+0)+\bar{k}(3) = 2\bar{i}+\bar{j}+3\bar{k}$$

$\therefore$ Directional Derivative (D.D) $= \bar{e}\cdot\nabla\phi$

$$= \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})\cdot(2\bar{i}+\bar{j}+3\bar{k})$$

$$= \frac{1}{3}(2+2+6) = 10/3$$

2) find the D.D of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where $Q = (5,0,4)$

Sol:

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= i(2x) + \bar{j}(-2y) + \bar{k}(4z)$$

$$\nabla f_{(1,2,3)} = 2(1)\bar{i} - 4\bar{j} + 12\bar{k}$$

$$\overline{OP} = \bar{i} + 2\bar{j} + 3\bar{k}$$

$$\overline{OQ} = 5\bar{i} + 0\bar{j} + 4\bar{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP}$$

$$= 5\bar{i} - \bar{i} - 2\bar{j} + 4\bar{k} - 3\bar{k}$$

$$= 4\bar{i} - 2\bar{j} + \bar{k}$$

$$\bar{e} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{4^2 + (-2)^2 + 1^2}} = \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k})$$

Directional derivative $= \bar{e}\,\nabla f$

$$= \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k})(2\bar{i} - 4\bar{j} + 12\bar{k})$$

$$= \frac{28}{\sqrt{21}}$$

3) find the directional derivative of the function $xyz^2 + xz$ at $(1,1,1)$ in a direction of the normal to the surface $3xy^2 + y = z$ at $(0,1,1)$

Sol:

let $f = xyz^2 + xz$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}(yz^2 + z) + \bar{j}(xz^2) + \bar{k}(2xyz + x)$$

$$\nabla f_{(1,1,1)} = 2\bar{i} + \bar{j} + 3\bar{k}$$

Normal to surface 'ϕ' is '$\nabla\phi$'.

let $\phi \Rightarrow 3xy^2 + y - z = 0$

$\phi = 3xy^2 + y - z$

$\therefore$ Normal vector is $\nabla\phi$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(3y^2) + \bar{j}(6xy + 1) + \bar{k}(-1)$$

$$\nabla\phi_{(0,1,1)} = 3\bar{i} + \bar{j} - \bar{k}$$

$\therefore$ unit normal $(\bar{e}) = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{\bar{n}}{|\bar{n}|}$

$$= \frac{3\bar{i} + \bar{j} - \bar{k}}{\sqrt{3^2 + 1^2 + 1^2}} = \frac{3\bar{i} + \bar{j} - \bar{k}}{\sqrt{11}}$$

Directional derivative $= \bar{e} \cdot \nabla f$

$$= \frac{1}{\sqrt{11}}(3\bar{i} + \bar{j} - \bar{k})(2\bar{i} + \bar{j} + 3\bar{k})$$

$$= \frac{4}{\sqrt{11}}$$

4) find a unit normal vector to the given surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$

Sol:

$$\phi = x^2y + 2xz - 4 \qquad \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y}$$

$$\bar{n} = \nabla\phi = \bar{i}(2xy + 2z) + \bar{j}(x^2) + \bar{k}[2x]$$

$$\nabla\phi_{(2,-2,3)} = -2\bar{i} + 4\bar{j} + 4\bar{k}$$

$$\text{Unit normal} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\bar{i} + 4\bar{j} + 4\bar{k}}{\sqrt{4+16+16}}$$

$$= \frac{-2\bar{i} + 4\bar{j} + 4\bar{k}}{6}$$

5) find the greatest value of the directional derivate of the function $f = x^2yz^3$ at $(2, 1, -1)$

Sol: The greatest value of $D \cdot D = |\nabla\phi|$

$$\nabla\phi \text{ (or) } \nabla f = \bar{i}(2xyz^3) + \bar{j}(x^2z^3) + \bar{k}(3x^2yz^2)$$

$$\nabla f_{(2,1,-1)} = -4\bar{i} - 4\bar{j} + 12\bar{k}$$

$\therefore$ Greatest value of Directional derivative

$$|\nabla f| = \sqrt{(-4)^2 + (-4)^2 + (12)^2} = \sqrt{176}$$

$$= 4\sqrt{11}$$

6) find the directional derivative of $2xy + z^2$ at $(1, -1, 3)$ in the direction of a vector $\bar{i} + 2\bar{j} + 3\bar{k}$

Sol:

$$f = 2xy + z^2$$

$$\nabla f = i(2y) + \bar{j}(2x) + \bar{k}(2z)$$

$$\nabla f_{(1,-1,3)} = -2\bar{i} + 2\bar{j} + 6\bar{k}$$

$$\bar{e} = \frac{\bar{a}}{|a|} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{3^2 + 2^2 + 1^2}} = \frac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{14}}$$

$$\therefore D \cdot D = \bar{e}\,\nabla f$$

$$= \frac{1}{\sqrt{14}}(\bar{i} + 2\bar{j} + 3\bar{k})(-2\bar{i} + 2\bar{j} + 6\bar{k})$$

$$= \frac{1}{\sqrt{14}}(-2 + 4 + 18) = \frac{20}{\sqrt{14}}$$

7) Find a unit normal vector to the surface $x^2 + y^2 + 2z^2 - 26$ at point $(2,2,3)$.

Sol:

$$\phi = x^2 + y^2 + 2z^2 - 26$$

$$\bar{n} = \nabla\phi = \bar{i}(2x) + \bar{j}(2y) + \bar{z}(4z)$$

$$\nabla\phi_{(2,2,3)} = 4\bar{i} + 4\bar{j} + 12\bar{k}$$

$$\text{Unit normal} = \frac{4(\bar{i} + \bar{j} + 3\bar{k})}{4\sqrt{11}} = \frac{\bar{i} + \bar{j} + 3\bar{k}}{\sqrt{11}}$$

8) Find the max. value of Directional derivative $\phi = x^2yz$ at $(1,4,1)$

Sol:

$$\nabla\phi = \bar{i}(2xyz) + \bar{j}(x^2z) + \bar{k}(x^2y)$$

$$= 8\bar{i} + \bar{j} + 4\bar{k}$$

$$\text{Max. value} = |\nabla\phi| = \sqrt{64 + 1 + 16} = \sqrt{81} = 9$$

9) Find the D.D of $x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the normal to the surface $x\log z - y^2$ at $(-1,2,1)$

Sol:

Let $f = x^2yz + 4xz^2$

$\nabla f = \bar{i}(2xyz + 4z^2) + \bar{j}(x^2z + 0) + \bar{k}(x^2y + 8xz)$

$\nabla f_{(1,-2,-1)} = 8\bar{i} - \bar{j} - 10\bar{k}$

let $\phi = x\log z - y^2$

$\nabla\phi = \bar{i}(\log z) - \bar{j}(2y) + \bar{k}\left(\frac{x}{z}\right)$

$\nabla\phi_{(-1,2,1)} = -4\bar{j} - \bar{k}$

$\therefore$ unit normal $(\bar{e}) = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\bar{j} - \bar{k}}{\sqrt{17}}$

$\therefore$ D.D $= \bar{e} \cdot \nabla f$

$= \frac{-4\bar{j} - \bar{k}}{\sqrt{17}}(8\bar{i} - \bar{j} - 10\bar{k})$

$= \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}$ ~~$\frac{22}{\sqrt{17}}$~~

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Divergence and Angle between normals:

Divergence: Let $\bar{f}$ $=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$ be any continuously differentiable vector point function. Then $\bar{i}\frac{\partial \bar{f}}{\partial x}+\bar{j}\frac{\partial \bar{f}}{\partial y}+\bar{k}\frac{\partial \bar{f}}{\partial z}$

is called divergence of $\bar{f}$ and it is written as

div $\bar{f}$ (or) $\nabla\cdot\bar{f}$

$$\therefore \text{div}\,\bar{f} = \nabla\bar{f} = \bar{i}\frac{\partial \bar{f}}{\partial x} + j\frac{\partial \bar{f}}{\partial y} + k\frac{\partial \bar{f}}{\partial z}$$

NOTE:

$$= \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z} = \Sigma\frac{\partial f_1}{\partial x}$$

If $\bar{f} = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$ then $\nabla\cdot\bar{f} = \text{div}\,\bar{f} =$

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

$$\Rightarrow \nabla\bar{f} = \Sigma\frac{\partial f_1}{\partial x}$$

Solenoidal vectors:

A vector point function '$\bar{f}$' is said to be solenoidal if $\nabla\cdot\bar{f}=0$ (or) div $\bar{f}=0$

Problems:

1) If $\bar{f} = xy^2\bar{i}+2x^2yz\bar{j}-3yz^2\bar{k}$, find div $\bar{f}$ at the point $(1,-1,1)$

ϕ = ______

grad ϕ =

Sol: Given

$$\bar{f} = xy^2\bar{i}+2x^2yz\bar{j}-3yz^2\bar{k}$$

$$\text{div.}\bar{f} = \nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla f_{(1,-1,1)} = 9$$

2) find $\text{div.}\bar{f}$ when $\bar{f} = \text{grad}\,(x^3+y^3+z^3-3xyz)$

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$

$$\nabla\phi = \bar{i}\left(\frac{\partial\phi}{\partial x}\right) + \bar{j}\left(\frac{\partial\phi}{\partial y}\right) + \bar{k}\left(\frac{\partial\phi}{\partial z}\right)$$

$$\text{grad}\,(\phi) = \bar{i}\,\underbrace{(3x^2-3yz)}_{f_1} + \bar{j}\,\underbrace{(3y^2-3xz)}_{f_2} + \bar{k}\,\underbrace{(3z^2-3xy)}_{f_3}$$

$$\text{div.}\bar{f} = 6x+6y+6z = 6(x+y+z)$$

3) If $\bar{f} = \underbrace{(x+3y)}_{f_1}\bar{i} + \underbrace{(y-2z)}_{f_2}\bar{J} + \underbrace{(x+pz)}_{f_3}\bar{K}$ is solenoidal. find 'p'.

Sol:- $\nabla\cdot\bar{f} = 0$ (for solenoidal)

$$\nabla\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1 + 1 + p = 0$$

$$\boxed{p = -2}$$

$\text{div}\,\bar{f} = 0$

4) Find div. $\bar{f}$ when $\bar{f} = r^n \bar{r}$. find 'n' if it is solenoidal.

Sol:-

$$\bar{f} = r^n \bar{r} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$$

$$\therefore \bar{f} = \underbrace{(r^n x)}_{f_1}\bar{i} + \underbrace{(r^n y)}_{f_2}\bar{j} + \underbrace{(r^n z)}_{f_3}\bar{k}$$

$$\text{div.}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \Sigma \frac{\partial f_1}{\partial x}$$

$$= \Sigma\, r^n + x\, n r^{n-1} \frac{\partial r}{\partial x} \quad \left(\because \frac{dr}{dx} = \frac{x}{r}\right)$$

$$= \Sigma \left[r^n + n r^{n-1} \cdot \frac{x^2}{r} \right]$$

$$= \Sigma r^n + \Sigma n r^{n-2} x^2$$

$$= 3r^n + n r^{n-2} [x^2 + y^2 + z^2]$$

$$= 3r^n + n r^n$$

$$= r^n (3+n)$$

If $\bar{f}$ is solenoidal $\Rightarrow r^n (3+n) = 0$

$$\boxed{n = -3}$$

NOTE: The angle between $\bar{n}_1$ and $\bar{n}_2$ is given by $\cos\theta = \dfrac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$

1) find the angle b/w the normals to the surface $xy = z^2$ at the points $(4,1,2)$ & $(3,3,-3)$

Sol:

$$xy - z^2 = 0$$

$$\nabla\phi = xy - z^2 = 0$$

$$\nabla\phi = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= i(y) + j(x) + k(-2z)$$

at point $(4,1,2)(n_1) = \bar{i} + 4\bar{j} - 4\bar{k}$.

at point $(3,3,-3) \Rightarrow (n_2) = 3\bar{i} + 3\bar{j} + 6\bar{k}$

$$\cos\theta = \frac{(\bar{i}+4\bar{j}-4\bar{k})(3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{1^2+4^2+4^2}\sqrt{3^2+3^2+6^2}}$$

$$= \frac{3+12-24}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{33}\sqrt{54}}\right)$$

2) find the values of a & b so that surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point $(1,-1,2)$

Sol

$$ax^2 - byz - (a+2)x$$

$$(1,-1,2) \Rightarrow a + 2b - (a+2) = 0$$

$$\boxed{b=1} \Rightarrow 2a - b = 0$$

$$a - 2b = 0$$

$$(a, b)$$

$4x^2y + z^3 - 4 = 0$

$(1,-1,2) \rightarrow 4(-1)+8-4=0$

$0 = 0$

Let $f = ax^2 - yz - (a+2)x$ and $g = 4x^2y + z^3 - 4$

Normal $= \nabla f = i(2ax - (a+2)) + \bar{j}(-z) + \bar{k}(-y)$

$(1,-1,2)\; n_1 = \bar{i}(2a - (a+2)) + \bar{j}(-2) + \bar{k}$

$= \cancel{-8\bar{a} + 4\bar{j} + k}$

$= (a-2)\bar{i} - 2\bar{j} + \bar{k}$

$\nabla g = \bar{i}(8xy) + \bar{j}(4x^2) + \bar{k}(3z^2)$

$(1,-1,2) \Rightarrow -8\bar{i} + 4\bar{j} + 12\bar{k}$

$$\cos\theta = \frac{((a-2)\bar{i} - 2\bar{j} + \bar{k})(-8\bar{i} + 4\bar{j} + 12\bar{k})}{\sqrt{(a-2)^2+4+1}\;\sqrt{64+16+144}}$$

$n_1 n_2 = 0$

$= (a-2)(-8) - 2(4) + 12 = 0$

$\boxed{a = 5/2}$

$\cancel{z}$ (a, b)

3) Find the angle between spheres $x^2+y^2+z^2=29$, $x^2+y^2+z^2+4x-6y-8z-47=0$ at $(4,-3,2)$

4) find div. $\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

5) Prove that $\operatorname{div}\left(\frac{\bar{r}}{r}\right) = \frac{2}{r}$

6) Prove that $\bar{F} = 3y^4z^2\bar{i} + z^3x^3\bar{j} - 3x^2y^2\bar{k}$ is solenoidal.

Sol: 3) $x^2+y^2+z^2=29$

$x^2+y^2+z^2+4x-6y-8z-47=0$

$n_1 = \nabla\phi = i(2x)+j(2y)+\bar{k}(2z)$

$(4,-3,2) \Rightarrow 8\bar{i}-6\bar{j}+4\bar{k}$

$n_2 = \nabla\phi = i(2x+4)+\bar{j}(2y-6)+\bar{k}(2z-8)$

$(4,-3,2) \Rightarrow 12\bar{i}-12\bar{j}-4\bar{k}$

$$\cos\theta = \frac{(8\bar{i}-6\bar{j}+4\bar{k})(12\bar{i}-12\bar{j}-4\bar{k})}{\sqrt{8^2+6^2+4^2}\sqrt{12^2+12^2+4^2}}$$

$$= \frac{152}{\sqrt{116}\sqrt{304}} = \sqrt{\frac{19}{29}}$$

$$\theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

4) Find $\operatorname{div}\bar{r} = \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_2}{\partial y} + \bar{k}\frac{\partial f_3}{\partial z}$

$\bar{r} = \underset{f_1}{x}\bar{i}+\underset{f_2}{y}\bar{j}+\underset{f_3}{z}\bar{k}$, z ~~(x) + y + z~~

$\operatorname{div}\bar{r} = 1+1+1$

$\operatorname{div}\bar{r} = 3$

6) If $f = 3y^4 z^2 \bar{i} + z^3 x^2 \bar{j} - 3x^2 y^2 \bar{k}$, find ∇f

$$\nabla f = \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_2}{\partial y} + \bar{k}\frac{\partial f_3}{\partial z}$$

$$= 0 + 0 + 0$$

$$\nabla f = 0$$

$\therefore f$ is solenoidal

5) PT $div\left(\frac{\bar{r}}{r}\right) = \frac{2}{r}$

$$div\left(\frac{\bar{r}}{r}\right) = \nabla \cdot \frac{\bar{r}}{r}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial y}\right)\left(\frac{x\bar{i} + y\bar{j} + z\bar{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \left[\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right] + \left[\frac{r - y\frac{\partial r}{\partial y}}{r^2}\right] + \left[\frac{r - z\frac{\partial r}{\partial z}}{r^2}\right]$$

$$= \frac{r - \frac{x^2}{r}}{r^2} + \frac{r - \frac{y^2}{r}}{r^2} + \frac{r - \frac{z^2}{r}}{r^2}$$

$$= \frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^3} = \frac{3r^2 - r^2}{r^3}$$

$$= \frac{2}{r}$$

6) If $f = (x^2+y^2+z^2)^{-n}$, find $div(grad f)$ and determine n if $div(grad f) = 0$

Curl and Scalar potential

Curl of a vector: Let $\bar{f}$ be any continuously diff. vector point function. Then the curl of a vector $\bar{f}$ is denoted by curl $\bar{f}$ (or) $\nabla \times \bar{f}$ and is given by curl $\bar{f} = \nabla \times \bar{f}$

$$= \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z} = \sum \bar{i} \times \frac{\partial \bar{f}}{\partial x}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

here $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$

Note: A vector $\bar{f}$ is said to be irrotational if curl $\bar{f} = 0$. In this case ∃ a fn ϕ such that $\bar{f} = \nabla\phi$ then 'ϕ' is called 'Scalar potential'.

Problems:

1) Find curl $\bar{f}$ where $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol: $\text{grad}\,\phi = \nabla\phi = \bar{i}(\underbrace{3x^2-3yz}_{f_1}) + \bar{j}(\underbrace{3y^2-3xz}_{f_2}) + \bar{k}(\underbrace{3z^2-3xy}_{f_3})$

Now, curl $\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3zx & 3z^2-3xy \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right]$$

$$-\bar{j}\left[\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right]$$

$$+\bar{k}\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$= \bar{i}\,[0-3x+3x] - \bar{j}\,[0-3y+3y] + \bar{k}\,[-3z+3z]$$

$\therefore$ curl $\bar{f} = 0$

$\therefore \bar{f}$ is irrotational vector

5) Prove that $r^n \bar{r}$ is irrotational.

Sol: let $\bar{f} = r^n \bar{r} = r^n(x\bar{i}+y\bar{j}+z\bar{k})$

$$= \underbrace{r^n x}_{f_1}\,\bar{i} + \underbrace{r^n y}_{f_2}\,\bar{j} + \underbrace{r^n z}_{f_3}\,\bar{k}$$

$$\therefore \text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] - \bar{j}\left[\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\right] + \bar{k}\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right]$$

$$\Sigma \bar{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right]$$

$= \sum \hat{i} \left[z\, n r^{n-1} \frac{\partial r}{\partial y} - y\, n r^{n-1} \frac{\partial r}{\partial z} \right]$

$= \sum \hat{i} \left[z\, n r^{n-1} \frac{y}{r} - y\, n r^{n-1} \left(\frac{z}{r}\right) \right]$

$= n r^{n-2} \sum \hat{i} \, (zy - yz) = 0$

curl $\vec{F} = 0$

$\therefore$ $\vec{F}$ is irrotational

3) Show that the vector $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ is irrotational and also find its scalar potential function.

Sol:

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2 - xy) - \frac{\partial}{\partial z}(x^2 - yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right]$$

$$= \hat{i}[-x + x] - \hat{j}[-y + y] + \hat{k}[-z + z]$$

curl $\vec{F} = 0$

$\therefore$ $\vec{F}$ is irrotational

Here $\bar{f} = \nabla\phi$, ϕ is scalar potential

$\Rightarrow (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

By equating :

$$\frac{\partial\phi}{\partial x} = x^2 - yz, \quad \frac{\partial\phi}{\partial y} = y^2 - zx, \quad \frac{\partial\phi}{\partial z} = z^2 - xy$$

Total derivative : $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$\Rightarrow d\phi = (x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz$$

$$= x^2dx + y^2dy + z^2dz - (yzdx + zxdy + xydz)$$

Integrating $\Rightarrow \int d\phi = \int x^2dx + \int y^2dy + \int z^2dz - \int d(xyz)$

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz$$

4) find the constants a, b, c if the vector

$\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$

is irrotational. also find its scalar potential $\left(\text{Ans: } \phi = x^2 + y^2 + \frac{3z^2}{2} + 3xy + 2zx + 3zy\right)$

Sol: , Curl $\bar{f}$ $= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = 0$

$= \bar{i}\left[\frac{\partial}{\partial y}(2x + cy + 3z) - \frac{\partial}{\partial z}(bx + 2y + 3y)\right]$

$- \bar{j}\left[\frac{\partial}{\partial x}(2x + cy + 3z) - \frac{\partial}{\partial z}(2x + 3y + az)\right]$

$+ \bar{k}\left[\frac{\partial}{\partial x}(bx + 2y + 3y) - \frac{\partial}{\partial y}(2x + 3y + az)\right] = 0$

$= \bar{i}(c - 3) - \bar{j}(2 - a) + \bar{k}(b - 3) = 0$

$\therefore\ c-3=0,\ a-2=0,\ b-3=0$

$$\boxed{a=2,\ b=3,\ c=3}$$

$\int d\phi = \int (x+3y+2z)\,dx + (3x+2y+3z)\,dy + (2x+3y+3z)\,dz$

$\Rightarrow \phi = x^2+y^2+\frac{3}{2}z^2+3xy+2zx+3yz$

5) Prove that $\bar{f} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$ is irrotational.

Sol:

$$\operatorname{curl}\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \bar{i}[1-1] - \bar{j}[1-1] + \bar{k}[1-1]$$

$$\operatorname{curl}\bar{f} = 0$$

$\therefore\ \bar{f}$ is irrotational.

6) If $\bar{f} = e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$ then find curl $\bar{f}$

7) If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$. find curl $\bar{f}$ at $P(1,-1,1)$ (Ans: $-\bar{i}-2\bar{k}$)

(Ans: $e^{xyz}(x(z-y)\bar{i} + y(x-z)\bar{j} + z(y-x)\bar{k})$

8) Show that the vector $\bar{f} = 2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\bar{j} + (2x^2yz + y\cos yz)\bar{k}$ is irrotational. Find its scalar potential. (Ans: $x^2yz^2 + \sin yz$)

Sol: 6)

$$\operatorname{curl}\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y+z} & e^{x+y+z} & e^{x+y+z} \end{vmatrix}$$

$$= \bar{i}\left[e^{x+z}\frac{\partial}{\partial y}(e^y) - e^{x+y}\frac{\partial}{\partial z}(e^z)\right]$$

$$-\bar{j}\left[e^{y+z}\frac{\partial}{\partial x}e^{x} - e^{x+y}\frac{\partial}{\partial z}e^{z}\right) + \bar{k}\left[e^{y+z}\frac{\partial}{\partial x}(e^{x}) - e^{x+z}\frac{\partial}{\partial y}(e^{y})\right]$$

$$= \bar{i}\left(\cancel{e^{x+y+z}} - \cancel{e^{x+y+z}}\right) - \bar{j}\left(\cancel{e^{x+y+z}} - \cancel{e^{x+y+z}}\right) + \bar{k}\left(\cancel{e^{x+y+z}} - \cancel{e^{x+y+z}}\right)$$

$$= 0$$

$\text{curl}\,\bar{f} = 0$

$\therefore \bar{f}$ is irrotational.

7) $$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left[-3z^2 - 2x^2y\right] - \bar{j}\left[0-0\right] + \bar{k}\left[4xyz - 2xy\right]$$

$$\text{curl}\,\bar{f}_{(1,-1,1)} = -\bar{i} - 2\bar{k}$$

8) $$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2+z\cos yz & 2x^2yz+y\cos yz \end{vmatrix}$$

$$= \bar{i}\left(2x^2z + \cos yz - y\sin yz \cdot z - 2x^2z - \cos yz + zy\sin yz\right) - \bar{j}\left(4xyz - 4xyz\right) + \bar{k}\left(2xz^2 - 2xz^2\right)$$

$$= 0$$

$\text{curl}\,\bar{f} = 0$

$\therefore \bar{f}$ is irrotational

Here $\bar{f} = \nabla\phi$, ϕ is a scalar potential

$$2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\bar{j} + (2x^2yz + y\cos yz)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

By equating, $\frac{\partial\phi}{\partial x} = 2xyz^2$, $\frac{\partial\phi}{\partial y} = x^2z^2 + z\cos yz$

$$\frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz$$

Total derivative $\Rightarrow d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$d\phi = (2xyz^2)dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$$

Integrating on both sides

$$\int d\phi = \int 2xyz^2dx + \int x^2z^2dy + \int 2x^2yz\,dz + \int z\cos yz\,dy + \int y\cos yz\,dz$$

$$\phi = x^2yz^2 + x^2yz^2 + x^2yz^2 + \frac{z\sin yz}{z} + \frac{y\sin yz}{y}$$

$$= 3x^2yz^2 + 2\sin yz$$

28-6-21

Laplacian operator:

The laplacian operator is denoted with '∇^2' and it is defined on the scalar function 'ϕ' and is given by

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) = \text{div}(\text{grad}\,\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \Sigma \frac{\partial^2\phi}{\partial x^2}$$

NOTE: If $\nabla^2\phi = 0$ then 'ϕ' is said to satisfy - Laplacian equation.

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Problems:

1) Prove that $\text{div}(\text{grad}\, r^m) = m(m+1)r^{m-2}$ (or)

$\nabla^2(r^m) = m(m+1)r^{m-2}$.

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $r = |\bar{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

W.k.t $\text{grad}(\phi) = \nabla\phi = \Sigma \bar{i}\frac{\partial\phi}{\partial x}$

Let $\phi = r^m \Rightarrow \text{grad}(r^m) = \nabla(r^m) = \Sigma \bar{i}\frac{\partial}{\partial x}(r^m)$

$= \Sigma \bar{i}\left[m r^{m-1}\frac{\partial r}{\partial x}\right]$

$= \Sigma \bar{i}\left[m r^{m-1}\left(\frac{x}{r}\right)\right]$

$= \Sigma \bar{i}\left(m r^{m-2}(\bar{x})\right)$

$\text{div} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Δ

$$m r^{m-2} (x\bar{i} + y\bar{j} + z\bar{k})$$

$$= \underbrace{(m r^{m-2} x)}_{f_1}\bar{i} + \underbrace{(m r^{m-2} y)}_{f_2}\bar{j} + \underbrace{(m r^{m-2} z)}_{f_3}\bar{k}$$

Now $\operatorname{div}(\operatorname{grad}(r^m)) = \operatorname{div}(\bar{f}) = \sum \frac{\partial f_1}{\partial x}$

$$\Rightarrow \operatorname{div}(\operatorname{grad} r^m) = \sum \frac{\partial}{\partial x} \left[m r^{m-2} x \right]$$

$$= m \sum \left\{ r^{m-2} + x(m-2) r^{m-3} \frac{\partial r}{\partial x} \right\}$$

$$= m \sum \left\{ r^{m-2} + (m-2) r^{m-3} \left(\frac{x^2}{r} \right) \right\}$$

$$= m \sum r^{m-2} + m(m-2) r^{m-4} \sum x^2$$

$$= 3 m r^{n-2} + m(m-2) r^{m-4} \underbrace{(x^2+y^2+z^2)}_{r^2}$$

$$\therefore \operatorname{div}(\operatorname{grad} r^m) = 3 m r^{n-2} + m(m-2) r^{m-2}$$

$$\nabla(\nabla r^m) = m r^{m-2} [3 + m - 2]$$

$$\therefore \nabla^2 (r^m) = m(m+1) r^{m-2}$$

2) Show that $\nabla^2 (f(r)) = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} = f''(r) + \frac{2}{r} f'(r)$

Sol:-

Let $\phi = f(r)$, $\operatorname{grad}(\phi) = \nabla \phi = \sum \bar{i} \frac{\partial \phi}{\partial x}$

$$\Rightarrow \nabla \phi = \sum \bar{i} \frac{\partial}{\partial x} (f(r))$$

$$\nabla\phi = \Sigma \bar{i}\, f'(r) \frac{\partial r}{\partial x}$$

$$= \Sigma \bar{i}\, f'(r) \left(\frac{x}{r}\right)$$

$$= \frac{f'(r)}{r} \Sigma \bar{i}(x)$$

$$\nabla(f(r)) = \nabla\phi = \text{grad}\,\phi = \frac{f'(r)}{r}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\therefore \nabla\phi = \frac{f'(r)}{r}(\bar{r}) = \frac{f'(r)}{r}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\Rightarrow \nabla\phi = \bar{f} = \left[\frac{f'(r)}{r}x\right]\bar{i} + \left[\frac{f'(r)}{r}y\right]\bar{j} + \left[\frac{f'(r)}{r}z\right]\bar{k}$$

$$\text{grad}\,\phi = \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$$

$$\nabla^2\phi = \nabla^2(f(r)) = \nabla\cdot(\nabla\phi) = \text{div}(\text{grad}\,\phi)$$

$$= \text{div}(\bar{f})$$

$$= \Sigma \frac{\partial f_1}{\partial x}$$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{f'(r)}{r} x\right)$$

$$\Rightarrow \nabla^2(f(r)) = \Sigma \left\{ \frac{r\frac{\partial}{\partial x}(f'(r)x) - f'(r)\, x \frac{\partial r}{\partial x}}{r^2} \right\}$$

$$= \Sigma \left\{ \frac{r\left(f'(r) + x f''(r)\frac{\partial r}{\partial x}\right) - f'(r).\frac{xx}{r}}{r^2} \right\}$$

$$\nabla^2 f(r) = \Sigma \frac{f'(r)}{r} + r\Sigma\left[\frac{x}{r^2} f''(r)\left(\frac{x}{r}\right)\right] - \Sigma\left[\frac{f'(r)x^2}{r^3}\right]$$

$$= \frac{3f'(r)}{r} + r\Sigma f''(r)\frac{x^2}{r^3} - \frac{f'(r)}{r^3}\Sigma x^2$$

$= \frac{3}{r} f'(r) + \frac{r f''(r)}{r^3} \sum x^2 - \frac{f'(r)}{r^3} (r^2)$

$= \frac{3}{r} f'(r) + \frac{r f''(r)}{r^3} (r^2) - \frac{f'(r)}{r}$

$\therefore \nabla^2 (f(r)) = \frac{2}{r} f'(r) + \left(\frac{f''(r)}{r}\right) r = \frac{2}{r} f'(r) + f''(r)$

3) Prove that i) $\nabla^2 \left(\frac{1}{r}\right) = 0$ ii) $\nabla^2 (\log r) = \frac{1}{r^2}$

4) If $f = (x^2+y^2+z^2)^{-n}$ then find div(grad f) and find 'n' if div (grad f) = 0 (Ans: $2n(2n-1)r^{-2n-2}$)

(Hint: $r^2 = x^2+y^2+z^2$, $f = (r^2)^{-n} = r^{-2n}$)

Sol:- i) Let $\phi = \frac{1}{r}$, grad $(\phi) = \nabla \phi = \sum \bar{i} \frac{\partial \phi}{\partial x}$

$\nabla \phi = \sum \bar{i} \frac{\partial}{\partial x} \left(\frac{1}{r}\right)$

$= \sum \bar{i} \left(-\frac{1}{r^2} \frac{\partial r}{\partial x}\right)$

$= \sum \bar{i} \left(\frac{-1}{r^2} \left(\frac{x}{r}\right)\right)$

$= \frac{-1}{r^3} \sum x \bar{i}$

$= -\frac{1}{r^3} (x\bar{i} + y\bar{j} + z\bar{k})$

$= -\frac{1}{r^3} (\bar{r})$

$\nabla \phi = -\frac{x}{r^2} \bar{i} - \frac{y}{r^2} \bar{j} - \frac{z}{r^2} \bar{k}$

grad $\phi = \bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$

$$\nabla^2\phi = \nabla^2\left(\frac{1}{r}\right) = \operatorname{div}(\bar{f})$$

$$= \sum \frac{\partial f_1}{\partial x}$$

$$= \sum \frac{\partial}{\partial x}\left(-\frac{x}{r^2}\right)$$

$$= -\sum \frac{r^2 - 2r\cdot x\frac{\partial r}{\partial x}}{r^4}$$

$$= -\sum \frac{r^2}{r^4} + \sum \frac{2x^2}{r^4}$$

(or)

3) i) $\nabla^2(f(r)) = \frac{2}{r}f'(r) + \left(\frac{f''(r)}{\cancel{r}}\right)\cancel{r}$

$f(r) = \frac{1}{r}$

$$\nabla^2\left(\frac{1}{r}\right) = \frac{2}{r}\left(-\frac{1}{r^2}\right) + \frac{2}{r^3}$$

$$= \frac{-\cancel{2}}{\cancel{r^3}} + \frac{\cancel{2}}{\cancel{r^3}} = 0$$

$\therefore \nabla^2\left(\frac{1}{r}\right) = 0$

ii) $f(r) = \log r$

$$\nabla^2(\log r) = \frac{2}{r}f'(\log r) + f''(\log r)$$

$$= \frac{2}{r}\left(\frac{1}{r}\right) - \frac{1}{r^2} = \frac{1}{r^2}$$

$\therefore \nabla^2(\log r) = \frac{1}{r^2}$

4)

$$f = (x^2+y^2+z^2)^{-n}$$

$$f = r^{-2n}$$

$$\text{div}(\text{grad } f) = \text{div}(\text{grad } r^{-2n})$$

$$\text{grad}(r^{-2n}) = \nabla r^{-2n} = \sum \hat{i} \frac{\partial}{\partial x}(r^{-2n})$$

$$= \sum \hat{i}\left[-2n\, r^{-2n-1} \frac{\partial r}{\partial x}\right]$$

$$= \sum \hat{i}\left[-2n\, r^{-2n-1}\left(\frac{x}{r}\right)\right]$$

$$= -2n \sum \hat{i}\, r^{-2n-2}\, x$$

$$= -2n\, r^{-2(n+1)} \sum \hat{i}\, x$$

$$\text{grad}(r^{-2n}) = -2n\, r^{-2(n+1)} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\text{div}(\text{grad } r^{-2n}) = \sum \frac{\partial}{\partial x}\left(-2n\, r^{-2(n+1)}\, x\right)$$

$$= -2n \sum \left\{ \underbrace{r^{-2(n+1)}}_{u}\ \underbrace{x}_{v} \right\}$$

$$= -2n\left\{ \sum r^{-2(n+1)} - 2(n+1) r^{-2n-4} \underbrace{\sum x^2}_{r^2} \right\}$$

$$= -2n\left\{ 3r^{-2(n+1)} - 2(n+1) r^{-2(n+1)} \right\}$$

$$= -2n\, r^{-2(n+1)} \{3 - 2n - 2\}$$

$$= -2n\, r^{-2(n+1)} \{1 - 2n\}$$

$$\operatorname{div}(\operatorname{grad} r^{-2n}) = 2n r^{-2n-2}(2n-1)$$

$$\operatorname{div}(\operatorname{grad} r^{-2n}) = 0 \quad \text{if } 2n-1=0, \; n = \tfrac{1}{2} \text{ (or) } n=0$$

Physical Interpretations and vector Identities:

Physical Interpretation of gradient: The gradient of a scalar function 'ϕ' is a vector along normal '$\bar{n}$' to the level surface $\phi(x, y, z) = c$ and is in the direction of increasing ϕ and its magnitude is equal to the greatest rate of increase of 'ϕ'.

Physical Interpretation of divergence: Let '$\bar{F}$' is the velocity field given by $\bar{F} = V_x\bar{i} + V_y\bar{j} + V_z\bar{k}$, then div $\bar{F}$ gives the rate at which fluid is originating at a point of space (or) region per unit volume. i.e. it gives the rate of loss of fluid per unit volume.

If $\bar{V}$ represents electric flux then div $\bar{V}$ is the amount flux which diverges per unit volume.

Solenoidal physical Interpretation: when the flux entering any element of space is same as the flux leaving from that space then we have div $\bar{V} = 0$

Physical Interpretation of curl: The curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Vector Identities:

1) $\text{grad}(\phi).\bar{a} + \phi\, \text{div}(\bar{a}) = \text{div}(\phi\bar{a})$

(or)

$\text{div}(\phi\bar{a}) = (\nabla \cdot \{\phi\bar{a}\}$

$= (\nabla\phi)\bar{a} + \phi(\nabla\cdot\bar{a})$

2) $\text{curl}(\phi\bar{a}) = (\text{grad}\,\phi)\times\bar{a} + \phi\, \text{curl}(\bar{a})$ (or)

$\nabla\times(\phi\bar{a}) = (\nabla\phi)\times\bar{a} + \phi(\nabla\times\bar{a})$

3) $\text{grad}(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times\text{curl}\,\bar{a}$

$+ \bar{a}\times\text{curl}\,\bar{b}$

4) $\text{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\text{curl}\,\bar{a} - \bar{a}\cdot\text{curl}\,\bar{b}$ (or)

$\nabla(\bar{a}\times\bar{b}) = \bar{b}\cdot(\nabla\times\bar{a}) - \bar{a}(\nabla\times\bar{b})$

5) $\text{curl}(\bar{a}\times\bar{b}) = \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\bar{a} + (\bar{b}\cdot\nabla)\times\bar{a} -$

$(\bar{a}\cdot\nabla)\bar{b}$

6) $\text{div}(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$

7) $\nabla\times(\nabla\times\bar{a}) = \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}$

(or)

$\text{curl}(\text{curl}\,\bar{a}) = \text{grad}(\text{div}\,\bar{a}) - \text{div}(\text{grad}\,\bar{a})$

8) Prove that curl(grad ϕ) = 0

Proof:

let $\bar{f}$ = grad $\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right] - \bar{j}\left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$+ \bar{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right]$$

$$= 0$$

9) Prove that div(curl $\bar{f}$) = 0 (or) $\nabla \cdot (\nabla \times \bar{f}) = 0$

Sol: let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl } \bar{f} = \bar{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \bar{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right]$$

$$+ \bar{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$\text{div}(\text{curl } \bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} + \frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial z \partial x}$$

$$- \frac{\partial^2 f_1}{\partial z \cdot \partial y} = 0$$

$\therefore$ $\text{div}(\text{curl } \bar{f}) = 0$ (or) $\nabla \cdot (\nabla \times \bar{f}) = 0$

$\Rightarrow$ If the temperature at any point in space is given by $T = xy + yz + zx$, find the direction in which the temperature changes most rapidly with distance from the point $(1, 1, 1)$. and also determine the max. rate of change.

(Hint: $\underset{\downarrow}{\nabla T}$ (or) grad (T) and max $= |\nabla T| = 2\sqrt{3}$)

$2\bar{i} + 2\bar{j} + 2\bar{k}$

$\Rightarrow$ Prove that $\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = -\frac{2}{r^3}\bar{r}$

(Hint: $\nabla\left(\frac{\bar{r}}{r}\right) = ?$ (i.e $\text{div}\left(\frac{\bar{r}}{r}\right)$) ;

$\downarrow \phi$

$\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = \text{grad}\left(\nabla \cdot \frac{\bar{r}}{r}\right))$

ϕ $\qquad \downarrow \phi$

$\Rightarrow$ If 'ϕ' satisfies the laplacian equation. Show that $\nabla\phi$ is both solenoidal and irrotational.

Sol:

Given: $\nabla^2 \phi = 0$

we have to p.T $\text{div}(\nabla\phi) = 0$

& $\text{curl}(\nabla\phi) = 0$

let $\text{div}(\nabla\phi) = \nabla(\nabla\phi)$

$$= \nabla^2\phi$$

$$= 0$$

$\therefore$ $\nabla\phi$ is solenoidal

Let $\text{curl}(\nabla\phi) = \nabla\times(\nabla\phi)$

$$= \text{curl}(\text{grad}\ \phi)$$

$$= 0$$

$\therefore$ $\nabla\phi$ is irrotational.

$\Rightarrow$ $T = xy + yz + zx = \phi$, let.

$$\text{grad}(T) = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y + z$$

$$\frac{\partial\phi}{\partial y} = x + z$$

$$\frac{\partial\phi}{\partial z} = y + x$$

$\Rightarrow$ $\text{grad}(T) = \bar{i}(y+z) + \bar{j}(x+z) + \bar{k}(y+x)$

$\Rightarrow$ $\text{grad}(T)_{(1,1,1)} = 2\bar{i} + 2\bar{j} + 2\bar{k}$

$$\max = |\nabla T| = \sqrt{2^2+2^2+2^2} = \sqrt{12} = 2\sqrt{3}$$

$\Rightarrow$ Find the scalar point f'n whose gradient is $2xyz\bar{i} + x^2z\bar{j} + x^2y\bar{k}$ (Ans: $x^2yz + C$)

Sol $\nabla\phi = 2xyz\bar{i} + x^2z\bar{j} + x^2y\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$.

$d\phi = (2xyz)dx + (x^2z)dy + (x^2y)dz \Rightarrow \phi = \int d(x^2yz) = x^2yz + c$