Math 106 – Notes
Week 4: January 26, 2026

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General continuous time Markov processes (5.3)

Now we switch back to talking about Markov processes. Eventually we will answer the question: Which Gaussian processes are also Markov processes. A Markov process is defined as follows.

Definition 1 (Markov processes)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A process* X_t *is a Markov processes if it is* $\mathcal{F}_t$*-adopted processes and for* $s \leq t$ *and* $B \in \mathcal{R}$

$$\mathbb{P}(X_t \in B|\mathcal{F}_s) = \mathbb{P}(X_t \in B|X_s) \tag{1}$$

We will consider homogenous Markov processes which are defined by a transition function

$$p(t, x, B) = \mathbb{P}(X_t \in B|X_0 = x) \tag{2}$$

we also define an linear operator

$$T_t f(x) = \mathbb{E}[f(X_t)|X_0 = x] \tag{3}$$

If there is a density $\rho(t, x, y)$ such that $p(t, x, [y, y + dy)) \approx \rho(t, x, y)dy$, then this becomes

$$T_t f(x) = \int_B f(y)\rho(t, x, y)dy \tag{4}$$

A considerable amount of time will later be spent studying the PDE for ρ, but first we spend a bit more time on the general case. Before proceeding, it is helpful to note that $p(t, x, B)$ and $T_t f(x)$ correspond respectively to the notation $P_{i,j}(t)$ and $h_i(t)$ which was introduced for Q-processes.