Math 106 – Notes
Week 1: January 9, 2026

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Review

On Thursday's class we reviewed some basic probability theory (probability spaces, probability measures etc.) and I introduced Markov chains. We saw the Chapman-Kolmogorov equation. You should try to prove it yourself.

Invariant Distributions

Let P be a stochastic matrix on a finite state space $\Omega = \{1, \dots, I\}$ (in the book they use S).

Definition. *A probability vector* π *is an* invariant distribution *if*

$$\pi = \pi P. \tag{1}$$

Two basic questions:

(i) When does an invariant distribution exist? (We know that 1 is an eigenvalue since $P\mathbf{1} = \mathbf{1}$, thus the question is about whether the left eigenvector for this eigenvalue has positive entries and thus can be normalized).

(ii) When is it unique?

In Thursday's class I proved L3.5, which says the spectral radius is 1 for a stochastic matrix.

Permutation Matrices and Relabeling

Definition (Not in book)**.** *A* permutation matrix $R \in \mathbb{R}^{I \times I}$ *has exactly one entry equal to* 1 *in each row and column, and zeros elsewhere. In other words,* $R_{i,j} \in \{0, 1\}$ *and* $\sum_{i\in\Omega} R_{i,j} = \sum_{j\in\Omega} R_{i,j} = 1$.

Such a matrix corresponds to a permutation, that is, a bijection $\sigma : \Omega \to \Omega$ via

$$R = \begin{bmatrix} \mathbf{e}_{\sigma(1)} & \cdots & \mathbf{e}_{\sigma(I)} \end{bmatrix} \tag{2}$$

Note that $R\mathbf{e}_i = \mathbf{e}_{\sigma(i)}$ and $R^T\mathbf{e}_{\sigma(i)} = \mathbf{e}_i$. Thus, Permutation matrices are orthogonal: $R^{-1} = R^T$. Note that R^T is a permutation matrix as well. A permutation matrix corresponds to a relabeling of the states

$$\tilde{P} = R^T P R = R^{-1} P R. \tag{3}$$

Here, R permutes the states, then P is applied, then R^T returns it to the permuted states.

Reducibility and Irreducibility

Definition (D3.6)**.** *A Markov chain with transition matrix P is* reducible *if there exists a permutation matrix R such that*

$$RPR^T = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}, \tag{4}$$

where A_1, A_2 are square stochastic matrices of the appropriate dimensions. Otherwise, the chain is irreducible*.*

We can think of the block decomposition as corresponding to a decomposition of the state space $\Omega = \Omega_1 \cup \Omega_2 = \{1, \ldots, I_1\} \cup \{I_1 + 1, \ldots, I\}$. Then, for example, A_1 would be a $I_1 \times I_1$ matrix representing Markov transitions in A_1.

Graph interpretation

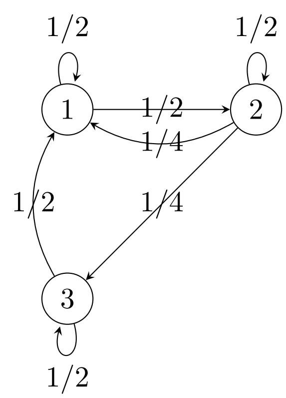
A Markov chain is irreducible if and only if, in the directed graph with edges $i \to j$ whenever $P_{ij} > 0$, there exists a path from any state to any other state.

Example

Consider

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \tag{5}$$

The associated transition graph is:



Perron–Frobenius Theorem

Theorem (T3.8)**.** *Let A be an irreducible nonnegative matrix, and let*

$$\rho = \max\{|\lambda| : \lambda \text{ eigenvalue of } A\}. \tag{6}$$

Then:

(1) There exist strictly positive left and right eigenvectors x, y such that

$$Ax = \rho x, \qquad y^T A = \rho y^T. \tag{7}$$

(2) The eigenvalue ρ has algebraic multiplicity one.

Applied to an irreducible stochastic matrix P, this implies existence (first part) and uniqueness (second part) of the invariant distribution π.