Math 106 – Notes
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General continuous time Markov processes (5.3)

Now we switch back to talking about Markov processes. Eventually we will answer the question: Which Gaussian processes are also Markov processes. A Markov process is defined as follows.

Definition 1 (Markov processes)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A process X_t is a Markov processes if it is $\mathcal{F}_t$-adopted processes and for $s \le t$ and $B \in \mathcal{R}$*

$$\mathbb{P}(X_t \in B|\mathcal{F}_s) = \mathbb{P}(X_t \in B|X_s) \tag{1}$$

We will consider homogenous Markov processes which are defined by a transition function

$$p(t, x, B) = \mathbb{P}(X_t \in B|X_0 = x) \tag{2}$$

we also define an linear operator

$$T_t f(x) = \mathbb{E}[f(X_t)|X_0 = x] \tag{3}$$

If there is a density $\rho(t, x, y)$ such that $p(t, x, [y, y + dy)) \approx \rho(t, x, y)dy$, then this becomes

$$T_t f(x) = \int f(y)\rho(t, x, y)dy \tag{4}$$

A considerable amount of time will later be spent studying the PDE for ρ, but first we spend a bit more time on the general case.

Note that $p(t, x, B)$ and $T_t f(x)$ correspond respectively to the notation $P_{i,j}(t)$ and $h_i(t)$ which was introduced for Q-processes. Let f be bounded and measurable. Using time–homogeneity and the Markov property,

$$\begin{aligned}
T_t(T_s f)(x) &= \mathbb{E}\big[T_s f(X_t) \mid X_0 = x\big] &(5)\\
&= \mathbb{E}\big[\mathbb{E}[f(X_{t+s}) \mid X_t] \mid X_0 = x\big] &(6)\\
&= \mathbb{E}[f(X_{t+s}) \mid X_0 = x] &(7)\\
&= T_{t+s} f(x). &(8)
\end{aligned}$$

Moreover $T_0 f(x) = f(x)$. Thus $\{T_t\}_{t\ge0}$ satisfies

$$T_0 = I, \qquad T_{t+s} = T_t T_s, \tag{9}$$

and is a semigroup of linear operators.

A brief detour: semigroup theory

We now pause to introduce the basic semigroup language needed later. This material provides the abstract framework that connects Markov processes, PDEs, and generators.

Definition 2 (Strongly continuous semigroup)**.** *Let X be a Banach space. A family of linear operators $\{T_t\}_{t\geq 0}$ on X is called a (strongly continuous) semigroup if*

$$T_0 = I, \tag{10}$$

$$T_{t+s} = T_t T_s, \qquad t, s \geq 0, \tag{11}$$

$$\lim_{t\downarrow 0} \|T_t f - f\| = 0 \quad \textit{for all } f \in X. \tag{12}$$

The continuity condition (12) *is called* strong continuity*.*

The Markov operators $\{T_t\}$ constructed above always satisfy the algebraic properties (10)–(11). The additional requirement (12) expresses the idea that the process evolves continuously in time when tested against observables f.

Why strong, not uniform, continuity? A stronger notion would be *uniform continuity*, which requires

$$\|T_t - I\| \to 0 \quad \text{as } t \downarrow 0, \tag{13}$$

where $\|\cdot\|$ denotes the operator norm on X. Recall that for a bounded linear operator $S : X \to X$,

$$\|S\| \;=\; \sup_{\|f\|\leq 1} \|Sf\| \;=\; \sup_{f\neq 0} \frac{\|Sf\|}{\|f\|}. \tag{14}$$

Thus (13) means that

$$\sup_{\|f\|\leq 1} \|(T_t - I)f\| \longrightarrow 0 \qquad \text{as } t \downarrow 0, \tag{15}$$

i.e. *every* function in the unit ball of X is moved only a small amount, uniformly over all such functions, for sufficiently small times.

This requirement is too restrictive for most semigroups arising in probability. A canonical example is the translation semigroup on functions on $\mathbb{R}$,

$$(T_t f)(x) = f(x + t). \tag{16}$$

On spaces such as $C_0(\mathbb{R})$ or $L^p(\mathbb{R})$, this semigroup is strongly continuous: for each fixed f, small translations produce small changes in norm. However, it is *not* uniformly continuous. Intuitively, one can construct functions with arbitrarily sharp spatial features, for which even a tiny translation produces an $O(1)$ change in norm. Thus (13) fails, even though (12) holds.

This example shows that strong continuity is the correct level of regularity if one wants a theory broad enough to include translation, diffusion, and Markov semigroups.

Infinitesimal generator. Given a strongly continuous semigroup $\{T_t\}$, its infinitesimal generator $\mathcal{A}$ is defined by

$$\mathcal{A}f = \lim_{t\downarrow 0} \frac{T_t f - f}{t}, \tag{17}$$

for all f in the domain $\mathcal{D}(\mathcal{A})$ where the limit exists. The generator describes the instantaneous rate of change of observables.

For the translation semigroup (14), a direct calculation shows that

$$\mathcal{A}f = f', \tag{18}$$

with domain consisting of (say) continuously differentiable functions vanishing at infinity. Thus the semigroup of translations is generated by the first derivative operator.

From generators to semigroups. Formally, the semigroup solves the abstract evolution equation

$$\partial_t u_t = \mathcal{A}u_t, \qquad u_0 = f, \tag{19}$$

with solution $u_t = T_t f$. For Markov processes, $\mathcal{A}$ is the object that appears most naturally:

- for Q-processes, $\mathcal{A} = Q$;
- for diffusions, $\mathcal{A}$ is a second–order differential operator;
- for jump processes, $\mathcal{A}$ is an integro–differential operator.

Resolvent operator. The resolvent associated with $\mathcal{A}$ is defined, for $\lambda > 0$, by

$$R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f \, dt. \tag{20}$$

It satisfies the resolvent identity

$$(\lambda I - \mathcal{A})R_\lambda f = f, \tag{21}$$

at least on a suitable domain. The resolvent provides a bridge between the generator and the semigroup via Laplace transform in time.

Why generators are central. The generator $\mathcal{A}$ is local in time and typically has a simple analytic form, whereas the full semigroup $\{T_t\}$ encodes global-in-time behavior. Much of the theory of continuous-time Markov processes proceeds by specifying $\mathcal{A}$, proving it generates a strongly continuous semigroup, and then deducing properties of the process from the generator rather than from T_t directly.