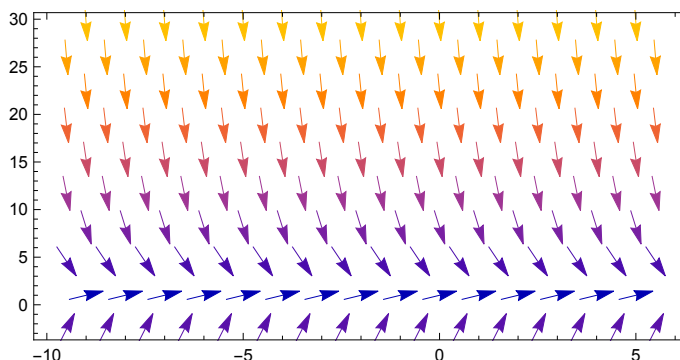


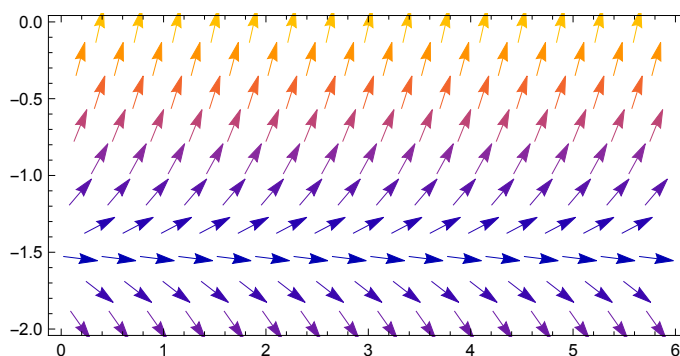
Section 1.1

1 GRADED [2 points]



[Correct drawing of vector field + 1 point] For [Long-term behavior + 1 point]

3 GRADED [2 points]



[Correct drawing of vector field + 1 point] For $y(0) > -3/2$ the solution will tend towards ∞ , while for $y(0) < -3/2$ for the solution will tend towards $-\infty$. If $y(0) = -3/2$ the solution will stay constant. [Long-term behavior + 1 point]

22 GRADED [6 points]

Let $V(t)$ be the volume of the raindrop and $S(t)$ the surface area. The question tells us that

$$\frac{d}{dt}V(t) = -\gamma S(t) \quad [+1 \text{ points}] \quad (1)$$

for some number γ (the rate per area). If R is the radius, the formula for the volume of a sphere is $V(t) = \frac{4}{3}\pi R^3$ and $S(t) = 4\pi R^2$ [These formulas +2 points] it follows that

$$S(t) = 4\pi \left(\frac{3}{4\pi} \right)^{2/3} V^{2/3} \quad [+1 \text{ points}] \quad (2)$$

hence

$$\frac{d}{dt}V(t) = -\mu V^{2/3} \quad [+2 \text{ points}] \quad (3)$$

for some constant μ .

Section 1.2

7 GRADED [9 points]

(a) The solution is

$$p(t) = 900 + ce^{t/2} \quad [\text{identify solution +2 points}] \quad (4)$$

To find c we set $p(0) = 850$:

$$850 = 900 + c \implies c = -50 \quad [\text{solve for } c +1 \text{ point}] \quad (5)$$

To find the time the population goes extinct, we set $p(t_{\text{ex}}) = 0$ and solve for t_{ex} :

$$900 - 50e^{t_{\text{ex}}/2} = 0 \implies e^{t_{\text{ex}}/2} = \frac{900}{50} = 18 \quad (6)$$

$$\implies t_{\text{ex}} = 2 \ln 18 \quad [\text{solve for } t_{\text{ex}} +2 \text{ point}] \quad (7)$$

(b) The same idea as (a): First find c

$$p_0 = 900 + c \implies c = p_0 - 900 \quad (8)$$

and find t_{ex}

$$900 - (p_0 - 900)e^{t_{\text{ex}}/2} = 0 \implies e^{t_{\text{ex}}/2} = \frac{900}{p_0 - 900} \quad (9)$$

$$\implies t_{\text{ex}} = 2 \ln \frac{900}{p_0 - 900} \quad [\text{repeat (a) with general } p_0 +2 \text{ point}] \quad (10)$$

(c) If $t_{\text{ex}} = 1$,

$$t_{\text{ex}} = 1 = 2 \ln \frac{900}{p_0 - 900} \implies \frac{900}{p_0 - 900} = e^{1/2} \quad (11)$$

$$\implies p_0 - 900 = \frac{900}{e^{1/2}} \quad (12)$$

$$\implies p_0 = 900 + \frac{900}{\ln 2} = 900 \left(1 + \frac{1}{e^{1/2}} \right) \quad [+2 \text{ point}] \quad (13)$$

12 GRADED [6 points]

(a) Solving the differential equation for Q yields exponential decay

$$Q = Q(0)e^{-rt} \quad [+1 \text{ point}] \quad (14)$$

For this problem $Q(0) = 100\text{mg}$ and at $t = 1$ (assuming we measure t in weeks) $Q(t) = 82.04$. Plugging these values in to the solutions yields an equation for r :

$$82.04 = 100e^{-r} \implies r = \ln \frac{100}{82.04} \approx 1.97 \quad [+2 \text{ point}] \quad (15)$$

(b) This is given by Equation (14) with the values found above:

$$Q = 100e^{-1.97t} \quad [+1 \text{ point}] \quad (16)$$

(c) Let $t_{1/2}$ be the time to decay to have the original amount. To find $t_{1/2}$ we solve

$$Q(t) = \frac{1}{2}Q(0) = Q(0)e^{-rt_{1/2}} \quad [+1 \text{ point}] \quad (17)$$

which yields

$$t_{1/2} = -\frac{1}{r} \ln \frac{1}{2} = \frac{1}{r} \ln 2 = \frac{1}{1.97} \ln(2) \quad [+1 \text{ point}] \quad (18)$$

Section 1.3

11 GRADED [6 points]

Starting with y_1 , the derivatives are

$$y' = \frac{1}{2}t^{1/2-1} = \frac{1}{2t^{1/2}} \quad [+1 \text{ point}] \quad (19)$$

$$y'' = -\frac{1}{2} \frac{1}{2t^{1/2+1}} = -\frac{1}{4t^{3/2}} \quad [+1 \text{ point}] \quad (20)$$

Hence

$$-2t^2 y'' + 3ty' - y = 2 \frac{t^2}{4t^{3/2}} + \frac{3t}{2t^{1/2}} - y \quad (21)$$

$$= -\frac{1}{2}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} = t^{1/2} - t^{1/2} \quad [+1 \text{ point}] \quad (22)$$

For t_2 we have

$$y' = -t^{-2} \quad [+1 \text{ point}] \quad (23)$$

$$y'' = 2t^{-3} \quad [+1 \text{ point}] \quad (24)$$

hence

$$-4t^{-1} - 3t^{-1} - t^{-1} = 0 \quad [+1 \text{ point}] \quad (25)$$

1 Additional Questions

Two State Markov Chain

A very important application of ODEs is to describe how probabilities change of time in random, or *stochastic*, processes called continuous time Markov Chains (CTMC). Perhaps the simplest example is a systems (say the confirmation of a molecule) which can be in one of two states, 0 or 1, and switches between them at a constant rate h . Then the probabilities to be in states 0 and 1 (p_0 and p_1 respectively) obey the system of two differential equations:

$$\begin{aligned} \frac{d}{dt}p_0 &= -hp_0 + hp_1 \\ \frac{d}{dt}p_1 &= hp_0 - hp_1. \end{aligned} \quad (26)$$

These equations are subject to the conservation of probability constraint $p_0 + p_1 = 1$. Solve the differential equations (hint: rewrite the system as a single ODE) and describe the long-term behavior.

Gene expression

A common model for the production of a protein regulated by a transcription factor is to model the transcription factor by a 2 state Markov chain in Equation (26). The production of the protein is proportional to the fraction of time the TF is bound to the DNA.

$$\frac{d}{dt}x(t) = \alpha p_1(t) - \beta x(t) \tag{27}$$

What is the long-term behavior of $x(t)$?