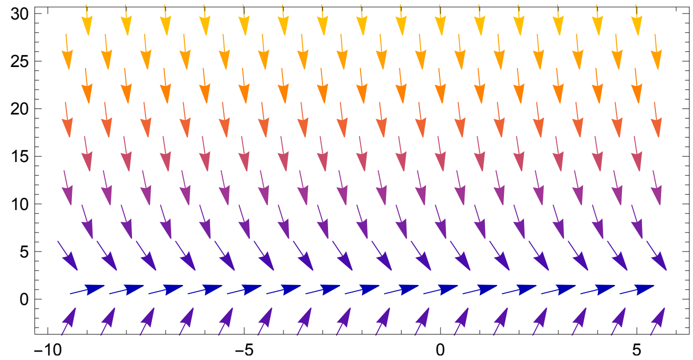
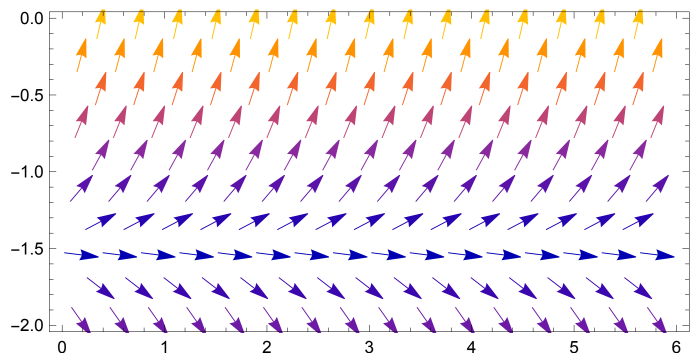
Section 1.1

1 GRADED [2 points]



[Correct drawing of vector field + 1 point] For any initial conditions to the solutions tends towards 3/2. [Long-term behavior + 1 point]

3 GRADED [2 points]



[Correct drawing of vector field + 1 point] For $y(0) > -3/2$ the solution will tend towards ∞, while for $y(0) < -3/2$ for the solution will tend towards $-\infty$. If $y(0) = -3/2$ the solution will stay constant. [Long-term behavior + 1 point]

22 GRADED [6 points]

Let $V(t)$ be the volume of the raindrop and $S(t)$ the surface area. The question tells us that

$$\frac{d}{dt}V(t) = -\gamma S(t) \qquad \text{[+1 points]} \tag{1}$$

for some number γ (the rate per area). If R is the radius, the formula for the volume of a sphere is $V(t) = \frac{4}{3}\pi R^3$ and $S(t) = 4\pi R^2$ [These formulas +2 points] it follows that

$$S(t) = 4\pi \left(\frac{3}{4\pi}\right)^{2/3} V^{2/3} \qquad \text{[+1 points]} \tag{2}$$

hence

$$\frac{d}{dt}V(t) = -\mu V^{2/3} \qquad \text{[+2 points]} \tag{3}$$

for some constant μ.

Section 1.2

7 GRADED [9 points]

(a) The solution is

$$p(t) = 900 + ce^{t/2} \qquad \text{[identify solution +2 points]} \tag{4}$$

To find c we set $p(0) = 850$:

$$850 = 900 + c \implies c = -50 \qquad \text{[solve for } c \text{ +1 point]} \tag{5}$$

To find the time the population goes extinct, we set $p(t_{\text{ex}}) = 0$ and solve for t_{ex}:

$$900 - 50e^{t_{\text{ex}}/2} = 0 \implies e^{t_{\text{ex}}/2} = \frac{900}{50} = 18 \tag{6}$$

$$\implies t_{\text{ex}} = 2\ln 18 \qquad \text{[solve for } t_{\text{ex}} \text{ +2 point]} \tag{7}$$

(b) The same idea as (a): First find c

$$p_0 = 900 + c \implies c = p_0 - 900 \tag{8}$$

and find t_{ex}

$$900 - (p_0 - 900)e^{t_{\text{ex}}/2} = 0 \implies e^{t_{\text{ex}}/2} = \frac{900}{p_0 - 900} \tag{9}$$

$$\implies t_{\text{ex}} = 2\ln\frac{900}{p_0 - 900} \qquad \text{[repeat (a) with general } p_0 \text{ +2 point]} \tag{10}$$

(c) If $t_{\text{ex}} = 1$,

$$t_{\text{ex}} = 1 = 2\ln\frac{900}{p_0 - 900} \implies \frac{900}{p_0 - 900} = e^{1/2} \tag{11}$$

$$\implies p_0 - 900 = \frac{900}{e^{1/2}} \tag{12}$$

$$\implies p_0 = 900 + \frac{900}{\ln 2} = 900\left(1 + \frac{1}{e^{1/2}}\right) \qquad \text{[+2 point]} \tag{13}$$

12 GRADED [6 points]

(a) Solving the differential equation for Q yields exponential decay

$$Q = Q(0)e^{-rt} \qquad \text{[+1 point]} \tag{14}$$

For this problem $Q(0) = 100$mg and at $t = 1$ (assuming we measure t in weeks) $Q(t) = 82.04$. Plugging these values in to the solutions yields an equation for r:

$$82.04 = 100e^{-r} \implies r = \ln\frac{100}{82.04} \approx 1.97 \qquad \text{[+2 point]} \tag{15}$$

(b) This is given by Equation (14) with the values found above:

$$Q = 100e^{-1.97t} \qquad \text{[+1 point]} \tag{16}$$

(c) Let $t_{1/2}$ be the time to decay to have the original amount. To find $t_{1/2}$ we solve

$$Q(t) = \frac{1}{2}Q(0) = Q(0)e^{-rt_{1/2}} \qquad [\ +1 \text{ point}] \tag{17}$$

which yields

$$t_{1/2} = -\frac{1}{r}\ln\frac{1}{2} = \frac{1}{r}\ln 2 = \frac{1}{1.97}\ln(2) \qquad [\ +1 \text{ point}] \tag{18}$$

Section 1.3

11 GRADED [6 points]

Starting with y_1, the derivatives are

$$y' = \frac{1}{2}t^{1/2-1} = \frac{1}{2t^{1/2}} \qquad [+1 \text{ point}] \tag{19}$$

$$y'' = -\frac{1}{2}\frac{1}{2t^{1/2+1}} = -\frac{1}{4t^{3/2}} \qquad [+1 \text{ point}] \tag{20}$$

Hence

$$-2t^2y'' + 3ty' - y = 2\frac{t^2}{4t^{3/2}} + \frac{3t}{2t^{1/2}} - y \tag{21}$$

$$= -\frac{1}{2}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} = t^{1/2} - t^{1/2} \qquad [+1 \text{ point}] \tag{22}$$

For t_2 we have

$$y' = -t^{-2} \qquad [+1 \text{ point}] \tag{23}$$

$$y'' = 2t^{-3} \qquad [+1 \text{ point}] \tag{24}$$

hence

$$-4t^{-1} - 3t^{-1} - t^{-1} = 0 \qquad [+1 \text{ point}] \tag{25}$$

1 Additional Questions

Two State Markov Chain GRADED [10 points]

A very important application of ODEs is to describe how probabilities change of time in random, or *stochastic*, processes called continuous time Markov Chains (CTMC). Perhaps the simplest example is a systems (say the confirmation of a molecule) which can be in one of two states, 0 or 1, and switches between them at a constant rate h. Then the probabilities to be in states 0 and 1 (p_0 and p_1 respectively) obey the system of two differential equations:

$$\begin{aligned} \frac{d}{dt}p_0 &= -hp_0 + hp_1 \\ \frac{d}{dt}p_1 &= hp_0 - hp_1. \end{aligned} \tag{26}$$

These equations are subject to the conservation of probability constraint $p_0 + p_1 = 1$. (a) Solve the differential equations (hint: rewrite the system as a single ODE) and describe the long-term behavior.

(b) For the more general Markov chain where the rates are different, the equations are

$$\frac{d}{dt}p_0 = -hp_0 + wp_1$$
$$\frac{d}{dt}p_1 = hp_0 - wp_1. \tag{27}$$

Without solving, describe the long-term behavior of this ODE.

SOLUTION

(a) Let $p_0 = p$ therefore $p_1 = 1 - p$. The first equation (we could equivalent look at the second) becomes

$$\frac{d}{dt}p = -hp + h(1-p) = h - 2hp \qquad \text{[Identify equation +2]} \tag{28}$$

the solution is

$$p = Ce^{-2th} + \frac{1}{2}\text{[Solve +2]}. \tag{29}$$

In the long term p will tend towards 1/2 [Correct long-term behavior +2].

(b) For the more general situation,

$$\frac{d}{dt}p = w - (w+h)p \qquad \text{[Identify equation +2]} \tag{30}$$

We can see that if $p > w/(w-h)$ the probability to be in state 0 will decrease, while if $p > w/(w+h)$ the probability will increase. Thus p will tend towards $w/(w+h)$ [Identify equation +2].

Gene expression

A common model for the production of a protein regulated by a transcription factor is to model the transcription factor by a 2 state Markov chain in Equation (27). The production of the protien is proportional to the fraction of time the TF is bound to the DNA.

$$\frac{d}{dt}x(t) = \alpha p_1(t) - \beta x(t) \tag{31}$$

Wha is the long-term behavior of $x(t)$?