# Math 50 Final – Practice

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Name: _____ Section: _____

## Instructions

- You have 3 hours to complete the exam.
- You may have a one page (single-sided) "cheat sheet" which must be turned in with the exam, but no electronics (including calculators).
- Each problem is worth 5 points.
- Write your solutions in the boxes.
- Don't cheat.

**Exercise 1** (Converting code to math)**:** Consider the following code

```
x1 = np.random.normal(0,1,100)
x2 = 2*x1 + np.random.normal(0,1,100)
y = x1 + x2 + np.random.normal(0,1,100)
b1 = np.cov(y,x1)/np.var(x1)
b2 = np.cov(y,x2)/np.var(x2)
```

What are b1 and b2 (approximately)?

**Solution:**

**Exercise 2** (Joint distribution)**:** Consider the model of a time series $X_1, X_2, X_3, \ldots$:

$$X_{i+1}|X_i \sim \text{Normal}(1 + X_i/2, 1/3)$$

(a) Write a Python function generatesim(L) to generates a simulation of $L$ steps of this time series starting with $X_i = 0$. The function should return a length $L$ numpy array.

(b) After many steps, the process reaches a steady state where $E[X_{i+1}] = E[X_i]$. What is the distribution of $X_i$ in steady-state?

**Exercise 3** (Regression model comparison)**:** $(X, Y)$ data is fit to a single-predictor regression model in statsmodels using OLS, yielding the following output:

| | coef | std err | t | P>\|t\| | [0.025 | 0.975] |
|---|---|---|---|---|---|---|
| const | 1.0685 | 0.186 | 5.756 | 0.000 | 0.700 | 1.437 |
| x1 | 1.9622 | 0.177 | 11.062 | 0.000 | 1.610 | 2.314 |

A second predictor $X_2$ is then included in the model, which yields the following output:

| | coef | std err | t | P>\|t\| | [0.025 | 0.975] |
|---|---|---|---|---|---|---|
| const | 1.0001 | 0.011 | 92.003 | 0.000 | 0.979 | 1.022 |
| x1 | 0.9810 | 0.012 | 82.470 | 0.000 | 0.957 | 1.005 |
| x2 | 2.0122 | 0.012 | 168.877 | 0.000 | 1.989 | 2.036 |

If $X_2$ and $X_1$ were fit to a linear regression model with $X_2$ as the response variable, what would be the regression slope?

**Solution:**

**Exercise 4** (Sample distribution)**:** Consider the model

$$Y_1 \sim \text{Normal}(\beta_1 X_1 + \beta_2 X_2, \sigma_\epsilon^2)$$

After fitting the model, we find $\hat{\beta}_1 = 100$, $\hat{\beta}_2 = -101$, $\hat{\sigma}_\epsilon^2 = 1/4$. The model is then fit to a different data set and it is found that $\hat{\beta}_1 = -100$, $\hat{\beta}_2 = 100.4$.

(a) Is $\text{cov}(X_1, X_2)$ likely to be positive or negative for the fitted data?

**Solution:**

(b) Based on this information is it possible that $R^2$ is very close to 1?

**Solution:**

**Exercise 5** (Bernoulli regression model)**:** Consider the the two predictor linear regression model with an interaction term:

$$Y = \beta_1 X_1 + \beta_2 X_2 + J_{1,2} X_1 X_2 + \epsilon$$

The following plot shows data generated from such a model.

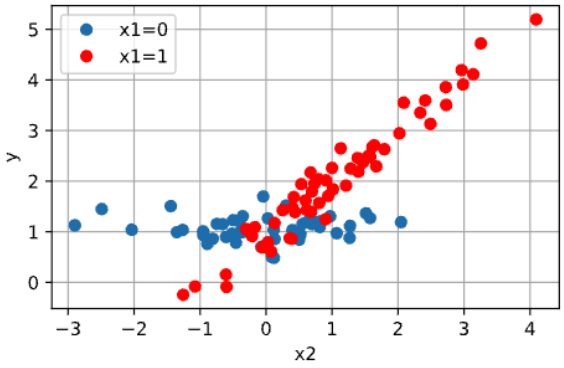



Figure 1:

What are the values of $\beta_1$, $\beta_2$ and $J_{1,2}$?

**Solution:**

**Exercise 6** (Orthogonality)**:** Consider the features $\phi_1(x) = x^3$ and $\phi_2(x) = x^2$.

(a) Are these orthogonal with respect to $X \sim \text{Uniform}(-1, 1)$? (in the sense that $E[\phi_1(X)\phi_2(X)] = 0$)

(b) What is an example of a distribution for $X$ such that $\phi_1$ and $\phi_2$ are not orthogonal?

**Exercise 7** (Bayesian posterior)**:** Consider a the Bayesian model $\sigma_\epsilon$ and $\beta_1$, but unknown intercept with Normal priors:

$$\beta_0 \sim \text{Normal}(0, \tau_0^2) \tag{1}$$

$$Y|X, \beta \sim \text{Normal}(\beta_0 + \beta_1 X, \sigma_\epsilon^2) \tag{2}$$

Calculate the posterior of $\beta_0$.

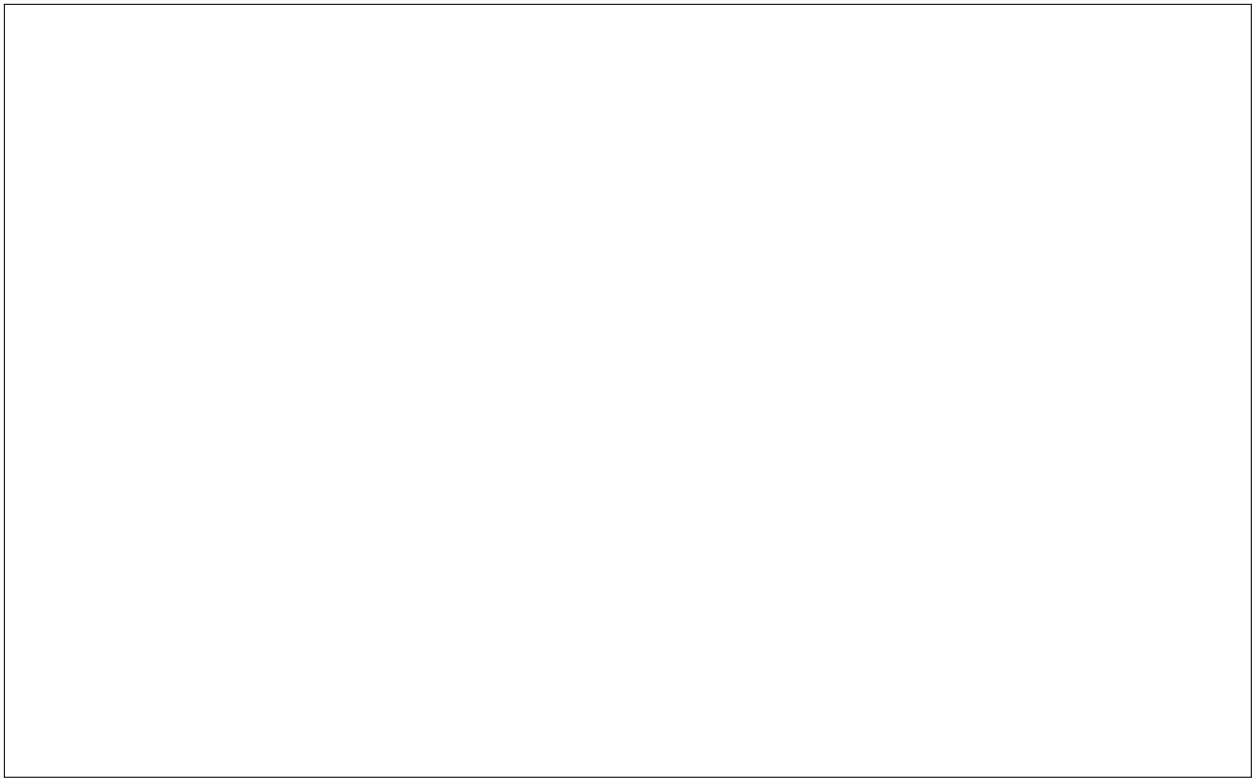
**Exercise 8** (Missing data)**:** You are given data with predictors $X_1$, $X_2$. You want to fit a linear regression model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$$

but some of the $X_2$ values are have been corrupted and are not reliable. One idea to handle this is called *imputation* and involves generating the missing values $X_2$.

Explain how you could implement imputation assuming you are a given dataframe with rows $Y, X_1, X_2, C$ where $C = 1$ for the rows with corrupted data and $C = 0$ otherwise. What assumptions are being made for the procedure to be unbiased?

**Solution:**



## Solutions

1. $b_1$ is the regression coefficient with only $X_1$ as a predictor and the regression coefficient $\beta_{1,2}$ of $X_1$ on $X_2$ is given to be 2 as well as $\beta_1 = \beta_2 = 1$. Thus we have $b_1 = 1+2 = 3$. The regression coefficient of $X_2$ on $X_1$ is $\beta_{2,1} = \text{cov}(X_1, X_2)/\text{var}(X_2) = \text{var}(X_1)\beta_{1,2}/(2^2+1) = 1 \times 2/5 = 2/5$. Hence $b_2 = 1 + 2/5 = 7/5$.

2. (a) One valid implementation:

```
def generatesim(L):
    X = np.zeros(L)
    for i in range(1,L):
        X[i] = 1.0 + 0.5*X[i-1] + np.random.normal(0.0, np.sqrt(1/3))
    return X
```

   (b) In steady-state we have the marginals are equal so $\mu_X = E[X_{i+1}] = 1 + E[X_i]/2 = 1 + \mu_X/2$. Thus $\mu_X = 2$. Similarly $\sigma_X^2 = \sigma_X^2/4 + 1/3$. Thus $\sigma_X^2 = 4/9$ and $X_i \sim \text{Normal}(2, 4/9)$.

3. We are given the single predictor regression coefficient $\hat{\beta}_1' \approx 2$ and the two predictor model coefficients $\hat{\beta}_1 \approx 1$, $\hat{\beta}_2 \approx 2$. Using $\hat{\beta}_1' = \hat{\beta}_1 + \hat{\beta}_2\beta_{1,2}$ we deduce the coefficient of $X_1$ with $X_2$ as the response variable, denoted $\beta_{1,2}$ here, is $\beta_{1,2} \approx 1/2$.

4. (a) This suggests the $\hat{\beta}_1$ and $\hat{\beta}_2$ are negatively correlated and therefore $X_1$ and $X_2$ are positively corrected, hence $\text{cov}(X_1, X_2) > 0$. (b) Yes, because even though we know $\hat{\sigma}_\epsilon^2$ we aren't given $\sigma_Y^2$, which could be quite large especially with such large values of $\hat{\beta}_i$.

5. $\beta_2$ is the slope of $X_2$ vs. $Y$ when $X_1 = 0$, which is approximately 0 in the plot. $\beta_2 + J_{1,2}$ is the slope when $X_1 = 1$, which appears to be 1. $\beta_1$ is the expected difference in $Y$ between $X_1 = 0$ and $X_1 = 1$ groups when $X_2 = 0$, which appears to be 0. In summary, $\beta_1 = \beta_2 = 0$ and $J_{1,2} = 1$.

6. (a) With $X \sim \text{Unif}(-1, 1)$,

$$E[\phi_1(X)\phi_2(X)] = E[X^5] = \int_{-1}^{1} x^5 \cdot \tfrac{1}{2}\, dx = 0, \tag{3}$$

   so they are orthogonal. (b) Any non-symmetric distribution makes $E[X^5] \neq 0$. For example, if $X \sim \text{Unif}(0, 1)$,

$$E[X^5] = \frac{1}{6} \neq 0, \tag{4}$$

   so they are not orthogonal.

7. The (frequentist) estimator $\hat{\beta}_0 = \overline{Y} - \beta_1\overline{X}$. This is Normal when conditioned on $\beta_0$: $\hat{\beta}_0|\beta_0 \sim \text{Normal}(\beta_0, \hat{\sigma}_\epsilon^2/N)$. If we invert this regression model we get

$$\beta_0|\hat{\beta}_0 \sim \text{Normal}\left(\beta_0 \frac{\tau^2}{\tau^2 + \sigma_\epsilon^2/N}, \frac{\tau^2\sigma_\epsilon^2/N}{\tau^2 + \sigma_\epsilon^2/N}\right) \tag{5}$$

8. Using rows with $C = 0$, fit

$$X_2 = \alpha_0 + \alpha_1 X_1 + \zeta, \tag{6}$$

   and estimate $\hat{\sigma}_\zeta^2 \approx \text{var}(\zeta)$. For each $C = 1$ row,

$$\tilde{X}_2 = \hat{\alpha}_0 + \hat{\alpha}_1 X_1 + \tilde{\zeta}, \qquad \tilde{\zeta} \sim \text{Normal}(0, \hat{\sigma}_\zeta^2). \tag{7}$$

   Then we fit

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 \tilde{X}_2 + \epsilon \tag{8}$$

   on the completed data. We have assumed $C$ is independent of $X_1$ and $X_2$.