# Unit 6: regularization, priors and Bayesian inference

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## 1.1 Introduction to Bayesian inference: some simple examples

So far, we think of models encode our assumptions about how data was generate. Our models depend on parameters for which the true values are fixed. When use data to <u>fit</u> or <u>infer</u> parameters, obtain a sample distribution which roughly speaking quantifies uncertainty in our estimates. This way of performing statistics, where the uncertainty is thought of as a distribution over repeated experiments, is <u>frequentist</u> approach to statistics. In this formulation of statistical inference we are pretending to have complete ignorance of the parameters before we see the data, but in reality, this is never true. For example, if we flip a coin $N = 3$ times and get $Y = 3$ heads, our estimate of the probability this coin will land on heads in the next flip is $\hat{q} = 1$, which is clearly not in line with out understanding that both sides are at least possible. It is also worth noting that the usual estimate standard error, $\text{se}(\hat{q}) = \sqrt{q(1-q)/N} \approx \sqrt{\hat{q}(1-\hat{q})/N}$ gives zero, but this is obviously not a good estimate of the uncertainty.

Similar issues emerge in the context of more machine learning-style data analysis, sometimes we want to incorporate vague information, such as "the function describing my data is very smooth and changes roughly on a time-scale of 5 hours". Or, we might want to include many predictors, but to avoid overfitting penalize high values of the predictors. For example, we might believe there is an interaction term, but suspect it is much smaller than the additive terms.

While there are ways handle these problems within the frequentist framework, they are handled more naturally by taking an entirely different treatment of parameters and statistical inference. This is called <u>Bayesian statistics</u>. In the Bayesian formulation of statistics, we will think of parameters as themselves random variables which are given a distribution before we have seen the data. Mathematically, this means instead of having a model with fixed parameter:

$$X \sim \text{ModelDistribution}(\theta) \tag{1}$$

we think of our model as a conditional on $\theta$

$$X|\theta \sim \text{ModelDistribution}(\theta). \tag{2}$$

We call $X|\theta$ the <u>likelihood</u> (this term is used in both frequentists and Bayesian approaches). Then, we add a new distribution for $\theta$, called the prior. The goal is then to condition on the observed data $X_1, \ldots, X_N$ to find a new distribution $\theta|X_1, \ldots, X_N$, called the posterior. The posterior plays a similar role to the sample distribution, and naturally we often use its mean as an estimate of $\theta$. However, there are both philosophical and mathematical differences between these two approaches which make the comparison imperfect. We will focus less on the philosophical aspect and see what is practically different about the approaches. This is best illustrated with the following example.

### 1.1.1 Bayesian inference for Bernoulli trails

Consider the model

$$X \sim \text{Bernoullli}(q)$$

for the outcome of a coin flip. If we perform $N$ flips, then recall $Y = \sum_{i=1}^{N} X_i$ is binomial. In a Bayesian model, we would specify the prior distribution of $q$. This should reflect what we know about the coin, which in reality

is that it is very likely to be fair. However, to make calculations simpler and better compare the frequentist approach we will take a prior where any value of $q$ is equally likely. This seems to be more similar to what we are doing in the frequentist approach. Our Bayesian model would then me

$$q \sim \text{Uniform}(0, 1)$$
$$X|q \sim \text{Bernoullli}(q)$$

Our goal is to find the posterior distribution

$$q|X_1, \ldots, X_N$$

Since the flips out independent, this distribution depends only on $Y$, so we can alternative obtain the posterior as the distribution of $q|Y$. Recall that the density of the uniform is simply $f(q) = 1$ (I'll use $f$ for a density and $P$ for probabilities, but sometimes we have to mix them). Using Bayes' theorem (or simply using the definition of conditional probability twice), we have

$$f(q|Y = y) = \frac{f(q, \{Y = y\})}{P(Y = y)} = \frac{P(Y = y|q)f(q)}{P(Y = y)} \tag{3}$$

where $f(q, \{Y = y\})$ means the joint density/probability of $q$ and $Y$. Recall the binomial distribution is

$$P(Y = y|q) = \binom{N}{y}q^y(1 - q)^{N-y} \tag{4}$$

The important thing to realize is that all the $q$ dependence comes from $P(Y = y|q)$ and since our goal is to find a distribution of $q$, we can write

$$f(q|Y = y) = C(N, y)q^y(1 - q)^{N-y} \tag{5}$$

where $C(N, y)$ is a constant which ensures $\int f(q|Y = y)dq = 1$. That is,

$$C(N, y)^{-1} = \int_0^1 q^y(1 - q)^{N-y}dq \tag{6}$$

$f(q|Y = y)$ is an example of <u>Beta distribution</u>. More generally, we say a variable $q$ is

$$q \sim \text{Beta}(a, b) \tag{7}$$

if $q$ has the density

$$f(q) = q^{a-1}(1 - q)^{b-1} \tag{8}$$

hence $q|Y \sim \text{Beta}(y + 1, N - y + 1)$. You don't need to memorize this, but you should look on Wikipedia or in the colab notebook to get a sense of what this function looks like.

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**Example 1** (Laplace Rule of Succession of Bayesian view). Consider Bayesian inference for Bernoulli trials with uniform priors as described above. Recall that Laplace's Rule of Succession gives the estimator

$$\hat{q}_L = \frac{Y + 1}{N + 2} \tag{9}$$

<u>Question</u>: Show that when $Y = N$

$$E[q|Y] = \frac{N + 1}{N + 2} = \hat{q}_L$$

<u>Solution</u>: In this case

$$C(N, y)^{-1} = C(N, N)^{-1} = \int_0^1 q^N dq = \frac{1}{N + 1} \tag{10}$$

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and

$$E[q|Y] = (N+1)\int_0^1 q \times q^N dq = (N+1)\int_0^1 q^{N+1}dq = \frac{N+1}{N+2}. \tag{11}$$

To summarize the idea of Bayesian inference, If $Y$ is a our data and $\theta$ is our parameter(s) Bayes' theorem tells us that

$$P(\theta|Y) = \frac{P(Y|\theta)P(\theta)}{P(Y)} \tag{12}$$

The distributions appearing in Equation (12) have the following names and interpretations:

- $P(\theta|Y)$ is the <u>posterior</u> (a beta distribution in Example 1).

- $P(Y|\theta)$ is the <u>likelihood</u> (Binomial distribution in Example 1).

- $P(\theta)$ is the prior distribution (Uniform distribution in Example 1)

- $P(Y)$ is the evidence. It represents the chance we observe the data *unconditional* on the parameters. This can be obtained by marginalizing over the priors.

While in classical statistics, the objective is to determine (estimate) a parameter $\theta$ and measure its uncertainty (with the sample distribution), in Bayesian statistics our objective is to compute the posteior distribution. Once we have this, we can obtain so-called <u>point estimates</u>, for example, by taking the average of $\theta$ or maximum of $P(\theta|Y)$ (maximum likelihood). The point estimates are however less central in Bayesian inference than the posterior.

### 1.1.2 Selecting priors with a beta distribution (optional)

The beta distribution will play an important role in what follows, so we note that if

$$q \sim \text{Beta}(a, b)$$

then (using calculus) it can be shown that

$$E[q] = \frac{a}{a+b}$$

$$\text{var}(q) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Notice that this implies the estimator $\hat{q}_b$ in Equation **??** becomes

$$\hat{q}_b = \frac{Y+1}{Y+1+N-Y+1} = \frac{Y+1}{N+2}$$

This is in-fact the estimator $\hat{q}_L$ from the week 7 exercise set!

The <u>mode</u> – the value which maximizes the probability density – for the $\beta$-distribution is

$$\frac{a-1}{a+b-2}$$

We can see that the estimator $\hat{q} = Y/N$ comes comes from the mode $f(q|Y)$ – recall that such an estimator is called the maximum likelihood estimator.

A very important observation is that adding priors often has the effect of introducing a bias while reducing the variance.

**Example 2** (Selecting priors). Suppose we are given a coin. We are trying to determine whether it is biased based on the outcome of $N$ flips. We are pretty sure that, like most coins, it is not biased. To be precise, let's say you are 95% confident that the bias of the coin is less than 10% biased towards heads or tails.

Question:

  (a) Select a prior distribution for $q$.

  (b) What is the posterior expectation if we flip the coin 5 times and see 5 heads?

Solution:

  (a) We know that a $\beta$-distribution is approximately Normal if $a$ and $b$ are large enough. Assuming we can make a Normal approximation, we can use the formulas for the mean and variance of the $b$ distribution to select parameters such that

$$P(|q - 0.5| < 0.1) = P(0.4 < q < 0.6) \approx 0.95 \tag{13}$$

Since we would like the mean of our prior distribution to be $1/2$, we select

$$q \sim \text{Beta}(a, a).$$

If $q$ is approximately Normal, then Equation 13 will hold when

$$1.96\sqrt{\text{var}(q)} = 0.1$$

$$\implies \sqrt{\frac{a^2}{4a^2(2a+1)}} = \frac{1}{2}\sqrt{\frac{1}{2a+1}} = \frac{0.1}{1.6}$$

$$\implies a \approx 31.5$$

  (b) In order to compute the posterior, we use Bayes' theorem

$$f(q|X_1, \ldots, X_N) \propto q^Y(1-q)^{N-Y} \times q^{a-1}(1-q)^{a-1} = q^{Y+a-1}(1-q)^{N-Y+a-1}$$

where $Y = \sum_{i=1}^{N} X_i$. Again, the $q$ dependence uniquely determines the distribution, since the area under this curve must be one. We can therefore say that

$$q|X_1, \ldots, X_N \sim \text{Beta}(Y + a, N - Y + a)$$

For the mean and variance we have

$$E[q|X_1, \ldots, X_N] = \frac{Y+a}{N+2a} = \frac{5+31.5}{5+2\times 31.5} \approx 0.54$$

Note that if we used usual estimator we would have found $\hat{q} = 1$, while if we used uniform priors and taken the posterior expectation $\hat{q}_L = 0.85$. Yet another approach would be to take the posterior maximum (the mode) of the posterior in using the $\beta$-distribution priors.

Something quite nice about using $\beta$-distribution priors for the Bernoulli model is that BOTH the prior and the posterior have a $\beta$-distribution, albeit with difference parameters. In general, when this is the case we say that the priors are conjugate.

Suppose we have a Normal model for $Y$

$$Y|(\mu, \sigma) \sim \text{Normal}(\mu, \sigma).$$

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Assume that $\sigma$ is known and take our priors on $\mu$ to be

$$\mu \sim \text{Normal}(\mu_0, \sigma_\mu)$$

What is the posterior distribution of $\mu$?

The likelihood is

$$p(D|\theta) = \prod_i \frac{1}{\sqrt{2\sigma^2\pi}} e^{-(Y_i-\mu)^2/2\sigma^2}$$

$$= \frac{1}{(2\sigma^2\pi)^{N/2}} e^{-\sum_i(Y_i-\mu)^2/2\sigma^2}$$

The posterior is *proportional to*

$$p(D|\theta)p(\theta) = \frac{1}{(2\sigma^2\pi)^{N/2}} e^{-\sum_i(Y_i-\mu)^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma_\mu^2}} e^{-(\mu-\mu_0)^2/2\sigma_\mu^2}$$

$$= \frac{1}{(2\sigma^2\pi)^{N/2}} \frac{1}{\sqrt{2\pi\sigma_\mu^2}} e^{-\sum_i(Y_i-\mu)^2/2\sigma^2-(\mu-\mu_0)^2/2\sigma_\mu^2}$$

On this surface this looks a bit complicated as a function of $\mu$, but there is a trick: Notice that all the dependence on $\mu$ comes from the exponent. We can rewrite this as

$$A\mu^2 + B\mu + C$$

where

$$A = \frac{1}{2}\left(\frac{N}{\sigma^2} + \frac{1}{\sigma_\mu^2}\right)$$

$$B = -\frac{\sum_i Y_i}{\sigma^2} - \frac{\mu_0}{\sigma_\mu^2}$$

$$C = \frac{\sum_i Y_i^2}{2\sigma^2} + \frac{\mu_0^2}{\sigma_\mu^2}$$

If we factor this quadratic equation, we find that is can be written

$$A(\mu - B/2A)^2 + \text{const.}$$

We don't care what the constant terms is, since the it is the first term which tells us the mean and standard deviation of $\mu$. Now observe that

$$-\frac{B}{2A} = \bar{Y}\frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma^2/N} + \mu_0\frac{\sigma^2/N}{\sigma_\mu^2 + \sigma^2/N} \equiv \mu_b$$

and

$$\frac{1}{A} = 2\frac{\sigma_\mu^2\sigma^2/N}{\sigma_\mu^2 + \sigma^2/N} \equiv 2\sigma_b^2$$

In particular, we can deduce that

$$\mu|D \sim \text{Normal}(\mu_b, \sigma_b)$$

where $\mu_b$ and $\sigma_b^2$ defined above are the mean and variance of the posterior distribution.

Importantly, for the Normal distribution the mean and mode are the same, so there is really only one choice for the point estimate of a parameter which has a Normal posterior.

## 1.2 Bayesian inference for linear regression models

We now discuss Bayesian inference for the general linear regression model with features. Let

$$f(x) = \sum_{i=1}^{K} \beta_i \phi_i(x)$$

and suppose that our priors are

$$\beta_i \sim \text{Normal}(0, \tau_i^2).$$

We will assume that $E[\phi_i(X)] = 0$. We will also assume, for simplicity, that $\sigma$ **is known. Hence, we do not need to consider estimating it.** This simplifies the conceptual picture considerably.

In this case, the Posterior distribution of the $\beta_i$ can be understood analytically. It turns out that the marginal posterior distribution of each $\beta_i$ is again Normal – thus Normal priors on $\beta_i$ are conjugate priors!. The marginal mean of each $\beta_i$ are determined by the system of questions given in the following Theorem.

**Theorem 1** (Posterior mean for regression coefficients). *Define a $K \times K$ matrix $\Omega$ called the <u>empirical covariance matrix</u> which has entries*

$$\Omega_{i,j} = N\overline{\phi_i(X)\phi_j(X)} = \sum_{k=1}^{N} \sum_{z=1}^{N} \phi_i(X_k)\phi_j(X_z)$$

*and let $\bar{\beta}_i$ denote the posterior mean of $\beta_i$; that is,*

$$\bar{\beta}_i = E[\beta_i | D].$$

*Then $\bar{\beta}_1, \ldots, \bar{\beta}_K$ satisfy the $K$ equations*

$$\left(\frac{\sigma}{\tau_i}\right)^2 \bar{\beta}_i + \sum_{j=1}^{K} \Omega_{i,j} \bar{\beta}_j = \sum_{j=1}^{N} \phi_i(X_j) Y_j \tag{14}$$

We make a few remarks on this Theorem:

- If $N > K$, then $\bar{\beta}_i$ actually have a solution. In-fact, depending on $\Omega$, there may not be a solution. For our discussion, we will assume there is however.

- Under the assumption that $E[\phi_i(X)] = 0$, the entries of $\Omega_{i,j}$ approximate the covariances between the predictor features. Moreover, if we take the predictors to be drawn from some distribution and average over the data (both $X$ and $Y$), we get

$$\frac{E[\Omega_{i,j}]}{N} = \text{cov}(\phi_j(X), \phi_i(X)).$$

and

$$\frac{1}{N} \sum_{j=1}^{N} E[\phi_i(X_j) Y_j] = \text{cov}(Y, \phi_i(X)).$$

Hence, we have the "math world" version of Equations 14, which is obtained by averaging over the data:

$$\frac{1}{N} \left(\frac{\sigma}{\tau_i}\right)^2 E[\bar{\beta}_i] + \sum_{j=1}^{K} \text{cov}(\phi_i(X), \phi_j(X)) E[\bar{\beta}_j] = \text{cov}(Y, \phi_i(X)).$$

In the limit $N \to \infty$ and with $K = 2$, we obtain a system of equations recognizable from Week 5 notes for the regression coefficients in the two predictor model in terms of the covariances. In this context, the role of the regression coefficients (which we previously took to be fixed numbers), is now played by the averages $E[\bar{\beta}_j]$, which are the average values of $\beta_j$ with respect to both the posterior and the data.

- Notice that the first term in Equation 14 vanishes as either $\tau_i \to \infty$ (very broad priors) or $\sigma \to 0$ (no variance in $Y$ conditional on the predictors). In both cases, the vales of $\bar{\beta}_i$ are entirely determined by the data, and the priors play no role. When $\tau_i \to 0$ or $\sigma \to \infty$, the priors entirely determine determine $\bar{\beta}_i$ and in fact $\bar{\beta}_i = 0$.

- The first term in Equation 14 is an example <u>regularization</u>.

Theorem 1 can be elegantly stated using matrices (see Linear algebra review note for intro to matrix multiplication). This is import if we want to implement these computations in Python. Observe that one way to construct $\Omega_{i,j}$ is to define the matrices

$$A = \left[\; \phi_1(X) \quad \phi_2(X) \quad \cdots \quad \phi_K(X) \;\right], \quad \Lambda_0 = \begin{bmatrix} \tau_1 & 0 & \dots & 0 \\ 0 & \tau_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tau_K \end{bmatrix}$$

Then $\Omega = A^T A$. Now define another matrix $K \times K$ matrix

$$M = \Omega + \sigma^2 \Lambda_0^{-1}$$

Then Equations 14 can be written as
$$M\bar{\beta} = A^T y$$

where $\bar{\beta}$ and $y$ are respectively a $K$-vector and $N$-vector

$$\bar{\beta} = \begin{bmatrix} \bar{\beta}_1 \\ \vdots \\ \bar{\beta}_K \end{bmatrix}, \quad y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}.$$

Then $\bar{\beta}$ satisfies
$$M\bar{\beta} = A^T y \implies \bar{\beta} = \left(A^T A + \sigma^2 \Lambda_0^{-1}\right)^{-1} A^T y$$

**Example 3** (Calculating posterior mean in python). Consider the model with two predictors:

$$f(x) = \beta_1 X_1 + \beta_2 X_2$$

<u>Question</u>: Generate some simulated data and compute the posterior means of $\beta$ for different choices of $\tau$.

<u>Solution</u>: See colab notebook.

### 1.2.1 Posterior prediction

We will mostly focus on how we can use the posterior estimates to make predictions, although there is an entirely different set of questions we could ask concerning model assessment and hypothesis testing. Recall that $\hat{y}(x, D)$ is defined as our prediction, using the regression model fitted to the data $D$, of $E[Y|X = x]$. In the Bayesian context, we can define the posterior predictive variable $Y|X, D$ as the random representing the distribution of our response variable conditioned on the data. We can obtain this distribution by replacing the $\beta_i$s in

$$Y = \sum_{i=1}^{K} \beta_i \phi_i(X)$$

with samples from the posterior distribution. In the Bayesian framework we set

$$\hat{y}(x, D) = E[Y|D, X] = \sum_{i=1}^{K} E[\beta_i|D]\phi_i(X) = \sum_{i=1}^{K} \bar{\beta}_i\phi_i(X)$$

We can write this in matrix notation by defining the vector

$$a = \begin{bmatrix} \phi_1(X) \\ \vdots \\ \phi_K(X) \end{bmatrix}.$$

Then

$$\hat{y}(x, D) = a^T\bar{\beta} = a^T M^{-1} A^T y \tag{15}$$

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**Example 4** (Posterior predictions for a regression model). Take the Fourier model

$$f(x) = \beta_0 + \sum_{i=1}^{K} \beta_i \sin\left(\frac{2\pi i x}{L}\right) + \alpha_i \cos\left(\frac{2\pi i x}{L}\right).$$

Implement a function to compute posterior predictions with different priors $\tau$ on the coefficients. In this example, we will fit this model to data generated by adding noise to the function

$$f_{\text{true}}(x) = \sin(2\pi 3x) + 5\sin(2\pi 6x^2) + 20x^2$$

with $x \in [0, 1]$. Note that for $K < \infty$, we can not pick any values of the $\beta_i$s and $\alpha_i$s such that $f = f_{\text{true}}$.

We will assume that $\beta_i$ and $\alpha_i$ have the same prior distribution:

$$\beta_i \sim \text{Normal}(0, \tau_i)$$
$$\alpha_i \sim \text{Normal}(0, \tau_i)$$

Question: Compute the posterior mean $\hat{y}(x, D)$ for the following choices of priors and $K = 50$.

(a) All $\tau_i$ are the same, $\tau_i = \tau_0$.

(b) The $\tau_i$ decay with $i$ as $\tau_i = \tau_0/i$

In each case, experiment with different choices of $\tau_0$ and pay attention to whether the model is overfitting (high variance, low bias) or under fitting (low variance, high bias).

Solution: See colab notebook.

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## 1.3   Regularization view of priors

- Recall that in standard least squares linear regression, the regression coefficients come from minimizing the sum of squared residuals. That is, they are the values that minimize the function

$$L(\beta) = \sum_{j=1}^{N} (\hat{y}(X_j, D) - Y_j)^2$$

This is an example of a <u>loss function</u>, which is simply something we want to minimize to solve our inference problem. This means that the derivative of $L$ with respect to each $\beta_i$ is zero.

$$
\begin{aligned}
\frac{\partial}{\partial \beta_i} L(\beta) &= \sum_{z=1}^{N} 2(\hat{y}(X_z, D) - Y_z) \frac{\partial}{\partial \beta_i} \hat{y}(X_j, D) \\
&= \sum_{z=1}^{N} 2(\hat{y}(X_z, D) - Y_z) \phi_i(X_z) \\
&= 2 \sum_{z=1}^{N} \sum_{j=1}^{K} \beta_j \phi_i(X_z) \phi_j(X_z) - 2 \sum_{z=1}^{N} Y_z \phi_i(X_z) \\
&= 2 \sum_{z=1}^{K} \beta_j \sum_{z=1}^{N} \phi_i(X_z) \phi_j(X_z) - 2 \sum_{z=1}^{N} Y_z \phi_i(X_z) \\
&= 2 \sum_{j=1}^{K} \beta_j \Omega_{i,j} - 2 \sum_{z=1}^{N} Y_z \phi_i(X_z)
\end{aligned}
$$

Setting

$$
\frac{\partial}{\partial \beta_i} L(\beta) = 0
$$

leads to Equation 14 without the prior term.

- The regularization view of Equation 14 is that, instead of thinking of the additional term as coming from priors, we think about adding a term to our loss function which penalizes models with too much flexibility. A common choice is

$$
L(\beta) = \sum_{j=1}^{N} (\hat{y}(X_j, D) - Y_j)^2 + \lambda \sum_{j=1}^{K} \beta_j^2
$$

This is called <u>ridge regression</u> and $\lambda$ is a parameter controlling how large a penalty we place on large values of $\beta_j$. If you compute the partial derivatives and equate it to zero, you will find that the equations $\beta_j$ satisfy are exactly Equation 14 with $\sigma/\tau_i = \sqrt{\lambda}$. More generally, we could use the loss function

$$
L(\beta) = \sum_{j=1}^{N} (\hat{y}(X_j, D) - Y_j)^2 + \sum_{j=1}^{K} \left(\frac{\sigma}{\tau_i}\right)^2 \beta_j^2
$$

we would obtain exactly Equation 14.

## 1.4   The kernel trick

- The problem of computing $\hat{y}(x, D)$ is an example of smoothing, or interpolating. Smoothing refers to the situation where we are given noisy measurements of a function $f(x)$ and want to "smooth out the noise". One way to do this is take the weighted averaging of neighboring values of $y$. This following examples illustrates how regression modeling is related to smoothing.

> **Example 5** (Orthogonal empirical covariance matrix)**.** Consider the special case where the empirical covariance matrix is diagonal − that is, $\Omega_{i,j} = 1$ if $i = j$ and 0 if $i \neq j$. Recall that this is the case when we use the fourier model and the $X_i$ are equally spaced.
> <u>Question:</u>
>
>  (a) Write down a formula for $\hat{y}(x, D)$ in this case. In particular, show how to express $\hat{y}(x, D)$ in

the form

$$\hat{y}(x, D) = \sum_{j=1}^{N} Y_j \kappa(x, X_j) \tag{16}$$

for some function $\kappa(x, x')$.

(b) What does this function $\kappa(x, x')$ look like when $f(x)$ is a Fourier model?

Solution:

(a) In this case,

$$\bar{\beta}_i = \frac{1}{1 + \left(\frac{\sigma}{\tau_i}\right)^2} \sum_{j=1}^{N} \phi_i(X_j) Y_j$$

and hence

$$\hat{y}(x, D) = \sum_{i=1}^{K} \frac{\phi_i(x)}{1 + \left(\frac{\sigma}{\tau_i}\right)^2} \sum_{j=1}^{N} \phi_i(X_j) Y_j$$

$$= \sum_{j=1}^{N} Y_j \left( \sum_{i=1}^{K} \frac{\phi_i(x)}{1 + \left(\frac{\sigma}{\tau_i}\right)^2} \phi_i(X_j) \right)$$

so

$$\kappa(x, x') = \sum_{i=1}^{K} \frac{\phi_i(x) \phi_i(x')}{1 + \left(\frac{\sigma}{\tau_i}\right)^2}$$

(b) For the Fourier model with $L = 1$,

$$\kappa(x, x') = \sum_{i=1}^{K} \frac{\sin(2\pi i x) \sin(2\pi i x') + \cos(2\pi i x) \cos(2\pi i x')}{1 + \left(\frac{\sigma}{\tau_i}\right)^2}$$

Using the identity

$$\cos(x) \cos(y) + \sin(x) \sin(y) = \cos(x - y)$$

and the fact that $\cos(x - y) = \cos(y - x)$, we get can rewrite this as

$$\kappa(x, x') = \eta(|x - x'|) = \sum_{i=1}^{K} \frac{\cos(2\pi i |x - x'|)}{1 + \left(\frac{\sigma}{\tau_i}\right)^2}$$

- We actually don't need the empirical covariance matrix to be diagonal to "kernalize" our predictions (meaning express $\hat{y}(x, D)$ in the form of Equation 16. Equation 15 also has this form, although we can not determine the kernel without inverting a very large matrix.