Computational Finance Assignment 3

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1 Introduction

There are many types of options in the financial market, it's important to price these options correctly such that there is no arbitrage in the market. For European options, we can use the Black-Scholes model to calculate its analytical price. However, the Black-Scholes model is not applicable for other options. For these options, we can only estimate their value. There are many ways to estimate the price of an option, among which the binomial tree model, Monte Carlo method, and finite difference method are three popular ways to do that. In this report, we will focus on the finite difference method.

The finite difference method is based on the Black-Scholes Partial Differential Equation(BS-PDE). We first introduce how to transform this PDE such that we can solve it backward in time and get the option price at $t = 0$. We implement two different scheme to estimate the first and the second order derivative in the equation, the Forward Time Centered Space (FTCS) scheme and Crank-Nicolson(CN) scheme. We derive the discrete Finite-Difference approximation of the FTCS and CN scheme for the PDE and study the order of the errors in the CN scheme. After that, we use this method to estimate the price of a European option. We first present these two schemes in matrix vectors and write a program to solve the equations in both schemes. Then we compare the results we get from these two schemes with the analytical value of the Black-Schole model and check the stability conditions of these two schemes. Then we check if the order of the errors of the CN scheme in our program is in line with the order of the error that we derived analytically. After that, we study how the error changes with different mesh size and determined our own optimal mesh size. Based on this mesh size, we calculate the hedge parameter δ of the option at $t = 0$ and compare it with the analytical δ we get in the Black-Scholes model.

This report is organized as follows: Section 2 will present the process of deriving the formulas of the CN scheme and FTCS scheme, present these two schemes in matrix vectors and study the stability condition of the FTCS scheme. Section 3 presents the results from our program and gives further discussions about these results.

2 Methods

2.1 Part I:Background of PDE approach

2.1.1 Transformed BS-PDE

The Black-Scholes partial differential equation has the following form:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV \tag{1}$$

let $X = \ln S$ and $\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$,then we get:

$$\frac{\partial V}{\partial S} = \frac{1}{S}\frac{\partial V}{\partial X} \tag{2}$$

and

$$\begin{aligned}\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}(\frac{1}{S}\frac{\partial V}{\partial X}) \\ &= \frac{1}{S^2}\frac{\partial^2 V}{\partial X^2} - \frac{1}{S^2}\frac{\partial V}{\partial X}\end{aligned} \tag{3}$$

We can now substitute the new expressions into PDE equation 1 and attain:

$$\begin{aligned}
\frac{\partial V}{\partial t} &= -\frac{\partial V}{\partial \tau} + rS\frac{1}{S}\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 S^2 \left(\frac{1}{S^2}\frac{\partial^2 V}{\partial X^2} - \frac{1}{S^2}\frac{\partial V}{\partial X}\right) \\
&= -\frac{\partial V}{\partial \tau} + r\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 V}{\partial X^2} - \frac{\partial V}{\partial X}\right) \\
&= -\frac{\partial V}{\partial \tau} + (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial X^2}
\end{aligned} \tag{4}$$

by rearranging the terms above, we have the new PDE:

$$\frac{\partial V}{\partial \tau} = (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial X^2} - rV \tag{5}$$

2.1.2 FTCS scheme

Here we introduce two finite difference scheme to approach the option price. Figure 1 and Figure 2 (Recktenwald,2014) show how FTCS scheme and CN scheme work respectively. First recalling the following Taylor

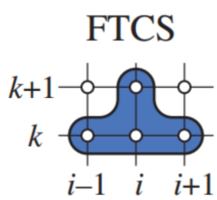

Figure 1: FTCS scheme

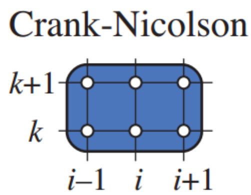

Figure 2: CN scheme

Expansion:

$$V(X + \Delta X, \tau) = V(X, \tau) + \Delta X\frac{\partial V(X,\tau)}{\partial X} + \frac{1}{2}\Delta X^2\frac{\partial^2 V(X,\tau)}{\partial X^2} + \cdots \tag{6}$$

$$V(X - \Delta X, \tau) = V(X, \tau) - \Delta X\frac{\partial V(X,\tau)}{\partial X} + \frac{1}{2}\Delta X^2\frac{\partial^2 V(X,\tau)}{\partial X^2} + \cdots \tag{7}$$

$$V(X, \tau + \Delta\tau) = V(X, \tau) + \Delta\tau\frac{\partial V(X,\tau)}{\partial \tau} + \cdots \tag{8}$$

Using these expressions, we can get the approximations for FTCS and then change equation 5 into our latest PDE:

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = (r - \frac{1}{2}\sigma^2)\frac{V_{i+1}^n - V_{i-1}^n}{2\Delta X} + \frac{1}{2}\sigma^2\frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta X^2} - rV_i^n \tag{9}$$

by rearranging the terms above, we have the new equation:

$$\begin{aligned}
V_i^{n+1} = {} & V_i^n + (r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{2\Delta X}(V_{i+1}^n - V_{i-1}^n) + \frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2}(V_{i+1}^n - 2V_i^n + V_{i-1}^n) \\
& - r\Delta\tau V_i^n
\end{aligned} \tag{10}$$

2.1.3 CN scheme

For the Forward Euler method, we have equation 9, and for the Backward Euler method, we have:

$$BE = \frac{V_i^{n+1} - V_i^n}{\Delta\tau} = (r - \frac{1}{2}\sigma^2)\frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2\Delta X} + \frac{1}{2}\sigma^2\frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{\Delta X^2} - rV_i^{n+1} \tag{11}$$

For CN scheme, we define:

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \frac{1}{2}(FE + BE) \tag{12}$$

Rearrange the equation, then we get:

$$\begin{aligned}
V_i^{n+1} = {} & V_i^n + (r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X}(V_{i+1}^n - V_{i-1}^n + V_{i+1}^{n+1} - V_{i-1}^{n+1}) \\
& + \frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2}(V_{i+1}^n - 2V_i^n + V_{i-1}^n + V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}) - \frac{r\Delta\tau}{2}(V_i^n + V_i^{n+1})
\end{aligned} \tag{13}$$

To analyse the CN scheme is second order in space, we need Taylor Expansions as follows:

$$V_{i+1}^n = V_i^n + \Delta X \frac{\partial V}{\partial X} + \frac{1}{2}\Delta X^2 \frac{\partial^2 V}{\partial X^2} + \frac{1}{3!}\Delta X^3 \frac{\partial^3 V}{\partial X^3} + O(\Delta X^4) \tag{14}$$

$$V_{i-1}^n = V_i^n - \Delta X \frac{\partial V}{\partial X} + \frac{1}{2}\Delta X^2 \frac{\partial^2 V}{\partial X^2} - \frac{1}{3!}\Delta X^3 \frac{\partial^3 V}{\partial X^3} + O(\Delta X^4) \tag{15}$$

For the Forward Euler method we have:

$$\begin{aligned} \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta X} &= \frac{\partial V}{\partial X} + \frac{1}{3!}\Delta X^2 \frac{\partial^3 V}{\partial X^3} \\ &= \frac{\partial V}{\partial X} + O(\Delta X^2) \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta X^2} &= \frac{\partial^2 V}{\partial X^2} + \frac{1}{12}\Delta X^2 \frac{\partial^4 V}{\partial X^4} \\ &= \frac{\partial^2 V}{\partial X^2} + O(\Delta X^2) \end{aligned} \tag{17}$$

For the Backward Euler method we have:

$$\begin{aligned} \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2\Delta X} &= \frac{\partial V}{\partial X} + \frac{1}{3!}\Delta X^2 \frac{\partial^3 V}{\partial X^3} \\ &= \frac{\partial V}{\partial X} + O(\Delta X^2) \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{\Delta X^2} &= \frac{\partial^2 V}{\partial X^2} + \frac{1}{12}\Delta X^2 \frac{\partial^4 V}{\partial X^4} \\ &= \frac{\partial^2 V}{\partial X^2} + O(\Delta X^2) \end{aligned} \tag{19}$$

In conclusion, we can see that $\frac{\partial V}{\partial X}$ and $\frac{\partial^2 V}{\partial X^2}$ are second order both in Forward Euler and Backward Euler method. Hence we say that CN scheme is second order in space.

2.2 FD-Schemes for European call

2.2.1 FTCS scheme

Now we compute the following matrix vector notation for FTCS scheme:

$$B\vec{V}^{n+1} = A\vec{V}^n \tag{20}$$

where A and B are sparse matrix with the three non-zero diagonals represented by the vectors:

$$B = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ & b_{-1} & b_0 & b_1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \, and \, A = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ & a_{-1} & a_0 & a_1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{21}$$

by rearranging equation 10, we get:

$$a_{-1} = -(r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{2\Delta X} + \frac{1}{2}\sigma^2 \frac{\Delta\tau}{\Delta X^2} \tag{22}$$

$$a_0 = 1 - \sigma^2 \frac{\Delta\tau}{\Delta X^2} - r\Delta\tau \tag{23}$$

$$a_1 = (r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{2\Delta X} + \frac{1}{2}\sigma^2 \frac{\Delta\tau}{\Delta X^2} \tag{24}$$

For FTCS scheme, our boundary conditions are: $V(0,\tau) = 0$, $V(X_{max},\tau) = e^{X_{max}}$ and $V(X,0) = max\{e^X - K, 0\}$ (i.e. $V_0^n = 0$,$V_N^n = e^{X_{max}}$ and $V_i^0 = max\{e^{i\Delta X} - K, 0\}$).Then equation 20 can also be written in the following way:

$$B\vec{V}^{n+1} = A\vec{V}^n + \vec{k}^n \tag{25}$$

Hence the matrix will be:

$$\begin{bmatrix} V_1^{n+1} \\ V_2^{n+1} \\ \vdots \\ V_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & 0 & \cdots & 0 & 0 \\ a_{-1} & a_0 & a_1 & \cdots & 0 & 0 \\ 0 & a_{-1} & a_0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{-1} & a_0 & a_1 \\ 0 & 0 & \cdots & 0 & a_{-1} & a_0 \end{bmatrix} \begin{bmatrix} V_1^n \\ V_2^n \\ \vdots \\ V_{N-1}^n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1 V_N^n \end{bmatrix} \tag{26}$$

2.2.2 CN scheme

For CN scheme, we also have equation 20 and the same boundary conditions as those in FTCS scheme. Then equation 20 can be written as:

$$B\vec{V}^{n+1} + \vec{k}^{n+1} = A\vec{V}^n + \vec{k}^n \tag{27}$$

by rearranging equation 13, we get:

$$b_{-1} = (r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X} - \frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2} \tag{28}$$

$$b_0 = 1 + \frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2} + \frac{r\Delta\tau}{2} \tag{29}$$

$$b_1 = -(r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X} - \frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2} \tag{30}$$

$$a_{-1} = -(r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X} + \frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2} \tag{31}$$

$$a_0 = 1 - \frac{1}{2}\sigma^2\frac{\Delta\tau}{\Delta X^2} - \frac{r\Delta\tau}{2} \tag{32}$$

$$a_1 = (r - \frac{1}{2}\sigma^2)\frac{\Delta\tau}{4\Delta X} + \frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2} \tag{33}$$

Hence the matrix will be:

$$\begin{bmatrix} b_0 & b_1 & 0 & \cdots & 0 & 0 \\ b_{-1} & b_0 & b_1 & \cdots & 0 & 0 \\ 0 & b_{-1} & b_0 & b_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{-1} & b_0 & b_1 \\ 0 & 0 & \cdots & 0 & b_{-1} & b_0 \end{bmatrix} \begin{bmatrix} V_1^{n+1} \\ V_2^{n+1} \\ \vdots \\ V_{N-1}^{n+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 V_N^n \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & 0 & \cdots & 0 & 0 \\ a_{-1} & a_0 & a_1 & \cdots & 0 & 0 \\ 0 & a_{-1} & a_0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{-1} & a_0 & a_1 \\ 0 & 0 & \cdots & 0 & a_{-1} & a_0 \end{bmatrix} \begin{bmatrix} V_1^n \\ V_2^n \\ \vdots \\ V_{N-1}^n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_1 V_N^n \end{bmatrix} \tag{34}$$

2.2.3 Stability of FTCS Scheme

Here we introduce a theorem to show what is the relationship between ΔX and $\Delta\tau$ when FTCS scheme is stable.

Theorem Let $\alpha = \frac{\sigma^2\Delta\tau}{\Delta X^2}$. If $\alpha \leq 1$ and $1 - \frac{1}{\sigma^2}|r - \frac{\sigma^2}{2}|\Delta X \geq 0$, then FTCS scheme is stable(de Graaf,2012).

3 Results and Discussion

3.1 Results of FTCS-scheme and Crank-Nicolson scheme

We first implement a program using the matrix vectors derived above to obtain the European call option prices for both the FTCS-scheme and the Crank-Nicolson scheme. As can be seen in Figure 3 and Figure 4, there's no significant difference between the curved surfaces on which the value of Z-axis represents the option value. And the European call option price obtained at t=0 by two schemes are shown in Table 1. Thus, we can tell that the result of CN scheme is closer to the analytical value calculated by the Black-Scholes model compared to the FTCS scheme. This mainly because the spatial truncation error for both schemes is $O(\Delta X^2)$ but the temporal truncation error for the CN scheme is $O(\Delta\tau^2)$, smaller than that of $O(\Delta\tau)$ for the FTCS scheme.

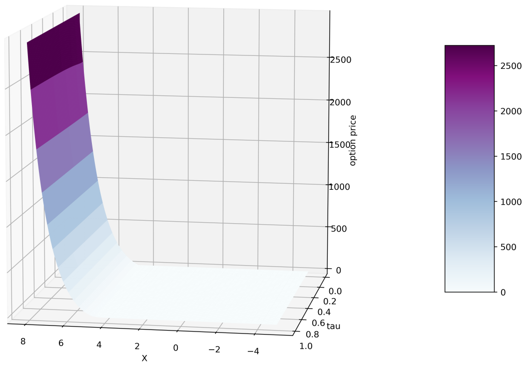

Figure 3: The 3D plot of European call option for the FTCS scheme. $X_{min} = -5$,$X_{max} = 8$,$\Delta\tau = 0.001$,$\Delta x = 0.013$

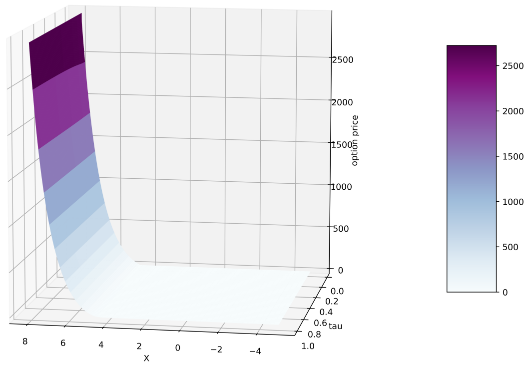

Figure 4: The 3D plot of European call option for the CN scheme.$X_{min} = -5$,$X_{max} = 8$,$\Delta\tau = 0.001$,$\Delta x = 0.013$

Table 1: The European call option of FTCS and CN at t=0

Category	$S_0 = 100$	$S_0 = 110$	$S_0 = 120$
Black-Scholes	9.6254	15.1286	21.7888
FTCS scheme	9.6278	15.1320	21.7914
CN scheme	9.6264	15.1303	21.7898

3.2 Second order convergence

As for the convergence, we plot the absolute error between the BS analytical value and the option price obtained by the CN scheme when the parameter setting is $S_0 = 100, \Delta\tau = 0.001$. As shown in Figure 5, the absolute error increases as ΔX grows from 0.0065 to 0.065. We could tell they follow a quadratic trend by fitting the values in a second order polynomial formula and the result proves that our program corresponds to the theoretical analysis mentioned above. The CN scheme indeed shows the second order convergence behavior as the number of grid points in space increases. The fluctuations are due to the error generated by using the interpolation method to get the estimated option price and the truncation error of ignoring higher order derivatives.

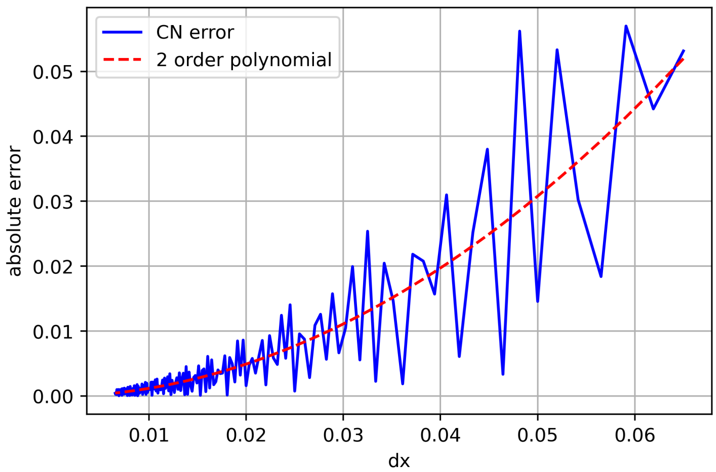

Figure 5: Second order convergence

3.3 Optimal mesh size and delta

Regarding the mesh size, we first fix $\Delta\tau = 0.001$ and investigate the relationship between the absolute error and ΔX for both schemes. Figure 6 and Figure 7 show that for the FTCS scheme, the errors drop significantly as ΔX increases from 0.008 to 0.0094 under unstable condition and the errors grow steadily as ΔX increases from 0.01 to 0.065 under stable condition. This result corresponds to the stability analysis that the FTCS scheme is conditionally stable when the number of ΔX is larger than 1370 when $\Delta\tau = 0.001$. Regarding the CN scheme, Figure 5 already tell that the absolute error increases as ΔX grows from 0.0065 to 0.065.

Thus, for the sake of stability of FTCS and small errors, we determine the optimal $\Delta X = 0.01$ and study the effects of $\Delta\tau$. We can tell from Figure 9 and Figure 10 that the errors for the FTCS scheme under unstable condition are ridiculously large when $\Delta\tau$ is larger than 0.0011 and the errors under stable condition increase as $\Delta\tau$ grow from 0.0005 to 0.0011. This is due to the fact that the FTCS scheme is conditionally stable when the number of $\Delta\tau$ is larger than 900 when $\Delta X = 0.01$. Thus, our experiment result corresponds to the stability analysis mentioned above. On the other hand, Figure 8 shows that the errors for the CN scheme increases as $\Delta\tau$ increases from 0.0005 to 0.005 since the CN scheme is unconditionally stable and the result would be more accurate given more grid points. To satisfy the stability requirement of FTCS and obtain more accurate results, the optima $\Delta\tau$ we decide is 0.0005.

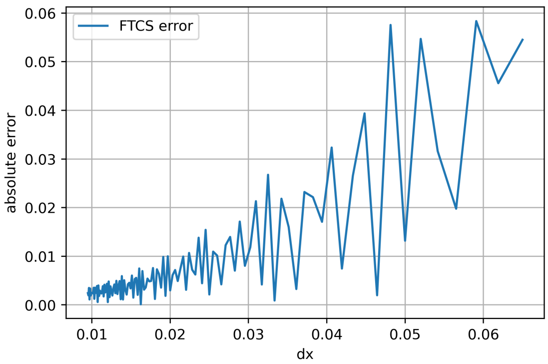

Figure 6: Absolute error of FTCS scheme with variation of dx-stable condition

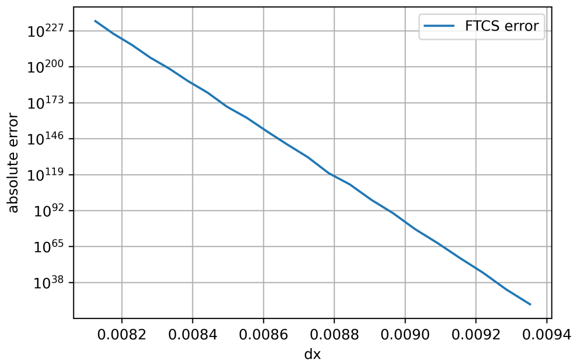

Figure 7: Absolute error of FTCS scheme with variation of dx-unstable condition

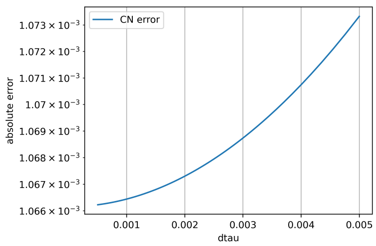

Figure 8: Absolute error of CN scheme with variation of dtau

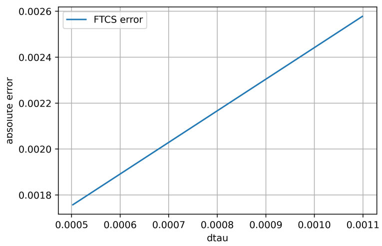

Figure 9: Absolute error of FTCS scheme with variation of dtau-stable condition

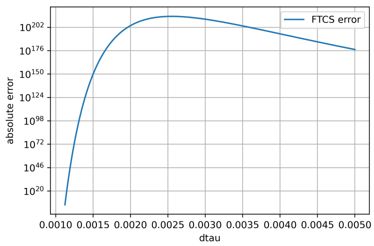

Figure 10: Absolute error of FTCS scheme with variation of dtau-unstable condition

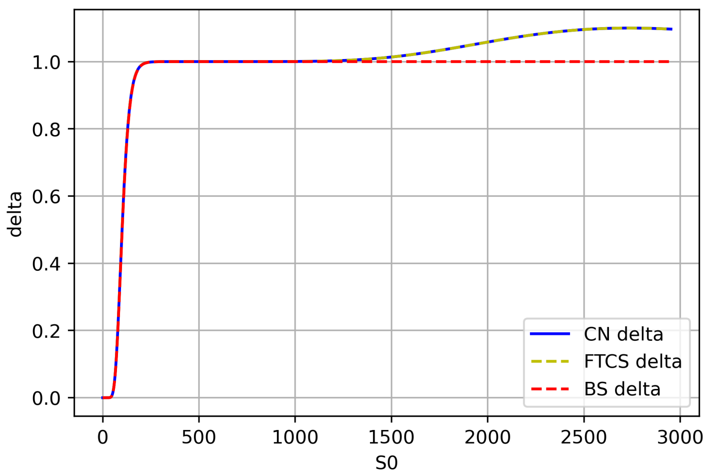

Figure 11: Delta of two schemes with optimal mesh size $\Delta x = 0.01$, $\Delta\tau = 0.0005$

Then we use the determined mesh size to study the estimated delta. As shown in Figure 11, delta of both schemes shows no significant differences to the analytical value obtained by Black-Sholes model when S_0 is smaller than 1400 approximately. However, the estimated delta grows larger than 1 as S_0 increases from 1400 to 3000. Since the option price would be incorrect when S_0 is close to S_{max}, delta of both schemes shows a slight deviation from the analytical value.

References

CSL de Graaf. 2012. Finite difference methods in derivatives pricing under stochastic volatility models. *Master's thesis, Leiden University* (2012).

Gerald Recktenwald. 2014. Crank Nicolson Solution to the Heat Equation.