Computational Finance Assignment 1

Group 9: Chun Cheng-13137727, Meifang Li-13043390, Anzhen Zhang-13012843

February 2021

1 Introduction

Options are contracts that give the holder the right to buy or sell the underlying asset at a fixed price on or before a certain date in the future. A call option gives the holder the right to buy a stock, while a put option gives the holder the right to sell the stock. Options have several different styles, American style, European style and Asian style , etc. In this report, we mainly focus on the first two styles. American option can be exercised before its expiration date whereas European option can only be exercised on its specific expiration date(Dar and Anuradha,2018). Option premium is how much we need to pay for purchasing the option. Options are very attractive to both speculators and hedgers in the financial market, it's important that the options are priced accurately(Higham,2004).

There are many theoretical works on option pricing, among which the binomial tree model and the Black-Scholes model (BS model) are very popular. We will only study these two models in this report. There are some differences between these two models: BS model is a continuous model, while binomial tree is a discrete model which breaks a period into many time intervals. BS model requires a solution of a stochastic differential equation and can not handle early exercise opportunities. Binomial model is simple and can be applied without too much mathematical knowledge.

In this report, we first use a binomial tree model to estimate the price of a European call option and investigate the difference between the option prices we get from both models with different values of the volatility. Then we study the convergence behavior of the option price when increasing the number of steps in the binomial tree model. We derive the formula of the hedge parameter in BS model and compare the hedge parameter in both models at $t = 0$ with different values of the volatility. We also study the relationship between the American call/put option and the European call/put option. Furthermore, we derive the risk-neutral pricing formula of a European put option in the BS model and perform a hedging simulation to study the results under different conditions, including volatility matching and unmatching, and hedging frequency changing from daily to weekly.

This report is organized as follows: Section 2 will introduce the theoretical interpretation including Binomial tree model, the BS model, put-call parity and Euler method. Section 3 present the results that our experiments obtained and give further discussions about these results.

2 Methods

2.1 Binomial tree

A binomial tree is a graphical representation with at most two branches per node. At each node and different periods, the option price which depends on the underlying asset can be shown intuitively. In binomial tree model, we assume that there is no arbitrage opportunity in the market, the interest rate and the volatility of the underlying asset are constant, the price only either increase or decrease in a small time interval, there is no transaction cost and the underlying asset do not pay a dividend. A given period can be subdivided into smaller time intervals and the model can be applied to dealing with many complex options. Under these assumptions, a risk-free hedging transaction can be established and a portfolio can be used to simulate the value of options(Cox et al.,1979).

For a European call option whose underlying asset is stock, we know T(maturity of the option), K(strike price) and denote r as risk-free rate, σ as the volatility of the stock price, S_0 as the stock price at t=0, S_T as the stock price at t=T. Then we set the number of steps in the binomial tree model to be N, thus each small time interval $\Delta t = \frac{T}{N}$. We assume the stock price can either be Su $(u > 1)$ or Sd $(d < 1)$ after Δt. To make sure these value of u and d can represent the stock price with volatility σ, it's assumed that $u \times d = 1$ and the variance of ΔS is $S^2\sigma^2\Delta t$ in Δt with S being the stock price. Based on these assumptions, we have $u = e^{\sigma\sqrt{\frac{T}{N}}}, d = e^{-\sigma\sqrt{\frac{T}{N}}}$ and the probability of the stock price going up at each node is $p = e^{\frac{\frac{T}{N}r-d}{u-d}}$ (Hull,2008).

We then calculate all possible payoffs of the option at maturity and use these values to derive the option premium. We denote the stock price after n time interval as $S_{i,n-i}$ with i being the number of times that the stock price increases.

$$S_{i,j} = S_0 u^i d^{n-i} \tag{1}$$

When $n = N$, we calculate the payoff of the option f for each possible stock price at maturity:

$$f_{i,N-i} = max\{S_{i,N-i} - K, 0\} \tag{2}$$

For $n < N$, we use backward induction and calculate the value of the option $f_{i,n-i}$ as below:

$$f_{i,n-i} = (pf_{i+1,n-i} + (1-p)f_{i,n+1-i})e^{-r\frac{T}{N}} \tag{3}$$

It's worthwhile to mention that equation 3 is only suitable for European options. For American call options, we need to compare $f_{i,n-i}$ that we calculated with $max\{S_{i,n-i}, 0\}$, and choose the larger one as the value of $f_{i,n-i}$.

As for the algorithm, we first build the binomial tree as a matrix, each node representing the stock price computed by the Equation 1. Then we construct another matrix of the same structure to compute the value of the option at each node, using the stock price at maturity S_T to calculate the payoff of the option at maturity. If it's a call option, then the payoff is $max\{0, S_T - K\}$ and if it's a put option, then the payoff is $max\{0, K - S_T\}$. The values of the payoffs are in the last row of the matrix. Then we use the Equation 3 to calculate the previous option prices through a backward induction scheme and get the option premium at $t = 0$.

2.2 Black-Scholes model

In the binomial tree model, the price of the underlying asset can either increase or decrease at each time interval. When we decrease the time interval to nearly 0 and assume that the price of the underlying asset change continuously, we get the BS model. To make sure that our price will not be negative, BS model assumes that the price of the underlying asset S follows a geometric Brownian motion $dS_t = rS_t dt + \sigma S_t dz_t$, z_t is standard Brownian motion (Hull,2008). It's worthwhile to mention that both binomial tree model and BS model assume that the volatility of the underlying asset is constant, this might not be true in real life.

The Black–Scholes partial differential equation is:

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + rS_t \frac{\partial V_t}{\partial S} = rV_t \tag{4}$$

where V_t denotes the price of the financial derivatives at time t, S_t means the stock price at time t, K is the strike price, σ is volatility of the stock price and r is the risk-free rate. This equation holds for all the financial derivatives whose price depends on the stock price. For a European call option, we can get

$$V_T = max\{S_T - K, 0\} \tag{5}$$

Combine the equation (4) and (5), we can get the price formula of a European call option, this is known as the Black-Scholes formula for a call option:

$$V_t = S_t N(d_1) - e^{-r(T-t)} K N(d_2) \tag{6}$$

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{7}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{8}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{w^2}{2}} dw \tag{9}$$

The Black-Scholes formula is based on equation (5), thus it is not suitable for American options because they can be exercised before maturity.

Regarding the algorithm of the BS model, we use the Black-Scholes formulas mentioned above to compute the price of a European call option.

2.3 Put-call parity

When there is no arbitrage opportunity in the market, the price of the European put option should be correlated to the price of the European call option with the same underlying asset, maturity and strike price. The relationship between them is called Put-call parity (Stoll,1969). We denote C_t as the price of call option at time t, P_t as the price of the put option at time t, S_t is the price of the underlying asset at time t. T is the maturity and K is the strike price of the two options. r is the interest rate. We can express put-call parity as the equation below:

$$C_t + Ke^{-r(T-t)} = P_t + S_t \tag{10}$$

We can prove this by considering two portfolios, portfolio A contains a call option and $Ke^{-r(T-t)}$ cash in the bank, portfolio B contains the corresponding put option ad the underlying asset S_t. At time t, the value of portfolio A is $C_t + Ke^{-r(T-t)}$ and the value of portfolio B is $P_t + S_t$. When there is no early exercise opportunity, the value of these two portfolios at maturity T are equal. When $S_T > K$, the call option will be exercised and the put option will not be exercised, the value of portfolio A and portfolio B are both S_T. When $S_T < K$, the put option will be exercised and the call option will not be exercised, the value of both portfolios are K. Since we assume no arbitrage opportunity in the market, the value of these two portfolios should be the same at time t, thus we can get the equation shown above.

It's worthwhile to mention that put-call parity might not hold for options with early exercise opportunities like American options.

2.4 Euler method

In BS model, we assume that the stock price follows a geometric Brownian motion. Euler method is a good way to approximate this process. In Euler method, the process is firstly discretized into M small time intervals δt and we assume that the stock price S_t don't change within the small-time intervals, then we calculate the stock price based on these assumptions (Kloeden and Platen,1992).

The discretization of the change in stock price is shown below. r denotes the interest rate and σ denotes the volatility of the stock. R is a random number from the normal distribution with $mean = 0$ and $variance = \delta t$.

$$\Delta S_t = rS_t\Delta t + \sigma S_t \Delta R \tag{11}$$

Based on the equation above, we simulate the stock price from $t = 0$ to maturity T at M steps with $\delta t = \frac{T}{M}$.

3 Results and Discussion

3.1 Part I-The binomial tree: option valuation

In this part we mainly focus on a European call option on a stock which do not pay dividend. The default parameters of this option are strike price $K = €99$, interest rate $r = 0.06$, initial stock price $S_0 = €100$, volatility $\sigma = 0.2$, maturity $T = 1$.

3.1.1 option price approximation

We construct a binomial tree with $N = 50$ steps to approximate the price of the option mentioned above, the approximated option price is €11.5464.

3.1.2 Compare option premiums from Binomial tree model and BS model

We build a binomial tree with 50 steps to compute the option premium and compare the results with the analytical value from the BS model with the same parameter setting. We change the volatility from 0.1 to 1 and calculate the difference between the estimated value we get from Binomial tree model and the analytical value from BS model. Figure 1 shows how option prices from both models change as we increase the volatility. We can see the option prices from these two models grow linearly as the volatility increases. The differences in the values from these two models are almost invisible because the differences are too small compared to the ticks of the y-axis. We calculate the error as the estimated value from the binomial tree model minus the analytical value from the BS model, the result is shown in Figure 2. We could tell that the absolute value of the error decreases as the volatility increases from 0.1 to 0.2 and increases as the volatility increases from 0.2 to 1.

There are two possible reasons why the option premium we get in both models increases as the volatility increase. The first one is that options are often used to reduce risks, increased volatility means increased risks, investors want to get a higher return for higher risk, thus the price of the option increases. The second one is that the chance that the option is in the money at maturity and the highest possible payoff of the option increases as the volatility increases (San and Brythan, 2014). Since the payoff when the option is out of money is always 0, the option premium should be higher as the volatility increases.

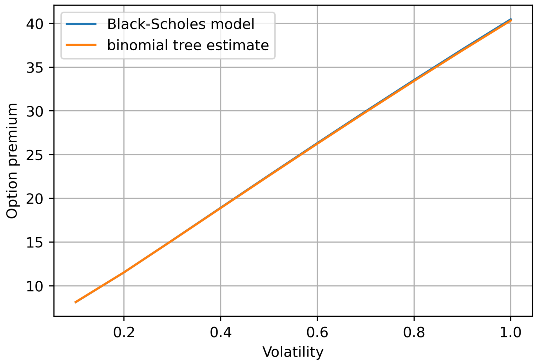

Figure 1: option price with volatility variation

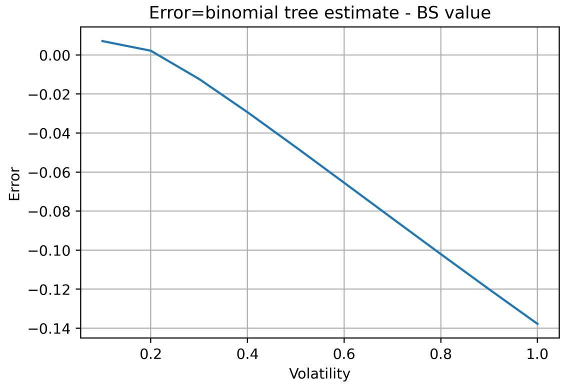

Figure 2: errors with volatility variation

3.1.3 convergence behavior of the Binomial tree model

We increase the number of steps in the binomial tree model and compute the estimated European call option price, the result is shown in Figure 3. We can see that the estimated option price generally converges to a certain value as we increase the number of steps and this value is the analytical value of the option we get from BS model. The difference between these two values is almost invisible as the number of steps exceeds 100.

We also consider the computational complexity of the binomial tree algorithm. The computational complexity of the binomial tree algorithm is $O(N^2)$. Since calculating the values inside the stock price matrix is $O(N^2)$ and calculating the value inside the option price matrix is also $O(N^2)$, the computational complexity of the whole algorithm is $O(N^2)$.

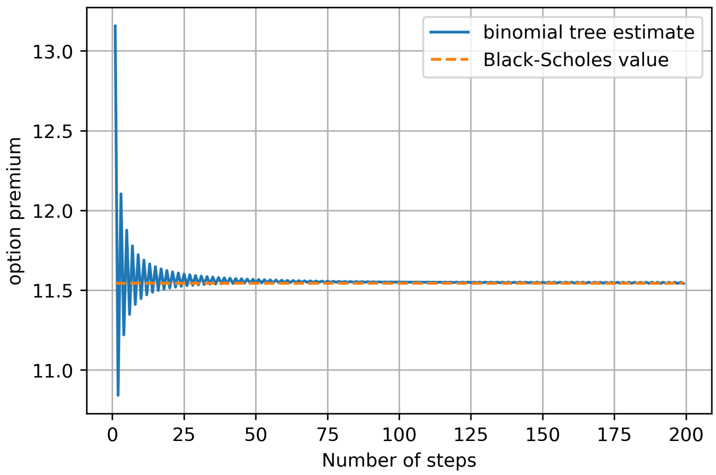

Figure 3: convergence behavior with increasing number of steps

We assume that the change in the stock price follows binomial distribution in binomial tree whereas it follows a normal distribution in BS model. When the number of trials n in the binomial distribution increases, the distribution becomes close to normal distribution. We believe this is the reason why our estimated option price from binomial tree model converges to the analytical value from the BS model as we increase the number of steps n. Even though the binomial tree model is a discrete model and BS model is a continuous model, the main assumptions of these two models are similar, we consider the binomial tree model as the discrete counterpart of the continuous BS model.

3.1.4 delta in the BS model

In BS model, we assume $\Delta_t = \frac{\partial C_t}{\partial S_t}$, C_t denotes the price of the option and S_t denotes the price of the stock at time t. T represents the maturity, r is the interest rate and σ is the volatility. The formula of C_t is given by equation (6), we can derive the formula of Δ_t below:

$$\begin{aligned}\Delta_t = \frac{\partial C_t}{\partial S_t} &= \frac{\partial}{\partial S_t}(S_t N(d_1) - Ke^{-r(T-t)}N(d_2)) \\ &= N(d_1) + S_t \frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S_t} - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S_t}\end{aligned} \tag{12}$$

$N(x)$ in the equation above is cumulative distribution function of the standard normal distribution, we can get $\frac{\partial N(x)}{\partial x} = n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. According to the equation (7) and (8), we can get:

$$\frac{\partial d_1}{\partial S_t} = \frac{\partial d_2}{\partial S_t} = \frac{1}{S_t\sigma\sqrt{T-t}}, n(d_2) = \frac{\partial N(d_2)}{\partial d_2} = n(d_1)\frac{S_t}{K}e^{r(T-t)} \tag{13}$$

Thus we can write the equation (12) as follows:

$$\Delta_t = N(d_1) + (S_t n(d1) - Ke^{-r(T-t)}n(d_1)\frac{S_t}{K}e^{r(T-t)})\frac{\partial d_1}{\partial S_t} = N(d_1) \tag{14}$$

We can see that in the BS model, $\Delta_t = N(d_1)$.

3.1.5 hedge parameter

We study the hedge parameter Δ from the binomial tree with 100 steps at $t = 0$ and compare it to Δ_0 in the BS model that is calculated using equation (14). As shown in Figure 4, both hedge parameters drop significantly as volatility increases from 0.1 to 0.4 and grow slowly as volatility increases from 0.4 to 1.0. Figure 5 shows how the difference between these two hedging parameters changes as volatility increases. We calculate the difference as the hedging parameter from binomial tree minus that from the BS model. The absolute value of the difference decreases as volatility grows from 0.1 to 0.6 and increases afterwards.

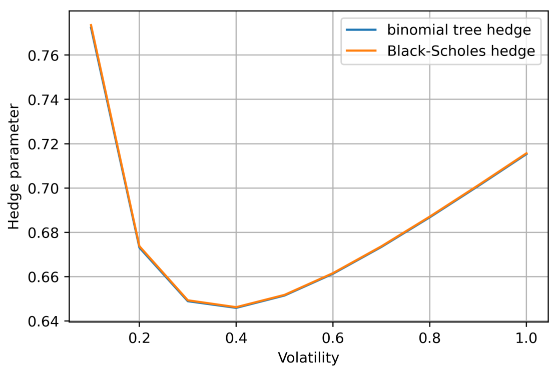

Figure 4: Hedge parameter with volatility variation, N=100

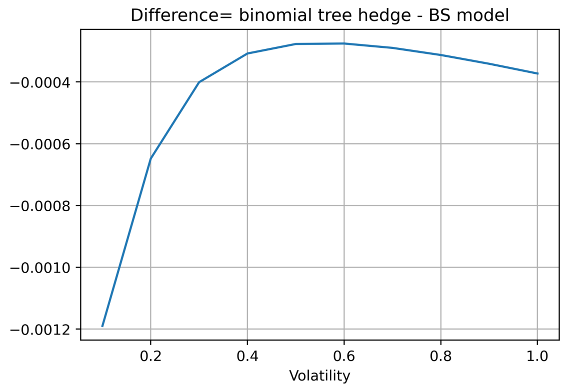

Figure 5: difference with volatility variation

The formulas of the hedging parameter from both models are presented below. Δf denotes the change in the option price and ΔS denotes the change in the stock price in the binomial tree model.

$$\delta_{binomial} = \frac{\Delta f}{\Delta S} = \frac{\Delta f}{100(e^{0.1\sigma} - e^{-0.1\sigma})} \tag{15}$$

$$\delta_{bsmodel} = N(d_1), d_1 = \frac{log\frac{100}{99} + 0.06 + 0.5\sigma^2}{\sigma} \tag{16}$$

For the BS model, the hedge parameter increases as d_1 increases and decreases as d_1 decreases. When $\sigma = 0.374$, the first-order derivative of d_1 equals 0 and d_1 reaches its minimum. Thus the hedge parameter decreases first and then increases. Since binomial tree model is the discrete counterpart of the BS model, we believe their hedging parameters follow a similar trend. Thus the hedging parameters in both models first decrease then increase as the volatility changes from 0.1 to 1.0.

3.1.6 American and European, put and call option

We change our binomial tree model such that it can deal with early exercise opportunities and compute the price of American call and put option with the same parameters mentioned above. Then we compare them to the price of the corresponding European call and put option. Figure 6 shows how these prices change as volatility increases. We can see that in general, the prices of all options grow as volatility increases. The price of the American call option is equal to that of the European call option whereas the price of the American put option is higher than that of the European put option. We could also tell that the call option price is always larger than the put option price both in the American and European scenario.

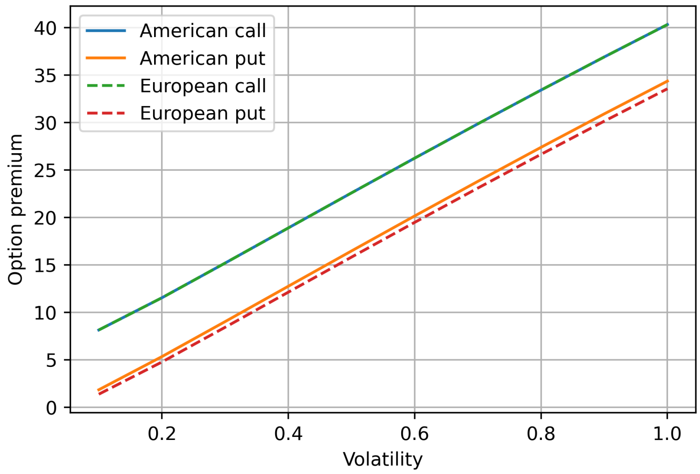

Figure 6: American/European call and put option with volatility variation, N=50

For the holder of an American call option, it's always better to hold the option until maturity to decide whether to exercise it rather than exercising it early because of the time value of money. If the holder exercises the call option early, it will pay the seller early and lose the interest that it can get if it puts the money in the bank. Thus, the price of American call option is the same as the European call option with the same parameters. For a put option, this is different. The holder of the option can get more than the strike price at maturity if it exercises the put option early. Thus the prices of the call option in both scenarios are the same whereas the price of the American put option is higher than that of the European put option. The fact that the prices of the call options are larger than the corresponding put options is due to the fact that the discounted strike price is smaller than the initial stock price.

3.2 Part II-Black-Scholes model:hedging simulations

The default parameters of European call option we use in this part is strike price $K = €99$, interest rate $r = 0.06$, initial stock price $S_0 = €100$, volatility $\sigma = 0.2$, maturity $T = 1$.

3.2.1 European put option

Using the put-call parity equation 10 and the call option price formulas from section 2.2, we could get the risk-neutral price of the European put option as below:

$$\begin{aligned} P_t &= C_t + e^{-r(T-t)}K - S_t \\ &= S_t N(d_1) - e^{-r(T-t)}KN(d_2) + e^{-r(T-t)}K - S_t \\ &= e^{-r(T-t)}K[1 - N(d_2)] - S_t[1 - N(d_1)] \end{aligned} \tag{17}$$

Due to the symmetric property of standard normal distribution, $1 - N(d_1) = N(-d_1)$ and $1 - N(d_2) = N(-d_2)$, we could finally obtain the pricing formula of a European put option:

$$P_t = e^{-r(T-t)}KN(-d_2) - S_t N(-d_1) \tag{18}$$

3.2.2 hedging simulation

We use the Euler method mentioned in section 2.4 to simulate the stock price process and perform a hedging simulation for the BS model. We first set the volatility of the stock price and the volatility that we use in the hedging to be 0.2. One simulation of the stock-price process with daily variation is shown in Figure 7. Suppose the option is hedged daily, the cumulative hedging cost including interest and the hedge parameter-delta in this stock price process is shown in Figure 8 and Figure 9 respectively. In these plots, we could see the change of the hedging parameter and cost correspond with the change of stock price. Besides, we can see that when the stock price at maturity is smaller than the strike price, the European call option will not be exercised, thus we do not need to hold the stock to hedge the option anymore, so the hedging parameter is 0 at maturity. If the stock price at maturity is higher than the strike price, the call option will be exercised, the delta value at maturity will be 1.

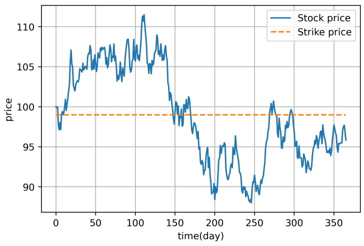

Figure 7: Stock price variation

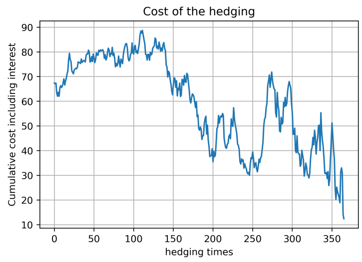

Figure 8: cost of hedging

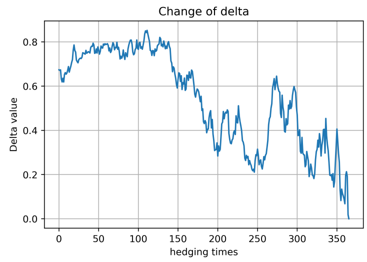

Figure 9: change of delta

To further study the difference between daily hedging and weekly hedging when the volatility is matching, we run 3000 simulations for each hedging. We calculate the error as the analytical option premium from BS model minus the discounted cost of hedging the option. We could tell from Figure 10 that the distribution of weekly error spread wider than that of daily error and the peak of daily error was much higher than that of the weekly error.

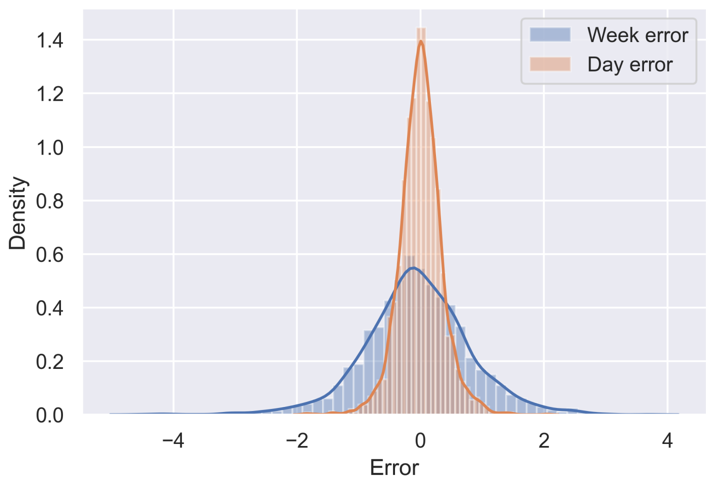

Figure 10: Distribution of errors

The mean of the weekly error is slightly larger than the daily error. We believe this is due to the time value of money. The more frequently we hedge, the less total expected hedging cost we need to pay. The BS model assumes that the hedging process is continuous and computes the analytical value of the option. When we increase the hedging frequency, the discounted cost of hedging will generally become closer to the analytical value and thus the absolute value of the error generally becomes smaller. This generally leads to a smaller variance in the daily hedging error in Figure 10.

We also study the weekly and daily hedging error when the volatility used in the delta valuation is not matching the volatility in the stock price process. We fix the volatility used in the delta valuation as 0.2 and change the volatility in the stock price process from 0.05 to 0.35. For each parameter setting, we repeat the experiment 1000 times. The mean and confidence interval of the errors are shown in Figure 11, Figure 12 and Figure 13. The absolute value of the errors in both weekly and daily hedging decrease as the volatility of the stock price changes from 0.05 to 0.2 and increase as the volatility change from 0.2 to 0.35. We can see in Figure 13 that the differences of the mean errors in weekly and daily hedging are very small, just like what we observe in Figure 10.

We can see that when the volatility of the stock price is smaller than 0.2, the error is positive, this means the option price is too high and the discounted cost of hedging is smaller than the option price, thus the seller of the option made a profit at maturity. When this volatility is larger than 0.2, the error is negative, the seller

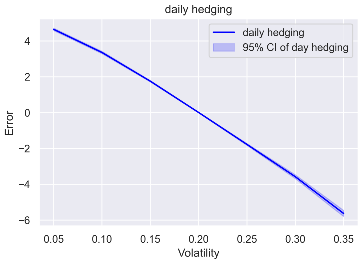

Figure 11: Errors with volatility variation when daily hedging

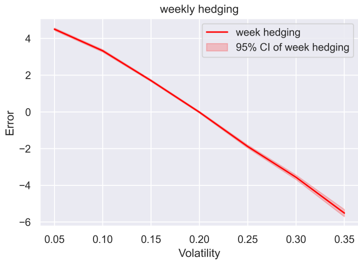

Figure 12: Errors with volatility variation when weekly hedging

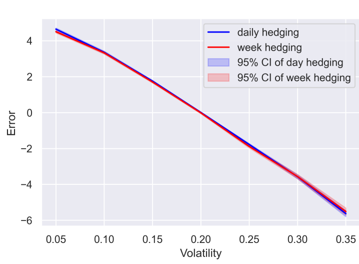

Figure 13: comparison of errors with volatility variation

of the option lost money at maturity. As we have seen in Figure 1, the option premium increase as the volatility increases. When the volatility in the stock price process is smaller than the estimated volatility used in delta hedging, the option is overpriced. When it's larger than the estimated volatility used in delta hedging, the option premium is underestimated.

References

John C. Cox, Stephen A. Ross, and Mark Rubinstein. 1979. Option pricing: A simplified approach. *Journal of Financial Economics* 7, 4 (1979), 229–263.

Amir Ahmad Dar and N. Anuradha. 2018. Comparison: Binomial model and Black Scholes model. *Quantitative Finance and Economics* 2, 1 (2018), 230–245.

Desmond J. Higham. 2004. *An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation.* Cambridge University Press, 2–3.

John C. Hull. 2008a. *Options, Futures, and Other Derivatives.* Prentice Hall, 400–405.

John C. Hull. 2008b. *Options, Futures, and Other Derivatives.* Prentice Hall, 285–288.

Peter E. Kloeden and Eckhard Platen. 1992. *Stochastic Differential Equations.* Springer Berlin Heidelberg, Berlin, Heidelberg, 103–160.

Nam San and Brythan. 2014. Why does a call option's price increase with higher volatility? `https://money.stackexchange.com/a/78234`

Hans R. Stoll. 1969. THE RELATIONSHIP BETWEEN PUT AND CALL OPTION PRICES. *The Journal of Finance* 24, 5 (1969), 801–824.