Introduction

The terminology degradation process refers to many types of models in reliability, which correspond to various kinds of stochastic processes used for deterioration modeling. This book is restricted to the case of a continuous set of possible deterioration levels, to be opposed for instance to multi-state models, where only a finite number of deterioration levels is possible (see, for instance, [COM 99, HOU 99, AND 02] for reviews on these multi-state models with application in epidemiology). Also, only univariate models are considered, which means that one single measurable quantity is assumed to be observed over time. In this book, the terms degradation model or degradation process refer to this context.

Even within this apparently restrictive context, degradation models have been much studied for several decades. Nowadays, there exists a large number of publications on this topic in scientific journals. However, only a few books deal with it. Some monographs contain one single chapter on degradation data and cannot cover a wide range of degradation models. For instance, Chapter 13 in [MEE 98] deals with the degradation path model (see section I.3 below for details) while Chapter 11 in [NEL 02] and Chapter 3 in [BAG 04] focus on some accelerated degradation models. One can however quote [FIN 13], which is mostly devoted to shocks and burn-in models and does not cover the range of this book. Beside these monographs, some books gather some material from conferences in mathematical reliability (see [NIK 10] or [RYK 10], for instance). They clearly contain interesting papers on degradation models but they do not share the same unity as a monograph and they do not allow us either to go as deeply into these models. Other books exist that mostly focus on the study of maintenance policies and their optimization, see [NAG 07].

Consequently, a monograph uniquely devoted to this topic is lacking. This book aims at filling this gap. Well-known results together with new results are provided here in a unique rigorous mathematical framework. We hope that it will be useful for researchers and PhD students in applied mathematics, but also for research engineers dealing with degradation modeling and maintenance. It can also be used as a basis for teaching applied stochastic processes.

After this brief overview, we now go on with the core of this introduction. We first explain why it is interesting to study degradation models. A few real data sets are presented in the following for illustrative purposes. The most classical degradation models are then introduced, which are nearly all based on specific Lévy processes or on some of their extensions. Basic definitions and properties are next provided for such mathematical objects. Finally, an outline of the book is sketched in the last section.

I.1. From lifetime data to degradation path observations

Historically, the first reliability studies [BAR 65, BAR 75] were focused on lifetime data analysis, where equipment (or component) lifetimes were directly modeled through random variables. These studies belong to the so-called survival analysis that had (and still has) a major role in reliability engineering [MEE 98] but also in many other fields such as biology and health [KLE 03], demography, actuarial science, economics and social sciences (see, for instance, the bibliographical notes of Chapter 1 in [LAW 03] for further references).

However, purely lifetime-based reliability analysis is not always an effective means to analyze the available data. First, samples of lifetime data can be very small or highly censored. Indeed, observing an item up to failure can be very long and costly, especially for highly reliable products. For some applications, lifetime data are even not observed at all due to the severity of the failure (such as in nuclear engineering, for instance). Second, the development of on-line monitoring together with the increasing use of sensors for safety assessment now allow us to have more and more available information. These data collected on-line can correspond to some covariate process, which has some influence on the lifetime data. Most of the time, they are however the reflection of some deterioration (or degradation) mechanism, which cannot be handled with lifetime models. This includes, for instance, processes of wear, corrosion, crack growth, vibration, etc. Often, this degradation is measured through a scalar indicator, which tends to increase

over time. This scalar indicator is hereafter called degradation measurement. In most cases, there exists a prescribed critical level above which a piece of equipment is considered not to be able to consistently perform its intended mission any more, either for safety or technical reasons. The time required to reach this critical level corresponds to the equipment lifetime, which of course remains of the first interest, even considering degradation data. However, the degradation data bring some additional information on the equipment status when compared to dealing only with failure data. The advantage of considering degradation data toward lifetime data has, for instance, been highlighted from an example by Meeker and Escobar [MEE 98, page 335].

From a probabilistic point of view, although lifetime data stand for a sample of non-negative random variables, successive degradation data represent observations of a stochastic process, whose parameters (eventually functional) can be fit from the data. A main interest is that once fitted, it next allows us to make some prediction over the future of the equipment. As an example, given its present deterioration level, the distribution of its (future) residual life can be derived (as well as many other quantities of interest in an industrial context). This is of major importance to define complex and powerful preventive maintenance policies that can now be based on the effective status of the equipment and on its known future (random) evolution. See [VAN 09] for a large survey on maintenance policies devoted to the gamma process (a specific degradation process studied in Chapter 2), with also some references on other processes such as the Wiener process (studied in Chapter 1). See also [AHM 12] and [JAR 06] for an overview on condition-based maintenance policies in a more general setting. Beyond the definition of efficient maintenance policies, a last interest of degradation models is that they can be used for test planning and burn-in modeling, see sections 5 and 6 in [YE 15] for a survey on these two topics as well as [FIN 13] for the second topic.

For a better understanding of which kind of data sets we are thinking about, we now introduce a few real data sets from the literature, which will be analyzed in the final chapter of the book, with the tools developed therein.

I.2. A few real data sets

The first data set is called Takeda device data in the literature and it has been first studied by Takeda and Suzuki [TAK 83] (see also [STI 89] or [LU 97], for instance). These authors have measured the performances of a certain kind of semiconductors, called metallized and oxidized

semiconductors field-effect transistors. Such semiconductors are subjected to one particular type of degradation, namely, the hot carrier degradation. For more details, the reader could refer to the monograph by Leblebici and Kang [LEB 93], especially Chapters 1 and 2. The critical level is fixed at 15%. Five specimens were observed at 35 instants, giving a total of 170 increments. All the observation times (and thus, in particular, the first ones) are the same for all units. The data set contains positive (126), null (32) and negative (12) increments. The data has been reproduced in Figure I.1 in a log-log scale, together with the critical level (dot line). The critical level is assumed to be equal to 15% of transconductance degradation [LU 97]. Clearly, after a log-log transformation, degradation appears to be linearly increasing with respect to the time in average. We can notice that three units among the five specimens are unfailed at the end of the experience. Hence, an interesting problem is the estimation of the remaining lifetime distribution.

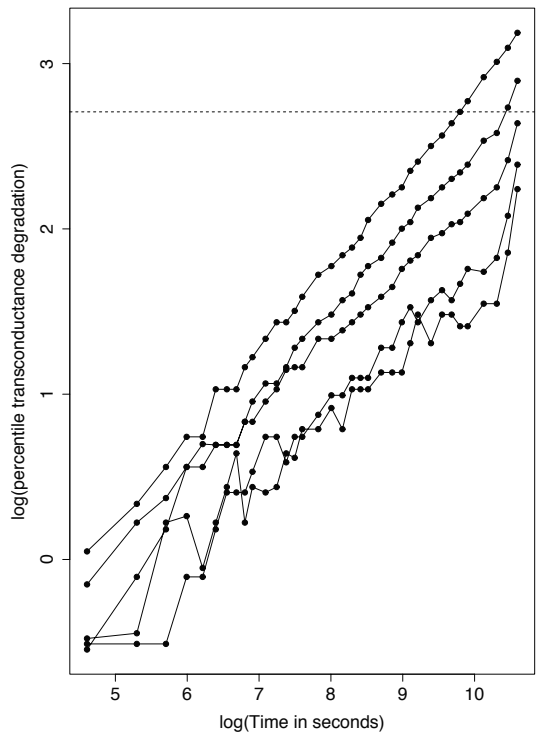

Figure I.1. *Takeda device data*

The second data set has been introduced by Meeker and Escobar [MEE 98], known in the literature as laser data. A set of 15 gallium arsenide laser devices is tested at 80° and periodically observed. More precisely, the performance characteristics of the devices are observed each 250 h until 4,000 h (observation times are identical for all units). A device is considered to be failed when its performance characteristic reaches the predefined critical level equal to 10. The data are plotted in Figure I.2. For this data set, we will be able to fit two different degradation parametric models. Hence, naturally, the problem of selection model arises: which model seems to be more suitable? After estimating the parameters for these two models, we will offer a discussion about this problem.

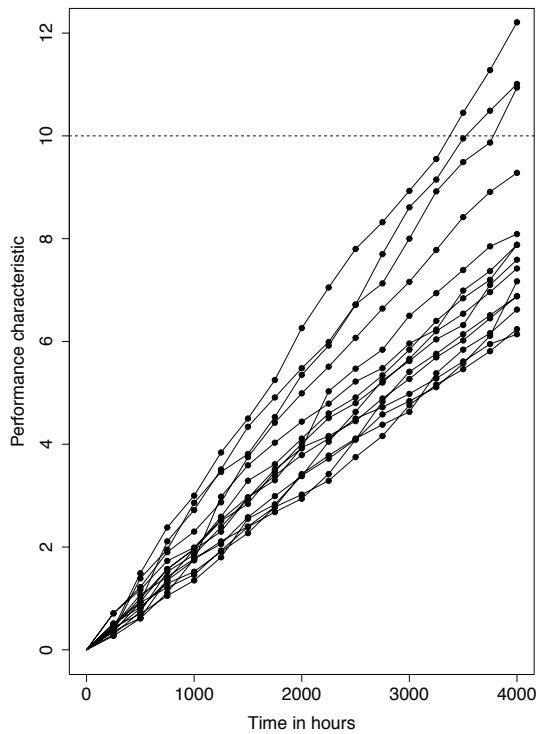

Figure I.2. *Laser data*

The third and last data set deal with crack growth. Hudak *et al.* [HUD 78] have observed 21 metallic specimens, each subjected to loading cycles with crack length recorded every 10^4 cycles. Initial crack length was 0.9 inches for each specimen. The average degradation is clearly nonlinear *a contrario* to the previous data set. The data are plotted in Figure I.3. Such a behavior is typical for crack growth (see, for instance, another famous data which is the Virkler data, [VIR 79]). Hence, the choice of a parametric degradation model is not obvious. We will see that, for one degradation model, semi-parametric estimators are available. Hence, we will be able to compare parametric models to the semi-parametric model.

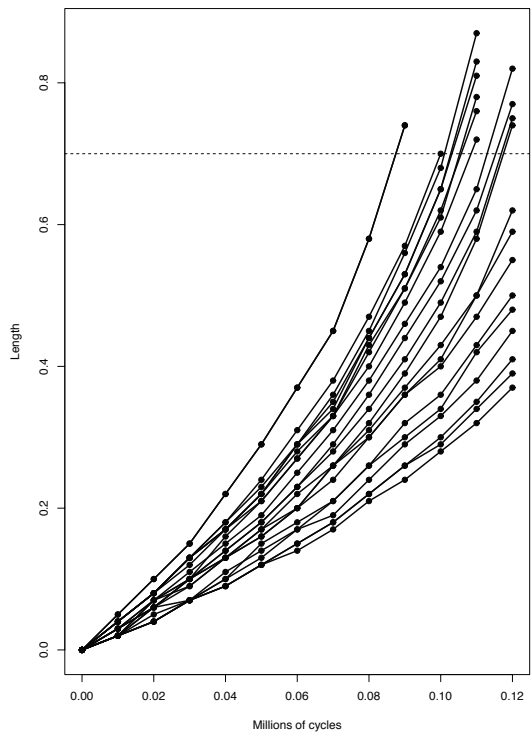

Figure I.3. *Hudak crack growth data*

I.3. A brief overview of classical degradation models

As already mentioned, this book is devoted to the case of a univariate degradation with a continuous set of possible outcomes. The degradation process is assumed to take range in an interval or half-line of $\mathbb{R}$, or in $\mathbb{R}$ itself. Hence, only this case is considered in this brief overview.

A first approach for modeling this kind of deterioration relies on the assumption that the degradation is a monotone and deterministic phenomenon, but that there exists a unit-to-unit variability, which can be translated through some random coefficients. More specifically, the assumption is that the degradation X_{ij} of unit i at time t_{ij} is given by

$$X_{ij} = \mathcal{D}(t_{ij}, \beta_{1i}, \ldots \beta_{ki}) + \epsilon_{ij}, \quad i \in \{1, \ldots, n\}, \quad j \in \{1, \ldots, m_i\},$$

where $\mathcal{D}$ is a deterministic function, ϵ_{ij} is Gaussian distributed with mean 0 and variance σ_ϵ^2, and β_{li}'s are random parameters. The model is called a degradation path model, see [MEE 98, Chapter 13] for a comprehensive introduction to this approach. The β_{li}'s can differ from one unit to the other or (some of them) can be common across all units. Degradation path models are very similar to well-known generalized linear models with fixed and random effects, which have already been the object of several books, such as [MCC 01] for instance. We consequently do not consider them in the present book. In a degradation path model, given the realization of the β_{li}'s for one specific item, all the randomness is contained in the ϵ_{ij}'s, which can be seen as measurement errors. These models are useful when a deterministic description of the degradation is available, based on mechanical or physical considerations. When it is not the case, other models are required.

A second approach for modeling deterioration (with continuous range) is based on the use of specific Lévy processes and on some of their extensions. This approach seems to date from the middle of 1970s (see the survey by van Noortwijk [VAN 09]). The most classical Lévy processes used for deterioration modeling are the Wiener and the gamma processes, which are dealt with in Chapters 1 and 2 of this book, respectively. Note that these two processes are often used in their non-homogeneous versions, which are not Lévy any more but direct extensions. To our knowledge, Abdel-Hameed was the first to consider the gamma process as a wear model in 1975 [ABD 75]. Çinlar and co-authors justify its use for extrapolating concrete creep in 1977, based on physics considerations and on the mathematical properties of the process. From an applicative perspective, the gamma process is well suited to

model some non-decreasing deterioration, which accumulates over time through many tiny increments. Indeed, it is a pure jump process with an infinitely number of jumps over any finite time interval (see Chapter 2 for details). On the contrary, a Wiener process models a continuous deterioration with an increasing trend but a non-monotonous trajectory (see Chapter 1). It seems that the use of the Wiener process for deterioration modeling started around the 1990s, see for instance [DOK 92] or [DOK 95]. During the last two decades, both gamma and Wiener processes has received much consideration in the literature. These two models correspond to some mostly continuous deterioration (which is the fruit of an accumulation of a large amount of tiny increments for the gamma process). However, the deterioration can also be due to isolated shocks on a system, where each shock induces a random amount of damage that is accumulating over time. In that case, the trajectory of the deterioration is piecewise constant and the two previous models are not adapted.

In the case of a piecewise constant deterioration trajectory, the classical approach is to consider a cumulative shock model, such as the compound-renewal damage model introduced by Morey in 1966 [MOR 66]. In the specific case where the renewal process is a homogeneous Poisson process, we obtain a compound Poisson process, which is a Lévy process again. There are a lot of papers dealing with shock models under various assumptions, such as [ÇIN 77a, ESA 73, FEN 94, SHA 83, SOB 87] and we do not intend to review all the possible models here, neither to describe them all in the book. We concentrate in Chapter 3 on the specific case where the arrival process for the shocks is a doubled-stochastic Poisson process, namely, where the intensity of the Poisson process is random. This process was introduced by Cox [COX 55]. The random amounts of damage are often considered as independent from their arrival times in the literature. This assumption is relaxed in Chapter 3, which hence appears to be dealt in a quite general setting.

I.4. The mathematical framework

All the degradation models studied in the book are stochastic processes. We here remind well-known facts on this notion and provide some of our notations. All of the stochastic processes considered hereafter take range in $\mathbb{R}^d$ (or $\mathbb{R}^d_+$), where $d \in \mathbb{N}^*$ and we set $\mathcal{B}(\mathbb{R}^d)$ to stand for the Borel σ-algebra on $\mathbb{R}^d$. Let $(\Omega, \mathcal{A})$ be a measurable space and let $T = \mathbb{R}^+$ or $T = \mathbb{N}$. Following [SAT 99], we call stochastic process with state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and parameter set T, any collection $X = (X_t)_{t\in T}$, where, for each $t \in T$, X_t

is a random variable defined on $(\Omega, \mathcal{A})$ which takes values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. The function $\omega \to X(\omega) = (X_t(\omega))_{t\in T}$ is then a random variable from $(\Omega, \mathcal{A})$ to $(E^T, \mathcal{E}^T)$, where E^T stands for the set of all functions from T to $E = \mathbb{R}^d$ and $\mathcal{E}^T$ is the σ-field generated by all finite sets of such functions.

Now, let us recall that a stochastic process X is said to be càdlàg ("*continu à droite avec des limites à gauche*" in French, namely, right-continuous with left-side limits), if the trajectory $t \to X_t(\omega)$ is right-continuous with left-side limits, for each $\omega \in \Omega$. Let us equip the measurable space $(\Omega, \mathcal{A})$ with a probability measure $\mathbb{P}$ so that $(\Omega, \mathcal{A}, \mathbb{P})$ becomes a probability space. Then, it is known that each stochastic process X on $(\Omega, \mathcal{A}, \mathbb{P})$ with state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ admits a càdlàg modification, which means that there exits a càdlàg process Y such that

$$\mathbb{P}(\omega \in \Omega : X_t(\omega) = Y_t(\omega) \text{ for all } t \in T) = 1.$$

As a result, there is no loss of generality assuming the process X to be càdlàg, which is done from now on and throughout the book.

In the specific context of deterioration modeling, additive accumulation of degradation and independence of degradation increments at disjoint time intervals are natural assumptions. These properties lead immediately to the conclusion that a suitable class of processes for degradation modeling are additive processes and among them, Lévy processes, see [SAT 99] for more details on these processes and for the following definitions.

DEFINITION I.1.– *Let $X = (X_t)_{t\geq 0}$ be a (càdlàg) stochastic process from $(\Omega, \mathcal{A}, \mathbb{P})$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then, X is said to be an additive process if the following conditions are satisfied:*

1) $X_0 = 0$ $\mathbb{P}$*-a.s.;*

2) *X has independent increments, i.e. for all $n \in \mathbb{N}^*$, for all $t_1 < \cdots < t_n$, the random variables X_{t_1}, $X_{t_2} - X_{t_1}$, $\cdots$, $X_{t_n} - X_{t_{n-1}}$ are independent;*

3) *X is stochastically continuous, i.e. for any $s, t \geq 0$ and for any $\varepsilon > 0$, $\lim_{t\to s} \mathbb{P}(|X_s - X_t| > \varepsilon) = 0$.*

The stochastic process X is said to have stationary increments, if, for any $s, t \geq 0$, the distribution of $X_{t+s} - X_t$ does not depend on t. An additive process with stationary increments is called a Lévy process.

Note that additive processes are also called non-homogeneous Lévy processes in the literature. Also, if $X = (X_t)_{t\geq 0}$ is a Lévy process, then both

mean and covariance matrix of X_t are linear functions of the time t (whenever they exist). Finally, starting from a Lévy process X and an increasing function $A : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $A(0) = 0$, then $\left(Y_t = X_{A(t)}\right)_{t \geq 0}$ is an additive process, see [SAT 99, example 1.7].

Now, let us consider instants $0 = t_0 < t_1 < \cdots < t_n$ and, for any $i \in \{1, \ldots, n\}$, let X_{t_i} stand for the deterioration level at time t_i and let $Y_i = X_{t_i} - X_{t_{i-1}}$ be the degradation increment between t_{i-1} and t_i. Based on the additive assumption of the deterioration, we should have $X_{t_n} = Y_1 + \cdots + Y_n$ (roughly speaking, this means that there is no conflict between measurements of deterioration at different time points). A necessary condition is that the distribution of X_{t_n} should be the same as the distribution of $Y_1 + \cdots + Y_n$. Assuming furthermore the increments to be independent, this mostly allows us to the requirement that the distribution of X_t should be infinitely divisible.

DEFINITION I.2.– *The distribution $\mathbb{P}_X$ of a random variable X is infinitely divisible if, for any $n \in \mathbb{N}^*$, there exists i.i.d. random variables $X_1^{(n)}, \ldots, X_n^{(n)}$ such that*

$$X \stackrel{d}{=} X_1^{(n)} + \cdots + X_n^{(n)}$$

where $\stackrel{d}{=}$ stands for equality in distribution.

It can be proved [SAT 99, theorem 9.1] that for any additive process $X = (X_t)_{t \geq 0}$ (and hence for any Lévy process), the distribution of X_t is infinitely divisible, for all $t \geq 0$. Examples of Lévy processes are the Wiener and the homogeneous gamma process described in Chapters 1 and 2. Gaussian and gamma distributions are consequently infinitely divisible. Examples of additive processes are the non-homogeneous gamma process described in Chapter 2 and the non-homogeneous compound Poisson process, which is a specific case of the doubly stochastic marked Poisson process considered in Chapter 3.

I.5. Outline of the book

The book contains three main chapters, each devoted to one degradation model. The last chapter contains some illustrative studies, based on the previously described real data sets. For each degradation model, we begin by providing probabilistic results (moments, time-to-failure distributions, etc.) and exploring simulation tools for sample path generation. Next, various estimation procedures are developed, depending on the available data

(degradation measurements and/or failure times). In each case, both maximum likelihood and moments methods are considered and asymptotic properties provided.

In Chapter 1, the Wiener process is considered as a candidate for a degradation model. After some recall on Gaussian random vectors, the Brownian motion is introduced and its construction provided, together with different simulation methods. Next, the Wiener process is defined and its first-passage time distribution is studied. For this model, the latter is available in full form and known as the inverse Gaussian distribution. Statistical inference is next considered for various types of sampling schemes. Since the Wiener process has a time-linear expectation (as any Lévy process), it is not always a suitable approach for deterioration modeling. Hence, to conclude this chapter, we provide a short review of some extensions of the Wiener process and some related models, which are not Lévy processes any more.

Chapter 2 is devoted to the gamma process, which is considered in his most general non-homogeneous version. We first recall basic facts on the gamma distribution. For a better understanding of the jump structure of a gamma process, we next introduce Poisson random measures on $\mathbb{R}^d$, from where gamma processes are next constructed. The section on Poisson random measures is intended to be self-contained since it is not necessarily part of the usual basic knowledge of reliability researchers. It can be skipped in a first reading. We next detail the presentation of the gamma process and provide several simulation tools, whose efficiency is studied both theoretically and numerically. Time-to-failures are next investigated from a probabilistic point of view and simulation tools are numerically studied. Just as for the Wiener process, statistical inference is next considered for various sampling schemes. Random effects (or frailty) are also envisioned, which allows us to introduce some variability between individuals. Finally, a brief review is proposed about extensions of the gamma process and related models.

In Chapter 3, a cumulative shock model is studied, where the shocks arrive according to a doubly stochastic Poisson process (or Cox process), with intensity assumed to be a random multiple of a deterministic function. Just as for the gamma process, this allows us to introduce some variability between individuals. Each shock induces an instantaneous increment of the degradation level, which is allowed to depend on the time of the shock, leading to a so-called position-dependent marking of the doubly stochastic Poisson process. Different parametric models are envisioned, for which probabilistic results are provided (including on the first passage time distribution) and estimation procedures developed.

Finally, Chapter 4 illustrates the use of the previously described degradation models on the real data sets from section I.2 (except however for the last model from Chapter 3, for which we had no real data at our disposal). This chapter allows us to highlight some differences between the Wiener and the gamma processes for modeling purposes and to illustrate the use of the previously developed probabilistic and statistical tools. Some procedures for selecting a model and testing its goodness of fit are also proposed in the last chapter.