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Paley–Wiener Type Theorems by Transmutations

A. Boumenir and M.Z. Nashed

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ABSTRACT. The classical Paley–Wiener theorem for functions in L^2_{dx} relates the growth of the Fourier transform over the complex plane to the support of the function. In this work we obtain Paley–Wiener type theorems where the Fourier transform is replaced by transforms associated with self-adjoint operators on $L^2_{d\mu}$, with simple spectrum, where $d\mu$ is a Lebesgue–Stieltjes measure. This is achieved via the use of support preserving transmutations.

1. Introduction

The Paley–Wiener theorem, see [20], states that if F is an entire function, then

$$\left.\begin{array}{l} | F(\lambda) | \leq M \exp(a\,|\lambda|) \\ F \in L^2_{d\lambda} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} F(\lambda) = \int_{-a}^{a} f(x) \exp(ix\lambda)\,dx \\ f \in L^2_{dx}(-a, a)\,. \end{array}\right.$$

This result shows that any entire function of finite type and order one is the Fourier transform of an L^2_{dx} function with compact support. For each $\lambda \in R$, the kernel of the Fourier transform, $\exp(ix\lambda)$, is an eigenfunctional of the self-adjoint differential operator $\mathcal{D} := -i\frac{d}{dx}$ acting in L^2_{dx}:

$$\mathcal{D} \exp(ix\lambda) = \lambda \exp(ix\lambda), \qquad x \in R\,.$$

Here $\exp(ix\lambda)$ are called eigenfunctionals rather than eigenfunctions, since they do not belong to L^2_{dx}, and therefore are treated as functionals, see [15].

In this work, we show that the Fourier transform appearing in the Paley–Wiener theorem can be replaced by a transform associated with a self-adjoint operator acting on some Hilbert space $L^2_{d\mu}$,

$$L^2_{d\mu} := \left\{ f \quad \text{measurable} \ : ||f||^2 = \int |f(x)|^2\, d\mu < \infty \right\}$$

where $d\mu$ is a Lebesgue–Stieltjes measure generated by a nondecreasing function μ, and supp $d\mu = R$.

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Statement of the Problem

Assume that the eigenfunctionals, $y(x, \lambda)$, of a self-adjoint operator L acting on $L^2_{d\mu}$, are entire functions of λ. We would like to determine nondecreasing functions Γ such that the following Paley–Wiener type theorem holds. The notion of a spectral function is recalled in the preliminaries.

Let Γ be a given (*spectral*) function. Is the following true for entire functions F?

$$\left.\begin{array}{l} \mid F(\lambda) \mid \leq M \exp(a \mid\lambda\mid) \\ F \in L^2_{d\Gamma}, \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} F(\lambda) = \int_{-a}^{a} f(x) y(x, \lambda)\, d\mu \\ f \in L^2_{d\mu} \end{array}\right.$$

The classical Paley–Wiener theorem plays an important role in Fourier analysis, sampling theorems, and harmonic analysis. When λ is frequency, and t is time, the Paley–Wiener theorem provides a direct characterization of the space of band-limited signals. Such signals can be recovered from their samples via the well-known Shannon–Whittaker sampling theorem.

We now briefly review the literature on this subject. Flensted–Jensen in [13] considers the differential operator defined by

$$\Delta_{pq} := -\frac{d^2}{dx^2} + (p \coth(x) + 2q \coth(2x)) \frac{d}{dx} \qquad x > 0$$

where p and q are positive constants. Relying on properties of special functions associated with this operator, he proves a Paley–Wiener theorem. Similarly, Koornwinder in [17] gives shorter proofs for Paley–Wiener theorems associated with the operator defined by

$$\frac{-1}{v_{\alpha,\beta}(x)} \frac{d}{dx} v_{\alpha,\beta}(x) \frac{d}{dx} + (\alpha + \beta + 1)^2 \qquad x > 0$$

where $v_{\alpha,\beta}(x) = \sinh^{2\alpha+1}(x) \cosh^{2\beta+1}(x)$ by exploiting basic properties of the Jacobi functions in the generalized Mehler formula defining the transmutation. Chebli in [8] considers a more general operator defined by

$$\Delta := \frac{-1}{A(x)} \frac{d}{dx} A(x) \frac{d}{dx} + q(x) \qquad x > 0$$

where $A(x) = x^{2\alpha+1} c(x)$, $2\alpha > 1$, $c(x) > 0$ and smooth. Estimates on the solutions allow for a detailed study of the spectral theory of Δ. Analytic properties of the eigenfunctionals and contour integration are then used to prove a Paley–Wiener theorem. Independently, Trimeche in [24] develops a transmutation theory for Δ where he extends classical results associated with the Riemann–Liouville, Weyl, and Hankel transforms. Here the transmutation is a Volterra operator which is expressed by a generalized Mehler formula

$$Tf(x) = \int_0^x K(x, t) f(t)\, dt\,.$$

Interestingly the inverse transmutation turns out to be an integro-differential operator.

It is fairly acknowledged that the above works pioneered many results in the area of harmonic analysis. We also note that the main tools were direct spectral theory, analytic function theory, special functions, and perturbation theory. However, in our work we take a different approach to the problem, namely the inverse spectral theory, support preserving operators, and related operational calculus. A key point in our analysis is to characterize the operator L by the growth and certain properties of its spectral function. Indeed, we recall that transmutations, say V, satisfy the factorization theorem, see [2]

$$\frac{d\Gamma_1}{d\Gamma_2}(L_2) = V V'$$

where Γ are spectral functions and V' is the adjoint of V. This leads to conditions of absolute continuity of the spectral function such as found in [7] and transmutations inverses expressed by pseudo-differential operators such as found in [24]. Observe that in order to use transmutations beyond the class of special functions, one first needs to show their existence, which is a major obstacle. To this end, we define transmutations in rigged spaces, or Gelfand's triplet, which would then lead us to a simpler operator-theoretic framework. In short, although we have used new tools such as the factorization of pseudo-differential operators and support preserving operators, our work is an attempt to unify and extend the existing theories related to the Paley–Wiener theorem.

2. Preliminaries

In this section we shall recall the main ingredients needed for our analysis. Let L be a linear self-adjoint operator acting in the Hilbert space $L^2_{d\mu}$, with simple spectrum, σ say. A point λ is in the spectrum of L, if and only if

$$\exists \{\varphi_n\} \in D_L \subset L^2_{d\mu}, \quad \|\varphi_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|L\varphi_n - \lambda\varphi_n\| = 0 \,. \tag{2.1}$$

Depending on the convergence behavior of the sequence $\{\varphi_n\}$ in the Hilbert space, $L^2_{d\mu}$, λ is in the continuous or discrete part of the spectrum. Indeed, if the sequence $\{\varphi_n\}$ happens to be compact in $L^2_{d\mu}$, then $\varphi_n \overset{L^2_{d\mu}}{\to} y(x,\lambda)$, which means $y(.,\lambda) \in L^2_{d\mu}$ is an eigenfunction and λ is an eigenvalue, i.e., a point of the discrete spectrum. In case $\{\varphi_n\}$ is not compact in $L^2_{d\mu}$, then λ is in the continuous spectrum. Usually the existence of a countably normed space Φ, which is compactly embedded in $L^2_{d\mu}$, is assumed (see [15]). This embedding is symbolically denoted by $\Phi \hookrightarrow L^2_{d\mu}$. By duality we obtain $L^2_{d\mu} = (L^2_{d\mu})' \hookrightarrow \Phi'$, i.e., $L^2_{d\mu}$ is also compactly embedded in Φ', and hence the Gelfand triplet or rigged spaces

$$\Phi \hookrightarrow L^2_{d\mu} \hookrightarrow \Phi' \,. \tag{2.2}$$

Thus bounded sets in $L^2_{d\mu}$ are relatively compact in Φ' and since the sequence $\{\varphi_n\}$ in (2.1) is a bounded sequence in $L^2_{d\mu}$, it is compact in Φ'. Here the simple spectrum means that the sequence has only one accumulation point in Φ' which we denote by $y(.,\lambda)$. There are numerous ways to construct these rigged spaces, see [1], and in what follows we shall provide a simple construction of Φ for which (2.2) holds.

A crucial tool in the formulation of Paley–Wiener type theorems of this article is the notion of spectral function. We recall that a spectral function Γ associated with L, and more precisely with the set $y(.,\lambda)$, is a nondecreasing right-continuous function defined for all $\lambda \in R$. The connection between L and Γ is best described when the spectrum is simple and discrete. Indeed, if $\{\phi_n\}_{n\geq 0}$ and $\{\lambda_n\}_{n\geq 0}$ denote the sets of eigenfunctions and eigenvalues respectively, then the classical eigenfunction expansion for $f \in L^2_{d\mu}$ is given by

$$f = \sum_{n\geq 0} (f,\phi_n)\phi_n \, ||\phi_n||^{-2} \,.$$

The spectral function in this simple case is given by

$$\Gamma(\lambda) = \sum_{\lambda \leq \lambda_n} ||\phi_n||^{-2}$$

and so the jumps of Γ occur only at the eigenvalues. It is continuous and increasing over the continuous spectrum. Finally, the support of the Lebesgue–Stieltjes measure $d\Gamma$ is the entire spectrum σ. Thus the decomposition of the spectrum into discrete and continuous parts is described by the growth of Γ.

It is well known that any self-adjoint operator L acting in a separable Hilbert space and with simple spectrum, generates a transform that is constructed by means of the eigenfunctionals of L and duality pairing in rigged spaces, see [15]. For the sake of simplicity we shall assume that the eigenfunctionals $y(x,\lambda) \in L^{1,\mathrm{loc}}_{d\mu}$ and so the duality pairing can be represented with the help of the Lebesgue–Stieltjes integrals. Define $\mathcal{F}_y : \Phi \subset L^2_{d\mu} \longrightarrow L^2_{d\Gamma}$ by

$$\mathcal{F}_y(f)(\lambda) := \langle f(x), y(x,\lambda)\rangle_{\Phi\times\Phi'} = \int f(x)\, y(x,\lambda)\, d\mu(x)$$

and extend it to the whole of $L^2_{d\mu}$ by continuity. The inverse y-transform is defined by $\mathcal{F}_y^{-1}$: $L^2_{d\Gamma} \to L^2_{d\mu}$

$$f(x) = \int_\sigma \mathcal{F}_y(f)(\lambda)\overline{y(x,\lambda)}\, d\Gamma(\lambda)\,.$$

Parseval's equality expresses the fact that the y-transform is an isometry between $L^2_{d\mu}$ and $L^2_{d\Gamma}$:

$$\forall f, g \in L^2_{d\mu} \qquad \int f(x)\overline{g(x)}\, d\mu = \int \mathcal{F}_y(f)(\lambda)\overline{\mathcal{F}_y(g)(\lambda)}\, d\Gamma(\lambda)\,.$$

A simple case is provided by the operator $\mathcal{D}$. We clearly have $S \hookrightarrow L^2_{dx} \hookrightarrow S'$, where S is the space of rapidly decreasing infinitely differentiable functions, i.e., the Schwartz space.

Recall that S is a perfect space, that is a countably normed space where bounded sets are relatively compact. Also it is invariant under multiplication by the independent variable. The Fourier transform, which is generated by $\exp(ix\lambda)$, is defined by

$$L^2_{dx} \xrightarrow{\mathcal{F}_e} L^2_{d\frac{\lambda}{2\pi}}, \qquad f(x) \mapsto \mathcal{F}_e(f)(\lambda)$$

where

$$\mathcal{F}_e(f)(\lambda) := \int f(x)\exp(ix\lambda)\, dx \qquad \text{and} \qquad f(x) = \frac{1}{2\pi}\int \mathcal{F}_e(f)(\lambda)\exp(-ix\lambda)\, d\lambda\,.$$

The spectral function is simply $\frac{\lambda}{2\pi}$ and thus the spectrum is continuous and fills the whole real line.

The question of the structure of eigenfunctionals of arbitrary self- adjoint operators in $L^2_{dx}(R^n)$ is examined in [15], Vol. 3 and in more details in [1]. There are various ways of constructing the spaces Φ', which can be chosen to be Hilbert spaces with "negative norm" see [1]. However, we can use the y-transform to construct the rigged spaces in a very simple way. We define

$$\Phi := \mathcal{F}_y^{-1}S := \left\{f \in L^2_{d\mu} :\ \mathcal{F}_y(f) \in S\right\}\,. \tag{2.3}$$

Proposition 1.

If S is compactly embedded in $L^2_{d\Gamma}$, then $\Phi := \mathcal{F}_y^{-1}S$ is a perfect space, invariant under L and $\Phi \hookrightarrow L^2_{d\mu} \hookrightarrow \Phi'$.

Proof. Since the y-transform is an isometry, Φ is a countably normed space if we define semi-norms by

$$\| f \|_p := \sup\left\{\left|\left(1+\lambda^{2p}\right)D^q\mathcal{F}_y(f)(\lambda)\right| \ :\ q \le p, \lambda \in R\right\}\,.$$

If $f \in \Phi$, then $\mathcal{F}_y(f) \in S$, and so $\lambda\mathcal{F}_y(f) \in S$. In other words, $\mathcal{F}_y(Lf) \in S$, i.e., $Lf \in \Phi$, and thus Φ is invariant under L. From the compact embedding $S \hookrightarrow L^2_{d\Gamma}$, it follows that

$$S \hookrightarrow L^2_{d\Gamma} \hookrightarrow S'$$

where the duality pairing for $\psi \in L^2_{d\Gamma(\lambda)}$ is defined by

$$\langle f, \psi \rangle_{S \times S'} := \int f(\lambda)\overline{\psi(\lambda)}\, d\Gamma(\lambda)\,.$$

From the assumption that the embedding $S \hookrightarrow L^2_{d\Gamma}$ is compact, we deduce that the embedding $\Phi \hookrightarrow L^2_{d\mu} \hookrightarrow \Phi'$ are compact. $\square$

Definition 1. A linear operator $V : S' \to \Phi'$ is called a transmutation if

$$V(\exp(ix\lambda)) = y(x, \lambda) \qquad \forall \lambda \in R\,.$$

We need to justify the term operator in the above definition. First V can be extended linearly to the algebraic span of $\{\exp(ix\lambda)\}_{\lambda \in R}$ and so V maps a dense subset of S' into a dense subset of Φ'. Therefore V is densely defined in S'. If V is a transmutation, we formally have in Φ'

$$LV(\exp(ix\lambda)) = Ly(x, \lambda) = \lambda y(x, \lambda) = V\lambda \exp(ix\lambda) = V\mathcal{D}\exp(ix\lambda) \quad \text{for } \lambda \in R\,.$$

Since $\{\exp(ix\lambda) : \lambda \in R\}$ form a complete set in S', we obtain

$$LV = V\mathcal{D}\,,$$

and thus we say that V transmutes L into $\mathcal{D}$. If V has an inverse then the above relation yields

$$L = V\mathcal{D}V^{-1}\,.$$

This construction of the transmutation is standard. It is called "spectral pairing" see [6], and was used by Gelfand and Levitan in their work on the inverse spectral problem, see [16]. Indeed, the transmutation there is given by

$$y(x, \lambda) = \cos(x\sqrt{\lambda}) + \int_0^x K(x, t)\cos\left(t\sqrt{\lambda}\right) dt \qquad x > 0, \quad \forall \lambda \in R$$

where $y(., \lambda)$ and $\cos\left(x\sqrt{\lambda}\right)$ are eigenfunctionals of a self-adjoint extension associated respectively with $\mathcal{D}^2 + q(x)$ and $\mathcal{D}^2$ in $L^2_{dx}(0, \infty)$.

The representation of V depends on the behavior of $y(., \lambda)$ as $\lambda \longrightarrow \infty$.

Example: Assume that

$$\exists\ n \quad \text{such that} \quad \forall x \in R \quad \frac{y(x, \lambda)}{1 + \lambda^{2n}} \in L^2_{d\lambda} \text{ and } y(., \lambda) \in L^2_{dx}\,.$$

Then for each x, there exists $K(x, .) \in L^2_{dt}$ such that

$$\frac{y(x, \lambda)}{1 + \lambda^{2n}} = \int_R K(x, t)\exp(it\lambda)\, dt$$

and we have

$$\begin{aligned} y(x, \lambda) &= \int_R K(x, t)\left(1 + \lambda^{2n}\right)\exp(it\lambda)\, dt \\ &= \int_R K(x, t)\left(1 + [\mathcal{D}_t]^{2n}\right)\exp(it\lambda)\, dt\,. \end{aligned}$$

From the last equation we deduce that the operator V is densely defined in L^2_{dx} by

$$(Vf)(x) = \int_R K(x,t)\left(1 + \mathcal{D}^{2n}\right) f(t)\, dt \qquad \forall f \in C_0^\infty$$

where

$$C_0^\infty = \left\{ f \in C^\infty(R) \text{ and support of } f \text{ is compact} \right\} .$$

The above example gives rise to the following fairly general sufficient condition for the existence of the operator V. Indeed using the fact that Fourier transform is an isomorphism in S', we obtain the following.

Proposition 2.

Assume that for all $x \in R$, $y(x,.) \in S'_\lambda$, then there exists a kernel $H(x,.) \in S'$ such that

$$y(x,\lambda) = \int H(x,t) \exp(it\lambda)\, dt .$$

The existence of V in general is a difficult problem (see [6]) and is due to the fact that mapping eigenfunctionals means acting "outside" the Hilbert spaces L^2_{dx} and $L^2_{d\mu}$ in case the continuous spectrum is present. We shall overcome this difficulty by obtaining the adjoint operator V', by a direct method. To this end we use the identity operator, which we denote by $\mathcal{J} : L^2_{d\Gamma} \to L^2_{d\lambda}$, i.e.,

$$L^2_{d\Gamma} \xrightarrow{\mathcal{J}} L^2_{d\lambda}, \qquad f \mapsto \mathcal{J}f = f .$$

$\mathcal{J}$ is a densely defined operator, since $C_0^\infty \subset D_{\mathcal{J}} := L^2_{d\Gamma} \cap L^2_{d\lambda}$.

We recall that by the Radon–Nikodym theorem (see [9]) if $d\lambda$ is absolutely continuous with respect to $d\Gamma$, denoted by $d\lambda$ is abs-$d\Gamma$, then

$$\text{there exists } g \geq 0 \text{ and } g \in L^{1,\text{loc}}_{d\Gamma} \text{ such that } \quad d\lambda = g(\lambda)\, d\Gamma(\lambda) . \tag{2.4}$$

Proposition 3.

If $d\lambda$ is abs-$d\Gamma$ continuous then $\mathcal{J} : L^2_{d\Gamma} \overset{\mathcal{J}}{\to} L^2_{d\lambda}$ is closable.

Proof. We only need to show that if $f_n \in C_0^\infty$ such that

$$f_n \overset{L^2_{d\Gamma}}{\to} 0 \text{ and } f_n \overset{L^2_{d\lambda}}{\to} c \text{ then } \quad c = 0 \text{ in } L^2_{d\lambda} . \tag{2.5}$$

Let g be the Radon–Nikodym derivative defined in (2.4) and define the locally $d\Gamma$−measurable sets

$$A_m = \{x / \, g(x) \leq m\}$$

where $m \geq 0$. Obviously the integral

$$\int_{A_m \cap \text{compact}} d\lambda = \int_{A_m \cap \text{compact}} g(\lambda)\, d\Gamma(\lambda)$$

exists and since $f_n \in C_0^\infty$ we have

$$\int_{A_m} |f_n|^2\, d\lambda = \int_{A_m} |f_n|^2 g(\lambda)\, d\Gamma(\lambda) \leq m \int_{A_m} |f_n|^2\, d\Gamma(\lambda) .$$

Upon using (2.5) we deduce that as $n \to \infty$ $f_n \overset{L^2_{d\lambda}}{\to} 0$ in A_m , which means that $c = 0$ a.e. in each A_m. Since m is arbitrary we have $c = 0$ in $L^2_{d\lambda}$.

We now summarize our notation: $y(x,\lambda)$ are the eigenfunctionals of an arbitrary self-adjoint operator L, with simple spectrum, acting in the Hilbert space $L^2_{d\mu}$ and Γ denotes its spectral function. Let $\sigma := \operatorname{supp} d\Gamma = R$, be the spectrum. Observe that since a Paley–Wiener theorem deals with conditions for $\int_{-a}^{a} f(x) y(x,\lambda)\, d\mu(x)$ to be an entire function of λ when $f(x) \in L^2_{d\mu}$, we need $y(x,\lambda)$ to be an entire function in λ. $\square$

3. Existence of the Transmutation Operator

In all that follows we shall assume the following.

Condition [A]:

- $S \hookrightarrow L^2_{d\Gamma}$ is a compact embedding
- The spectrum $\sigma = R$ is simple and $d\lambda$ is abs-$d\Gamma$, i.e., (2.4) holds.
- $y(x,.)$ is an entire function of λ, and $y(.,\lambda) \in L^{1,\mathrm{loc}}_{d\mu}$.

In this section we shall prove the existence of the transmutation. It is more convenient to first establish the existence of its adjoint.

Proposition 4.
Assume that Condition [A] holds, and define the W operator as follows:

$$W : \Phi \to S \qquad Wf := \mathcal{F}_e^{-1} \mathcal{J}\, \mathcal{F}_y(f)\,. \tag{3.1}$$

Then

- $WLf = \mathcal{D}Wf \qquad \forall\, f \in \Phi$
- $y(x,\lambda) = W' \exp(ix\lambda) \qquad \forall \lambda \in R.$

Proof. Clearly the operator W is well defined by (3.1) and its range is S. We now show that it is a transmutation. For $f \in \Phi$, we have

$$\begin{aligned}
\mathcal{D}Wf &= \frac{1}{2\pi} \int \mathcal{J}\mathcal{F}_y(f)(\lambda)\mathcal{D}\exp(-ix\lambda)\,d\lambda \\
&= \frac{1}{2\pi} \int \mathcal{J}\mathcal{F}_y(f)(\lambda)\lambda\exp(-ix\lambda)\,d\lambda \\
&= \frac{1}{2\pi} \int \mathcal{J}\mathcal{F}_y(Lf)(\lambda)\exp(-ix\lambda)\,d\lambda \\
&= WLf
\end{aligned}$$

and therefore

$$\mathcal{D}Wf = WLf \qquad \forall f \in \Phi\,.$$

The mapping between eigenfunctionals can be obtained by using the adjoint operator W'. To this end we use the duality pairing

$$\begin{aligned}
\langle Wf(x), \exp(ix\lambda)\rangle_{S\times S'} &= \mathcal{J}\langle f(x), y(x,\lambda)\rangle_{\Phi\times\Phi'} \\
\langle f(x), W'\exp(ix\lambda)\rangle_{\Phi\times\Phi'} &= \mathcal{J}\langle f(x), y(x,\lambda)\rangle_{\Phi\times\Phi'} \\
&= \langle f(x), y(x,\lambda)\rangle_{\Phi\times\Phi'}
\end{aligned}$$

where the fact $\langle f, y(x,.)\rangle \in S \subset L^2_{d\Gamma} \cap L^2_{d\lambda}$, is used. Therefore we deduce

$$W'\exp(ix\lambda) = y(x,\lambda) \qquad \forall\lambda \in R\,.$$

Using the fact that $\Phi \hookrightarrow L^2_{d\mu}$ and $S \hookrightarrow L^2_{dx}$, we can look at W as a densely defined operator acting $L^2_{d\mu} \longrightarrow L^2_{dx}$. Since both transforms are unitary, W is closable if $\mathcal{J}$ is closable and by Proposition 3, this is indeed the case when $d\lambda$ is abs-$d\Gamma$ by Condition [A]. Hence W is closable when acting between $L^2_{d\mu}$ and L^2_{dx}, which we denote its closure by $\overline{W}$. □

Proposition 5.

Assume that Condition [A] holds and $f \in D_{\overline{W}}$. Then $\mathcal{J}\mathcal{F}_y(f) = \mathcal{F}_e(\overline{W}f)$ and $\|\overline{W}\| = \|\mathcal{J}\|$, that is W and $\mathcal{J}$ are metrically equal.

Proof. If $f \in D_{\overline{W}}$, then there exists a sequence of functions $\varphi_n \in D_W$ such that $\varphi_n \longrightarrow f$. By Proposition 2, for each φ_n we have $\mathcal{J}\mathcal{F}_y(\varphi_n)(\lambda) = \mathcal{F}_e(\overline{W}(\varphi_n))(\lambda)$. Since $\overline{W}$ and $\mathcal{J}$ are closed, and $\mathcal{F}$ and $\mathcal{F}_y$ are bounded operators, it follows that $\mathcal{J}\mathcal{F}_y(f)(\lambda) = \mathcal{F}_e(\overline{W}f)(\lambda)$. □

Proposition 6.

Assume that Condition [A] holds. Then

(i) $\overline{W}$ *is densely defined in* $L^2_{d\mu}$;

(ii) $D_{\overline{W}} = \{f \in L^2_{d\mu} : \int_\sigma \mid \mathcal{F}_y(f)(\lambda) \mid^2 \, d\lambda < +\infty\} := \mathcal{F}_y^{-1}\{L^2_{d\Gamma} \cap L^2_{d\lambda}\}$;

(iii) $\overline{W}$ *is bounded* $\Longleftrightarrow$ $\operatorname{esssup} \frac{d\lambda}{d\Gamma}(\lambda) < \infty \quad \Longrightarrow \quad L^2_{d\Gamma} \subset L^2_{d\lambda}$;

(iv) $\overline{W}^{-1}$ *exists* $\Longleftrightarrow$ $\{\int \mid \mathcal{F}_y(f)(\lambda) \mid^2 \, d\lambda = 0 \Longrightarrow f = 0 \text{ in } L^2_{d\mu}\}$; *and*

(v) $\overline{W}^{-1}$ *is bounded* $\Longleftrightarrow$ $0 < m \le \operatorname{essinf} \frac{d\lambda}{d\Gamma}(\lambda)$.

Proof. To prove (i), observe that from Proposition 5, we have $\overline{W}f = \mathcal{F}_e^{-1}\mathcal{J}[\mathcal{F}_y(f)]$ which is well defined if $\mathcal{F}_y(f) \in L^2_{d\lambda}$. It is also clear that $\overline{W}$ is densely defined, since the space of continuous functions with compact support is contained in $\mathcal{F}_y\{D_{\overline{W}}\} := \{L^2_{d\Gamma} \cap L^2_{d\lambda}\}$ and dense in $L^2_{d\lambda}$. Thus we deduce that $D_{\overline{W}}$ is also dense in $L^2_{d\mu}$.
(ii) is a simple consequence of Proposition 4.
(iii) is proved by the fact

$$\begin{aligned} &\int \left|\overline{W}f(x)\right|^2 dx \le c \int |f(x)|^2 \, d\mu \\ \Rightarrow \; &\frac{1}{2\pi}\int \left|\mathcal{F}_e\overline{Wf}(\lambda)\right|^2 d\lambda \le c \int |f(x)|^2 \, d\mu \\ \Rightarrow \; &\frac{1}{2\pi}\int \left|\mathcal{F}_y f(\lambda)\right|^2 d\lambda \le c \int \left|\mathcal{F}_y f(\lambda)\right|^2 d\Gamma(\lambda) \, . \end{aligned}$$

Hence we have proved

$$\forall \mathcal{F}_y f \in C_0^\infty \qquad \int \left|\mathcal{F}_y f(\lambda)\right|^2 d\lambda \le 2\pi c \int \left|\mathcal{F}_y f(\lambda)\right|^2 d\Gamma(\lambda) \, ,$$

from which it follows that $d\lambda$ is absolutely continuous with respect to $d\Gamma$ and

$$\operatorname{esssup} \frac{d\lambda}{d\Gamma}(\lambda) \le 2\pi c < \infty \, .$$

Also the boundedness of $\overline{W}$ means $D_{\overline{W}} := L^2_{d\mu} \Longleftrightarrow \mathcal{F}_y\{D_{\overline{W}}\} := L^2_{d\Gamma}$ and by using (ii) $L^2_{d\Gamma} \cap L^2_{d\lambda} := L^2_{d\Gamma}$ we deduce that $L^2_{d\Gamma} \subset L^2_{d\lambda}$. The proofs of (iv) and (v) are obvious. □

We now prove a "crude" version of a Paley–Wiener type theorem.

Proposition 7.

Assume that Condition [A] holds, and let $F(\lambda)$ be an entire function. Then

$$\left.\begin{array}{l} \mid F(\lambda) \mid \le M \exp(a\,|\lambda|) \\ F \in L^2_{d\Gamma} \cap L^2_{d\lambda} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} F(\lambda) = \int f(x) y(x,\lambda)\, d\mu \\ \operatorname{supp} \overline{W}f \subset [-a,a] \quad f \in D_{\overline{W}} \, . \end{array}\right.$$

Proof. Let $f \in D_{\overline{W}}$ with $\operatorname{supp}\overline{W}f \subset [-a,a]$. Thus $\overline{W}f \in L^2_{dx}$ and clearly by the classical Paley–Wiener theorem we have $| \mathcal{F}_e(\overline{W}f)(\lambda) | \leq M\exp(a\,|\lambda|)$ and $\mathcal{F}_e(\overline{W}f(\lambda)) \in L^2_{d\lambda}$, so by Proposition 5, we obtain

$$| \mathcal{F}_y(f)(\lambda) | = | \mathcal{F}_e\left(\overline{W}f\right)(\lambda) | \leq M\exp(a\,|\lambda|)$$

and $\mathcal{F}_y(f) = \mathcal{F}_e(\overline{W}f) \in L^2_{d\lambda}$. Thus $| \mathcal{F}_y(f)(\lambda) | \leq Me^{a|\lambda|}$ and $\mathcal{F}_y(f)(\lambda) \in L^2_{d\lambda}$. Recall that $f \in D_{\overline{W}} \subset L^2_{d\mu}$ and so $\mathcal{F}_y(f) \in L^2_{d\Gamma}$ from which it follows that

$$\mathcal{F}_y(f) \in L^2_{d\Gamma} \cap L^2_{d\lambda}\,.$$

Conversely, if $F \in L^2_{d\Gamma} \cap L^2_{d\lambda}$, then there exists $f(x) \in D_{\overline{W}} \subset L^2_{d\mu}$ such that $F = \mathcal{F}_y(f)$. From the fact that

$$F = \mathcal{F}_y(f) = \mathcal{F}_e\left(\overline{W}f\right)$$

and F is entire it follows that $\mathcal{F}_e(\overline{W}f)$ is entire, $|\mathcal{F}_e(\overline{W}f)(\lambda)| \leq M\exp(a\,|\lambda|)$ and $\mathcal{F}_e(\overline{W}f)(\lambda) \in L^2_{d\Gamma}$. As a consequence of the Paley–Wiener theorem we obtain $\operatorname{supp}\overline{W}f \subset [-a,a]$ and $F(\lambda) = \mathcal{F}_y(f)(\lambda) = \int f(x)y(x,\lambda)\,d\mu$.

It will be interesting to determine when we can replace in Proposition 5 the condition

$$\mathcal{F}_y(f) \in L^2_{d\lambda} \bigcap L^2_{d\Gamma} \qquad \text{by} \qquad \mathcal{F}_y(f) \in L^2_{d\Gamma}\,.$$

In order to improve the above propositions we shall need to use the support of functions in the spaces L^2_{dx} and $L^2_{d\mu}$. The support of functions shall be denoted by

$$\operatorname{supp} f := \text{Closure}\,\{x :\quad f(x) \neq 0\} \qquad \text{if} \quad f \in L^2_{dx}\,.$$

In the space $L^2_{d\mu}$ the support of a function is defined with the help of the measure $d\mu$. For $f \in L^{1,\text{loc}}_{d\mu}$ consider the support of the measure defined by $f(x)\,d\mu$

$$\operatorname*{supp}_{\mu} f := \operatorname{supp}\ f(x)\,d\mu$$

which is the complement of the largest $f(.)\,d\mu$ negligeable open set in R, see [9]. In case $d\mu$ is abs$-dx$, then $\operatorname*{supp}_{\mu} f = \operatorname{supp} f(.)\mu'(.)$. It follows that if $\int f(x)\overline{\psi(x)}\,d\mu = 0$ for any $\psi \in C_0$ where $\operatorname*{supp}_{\mu} \psi \subset R\backslash(a,b)$ and C_0 is the space of continuous functions with compact support. then $\operatorname*{supp}_{\mu} f \subset [a,b]$. For $g \in L^{1,\text{loc}}_{d\Gamma}$, consider the operator acting in $L^2_{d\mu}$ and defined by:

$$g(L)f(x) := \int g(\lambda)\mathcal{F}_y(f)(\lambda)\overline{y(x,\lambda)}\,d\Gamma(\lambda) \qquad \text{where } \mathcal{F}_y(f) \in C_0\,. \tag{3.2}$$

When g is real valued it then defines a self-adjoint operator in $L^2_{d\mu}$, and by Parseval equality we have

$$\|g(L)f\| = \|g\mathcal{F}_y(f)\|\,.$$

Before we proceed further we need the following spaces. Let (a,b) be a given interval and let

$$L^2_{d\mu}(a,b) := \left\{ f \in L^2_{d\mu} \;:\quad \operatorname*{supp}_{\mu} f \subset (a,b) \right\}\,,$$

and

$$D_\infty := \left\{ f \in L^2_{d\mu} \ : \quad \mathcal{F}_y(f) \in C_0 \right\} . \qquad \square$$

Proposition 8.

$D_\infty \cap L^2_{d\mu}(a,b)$ is dense in $L^2_{d\mu}(a,b)$.

Proof. Since C_0 is dense in $L^2_{d\Gamma}$, it follows that D_∞ is dense in $L^2_{d\mu}$. It is also clear that $D_\infty \subset D_{L^n}$ for any n. Observe that $L^2_{d\mu} = A + B$, where B is the orthogonal complement of $A := L^2_{d\mu}(a,b)$ in $L^2_{d\mu}$. Assume that $f_A \in D_\infty^\perp \cap A$ and $f_A \neq 0$, then $f_A + 0 \in D_\infty^\perp$, which is impossible since $D_\infty^\perp = \{0\}$, and so $f_A = 0$ in A. $\square$

4. Main Results

It is readily seen that in order to obtain a genuine generalization of the Paley–Wiener theorem, we need to express supp $\underset{\mu}{\overline{W}} f$ in terms of supp f, so that it can be used in Proposition 7. This would be a simple task if the transmutations V and V^{-1} were known explicitly. The following proposition will help us make use of a well-known local property for pseudodifferential operators.

Proposition 9.

Let $a(x,.)$ be an entire function of λ for all x almost everywhere with respect to the measure $d\mu$. Assume further that $a(x,.)$ is of order 1, minimal type, and of polynomial growth for real λ, i.e., $a(x,\lambda) = O(|\lambda|^p)$ as $\lambda \to \pm\infty$. Then $a(x,D)$ is local, i.e.,

$$\forall \ f \in C_0^\infty \qquad \text{supp } a(x,D)f(x) \subset \text{supp } f(x) .$$

Proof.

$$\begin{aligned} a(x,D)f(x) &= \frac{1}{2\pi} \int a(x,\lambda) \mathcal{F}_e f(\lambda) e^{-ix\lambda}\, d\lambda \\ &= \frac{1}{2\pi} \int \mathcal{F}_e \left[\mathcal{F}_e^{-1} a(x,t) \right] \mathcal{F}_e f(\lambda) e^{-ix\lambda}\, d\lambda \\ &= \int \mathcal{F}_e^{-1} a(x, x-t) f(t)\, dt . \end{aligned}$$

From the Paley–Wiener–Schwartz theorem, see [15], it follows that supp $\mathcal{F}_e^{-1} a(x,t) \subset [-\varepsilon, +\varepsilon]$ where ε is arbitrary small. Therefore we obtain

$$\text{supp } a(x,D)f \subset \text{supp } f \ + \ [-\varepsilon, \varepsilon]$$

where the + denotes the vector sum of sets. $\square$

Similarly one has the following.

Proposition 10.

Let $K(x,t,\lambda)$ be an entire function of λ, of order one, minimal type and of a polynomial growth on the real line. If $f \in C_0^\infty$ and supp $f \cap [-a,a] = \emptyset$ then

$$\text{supp} \int_{-|x|}^{|x|} K(x,t,D) f(t)\, dt \cap [-a,a] = \emptyset .$$

Proof. Indeed, for $f \in S$

$$K(x,t,D)f(t) := \frac{1}{2\pi}\int K(x,t,\lambda)\,\mathcal{F}_e(f)(\lambda)e^{-it\lambda}\,d\lambda$$
$$= \int \mathcal{F}_e^{-1}K(x,t,t-\eta)f(\eta)d\eta\,.$$

Now we examine the relation between the supports. We have

$$\operatorname{supp} K(x,t,D)f(t) \subset \operatorname{supp} \mathcal{F}_e^{-1}K(x,t,t-\eta) + \operatorname{supp} f(t)$$
$$\operatorname{supp} K(x,t,D)f(t) \subset [-\epsilon,\epsilon] + \operatorname{supp}\ f(t)\,.$$

Since ϵ is arbitrary small

$$\operatorname{supp}\int_{-|x|}^{|x|} K(x,t,D)f(t)\,dt \subset (-\infty,-a\,)\cup(a\,,\infty)\,. \qquad \square$$

Remark. Observe that if the growth is not polynomial on the real line or the type is not minimal, then the support cannot be preserved. Indeed a simple example is provided by the translation operator

$$a(x,\mathcal{D})\ f(x) := e^{ib\mathcal{D}}f(x) = e^{b\frac{d}{dx}}f(x) = \sum_{n\geq 0}\frac{b^n}{n!}\frac{d^n}{dx^n}f(x) = \ f(x+b)$$

from which it follows that

$$\operatorname{supp}\ a(x,\mathcal{D})\ f(x)\ = b + \operatorname{supp}\ f$$

and hence the support need not be preserved unless $b = 0$. We now illustrate a very simple application of the concept of the transmutation. Assume first that

$$\exp(ix\lambda) = b(x,\lambda)y(x,\lambda) + \int_{-|x|}^{|x|} H(x,t,\lambda)y(t,\lambda)\,d\mu(t) \tag{4.1}$$

where $b(x,\lambda)$, and $H(x,t,\lambda)$ are polynomials in λ. In this case (4.1) is a Volterra integral equation and under certain conditions, see [22], the inverse operator is also a Volterra operator

$$y(x,\lambda) = a(x,\lambda)\exp(ix\lambda) + \int_{-|x|}^{|x|} K(x,t,\lambda)\exp(it\lambda)\,dt \tag{4.2}$$

where $a(x,\lambda)$, $K(x,t,\lambda)$, are entire in λ. If $b(x,\lambda) := b(x) \geq \delta > 0$, then (4.1) implies (4.2). In differential equations, $K(x,t,\lambda)$ can be obtained from the Green's function of the operator L. In our work we shall use the symbol of a pseudodifferential operator to generate the transmutation. To this end the previous propositions will help us prove the following Paley–Wiener type theorem.

Theorem 1.
Assume that Condition [A] holds and

(i) $a(x,\lambda)$, $K(x,t,\lambda)$, *are entire functions of* λ, *order 1, minimal type and have polynomial growth for real* λ, *i.e.,* $O(\lambda^{2p})$ *as* $\lambda \to \pm\infty$;

(ii) $b(x,\lambda)$, *and* $H(x,t,\lambda)$ *are polynomials in* λ; *and*

(iii) $\operatorname{supp}_\mu Lf \subset \operatorname{supp}_\mu f$ *for* $f \in D_L$.

Under the above assumptions if F is an entire function, then

$$\left.\begin{array}{l} | F(\lambda) | \leq M \exp(a|\lambda|) \\ F \in L^2_{d\lambda} \cap L^2_{d\Gamma} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} F(\lambda) = \int_{-a}^{a} f(x) y(x,\lambda)\, d\mu \\ \quad f \in D_{\overline{W}} \subset L^2_{d\mu}\,. \end{array}\right.$$

Proof. By Proposition 7, we already have

$$\left.\begin{array}{c} | F(\lambda) | \leq M \exp(a|\lambda|) \\ F \in L^2_{d\lambda} \cap L^2_{d\Gamma} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} F(\lambda) = \int f(x) y(x,\lambda)\, d\mu \\ \operatorname{supp} \overline{W} f \subset [-a,a] \quad f \in D_{\overline{W}} \end{array}\right.$$

where $D_{\overline{W}} := \mathcal{F}_y^{-1}(L^2_{d\lambda} \cap L^2_{d\Gamma}) \subset L^2_{d\mu}$.

We only need to show that assumptions (4.2) and (4.1) imply

$$\operatorname*{supp}_{\mu} f(x) \subset [-a,a] \Longrightarrow \operatorname{supp} \overline{W} f(x) \subset [-a,a] \tag{4.3}$$

and

$$\operatorname*{supp}_{\mu} f(x) \subset [-a,a] \Longleftarrow \operatorname{supp} \overline{W} f(x) \subset [-a,a]\,, \tag{4.4}$$

respectively, for any $f \in D_{\overline{W}}$.

In order to verify the above conditions we first need to construct the operator W by means of Proposition 4. From (4.2), we obtain for $f \in D_{\overline{W}}$,

$$\mathcal{F}_e\left(\overline{W} f\right)(\lambda) = \mathcal{F}_y(f)(\lambda) = \int f(x) y(x,\lambda)\, d\mu \tag{4.5}$$

$$\mathcal{F}_e\left(\overline{W} f\right)(\lambda) = \int a(x,\lambda) e^{ix\lambda} f(x)\, d\mu + \int\int_{-|x|}^{|x|} K(x,t,\lambda) e^{it\lambda}\, dt f(x)\, d\mu\,.$$

Let $f \in D_{\overline{W}}$ with $\operatorname*{supp}_{\mu} f \subset [-a,a]$ and $\psi \in C_0^\infty$ be such that $[-a,a] \cap \operatorname{supp} \psi = \emptyset$. For (4.3) to be satisfied we only need to show that $\operatorname{supp} \overline{W} f \cap \operatorname{supp}\ \psi = \emptyset$, or equivalently

$$\int \overline{W} f(x) \overline{\psi(x)}\, dx = 0\,. \tag{4.6}$$

Using (4.5), (4.6) can be written,

$$\int \overline{W} f(x) \overline{\psi(x)}\, dx = \frac{1}{2\pi} \int \mathcal{F}_e(\overline{W} f)(\lambda) \overline{\mathcal{F}_e \psi(\lambda)}\, d\lambda$$

$$= \frac{1}{2\pi} \int\int a(x,\lambda) e^{ix\lambda} f(x)\, d\mu(x) \overline{\mathcal{F}_e \psi(\lambda)}\, d\lambda \tag{4.7}$$

$$+ \frac{1}{2\pi} \int\int_{-|x|}^{|x|} k(x,t,\lambda) e^{it\lambda}\, dt f(x)\, d\mu(x) \overline{\mathcal{F}_e \psi(\lambda)} d\lambda\,. \tag{4.8}$$

For the sake of simplicity, we shall examine (4.7) and (4.8) separately. Since $\mathcal{F}_e(\psi) \in S$ and

$\operatorname{supp}_{\mu} f \subset [-a, a]$, Fubini's theorem is applicable and (4.7) yields

$$\begin{aligned}
&\frac{1}{2\pi}\int\int a(x,\lambda)e^{ix\lambda}f(x)\,d\mu(x)\overline{\mathcal{F}_e\psi(\lambda)}d\frac{\lambda}{2\pi} \\
&= \frac{1}{2\pi}\int_{-a}^{a}\int a(x,\lambda)e^{ix\lambda}\overline{\mathcal{F}_e\psi(\lambda)}\,d\lambda f(x)\,d\mu \\
&= \frac{1}{2\pi}\int\int\sum_{n\geq 0}a_n(x)\lambda^n e^{ix\lambda}\overline{\mathcal{F}_e\psi(\lambda)}\,d\lambda f(x)\,d\mu \\
&= \frac{1}{2\pi}\int\sum_{n\geq 0}a_n(x)\overline{\int\lambda^n e^{-ix\lambda}\mathcal{F}_e\psi(\lambda)d\frac{\lambda}{2\pi}}f(x)\,d\mu \\
&= \frac{1}{2\pi}\int\sum_{n\geq 0}a_n(x)\overline{\int e^{-ix\lambda}\mathcal{F}_e(\mathcal{D})^n(\psi)(\lambda)\,d\lambda}f(x)\,d\mu \\
&= \int\sum_{n\geq 0}a_n(x)\ \overline{\mathcal{D}^n\psi(x)}\ f(x)\,d\mu \\
&= 0\,.
\end{aligned}$$

Since $\mathcal{F}_e\psi \in S$ and $|\sum_{n\geq 0}a_n(x)\lambda^n| = O(\lambda^{2p})$, the series converges uniformly, and so

$$\int\sum_{n\geq 0}a_n(x)\lambda^n e^{ix\lambda}\mathcal{F}_e\psi(\lambda)d\frac{\lambda}{2\pi} = \sum_{n\geq 0}a_n(x)\int\lambda^n e^{ix\lambda}\mathcal{F}_e\psi(\lambda)d\frac{\lambda}{2\pi}\,.$$

Proposition 7 implies that

$$\operatorname{supp}\int\sum_{n\geq 0}a_n(x)\lambda^n e^{ix\lambda}\mathcal{F}_e\psi(\lambda)d\frac{\lambda}{2\pi} \subset \operatorname{supp}\psi\,.$$

Next we examine the expression in (4.8). Like (4.7), it also vanishes because we have

$$\begin{aligned}
&\frac{1}{2\pi}\int\int\int_{-|x|}^{|x|}k(x,t,\lambda)e^{it\lambda}\,dtf(x)\,d\mu(x)\overline{\mathcal{F}_e\psi(\lambda)}\,d\lambda \\
&= \frac{1}{2\pi}\int\int\int_{-|x|}^{|x|}k(x,t,\lambda)e^{it\lambda}\,dt\overline{\mathcal{F}_e\psi(\lambda)}d\frac{\lambda}{2\pi}f(x)\,d\mu \\
&= \frac{1}{2\pi}\int\int_{-|x|}^{|x|}\int\sum_{n\geq 0}k_n(x,t)\lambda^n e^{it\lambda}\overline{\mathcal{F}_e\psi(\lambda)}\,d\lambda\,dtf(x)\,d\mu \\
&= \frac{1}{2\pi}\int\int_{-|x|}^{|x|}\sum_{n\geq 0}k_n(x,t)\int\lambda^n e^{it\lambda}\overline{\mathcal{F}_e\psi(\lambda)}\,d\lambda\,dtf(x)\,d\mu \\
&= \frac{1}{2\pi}\int\int_{-|x|}^{|x|}\sum_{n\geq 0}k_n(x,t)\overline{\int\lambda^n e^{-it\lambda}\mathcal{F}_e\psi(\lambda)\,d\lambda}\,dtf(x)\,d\mu \\
&= \frac{1}{2\pi}\int\int_{-|x|}^{|x|}\sum_{n\geq 0}k_n(x,t)\overline{\int e^{-it\lambda}\mathcal{F}_e(\mathcal{D})^n\psi(\lambda)\,d\lambda}\,dtf(x)\,d\mu \\
&= \int\int_{-|x|}^{|x|}\sum_{n\geq 0}k_n(x,t)\overline{\mathcal{D}^n\psi(t)}\,dtf(x)\,d\mu \\
&= 0\,,
\end{aligned}$$

where the fact that $\operatorname{supp} \int_{-|x|}^{|x|} \sum_{n\geq 0} k_n(x,t)\overline{\mathcal{D}^n\psi(t)}\,dt \cap \underset{\mu}{\operatorname{supp}} f = \emptyset$ was used. This shows that the integrals (4.7) and (4.8) are zero and so (4.6) holds, and consequently (4.3) also holds.

We now show that condition (ii) implies (4.4). Thus we need to show that

$$\forall f \in D_W \quad \operatorname{supp} Wf \subset [-a,a] \quad \forall \psi \in D_\infty \quad \operatorname{supp}\psi \cap [-a,a] = \emptyset \Longrightarrow \int f(x)\overline{\psi(x)}\,d\mu = 0\,.$$

From Parseval equality and the definition of the operator W it follows that

$$\begin{aligned}
\int f(x)\overline{\psi(x)}\,d\mu &= \int \mathcal{F}_y(f)(\lambda)\overline{\mathcal{F}_y(\psi)(\lambda)}\,d\Gamma(\lambda)\\
&= \int \mathcal{F}_e\left(\overline{W}f\right)(\lambda)\overline{\mathcal{F}_y(\psi)(\lambda)}\,d\Gamma(\lambda)\\
&= \int\int \overline{W}f(x)e^{ix\lambda}\,dx\overline{\mathcal{F}_y(\psi)(\lambda)}\,d\Gamma(\lambda)\\
&= \int \overline{W}f(x)\int \overline{\mathcal{F}_y(\psi)(\lambda)}e^{ix\lambda}\,d\Gamma(\lambda)\,dx\\
&= \int \overline{W}f(x)\int \overline{\mathcal{F}_y(\psi)(\lambda)}b(x,\lambda)y(x,\lambda)\,d\Gamma(\lambda)\,dx\\
&\quad + \int \overline{W}f(x)\int \overline{\mathcal{F}_y(\psi)(\lambda)}\int_{-|x|}^{|x|} H(x,t,\lambda)y(t,\lambda)\,d\mu\,d\Gamma(\lambda)\,dx\,.
\end{aligned}$$

We only need to show that

$$\operatorname{supp}\int \overline{\mathcal{F}_y(\psi)(\lambda)}b(x,\lambda)y(x,\lambda)\,d\Gamma(\lambda)\cap[-a,a] = \emptyset \tag{4.9}$$

$$\operatorname{supp}\int \overline{\mathcal{F}_y(\psi)(\lambda)}\int_{-|x|}^{|x|} H(x,t,\lambda)y(t,\lambda)\,d\mu(t)\,d\Gamma(\lambda)\cap[-a,a] = \emptyset \tag{4.10}$$

to deduce $\int f(x)\overline{\psi(x)}\,d\mu(x) = 0$.

It is readily seen that (4.9) follows from (iii)

$$\begin{aligned}
&\int b(x,\lambda)y(x,\lambda)\overline{\mathcal{F}_y(\psi)(\lambda)}\,d\Gamma(\lambda)\\
&\qquad = \sum_{n=0}^{n=p} b_n(x)\int \lambda^n y(x,\lambda)\overline{\mathcal{F}_y(\psi)(\lambda)}\,d\Gamma(\lambda)\\
&\qquad = \sum_{n=0}^{n=p} b_n(x)\int y(x,\lambda)\overline{\mathcal{F}_y\left(L^n\psi\right)(\lambda)}\,d\Gamma(\lambda)\\
&\qquad = \overline{\sum_{n=0}^{n=p} b_n(x)\int \overline{y(x,\lambda)}\mathcal{F}_y\left(L^n\psi\right)(\lambda)\,d\Gamma(\lambda)}\\
&\qquad = \overline{\sum_{n=0}^{n=p} b_n(x)\left(L^n\psi\right)(x)}
\end{aligned}$$

and $\operatorname{supp}\psi \cap [-a, a] = \emptyset$. Similarly we have

$$\begin{aligned}
&\int \int_{-|x|}^{|x|} H(x,t,\lambda) y(t,\lambda)\, d\mu(t) \overline{\mathcal{F}_y(\psi)(\lambda)}\, d\Gamma(\lambda) \\
&= \int_{-|x|}^{|x|} \int \sum_{n=0}^{n=p} H_n(x,t) \lambda^n y(t,\lambda)\, d\mu(t) \overline{\mathcal{F}_y(\psi)(\lambda)}\, d\Gamma(\lambda) \\
&= \int_{-|x|}^{|x|} \sum_{n=0}^{n=p} H_n(x,t) \overline{\int \lambda^n \overline{y(t,\lambda)} \mathcal{F}_y(\psi)(\lambda)\, d\Gamma(\lambda)}\, d\mu(t) \\
&= \int_{-|x|}^{|x|} \sum_{n=0}^{n=p} H_n(x,t) \overline{\int \overline{y(t,\lambda)} \mathcal{F}_y\left(L^n\psi\right)(\lambda)\, d\Gamma(\lambda)}\, d\mu(t) \\
&= \int_{-|x|}^{|x|} \sum_{n=0}^{n=p} H_n(x,t) \overline{L^n\psi(t)}\, d\mu(t) = 0\,.
\end{aligned}$$

Since $\operatorname{supp}_\mu L\psi(x) \subset \operatorname{supp}_\mu \psi(x)$ and the sum being finite we obtain

$$\operatorname{supp} \int_{-|x|}^{|x|} \sum_{n\geq 0} H_n(x,t) \overline{L^n\psi(t)}\, d\mu(t) \cap [-a, a] = \emptyset\,.$$

In other words, (4.10) holds. $\square$

The next result establishes a simple connection between $\operatorname{supp}_\mu f$ and $\operatorname{supp} \overline{W} f$.

Theorem 2.
Recall that $g(\lambda) = \frac{d\lambda}{d\Gamma}(\lambda) \in L^{1,\mathrm{loc}}_{d\Gamma}$ and assume that Condition [A] holds together with

- *$\operatorname{essinf} \frac{d\lambda}{d\Gamma}(\lambda) \geq a > 0$, i.e., $\overline{W}^{-1}$ is bounded;*
- *$\forall f \in D_{\overline{W}} \quad \operatorname{supp} \overline{W} f \subset \operatorname{supp}_\mu f$; and*
- *$\forall \psi \in D_{g(L)} \quad \operatorname{supp}_\mu \psi \subset \operatorname{supp}_\mu g(L)\psi$, where $g(L)$ is given by (3.2).*

Let F be an entire function. Then

$$\left.\begin{array}{l} |F(\lambda)| \leq M e^{a|\lambda|} \\ F \in L^2_{d\lambda} \subset L^2_{d\Gamma} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} F(\lambda) = \int_{-a}^{a} f(x) y(x,\lambda)\, d\mu(x) \\ f \in D_{\overline{W}}\,. \end{array}\right.$$

Proof. It suffices to show that $\operatorname{supp}_\mu f \subset \operatorname{supp} \overline{W} f$ and then use Proposition 5. Let Ω denote an arbitrary open set in R and denote by

$$Z := D_{\overline{W}} \cap D_{g(L)} \cap L^2_{d\mu}(\Omega)\,,$$

where

$$L^2_{d\mu}(\Omega) := \left\{ f \in L^2_{d\mu} : \quad \operatorname{supp}_\mu f \subset \Omega \right\}\,.$$

We now show that the set Z is dense in $L^2_{d\mu}(\Omega)$. We begin by showing that $D_{\overline{W}} \cap D_{g(L)}$ is dense in $L^2_{d\mu}$. From Proposition 6, we deduce that

$$\left\|\overline{W}\psi\right\|^2 = \int g(\lambda) \left|\mathcal{F}_y \psi(\lambda)\right|^2 d\Gamma(\lambda)\,,$$

and similarly from the definition of $g(L)$,

$$||g(L)\psi||^2 = \int |g(\lambda)|^2 \left|\mathcal{F}_y\psi(\lambda)\right|^2 d\Gamma(\lambda)$$

and so $\overline{W}\psi$ and $g(L)\psi$ are both defined if $\mathcal{F}_y(\psi)(\lambda) \in C_0^\infty$. From the density of the space C_0^∞ in $L^2_{d\Gamma}$, we deduce that $D_{\overline{W}} \cap D_{g(L)}$ is dense in $L^2_{d\mu}$. If Z is not dense in $L^2_{d\mu}(\Omega)$ then there exists $k \in L^2_{d\mu}(\Omega)$ such that $k \in Z^\perp$ and $k(x) \neq 0$. If we set

$$\tilde{k}(x) := \begin{cases} k(x) & \text{if } x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

then $\forall \psi \in D_{\overline{W}} \cap D_{g(L)}$

$$\int_\Omega k(x)\overline{\psi(x)}\, d\mu = \int_R \tilde{k}(x)\overline{\psi(x)}\, d\mu = 0\,,$$

which means $\tilde{k}(x) = 0$ in $L^2_{d\mu}$, and so $k(x) = 0\ \ d\mu$ a.e.. Thus Z is dense in $L^2_{d\mu}(\Omega)$.

Let $f \in D_{\overline{W}}$ be given and denote by $\Omega := R - \operatorname{supp} \overline{W} f$. The case $\operatorname{supp} \overline{W} f = R$ is trivial. To prove the result in the case $\Omega \neq \emptyset$, we only need to show that $f \in Z^\perp$. Indeed, the density of Z in $L^2_{d\mu}(\Omega)$ will imply $f_{|\Omega} = 0$ and so $\operatorname{supp}_\mu f \subset \Omega^c = \operatorname{supp} \overline{W} f$. Since for $\psi \in Z$ the supports of the functions ψ and $\overline{W} f$ are disjoint, we have

$$\int \overline{W} f(x)\overline{\psi(x)}\, dx = 0 \qquad \forall \psi \in Z\,.$$

Since, in addition the space Z is contained in $D_{\overline{W}}$, from the second assumption, i.e., from the inclusion $\operatorname{supp} \overline{W}\psi \subset \operatorname{supp}_\mu \psi,\ \ \psi \in Z$, we deduce that

$$\int \overline{W} f(x)\overline{\overline{W}\psi(x)}\, dx = 0, \qquad \forall \psi \in Z\,. \tag{4.11}$$

By Proposition 5 and the Parseval equality we obtain,

$$\begin{aligned} \int \overline{W} f(x)\overline{\overline{W}\psi(x)}\, dx &= \int \mathcal{F}_e\left(\overline{W} f\right)(\lambda)\overline{\mathcal{F}_e\left(\overline{W}\psi\right)(\lambda)}d\frac{\lambda}{2\pi} \\ &= \int \mathcal{F}_y(f)(\lambda)\overline{\mathcal{F}_y(\psi)(\lambda)}d\frac{\lambda}{2\pi} \\ &= \int g(\lambda)\mathcal{F}_y(f)(\lambda)\overline{\mathcal{F}_y(\psi)(\lambda)}\, d\Gamma(\lambda) \\ &= \int f(x)\overline{g(L)\psi(x)}\, d\mu\,. \end{aligned}$$

Hence from (4.11),

$$\forall \psi \in Z \qquad \int f(x)\overline{g(L)\psi(x)}\, d\mu = 0\,. \tag{4.12}$$

It remains to show that the set $\{g(L)\psi(x) :\ \ \psi \in Z\}$ is dense in $L^2_{d\mu}(\Omega)$. For this we only need to prove that for all $v \in L^2_{d\mu}(\Omega)$, the equation $g(L)\psi = v$ has a solution in the space Z. Define

$$\psi(x) := \int \frac{\mathcal{F}_y(v)(\lambda)}{g(\lambda)}\overline{y(x,\lambda)}\, d\Gamma(\lambda)\,.$$

Clearly ψ is well defined, $g(L)\psi = v$, and

$$\|\psi\|^2 = \int \frac{\left|\mathcal{F}_y(v)(\lambda)\right|^2}{|g(\lambda)|^2}\, d\Gamma(\lambda) \le \frac{\|v\|^2}{\operatorname*{essinf}_{\lambda\in\sigma} |g(\lambda)|^2} \le \frac{\|v\|^2}{a}\,.$$

Therefore $\psi \in L^2_{d\mu}$. To show that $\psi \in D_{\overline{W}}$ observe that if set $f = \psi$ in (4.12), then

$$\left\|\overline{W}\psi\right\|^2 = \int g(\lambda)\left|\mathcal{F}_y\psi(\lambda)\right|^2 d\Gamma(\lambda) = \int \frac{1}{g(\lambda)}\left|\mathcal{F}_y v(\lambda)\right|^2 d\Gamma(\lambda) \le \frac{\|v\|^2}{\operatorname{essinf}|g(\lambda)|} < \infty$$

thus $\psi \in D_{\overline{W}}$. Since $\|g(L)\psi\|^2 = \|v\|^2 < \infty$ we deduce that $\operatorname*{supp}_{\mu}\psi \subset \operatorname*{supp}_{\mu} g(L)\psi = \operatorname*{supp}_{\mu} v \subset \Omega$. Therefore $\psi \in L^2_{d\mu}(\Omega) \cap D_{\overline{W}} \cap D_{g(L)}$, that is $\psi \in Z$, and so (4.12) can be written

$$\int f(x)\overline{v(x)}\, d\mu = 0 \quad \forall v \in Z$$

which means $\operatorname*{supp}_{\mu} f \subset (\Omega)^c = \operatorname{supp}\overline{W}f$. This allows us to use Proposition 5, and so the result is proved. □

Remarks.

1. If $(Vf)(x) = \int_{x-a}^{x} k(x,t) f(t)\, dt$, where $a > 0$, then formally $(Wf)(t) = \int_t^{t+a} \overline{k(x,t)}\, f(x)\, d\mu(x)$ and $\operatorname{supp} Wf$ is compact if $\operatorname*{supp}_{\mu} f$ is compact.
2. If $y(x,\lambda)$ is a solution of a differential equation, then usually the following conditions are verified, see [19]:

   - $y(x,\lambda)$ is entire in λ;
   - $|y(x,\lambda)| \le M \exp(|x||\lambda|)$ as $|\lambda| \to \infty$; and
   - $y(x,\lambda) = O(\lambda^{2n})$ as $\lambda \to \infty$.

   The Paley–Wiener–Schwartz theorem, see [15], implies the existence of $K(x,.) \in S'$ for each x, such that

   $$y(x,\lambda) = \int_{-|x|}^{|x|} K(x,t)\left(1+\lambda^{2n}\right) e^{i\lambda t}\, dt\,.$$

   The transmutation operator V is then given by

   $$V\left(e^{i\lambda x}\right) = y(x,\lambda) = \int_{-|x|}^{|x|} K(x,t)\left(1+\mathcal{D}_t^{2n}\right) e^{i\lambda t}\, dt\,.$$

   The operator

   $$\overline{W}f(x) = \left(1+\mathcal{D}_x^{2n}\right)\left\{\int_{-\infty}^{-|x|} \overline{K(t,x)} f(t)\, d\mu + \int_{|x|}^{-\infty} \overline{K(t,x)} f(t)\, d\mu\right\} \tag{4.13}$$

   may not be local, since for example we may have $\operatorname*{supp}_{\mu} f \subset [1/2, 1]$ while $\operatorname{supp} Wf \subset [-1, 1]$. Observe that (4.3) holds.

5. Support of $\overline{W}^{-1} f$

It is readily seen that in order to obtain a Paley–Wiener type theorem, one needs both (4.4) and (4.3) to be satisfied. It is also clear that from the remark of Theorem 2, that (4.3) is a soft part and the difficult part is to obtain (4.4). In this section we shall examine this question and give sufficient conditions for the existence of transmutations that satisfy (4.4). Observe that the operators V and V^{-1} are related by

$$\mathcal{F}_e(H)(\lambda, t) := 2\pi \frac{d\Gamma(\lambda)}{d\lambda} y(t, \lambda) \qquad \lambda \in R$$

where $H(x, t)$ is the kernel of the integral representation of V^{-1} and $y(x, \lambda) = V(e^{i\lambda x})$. It is clear that $H(., t) \in S'$ exists if and only if $\frac{d\Gamma}{d\lambda}(\lambda) y(t, \lambda) \in S'_\lambda$.

One simple way of finding out the support of $H(x, t)$ is to use the space of analytic functions $\mathcal{H}^2$ in the upper-half of the complex plane:

$$\mathcal{H}^2 := \left\{ F(z) \text{ analytic for } \operatorname{Im} z > 0 : \sup_{y \geq 0} \int_{-\infty}^{\infty} | F(x + iy) |^2 \, dx < \infty \right\} .$$

For properties of $\mathcal{H}^2$ see [10]. Let $y(t, \lambda) \frac{d\Gamma(\lambda)}{d\lambda}$ have an analytic extension in $\{\lambda \in C \ : \operatorname{Im} \lambda \neq 0\}$. In this case we have the following.

Proposition 11.
If $\exp(-it\lambda) y(t, \lambda) \frac{d\Gamma}{d\lambda}(\lambda) \in \mathcal{H}^2$, *then* $\operatorname{supp} H(., t) \subset [t, \infty)$.

Proof. Since for $\operatorname{Im} \lambda > 0$, the function $\exp(-it\lambda) y(t, \lambda) \frac{d\Gamma(\lambda)}{d\lambda} \in \mathcal{H}^2$ and there exists a function $f(t, \eta)$ such that

$$\begin{aligned} e^{-i\lambda t} y(t, \lambda) \frac{d\Gamma(\lambda)}{d\lambda} &= \int_0^\infty f(t, \eta) e^{i\lambda\eta} d\eta \\ y(t, \lambda) \frac{d\Gamma(\lambda)}{d\lambda} &= \int_0^\infty f(t, \eta) e^{i\lambda\{t+\eta\}} d\eta \\ y(t, \lambda) \frac{d\Gamma(\lambda)}{d\lambda} &= \int_t^\infty f(t, \xi - t) e^{i\lambda\xi} d\xi \end{aligned}$$

which means that $H(x, t) := \begin{cases} f(t, x - t) & \text{if} \quad x > t \\ 0 & \text{if} \quad x < t. \end{cases}$ □

Remark. We end this section by giving one more sufficient condition for W and W^{-1} to preserve compact supports. It is readily seen that a sufficient condition is for W and W^{-1} to be bounded operators in C_0^∞, the space of compactly supported functions. To this end we only need to require V to be a bounded operator in $(C_0^\infty)'$. Indeed if V^{-1} exists it will also be a bounded operator, by the open mapping theorem, see [15]. Thus W will satisfy a condition similar to (4.3). That is if V is a bounded operator $(C_0^\infty)' \xrightarrow{V} (C_0^\infty)'$ and V^{-1} exists, then it follows that

$$\operatorname{supp}_{\mu} f \text{ is compact} \iff \operatorname{supp} \overline{W} f \text{ is compact}.$$

Moreover, operators P that satisfy $\operatorname{supp} \ Pu \subset \operatorname{supp} \ u$, as in Theorem 2, are completely characterized, see [21]. Indeed they are differential operators, i.e., $P(u) = \sum a_\alpha D^\alpha u$, and in our setting, see Theorem 2, this means $[g(L)]^{-1} = \sum b_\alpha D^\alpha$. This approach needs the apparatus of pseudodifferential operators or Fourier integral operators, which unfortunately falls outside the scope of this work, but is worth a separate investigation, see [12]. We now illustrate the above concepts by the following.

Theorem 3.

Assume that $\frac{d\Gamma}{d\lambda}(\lambda)$ *is an entire function such that,* $|\exp(ix\lambda)\overline{y(x,\lambda)}\frac{d\Gamma(\lambda)}{d\lambda}| < C\exp(\epsilon|\lambda|)$, *where* ϵ *is arbitrarily small. Then* W^{-1} *is a support preserving operator.*

Proof. From (3.1) we formally have

$$\begin{aligned} W^{-1}f(x) &= \mathcal{F}_y^{-1}\mathcal{F}_e(f)(\lambda) \\ &= \int \mathcal{F}_e(f)(\lambda)\overline{y(x,\lambda)}\,d\Gamma(\lambda) \\ &= \int \mathcal{F}_e(f)(\lambda)\overline{y(x,\lambda)}\frac{d\Gamma(\lambda)}{d\lambda}d\lambda \\ &= \int \exp(ix\lambda)\overline{y(x,\lambda)}\frac{d\Gamma(\lambda)}{d\lambda}\mathcal{F}_e(f)(\lambda)\exp(-ix\lambda)d\lambda\,. \end{aligned}$$

If $\operatorname{supp} f \subset [-a,a]$ then $|\exp(ix\lambda)\overline{y(x,\lambda)}\frac{d\Gamma}{d\lambda}(\lambda)\mathcal{F}_e(f)(\lambda)| < C\exp(|\lambda|(\epsilon+a))$ which means $\operatorname{supp} W^{-1}f \subset [-a-\epsilon, a+\epsilon]$ and since ϵ is arbitrary, we have $\operatorname{supp} W^{-1}f \subset [-a,a]$. □

6. Examples

Below we shall illustrate the above theory starting with simple examples and then providing a different treatment of known examples, such as those in [7].

1. Consider the following self-adjoint operator in L^2_{dx}

$$Ly := -i\frac{dy}{dx} + qy \qquad x \in (-\infty,\infty)$$

where $q \in L^{1,\mathrm{loc}}_{dx}$. The normalized eigenfunctionals are solutions of

$$\begin{cases} -iy'(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda) \\ y(0,\lambda) = 1 \end{cases}$$

that is

$$y(x,\lambda) = \exp(ix\lambda)\exp\left(-i\int_0^x q(t)\,dt\right)$$

and so the operator V is defined by

$$(Vf)(x) := \exp\left(-i\int_0^x q(t)\,dt\right)f(x)\,.$$

The operator $Wf = \exp\left(i\int_0^x q(t)\,dt\right)f(x)$ is obviously bounded and (4.3) and (4.4) hold, i.e.,

$$\operatorname{supp} f \subset [-a,a] \Longleftrightarrow \operatorname{supp} \overline{W}f \subset [-a,a]\,.$$

2. Consider the operator defined by

$$Ly = \frac{1}{w(x)}\frac{dy}{idx}(x) + q(x)y(x) \qquad\qquad x \in (-\infty,\infty)$$

where $w(x) \geq 0$, and $w(x), q(x) \in L^{1,\mathrm{loc}}_{dx}$. L is self-adjoint in L^2_{wdx} and has a simple spectrum. The eigenfunctionals are entire functions of λ and are given by

$$\begin{aligned} y(x,\lambda) &= \exp\left(i\lambda \int_0^x w(s)ds\right)\exp\left(-i\int_0^x q(t)w(t)\,dt\right) \\ &= s(x)\exp\left(i\lambda b(x)\right) \\ &= V\left(e^{ix\lambda}\right) \end{aligned}$$

where $s(x) := \exp\left(-i\int_0^x q(t)w(t)\,dt\right)$ and $b(x) := \int_0^x w(s)\,ds$. The operator V is defined by $(Vf)(x) = s(x)f(b(x))$ and possesses the following properties:

$$\begin{aligned} \int_0^\infty Vf(x)\overline{\psi(x)}w(x)\,dx &= \int f(x)\overline{W\psi(x)}\,dx \\ \int_0^\infty s(x)f(b(x))\overline{\psi(x)}db(x) &= \int f(x)\overline{W\psi(x)}\,dx \\ \int_0^{b(\infty)} f(\eta)s\left(b^{-1}(\eta)\right)\overline{\psi(b^{-1}(\eta))}d\eta &= \int f(x)\overline{W\psi(x)}\,dx\,. \end{aligned}$$

Hence $W\psi(t) = \overline{s(x(t))}\psi(x(t))$ where $b(x(t)) = t$. In this case the support is not preserved, but

$$\operatorname{supp} f \subset [-a,a] \Longleftrightarrow \operatorname{supp} Wf \subset \left[b^{-1}(-a), b^{-1}(a)\right]\,.$$

We now consider examples of second order differential operators. Since the multiplicity is two, we can obtain a simple spectrum just by restricting the operators to $L^2_{dx}(0,\infty)$. In this case the transmutation operator will be of the form

$$y(x,\lambda) = V(\cos(x\lambda)) \qquad \lambda \geq 0$$

and Paley–Wiener theorem will use the cosine transform.

3. Second-order differential operators. It is well known that

$$\begin{cases} Lf := f''(x) + q(x)f(x) & x \geq 0 \\ nf(0) - f'(0) = 0, & n \in R \end{cases}$$

defines a self-adjoint operator in $L^2_{dx}[0,\infty)$. The eigenfunctionals are solutions of

$$\begin{cases} -y''(x,\lambda) + q(x)y(x,\lambda) = \lambda^2 y(x,\lambda) \\ y(0,\lambda) = 1 \text{ and } y'(0,\lambda) = n\,. \end{cases}$$

In [16] the existence of two functions $H(x,t)$ and $K(x,t)$ is shown, such that

$$\begin{cases} y(x,\lambda) = \cos(x\lambda) + \int_0^x K(x,t)\cos(t\lambda)\,dt \\ \cos(x\lambda) = y(x,\lambda) + \int_0^x H(x,t)y(t,\lambda)\,dx \end{cases}.$$

Here conditions of Theorem 1 are satisfied.

4. Generalized second order differential operators.
Consider the self-adjoint operator acting in the Hilbert space $L^2_{w(x)\,dx}(0,\infty)$ and defined by

$$\begin{cases} Lf(x) := \frac{-1}{w(x)}f''(x) & x \geq 0 \\ f'(0) = 0 \end{cases}$$

where

$$w(x) \geq 0, \quad w(x) \asymp x^{\alpha} \text{ as } x \to 0, \quad \text{and} \quad w(x) \in L^{1,\mathrm{loc}}_{dx} .$$

Since the spectrum is positive the eigenfunctionals $y(x,\lambda)$ are solutions of

$$\begin{cases} y''(x,\lambda) + \lambda^2 w(x) y(x,\lambda) = 0 \\ y(0,\lambda) = 1 \qquad\qquad y'(0,\lambda) = 0 \end{cases} .$$

Clearly $y(x,.)$ is entire, and satisfies $\mid y(x,\lambda) \mid \leq \exp(|\lambda| t(x))$ where $t(x) = \sqrt{2x \int_0^x w(s)ds}$, see [18]. As $\lambda \to \infty$ we have the following asymptotics,

$$y(x,\lambda) \asymp c\sqrt{\frac{\xi(x)}{p(x)}}\, \lambda^{\frac{-1}{\alpha+2}}\, \mathrm{J}_{\frac{1}{\alpha+2}}(\xi(x))$$

where $\alpha + 2 > 0$, $p(x) = |\lambda|\sqrt{w(x)}$, $\xi(x) = \int_0^x p(t)\,dt$, J_ν is the Bessel function of the first kind and c_1 and c_2 are constants. For fixed x, the above estimate shows that $y(x,\lambda)$ increases as a power as $|\lambda| \to \infty$. Thus we can use the Paley–Wiener–Schwartz theorem, see [15, Volume 2, p. 162], to show the existence of $K(x,t) \in S'$ such that

$$y(x,\lambda) = \int_0^{t(x)} K(x,\eta)\cos(\lambda\eta)d\eta$$

where $t(x) = \sqrt{2x \int_0^x w(s)ds}$. The above estimate can be found in [18] and the asymptotics is a simple result of the semi-classical approximation method. Clearly the first assumption of theorem (2.2) holds, for all functions w. In order to satisfy the second assumption, one can select a class of spectral functions Γ, i.e., symbols $g(\lambda)$, such that $\operatorname{supp} f \subset \operatorname{supp} g(L)f$, see [21]. Then solve the inverse spectral problem, see [11] to recover w. This way one can determine a set of weights w for which the Paley–Wiener holds for the transform associated with the above generalized Sturm–Liouville operator.

5. General transforms.

Assume that the eigenfunctionals of a self adjoint operator have the form

$$y(x,\lambda) := u(\lambda x) ,$$

where u is a given function. For example they can be generated by

$$Lf(x) := w(x)^{-1} \sum_{j=0}^{n} a_j(x) f^{(j)}(x)$$

where $a_j(\lambda x) = \lambda^j a_j(x)$ and $w(\lambda x) = \lambda w(x)$, for further details we refer to [23]. Under certain conditions on the real function u, the inverse transform is defined by

$$f(x) = \int \hat{f}(\lambda) u(x\lambda)\, d\lambda$$

where $\hat{f}(\lambda) = \int f(x) u(\lambda x)\, dx$ and this means $\Gamma(\lambda) = \lambda$. For the transmutation operator to be in the form

$$y(x,\lambda) = \int_{-x}^{x} K(x,t) e^{i\lambda t}\, dt$$

we need

$$\begin{aligned}K(x,t) &= \frac{1}{2\pi}\int u(\lambda x)e^{-i\lambda t}\,d\lambda \\ &= \frac{1}{2\pi}\int u(s)e^{-is\frac{t}{x}}ds \\ &= \frac{1}{x}d\left(\frac{t}{x}\right)\end{aligned}$$

to vanish for $|t| > x$ i.e., $\operatorname{supp} d(x) = [-1, 1]$. To this end we require $u(\lambda)$ to be entire of order one, type 1 and to grow like a polynomial as $\lambda \to \infty$. Therefore

$$y(x,\lambda) = \int_{-x}^{x} \frac{1}{x}d\left(\frac{t}{x}\right)\exp(it\lambda)\,dt\ .$$

The spectral function $\Gamma(\lambda) := \lambda$ implies $g(\lambda) = \frac{1}{2\pi}$ and it follows that $\operatorname{supp} g(L)f = \operatorname{supp} \frac{1}{2\pi}f(x)$ $= \operatorname{supp} f$. Thus the conditions of Theorem 2 hold.

6. In [14], the transmutation associated with the following differential operator is studied:

$$Lf := \left[D^2+\lambda^2\right]^m f(x) + \sum_{0\le k\le m-1} P_k(x,\lambda)\frac{d^k}{dx^k}f(x) \qquad x>0$$

where $P_k(x,\lambda) = \sum_{0\le j\le k}\lambda^j p_{k,j}(x)$. It is shown that the existence of kernels K_j such that

$$y_j(x,\lambda) = x^j e^{i\lambda x} + \int_{-x}^{x} K_j(x,t)t^j e^{i\lambda t}\,dt$$

where $y_j(x,\lambda)$ are independent solutions of $Ly = 0$. This shows that (4.2) holds and since we have an equation of Volterra type we deduce that

$$x^j e^{i\lambda x} = y_j(|x|,\lambda) + \int_{-x}^{x} H_j(x,t)y_j(|t|,\lambda)\,dt\ .$$

Hence (4.1) also holds.

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Department of Mathematics, Moravian College, Bethlehem, PA 18018

Department of Mathematics, University of Delaware, Newark, DE 19716-2553
e-mail: nashed@math.udel.edu