LMAFY1101 - Solutions - Série 4

Variables aléatoires: Généralités

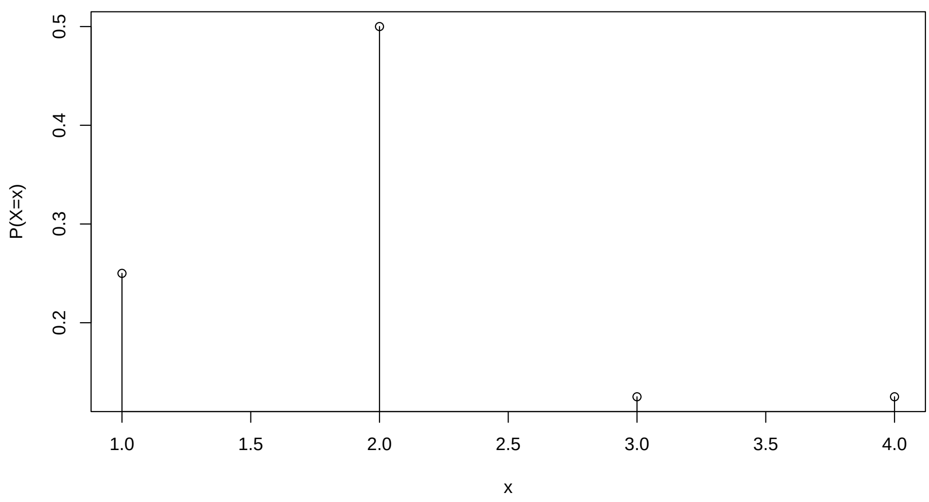
Variables aléatoires discrètes

Exercice 1

1.

Pour déterminer la distribution d'une variable aléatoire discrète, on doit calculer la probabilité d'observer chacune de ses valeurs possibles. On a que
$P[X = 1] = 0.25$
$P[X \leq 2] = P[X = 1] + P[X = 2] = 0.25 + P[X = 2] = 0.75 \rightarrow P[X = 2] = 0.5$
$P[X \leq 3] = P[X \leq 2] + P[X = 3] = 0.75 + P[X = 3] = 0.875 \rightarrow P[X = 3] = 0.125$
$P[X = 4] = 1 - P[X \leq 3] = 1 - 0.875 = 0.125$

```
plot(x = c(1, 2, 3, 4), y = c(0.25, 0.5, 0.125, 0.125), type = "h",
  main = "", xlab = "x", ylab = "P(X=x)")
points(x = c(1, 2, 3, 4), y = c(0.25, 0.5, 0.125, 0.125))
```

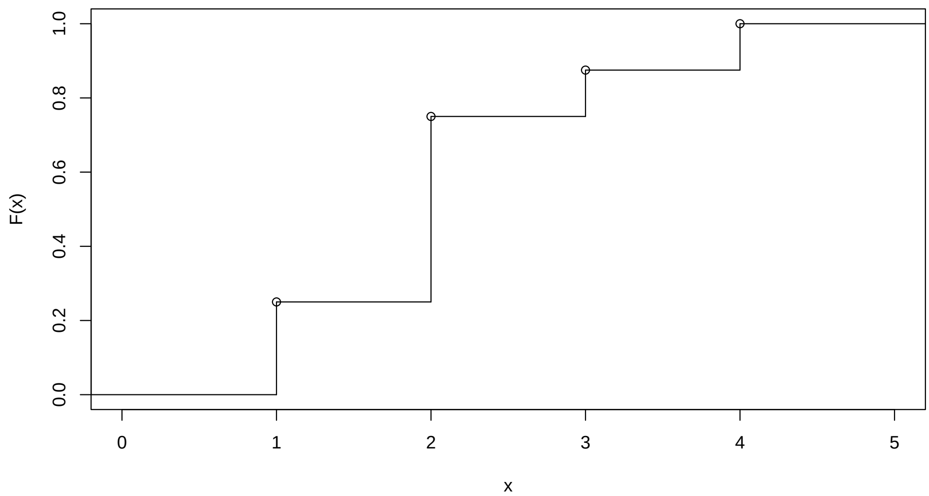


2.

$P[X \leq 1] = 0.25$
$P[X \leq 2] = 0.75$
$P[X \leq 3] = 0.875$
$P[X \leq 4] = 1$

$$\Rightarrow F(x) = \begin{cases} 0 & \text{si } x < 1 \\ 0.25 & \text{si } 1 \leq x < 2 \\ 0.75 & \text{si } 2 \leq x < 3 \\ 0.875 & \text{si } 3 \leq x < 4 \\ 1 & \text{si } x \geq 4 \end{cases}$$

```
plot(stepfun(1:4, c(0, 0.25, 0.75, 0.875, 1)), main = "", xlab = "x",
  ylab = "F(x)")
```



3.

$$\begin{aligned} E(X) &= \sum_{x=1}^{4} xP(X=x) = 1 \times 0.25 + 2 \times 0.5 + 3 \times 0.125 + 4 \times 0.125 \\ Var(X) &= E(X^2) - (E(X))^2 \\ E(X^2) &= \sum_{x=1}^{4} x^2 P(X=x) = 1^2 \times 0.25 + 2^2 \times 0.5 + 3^2 \times 0.125 + 4^2 \times 0.125 \\ CV &= \sqrt{Var(X)}/E(X) \end{aligned}$$

```
x <- 1:4
p <- c(0.25, 0.5, 0.125, 0.125)
E <- sum(x * p)
E
```

```
[1] 2.12
```

```r
E2 <- sum(x^2 * p)
Var <- E2 - E^2  # ou sum((x-E)^2*p)
Var
```

```
[1] 0.859
```

```r
CV <- sqrt(Var)/E
CV
```

```
[1] 0.436
```

4.

$$P(X \geq 2 \text{ ou } X \geq 3) = P(X \geq 2) = 1 - F(1) = 0.75$$
$$P(X \geq 3 | X \geq 2) = \frac{P(X \geq 3)}{P(X \geq 2)} = \frac{1 - F(2)}{1 - F(1)} = 0.25/0.75 \approx 0.333$$

5.

$$E(1/X) = \sum_{x=1}^{4} \frac{1}{x} P(X = x) = 1/1 \times 0.25 + 1/2 \times 0.5 + 1/3 \times 0.125 + 1/4 \times 0.125$$
$$E(3X^2 - X + 1) = 3E(X^2) - E(X) + 1$$

```r
sum((1/(1:4)) * p)
3 * E2 - E + 1
```

```
[1] 0.573
[1] 15
```

Exercice 2

1.

$$\sum_x p(x) = 1 \Leftrightarrow \frac{13c}{5} = 1 \Leftrightarrow c = 5/13$$

2.

$$P(Y = y) = P(X = y^{1/3}) = \frac{c}{1 + y^{2/3}}, \; y = 0, \pm 1, \pm 8, \pm 27$$

3.

Les valeurs possibles de Z sont

```r
sin((pi / 2) * c(-3, -2, -1, 0, 1, 2, 3)) |> round(10)
```

```
[1]  1  0 -1  0  1  0 -1
```

$\rightarrow$
$$
\begin{aligned}
P(Z=-1) &= P(X=-1)+P(X=3)=3/13\\
P(Z=0) &= P(X=-2)+P(X=0)+P(X=2)=7/13\\
P(Z=1) &= P(X=-3)+P(X=1)=3/13
\end{aligned}
$$

Exercice 3

Ensemble des valeurs de X : $\{1,2,3,4\}$:

k	1	2	3	4
$P(X=k)$	$\frac{4\times 6!}{7!}=4/7$	$\frac{3\times 4\times 5!}{7!}=2/7$	$\frac{3\times 2\times 4\times 4!}{7!}=4/35$	$\frac{3!\times 4!}{7!}=1/35$

Notez qu'on peut calculer les factorielles en R avec la fonction `factorial`:

```r
factorial(6)
```

```
[1] 720
```

Et voici comment calculer l'écart-type

```r
E <- sum((1:4) * c(4/7, 2/7, 4/35, 1/35))
E2 <- sum((1:4)^2 * c(4/7, 2/7, 4/35, 1/35))
Var <- E2 - E^2
sqrt(Var)
```

```
[1] 0.8
```

ou comme ceci

```r
sum((1:4 - E)^2 * c(4 / 7, 2 / 7, 4 / 35, 1 / 35)) |> sqrt()
```

```
[1] 0.8
```

Exercice 4

k	1	0.5	-0.5
$P(X=k)$	0.5	0.25	0.25

$\Rightarrow E[X]=0.5$ est le gain espéré suite à une participation. Pour n participants le gain espéré et de $E(nX)=nE(X)=n/2$. Donc pour espérer obtenir 250 euros, il faut $n=500$ participants.

Exercice 5

1.

Ensemble des valeurs de X : $\{0, 1, 2, \ldots\}$:

$$P(X = x) = (1/2)^{x+1},\ x = 0, 1, 2, \ldots$$

2.

$$P(X > 3) = 1 - P(X \leq 3) = 1 - \sum_{x=0}^{3} P(X = x)$$

```
1 - sum(0.5^(1:4))
```

```
[1] 0.0625
```

3.

$$E(X) = \sum_{x=0}^{\infty} x0.5^{x+1}$$

On peut montrer, par des calculs mathématiques relativement simple, que cela vaut 1.

On peut approximer cette somme comme suite.

```
x <- 0:1000
sum(x * (0.5)^(x + 1))
```

```
[1] 1
```

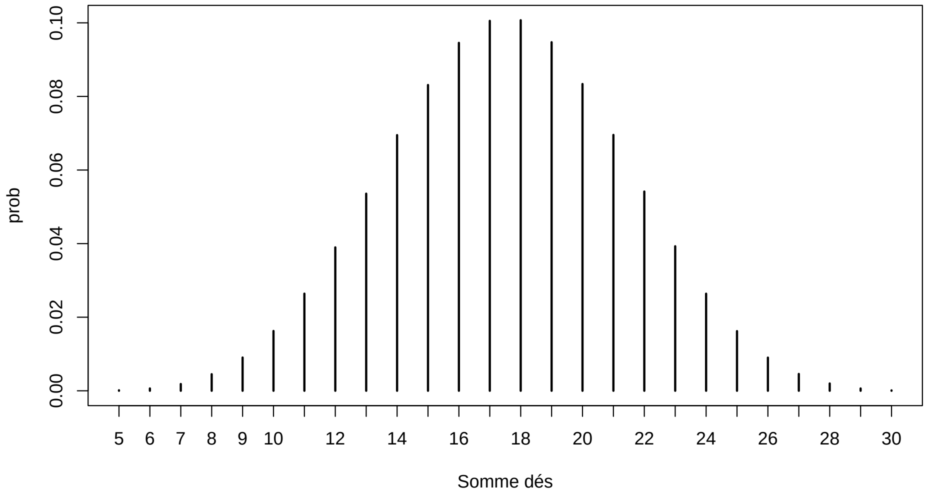
Exercice 6

```
exp <- replicate(10^6, {
  de <- sample(1:6, 5, replace = TRUE)
  sum(de)
})
```

```
prp <- table(exp) |> proportions()
prp
```

```
exp
       5        6        7        8        9       10       11       12 
0.000149 0.000648 0.001852 0.004521 0.009050 0.016266 0.026409 0.038983 
      13       14       15       16       17       18       19       20 
0.053578 0.069494 0.083116 0.094565 0.100544 0.100714 0.094738 0.083384 
      21       22       23       24       25       26       27       28 
0.069546 0.054161 0.039286 0.026396 0.016225 0.009023 0.004587 0.002000 
      29       30 
0.000644 0.000121 
```

```r
plot(prp, xlab = "Somme dés", ylab = "prob")
```



Exercice 7

1.

```r
exp <- replicate(10^6, {
  de <- sample(1:6, size = 2, replace = TRUE)
  c(Z = max(de), W = prod(de))
})
```

```r
table(exp["Z", ]) |> proportions()
```

```
     1      2      3      4      5      6 
0.0277 0.0834 0.1388 0.1943 0.2497 0.3061 
```

```r
table(exp["W", ]) |> proportions()
```

```
     1      2      3      4      5      6      8      9     10     12     15 
0.0277 0.0553 0.0555 0.0838 0.0559 0.1114 0.0555 0.0276 0.0549 0.1111 0.0554 
    16     18     20     24     25     30     36 
0.0277 0.0557 0.0553 0.0557 0.0282 0.0555 0.0278 
```

2.

```r
mean(exp["Z", ] >= 6)
mean(exp["W", ] >= 6)
```

```
[1] 0.306
[1] 0.722
```

3.

```
mean(exp["Z", ])
mean(exp["W", ])
var(exp["Z", ])
var(exp["W", ])
```

```
[1] 4.47
[1] 12.3
[1] 1.97
[1] 80
```

4.

Il s'agit de résoudre l'équation en $x > 0$

$$1 - \frac{80}{x^2} = 0.8 \Leftrightarrow x = \sqrt{80/(1-0.8)} = 20$$

5.

```
mean(abs(exp["W", ] - 12.3) <= 20)
```

```
[1] 0.972
```

$\rightarrow P(|W - 12.3| \leq 20) \approx 0.972$ est bel et bien plus grande que 0.8, ce qui est en accord avec l'inégalité de Tchebychev. Sachez que cette dernière est connue pour être souvent trop conservatrice.

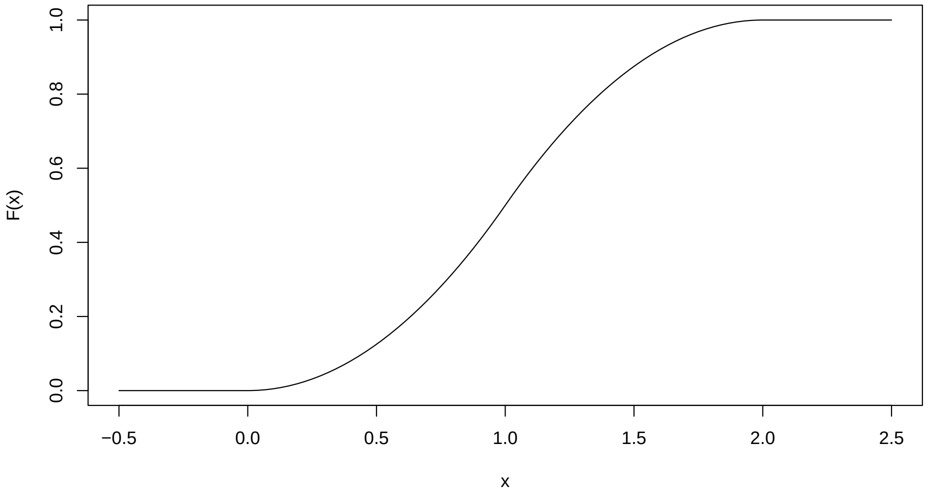
Variables aléatoires continues

Exercice 8

1.

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ \frac{x^2}{2} & \text{si } 0 \leq x < 1 \\ \frac{-x^2}{2} + 2x - 1 & \text{si } 1 \leq x < 2 \\ 1 & \text{si } x \geq 2 \end{cases}$$

```
curve((x^2/2) * (x >= 0 & x < 1) + (-x^2/2 + 2 * x - 1) * (x >=
  1 & x < 2) + I(x >= 2), from = -0.5, to = 2.5, ylab = "F(x)")
```



2.
$$P[0.8 \leq X \leq 1.2] = F(1.2) - F(0.8) = \left(\frac{-(1.2^2)}{2} + 2.4 - 1\right) - \frac{0.8^2}{2} = 0.36$$

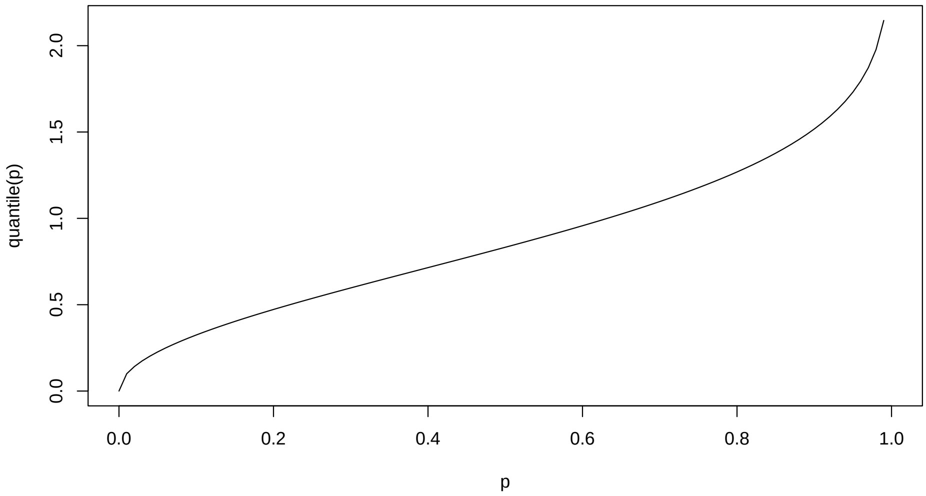
3.
$$P[X \leq 1.5 | X > 1] = \frac{F(1.5) - F(1)}{1 - F(1)} = \frac{0.375}{0.5} = 0.75$$

Exercice 9

1.
$$P(Y \geq 2) = 1 - F(2) = 1 - (1 - e^{-4}) = 0.0183.$$

2.
$$1 - e^{-y^2} = p,\ p \in [0, 1] \Leftrightarrow y = \sqrt{-\log(1-p)}.$$

```
curve(sqrt(-log(1 - x)), from = 0, to = 1, xlab = "p", ylab = "quantile(p)")
```



3.

Il faut trouver y tel que $P(Y \geq y) = 0.935 \Leftrightarrow P(Y \leq y) = 0.065$

```
round(sqrt(-log(1 - 0.065)) * 100)
```

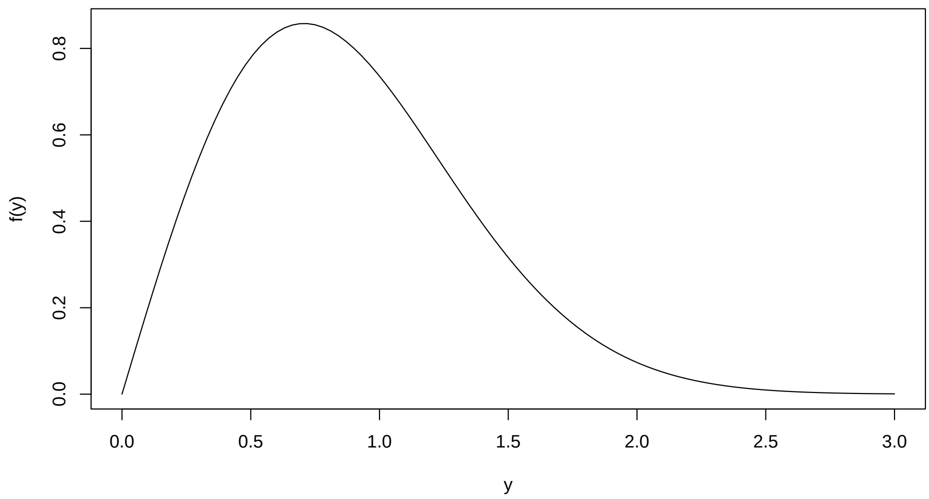
```
[1] 26
```

4.

La densité est donnée par

$$f(y) = \frac{d}{dy}F(y) = 2ye^{-y^2}I(y \geq 0)$$

```
curve(2 * x * exp(-x^2) * (x >= 0), from = 0, to = 3, xlab = "y",
  ylab = "f(y)")
```



4.

- moyenne:

```
E <- integrate(function(x) 2 * x^2 * exp(-x^2), lower = 0, upper = Inf)$value
E
```

```
[1] 0.886
```

- variance:

```r
integrate(function(x) 2 * x^3 * exp(-x^2), lower = 0, upper = Inf)$value -
  E^2
```

```
[1] 0.215
```

Exercice 10

1.

Pour que f soit une fonction de densité il faut que $f \geq 0$ et que $\int f = 1$. Dans notre cas cela revient à verifier que $c \int_0^1 1dx = 1 \rightarrow c = 1$.

2.

Soit $Y = X^2 + 1$. Notez que puisque $X \in [0,1]$, $Y \in [1,2]$.

$$
\begin{aligned}
& P(Y \leq y) = P(X \leq \sqrt{y-1}), \text{ pour } 1 \leq y \leq 2. \\
& \Rightarrow F(y) = \begin{cases} 0, & y < 1 \\ \sqrt{y-1}, & 1 \leq y \leq 2 \\ 1, & y > 2. \end{cases} \\
& \Rightarrow f(y) = \frac{d}{dy} F(y) = \frac{1}{2\sqrt{y-1}} I(1 \leq y \leq 2).
\end{aligned}
$$

3.

- méthode 1 (définition de l'espérance):

$$E(X^2 + 1) = E(Y) = \frac{1}{2} \int_1^2 \frac{y}{\sqrt{y-1}} dy = 1.333 \quad \text{(R–> fonction integrate)}$$

- méthode 2 (propriétés de l'espérance):

$$E(X^2 + 1) = E(X^2) + 1 = 1 + \int_0^1 x^2 dx = 1 + 1/3$$

Exercice 11

- Y = durée de vie dans le jeu en heures
- Coût d'inscription 25 euros
- Gain par minute 0.25 euros
- Gain par heure $0.25 \times 60 = 15$

$\rightarrow$ le gain net est la v.a. $Z = 15Y - 25$.

1.

Espérance du gain net est

$$E(Z) = 15 \times E(Y) - 25 = \frac{15}{8}\int_0^4 y^2 dy - 25 = 15.$$

2.

$$\begin{aligned} P(\text{perdre}) &= P(Z < 0) \\ &= P(15Y < 25) = P(Y < \frac{5}{3}) \\ &= \frac{1}{8}\int_0^{\frac{5}{3}} y dx \\ &= 0.1736 \end{aligned}$$

3.

La probabilité de gagner une partie (au hasard) est de $p = 1 - 0.1736 = 0.826$.

- La probabilité de gagner au moins une partie sur 5 est de

$$1 - (1-p)^5 = 0.999845.$$

- La probabilité de gagner deux parties sur 5 est de

$$C_5^2 \times p^2 \times (1-p)^3 = 0.035412.$$