THE REMAINDER IN TAYLOR SERIES

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1. Introduction

Let $f(x)$ be infinitely differentiable on an interval I around a number a. We want to bound the difference between $f(x)$ and its nth degree Taylor polynomial $T_{n,a}(x) = \sum_{k=0}^{n}(f^{(k)}(a)/k!)(x-a)^k$ at the numbers in I. To do this, we want to write a formula for the *remainder* $R_{n,a}(x) = f(x) - T_{n,a}(x)$, and will describe this remainder in two ways: using derivatives and using integrals.

Theorem 1.1 (Differential form of the remainder (Lagrange, 1797))**.** *With notation as above, for $n \geq 0$ and b in the interval I with $b \neq a$,*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} = T_{n,a}(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some c strictly between a and b.

The number c depends on a, b, and n. When $n = 0$ Theorem 1.1 says $f(b) = f(a) + f'(c)(b-a)$ for some c strictly between a and b. That is the Mean Value Theorem.

Theorem 1.2 (Integral form of the remainder (Cauchy, 1821))**.** *With notation as above, for $n \geq 0$ and b in the interval I with $b \neq a$,*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{f^{(n+1)}(t)}{n!}(b-t)^n\,dt = T_{n,a}(b) + \int_a^b \frac{f^{(n+1)}(t)}{n!}(b-t)^n\,dt.$$

If $n = 0$, the conclusion of Theorem 1.2 says $f(b) = f(a) + \int_a^b f'(t)\,dt$, which is the Fundamental Theorem of Calculus. Unlike the differential form of the remainder in Theorem 1.1, the integral form of the remainder involves no additional parameters like c.

Using these theorems we will prove *Taylor's inequality*, which bounds $|f(x) - T_{n,a}(x)|$ for $x \in I$ when the $(n+1)$th derivative of f is bounded on I: if $|f^{(n+1)}(x)| \leq M$ for all x in I then Taylor's inequality says

$$\text{for all } b \text{ in } I, |f(b) - T_{n,a}(b)| \leq M\frac{|b-a|^{n+1}}{(n+1)!}.$$

This is easy to see if $b = a$, since in this case $f(b) - T_{n,a}(b) = f(b) - T_{n,b}(b) = 0$. We'll use either Theorem 1.1 or 1.2 to treat $b \neq a$.

2. Differential (Lagrange) form of the remainder

To prove Theorem 1.1 we will use Rolle's theorem. Recall this theorem says if F is continuous on $[a,b]$, differentiable on (a,b), and $F(a) = F(b)$ then $F'(c) = 0$ where c is strictly between a and b.

Proof of Theorem 1.1. The following argument is based on a comment by Pieter-Jan De Smet on the page `https://gowers.wordpress.com/2014/02/11/taylors-theorem-with-the-lagrange-form-of-the-remainder/`.

There is a number C such that

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{C}{(n+1)!}(b-a)^{n+1} \tag{2.1}$$

since we can solve this equation for C (the factor $(b-a)^{n+1}$ is nonzero). We want to show $C = f^{(n+1)}(c)$ for some c strictly between a and b, and will do this by replacing a everywhere in (2.1) with a variable x to which we'll be able to use Rolle's theorem.

Consider the function

$$E(x) = f(b) - \left(\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{C}{(n+1)!}(b-x)^{n+1}\right),$$

for which $E(a) = 0$ by the choice of C and $E(b) = f(b) - f(b) = 0$. Then $E'(c) = 0$ for some c between a and b by Rolle's theorem. Let's now compute $E'(x)$ to see there is a lot of cancellation in it!

Using the product rule and some algebra,

$$\begin{aligned}
E'(x) &= -\left(\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{C}{(n+1)!}(b-x)^{n+1}\right)' \\
&= -\left(f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{C}{(n+1)!}(b-x)^{n+1}\right)' \\
&= -\left(f'(x) + \sum_{k=1}^{n}\left(\frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1}\right) - \frac{C}{n!}(b-x)^n\right) \\
&= -\left(f'(x) + \sum_{k=1}^{n} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1} - \frac{C}{n!}(b-x)^n\right) \\
&= -\left(f'(x) + \sum_{k=1}^{n} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{C}{n!}(b-x)^n\right) \\
&= -\left(f'(x) + \frac{f^{(n+1)}(x)}{n!}(b-x)^n - f'(x) - \frac{C}{n!}(b-x)^n\right) \\
&= \frac{-f^{(n+1)}(x) + C}{n!}(b-x)^n.
\end{aligned}$$

Therefore having $E'(c) = 0$ means $C = f^{(n+1)}(c)$, so (2.1) becomes

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}. \qquad \square$$

Remark 2.1. A variation on this has the variable x in place of b rather than a: we use

$$F(x) = f(x) - \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{C}{(n+1)!}(x-a)^{n+1} \right),$$

with C to be determined. Details are in [1].

Taylor's inequality is an immediate consequence of this differential form of the remainder: if $|f^{(n+1)}(x)| \leq M$ for all x in I and $b \neq a$ in I, then $|f^{(n+1)}(c)| \leq M$ since c is between a and b, so $|f(b) - T_{n,a}(b)| = |f^{(n+1)}(c)(b-a)^{n+1}/(n+1)!| \leq M|b-a|^{n+1}/(n+1)!$.

3. Integral (Cauchy) form of the remainder

Proof of Theorem 1.2. Start with the Fundamental Theorem of Calculus in the form

$$f(b) = f(a) + \int_a^b f'(t)\,dt.$$

Apply integration by parts with $u = f'(t)$ and $dv = dt$, so $du = f''(t)\,dt$ and take $v = t - b$ (*not* $v = t$) to get

$$\begin{aligned} f(b) &= f(a) + f'(t)(t-b)\Big|_a^b - \int_a^b (t-b)f''(t)\,dt \\ &= f(a) - f'(a)(a-b) - \int_a^b (t-b)f''(t)\,dt \\ &= f(a) + f'(a)(b-a) + \int_a^b (b-t)f''(t)\,dt. \end{aligned}$$

Apply integration by parts again with $u = f''(t)$ and $dv = (b-t)\,dt$, so $du = f'''(t)\,dt$ and take $v = -(b-t)^2/2$. Then

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + \int_a^b (b-t)f''(t)\,dt \\ &= f(a) + f'(a)(b-a) - \frac{f''(t)}{2}(b-t)^2\Big|_a^b + \int_a^b \frac{(b-t)^2}{2} f'''(t)\,dt \\ &= f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \int_a^b \frac{(b-t)^2}{2} f'''(t)\,dt. \end{aligned}$$

After another integration by parts with $u = f'''(t)$ and $dv = \frac{1}{2}(b-t)^2\,dt$ we get

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f'''(a)}{6}(b-a)^3 + \int_a^b \frac{(b-t)^3}{6} f^{(4)}(t)\,dt.$$

After repeated integration by parts we eventually get

$$f(b)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(b-a)^k+\int_a^b\frac{(b-t)^n}{n!}f^{(n+1)}(t)\,dt. \qquad \square$$

We will derive Taylor's inequality from Theorem 1.2 in two ways.

Method 1: Assume $|f^{(n+1)}(x)|\le M$ for all x in I and $b\neq a$ in I. For $a<b$,

$$|f(b)-T_{n,a}(b)|\le\int_a^b\frac{|b-t|^n}{n!}|f^{(n+1)}(t)|\,dt\le\int_a^b\frac{(b-t)^n}{n!}M\,dt=M\frac{(b-a)^{n+1}}{(n+1)!},$$

where the last calculation comes from the Fundamental Theorem of Calculus. For $b<a$, we get in a similar way $|f(b)-T_{n,a}(b)|\le M(a-b)^{n+1}/(n+1)!$. Putting both cases together,

$$\text{if } |f^{(n+1)}(x)|\le M \text{ for all } x \text{ in } I \text{ and } b\neq a \text{ in } I, \text{ then } |f(b)-T_{n,a}(b)|\le M\frac{|b-a|^{n+1}}{(n+1)!}.$$

Method 2: We make a change of variables in the integral to bypass the need for separate cases as in the first method. The integral is taken from a to b (whether $a<b$ or $a>b$), and numbers from a to b can be written in parametric form as $a+(b-a)u$ as u runs from 0 to 1. Therefore with the change of variables $t=a+(b-a)u$ the integral remainder equals

$$\begin{aligned}\int_a^b\frac{(b-t)^n}{n!}f^{(n+1)}(t)\,dt&=\int_0^1\frac{((b-a)(1-u))^n}{n!}f^{(n+1)}(a+(b-a)u)(b-a)\,du\\&=\frac{(b-a)^{n+1}}{n!}\int_0^1(1-u)^nf^{(n+1)}(a+(b-a)u)\,du,\end{aligned}$$

so if $|f^{(n+1)}(x)|\le M$ for all x from a to b then the absolute value of the integral remainder is at most

$$\begin{aligned}\frac{|b-a|^{n+1}}{n!}\int_0^1(1-u)^nM\,du&=M\frac{|b-a|^{n+1}}{n!}\int_0^1(1-u)^n\,du\\&=M\frac{|b-a|^{n+1}}{n!}\frac{1}{n+1}\\&=M\frac{|b-a|^{n+1}}{(n+1)!}.\end{aligned}$$

Remark 3.1. Taylor's inequality is not due to Taylor. In fact, Taylor's treatment of power series in his book [2], written in 1715, was not concerned with justifications of convergence or error estimates, and preceded by almost 80 years the work of Lagrange and by over 100 years the work of Cauchy.

4. Using Taylor's inequality

Having proved Taylor's inequality by using a differential and an integral form of the remainder for a Taylor polynomial approximation on an interval, we'll use this inequality to prove three standard power series formulas valid at all x by treating the real line as being built from intervals $[-R,R]$ as R gets bigger.

Theorem 4.1. *Let $f(x)$ be infinitely differentiable at all x. For each $R > 0$, suppose all the derivatives of $f(x)$ on $[-R, R]$ are bounded in a common way: there is an $M_R > 0$ such that $|f^{(k)}(x)| \leq M_R$ for all x in $[-R, R]$ and all $k \geq 0$. Then $f(x) = \sum_{k\geq 0}(f^{(k)}(0)/k!)x^k$ for all x.*

Proof. To prove $f(x) = \sum_{k\geq 0}(f^{(k)}(0)/k!)x^k$ for all x, we will prove this at all x in $(-R, R)$ for all $R > 0$.

Pick an $R > 0$. We are assuming that $|f^{(k)}(x)| \leq M_R$ for all x in $[-R, R]$ and all $k \geq 0$, so Taylor's inequality on $[-R, R]$ with $a = 0$ tells us that when x is in $(-R, R)$ and $n \geq 0$,

$$|f(x) - T_{n,0}(x)| \leq M_R \frac{|x|^{n+1}}{(n+1)!} \leq M_R \frac{R^{n+1}}{(n+1)!}.$$

The upper bound $M_R R^{n+1}/(n+1)!$ depends only on R, which is being fixed, and on n. As $n \to \infty$, $R^{n+1}/(n+1)! \to 0$. Thus for all x in $(-R, R)$, we have $|f(x) - T_{n,0}(x)| \to 0$ as $n \to \infty$, which shows

$$f(x) = \lim_{n\to\infty} T_{n,0}(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k\geq 0} \frac{f^{(k)}(0)}{k!} x^k. \qquad \square$$

To see e^x, $\sin x$, and $\cos x$ are each represented at all x by their Taylor series at 0, it suffices to show all three functions satisfy the hypotheses of the theorem. They are each infinitely differentiable at all x. When $f(x)$ is e^x, for $R > 0$ and x in $[-R, R]$ we have

$$f^{(k)}(x) = e^x \Longrightarrow |f^{(k)}(x)| = e^x \leq e^R,$$

so we can use $M_R = e^R$. When $f(x)$ is $\sin x$ or $\cos x$, for $R > 0$ and x in $[-R, R]$ we have

$$f^{(k)}(x) = \pm \sin x \text{ or } \pm \cos x \Longrightarrow |f^{(k)}(x)| \leq 1,$$

so we can use $M_R = 1$.

References

[1] J. Wolfe, "A proof of Taylor's formula," *Amer. Math. Monthly* **60** (1953), 415.

[2] B. Taylor, *Methodus Incrementorum Directa et Inversa*, Typis Pearsonianis, London, 1715. URL `https://archive.org/details/UFIE003454T00324PNI-2529000000`. English translation `http://www.17centurymaths.com/contents/taylorscontents.html`.