Network Topology and Integral Multicommodity Flow Problems

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ABSTRACT

In this paper we investigate the nature of integer solutions to multicommodity network flow problems from a graph-theoretic viewpoint. A sufficient condition for unimodularity is developed that is based upon the topological characteristics of the associated graph, and the results are applied to certain well-structured examples.

INTRODUCTION

Contrary to single commodity network flow problems, multicommodity flow problems do not possess totally unimodular constraint matrices. As a result, basic feasible solutions are often noninteger, and efficient labeling algorithms that take advantage of the basis structure of single commodity problems [15] are not available. Certain classes of multicommodity networks have been shown to be equivalent to single commodity flow problems and have unimodular constraint matrices [4,5,6,7, 9]. For these classes of problems, a simplex labeling algorithm can be described [8]. In this paper we investigate unimodularity in multicommodity networks from a graph-theoretic viewpoint, and develop a sufficient topological condition for unimodular bases. Although the problem of verifying this condition is not computationally tractable in general, the results do provide simple proofs of integrality theorems for certain classes of multicommodity networks, as illustrated by several examples.

Networks, Vol. 8 (1978) 107-119 0028-3045/78/0008-0107$01.00

Unimodularity and Basis Structure in Multicommodity Networks

A matrix A is said to be *totally unimodular* if every square submatrix has determinant +1, -1, or 0. A is *unimodular* if the determinant of every basis is +1 or -1. Several characterizations of unimodularity and total unimodularity have been presented [1,2,12,13,14,21]. Truemper [18,19,20] has provided a unified framework in an algebraic setting.

Multicommodity network flow problems over a (directed) graph $G = (N,A)$ with $|N| = m$ and $|A| = n$ have the following constraint matrix:

$$A_k \underline{x}^k = b^k; \qquad k = 1,2,\ldots,r \tag{1}$$

$$\sum_k \underline{x}^k \leq \underline{\alpha} \tag{2}$$

where k is the index for commodities, $\underline{b}^k$ is an integer supply-demand vector, $\underline{x}^k \geq \underline{0}$ the vector of flows on arcs for commodity k, $\underline{\alpha}$ is an n-vector of capacities, and A_k is an $(m-1) \times n$ matrix with rank equal to m-1, obtained by deleting one row from the $m \times n$ node-arc incidence matrix for commodity k. A_k is totally unimodular for each k, although the system (1) - (2) is generally not.

For each k, there exists a basis B_k corresponding to a spanning tree of G. Thus (1) may be expressed as

$$B_k \underline{x}_B^k + R_k \underline{x}_R^k = \underline{b}^k \tag{1'}$$

or

$$\underline{x}_B^k = B_k^{-1} \underline{b}^k - Y^k \underline{x}_R^k$$

where

$$Y^k = B_k^{-1} R_k \qquad \text{for } k = 1,2,\ldots,r.$$

Likewise, (2) may be expressed as

$$\sum_k \underline{x}_B^k + \sum_k \underline{x}_R^k \leq \underline{\alpha} \tag{2'}$$

Substituting $\underline{x}_B^k$ from (1') into (2') we obtain

$$\sum_k S_k \underline{x}_R^k \le \underline{\alpha} - \sum_k B_k^{-1} \underline{b}^k \tag{2"}$$

where

$$S_k = E_k - Y_k.$$

The columns of E_k are unit vectors corresponding to the components of $\underline{x}_R^k$. The columns of each S_k are vector representations of cycles in G. We shall define *cycle* as meaning elementary cycle, i.e., as one traverses the cycle, no vertex is repeated except the initial and final ones. Suppose that a basic solution to (2") is $\underline{x}_{R(B)}^k$ for each k. Clearly, if the $\underline{x}_{R(B)}^k$ are integer-valued, then so are $\underline{x}_B^k$ (from 1') since B_k is totally unimodular. The flows $\underline{x}_B^k$ are imposed over the cycles represented by the corresponding columns of S_k. While each S_k is totally unimodular, the set $\{S_1, S_2, \ldots, S_r\}$ is generally not. Thus an integer-valued basic solution to (1) - (2) may not be guaranteed except for the case where $B_1 = B_2 = \ldots = B_r$.

Nonunimodularity, then, arises from the fact that the basic cycles (corresponding to the columns of S_k) do not form a *fundamental set* of cycles relative to the same spanning tree (which occurs when some $B_i \neq B_j$). In graph theory terminology, a fundamental cycle relative to a spanning tree T is a cycle formed when an out-of-tree arc (chord) is added to T. The set of cycles formed by adding chords to T one at a time is called a fundamental set of cycles and can be shown to form a basis for the cycle subspace, i.e., any other cycle is a linear combination of the vector representations of a fundamental set of cycles. A fundamental cycle matrix is totally unimodular; however, a set of independent cycles that are fundamental relative to *different* trees may not have a unimodular matrix representation. These are precisely the types of cycles that appear in the set $\{S_1, S_2, \ldots, S_r\}$.

A Group Characterization of Directed Cycles

Let G be any directed graph. Define H as a matrix whose rows are the vector representations of *all* directed cycles and arc-disjoint unions of cycles (unions of cycles with no common arcs) of G. Each S_k, then, is a subset of rows of H. If H is totally unimodular, it follows that $\{S_1, S_2, \ldots, S_r\}$ is totally unimodular, and hence all basic solutions to the system (1) - (2) are integer valued.

Let f and g be the vectors corresponding to two rows of H. $f(e_j)$ is the coefficient of arc e_j in f. The coefficients are determined as follows. Give each cycle an arbitrary orientation on G. The coefficient $f(e_j)$ is +1 (-1) if e_j is a forward (reverse) arc with respect to the given orientation. We may represent the cycle corresponding to f by the expression $\sum_j f(e_j)e_j$. Let $|f|$ denote the set of arcs in G corresponding to the cycle, i.e., $|f| = \{e_j | f(e_j) \neq 0\}$. Define a binary operation $*$ as follows. Let e be any member of $|f| \cap |g|$. If $|f| \cap |g| = \emptyset$, then define $f*g = f+g$ where + is the usual addition operator. If $e \in |f| \cap |g|$ and $f(e) = +1$ and $g(e) = -1$ or vice-versa, then $f*g = f+g$. If $f(e) = g(e) = +1$ or $f(e) = g(e) = -1$, then $f*g = f+(-g)$.

The system $(H,*)$ may not satisfy the property of closure for an arbitrary graph. $(H,*)$ is closed if $f, g \in H$ implies $f*g \in H$, for all pairs f,g. Consider the graph in Figure 1. Let $f = e_1 - e_2 + e_4$ and $g = e_3 + e_5 + e_2$. Since $|f| \cap |g| = \{e_2\}$, $f(e_2) = -1$, and $g(e_2) = +1$, then $f*g = e_4 + e_1 + e_3 + e_5$ and $f*g \in H$. However, let $f = -e_1 - e_6 + e_5 + e_2$ and $g = e_1 + e_3 + e_4 + e_5$. Then $|f| \cap |g| = \{e_1, e_5\}$. Defining $f*g$ with respect to e_1 we obtain $f*g = e_3 - e_6 + 2e_5 + e_4 \notin H$; with respect to e_5 we obtain $f*g = -2e_1 - e_6 - e_3 + e_2 - e_4 \notin H$. Therefore $(H,*)$ is not closed on this graph. We remark that one may also interpret the operation $*$ in a graph-theoretic fashion. Assign a label of (+) or (-) to each arc e_j in each cycle depending on whether the coefficient $f(e_j)$ is +1 or -1. Let $E = \{e_j | e_j \in |f| \cap |g|\}$. Then if every member of E has either (i) identical labels, or (ii) opposite labels in both cycles, then $f*g \in H$ and is a cycle.

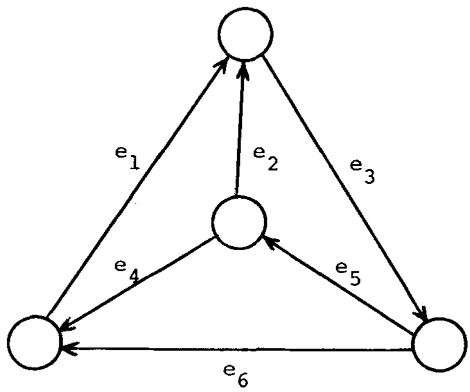

Fig. 1

Theorem 1: Let $P \subseteq H$. If $(P,*)$ is closed, then the algebraic system $(P,*)$ is a group.

Proof: Let f, g, $k \in P$. Since $f*g = h \in P$ and $g*k = \ell \in P$, it follows that

$$(f*g)*k = \pm f \pm g \pm k = f*(g*k)$$

since the operation * is simply the addition or subtraction of real vectors which is clearly associative. The zero vector is the identity element, and $f^{-1} = (-f)$ since $f*(-f) = 0$. Therefore, all axioms for a group are satisfied and the theorem is complete.

Q.E.D.

Theorem 2: If $(H,*)$ is a group, then H is totally unimodular.

Proof: The proof relies on a unimodularity condition developed by Commoner [2]. The reader is referred to the Appendix for a discussion of this result. Consider H^t (t denotes transpose). Each row corresponds to an arc of G and each column to a cycle of G. Construct the bipartite graph $K(H^t)$ according to Commoner's development. An elementary cycle of n arcs of $K(H^t)$ is represented by a submatrix A of H^t of the form:

$$
A = \begin{array}{c} \\ 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ n-1 \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1' & 2' & 3' & \cdots & (n-1)' & n' \end{array} \\
\left[\begin{array}{cccccc}
1 & a_{12} & & & & \\
 & a_{22} & a_{23} & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & a_{n-1,n-1} & a_{n-1,n} \\
a_{n1} & & & & & a_{nn}
\end{array}\right]
\end{array}
$$

where the a_{ij} entries specified are ± 1, and the remaining entries of A are 0. The row and column labels $1,\ldots,n$ and $1',\ldots,n'$ correspond to nodes of $K(H^t)$. We will show that if an elementary cycle of $K(H^t)$ has sign -1, then (H,*) is not a group. The submatrix A corresponds to a cycle of the form given in Figure 2, where arc (1,1') is directed from node 1 to node 1' and the remaining arc directions are determined by whether $a_{ij} = +1$ or -1. Traversing the cycle in the direction of arc (1,1') we find that it has sign -1 if there is an odd number of arcs oriented opposite to this direction. Now, if all arcs were oriented in the same direction as (1,1') then all diagonal elements of A would be +1 and all off diagonal elements would be -1. Therefore, any cycle with sign -1 has an odd number of negative diagonal and positive off diagonal elements. If the columns are multiplied by -1 in an attempt to obtain a +1 and a -1 in each row, there will always exist a row in which both nonzero elements have the same sign. But this contradicts the group characterization and the proof is complete.

Q.E.D.

Theorem 2, then, provides a sufficient condition for unimodularity in multicommodity networks based upon topological characteristics of networks. To determine if (H,*) is closed, one would have to examine every pair of cycles in the graph G = (N,A). This could be accomplished in a systematic fashion, though not very efficiently, by choosing a set of $q = |A| - |N| + 1$ fundamental cycles to generate all cycles in the network and then examining all pairs. Since the number of elementary cycles is

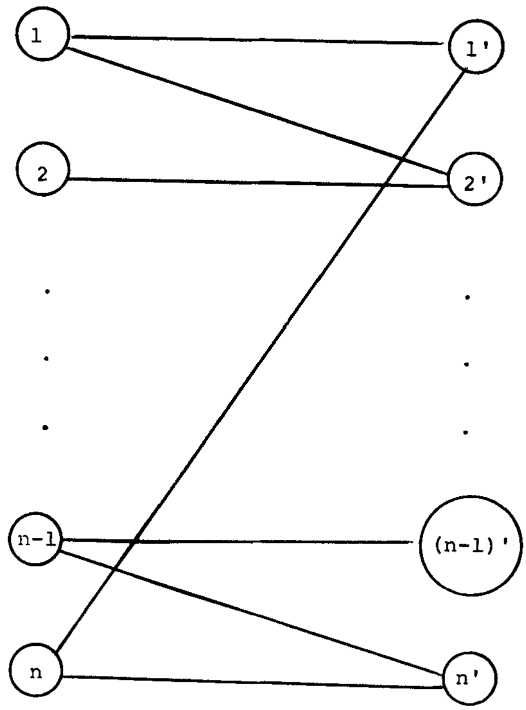

Fig. 2

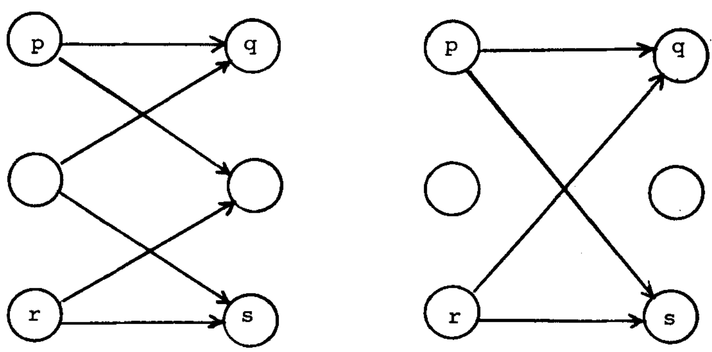

Fig. 3

bounded by $2^q - 1$, one would have to examine at most $(2^q-1)(2^{q-1}-1)$ pairs of cycles. In general, this is an extensive amount of computation. An alternate procedure is to search for a pair of cycles f and g such that $f*g \notin H$. If $(H,*)$ is not closed, then there must exist a pair of arcs with the following property. Let (p,q) and (r,s) be the pair of arcs. Then there exists a pair of arc-disjoint paths from p to s and q to r, and also a pair of arc-disjoint paths from p to r and q to s none of which use (p,q) or (r,s). This situation is illustrated in Figure 3 with a 3×3 transportation network. The fact that there may exist several paths with common arcs between pairs of nodes makes finding an appropriate pair of edge-disjoint paths rather difficult.

Despite the lack of a good algorithm for verifying the condition of Theorem 1, the characterization presented here can be used to prove unimodularity for certain classes of problems in which one can take advantage of special network structures and rather easily verify the condition for arbitrary pairs of cycles. This is illustrated in some examples in the next section.

EXAMPLES

Example 1

In [9], Evans, Jarvis, and Duke proved that a necessary and sufficient condition for a multicommodity transportation problem to always have integer solutions is that the number of sources or sink be not greater than two. We shall establish sufficiency for a two source problem by choosing an arbitrary pair of cycles and verifying that $f*g \in H$.

Consider the network in Figure 4, denoted by K(2,n).

Proposition: Any cycle of K(2,n) has exactly 4 arcs.

Proof: Obvious.

Q.E.D.

Proposition: If f and g are any two independent cycles of K(2,n), then $p = ||f| \cap |g||$ *is 0, 1, or 2.*

Proof: By the previous proposition, if $p = 4$ then $f = g$ and hence not independent. Since any set of three arcs that is a subset of a cycle of K(2,n) forms a tree on the subset of incident nodes of K(2,n), there is a unique arc that completes the cycle. Hence either $f = g$ or, f or g is not a cycle.

To establish the integrality property, note that if $p = 0$ or 1 then $f*g$ is clearly defined. (This is true for any graph.) We need only consider the case when $p = 2$. Let f be any cycle of $K(2,n)$ as shown in Figure 5. If $p = 2$, then any cycle g contains one of the following pairs of arcs: (1,2), (1,3), (1,4), (2,3), (2,4), (3,4). Elements within the sets $\{(1,2),(3,4)\}$, $\{(1,3),(2,4)\}$, and $\{(1,4),(2,3)\}$ are topologically indistinguishable. By considering each set, it is trivial to verify that $f*g \in H$.

Q.E.D.

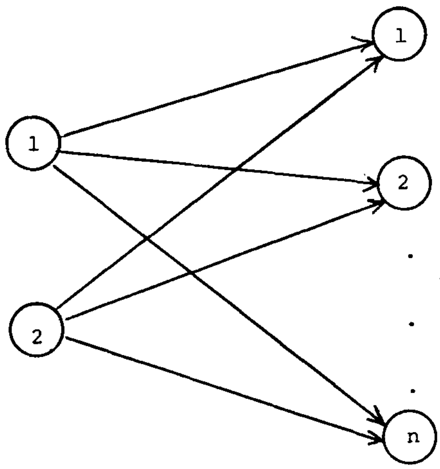

Fig. 4

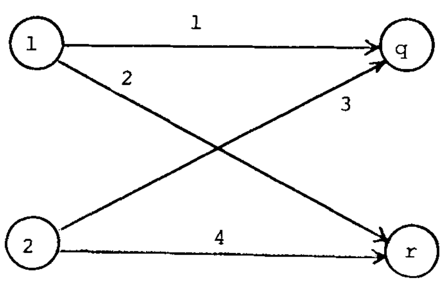

Fig. 5

Example 2

In [5] and [7], certain multiproduct dynamic production planning problems are investigated. A single stage problem is shown in Figure 6. Nodes 1 through T represent time periods, an arc from node 0 to node k represents production in period k, and arcs from nodes k to k+1 represent inventory carryover. All arcs are capacitated. In this case, one can readily see that $f*g \varepsilon H$ for any pair of cycles since all arcs of the form (k,j) where $k < j$ can be labeled (+) whereas the remaining arc in the cycle is labeled (-).

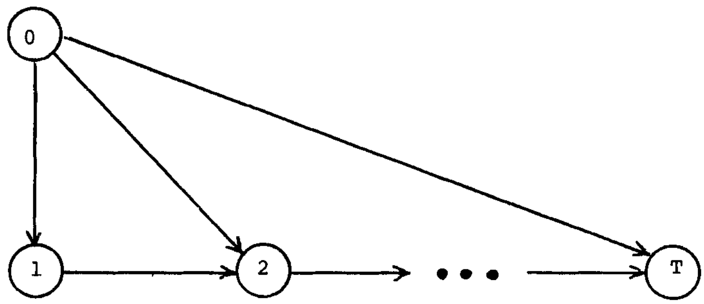

Fig. 6

Example 3

For the extension to multistage models, (H,*) is not closed. For the two-stage, three-period problem in Figure 7, one can easily find two arcs (0,1) and (2,5) such that there exists appropriate pairs of arc-disjoint paths between their pairs of nodes (Figure 8). Although integral solutions may exist, we have no guarantee.

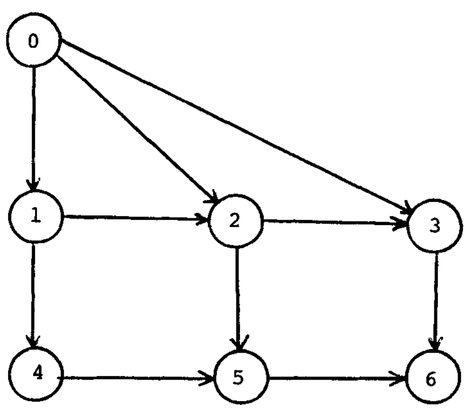

Fig. 7

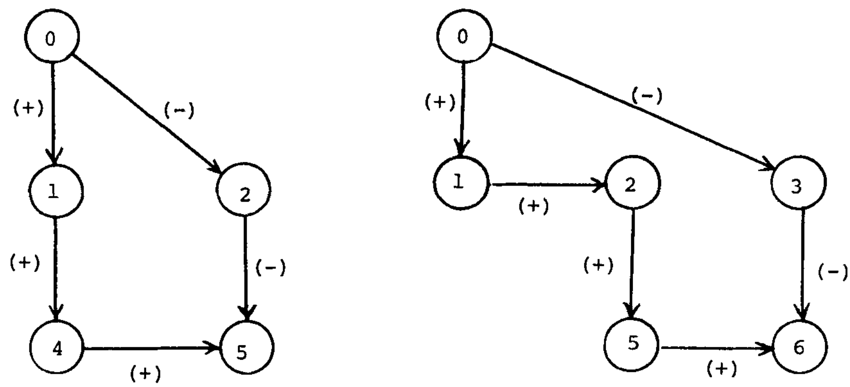

Fig. 8

APPENDIX

We briefly review the sufficient condition for total unimodularity given by Commoner [2]. Given a matrix A consisting of 1, -1 and 0, construct a directed bipartite graph K(A) by associating a vertex with every row and column, and drawing an arc between a row vertex and a column vertex if the corresponding entry in A is nonzero. The arc is directed from the row to column vertices if the entry is +1; and from column to row if it is -1. Give each cycle of K(A) an arbitrary orientation. Assign to each arc of the cycle a +1 or -1 depending on whether it is oriented in the same or opposite direction as the cycle. The product of these numbers, taken over the arcs of the cycle, is called the *sign* of the cycle. Commoner's main result is: If every elementary cycle of K(A) has sign 1, then A is totally unimodular.

ACKNOWLEDGEMENT

We are indebted to an anonymous referee for his comments and improvement of the organization of this paper.

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Paper received June 14, 1976.