For undirected graphs

Adjacency matrix $A = A^T$

If we sum all the rows of A and put them to the diagonal, we get Degree Matrix D

$$D_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{j=1}^{N} A_{ij} & \text{otherwise} \end{cases} \quad \text{for } i,j \in \mathbb{Z}_{[1,N]}$$

$D = D^T \because D$ is a diagonal matrix

"Blocked structures in A for disconnected graphs":

Has a big impact on matrix A's eigenvalues & eigenvectors.

$$A = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{0} \end{matrix} \end{bmatrix}$$

Eigendecomposition / Spectral Decomposition

Theorem: If a matrix X is REAL and SYMMETRIC, then $\exists\, P, \Lambda$ s.t. $X = P \Lambda P^{-1}$, where Λ is a diagonal matrix

$L_{un} := D - W$

$\Rightarrow L_{un} = L_{un}^T$

$\Rightarrow$ Eigenvalues of L_{un} are real & non-negative if all values in A are positive $\because$

$\Rightarrow$ For some vector $\underset{\sim}{x}$, $\quad \underset{\sim}{x}^T L_{un} \underset{\sim}{x} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{ij} (x_i - x_j)^2$

$\Rightarrow L_{un}$ is positive semi-definite if all values in A are positive

$\because x^T L_{un} x = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{ij} (x_i - x_j)^2, \quad \therefore \forall x, \; l_{ij} (x_i - x_j)^2 \geq 0$

$\Rightarrow$ Each row in L_{un} sums to $0 \Rightarrow L_{un} \underset{\sim}{1} = \underset{\sim}{0} = 0\,\underset{\sim}{1}$ $\Rightarrow \therefore$ There is an (Eigenvector, Eigenvalue) pair of $(\lambda = 0, \; v = \underset{\sim}{1})$

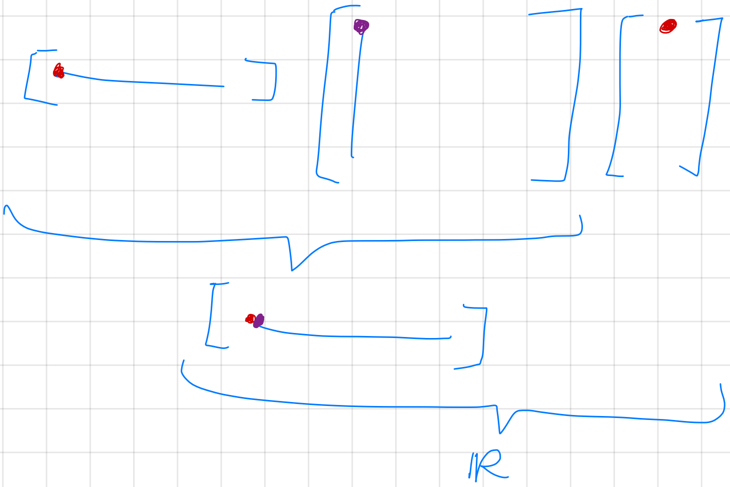
ones vector — zeros vector — scalar — ones vector

$\Rightarrow$ The multiplicity of the eigenvalue $\lambda = 0$ reveals the # of connected components in graph G.

$\Rightarrow$ The 2nd smallest eigenvalue (aka the Fiedler value) measures how well connected the graph is

$\Rightarrow$ The eigenvector associated with the Fiedler's value can be used to partition graph G. $\Leftarrow$ known as spectral clustering

$x^T A x =$



$$\boxed{x^T A x = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}\, x_i\, x_j}$$