

Thus our integral serves all the needs of integration with respect to distributions in finite-dimensional spaces.

However, such processes as infinite sequences of tosses of a coin occur in fairly elementary probability theory, and at a more advanced level we meet stochastic processes, or random functions. An infinite sequence of real numbers is a point of R^Z , where $Z = \{1, 2, 3, \dots\}$; a function on a set T is a point of R^T . So the probability theory of such processes calls for distributions and integration over infinite-dimensional spaces R^T (where T may be Z). If we define intervals and neighborhoods in R^T in the manner that has long been customary in topology, we find that Definition 2.1 applies to this case also.

By now it should be clear that the only thing keeping us from going on and on with the full development of the Lebesgue integration theory is the fact that we have reached the end of our program of fitting the theory into undergraduate instruction (and, perhaps, of the editor's patience). Anything that can be proved about the Lebesgue integral can be proved about this integral, because it is the Lebesgue integral. And for those students, such as engineers, who lack time or inclination to work through detailed proofs, we are at least asking them to believe unproved statements about an integral they know, rather than about an integral whose very definition is unfamiliar to them.

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HIGHLIGHTS IN THE HISTORY OF SPECTRAL THEORY

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Not least because such different objects as atoms, operators and algebras all possess spectra, the evolution of spectral theory is one of the most informative chapters in the history of contemporary mathematics. The central thrust of the modern spectral theorem is that certain linear operators on infinite dimensional

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spaces can be represented in a “diagonal” form. At the beginning of the twentieth century neither this spectral theorem nor the word “spectrum” itself had entered the mathematician’s repertoire. Thus, although it has deep roots in the past, the mathematical theory of spectra is a distinctly twentieth century phenomenon.

Today every student of mathematics encounters the spectral theorem not later than his first course in functional analysis and often as early as his first course in linear algebra. Usually he studies one specimen of the spectral theorem, plucked out of historical context and imbedded in the logical context of his particular course. Although this scheme is pedagogically efficient and logically aesthetic, it does often obscure the fact that the spectral theorem was (and perhaps still is) an evolving species. Its evolution is an outstanding example of the counterpoint between pure and applied mathematics, for while the motive force in its evolution was the attempt to provide adequate mathematical theories for various physical phenomena, the forms through which it evolved are precisely those which have marked the development of modern abstract analysis.

So we offer here an austere outline of the evolution of the spectral theorem as a microcosmic example of the history of twentieth century mathematics. To understand the significance of contemporary achievements and to recognize their continuity with the past, we begin with the principal historical roots of our subject.

1. Principal axes theorem. The only theorem available at the turn of the twentieth century which we can with hindsight recognize as a direct forerunner of the modern spectral theorem is the principal axes theorem of analytical geometry. It should not be surprising that the simplest form of this theorem is contained in the writings of the founders of analytical geometry, Pierre de Fermat (1601–1665) and René Descartes (1596–1650). For the Euclidean plane R^2 , this theorem says that a quadratic form $ax^2 + 2bxy + cy^2$ can be transformed by a rotation of the plane into the normal form $\alpha x^2 + \beta y^2$, where the principal axes of the normal form coincide with the new coordinate axes. The essential content of this theorem—that the algebraic reduction to normal form corresponds to the geometric rotation onto principal axes—is contained in Descartes’ *La Géométrie* [1637], and was known at about the same time by Fermat but not published until after his death [1679]. The term “principal axes” was introduced by Leonhard Euler (1707–1783) in his investigation of the mechanics of rotating bodies [1765]; Euler also discussed (in [1748]) the reduction of quadratic forms in two and three dimensions.

The general form of the principal axes theorem asserts that any symmetric quadratic form $(Ax, x) = \sum \alpha_{ij}x_i x_j$ on R^n can be rewritten by means of an orthogonal transformation $T: R^n \rightarrow R^n$ in the normal form $\sum \lambda_i x_i^2$. (A is **symmetric** if $\alpha_{ij} = \alpha_{ji}$, and T is **orthogonal** if it leaves invariant the Euclidean metric on R^n .) The generalization from R^3 to R^n of the algebraic part of this theorem (that a quadratic form can be written as a sum of squares) was discussed by Joseph Louis Lagrange (1736–1813) in a paper [1759] on the maxima and minima of functions of several variables. In

[1827] Carl Gustav Jacob Jacobi [1804–1851] investigated the principal axes of various quadratic surfaces, and about the same time Augustin-Louis Cauchy (1789–1867) showed in [1829] and [1830] that the coefficients λ_i of the normal form of a symmetric quadratic form must be real.

But it was not until the second half of the nineteenth century that the general form of the principal axes theorem was achieved when James Joseph Sylvester (1814–1897) and Arthur Cayley (1821–1895) used the notation of matrices to systematize the algebraic description of n -dimensional space. In [1852] Sylvester showed explicitly that the coefficients λ_i in the normal form of (Ax, x) are the roots of the characteristic polynomial $\det(\lambda I - A) = 0$; in [1858] Cayley inaugurated the calculus of matrices, in which the reduction to normal form corresponded to a diagonalization process on the matrix A . Specifically, the principal axes theorem says in the language of matrices that each symmetric real matrix A is orthogonally equivalent to a diagonal matrix D ; in other words, for some orthogonal matrix T , the matrix $D = T^{-1}AT$ is in diagonal form. The diagonal entries of D are the eigenvalues of A , that is, the roots of the polynomial equation $\det(\lambda I - A) = 0$.

Although the new concepts of matrix theory had an immediate and profound influence on British mathematics, their impact on the continent was relatively minor. Especially in Germany bilinear forms continued well into the twentieth century to be the principal tool of analytical geometry, and in [1878] Georg Frobenius (1849–1917) published a systematic account of matrix algebra entirely in the language of bilinear forms. So by the end of the nineteenth century we can discern two versions of the principal axes theorem: the reduction to normal form of a symmetric bilinear form, and the diagonalization of a real symmetric matrix.

2. Infinite systems of linear equations. The central fact of modern spectral theory is that certain linear operators on infinite dimensional spaces can also be presented in “diagonal” form. Thus the second historical taproot of spectral theory is the evolution of infinite dimensional theory from finite dimensional cases. This evolution occurred first in algebra—in the solution of systems of linear equations—and only much later in geometry. Finite systems of linear equations were solved most often throughout the eighteenth and nineteenth centuries by the method of elimination, as expounded, for instance, in [1770] and [1779] by Euler and Etienne Bézout (1730–1783). In [1750] Gabriel Cramer (1704–1752) introduced for 3×3 systems the rule which now bears his name, although he did not, of course, use the concept or notation of determinants.

Infinite systems of equations were used throughout the eighteenth and nineteenth centuries to obtain formal solutions to differential equations by the method of undetermined coefficients: if a formal power series with unknown coefficients is substituted for the unknown in a given differential equation, the task of solving the differential equation is reduced to that of determining the infinitely many unknown coefficients. (Of course few at that time worried very much about the convergence of

the power series thus obtained.) If all went well, the infinite system of equations in the unknown coefficients would exhibit a recursive pattern which made it possible to solve the infinite system by finite dimensional tools. But for this reason precisely, these recursive techniques contributed little to the development of a general theory of infinite dimensional systems.

Joseph Fourier (1768–1830) launched the first significant general attack on the problem of infinite systems of equations when he attempted to show [1822] that every function can be expressed as an infinite linear combination of trigonometric terms. The problem of determining the unknown coefficients in these linear combinations led him directly to the general problem of solving an infinite system of linear equations. Fourier's approach (called the *principe des réduites* by Frédéric Riesz [1913a]) was to solve the first $n \times n$ system by ordinary means and let $n \rightarrow \infty$.

Although Fourier's assertion about the expansion of “arbitrary” functions into trigonometric series stimulated intense work on the theory of integration, his method of solving infinite systems of linear equations was virtually ignored. More than fifty years passed before Theodor Kötteritzsch of Saxony reopened the investigation with a paper [1870] in which he attempted to extend Cramer's rule to infinite systems. Seven years later the American astronomer George William Hill (1838–1914) published in Cambridge, Massachusetts, a monograph [1877b] in which he successfully applied to the infinite dimensional case the theory of determinants which had at that time only been established for finite dimensional systems. Hill's work was first disseminated in Europe in [1886a] when G. Mittag-Leffler reprinted it in *Acta Mathematica* in the year following the appearance in France of a paper [1885a] by Paul Appell (1855–1930) in which he applied the *principe des réduites* to determine the coefficients of the power series expansion of elliptic functions.

At this point Henri Poincaré (1854–1912) entered the discussion with two papers ([1885b], [1886b]) in which he provided a rigorous definition for an infinite determinant in order to clarify the works of Hill and Appell. The work begun in Paris by Poincaré was continued in Stockholm by Helge von Koch (1870–1924) who developed between 1890 and 1910 an extensive theory of infinite determinants. Von Koch's first major papers on this subject appeared in [1891] and [1892]; his own survey of the field in [1910d] provides further references. The more recent survey [1968] by Michael Bernkopf includes a complete discussion of these fundamental papers.

3. Integral equations. The theory of infinite matrices and determinants might have led directly to an elementary spectral theorem if someone had generalized the diagonalization form of the principal axes theorem. But the road to spectral theory was not that straight: the first spectral theorem was achieved only after infinite determinants were applied to integral equations, thereby extending the theory from the countably to the uncountably infinite. The formal study of integral equations is usually traced back to [1823] and [1826] when the young Norwegian genius Niels Henrik Abel (1802–1829) used an integral equation to solve a generalized tautochrone

problem concerning the shape of a wire along which a frictionless bead slides under the influence of gravity. Somewhat later Joseph Liouville (1809–1882) introduced (in [1837]) the method of iteration to solve a specific type of integral equation; in [1877a] Carl Neumann (1832–1925) extended Liouville's iterative method to a more general setting while investigating a boundary value problem for harmonic functions.

Neumann's work precipitated considerable research in integral equations, especially by Poincaré in France and in Rome by Vito Volterra (1860–1940). But it was not until 1900 that the theory of integral equations became especially relevant to the history of spectral theory, for in that year the Swedish mathematician Ivar Fredholm (1866–1927), then a docent at the University of Stockholm, applied to integral equations the theory of infinite matrices and determinants as developed by his colleague von Koch. By mimicking von Koch's technique for expanding infinite determinants, Fredholm developed in [1900] his now famous “alternative” theorem concerning the solutions ϕ of the integral equation

$$(1) \quad \phi(x) + \int_0^1 K(x,y)\phi(y)dy = \psi(x), \quad (0 \leq x \leq 1).$$

Just as Daniel Bernoulli (1700–1784) nearly two centuries earlier had represented the vibrating string as the limit of n oscillating particles [1732], so Fredholm considered the integral equation (1) to be the limiting case of the corresponding linear system

$$(2) \quad \phi(x_i) + \sum_{j=1}^n K(x_i, y_j) \phi(y_j) = \psi(x_i), \quad (1 \leq i \leq n).$$

Fredholm defined a “determinant” D_K for the integral equation (1) which is the continuous analog of the classical determinant of the $n \times n$ system (2) and showed—in exact analogy to the classical theory for (2)—that the integral equation (1) has a unique solution which can be expressed as the quotient of two “determinants” whenever $D_K \neq 0$; or alternatively, if $D_K = 0$, then the transposed homogeneous equation $\phi(x) + \int_0^1 K(y, x) \phi(y) dy = 0$ has nontrivial solutions and (1) is solvable if and only if ψ is orthogonal to each of these solutions. Fredholm's major paper on this subject appeared in [1903a]; a summary of this work together with later developments is the substance of his survey article [1910e].

4. David Hilbert. Although there is very little in the papers of either von Koch or Fredholm that could be construed as a logical ancestor of the modern spectral theorem, we have discussed these developments for two particular reasons—one mathematical, the other historical. The twentieth century evolution of infinite dimensional spectral theory from the much simpler finite dimensional theory is foreshadowed by the nineteenth century development of linear equation and determinant theory, from the finite to the infinite (von Koch) to the continuous (Fredholm). But there is even a more direct connection, for when Fredholm's ideas were introduced (by Fredholm's colleague Eric Holmgren) in David Hilbert's 1900–01 seminar at

Göttingen, Hilbert, in the words of Hermann Weyl [1944], "caught fire at once". For the next ten years Hilbert (1862–1943) focused his impressive mathematical talent exclusively on integral equations, and through a series of six papers published in *Göttingen Nachrichten* from 1904 to 1910 (collected and published as one volume in [1912a]) he outlined the basic definitions and theorems of spectral theory (which he named) and Hilbert space theory (which he did not name, or even define directly).

Hilbert worked primarily with the integral equation

$$(3) \quad \phi(x) - \lambda \int_0^1 K(x, y) \phi(y) dy = \psi(x)$$

together with the analogous finite or infinite dimensional matrix equation

$$(4) \quad \phi(x_i) - \lambda \sum_j K(x_i, y_j) \phi(y_j) = \psi(x_i).$$

In the process of constructing the machinery necessary to solve these equations, Hilbert defined the spectrum of the quadratic form K , distinguished the point spectrum from the continuous spectrum, and defined the concept of complete continuity which served to separate those forms that had pure point spectra from those with more complicated spectra. But most important from the viewpoint of this essay, he formulated and proved the spectral theorem—not only for completely continuous forms, but for bounded forms as well.

Hilbert's papers on integral equations contain an astonishing quantity of what we now recognize as modern analysis in classical language. Because he was primarily concerned with solving integral equations, Hilbert never applied his results specifically to matrices or operators; furthermore, because of the position of the parameter λ in equation (3), all of Hilbert's eigenvalues and spectral points are reciprocals of those in use today. And while his theorems had a most modern thrust, his basic method of proof was that of Bernoulli and Fredholm—a laborious passage to the limit from the corresponding finite case.

Beginning in 1905 with his doctoral dissertation under Hilbert, Erhard Schmidt (1876–1959) generalized and simplified Hilbert's work by introducing the suggestive language of Euclidean geometry. In [1907a], [1907b] and [1908a] Schmidt presented a definitive theory of "Hilbert's space"—what we now call l^2 , the space of square summable sequences—replete with the language of norms, linearity, subspaces and orthogonal projections. (It was Schmidt who generalized to l^2 the iterative algorithm for orthonormalization first introduced in [1883] by Jørgen Pederson Gram of Copenhagen.) Schmidt's conceptual simplifications were immediately incorporated by Ernst Hellinger (1883–1950) and Hermann Weyl (1885–1955) in their 1907 and 1908 dissertations under Hilbert. In [1909a] Hellinger reformulated the theory of quadratic forms in the new language of Hilbert and Schmidt, and in the same year Weyl published an extensive study of bounded forms and their spectra [1909d]. So

by the end of the first decade of the twentieth century we can perceive in the writings of Hilbert and his pupils the major part of spectral theory for bounded linear transformation on l^2 .

5. Hilbert-Schmidt spectral theory. Recall, as did Hilbert at the beginning of his first paper on integral equations [1904a], the principal axes theorem for finite dimensional spaces. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the n (real) eigenvalues of the symmetric $n \times n$ matrix K , listed according to multiplicity. Let $\phi_1 \dots \phi_n$ be an orthonormal collection of corresponding eigenvectors, so $K\phi_i = \lambda_i\phi_i$ for $1 \leq i \leq n$. Then the action of K is represented, with respect to the basis $\phi_1 \dots \phi_n$, by the diagonal matrix L with entries λ_i on the main diagonal. The matrix T whose rows are the vectors $\phi_1 \dots \phi_n$ is an orthogonal transformation which maps the new basis vectors $\phi_1 \dots \phi_n$ back to the original (canonical) basis vectors. Thus $L = T^{-1}K T$ is the diagonalization of K by the orthogonal transformation T . The matrix L can be written as $\sum_{i=1}^n \lambda_i P_i$ where P_i is the **projection** (i.e., the transformation which projects R^n) onto the one dimensional subspace spanned by ϕ_i .

Hilbert's first step in extending this theorem was to generalize the concept of eigenvalue to the case of an infinite symmetric form K . His new concept was the **spectrum** of K , denoted by $\sigma(K)$, which is the set of λ for which the transformation $\lambda I - K$ is not invertible. (Actually Hilbert used $I - \lambda K$ while Schmidt used $\lambda I - K$.) The subset of $\sigma(K)$ consisting of those λ for which the equation $K\phi = \lambda\phi$ has non-trivial solutions is called the **point spectrum** of K ; this is the strict analog of the set of eigenvalues. The complement of the point spectrum in $\sigma(K)$ is called the **continuous spectrum**. Much of Hilbert's fourth paper [1906a] is devoted to a study of the relationships between a transformation K and its spectrum $\sigma(K)$.

One of the simplest relationships Hilbert discovered was that the spectrum of K is a bounded set whenever K is a bounded transformation—that is, whenever the set $S = \{\|Kx\| : \|x\| \leq 1\}$ is bounded, where the notation $\|\cdot\|$, due to Schmidt, is the l^2 norm. In fact, whenever K is symmetric, the least upper bound of S , called the bound (or norm) of K and denoted by $\|K\|$, is the same as the least upper bound of $\{|\lambda| : \lambda \in \sigma(K)\}$; this fact is now called the spectral radius theorem. The bounded linear transformations on l^2 are important from another point of view, also due to Hilbert: they are precisely the continuous linear transformations, in the sense that they preserve strong convergence (i.e., $\|Kx_n - Kx\| \rightarrow 0$ whenever $\|x_n - x\| \rightarrow 0$).

Hilbert extended the principal axes theorem to symmetric bounded linear transformations; the spectra of these transformations are bounded subsets of the real axis. Those λ in the point spectrum $\rho(K)$ of K are like eigenvalues since there exists an orthonormal collection of corresponding eigenvectors ϕ_λ satisfying $K\phi_\lambda = \lambda\phi_\lambda$. If P_λ denotes the projection onto the subspace generated by ϕ_λ , we can form the diagonal transformation $L = \sum \lambda P_\lambda$ where λ ranges over the point spectrum $\rho(K)$. The transformation L reflects accurately the action of K on the subspace generated by the eigenvectors ϕ_λ , but since this subspace will in general be strictly smaller than

l^2 —since we have omitted the continuous spectrum—we cannot say that L and K represent the same transformation.

To express the contribution of the continuous spectrum, Hilbert set up an integral patterned after one defined in [1894] by the Dutch mathematician Thomas-Jean Stieltjes (1856–1894). In his study of continued fractions, Stieltjes was led (via the problem of moments) to the integral $\int_a^b f(x) dg(x)$ as the limit of the sum $\sum f(\xi_i) [g(x_i) - g(x_{i-1})]$ (for continuous f and increasing g). By rewriting the sum $\sum \lambda_i P_{\lambda_i}$ as $\sum \lambda_i [E_{\lambda_i} - E_{\lambda_{i-1}}]$, where $E_{\lambda_i} = \sum_{j=1}^i P_{\lambda_j}$, Hilbert constructed for the continuous spectrum $s(K)$ the Stieltjes-type integral $\int_{s(K)} \lambda dE_\lambda$ as the limit of sums of the form $\sum \lambda_i [E_{\lambda_i} - E_{\lambda_{i-1}}]$. Then Hilbert's spectral theorem was that every symmetric bounded linear transformation on l^2 can be represented (by means of an orthogonal transformation) in the “diagonal” form

$$(5) \quad \sum_{\rho(K)} \lambda P_\lambda + \int_{s(K)} \lambda dE_\lambda$$

where the summation is over the point spectrum, and the integral is over the continuous spectrum.

Hilbert completed his spectral theory by identifying a large class of transformations whose continuous spectra were empty. He called these transformations **completely continuous**, and Schmidt characterized them by the property of mapping weakly convergent sequences to strongly convergent sequences. In other words, the linear transformation K is completely continuous if

$$\|Kx_n - Kx\| \rightarrow 0$$

whenever $(y, x_n) \rightarrow (y, x)$ for all y . The completely continuous transformations are the nearest infinite dimensional analog to the finite dimensional transformations, since their spectra consist entirely of eigenvalues with zero as the only possible accumulation point; furthermore, every completely continuous symmetric linear transformation K can be expressed (by an orthogonal transformation) in the diagonal form $\sum \lambda P_\lambda$ (since $s(K) = \emptyset$).

Although Hilbert originally used infinite matrices merely as convenient approximations to integral equations, he concluded his theoretical investigation by establishing a major link between these two theories, namely that of a **complete orthogonal system**. Such a system $\{\phi_n\}$, either of vectors in the sequence space l^2 or of continuous functions on the interval $[0, 1]$, is characterized by the orthogonality relation $(\phi_n, \phi_m) = 0$ if $n \neq m$, together with the fact that every vector (or continuous function) ϕ can be represented by the Fourier-type series $\phi = \sum_{n=1}^{\infty} a_n \phi_n$. The matrix equation (4) can then be derived (by mathematics, rather than by analogy) from the integral equation (3) by replacing each continuous function ϕ, ψ by its Fourier expansion with respect to the complete orthogonal system $\{\phi_n\}$. This application of a complete orthogonal system enabled Hilbert to derive Fredholm's alternative

theorem for the integral equation (1) directly from the corresponding theorem for the infinite linear system (2).

To keep the record straight, we should emphasize again that Hilbert introduced spectral theory in the language of quadratic forms, whereas we have reported his work primarily in the language of linear transformations on the infinite dimensional space l^2 . Barely fifty years had elapsed since Cayley in England and Hermann Grassman (1809–1877) in Germany had begun, in [1843] and [1844] the systematic study of Euclidean n -dimensional space for $n > 3$. Hilbert and Schmidt were the first to explore the totally unknown depths of an infinite dimensional space and it was not until other such spaces were studied that the broad outlines of a theory of linear transformations became clear. The early twentieth century development of infinite dimensional (function) spaces is recorded in [1966].

6. The Lebesgue Integral. At about the same time as Hilbert was creating his spectral theory for spaces of square summable sequences, Henri Lebesgue (1875–1941) was developing the new integral which now bears his name ([1901], [1904c]). In three brief papers in 1907 Friedrich Riesz (1880–1956) and Ernst Fischer (1875–1959) joined together the works of Hilbert and Lebesgue by showing that Hilbert's space l^2 is isomorphic to the space L^2 of functions whose square is Lebesgue integrable. In a subsequent paper [1910c] (in which he introduced the more general L^p spaces), Riesz derived a spectral theory for L^2 entirely analogous to that developed for l^2 by Hilbert and Schmidt.

In the year preceding the appearance of his paper on L^p spaces, Riesz proved in [1909b] his now famous representation theorem in which he solved a problem first studied in [1903b] by Jacques Hadamard (1865–1963). What Riesz showed was that every continuous linear functional on $C([a, b])$ is a Stieltjes integral $\int f dg$ with respect to some function g of bounded variation. Lebesgue then showed in [1910f], in direct response to Riesz's paper, that every Stieltjes integral can be interpreted as a Lebesgue integral under a proper interpretation of the heuristic formula

$$\int f(x)dg(x) = \int f(x)g'(x)dx.$$

This led Johann Radon (1887–1956) to develop (in [1913b]) integration with respect to a measure (i.e., a countably additive set function) thus encompassing the integrals of both Lebesgue and Stieltjes and providing the foundation for all modern theories of the abstract integral.

We can see from this digression that the evolution of the modern integral was closely connected to Hilbert's creation of spectral theory. Although neither theory depended logically on the other, the historical dependence of each on the other is quite clear: Hilbert used Stieltjes' integral to obtain the spectral theorem for l^2 , while Riesz, following Hilbert, used and thereby immortalized Lebesgue's integral by developing the spectral theory of L^2 .

The second decade of spectral theory was rather uneventful. In Göttingen, Hilbert had turned his attention to the axiomatization of physics, a task which he had proposed to the International Congress of Mathematicians in 1900 as the sixth of his famous 23 problems for twentieth century mathematics. "Physics," he said, "is much too hard for physicists" ([1970]). In the United States Eliakim Hastings Moore (1862–1932) at the University of Chicago developed a system of "general analysis" ([1908b], [1912b]) which was designed to include as special cases the work of Hilbert, Fredholm and Riesz. But Moore's results were constrained by the fact that European investigators were not then accustomed to receiving new mathematical ideas from America. So while Moore's research had a profound effect on the development of mathematics in the United States, it did not influence significantly the direction of research on spectral theory.

Many European efforts from 1910 to 1925 were devoted to exposition and recapitulation. Riesz [1913a], Fredholm [1910e] and von Koch [1910d] published surveys of the theory of infinitely many variables and integral equations, each of which contained various forms of Hilbert's spectral theory. Hilbert's collected papers on integral equations were themselves published in book form in [1912a]. But certainly the most impressive survey work of this period was the massive *Enzyklopädie der Mathematischen Wissenschaften* which contains in volume II.3.2. a comprehensive discussion of integral equations and spectral theory by Hellinger and Otto Toeplitz (1881–1940); this survey paper was also published separately [1928a].

7. Quantum mechanics. In Göttingen in 1925–26 Werner Heisenberg (1901–) and Erwin Schrödinger (1887–1961) created the theory of quantum mechanics. In Heisenberg's theory the physical fact that certain atomic observations cannot be made simultaneously was interpreted mathematically to mean that the operations which represented these observations were not commutative. Since the algebra of matrices is non-commutative, Heisenberg together with Max Born and Pascual Jordan ([1925a], [1926a]) represented each physical quantity by an appropriate (finite or infinite) matrix, called a transformation; the set of possible values of the physical quantity was the spectrum of the transformation. (So the spectrum of the transformation which represented the energy of an atom was precisely the spectrum of the atom.)

Schrödinger, in contrast, advanced a less unorthodox theory based on his partial differential wave equation. Following some initial surprise that Schrödinger's "wave mechanics" and Heisenberg's "matrix mechanics"—two theories with substantially different hypotheses—should yield the same results, Schrödinger unified the two approaches by showing, in effect, that the eigenvalues (or more generally, the spectrum) of the differential operator in Schrödinger's wave equation determine the corresponding Heisenberg matrix. Similar results were obtained simultaneously ([1925b], [1926b]) by the British physicist Paul A. M. Dirac (1902–). Thus interest in spectral theory once again became quite intense.

Hilbert himself was astonished that the spectra of his quadratic forms should come to be interpreted as atomic spectra. "I developed my theory of infinitely many variables from purely mathematical interests, and even called it 'spectral analysis' without any presentiment that it would later find an application to the actual spectrum of physics" [1970]. It quickly became clear, however, that Hilbert's spectral theory was the proper mathematical basis for the new mechanics. Finite and infinite matrices were interpreted as transformations on a Hilbert space (still thought of primarily as l^2 or L^2) and physical quantities were represented by these transformations. The mathematical machinery of quantum mechanics became that of spectral analysis and the renewed activity precipitated the publication by Aurel Wintner (1903–1958) of the first book [1929b] devoted to spectral theory.

Hilbert's original spectral theorem applied to real quadratic forms (or infinite matrices) that were bounded and symmetric. This theorem was quickly and easily extended (by Schmidt and others) to bounded complex matrices $A = (a_{ij})$ for which $a_{ij} = \bar{a}_{ji}$; such matrices are called **Hermitian** after the French mathematician Charles Hermite (1822–1901) who introduced them (in [1855]) and proved their eigenvalues real. Both symmetric and Hermitian forms may be characterized in terms of their respective inner product by the relation $(Ax, y) = (x, Ay)$ for all x, y . Like symmetric matrices, Hermitian transformations have real spectra and, more generally, play the role of the real number line in the algebra of transformations.

Almost miraculously, it was precisely the Hermitian transformations which qualified in the new mechanics to represent a physical quantity. One reason for this is that physical quantities are measured by real numbers, so it is natural to represent them by those transformations which behave like real numbers. Perhaps a more compelling justification is that the hypothesis that the transformations of mathematical physics are Hermitian implies certain fundamental laws (or assumptions) of physics: if A is Hermitian, the wave equation $\ddot{\phi} = A\phi$ implies the conservation of energy, a fundamental law of classical mechanics, and the solutions of Schrödinger's equation $\ddot{\phi} = iA\phi$ will have constant norm, which is a fundamental assumption of quantum mechanics.

Although every observable was represented in the new mechanics by a Hermitian transformation, it was not necessarily true that every such transformation represented an observable. Dirac [1930b] added the crucial hypothesis that a Hermitian transformation represents an observable if and only if its eigenvectors form a complete (orthogonal) system: his hypothesis was designed to insure that any vector (representing a quantum mechanical state) could be expressed as a (possibly infinite) linear combination of eigenvectors of any given observable. The identification of transformations with this property is part of the Hilbert-Schmidt spectral theory, but this theory provided only a partial answer: those Hermitian transformations which are completely continuous have a complete set of eigenvalues.

This theorem did not provide a satisfactory elucidation of Dirac's hypotheses

since the transformations of quantum mechanics are usually not completely continuous. Most of the important transformations in physics involve differentiation of, say, functions in L^2 . The theorem on integration by parts shows that differentiation is formally symmetric, for in this case $(Af, g) = (f, Ag)$ means $\int f'g = \int fg'$. But since the derivative of a function has practically no relation to the magnitude of the function, differentiation is neither continuous nor bounded, nor even defined everywhere. In fact, if a symmetric or Hermitian transformation (like differentiation) were defined everywhere, it would have to be bounded. This rather surprising result—which says, in effect, that a candidate for the spectral theorem which fails to be bounded must fail to be everywhere defined—was demonstrated as early as [1910b] by Hellinger and Toeplitz.

Thus many of the transformations of quantum mechanics, although Hermitian, failed nevertheless to satisfy the second of Hilbert's hypotheses, namely, that they be bounded. Like differentiation, they were unbounded and defined only on a dense subset of L^2 . Paul Dirac attempted to overcome the exceptional behavior of differentiation by introducing his δ -function to provide derivatives where none existed and thereby to enlarge the set of functions to which the differentiation transformation could be applied. Dirac's approach was highly successful in explaining the new quantum mechanics and led eventually to Laurent Schwartz' theory of distributions precisely because it lacked an adequate mathematical foundation. But in 1926 Dirac's approach represented more an alternative to spectral theory than an extension of it, and it did not really help to extend Hilbert's theory to unbounded transformations.

8. John von Neumann. After Hilbert, the only major study of unbounded transformations was that published in [1923] by Torsten Carleman (1892–1949) in Sweden. In this monograph Carleman showed that many of the results of Fredholm and Hilbert still hold under a weaker type of boundedness hypothesis. But from the viewpoint of spectral theory, the major breakthrough came in 1927–29 when the twenty-five year old Hungarian John von Neumann (1903–1957) revolutionized the study of spectral theory by introducing the abstract concept of a linear operator on Hilbert space. In [1927] von Neumann expressed the transformation theory of quantum mechanics in terms of operators on a Hilbert space, and explicitly recognized the need to extend from the bounded to the unbounded case the spectral theory of Hermitian operators. In [1929a] he carried out that extension.

Before von Neumann, the name “Hilbert space” had been applied principally to the space l^2 of square summable sequences (often called “Hilbert's space”) or to the space L^2 of Lebesgue square integrable functions which Riesz had proved isomorphic to l^2 . The essential properties of these spaces, widely recognized, were those of a vector space with an inner product which was complete and separable (i.e., which had a countable dense subset). Von Neumann's first step in his theory of linear operators was to define an (abstract) Hilbert space axiomatically as any

separable, complete inner product space. He then defined a general linear **operator** on the abstract Hilbert space H as a linear transformation defined on some subset of H . This subset, called the domain D_T of the operator T , is usually assumed to be a linear subspace of H , which, like the domain of the differentiation operator, is dense in H . Von Neumann's linear operators thus comprehend both the matrices and quadratic forms of Hilbert's theory, and the transformations of quantum mechanics.

A linear operator is continuous if and only if it is bounded, and a bounded linear operator with a dense domain can be uniquely extended to a bounded linear operator on the whole space H . Every linear operator T with a dense domain has a unique **adjoint** operator T^* defined by the relation

$$(6) \quad (Tx, y) = (x, T^*y)$$

for all $x \in D_T$; the domain of T^* is the set of $y \in H$ for which (6) holds for all x . An operator T is called **self-adjoint** if $T = T^*$, and **symmetric** if T^* is an extension of T , or equivalently, if $(Tx, y) = (x, Ty)$ whenever $x, y \in D_T$. (In von Neumann's papers, the self-adjoint operators were called **hypermaximal**.)

Every self-adjoint operator is clearly symmetric, and every symmetric operator which is everywhere defined must be self-adjoint. Thus for bounded linear operators (which either are everywhere defined or may be extended to become so) the concept of symmetric and self-adjoint coincide. The Hellinger-Toeplitz theorem, cited in section 7 above, can be extended to von Neumann's operators and shows that any symmetric operator which is everywhere defined must be bounded. (This result is closely related to a more general theorem due to Stefan Banach (1892–1945), now commonly known as the closed graph theorem [1932b].) Thus in von Neumann's theory there are precisely three types of symmetric operators:

- I. bounded, self-adjoint and everywhere defined;
- II. unbounded, self-adjoint and densely but not everywhere defined; and
- III. unbounded, not self-adjoint, and densely but not everywhere defined.

Hilbert's original theory applied to operators of type I, while von Neumann's spectral theorem encompassed those of type II as well since it applies to all self-adjoint operators. This theory, though initiated by von Neumann, was developed by Riesz [1930c] and more extensively, by Marshall H. Stone (1903–) at Yale University who expounded it in great detail in [1932a]. The combined (but largely independent) efforts of von Neumann and Stone for the five year period 1927–1932 provided for spectral theory the largest collection of new methods since Hilbert's five year effort of 1901–1906.

9. Von Neumann — Stone Spectral Theory. Hilbert's general spectral theorem says that every bounded symmetric linear transformation T can be written in the form

$$\sum_{\rho(T)} \lambda P_\lambda + \int_{s(T)} \lambda dE_\lambda.$$

By rewriting the first sum as a Stieltjes-type integral and combining it with the second integral, we may express Hilbert's spectral theorem in the concise form

$$(7) \quad T = \int_{\sigma(T)} \lambda dE_\lambda,$$

where the integral is over the entire (bounded) spectrum of T . The operators E_λ are projections with the following properties:

- (i) If $\lambda < \mu$, the range of E_λ is contained in the range of E_μ ;
- (ii) If $\varepsilon > 0$, $E_{\lambda+\varepsilon} \rightarrow E_\lambda$ as $\varepsilon \rightarrow 0$;
- (iii) $E_\lambda \rightarrow 0$ as $\lambda \rightarrow -\infty$;
- (iv) $E_\lambda \rightarrow I$ as $\lambda \rightarrow +\infty$.

Stone called such a family of operators a **resolution of the identity**; in more intuitive language, properties (i)–(iv) require that the function $\lambda \rightarrow E_\lambda$ be increasing, continuous from the right, with 0 and I as left and right limiting values.

The von Neumann-Stone extension of the spectral theorem for self-adjoint operators from the bounded to the unbounded case corresponds to the extension of (7) from bounded to unbounded spectra $\sigma(T)$. Specifically, it says that to each self-adjoint operator T there corresponds a unique resolution of the identity $\{E_\lambda\}$ such that (7) holds.

Despite the power of this theorem, many differential operators are not covered by it since they are rarely self-adjoint. For instance, to make the operator $D = d/dt$ symmetric on a dense subset Δ of the Hilbert space $L^2(0, 1)$, we should select for Δ the subset consisting of those continuously differentiable functions f which satisfy $f(0) = f(1) = 0$ (in order to insure that the relation $(Df, g) = (f, Dg)$ would follow by integration by parts). But the domain Δ is too small to permit D to be self-adjoint, for *every* continuously differentiable L^2 function is in the domain of D^* . To make D self-adjoint we would have to enlarge its domain appropriately—thereby risking a loss of symmetry. Each symmetric operator of type III suffers from the same disease: its domain is smaller than that of its adjoint. Moreover the cure—namely, extension of the domain—is often fatal since with a larger domain the operator may fail to be symmetric.

To apply his spectral theorem to symmetric operators von Neumann had to know which types of symmetric operators admit self-adjoint extensions. He [1929a] and Wintner [1929b] identified a large class of such operators, namely those operators T , called **semibounded**, for which there is a positive constant M satisfying either $(Tx, x) \leq M \|x\|^2$ for all $x \in D_T$ or $-M \|x\|^2 \leq (Tx, x)$ for all $x \in D_T$. The best statement of this result is due to Stone [1932a] and Kurt O. Friedrichs [1934]: every semibounded symmetric operator may be extended to a semibounded self-adjoint operator with the same bound.

Whereas the central focus of the von Neumann-Stone spectral theory (and of Hilbert's also) is on operators with real spectra, the spectral theorem does apply, at

least in two cases, to operators with more general spectra. The simplest case concerns *isometric* operators which leave the inner product on H invariant; from this definition it follows easily that the spectrum of an isometric operator is a subset of the unit circle. An isometric operator that maps H onto H is called *unitary* and is characterized by the fact that its adjoint is its inverse (i.e., $TT^* = T^*T = I$). Unitary operators were first studied in [1909c] by Isaac Schur following their introduction by Léon Autonne in [1902]. In [1929a] von Neumann employed the Cayley transform $C: T \rightarrow (T - iI)(T + iI)^{-1}$ to map symmetric operators T into isometric operators $C(T)$; he showed that T is self-adjoint if and only if $C(T)$ is unitary. Thus the spectral theory for unitary operators follows from that for self-adjoint operators by use of a spectral integral on the unit circle instead of on the real line.

Now every bounded linear operator T can be written in the “Cartesian” form $T = A + iB$, where A and B are bounded and self-adjoint; in fact,

$$A = \frac{1}{2}(T + T^*), \quad B = \frac{1}{2i}(T - T^*).$$

Thus it would appear likely that the spectral theorem could be extended to all bounded linear operators by using this decomposition. However, the details of that extension require that $AB = BA$ (or equivalently, that $TT^* = T^*T$). So the desired extension works only for those operators which commute with their adjoints: such operators are called **normal**, after Toeplitz [1918a]. Toeplitz extended Hilbert’s spectral theorem to completely continuous normal quadratic forms by showing that such a form was unitarily equivalent to a diagonal form. More generally, the spectral resolution

$$T = \int_{\sigma(T)} \lambda \, dE_\lambda$$

extends to bounded normal operators, where the integration is over the spectrum of T which is a compact subset of the complex plane contained in the disc of radius $\|T\|$. Von Neumann [1930a] and Stone [1932a] extended both the definition and spectral theory of normal operators to the unbounded case as well.

We have come a long way from the principal axes theorem, and the spectral theorems of von Neumann and Stone reflect far more analysis than geometry. The geometric content of the spectral theorem for finite dimensional space is that the entire space can be expressed as the direct sum of subspaces on each of which the given transformation acts like simple multiplication. But this theorem fails in the infinite dimensional cases as soon as the continuous spectrum appears. In a paper written in 1938 but not published until [1949a], von Neumann effectively resuscitated the geometrical spectral theorem by defining a direct integral of Hilbert spaces (in strict analogy with the direct sum). He then showed that the action of a self-adjoint operator on any Hilbert space could be represented as the accumulated effect of

simple multiplications on certain subspaces whose direct integral was (unitarily equivalent to) the original space.

10. Gelfand-Naimark Theorem. The collection of all operators on a Hilbert space forms a ring; such rings, with various topologies, were extensively investigated by von Neumann and Francis J. Murray in [1936a], [1937a] and [1940a]. During the same period 1936–40 S. W. P. Steen published in England a series of five papers ([1936b], [1937b], [1938a], [1939], [1940b]) devoted to an axiomatic theory of operators. But the papers that offered the most significant insight into the spectral theorem were [1941a], [1941b] and [1943] published in the U.S.S.R by Israel M. Gelfand, Mark A. Naimark and Georgii E. Silov. Gelfand and his colleagues created a theory of normed rings which not only subsumed much of the work of von Neumann, Murray and Steen on rings of operators, but also provided a beautiful general setting for the study of Fourier transforms and harmonic analysis. Related studies were carried out in the United States by Stone ([1940c], [1941c]) and Shizuo Kakutani [1941d].

Normed rings were first introduced in [1936c] by the Japanese mathematician, Mitio Nagumo under the name of “linear metric rings”. In [1946] Charles E. Rickart christened Gelfand’s normed rings “Banach algebras” to avoid misunderstanding due to the algebraic meaning of “ring”; as a consequence Russian mathematicians now use the former name, while Americans use the latter. But regardless of its name, the properties of a **Banach algebra** are those of a complete normed algebra (over the complex field C) satisfying the multiplicative triangle inequality $\|x\| \|y\| \leq \|xy\|$. We shall assume that each Banach algebra contains an identity element e , where $\|e\| = 1$. The set of all bounded linear operators on a Hilbert space is a Banach algebra, as is the set of all continuous complex-valued functions on a compact topological space X (with the sup norm $\|f\| = \sup \{|f(x)| : x \in X\}$). The part of Banach algebra theory germane to spectral theory is the relation between these two examples.

Gelfand’s theory of commutative Banach algebras depends on three fundamental concepts: homomorphisms, maximal ideals and spectra. A *homomorphism* of a commutative Banach algebra B is a non-zero multiplicative linear functional; its kernel is a **maximal ideal** since it is contained in no larger proper ideal. Moreover, every maximal ideal I is the kernel of some homomorphism for in this case the factor algebra B/I is the field C of complex numbers (according to a result announced by Stanislaw Mazur [1938b] and proved by Gelfand [1941a]) so the composite map $B \rightarrow B/I \rightarrow C$ is a homomorphism of B whose kernel is I . The set M_B of homomorphisms (or equivalently, of maximal ideals) of B is given the weakest topology relative to which all of the functions $\hat{x}: h \rightarrow h(x)$ are continuous, for all $x \in B$. Then the topological space M_B is compact and Hausdorff, and each element x of B is represented in $C(M_B)$ (the Banach algebra of continuous complex valued functions on M_B) by its “Gelfand transform” \hat{x} .

In strict analogy with the spectral theory of operators on a Hilbert space, Gelfand defines the *spectrum* $\sigma(x)$ of an element $x \in B$ to be the set of complex numbers λ for which the element $x - \lambda e$ has no inverse. The set $\sigma(x)$ is compact, non-empty and contained in the disc of radius $\|x\|$. Furthermore, $\sigma(x)$ happens to be precisely equal to the range of the Gelfand transform $\hat{x}: \sigma(x) = \{h(x) \mid h \in M_B\}$. For this reason the space M_B of maximal ideals is often called the *spectrum* of the Banach algebra B . If B is the algebra generated by a single element x (such as a particular operator on H), then the spectrum of the algebra B is mapped homeomorphically by \hat{x} onto the spectrum of x .

In [1943] Gelfand and Naimark showed that the commutative Banach algebra $C(M_B)$ is characterized by the presence of an *involution*, namely the operation of complex conjugation $*: f \rightarrow \bar{f}$. Specifically, they showed that any commutative Banach algebra with an involution (called a B^* -algebra) is isometrically isomorphic to the algebra $C(M_B)$ for some Banach algebra B . In particular, the commutative B^* algebra $B(T)$ generated by a given bounded normal operator T is isomorphic to the algebra $C(\sigma(T))$ of all continuous functions on $\sigma(T)$, the spectrum of $B(T)$; T is assumed normal in order that the presence of the involution $*: T \rightarrow T^*$ should not destroy the commutativity of the algebra $B(T)$.

The impact of the Gelfand-Naimark theorem on spectral theory is this: the spectral theorem for a bounded normal operator T can be inferred via the isomorphism between $B(T)$ and $C(\sigma(T))$ from a corresponding theorem concerning continuous complex valued functions on $\sigma(T)$. The required theorem is just that every continuous function f on $\sigma(T)$ (in particular, the identity function $f(\lambda) = \lambda$) can be approximated uniformly by measurable step functions of the form $\sum f(\lambda_i) \chi_{\Lambda_i}$, where χ_{Λ_i} is the characteristic function of the measurable set Λ_i . The translation of this theorem to the algebra $B(T)$ (in the special case $f(\lambda) = \lambda$) is the spectral resolution of the bounded normal operator $T: T = \int \lambda dE_\lambda$. In words instead of symbols, the approximation theorem says that a continuous function can be approximated by linear combinations of characteristic functions, while the spectral theorem says that bounded normal operators can be approximated by linear combinations of projections. Thus Gelfand's theory of Banach algebras revealed that the spectral theorem is in some fundamental sense equivalent to a most rudimentary fact in the theory of functions.

Gelfand's theory actually yields a spectral theorem far stronger than those which we have so far discussed. By translating the approximation theorem for an arbitrary continuous function f we obtain a spectral resolution of the form $f(T) = \int f(\lambda) dE_\lambda$. This formula was originally introduced by von Neumann and Stone as the basis of their "operational calculus". A related general spectral theorem, also due to von Neumann [1930a], can be inferred from the Gelfand-Naimark isomorphism: any commutative family of bounded normal operators admits a simultaneous diagonalization—that is, a single resolution of the identity which simultaneously represents all operators in the family by means of the integral $\int f(\lambda) dE_\lambda$ for various functions f .

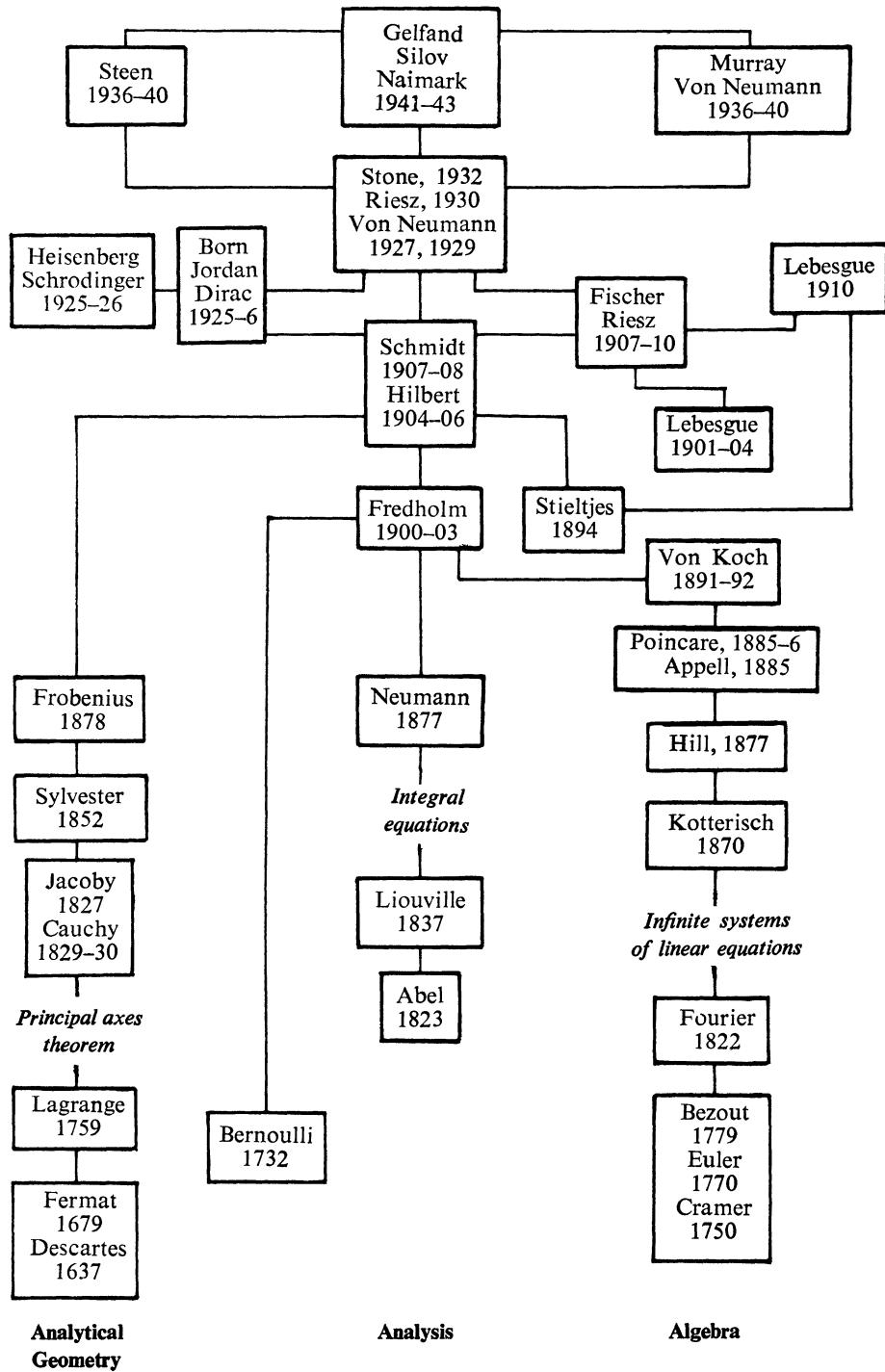


FIG. 1

11. Unfinished business. This concludes our saga of the spectral theorem. Our historical vision has been deliberately narrow, focused throughout on the evolution of just one theorem and only rarely have we glanced at the many fascinating applications and extensions of the basic theory. For example, spectral theory for spaces without inner products can be traced back to Riesz [1918b] and T.H. Hildebrandt [1931], while the rudiments of spectral theory for differential operators are contained in the work [1908c] of George Birkhoff; in [1928b] and [1930d] Norbert Wiener developed a theory of **spectral analysis** for functions in an attempt to analyze mathematically the spectrum of white light, while twenty years later Arne Beurling [1949b] inaugurated the complementary study of *spectral synthesis*; and in [1942] Edgar R. Lorch, continuing work begun in [1913a] by F. Riesz, investigated spectral sets in the plane by means of contour integrals.

Had we stopped to investigate each such offshoot our evolutionary tree (Figure 1) would have looked like a forest. Indeed, it took Nelson Dunford and Jacob Schwartz nearly 3000 pages to survey spectral theory ([1958a], [1963], [1971]). So any who are inspired to examine the fruits of spectral theory are invited to read this treatise or any of its many less ambitious companions ([1951], [1953], [1958b], [1962]). Our mission to describe the roots and main trunk of spectral theory is accomplished.

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