

Spectral Statistics of Random Matrices

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List of Abbreviations

r.v	Random Variable
i.i.d	Independent and Identically Distributed
pdf	Probability Distribution Function
cdf	Cumulative Distribution Function

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Table of Random Matrix Distributions			
Distribution	Notation (\mathcal{D})	Parameters	Class
Normal	$\mathcal{N}(\mu, \sigma)$	$\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$	Explicit
Uniform	$\text{Unif}(a, b)$	$a, b \in \mathbb{R}$	Explicit
Hermite- β	$\mathcal{H}(\beta)$	$\beta \in \mathbb{N}$	Implicit
Erdos- p	$\text{ER}(p)$	$p \in [0, 1]$	Implicit

Table of Spectrum Schema			
Scheme	Matrix	Notation	Ordering
Sign-Ordered	P	$\sigma_S(P)$	$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$
Norm-Ordered	P	$\sigma_N(P)$	$ \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N $
Singular	$P \cdot P^T$	$\sigma_+(P)$	$\sqrt{\lambda_1} \geq \sqrt{\lambda_2} \geq \dots \geq \sqrt{\lambda_N}$

Table of Dispersion Metrics				
Metric*	Notation	Formula	Symmetric	Parameters
Standard Norm	δ	$ z' - z $	True	-
β -Norm	δ_β	$ z' - z ^\beta$	True	$\beta \in \mathbb{N}$
Difference of Absolutes	δ_{abs}	$ z' - z $	False	-
Identity Difference	δ_{id}	$z' - z$	False	-

Table of Pairing Schema		
Scheme	Notation	Formula
Lower	$\Pi_<$	$\{(i, j) \mid i < j \text{ for } i, j \in \mathbb{N}_N\}$
Upper	$\Pi_>$	$\{(i, j) \mid i > j \text{ for } i, j \in \mathbb{N}_N\}$
Consecutive	Π_C	$\{(i, j) \mid i = j + 1 \text{ for } i, j \in \mathbb{N}_N\}$
All	Π_0	$\{(i, j) \mid i, j \in \mathbb{N}_N\}$

Abstract

On their own, random variables exude deterministic properties regarding their uncertainty. The same generalization can be made for random matrices, which are matrices whose entries are random variables. One particular statistic worth investigating is the distribution of a matrix ensemble's eigenvalues, or its spectrum. In this thesis, there will be an exploration of various classes of random matrices and relevant spectral statistics like their spectra and mixing times.

Dedication

For my mother.

Introduction

So, what are *spectral statistics*? Do they have to do with rainbows? Sceptres? No, they don't, but they're almost as colorful and regal. The word spectral is borrowed from the spectral-like patterns observed in statistical physics - whether it may be atomic spectra or other quantum mechanical phenomena. The borrowing is loose and not literal, but still somewhat well founded.

The field of Random Matrix Theory was extensively developed in the 1930s by the nuclear physicist Eugene Wigner. He found connections between the deterministic properties of atomic nuclei and their random and stochastic behaviors. The link? Random matrices.

So in the context of this thesis, *Spectral statistics* will be an umbrella term for random matrix statistics that somehow involve that matrix's eigenvalues and eigenvectors.

The RMAT Package

Tell Me and I Forget; Teach Me and I May Remember; Involve Me and I Learn. -Confucius

To explore these spectral statistics, this thesis will use the **RMAT** package. This package was developed alongside this thesis in order to facilitate the simulation of these random matrices and spectral statistics. As such, there is a large simulation component to this thesis. To showcase the methodology of the simulations, code snippets will be sprinkled about the thesis. It will use code derived from the package RMAT which can be found on GitHub; minimal source code will also be available in the appendix, as well as pre-simulated data available on the Reed database. All code examples in this thesis are reproducible. Simply use the RMAT package (or equivalent code) and use `set.seed(23)`.

You are strongly encouraged to try out these simulations yourself!

Instructions on how to get RMAT. Try CRAN then Github then source code in appendix. The source code was functional as of R 4.0.5.

Chapter 1

Random Matrices

Unfortunately, no one can be told what The Matrix is. You'll have to see it for yourself.

*Morpheus
The Matrix*

As discussed in the introduction, this thesis will be an exploration of spectral statistics of random matrices. This means that we must first be able to understand what random matrices are. At a fundamental level, random matrices are simply matrices whose entries are randomly distributed in accordance to some distribution or method. To formalize all these notions, we will define what random matrices are and what it means for them to be \mathcal{D} -distributed.

Prior to beginning the discussion on \mathcal{D} -distributions, the reader should be familiar or at least acquainted with the notion of random variables and what they are. A summary is available in the appendix in A.x.

When it comes to random simulation, there must always be a rule to which our randomness must conform, regardless of complexity. For example, sampling a vector from a distribution is a rudimentary example of this. For random matrices, there will be a few methods of generating their entries that are not just sampling from theoretical distributions. As such, we motivate the \mathcal{D} -distribution.

1.1 \mathcal{D} -Distributions

Now, we motivate the D -distribution. A formalization on how to initialize a random matrix.

Definition 1.1.1 (\mathcal{D} -distribution). *When we define a random matrix that is \mathcal{D} -distributed, we say that $P \sim \mathcal{D}$. In the simplest of terms, \mathcal{D} is essentially the algorithm that generates the random matrix P . We define two primary methods of distribution: explicit distribution and implicit distribution. If \mathcal{D} is an explicit distribution, then every entry of P is homogenously sampled from that distribution. Otherwise, if it is implicit, we utilize an algorithm that imposes an implicit distribution on the entries.*

1.1.1 Explicit Distributions

The simplest case is homogenous, explicitly distributed random matrices. If \mathcal{D} is an explicit distribution, then we overload the notation \mathcal{D} to mean a probability distribution in the classical sense (see Appendix A.x). So, if \mathcal{D} is a probability distribution, the matrix $P \sim \mathcal{D}$ when $p_{ij} \sim \mathcal{D}$. In otherwords, we simply perform entry-wise sampling from that distribution.

Take for example the following explicit distributions which we cover in this thesis.

1. If $\mathcal{D} = \mathcal{N}(0, 1)$, then $p_{ij} \sim \mathcal{N}(0, 1)$.
2. If $\mathcal{D} = \text{Unif}(0, 1)$, then $p_{ij} \sim \text{Unif}(0, 1)$.

Formalization. *Explicit distributions can be formalized in scope of the standard notation in probability theory. At the heart of the formalization is the usage of index hacking to collapse the array's indices from two dimensions to one. This way, our random matrix has a representation as a random vector, which we commonly encounter in probability theory as a (i.i.d) sequence of random variables! Generally, an $N \times N$ random matrix that is explicity and homogenously \mathcal{D} -distributed is essentially a sequence of N^2 i.i.d random variables sampled from \mathcal{D} .*

Example (Formalization). *For example, suppose P is a 2×2 random matrix with $\mathcal{D} = \mathcal{N}(0, 1)$. Then, we have four random variables to initialize. In the random matrix representation, we need to initialize P_{11}, P_{12}, P_{21} , and P_{22} by sampling them from \mathcal{D} . In the vector representation, we just say we are sampling four i.i.d random variables from \mathcal{D} . The matrix indexing is intrinsic and preservable by using index hacking with some modular math.*

1.1.2 Implicit Distributions

In the latter case, we are concerned less about the distribution of the matrix entries and moreso about its holisitic properties.

Consider for example, the following implicit distributions.

1. If $\mathcal{D} = \text{Stochastic}$, then the matrix is a row of random stochastic rows. (See Algorithm B.x)
2. If \mathcal{D} is any distribution (implicit or explicit), then \mathcal{D}^\dagger is the Symmetric/Hermitian version of \mathcal{D} . (See Algorithm B.x)

1.1.3 Random Matrices

With \mathcal{D} -distributions defined, we may finally define a random matrix.

Definition 1.1.2 (Random Matrix). *Assuming \mathcal{D} is an explicit distribution, a random matrix is any matrix over the field \mathbb{F} is a matrix $M \in \mathbb{F}^{N \times N}$ is a matrix whose entries are i.i.d random variables. So, if a random matrix $M = (m_{ij})$ is \mathcal{D} -distributed,*

then we say $m_{ij} \sim \mathcal{D}$. In the scope of this thesis, assume every random matrix to be homogenously distributed. Otherwise, if \mathcal{D} is an implicit distribution, then P is a matrix whose entries are determined by the algorithm imposed by \mathcal{D} .

Sometimes, we want our matrix to have complex entries. We note this by specifying $\mathbb{F} = \mathbb{C}$.

Remark (Complex Entries). *To say that a random matrix is explicitly \mathcal{D} -distributed over \mathbb{C} would mean that its entries take the form $a + bi$ where $a, b \sim \mathcal{D}$ are random variables. In other words, if we allow the matrix to have complex entries by setting $\mathbb{F} = \mathbb{C}$, then we must sample the real and imaginary component as \mathcal{D} -distributed i.i.d. random variables.*

Below, we can see code on how to generate a standard normal random matrix using the **RMAT** package.

Code Example (Standard Normal Matrix). *Let $\mathcal{D} = \mathcal{N}(0, 1)$. We can generate $P \sim \mathcal{D}$, a 4×4 standard normal matrix, as such:*

```
library(RMAT)
P <- RM_norm(N = 4, mean = 0, sd = 1)
# Outputs the following
P
      [,1]      [,2]      [,3]      [,4]
[1,]  0.1058257 -1.0835598 -0.7031727  1.01608625
[2,] -0.2170453  1.8206070 -0.4539230  0.06828296
[3,]  1.3002145  0.1254992 -0.5214005 -0.61516174
[4,] -1.0398587  0.1975445 -0.8511950  0.86366082
```

Summary Table of \mathcal{D} -Distributions

Table of Random Matrix Distributions			
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Normal	$\mathcal{N}(\mu, \sigma)$	$\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$	Explicit
Uniform	$\text{Unif}(a, b)$	$a, b \in \mathbb{R}$	Explicit
Hermite- β	$\mathcal{H}(\beta)$	$\beta \in \mathbb{N}$	Implicit
Erdos- p	$\text{ER}(p)$	$p \in [0, 1]$	Implicit

1.2 The Crew: Ensembles

With a random matrix well defined, we may now motivate one of the most important ideas - the random matrix ensemble. One common theme in this thesis will be that random matrices on their own provide little information. When we consider them at the ensemble level, we start to obtain more fruitful results. Without further ado, we motivate the random matrix ensemble.

Definition 1.2.1 (Random Matrix Ensemble). *A \mathcal{D} -distributed random matrix ensemble \mathcal{E} over $\mathbb{F}^{N \times N}$ of size K is defined as a set of \mathcal{D} -distributed random matrices $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N \times N}\}_{i=1}^K$. In simple words, it is simply a collection of K iterations of a specified class of random matrix.*

So, for example, we could compute a simple ensemble of matrices as follows.

Code Example (Standard Normal Hermitian Ensemble). *Let $\mathcal{D} = \mathcal{N}(0, 1)^\dagger$. We can generate $\mathcal{E} \sim \mathcal{D}$ over \mathbb{C} , an ensemble of 4×4 complex Hermitian standard normal matrices of size 10 as such:*

```
library(RMAT)
# By default, mean = 0 and sd = 1.
ensemble <- RME_norm(N = 4, cplx = TRUE, herm = TRUE, size = 10)
# Outputs the following
ensemble
...
[[10]]
      [,1]           [,2]           [,3]
[1,] -0.59931+1.24286i 1.29457+0.66058i 0.83539-0.16662i
[2,]  1.29457-0.66058i 0.78841+0.09818i -1.16592+1.14666i
[3,]  0.83539+0.16662i -1.16592-1.14666i -0.51256+0.17750i
```

With this in mind, we will gloss over, characterize, and briefly discuss a few special recurring ensembles in this thesis.

1.2.1 Erdos-Renyi p -Ensembles

The Hermite β -ensembles are a normal-like class of random matrices. Now, we will veer away from the normal distribution as a whole and switch to a different class of matrices: stochastic matrices. Stochastic matrices, in short, are matrices that represent Markov Chains (see A.x). We can also think of them as the matrix representation of a specific setup of a walk on a random graph.

A particular class of random graphs that we will consider are the Erdos-Renyi random graphs. Essentially, these are graphs whose vertices are connected with a uniform probability p . We can interpret this as saying an Erdos-Renyi graph is a simple random walk on a graph with parameterized sparsity (given by p). Without further ado, we motivate the Erdos-Renyi graph:

Definition 1.2.2 (Erdos-Renyi Graph). *An Erdos-Renyi graph is a graph $G = (V, E)$ with a set of vertices $V = \{1, \dots, N\}$ and edges $E = \mathbb{1}_{i,j \in V} \sim \text{Bern}(p_{ij})$. It is homogenous if $p_{ij} = p$ is fixed for all i, j .*

Essentially, an Erdos-Renyi graph is a graph whose 'connectedness' is parameterized by a probability p (assuming it's homogenous, which this document will unless otherwise noted). As $p \rightarrow 0$, we say that graph becomes more sparse; analogously, as $p \rightarrow 1$ the graph becomes more connected.

Recall from probability theory that a sum of i.i.d Bernoulli random variables is a Binomial variable. As such, we may alternatively say that the degree of each vertex v is distributed as $\text{deg}(v) \sim \text{Bin}(N, p)$ where N is the number of vertices. This makes simulating the graphs much easier.

Code Example (Erdos-Renyi $p = 0.5$ Ensemble). *Let $\mathcal{D} = ER(p = 0.5)$. We can generate $\mathcal{E} \sim \mathcal{D}$, an ensemble of 4×4 Erdos-Renyi matrices ($p = 0.5$) of size 10 as such:*

```
library(RMAT)
ensemble <- RME_erdos(N = 4, p = 0.5, size = 10)
# Outputs the following
ensemble
...
[[10]]
 [,1]      [,2]      [,3]      [,4]
[1,] 0.0000000 0.1729581 0.8270419 0.000000
[2,] 0.0000000 0.0000000 1.0000000 0.000000
[3,] 0.2557890 0.3766740 0.0000000 0.367537
[4,] 0.2151029 0.3929580 0.3919391 0.000000
```

1.2.2 Hermite β -Ensembles

The Hermite β -Ensembles will be one of the primary ensembles discussed in this thesis. This ensemble will be characterized, motivated, and defined more thoroughly in **Chapter 4**. However, we will give a brief introduction to the ensemble.

At a practical level, all one would need to know is that the matrices are generated in accordance to an algorithm found in the appendix (Algorithm B.x) and in Chapter 4.

Chapter 2

Spectra

Life is like a box of crayons.

Unknown

2.1 Introduction

So, what are *spectral statistics*? Do they have to do with rainbows? Sceptres? No, they don't, but they're almost as colorful and regal. The word spectral is borrowed from the spectral-like patterns observed in statistical physics - whether it may be atomic spectra or other quantum mechanical phenomena. The borrowing is loose and not literal, but still somewhat well founded. In fact, the field of Random Matrix Theory was extensively developed in the 1930s by the nuclear physicist Eugene Wigner. He found connections between the deterministic properties of atomic nuclei and their random and stochastic behaviors. The link? Random matrices.

So in the context of this thesis, *spectral statistics* will be an umbrella term for random matrix statistics that somehow involve that matrix's eigenvalues and eigenvectors. That being said, if we fix a *random matrix*, we can study its features by studying its eigenvalues - fundamental numbers that tell us a lot about the matrix. They are quite important for many reasons. For instance in statistical physics, many processes are represented by operators or matrices, and as such, their behaviours could be partially determined by the eigenvalues of their corresponding matrices. The study of eigenvalues and eigenvectors primarily falls in the scope of Linear Algebra, but their utility is far-reaching. So, what exactly are *eigenvalues* exactly?

2.1.1 The Quintessential Spectral Statistic: the Eigenvalue

Given any standard square matrix $P \in \mathbb{F}^{N \times N}$, its *eigenvalues* are simply the roots of the characteristic polynomial $\text{char}_P(\lambda) = \det(P - \lambda I)$. By the Fundamental Theorem of Algebra, we know that there are always as many complex eigenvalues $\lambda \in \mathbb{C}$ as the dimension of the matrix.

That being said, when our random matrix has a specified distribution (say, standard normal), we can see patterns in the eigenvalue distributions. So, an eigenvalue is a **spectral statistic** of a random matrix! To talk about a matrix's eigenvalues in a more formal and concise manner, we motivate what is the eigenvalue spectrum.

Definition 2.1.1 (Spectrum). *Suppose $P \in \mathbb{F}^{N \times N}$ is a square matrix of size N over \mathbb{F} . Then, the (eigenvalue) spectrum of P is defined as the multiset of its eigenvalues and it is denoted*

$$\sigma(P) = \{\lambda_i \in \mathbb{C} \mid \text{char}_P(\lambda_i) = 0\}_{i=1}^N$$

Note that it is important to specify that a spectrum is a multiset and not just a set; eigenvalues could be repeated due to algebraic multiplicity and we opt to always have N eigenvalues.

For example, consider the following code example from the RMAT package.

Code Example (Spectrum of a Standard Normal Matrix). *Let $P \sim \mathcal{N}(0, 1)$ be a 4×4 standard normal random matrix. We can generate the spectrum of P , $\sigma(P)$ as follows:*

```
library(RMAT)
P <- RM_norm(N = 5, mean = 0, sd = 1)
spectrum_P <- spectrum(P)
# Outputs the following
spectrum_P
...
      Re      Im   Norm Order
1 -0.5434  1.3539 1.4589    1
2 -0.5434 -1.3539 1.4589    2
3  0.2255  1.4250 1.4427    3
4  0.2255 -1.4250 1.4427    4
5 -0.8678  0.0000 0.8678    5
```

While our definition for spectrum is nice and clean, there are a few caveats that we must take care of. First, how are spectra statistics? So we can even call them statistics, so there needs to be some formalization as to why we consider eigenvalues statistics. In this dialogue, we will call back on the same sort of formalization dialogue in the Random Matrices section. Before beginning, the reader is encouraged to review what a statistic is in the review appendix (A.x).

Recall that when we defined and motivated the \mathcal{D} -distribution framework of simulating random matrices, we always had one thing - a vector representation of random variables. Using this framework, the formalization is trivial.

Formalization. *Suppose $P \sim \mathcal{D}$ is an $N \times N$ random matrix. Then, P has a representation as a sequence of N^2 random variables, denote it $\vec{X} = \{X_i \mid i = 1, 2, \dots, N^2 - 1, N^2\}$. Then, the spectrum of the matrix P is simply a function of the vector \vec{X} . We can denote this $\sigma(\vec{X})$, where the operator σ is overloaded to mean the spectrum of a matrix **with respect to the vector representation**. The actual*

process for σ is not necessary to explicitly write out, since characterizing it will be sufficient for now. Essentially, σ is a function that would parse the random variables into the array form by index hacking. Then, it must compute the determinant of the matrix $P - \lambda I$, and solve for its roots. To summarize this, consider the flow chart below.

To summarize, here is how we formalize the spectrum as a statistic. Suppose we sample $P \sim \mathcal{D}$. Then, we take its spectrum formally using σ as such:

$$P_{\text{array}} \rightarrow \vec{P} \rightarrow \sigma(\vec{P}) \rightarrow \text{index magic} \rightarrow \det(P - \lambda I) \rightarrow \sigma(P_{\text{array}})$$

2.1.2 Interlude: Ensembles

While the spectrum of a matrix provides a good summary of the matrix, a matrix is only considered a single point/observation in random matrix theory. Additionally, simulating large matrices and computing their eigenvalues becomes harder and more computationally expensive as $N \rightarrow \infty$. As such, to obtain more eigenvalue statistics efficiently, another dimension is introduced by motivating the *spectrum of a random matrix ensemble*. If we have an ensemble \mathcal{E} , then we can naturally extend the definition of $\sigma(\mathcal{E})$.

Definition 2.1.2 (Ensemble Spectrum). *Let $\mathcal{E} \sim \mathcal{D}$ be an ensemble of matrices $P_i \in \mathbb{F}^{n \times n}$. To take the spectrum of \mathcal{E} , simply take the union of the spectra of each of its matrices. In other words, if $\mathcal{E} = \{P_i \sim \mathcal{D}\}_{i=1}^K$, then we denote the spectrum of the ensemble*

$$\sigma(\mathcal{E}) = \bigcup_{i=1}^K \sigma(P_i)$$

For example, consider the following code example from the RMAT package.

Code Example (Spectrum of a Standard Normal Matrix Ensemble). *Let $\mathcal{E} \sim \mathcal{N}(0, 1)$ be an ensemble of 3×3 standard normal random matrices of size 3. We can generate the spectrum of \mathcal{E} , $\sigma(\mathcal{E})$ as follows:*

```
library(RMAT)
ens <- RME_norm(N = 3, mean = 0, sd = 1, size = 3)
spectrum_ens <- spectrum(ens)
# Outputs the following
spectrum_ens
...
      Re      Im   Norm Order
1  1.7581  0.0000  1.7581     1
2 -0.2614  1.0012  1.0347     2
3 -0.2614 -1.0012  1.0347     3
4  1.2327  0.4227  1.3032     1
5  1.2327 -0.4227  1.3032     2
6 -0.8504  0.0000  0.8504     3
```

7	-0.5296	1.0508	1.1767	1
8	-0.5296	-1.0508	1.1767	2
9	0.7357	0.0000	0.7357	3

A common theme in this thesis will be that singleton matrices do not provide insightful information on their own. Rather, it is the collective behavior of a \mathcal{D} -distributed ensemble that tells us about how \mathcal{D} impacts our spectral statistics. So in a way, ensemble statistics are the engine of this research.

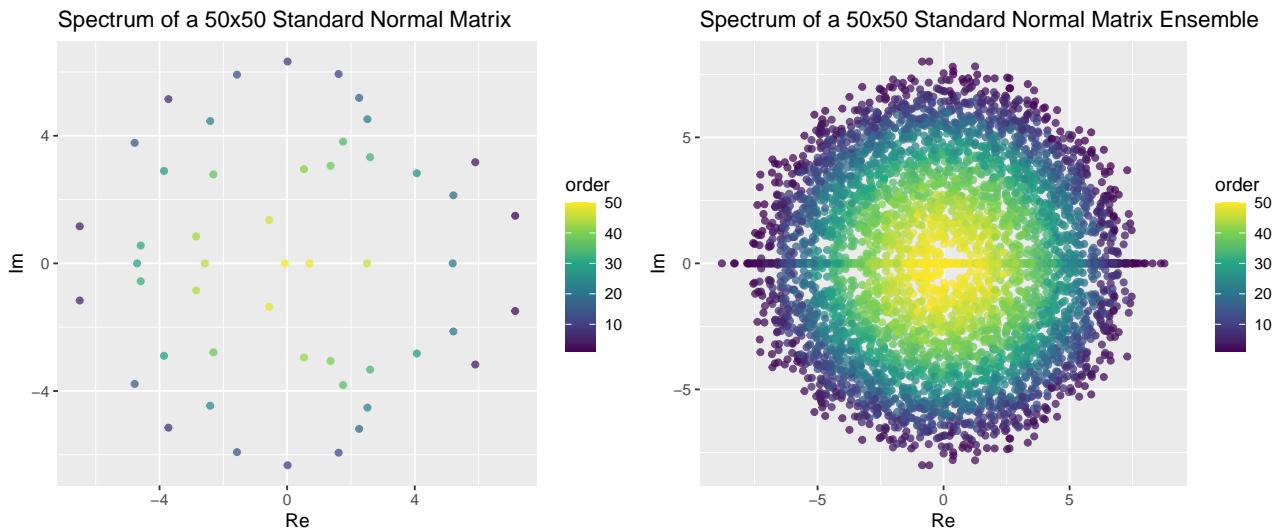


Figure 2.1: Spectrum of a Matrix versus an Ensemble

2.2 Spectrum Analysis

2.2.1 Ordered Spectra

When we motivate the idea of matrix dispersion in the next section, we will consider order statistics of that matrix's eigenvalues in tandem with its dispersion. However, to do so presupposes that we have a sense of what *ordered* eigenvalues means. Take a matrix P and its *unordered* spectrum $\sigma(P) = \{\lambda_j\}$. It is paramount to know what ordering scheme $\sigma(P)$ is using, because otherwise, the eigenvalue indices are meaningless! So, to eliminate confusion, we add an index to σ that indicates how the spectrum is ordered. Often, the ordering context will be clear and the indexing will be omitted.

Order Schemes: How to Order Eigenvalues

Consider the two following *ordering schema*:

Standard definitions of an ordered spectrum follows the standard ordering in the reals; denote this as the ordering by the **sign scheme**. Note that because total-ordering is only well-defined on the reals, we can only use this scheme when on a spectrum with real entries. So, we write the *sign-ordered spectrum* as follows:

$$\sigma_S(P) = \{\lambda_j : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}_{j=1}^N$$

Alternatively, we can motivate a different scheme that properly handles complex eigenvalues. We could sort the spectrum by the norm of its entries; denote this as ordering using the *norm scheme*. This way, all the eigenvalues are mapped to a real value, in which we could use the sign-scheme of ordering. Without further ado, we write the *norm-ordered spectrum* as follows:

$$\sigma_N(P) = \{\lambda_j : |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|\}_{j=1}^N$$

Note that when we take the norms of the eigenvalues, we essentially ignore "rotational" features of the eigenvalues. Signs of eigenvalues indicate reflection or rotation, so when we take the norm, we essentially become more concerned with scaling.

For example, consider the following plots, showing the difference in using the sign and norm ordering schemes for the same spectrum.

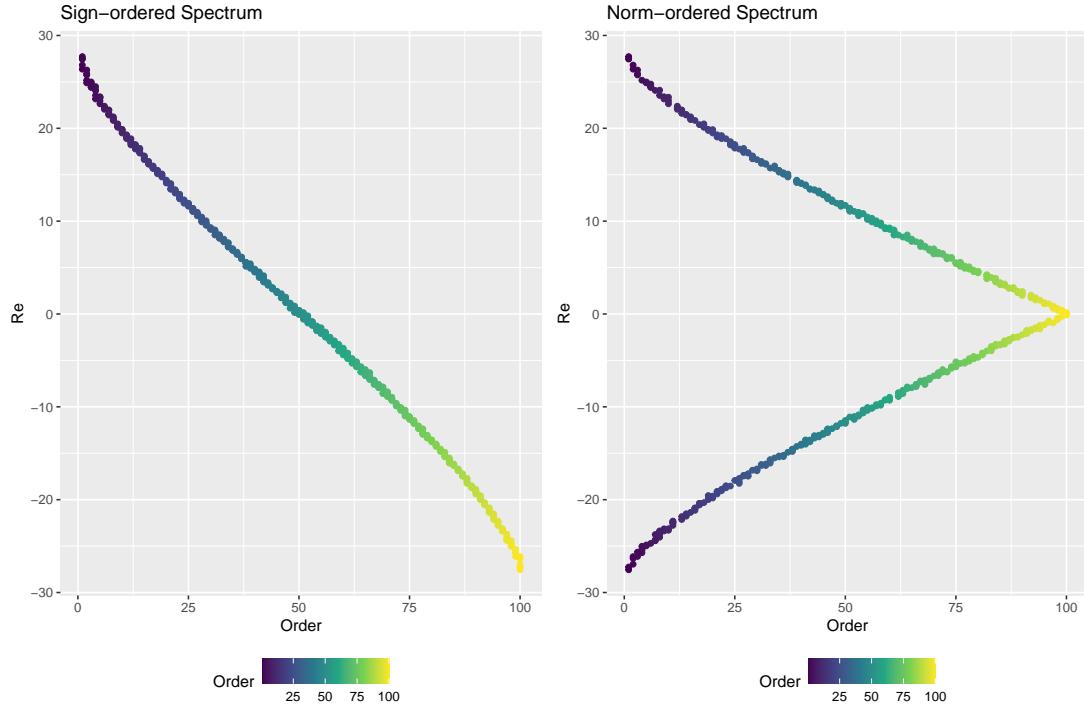


Figure 2.2: Spectrum displaying two different ordering scheme

Singular Values

An alternative to using the norm ordering scheme is using the singular values of the matrix. If a matrix is symmetric, the singular values are simply the norm of the eigenvalues. We ignore rotational features and focus solely on scale when we do so.

Suppose P is a random matrix. Then, we can take its singular values as such.

Definition 2.2.1 (Singular Values). *The singular values of a matrix P are given by the square root of the eigenvalues of the corresponding product of that matrix and its transpose. That is, $\sigma_+(P) = \sqrt{\sigma(P \cdot P^T)}$.*

Summary Table of Spectrum Schema

Table of Spectrum Schema			
Scheme	Matrix	Notation	Ordering
Sign-Ordered	P	$\sigma_S(P)$	$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$
Norm-Ordered	P	$\sigma_N(P)$	$ \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N $
Singular	$P \cdot P^T$	$\sigma_+(P)$	$\sqrt{\lambda_1} \geq \sqrt{\lambda_2} \geq \dots \geq \sqrt{\lambda_N}$

Order Statistics

With eigenvalue ordering unambiguous and well-defined, we may proceed to start talking about their order statistics. In short, given a random sample of fixed size,

order statistics are random variables defined as the value of an element conditioning on its rank within the sample. (See A.x)

In general, order statistics are quite useful and tell us a lot about how the eigenvalues distribute given a distribution. They tell us how the eigenvalues space themselves and give us useful upper and lower bounds.

For example, the maximum of a sample is an order statistic concerned with the highest ranked element. In our case, this could correspond the largest eigenvalue of a spectrum. After all, a spectrum is a random sample of fixed size, so this statistic is well-defined.

Example (The Largest Eigenvalue). *Suppose we have seek the largest eigenvalue distribution for a ensemble distribution \mathcal{D} , we would simulate an ensemble \mathcal{E} and observe λ_1 for each of its matrices. Then, we can set the distribution of the largest eigenvalue for \mathcal{D} by observing the distribution of λ_1 .*

So, we will consider the conditional order statistics $\mathbb{E}(\lambda_i \mid i)$ and $\text{Var}(\lambda_i \mid i)$.

2.2.2 Case Study: Perron-Frobenius Theorem

Definition 2.2.2 (Spectral Radius). *Let P be any matrix and $\sigma(P)$ be its ordered spectrum. Then, the spectral radius of P is defined as $\rho(P) = \|\sup \sigma(P)\|$ is the norm of its largest eigenvalue.*

Detour: The Perron-Frobenius Theorem

Using the toolkit we have now acquired, we can now discuss an elegant, visual representation of the Perron-Frobenius theorem (See A.x). To put it shortly, the Perron-Frobenius theorem, applied to stochastic matrices is a result that guarantees a stationary distribution to an ergodic Markov Chain (see A.x). Sparing the details, we can use the conjunction of two facts, that if true, yield the result for the Perron-Frobenius Theorem for ergodic Markov chains through stochastic matrices.

1. The largest eigenvalue of stochastic matrices is 1.
2. When multiplied by P^K (for large K), any point v enters the eigenspace of the matrix's largest eigenvalue. That is, vP^K is an eigenvector of P .

Again, those two statements were not formally proven in this thesis. However, there are large amounts of computational evidence for both of these results. Consider the first statement.

Below we have a stochastic ensemble spectrum. Notice how the largest eigenvalue is 1, living far from the island of **complex** eigenvalues.

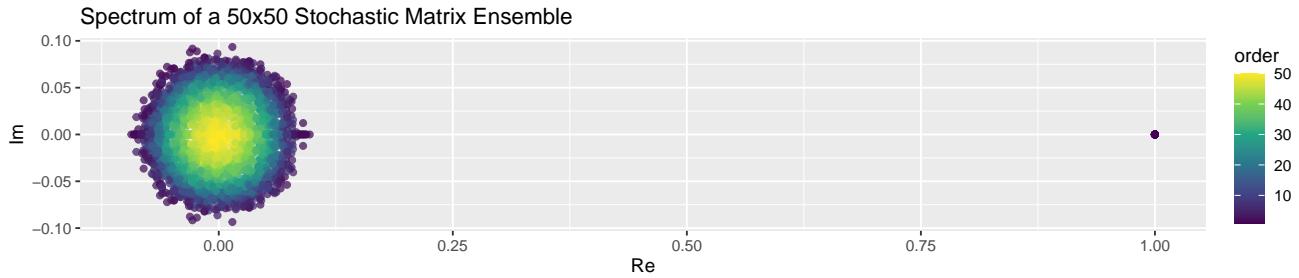


Figure 2.3: Spectrum of a Stochastic Matrix ensemble

This shows that the spectral of stochastic matrices is 1! What is left is the other fact. For the second statement, again, there is plenty of computational evidence to back this up. This is the reason why **Appendix D** is included. This is a side project that started in this thesis, simulating eigenvectors computationally. From the simulations, we find that there is overwhelming evidence that most matrices, if not all, satisfy the second statement.

Conclusions. So, to conclude, we can say that the computational evidence for these two facts support and provide an alternative method of seeing why the Perron-Frobenius theorem is true.

2.3 Symmetric and Hermitian Matrices

A very important class of matrices in Linear Algebra is that of Symmetric or Hermitian matrices (See A.x). Simply put, those are matrices which are equal to their conjugate transpose.

Note: Since real numbers are their own conjugate transpose, every Symmetric matrix is Hermitian. However, we will still delineate the two terms to avoid confusion.

In any case, one critical result in Linear Algebra that will be extensively wielded in this thesis is the fact that a matrix is Symmetric or Hermitian if and only if it has real eigenvalues. In other words:

$$P = \overline{P^T} \implies \sigma(P) = \{\lambda_i \mid \lambda_i \in \mathbb{R}\}$$

Having a complete set of real eigenvalues yields many great properties. For instance, if all eigenvalues are real, we have the option of observing either the sign-ordered spectrum or the norm-ordered spectrum. This way, we can preserve negative signs and we would not lose the rotational aspect of the eigenvalue when we study its statistics. That is just one reason out of many more why having real eigenvalues is quite nice.

Example: Stochastic Matrices

One very pleasing example to look at is stochastic matrices. As seen previously in **Section 2**, stochastic matrices tend to have two components: a complex disk of eigenvalues about the origin and singularity that is the largest eigenvalue.

Below we have a **symmetric** stochastic ensemble spectrum. Notice how the largest eigenvalue is 1, living far from the island of **real** eigenvalues. This can be compared to the spectrum of non-Symmetric stochastic matrices in the previous figure. The patterns are similar, but now, imposing symmetry on the matrices forces the eigenvalues to “collapse” onto the real line!

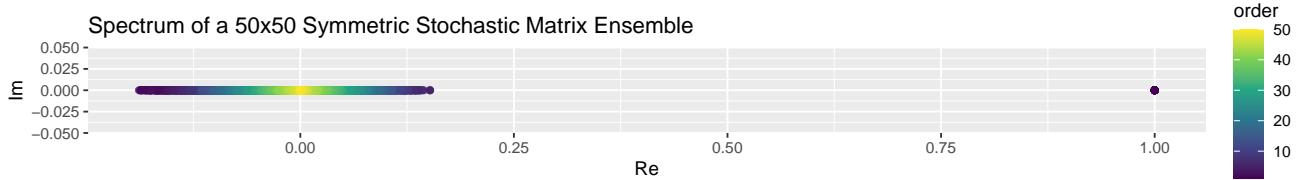


Figure 2.4: Spectrum of a Symmetric Stochastic Matrix ensemble

2.3.1 Wigner's Semicircle Distribution

The eigenvalues of Hermitian matrices obey Wigner's Semicircle distribution. Since Hermitian matrices have real eigenvalues, then we can be more precise and generally say that the real component of the eigenvalues follow the semicircle distribution. The distribution is named after physicist Eugene Wigner.

Definition 2.3.1 (Semicircle Distribution). *If a random variable X is semicircle distributed with radius $R \in \mathbb{R}^+$, then we say $X \sim SC(R)$. X has the following probability density function:*

$$P(X = x) = \frac{2}{\pi R^2} \sqrt{R^2 - x^2} \text{ for } x \in [-R, R]$$

Remark (Radius and Matrix Dimension). *The dimension of the matrix determines the radius of the eigenvalues. Namely, if a Hermitian matrix P is $N \times N$, then its eigenvalues are semicircle distributed with radius $R = 2\sqrt{N}$. That is, P^\dagger has a spectrum $\sigma(P) \sim SC(2\sqrt{R})$.*

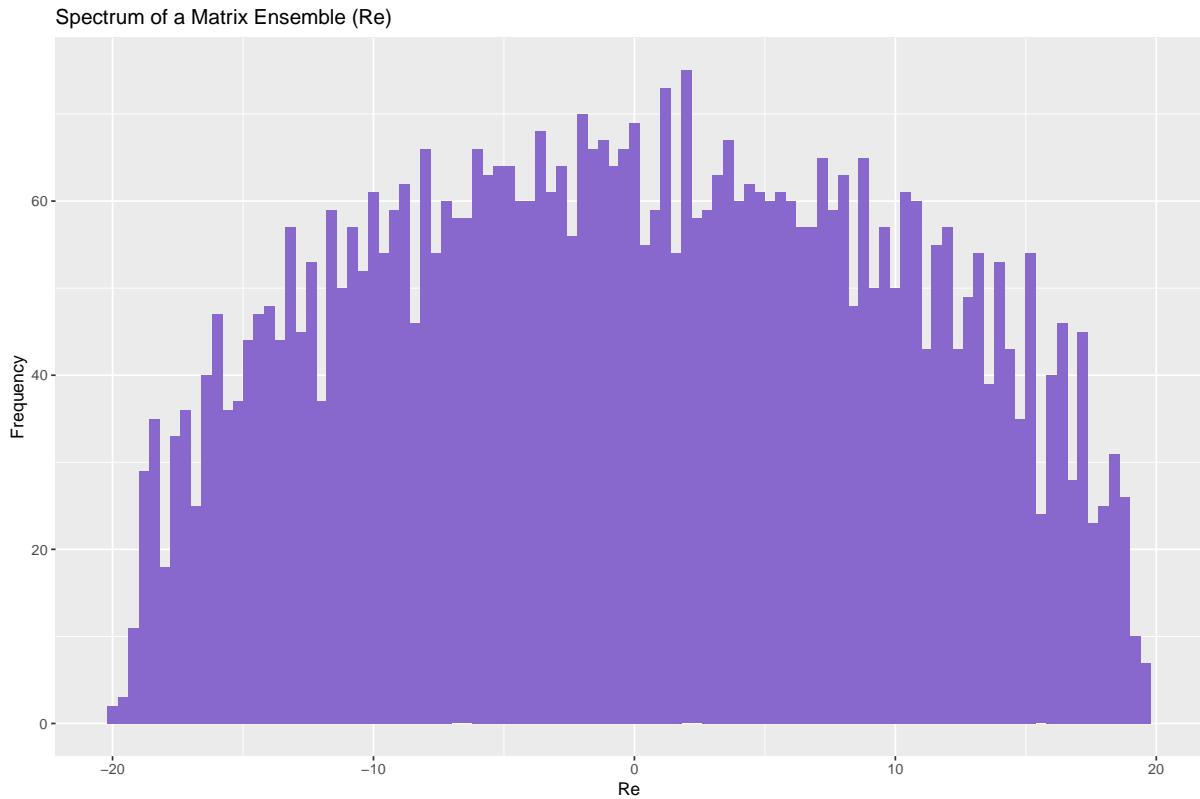


Figure 2.5: Eigenvalues of a Symmetric Matrix displaying the Semicircle Distribution

2.4 Findings

2.4.1 Uniform Ensembles

Here we have an ensemble of $\mathcal{E} \sim \text{Unif}(0, 1)$ matrices. We see a resemblance to stochastic matrices.

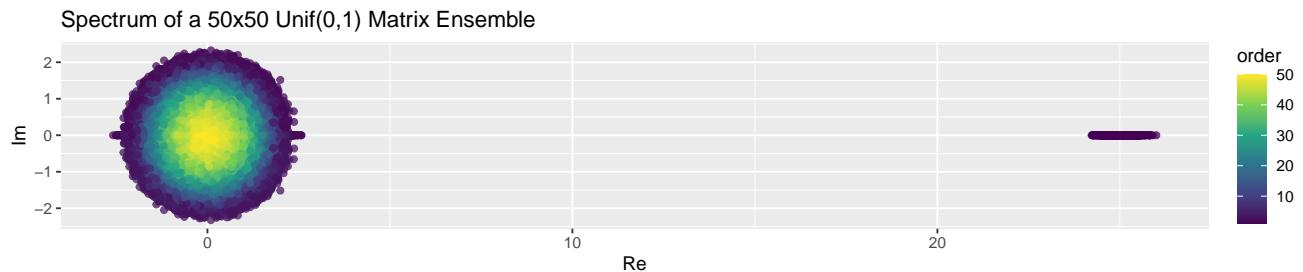


Figure 2.6: Spectrum of a $\text{Unif}(0,1)$ ensemble

2.4.2 Erdos p-Ensembles

Here we have the second largest eigenvalue of the Erdos-Renyi ensemble. We parameterize the eigenvalue by p and we observe a trend below.



Figure 2.7: 2nd Largest Eigenvalue of an Erdos-p ensemble

2.4.3 Normal Matrices

Normal matrices tend to have eigenvalues distributed about the complex disk.

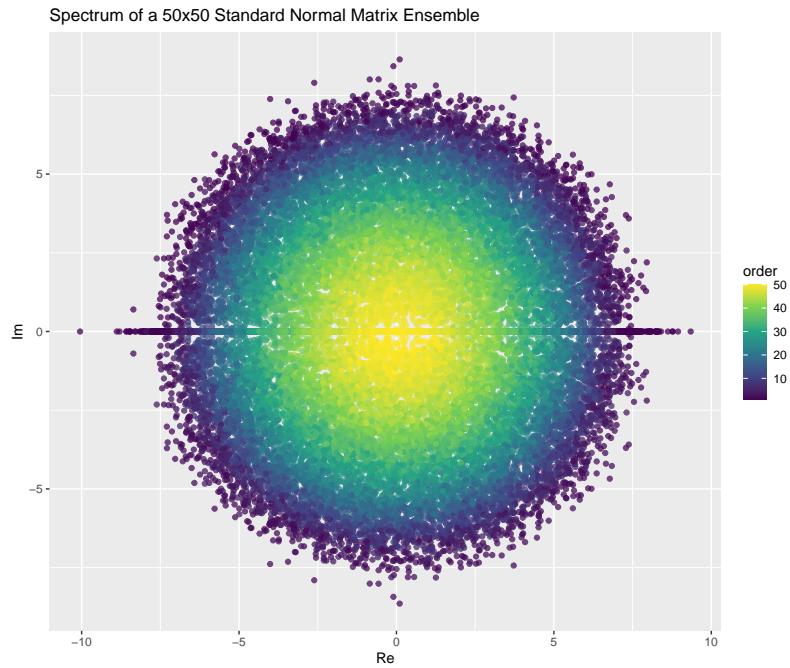


Figure 2.8: Spectrum of a Standard Normal Matrix ensemble

Hermitian Complex Normal matrices tend to have eigenvalues distributed about the complex disk.

Remark (Floating Point Errors). *Because the model involves finding coefficients of polynomials with rounded entries (by computing the determinant), floating point errors and algorithmic systemic error will yield small, but negligible imaginary components.*

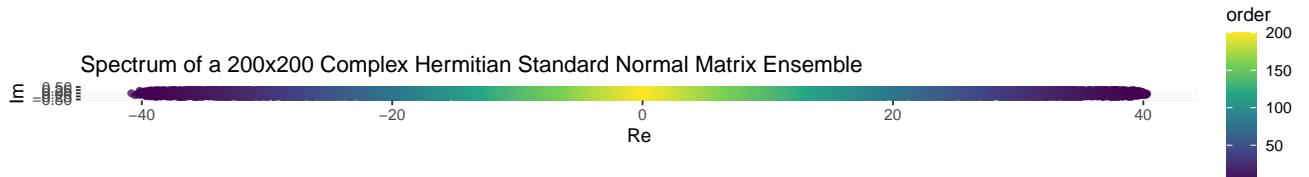


Figure 2.9: Spectrum of a Complex Hermitian Standard Normal Matrix ensemble

Chapter 3

Dispersions

Distance doesn't exist, in fact, and neither does time. Vibrations from love or music can be felt everywhere, at all times.

Yoko Ono

3.1 Introduction

In this section, we define the final spectral statistic studied in this chapter: eigenvalue dispersions. As the name suggests, these statistics are concerned with the distribution of the spacings between the eigenvalues. Interestingly, this is almost as literal as it gets when we use the word "spectral". In physics and chemistry, atomic spectra are essentially differences between energy levels or quanta, so the translation is close.

In any case, we will begin this chapter by first motivating a few definitions and formalisms in this section. Then, once our setup is ready, we will motivate the definition of a matrix's eigenvalue dispersion and formalize it as a statistic. To outline the section, we will first define two things: the dispersion metric and the pairing scheme. In simple terms, we formalize **what** "eigenvalue spacings" are and **which** eigenvalue pairs' spacings to consider. So, to begin, we first formally define an eigenvalue pair.

Definition 3.1.1 (Eigenvalue Pair). *Suppose P is a matrix and $\sigma(P)$ is its ordered spectrum. Then, an eigenvalue pair with respect to this ordered spectrum is denoted π_{ij} . It is defined as the ordered pair $\pi_{ij} = (\lambda_i, \lambda_j)$.*

Consecutive Pairs

In the future section where we define pairing schemes, we will introduce new notation of eigenvalue pair with one index is introduced when defining the consecutive pairing scheme Π_C , and it takes the form $\tilde{\pi}_j$. This notation is used to denote the consecutive eigenvalue pair for a given matrix. The consecutive eigenvalue pairs are

so special that we denote them with a unique notation for convenience. It also makes the discussion regarding order statistics more intrinsic.

Definition 3.1.2 (Consecutive Pairs). *Suppose P is a matrix and $\sigma(P)$ is its ordered spectrum. Then, let $\tilde{\pi}_j$ denote a pair of consecutive eigenvalues, the largest of the two being the j^{th} largest eigenvalue. So, $\tilde{\pi}_j = (\lambda_{j-1}, \lambda_j)$.*

Since the consecutive pair scheme can be sufficiently indexed by one index (j), we will use the convention of omitting the index of the smaller eigenvalue to be more concise and idiomatic. So, whenever one sees the notation $\tilde{\pi}_j$, think the j^{th} largest eigenvalue and its smaller neighbour.

Example (The Largest Eigenvalues). *With this notation at hand, we say that $\tilde{\pi}_1$ represents the pair of the two largest eigenvalues in an ordered spectrum.*

$$\tilde{\pi}_1 = (\lambda_2, \lambda_1)$$

Without further ado, we now motivate the dispersion metric.

3.1.1 Dispersion Metrics

Before we may even start to consider studying dispersions of eigenvalues, we must first formalize and make clear what “metric” of spacing we are using. To do so, we motivate the dispersion metric. In simple words, a dispersion metric is a function that takes in a pair of eigenvalues and returns a positive real number that represents some metric of dispersion or spacing.

Definition 3.1.3 (Dispersion Metric). *A dispersion metric $\delta : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}^+$ is defined as a function from the space of pairs of complex numbers to the positive reals. In simple terms, it is a way of measuring “space” between two complex numbers - our eigenvalues.*

In the scope of this thesis, we consider the following dispersion metrics below.

1. The standard norm: $\delta(z, z') = |z' - z|$
2. The β -norm: $\delta_\beta(z, z') = |z' - z|^\beta$
3. The difference of absolutes: $\delta_{\text{abs}}(z, z') = |z'| - |z|$

Remark (Symmetric Metrics). *Note that the standard norm and the β -norm are symmetric operations. This means that switching the order of arguments will have no effect on the function’s output. Otherwise, the difference of absolutes metric is not symmetric.*

While we have defined dispersion metrics to be functions from \mathbb{C}^2 , there is one special case where we can make an exception so that the domain of δ is not \mathbb{R}^+ .

Remark (Identity Difference Heuristic). *Suppose we take the arithmetic difference of two complex numbers. Then, the range of δ is \mathbb{C} . For this reason, we won't consider the arithmetic difference a formal dispersion metric, but we will honor it as a dispersion heuristics. As such, we will denote this as the "identity difference" heuristics and call it δ_{id} . So, we define $\delta_{id} : (z, z') \mapsto z' - z$*

Summary Table of Dispersion Metrics

For every dispersion metric, assume the functions' order of arguments is $\delta(z, z')$.

Table of Dispersion Metrics				
Metric*	Notation	Formula	Symmetric	Parameters
Standard Norm	δ	$ z' - z $	True	-
β -Norm	δ_β	$ z' - z ^\beta$	True	$\beta \in \mathbb{N}$
Difference of Absolutes	δ_{abs}	$ z' - z $	False	-
Identity Difference	δ_{id}	$z' - z$	False	-

*Note that the identity difference is a heuristic, and not a formal metric.

3.1.2 Pairing Schema

The next thing we need to motivate before talking about eigenvalue dispersions are pairing schema. In simple terms, pairing schema are templates (for some $N \in \mathbb{N}$) for which eigenvalue pairs to pick. There are many subtle reasons why this is important, which will be covered in detail later. Without further ado, we motivate the pairing schema.

Before we may begin talking about pairing scheme, we define an auxilliary object - the spectral pairs of a matrix, which we denote $\sigma^{(2)}$. Since we are now talking about eigenvalue pairs, it is helpful to define this object before proceeding.

Definition 3.1.4 (Spectral Pairs). *Suppose P is an $N \times N$ random matrix. Then, taking the spectral pair of P , denoted $\sigma^{(2)}(P)$ is equivalent to taking the Cartesian product of its ordered spectrum $\sigma(P)$. That is, $\sigma^{(2)}(P) = \sigma(P) \times \sigma(P) = \{(\lambda_i, \lambda_j) \mid i, j \in \mathbb{N}_N\}$.*

Now, to select eigenvalue pairs, we motivate the pairing scheme - this is what tells us which indices to select.

Definition 3.1.5 (Pairing Scheme). *Suppose P is any $N \times N$ matrix and $\sigma^{(2)}(P)$ are its spectral pairs. A pairing scheme for the matrix P is a subset of $\mathbb{N}_N \times \mathbb{N}_N$ - a subset of pairs of numbers from $\mathbb{N}_N = \{1, \dots, N\}$. In other words, it is a subset of pair indices for N objects - in our case, eigenvalues. We denote a pairing scheme as a set $\Pi = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{N}_N\} \subseteq \mathbb{N}_N \times \mathbb{N}_N$. To take a matrix's spectral pairs with respect to Π , we simply take the set of eigenvalue pairs with the matching indices, $\sigma^{(2)}(P \mid \Pi) = \{(\lambda_\alpha, \lambda_\beta) \mid (\alpha, \beta) \in \Pi\}$.*

The reader might find this definition slightly obscure, and rightfully so. The definition is a mere formality, as we will usually only consider a few specific pairing schema. Seeing the explicit examples will hopefully make things more clear. With all the technical details aside, we can just say that a pairing scheme tells us which subset of eigenvalue pairs to consider. If one visualizes an array with N objects on two axes, we are simply choosing a subset of that plane. In fact, consider the following.

Remark (Proper Subset). *If $\sigma^{(2)}(P)$ is the spectral pairs of the matrix P , then for any pairing scheme Π , we will find that $\sigma^{(2)}(P | \Pi) \subseteq \sigma^{(2)}(P)$. By definition, taking a spectral pair with respect to some pairing scheme constricts which pairs to select - meaning it is a proper subset of the general spectral pairs.*

Common Pairing Schema

Suppose P in an $N \times N$ square matrix, and $\sigma^{(2)}(P)$ are its spectral pairs. For every pairing scheme, assume $i, j \in \mathbb{N}_N$.

1. The unique pair combinations schema are two complementary pair schema. By specifying **either** $i > j$ or $i < j$, we characterize this scheme to entail all unique pair combinations of eigenvalues without repeats. The reason we call them upper and lower pair combinations is an allusion to the indices of the upper and lower triangular matrices.
 - (a) Let $\Pi_>$ be the lower-pair combinations of ordered eigenvalues. This will be the standard unique pair combination scheme used in lieu of the argument orders of our dispersion metrics (more later). In this pairing scheme, the eigenvalue with the lower rank is always listed first, and the higher rank second.
$$\sigma^{(2)}(P | \Pi_>) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i > j\}_{i=1}^{N-1}$$
 - (b) For completeness, we will also define the upper-pair scheme. $\Pi_<$ is the set of upper-pair unique combinations of ordered eigenvalues. Wont be used because we want bigger - smaller to make positive definite.
$$\sigma^{(2)}(P | \Pi_<) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i < j\}_{i=1}^{N-1}$$

Benefits: Solves the issue of repeated pairs for symmetric dispersion metrics.

2. Let Π_C be the consecutive pairs of eigenvalues in a spectrum.

$$\sigma^{(2)}(P | \Pi_C) = \{\tilde{\pi}_j = (\lambda_{j+1}, \lambda_j)\}_{j=1}^{N-1}$$

Benefits: This pairing scheme gives us the minimal information needed to express important bounds and spacings in terms of its elements.

3. Let Π_0 be all the pairs in a spectrum. For completeness, we define this pairing scheme as the implicit pairing scheme for the spectral pairs of a matrix.

$$\sigma^{(2)}(P | \Pi_0) = \sigma^{(2)}(P) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i, j \in \mathbb{N}_N\}$$

Consider some examples of generating pairing schemes in R. These methods are not exported in the **RMAT** package.

Code Example (Consecutive Pairing Scheme). *We generate the pairing scheme (indices) for what would be a 5×5 matrix.*

```
# Helper function in the source code
pair_indices <- .consecutive_pairs(N = 5)
# Outputs the following
pair_indices
...
      i   j
[1,] 2   1
[2,] 3   2
[3,] 4   3
[4,] 5   4
```

Code Example (Lower Pairing Scheme). *We generate the pairing scheme (indices) for what would be a 4×4 matrix.*

```
# Helper function in the source code
pair_indices <- .unique_pairs_lower(N = 4)
# Outputs the following
pair_indices
...
      i   j
[1,] 2   1
[2,] 3   1
[3,] 3   2
[4,] 4   1
[5,] 4   2
[6,] 4   3
```

Think of the pairing schemes as a conditional argument when considering dispersions. You ask yourself: which pairs do I want to look at? There will be a discussion in **Section 3.2** about why we have to formalize the notions of dispersion metrics and pairing schemes when we eventually talk about dispersions.

Additionally, making the pairing schemes independent from the matrices (taking only their dimension, $N \in \mathbb{N}$) was a difficult decision. It may cloud understanding initially but the payoff comes if you consider the implementation of the functions in **RMAT**. As a whole, you should not be bothered by the details of formalizing the pairing schemes but you should definitely understand what they do. In fact, it's totally fair that you do. The **RMAT** implementation abstracts this all away in the dispersion function. The user would input a string denoting which pairing scheme and the function would handle it. Similarly, you should do the same!

Summary Table of Pairing Schema

Suppose P is an $N \times N$ matrix. Then, its spectral pairs $\sigma^{(2)}(P)$ may take on the following pairing schemes.

Table of Pairing Schema		
Scheme	Notation	Formula
Lower	$\Pi_<$	$\{(i, j) \mid i < j \text{ for } i, j \in \mathbb{N}_N\}$
Upper	$\Pi_>$	$\{(i, j) \mid i > j \text{ for } i, j \in \mathbb{N}_N\}$
Consecutive	Π_C	$\{(i, j) \mid i = j + 1 \text{ for } i, j \in \mathbb{N}_N\}$
All	Π_0	$\{(i, j) \mid i, j \in \mathbb{N}_N\}$

Another way we can define the pairing scheme is defining an auxilliary object: the eigenvalue (pair) matrix.

An Alternative Represetation: Eigenvalue Matrix

Definition 3.1.6 (Eigenvalue Matrix). *Suppose P is an $N \times N$ square matrix and $\sigma^{(2)}(P)$ are its spectral pairs. Then, the eigenvalue matrix of P , given by $\Lambda(P)$ is the matrix with entries $\pi_{ij} = (\lambda_i, \lambda_j)$ for $\pi_{ij} \in \sigma^{(2)}(P)$. Again, it is given that the eigenvalues λ_i come from some **ordered** spectrum $\sigma(P)$ as per the definition of spectral pairs.*

With the matrix analogy, describing the pairing schemes can be much simpler. For instance, the **lower pairs** are given by the indices of the entries in the lower triangle of the matrix. Analogously, the **upper pairs** by those of the upper triangle of the matrix. The **consecutive pairs**, given our default lower pair scheme, are given by the lower main off-diagonal band of the matrix.

Example (Eigenvalue Matrix for a 5×5 Matrix). *Suppose we have a 5×5 matrix P and its corresponding spectral pairs $\sigma^{(2)}(P)$. Then, its eigenvalue matrix has the following strucutre:*

$$\begin{bmatrix} - & \pi_{12} & \pi_{13} & \pi_{14} & \pi_{15} \\ \tilde{\pi}_1 & - & \pi_{23} & \pi_{24} & \pi_{25} \\ \pi_{31} & \tilde{\pi}_2 & - & \pi_{34} & \pi_{35} \\ \pi_{41} & \pi_{42} & \tilde{\pi}_3 & - & \pi_{45} \\ \pi_{51} & \pi_{52} & \pi_{53} & \tilde{\pi}_4 & - \end{bmatrix}$$

3.1.3 Dispersions

Alas, with dispersion metrics and pairing schemes defined, we are finally able to motivate the definition of a matrix dispersion for both singleton matrices and their ensemble counterparts.

Definition 3.1.7 (Dispersion). Suppose P is an $N \times N$ matrix, and $\sigma^{(2)}(P)$ are its spectral pairs. The dispersion of P with respect to the pairing scheme Π and dispersion metric δ is denoted by $\Delta_\delta(P | \Pi)$ and it is given by the following:

$$\Delta_\delta(P | \Pi) = \{\delta(\pi_{ij}) \mid \pi_{ij} \in \sigma^{(2)}(P | \Pi)\}$$

Consider the following code example, generating the dispersion of standard normal 5×5 matrix with respect to the consecutive pairing scheme.

Code Example (Consecutive Pair Dispersion of a Standard Normal Matrix). In our notation, we are simulating the dispersion of $P \sim \mathcal{N}(0, 1)$ where $P \in \mathbb{R}^{5 \times 5}$. Specifically, we are simulating $\Delta(P | \Pi_C)$. In the code implementation, we obtain an array for every dispersion metric δ .

```
library(RMAT)
P <- RM_norm(N = 5, mean = 0, sd = 1)
disp_P <- dispersion(P, pairs = "consecutive")
# Outputs the following
disp_P
...
i j eig_i eig_j id_diff iddiff_norm abs_diff diff_ij
2 1 -0.54-1.35i -0.54+1.35i 0.00+2.71i 2.71 0.00 1
3 2 0.23+1.43i -0.54-1.35i -0.77-2.78i 2.88 0.02 1
4 3 0.23-1.43i 0.23+1.43i 0.00+2.85i 2.85 0.00 1
5 4 -0.87+0.00i 0.23-1.43i 1.09-1.43i 1.80 0.57 1
```

Next, we extend the definition of dispersion for an ensemble as we usually do.

Definition 3.1.8 (Ensemble Dispersion). If we have an ensemble \mathcal{E} , then we can naturally extend the definition of $\Delta_\delta(\mathcal{E} | \Pi)$. To take the dispersion of an ensemble, simply take the union of the dispersions of each of its matrices. In other words, if $\mathcal{E} = \{P_i \sim \mathcal{D}\}_{i=1}^K$, then its dispersion is given by:

$$\Delta_\delta(\mathcal{E} | \Pi) = \bigcup_{i=1}^K \Delta_\delta(P_i | \Pi)$$

Consider the following code example, generating the dispersion of a beta ensemble with respect to the consecutive pairing scheme.

Code Example (Consecutive Pair Dispersion of a Beta Ensemble). In our notation, we are simulating the dispersion of $\mathcal{E} \sim \mathcal{H}(\beta = 4)$ where $P \in \mathbb{R}^{5 \times 5}$. Specifically, we are simulating $\Delta(P | \Pi_C)$. In the code implementation, we obtain an array for every dispersion metric δ .

```
library(RMAT)
ens <- RME_beta(N = 4, beta = 4, size = 3)
disp_ens <- dispersion(ens, pairs = "consecutive")
# Outputs the following
```

```
disp_ens
...
i j eig_i     eig_j     id_diff iddf_norm abs_diff diff_ij
2 1 -3.78+0i 4.00+0i  7.78+0i 7.78      0.22      1
3 2 2.06+0i -3.78+0i -5.84+0i 5.84      1.72      1
4 3 0.19+0i  2.06+0i  1.88+0i 1.88      1.88      1
2 1 3.80+0i -4.00+0i -7.80+0i 7.80      0.20      1
3 2 -1.80+0i 3.80+0i  5.60+0i 5.60      2.00      1
4 3 0.89+0i -1.80+0i -2.69+0i 2.69      0.92      1
2 1 3.51+0i -3.53+0i -7.04+0i 7.04      0.03      1
3 2 1.35+0i  3.51+0i  2.16+0i 2.16      2.16      1
4 3 -0.67+0i 1.35+0i  2.02+0i 2.02      0.68      1
```

3.2 Dispersion Analysis

With dispersions well defined, we provide a few guidelines for analyzing a dispersion of a matrix. We will synthesize all the techniques, notations, and remarks in the previous section to provide a comprehensive manual on how to analyze the dispersion of a matrix.

3.2.1 Considerations

Here are some considerations that should be taken into account when studying dispersions.

Dispersion metrics, linear combinations, and the relation to pairing schemes.

Bounds and Considerations

One thing to consider from this dispersion set is what we could learn from it. The dispersion of a matrix P , $\Delta_d(P)$ could tell us quite a few things. One thing it could tell us is the distribution of the dispersion of eigenvalues given that we uniformly and randomly select a pair of eigenvalues; this works when d is a symmetric function. However, if we are concerned about spacings, then we must take into consideration a few facts.

Linear Combinations

The identity difference metric is not necessarily a "bad metric". In fact, it is one of the few metrics which allow transitivity under composition - which brings up an important detail. In the total dispersion $\Delta_d(P)$, there is some redundant information if we are totally concerned with spacings. Namely, consider the following fact:

$$\lambda_1 - \lambda_3 = (\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_2)$$

In other words, if d is the ID dispersion metric, then $d(\pi_{13}) = d(\pi_{12}) - d(\pi_{32})$. We could say the second eigenvalue is a pivot in which we could perform right-cancellation.

Triangle Inequality

However, as we said previously, only the identity difference metric gives us transitive properties. On the other hand, the other metrics provide us bounds when we compose them. They are given by a simple application of the reverse triangle inequality. Namely, consider the following fact:

$$||y| - |x|| \leq |y - x|$$

If we let d_1 be the difference of absolutes metric and d_2 be the standard norm, then we could rephrase this as saying $|d_1(\pi_{ij})| \leq d_2(\pi_{ij})$. This is just a rephrasing of the inequality. However, if we are careful enough, we can remove the absolute value to obtain a more meaningful upper bound. The value of $d_1(\pi_{ij})$ can be interpreted as the difference between the sizes of λ_j and λ_i . However, recall that in the standard definition of a dispersion, we use the **lower** pair combinations so that we get $i > j$, making the value $|\lambda_j|$ always greater than $|\lambda_i|$ by construction. As such, the left-hand value would always be positive, allowing us to drop the absolute value. So, we could say:

$$d_1(\pi_{ij}) \leq d_2(\pi_{ij}) \text{ given } \pi_{ij} \in \Pi_>$$

Sufficiency of Consecutive Pairs

Recall from previously that with respect to the identity difference, some elements of the dispersion $\Delta_d(P)$ can be expressed as a linear combination of other elements. Note that using those linear combinations and the triangle inequality, we could derive the analogous bounds. It is quite useful to impose a condition that none of the values are a linear combination of the other.

So, we motivate a more restrictive definition of matrix dispersion: the consecutive-values dispersion $\Delta_d(P, \Pi_c)$. This dispersion is essentially a special case of the regular dispersion, but with a particular pairing scheme Π_c . To avoid this issue of linear combinations, we simply take the pairs of consecutive eigenvalues. In other words, $\Pi_c = \{(\lambda_i, \lambda_j) \in \sigma_P \times \sigma_P \mid j = i + 1\}_{i=1}^{N-1}$. This pairing scheme enumerates the pairs such that the dispersion metric has a natural mapping - it is simply the "distance" between an eigenvalue and the one that is one rank smaller than it.

3.2.2 Order Statistics

With eigenvalue dispersions and eigenvalue orderings well-defined, we may proceed to start talking about their order statistics.

In **Chapter 2**, we observed simple order statistics regarding the eigenvalues. This time around, we have two eigenvalues being compared at once, so there needs to be further setup with regards to dispersion order statistics.

However, prior to introducing any new concepts, there is one scenario where using one index is sufficient in terms of discussing dispersion order statistics.

Remark (Consecutive Pair Dispersions). *When we consider the dispersion with respect to the consecutive pairs, things are much simpler since our pairs are intrinsically defined by one index (j). This is because we are observing the j^{th} eigenvalue and its smaller neighbour. As such, we will consider dispersion statistics in the form of $\mathbb{E}(\tilde{\pi}_j \mid j)$ and $\text{Var}(\tilde{\pi}_j \mid j)$.*

Otherwise, we synthesize a new order statistic that takes in two indices (from a given pair) called the **ranking difference class**. Since we are no longer observing a single eigenvalue at a given rank, we will need a way to standardize observing a pair of eigenvalues at a time. To do so, we introduce a new **equivalence relation** (see A.x) called the **ranking difference**. As the name suggests, it is precisely the integer difference of the eigenvalue orders.

Ranking Difference

Definition 3.2.1 (Ranking Difference). *The ranking difference is a function $\rho : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ which takes the index of two ordered eigenvalues and returns their difference. In other words, it is the function $\rho : (\lambda_i, \lambda_j) \mapsto (i - j)$.*

With the ranking difference, we may now take the spectral pairs of some random matrix with respect to either the lower ($\Pi_<$) or upper ($\Pi_>$) pairing scheme and partition the eigenvalues into distinct equivalence classes.

Remark (Equivalence Relation). *Formally speaking, we may define \sim_ρ as an equivalence relation. It satisfies all three properties of reflexivity, symmetry, and transitivity. More precisely, it is an equivalence class which partitions $\mathbb{N}_N \times \mathbb{N}_N$ into $N - 1$ equivalence classes given by $[r]_\rho$ for $r \in \mathbb{N}_{N-1}$. We would define the equivalence relation \sim_ρ as follows:*

$$(\lambda_a, \lambda_b) \sim_\rho (\lambda_c, \lambda_d) \iff (a - b) = (c - d)$$

So, two eigenvalue pairs would belong to the same class $[r]_\rho$ if their ranking difference is the same. That is,

$$[r]_\rho = \{(\lambda_\alpha, \lambda_\beta) \mid \alpha - \beta = r\}$$

Remark (Matrix Analogy). *Using the eigenvalue matrix analogy, \sim_ρ partitions the matrix into diagonal bands. The diagonal represents $[0]_\rho$, the first off-diagonal represents $[1]_\rho$, and so on... Notice that this means every equivalence class has a different size.*

With this partition in mind, we can consider various statistics conditioning on the value of r . Conditioning on r will be especially useful in the cases where we are considering matrices like the Hermite- β matrices; the eigenvalues of those matrices tend to *repel*, so to speak, and we can observe these patterns using r .

Consider the following example.

Example (Ranking Differences for a 4×4 Matrix). *Suppose P is a 4×4 random matrix. Then, if we take the spectral pairs with respect to the lower pairs $\sigma^{(2)}(P \mid \Pi_<)$ and partition it by ρ , we get the following classes:*

1. $[1]_\rho = \{\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3\}$
2. $[2]_\rho = \{\pi_{31}, \pi_{42}\}$
3. $[3]_\rho = \{\pi_{41}\}$

3.2.3 Conditional Statistics

We will consider the conditional statistics $\mathbb{E}(\rho_{ij} \mid \rho)$ and $\text{Var}(\rho_{ij} \mid \rho)$.
 [Plots]

3.3 Analytical Results

3.3.1 Wigner's Surmise

Wigner's surmise is a result found by Eugene Wigner regarding the limiting distribution of eigenvalue spacings of symmetric matrices. To start talking about this, we must talk about normalized spacings, which are the precise items considered in the distribution. Before, we can talk about the normalized spacing, we define the mean spacing.

Definition 3.3.1 (Mean Spacing). *Suppose P is an $N \times N$ symmetric matrix, and $\sigma(P)$ are its real, sign-ordered eigenvalues. Then, the mean (eigenvalue) spacing, denoted $\langle s \rangle$ is the average distance between two consecutive eigenvalues. That is,*

$$\langle s \rangle = \mathbb{E}[\Delta_\delta(P \mid \Pi_C)] = \mathbb{E}[\delta(\tilde{\pi}_j)]_{j=1}^{N-1}$$

So, with the mean spacing defined, we now define the normalized spacing between a pair of consecutive eigenvalues below.

Definition 3.3.2 (Normalized Spacing). *Suppose P is an $N \times N$ symmetric matrix, and $\sigma(P)$ are its real, sign-ordered eigenvalues. Then, the normalized spacing of the j^{th} pair of eigenvalues, denoted s_j is given by the following formula.*

$$s_j = \frac{(\lambda_j - \lambda_{j+1})}{\langle s \rangle} = \frac{\delta(\tilde{\pi}_j)}{\langle s \rangle}$$

Beta Ensembles

The analytical results for the Hermite β -ensembles for $\beta = 1, 2, 4$, denoted by w_β , are stated below.

$$\begin{aligned} w_1(s) &= \frac{\pi}{2} \exp\left(-\frac{\pi}{4}s^2\right) \\ w_2(s) &= \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi}s^2\right) \\ w_4(s) &= \frac{2^{18}}{3^6 \pi^3} s^4 \exp\left(-\frac{64}{9\pi}s^2\right) \end{aligned}$$

Additionally, their simulated densities are given in the plot below.

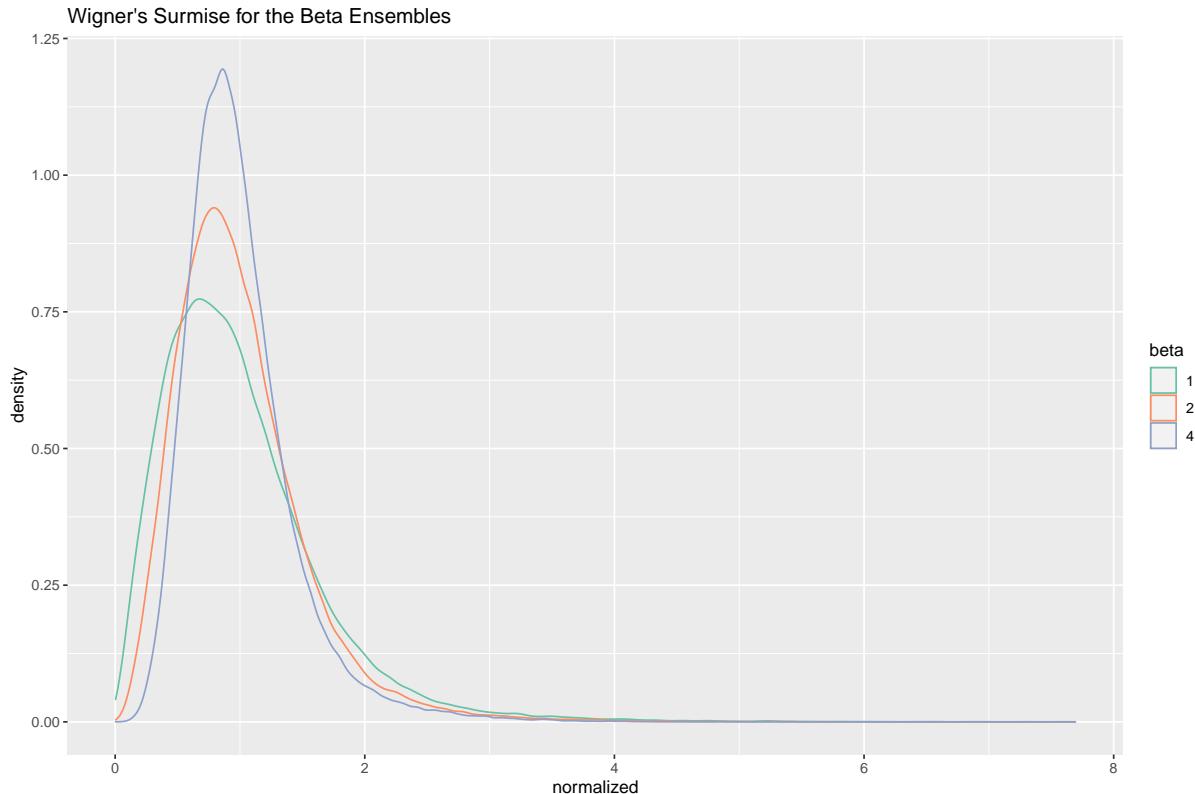


Figure 3.1: Wigner's Surmise for standard Beta Ensembles

Extending Beta Ensembles

Below, we extended the simulations beyond the standard values of $\beta = 1, 2, 4$. We notice a continuing pattern! The densities seem to have less variance as beta increases.

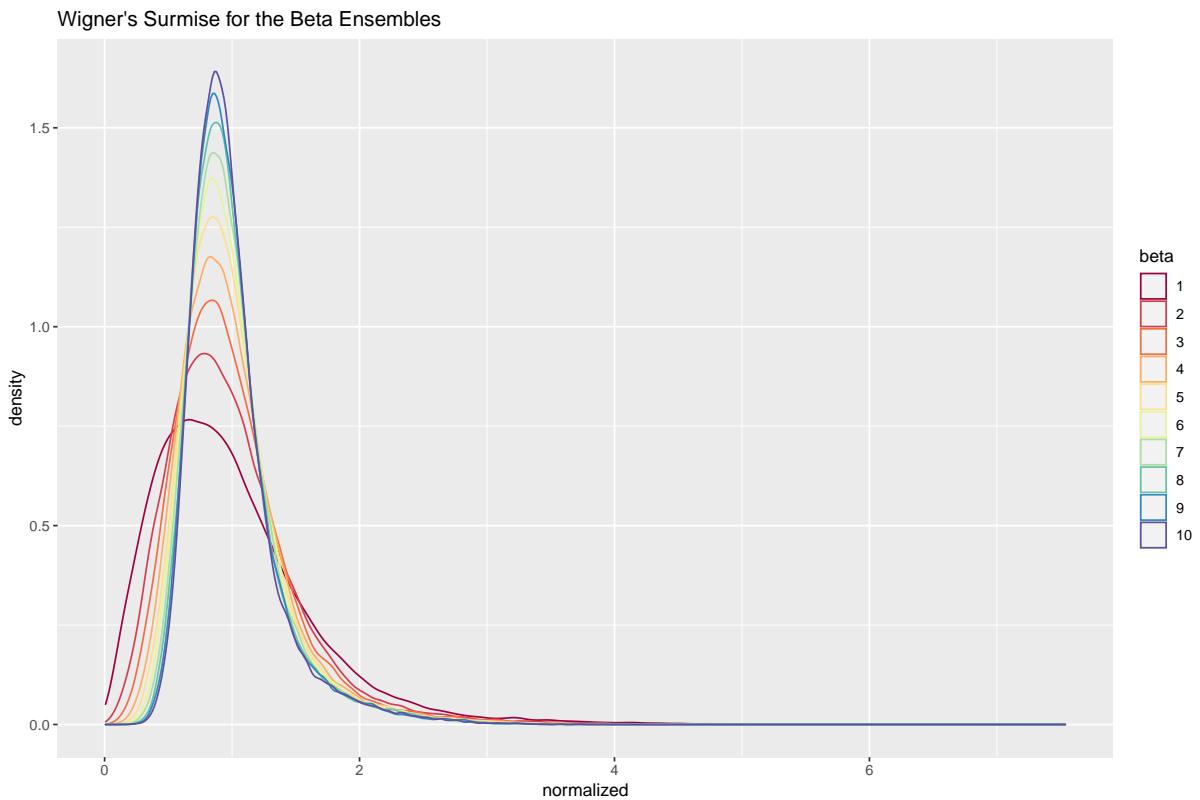


Figure 3.2: Wigner's Surmise for non-standard Beta Ensembles

3.4 Checkpoint

Here's what we've done so far.

Chapter 1

1. Formalized the following items
 - (a) Random Matrices using \mathcal{D} -distributions.
 - (b) Random Matrix Ensembles
2. Some item

Chapter 2

1. Formalized the following items
 - (a) Spectrum of a Random Matrix (Ensemble)
2. Some item

Chapter 3

1. Formalized the following items
 - (a) Spectral Pairs: the set of eigenvalue pair combinations
 - (b) Pair schemes: which eigenvalue pairs to choose
 - (c) Dispersion metrics: what distance metric to use on the pairs
 - (d) Dispersion of a Random Matrix (Ensemble)
2. Some item

Segway to Chapter 4

Here's how we'll segway to Chapter 4, which doesn't flow without this segway.

Chapter 4

β -Ensembles

With thermodynamics, one can calculate almost everything crudely; with kinetic theory, one can calculate fewer things, but more accurately; and with statistical mechanics, one can calculate almost nothing exactly.

Eugene Wigner

4.1 Introduction

In this chapter, we will talk about the Hermite β -ensembles more in depth. The beta ensembles have wide applications in statistical physics, engineering, and many other places. They are defined by the joint density of their eigenvalues, and have a special characterization discussed in the next subsection.

4.1.1 Hermite β -Ensembles

The Hermite β -ensembles, also called the Gaussian ensembles, are an important class of random matrix ensembles studied in engineering, statistical physics, and probability theory. Parameterized by $\beta \in \mathbb{N}$ through the Dyson index, this ensemble is characterized by a few things.

- The Dyson index β corresponds to the number of real number of components the subject matrices have.
- The subject matrices are classically defined for $\beta = 1, 2, 4$ and they correspond to matrices with real, complex, and quaternionic entries. The corresponding fields are \mathbb{R} , \mathbb{C} , and \mathbb{H} .

- The matrices in this ensemble, most importantly, have a feature called conjugation invariance. With respect to the conjugation by the respective group of matrices.
- Most importantly, the eigenvalues are determined by the joint probability density function given below.

Definition 4.1.1 (Hermite β -ensembles). *The Hermite β -ensembles, commonly known as the Gaussian ensembles, are an ensemble of random matrices parameterized by β , and their eigenvalues have the joint probability density function:*

$$f_\beta(\Lambda) = c_H^\beta \prod_{i < j} |\lambda_i - \lambda_j|^\beta e^{-1/2 \sum_i \lambda_i^2}$$

where the normalization constant c_H^β is given by:

$$c_H^\beta = (2\pi)^{-n/2} \prod_{j=1}^n \frac{\Gamma(1 + \beta/2)}{\Gamma(1 + \beta j/2)}$$

To simulate matrices from the β -ensemble, we will be using a recent result published in “Matrix Models for Beta Ensembles” Dumitriu & Edelman (2018). This makes the β -ensemble a canonical example of an implicitly distributed matrix; we do not care about the actual distribution of the entries, but rather the effect they have on the eigenvalues (trace) of the matrices. The algorithm used is directly cited from the results of Dumitriu’s paper, and can be found in Appendix B.X.

Code Example (Hermite Beta = 2 Ensemble). *Let $\mathcal{D} = \mathcal{H}(\beta = 2)$. We can generate $\mathcal{E} \sim \mathcal{D}$, an ensemble of 4×4 Hermite matrices ($\beta = 2$) of size 10 as such:*

```
library(RMAT)
ensemble <- RME_beta(N = 4, beta = 2, size = 10)
# Outputs the following
ensemble
...
[[10]]
 [,1]      [,2]      [,3]      [,4]
[1,] 0.7246302 1.8893868 0.00000000 0.000000
[2,] 1.8893868 1.5278221 0.68840045 0.000000
[3,] 0.0000000 0.6884004 -0.03876104 1.944495
[4,] 0.0000000 0.0000000 1.94449533 1.042741
```

Dumitriu’s Matrix Model

To generate Hermite β matrices, we consider the result given in the Matrix Models of Beta Ensembles paper Dumitriu & Edelman (2018). We obtain the following algorithm.

Algorithm (Dumitriu's Beta Matrix).

1. To simulate an $N \times N$ beta matrix, fix $N \in \mathbb{N}$.
2. Start by taking a diagonal of $\mathcal{N}(0, 2)$ variables.
3. Set both of the nearest off-diagonals to the row that samples from a $\chi(df = c_j)$ where $c_j = \beta \cdot j$ for columns spanning $j = 1, \dots, n - 1$.

4.1.2 A Physical Interpretation

4.2 Spectra

We know that β -matrices must be symmetric. So, their eigenvalues must be real. Any imaginary component we observe is simply computational error, and we may safely ignore it. It is good to see that the error is uniform and small.

Remark (Implicit Distribution). *Note that the beta ensemble is an ensemble characterized by some joint density function on its eigenvalues. So, this is a specific instance of an implicitly distributed matrix. There are many things to consider about **identifiability** that are subtle, but important. Recall that in our formalization of a specturm as a formal statistic, we vectorized the matrix then tooks the determinant of the characteristic polynomial that came about it. For this reason, we can see that while Dumitriu's model provides an explicit formula, there is an issue of identifiability. That is, given some eigenvalues, there is no injective function to the characteristic polynomial that produced it. Rather, there are infinitely many equivalence classes of characteristic polynomials (and such, random matrices) that surjectively produce a given multiset of eigenvalues.*

Remark (Floating Point Errors). *Because the model involves diving by $\sqrt{2}$, floating point errors and algorithmic systemic error will yield small, but negligible imaginary components. This is on top of the floating errors from rounding, as discussed in Section 2.3.*

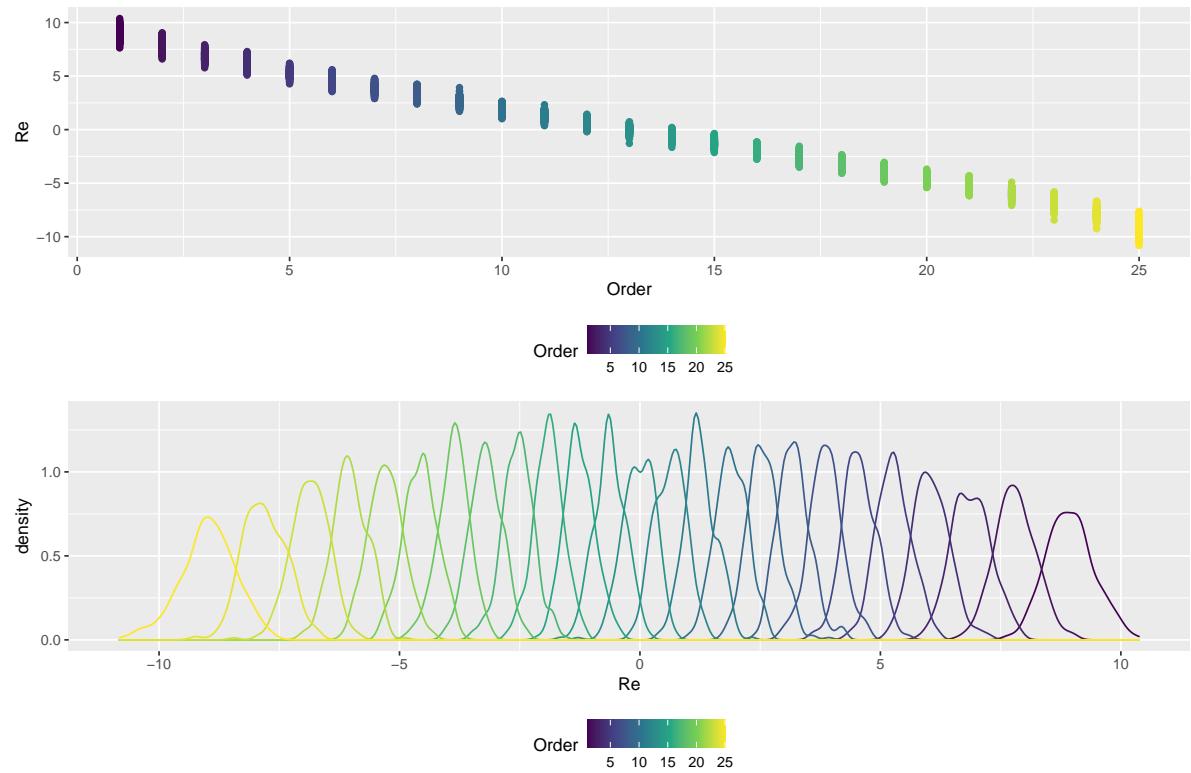


Figure 4.1: Wigner's Surmise for non-standard Beta Ensembles

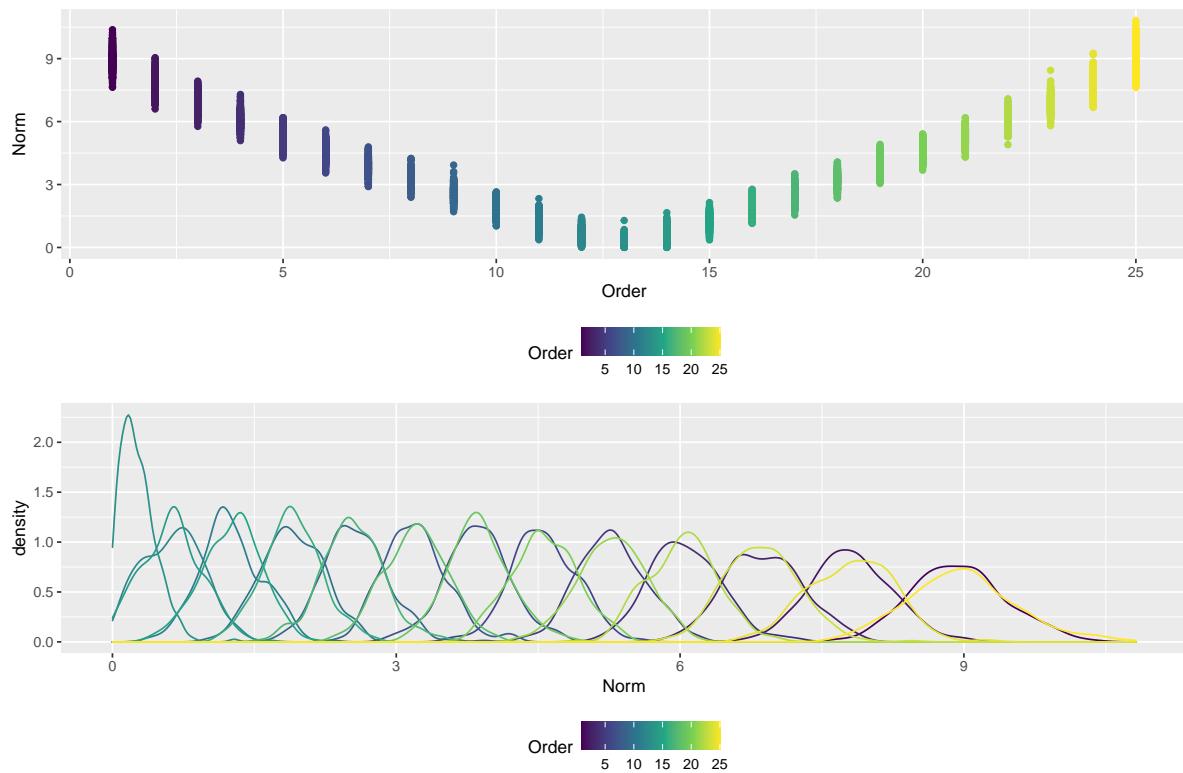


Figure 4.2: Wigner's Surmise for non-standard Beta Ensembles

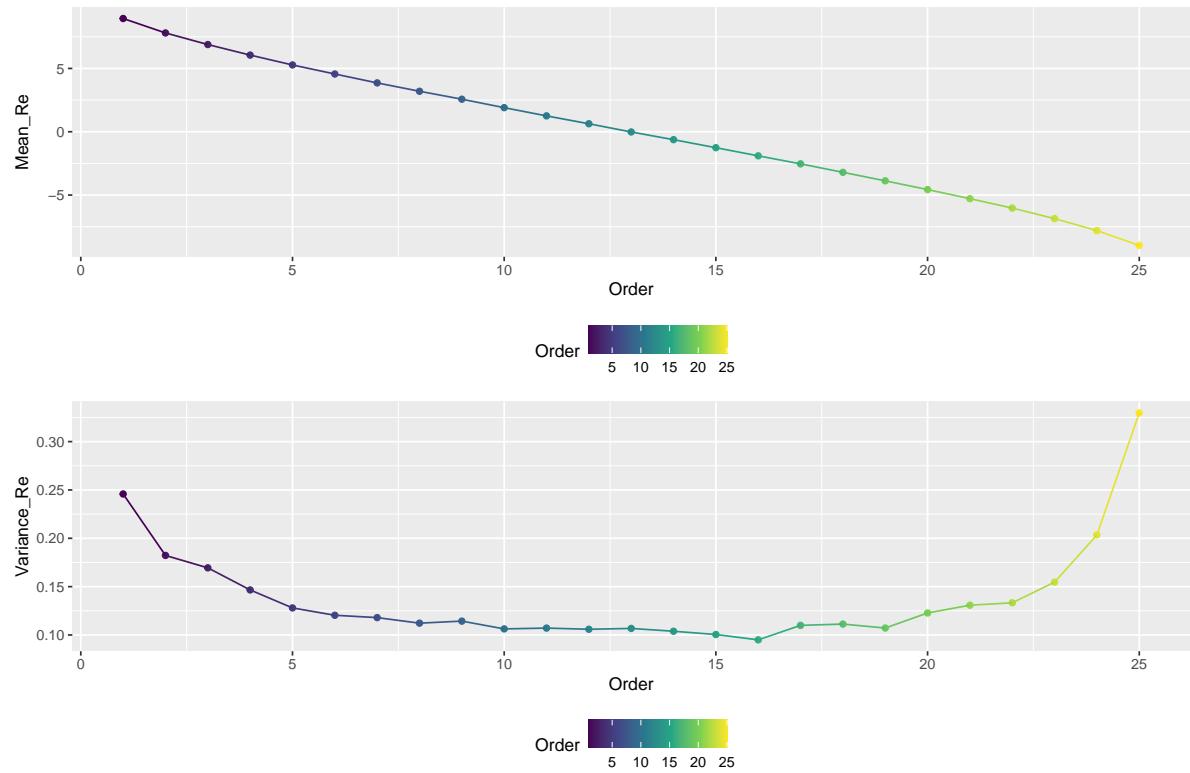


Figure 4.3: Wigner's Surmise for non-standard Beta Ensembles

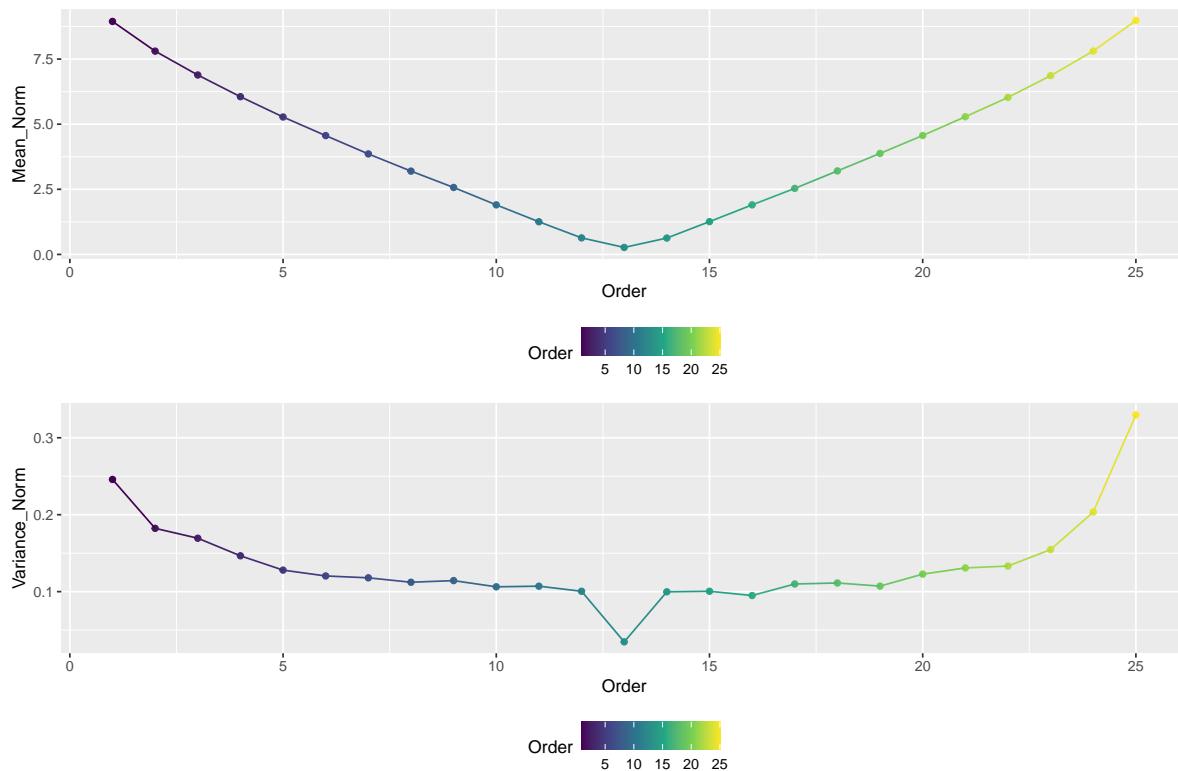


Figure 4.4: Wigner's Surmise for non-standard Beta Ensembles

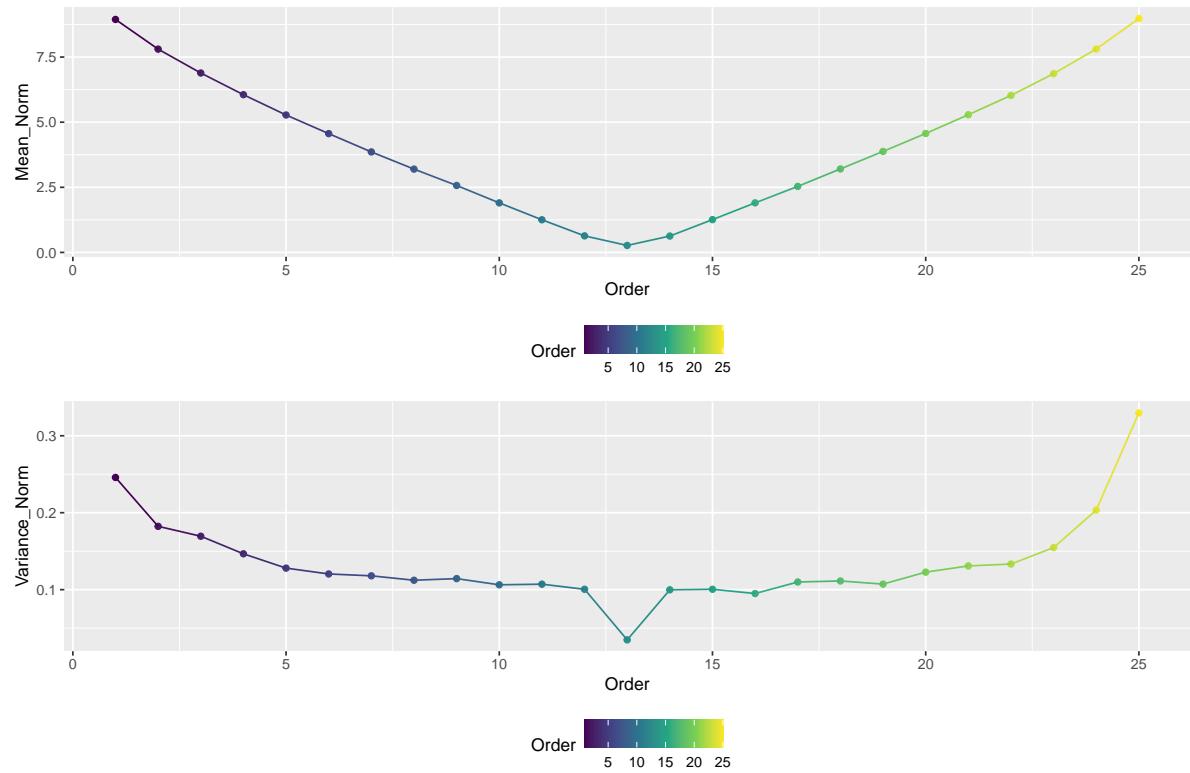


Figure 4.5: Wigner's Surmise for non-standard Beta Ensembles

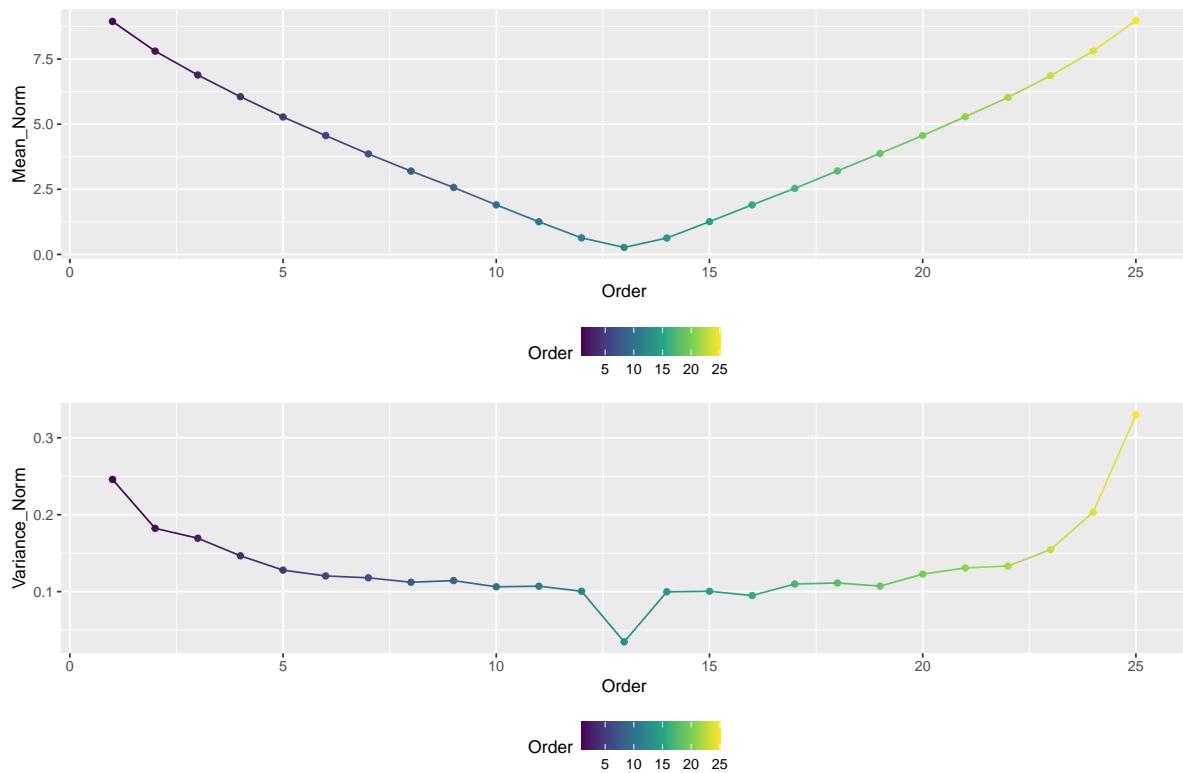


Figure 4.6: Wigner's Surmise for non-standard Beta Ensembles

4.3 Dispersions

Consider the following plot of the sign-sorted eigenvalue dispersions below.

Appendix A

Math Review

In the first section of this appendix, we list a few items in Linear Algebra worth reviewing to make sure there are no gaps in knowledge necessary to understand concepts in this thesis. Additionally, there is another subsection including a proof that was written in the research process of this thesis. Albeit this proof has no direct relation to the primary topic, spectral statistics, it is loosely relevant and displays the tools and techniques relevant to the material.

A.1 Linear Algebra

A.1.1 Matrices

Definition A.1.1 (Eigenvalue). *Suppose $P \in \mathbb{F}^{n \times n}$ is a square matrix. Then, the eigenvalues of the matrix P are precisely the roots of the characteristic polynomial of P , given by $\text{char}_P(\lambda) = \det(P - \lambda I)$. The polynomial $\text{char}_P(\lambda)$ has degree n . So by the Fundamental Theorem of Algebra, P has a multiset of n eigenvalues.*

Definition A.1.2 (Inverse Matrix). *Suppose there is a matrix $P \in \mathbb{F}^{m \times n}$. Then, P^{-1} is its inverse matrix iff multiplying it by P returns the identity matrix. That is, the inverse matrix must satisfy:*

$$P^{-1}P = I$$

Definition A.1.3 (Transpose Matrix). *Suppose there is a matrix $P = (p_{ij}) \in \mathbb{F}^{m \times n}$. Then, its transpose matrix, $P^T = (q_{ij}) = (p_{ji}) \in \mathbb{F}^{n \times m}$ matrix whose columns are the rows of the original matrix.*

Definition A.1.4 (Conjugate Transpose Matrix). *Suppose there is a matrix $P = (p_{ij}) \in \mathbb{C}^{m \times n}$. Then, its conjugate transpose matrix, $P^\dagger = (q_{ij}) = (\overline{p_{ji}}) \in \mathbb{F}^{n \times m}$ is a matrix whose columns are the rows of the original matrix.*

Definition A.1.5 (Symmetric Matrix). *A matrix P is symmetric iff it is equal to its transpose:*

$$P = P^T$$

Definition A.1.6 (Hermitian Matrix). *A matrix P is Hermitian iff it is equal to its conjugate transpose:*

$$P = P^\dagger$$

Definition A.1.7 (Orthogonal Matrix). *A matrix P is called orthogonal iff its transpose is its inverse:*

$$P^T = P^{-1}.$$

Theorem A.1.1 (Orthogonal Group). *The set of all unitary matrices in $\mathbb{F}^{n \times n}$ is a matrix group. (It is called the orthogonal group.)*

Definition A.1.8 (Unitary Matrix). *A matrix P is called unitary iff its conjugate transpose is its inverse:*

$$P^\dagger = P^{-1}.$$

Theorem A.1.2 (Unitary Group). *The set of all unitary matrices in $\mathbb{C}^{n \times n}$ is a matrix group. (It is called the unitary group.)*

A.1.2 Other

As mentioned in **Section 2.3**, here is the statement of the Perron-Frobenius Theorem. Specifically, we consider only the application to ergodic Markov Chains by means of limiting scope to stochastic/transition matrices.

Theorem A.1.3 (Perron-Frobenius Theorem). *bla bla*

A.1.3 Proof: Real Symmetric Matrices have Real Eigenvectors

How does this relate to the rest of the thesis?

Eigenvectors have a wide range of applications in various fields. As such, a result **guaranteeing** the existence of real-valued eigenvectors is a strong, non-trivial one. For instance, the stationary distribution of a Markov chain is an eigenvector of its transition matrix of eigenvalue 1. In other scenarios, right-eigenvectors represent the conditional expectation of a transition matrix, and so forth. Altogether, this proof showcases techniques and tools used in this related domains of study.

Notation. For notational convenience, for any $N \in \mathbb{N}$, let $\mathbb{N}_N = \{1, \dots, N\}$.

The Proof

In this section, we will prove that for any $M \times M$ real symmetric matrix, $S_M \in \mathbb{R}^{M \times M}$, there exists for some eigenvalue λ , a corresponding **real** eigenvector $\vec{v} \in \mathbb{R}^M$. Prior to starting the main proof, we begin by proving a auxilliary lemma.

Lemma. Suppose we have a $M \times M$ real symmetric matrix with a some eigenvalue λ . If there we have a corosponding eigenvector $v \in \mathbb{C}^M$, then every entry of v , say v_i is equal to a **real** linear combination of the other entries $v_j \mid j \neq i$. So, we will show that:

$$\forall i \in \mathbb{N}_M : v_i = \sum_{j \neq i} c_j v_j \quad (c_j \in \mathbb{R})$$

Proof of Lemma. Begin by taking a real symmetric matrix S_M for some $M \in \mathbb{N}$. Suppose we have an eigenvalue λ . Then, if we have some eigenvector v , we know that:

$$(1) : \forall i \in \mathbb{N}_M : a_1 v_1 + \dots + d_i v_i + \dots + a_{m-1} v_m = \lambda v_i \quad (a_j \in \mathbb{R})$$

We obtain (1) by expanding the equality $A\vec{v} = \lambda\vec{v}$ and noticing that every row of A is expressible as the sum of the non-diagonal entries multiplied by $v_j \mid j \neq i$ plus $d_i v_i$. Next, we collect the terms to solve for v_i :

$$\forall i \in \mathbb{N}_M : a_1 v_1 + \dots + a_{m-1} v_m = v_i(\lambda - d_i)$$

Since S_M is a real symmetric matrix, the a_j terms are real so we can say:

$$\forall i \in \mathbb{N}_M : v_i(\lambda - d_i) = \sum_{j \neq i} a_j v_j \quad (a_j \in \mathbb{R})$$

Finally, divide both sides by $(\lambda - d_i)$. On the right hand side, the coefficients of v_j become $\frac{a_j}{(\lambda - d_i)}$. Denote this coefficient $a'_j = \frac{a_j}{(\lambda - d_i)}$. Since S_M is a real symmetric matrix, we know its eigenvalues are real so $\lambda \in \mathbb{R}$ and that its entries are real so

$a_i, d_i \in \mathbb{R}$. Since a'_j is an arithmetic expression involving real numbers, then it follows that for every j , $a'_j \in \mathbb{R}$. As such, we can rewrite v_j as the following sum.

$$\forall i \in \mathbb{N}_M : v_i = \sum_{j \neq i} c_j v_j \quad (\forall j : a'_j \in \mathbb{R})$$

Thus, for any $M \in \mathbb{N}$, a real symmetric matrix $S_M \in \mathbb{R}^{M \times M}$ with eigenvalue λ must have a corresponding eigenvector v such that each of its entries is expressible as a real linear combination of the other entries. So, the proof is complete. \square

Now, leveraging the result of this lemma, we will prove the main theorem.

Theorem A.1.4 (Taqi). *Suppose we have a $M \times M$ real symmetric matrix, S_M . Then, we will show that there exists for some eigenvalue λ , a corresponding **real** eigenvector $\vec{v} \in \mathbb{R}^M$.*

Proof. For this proof we will induct on the dimension of the matrix, M . So, let the inductive statement be:

$$f(M) : S_M \text{ has a real eigenvector } v \text{ corresponding to an eigenvalue } \lambda$$

Base Case. Take the base case $M = 2$. This proof is left to the reader as an exercise. Begin by taking a 2×2 symmetric matrix, and show that there exist real coefficients for the eigenvector corresponding to λ by using Gaussian Elimination.

Inductive Step. For our inductive step, we need to show that $f(M) \Rightarrow f(M + 1)$. So, let us assume $f(M)$. This means that we can assume any real symmetric matrix S_M has a real eigenvector $v \in \mathbb{R}^M$ corresponding to λ .

Next, we will write S_{M+1} as the matrix S_M augmented by some $u \in \mathbb{R}^M$ as follows:

$$S_{M+1} = \left[\begin{array}{c|c} S_M & u \\ \hline u^T & d_{M+1} \end{array} \right]$$

From our lemma, we use the fact that S_{M+1} is symmetric and our lemma to obtain (1) and our assumption of $f(M)$ to obtain (2), listed below:

$$(1) : \forall i \in \mathbb{N}_{M+1} : v_i = \sum_{j \neq i} c_j v_j \quad (c_j \in \mathbb{R})$$

$$(2) : \forall i \in \mathbb{N}_M : v_i \in \mathbb{R}$$

In particular for (2), we know that $v_i = \left(\sum_{j \neq i} \frac{a_j}{d_i - \lambda} v_j \right)$. From (1), we know that for the $(m + 1)^{th}$ row, $v_{m+1} = \sum_{j \neq m+1} c_j v_j$ for real coefficients $c_j \in \mathbb{R}$. By (2), this is a linear combination of real entries v_i . Since $v_{m+1} \in \mathbb{R}$, this means we have shown that:

$$\forall i \in \mathbb{N}_{M+1} : v_i \in \mathbb{R}$$

In other words, we have established that $f(M) \Rightarrow f(M + 1)$. By induction, the proof of the theorem is complete. \square

A.2 Probability Theory

Please refer to Hwang & Blitzstein (2019) as a resource; the definitions were sourced from there. For the definitions and theorems covered in Section A.1, consider checking out Chapter 3: Random Variables and their distributions. For Order Statistics, consider Chapter 8, Section 6: Order Statistics.

A.2.1 Random Variables

Definition A.2.1 (Random Variable). *A random variable $X : \Omega \rightarrow \mathbb{R}$ is a function from some sample space $\Omega = \{s_i\}_{i=1}^n$ to the real numbers \mathbb{R} . The sample space is taken to be any set of events such that the probability function corresponding to the random variable, p_X exhausts over all the events in Ω . In other words, we expect $\int_{\Omega} p_X(s) = 1$.*

PDFs

Definition A.2.2 (Probability Density Function). *For a continuous r.v. X with CDF F , the probability density function (PDF) of X is the derivative f of the CDF, given by $f(x) = F'(x)$. The support of X , and of its distribution, is the set of all x where $f(x) > 0$.*

Theorem A.2.1 (Characterizing the PDF). *A probability density function is characterized by a few properties that are necessary for it to be valid. They are as follows:*

1. Non-negativity: the PDF must be a non-negative valued function everywhere.

$$f(x) \geq 0$$

2. Integrates to 1: the PDF must integrate to 1 when integrated over its entire support.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Example (Normal PDF). *For example, consider the probability density function for a r.v. $X \sim \mathcal{N}(\mu, \sigma)$.*

$$\mathbf{P}(X = x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2\right)$$

CDFs

Definition A.2.3 (Cumulative Distribution Function). *The cumulative distribution function (CDF) of an r.v. X is the function F_X given by $F_X(x) = \mathbf{P}(X \leq x)$. When there is no risk of ambiguity, we sometimes drop the subscript and just write F (or some other letter) for a CDF.*

Theorem A.2.2 (Characterizing the CDF). *A cumulative distribution function is characterized by a few properties that are necessary for it to be valid. They are as follows:*

1. *Monotonic function: the CDF must always be a monotonically increasing function. That is:*

$$x_0 \leq x_1 \implies F(x_0) \leq F(x_1)$$

2. *Right-continuous: The CDF is continuous except possibly for having some jumps. Wherever there is a jump, the CDF is continuous from the right. That is, for any a , we have:*

$$F(a) = \lim_{x \rightarrow a^+} F(x)$$

3. *Converges to 0 and 1 in its limits: since the CDF represents a cumulative probability, its limits must reflect that aspect of probability spaces. So, the CDF must satisfy:*

$$\lim_{x \rightarrow -\infty} F(x) = 0 \text{ and } \lim_{x \rightarrow \infty} F(x) = 1$$

Independence

Definition A.2.4 (Independence). *Random variables X and Y are said to be independent if $\forall x, y \in \mathbb{R}$:*

$$\mathbf{P}(X \leq x, Y \leq y) = \mathbf{P}(X \leq x)\mathbf{P}(Y \leq y)$$

If the variables are discrete, then this is equivalent to the condition:

$$\mathbf{P}(X = x, Y = y) = \mathbf{P}(X = x)\mathbf{P}(Y = y)$$

Definition A.2.5 (i.i.d.). *A vector of random variables $\vec{X} = (X_i)_{i=1}^N$ is said to be i.i.d (independent and identically distributed) if each of its entries X_i are exactly so.*

A.2.2 Statistics

Definition A.2.6 (Statistic). *A statistic is formally defined as a function of a vector of random variables. So, for instance f is a statistic of a vector of random variables \vec{X} . Its observed value is given by $f(\vec{X})$.*

Example (Mean Statistic). *For instance, the mean of a random sample \vec{X} is a statistic. Formally, we would define the statistic $f : \vec{X} \rightarrow \frac{\sum_i x_i}{N}$. So the mean of the random sample is defined as $\bar{x} = f(\vec{X})$.*

Order Statistics

Definition A.2.7 (Order Statistic). *Suppose $\vec{X} = \{X_i\}_{i=1}^N$ is an ordered vector of random variables. Then, the i^{th} order statistic of X is given by X_i .*

Example (Order Statistic). *Suppose $X = (20, 7, 2, 1)$. The smallest (fourth) order statistic is 1. The second order statistic is 7. The largest (first) order statistic is 20.*

Theorem A.2.3 (Order Statistics of Uniform i.i.d Variables). *Let U_0, U_1, \dots, U_n be i.i.d. $\text{Unif}(0, 1)$. Then, the j^{th} order statistic $U_{(j)}$, is distributed as $U_{(j)} \sim \text{Beta}(j, n-j+1)$.*

A.3 Markov Chains

How does this relate to the rest of the thesis?

Tie back the relevance of this section to the rest of this thesis. This will come up in Appendix D. It is also important because transition matrices are representations of markov chains. Please refer to Hwang & Blitzstein (2019), Chapter 11 for all this information.

Definition A.3.1 (Markov Chain). *Say a set of random variables X_i each take a value in a set, called the state space, $S_M = \{1, 2, \dots, M\}$. Then, a sequence of such random variables X_0, X_1, \dots, X_n is called a Markov Chain if the following conditions are satisfied:*

- $\forall X_i : X_i$ has support and range $S_M = \{1, 2, \dots, M\}$.
- (**Markov Property**) The transition probability from state $i \rightarrow j$, given by $\mathbf{P}(X_{n+1} = j | X_n = i)$ is conditionally independent from all past events in the sequence $X_{n-1} = i', X_{n-2} = i'', \dots, X_0 = i^{(n-1)}$, excluding the present/last event in the sequence. In other words, given the present, the past and the future are conditionally independent. So, $\forall i, j \in S_M$, we observe:

$$\mathbf{P}(X_{n+1} = j | X_n = i) = \mathbf{P}(X_{n+1} = j | X_n = i, X_{n-1} = i', \dots, X_0 = i^{(n-1)})$$

Transition Matrices

Definition A.3.2 (Transition Matrix). *Take a Markov Chain with states $\{1, \dots, M\}$. Letting $q_{ij} = \mathbf{P}(X_{n+1} = j | X_n = i)$ be the transition probability from $i \rightarrow j$, then the matrix $Q = (q_{ij})$ is the transition matrix of the chain.*

- Q is a non-negative matrix. So every entry $q_{ij} \in \mathbb{R}^+$ is non-negative. This follows because probabilities are necessarily non-negative values.
- For this transition matrix to be valid, its rows have to be stochastic, meaning their entries sum to 1. This may be understood as applying the law of total probability to the event of transitioning from any given state $i \in S_M$. In other words, the chain has to go somewhere with probability 1.

$$\forall i \in \mathbb{N}_M : \sum_{j \in \mathbb{N}_M} q_{ij} = 1$$

- Note, it is **not** necessary that the converse holds. The columns of our transition matrix need not sum to 1 for it to be a valid transition matrix.

Definition A.3.3 (n -step Transition Probability). *The n -step transition probability of $i \rightarrow j$ is the probability of being at j exactly n steps after being at i . We denote this value $q_{ij}^{(n)}$:*

$$q_{ij}^{(n)} : \mathbf{P}(X_n = j \mid X_0 = i)$$

Realize:

$$q_{ij}^{(2)} = \sum_{k \in S_M} q_{ik} \cdot q_{kj}$$

Because by definition, a Markov Chain is closed under a support/range of S_M so the event $i \rightarrow j$ may have taken any intermediate step $k \in S_M$. Realize by notational equivalence, $Q^2 = (q_{ij}^{(2)})$. Inducting over n , we then obtain that:

$$q_{ij}^{(n)} \text{ is the } (i, j) \text{ entry of } Q^n$$

Definition A.3.4 (Marginal Distribution of X_n). *Let $\mathbf{t} = (t_1, t_2, \dots, t_M)$ such that $\forall i \in S_M : t_i = \mathbf{P}(X_0 = i)$. So, $\mathbf{t} \in \mathbb{R}^{1 \times M}$. Then, the marginal distribution of X_n is given by the product of the vector $\mathbf{t}Q^n \in \mathbb{R}^{1 \times M}$. That is, the j^{th} component of that vector is $\mathbf{P}(X_n = j)$ for any $j \in S_M$. We may call \mathbf{t} an initial state distribution.*

Classification of states

- A state $i \in S_M$ is said to be **recurrent** if starting from i , the probability is 1 that the chain will *eventually* return to i . If the chain is not recurrent, it is **transient**, meaning that if it starts at i , there is a non-zero probability that it never returns to i .
- Caveat: As we let $n \rightarrow \infty$, our Markov chain will guarantee that all transient states will be left forever, no matter how small the probability is. This can be proven by letting the probability be some ε , then realizing that by the support of $\text{Geom}(\varepsilon)$ is always some finite value, then the equivalence between the Markov property and independent Geometric trials guarantees the existence of some finite value such that there is a success of never returning to i .

Definition A.3.5 (Reducibility). *A Markov chain is said to be **irreducible** if for any $i, j \in S_M$, it is possible to go from $i \rightarrow j$ in a finite number of steps with positive probability. In other words:*

$$\forall i, j \in S_M : \exists n \in \mathbb{N} : q_{ij}^{(n)} > 0$$

- From our quantifier formulation of irreducible Markov chains, note that we can equivalently say that a chain is irreducible if there is integer $n \in \mathbb{N}$ such that the (i, j) entry of Q^n is positive for any i, j .
- A Markov chain is **reducible** if it is not **irreducible**. Using our quantifier formulation, it means that it suffices to find transient states so that:

$$\exists i, j \in S_M : \nexists n \in \mathbb{N} : q_{ij}^{(n)} > 0$$

Appendix B

Algorithm Appendix

In this appendix, we will enumerate the various algorithms used in this thesis for the purpose of transparency and reproducibility. Every one of these algorithms have their code counterpart in the **RMAT** package. For the most part, the implicit \mathcal{D} -matrices are the non-trivial algorithms to peruse. Changing methodology could impact results, and as such, all results are tied to the algorithms in this appendix.

B.1 Implicit \mathcal{D} -Matrices

Algorithm B.1.1 (Stochastic Row).

1. To sample a row of size N , fix $N \in \mathbb{N}$.
2. Sample a vector X with N entries between $[0, 1]$. So, generate $X = (x_i)_{i=1}^N$ where $x_i \sim \text{Unif}(0, 1)$.
3. Normalize the rows by diving each entry by the row sum; so assign $x_i \leftarrow x_i \cdot \frac{1}{\sum x_i}$.
4. Return the stochastic row X .

Algorithm B.1.2 (Stochastic Matrix).

1. To generate a stochastic square matrix P of size N , fix $N \in \mathbb{N}$.
2. Then, for every row of P , randomly sample a stochastic row and assign it to P .
3. Return the stochastic matrix P .

Algorithm B.1.3 (Symmetric Stochastic Matrix).

1. To sample a symmetric stochastic matrix P of size N , fix $N \in \mathbb{N}$.
2. Sample a random stochastic matrix Q of size N .
3. Choosing one of the triangles of Q , set both the upper and lower of triangles of P to be that triangle.

4. Set the diagonal of the matrix P to be equal to 1 minus the sum of the non-diagonal entries.
5. Return the symmetric stochastic matrix P .

Algorithm B.1.4 (Transition Matrix of an Erdos-Renyi Graph).

1. Fix $N \in \mathbb{N}$ and $p \in [0, 1]$.
2. Generate a matrix Q such that every entry $i, j \in 1, \dots, N$ is $x_{ij} \sim \text{Unif}(0, 1)$.
3. For each v_i in $\{1, \dots, N\}$, generate $\deg(v_i) \sim \text{Bin}(N, p)$.
4. Randomly chose $1 - \deg(v_i)$ vertices, set the entries x_{ij} in the j columns to 0.
5. Renormalize the matrix by dividing each row by its sum; let $(x_i) \leftarrow (x_i) / \sum_j (x_i)$.

B.2 Explicit \mathcal{D} -Matrices

Algorithm B.2.1 (Explicit \mathcal{D} -Matrix).

1. To simulate a \mathcal{D} -distributed square matrix P of size N , fix $N \in \mathbb{N}$.
2. Sample a vector X with N entries from \mathcal{D} . So, generate $X = (x_i)_{i=1}^N$ where $x_i \sim \mathcal{D}$.
3. Assign the vector X as a row of the matrix P . Repeat for every other row.
4. Return the \mathcal{D} -distributed matrix P .

Algorithm B.2.2 (Dumitriu's Beta Matrix).

1. To simulate an $N \times N$ beta matrix, fix $N \in \mathbb{N}$.
2. Start by taking a diagonal of $\mathcal{N}(0, 2)$ variables.
3. Set both of the nearest off-diagonals to the row that samples from a $\chi(df = c_j)$ where $c_j = \beta \cdot j$ for columns spanning $j = 1, \dots, n - 1$.

Appendix C

Code Appendix

The RMAT Package.

The RMAT package is composed of two primary modules, the random matrix module (`matrices.R`) and the spectral statistics module (`spectrum.R`) and (`dispersion.R`). They can be further subdivided into smaller modules as follows.

1. Random Matrix Module

- (a) Explicitly Distributed Matrices
- (b) Implicitly Distributed Matrices
- (c) Ensemble Extensions

2. Spectral Statistics Module

- (a) Spectrum
- (b) Dispersions
- (c) Parallel Extensions

C.1 Matrix Module

C.1.1 Explicitly Distributed Matrices

```
# ... represents all the arguments taken in by the rdist function
RM_explicit <- function(rdist){
  function(N, ..., symm = FALSE){
    # Create an [N x N] matrix sampling the rows from rdist, passing ... to rdist
    P <- matrix(rdist(N^2, ...), nrow = N)
    # Make symmetric if prompted
    if(symm){P <- .makeHermitian(P)}
    # Return P
    P
  }
}

# A version where we add an imaginary component
RM_explicit_cplx <- function(rdist){
  RM_dist <- function(N, ..., symm = FALSE, cplx = FALSE, herm = FALSE){
    # Create an [N x N] matrix sampling the rows from rdist, passing ... to rdist
    P <- matrix(rdist(N^2, ...), nrow = N)
    # Make symmetric/hermitian if prompted
    if(symm || herm){P <- .makeHermitian(P)}
    # Returns a matrix with complex (and hermitian) entries if prompted
    if(cplx){
      # Recursively add imaginary components as 1i * instance of real-valued matrix.
      Im_P <- (1i * RM_dist(N, ...))
      # Make imaginary part hermitian if prompted
      if(herm){P <- P + .makeHermitian(Im_P)}
      else{P <- P + Im_P}
    }
    P # Return the matrix
  }
}
```

C.1.2 Implicitly Distributed Matrices

C.1.3 Ensemble Extensions

C.2 Spectral Statistics Module

C.2.1 Spectrum

C.2.2 Dispersions

C.2.3 Parallel Extensions

Appendix D

Mixing Time Simulations

D.1 Introduction

In this chapter, we'll talk about ratio-mixing time simulations. Essentially, these simulations are a method of approximating the distribution of a random transition matrix's mixing time. There will be a fun exploration of the Erdos-Renyi matrix ensembles and we will computationally show that the parameterized ensemble has a mixing time inversely proportional to graph sparsity.

D.2 Mixing Time Simulations

With the Erdos-Renyi graph defined, we may now motivate the simulation of random walks on them. First, however, we need to generate their corresponding transition matrices. An algorithm for this is outlined below.

Suppose we have simulated a transition matrix for an Erdos-Renyi graph called Q . Now, fixing some initial probability distribution $\vec{x} \in \mathbb{R}^M$, we may consider the evolution sequence of a random walk on this Erdos-Renyi graph by taking its evolution sequence $\mathcal{S}(Q, x)$.

Definition D.2.1 (Random Batches). *Let \mathbb{F} be a field, and fix some $M \in \mathbb{N}$. Let $\mathcal{B}_\lambda \subset \mathbb{F}^M$ be a uniformly random batch of points in the M -hypercube of length λ . That is,*

$$\mathcal{B}_\lambda = \{\vec{x} \mid x_i \sim \text{Unif}(-\lambda, \lambda) \text{ for } i = 1, \dots, M\}$$

Note: If $\mathbb{F} = \mathbb{C}$, then take $\vec{x} \in \mathcal{B}_\lambda$ to mean $\vec{x} = a + bi$ where $a, b \sim \text{Unif}(-\lambda, \lambda)$.

Definition D.2.2 (Evolution Sequence). *An evolution sequence of a vector $\vec{\pi}$ and a transition matrix Q is defined as the sequence $\mathcal{S}(Q, \pi) = (\pi'_n)_{n=1}^N$ where $\pi'_n = \pi Q^n$*

Definition D.2.3 (Finite Evolution Sequences). *Suppose we sample a random point from \mathcal{B}_λ , emulating a random point $\vec{v} \in \mathbb{F}^M$. Additionally, let $Q \in \mathbb{F}^{M \times M}$ be a transition matrix over \mathbb{F} . Fixing a maximum power ('time') $T \in \mathbb{N}$, define the evolution sequence of \vec{v} as follows:*

$$\mathcal{S}(v, Q, T) = (\alpha_n)_{n=1}^T \text{ where } \alpha_k = v Q^k$$

If we do not impose a finiteness constraint on the sequence, we consider powers for $n \in \mathbb{N}$ or $t = \infty$

Definition D.2.4 (Consecutive Ratio Sequences). Accordingly, define the consecutive ratio sequence (CST) of \vec{v} as follows:

$$\mathcal{R}(v, Q, T) = (r_n)_{n=2}^T \text{ where } (r_n)_j = \frac{(\alpha_n)_j}{(\alpha_{n-1})_j} \text{ for } j = 1, \dots, M$$

In other words, the consecutive ratio sequence of v can be obtained by performing **component-wise division** on consecutive elements of the evolution sequence of v .

Definition D.2.5 (Near Convergence). Because these sequences may never truly converge to eigenvectors of the matrix, we formalize a notion of "near convergence". As a preliminary, we first define ε -equivalence. Let \mathbb{F} be a field, and fix $\varepsilon \in \mathbb{R}^+$. Suppose we have vectors $v, v' \in \mathbb{F}^M$. Then, $v \sim_\varepsilon v'$ if $\|v - v'\| < \varepsilon$ where $\|\cdot\|$ is the norm on \mathbb{F} .

Let $\varepsilon \in \mathbb{R}^+$, and suppose we have an evolution sequence $(a[\vec{v}])_n$. Then, a_n ε -converges at $N \in \mathbb{N}$ if:

$$\forall n \geq N \mid a_N \sim_\varepsilon a_n$$

D.3 Erdos-Renyi Ensemble Simulations

Definition D.3.1 (Erdos-Renyi Graph). An Erdos-Renyi graph is a graph $G = (V, E)$ with a set of vertices $V = 1, \dots, M$ and edges $E = \mathbf{1}_{i,j \in V} \sim \text{Bern}(p_{ij})$. It is homogenous if $p_{ij} = p$ is fixed for all i, j .

Essentially, an Erdos-Renyi graph is a graph whose 'connectedness' is parameterized by a probability p (assuming it's homogenous, which this document will unless otherwise noted). As $p \rightarrow 0$, we say that graph becomes more sparse; analogously, as $p \rightarrow 1$ the graph becomes more connected.

Recall from probability theory that a sum of i.i.d Bernoulli random variables is a Binomial variable. As such, we may alternatively say that the degree of each vertex v is distributed as $\text{deg}(v) \sim \text{Bin}(M, p)$. This is helpful to know because the process of simulating graphs becomes much simpler.

D.4 Questions

1. How are the entries of the CRS distributed? Are they normal, and if so, what is its mean?
2. Are the entries of the CRS i.i.d as $t \rightarrow \infty$?
3. For an Erdos-Renyi matrix, is the mixing time t dependent on the parameter p ?
4. What impact does the running time parameter T have on σ (the variance of the distribution of the CRS entries)?

D.4.1 Cauchy Distributed Ratios

It seems to be the case that the **log-transformed** entries of the CRS are Cauchy distributed about $\log \lambda_1$ where $\lambda_1 = \max(\sigma(Q))$, the largest eigenvalue of Q . That is,

$$r_i \sim \text{Cauchy}(\ln \lambda_1) \text{ for } i = 1, \dots, M$$

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