Asymmetric Cryptography: A Deep Dive

Eli Holderness – @eli@hachyderm.io – they/them/theirs

Eli (pronounced /ˈiːlaɪ̯/) is a is a freelance developer advocate, recovering mathematician, and audience participator.

They like people, the web, and learning weird facts about computers.

They can be found on Mastodon at @eli@hachyderm.io, and — for now — on Twitter at @eliholderness.

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Agenda

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1. Brief history

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2. How RSA works

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3. How ECC works

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4. QC & Shor's Algorithms

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2. How RSA works
3. How ECC works
4. QC & Shor's Algorithms
5. What next?

1

A brief history of cryptography

Devoxx is great!

A	B	C	D	E	F	G	H	I	J	K
G	H	I	J	K	L	M	N	O	P	Q

Jkbudd oy mxkgz!

Devoxx is great!

+6	1	2	3	4	5	6	7	8	9	10	11
	7	8	9	10	11	12	13	14	15	16	17

Jkbudd oy mxkgz!

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DEVOXX 4 5 22 15 24 24

DEVOXX	4 5 22 15 24 24
+	+
SECRET	19 5 3 18 5 20

DEVOXX

+

SECRET

4 5 22 15 24 24

+

19 5 3 18 5 20

=

23 10 25 33 29 44

DEVOXX

+

SECRET

4 5 22 15 24 24

+

19 5 3 18 5 20

=

23 10 25 7 3 18

DEVOXX	4 5 22 15 24 24
+	+
SECRET	19 5 3 18 5 20
=	=
WJYGCR	23 10 25 7 3 18

symmetric cryptography requires both parties to know a specific secret

2

RSA & group theory

RSA cryptosystem

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security based on the difficulty of factoring large numbers N = pq where p, q prime

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worked example with $N = 323 = 17 * 19$

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We need to know $\lambda(N)$, the smallest number where $a^{\lambda(N)} \equiv 1 \bmod N$ for every a coprime to N

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$\lambda(N) = 144$

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$$\lambda(N) = \text{lcm}(\lambda(p), \lambda(q)) = \text{lcm}(p-1, q-1) = \text{lcm}(16, 18) = 144$$

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Choose e between 2 and N coprime to N; let's pick 5

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worked example with N = 323 = 17 * 19

$\lambda(N) = 144$ $e = 5; d = 29$

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Choose e between 2 and N coprime to N; let's pick 5

Find d such that $d * e \equiv 1 \bmod \lambda(N)$; this is 29

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Our public key is $(N, e) = (323, 5)$ and our private key is $d = 29$

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Someone wants to send us the message 14, 4, 3

To encrypt a number, they raise it to the power of $e = 5$:

$14^5, 4^5, 3^5 = 537824, 1024, 243$

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worked example with N = 323 = 17 * 19

$\lambda(N) = 144$ $\quad e = 5; d = 29$ $\quad m = (29, 55, 243)$

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Someone wants to send us the message 14, 4, 3

To encrypt a number, they raise it to the power of $e = 5$:

$$14^5, 4^5, 3^5 = 537824, 1024, 243$$

Then take the modulus of N:

$$14^5, 4^5, 3^5 \equiv 29, 55, 243 \pmod{N}$$

worked example with N = 323 = 17 * 19

$\lambda(N) = 144$ $e = 5; d = 29$ $m = (29, 55, 243)$

We received the message (29, 55, 243)

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Decode by raising each number to the power of $d = 29$, then taking the modulus of N

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We received the message (29, 55, 243)

Decode by raising each number to the power of $d = 29$, then taking the modulus of N

$29^{29}, 55^{29}, 243^{29} \equiv 14, 4, 3 \bmod N$

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$$a^{\lambda(N)+1} = a^{145} = a^{5 \times 29} = (a^5)^{29}$$

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$$a^{\lambda(N)+1} = a^{145} = a^{5 \times 29} = (a^5)^{29}$$

So $(a^5)^{29} \equiv a \bmod N$ and we can recover the original message from the encrypted intermediate

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limitations & considerations

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requires large prime numbers, which are expensive to find

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if e is small enough that $M = m^e < N$, an attacker can simply do $\sqrt[e]{M}$ to recover m

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without padding, messages can be vulnerable to chosen plaintext attacks

TURKEY TROTS TO WATER GG
FROM CINCPAC ACTION COM
THIRD FLEET INFO COMINCH
CTF SEVENTY-SEVEN X WHERE
IS RPT WHERE IS TASK FORCE
THIRTY FOUR RR **THE WORLD
WONDERS**

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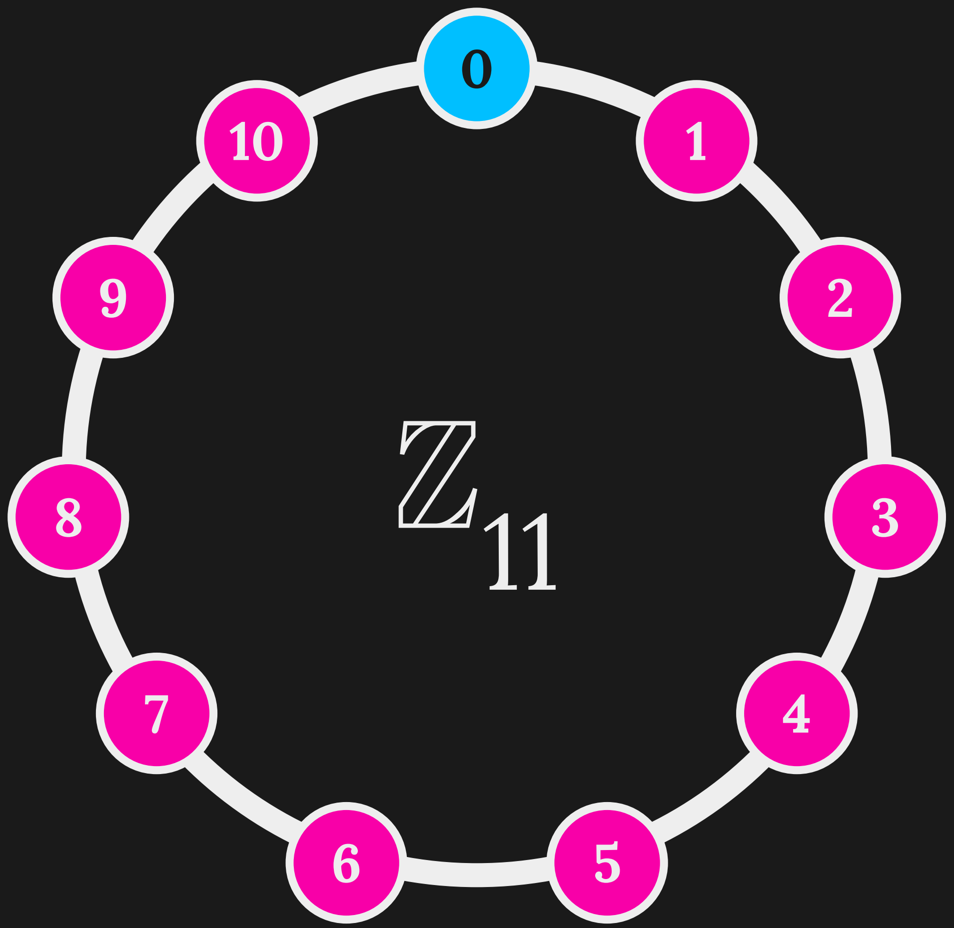

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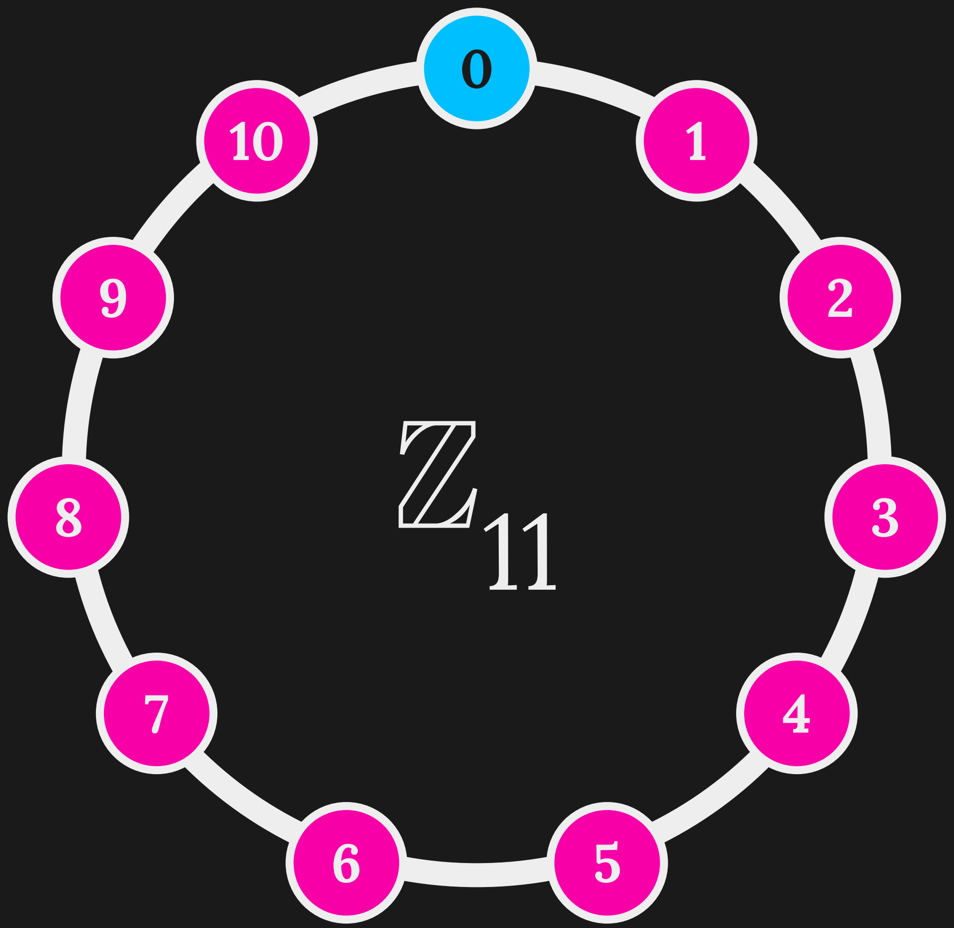
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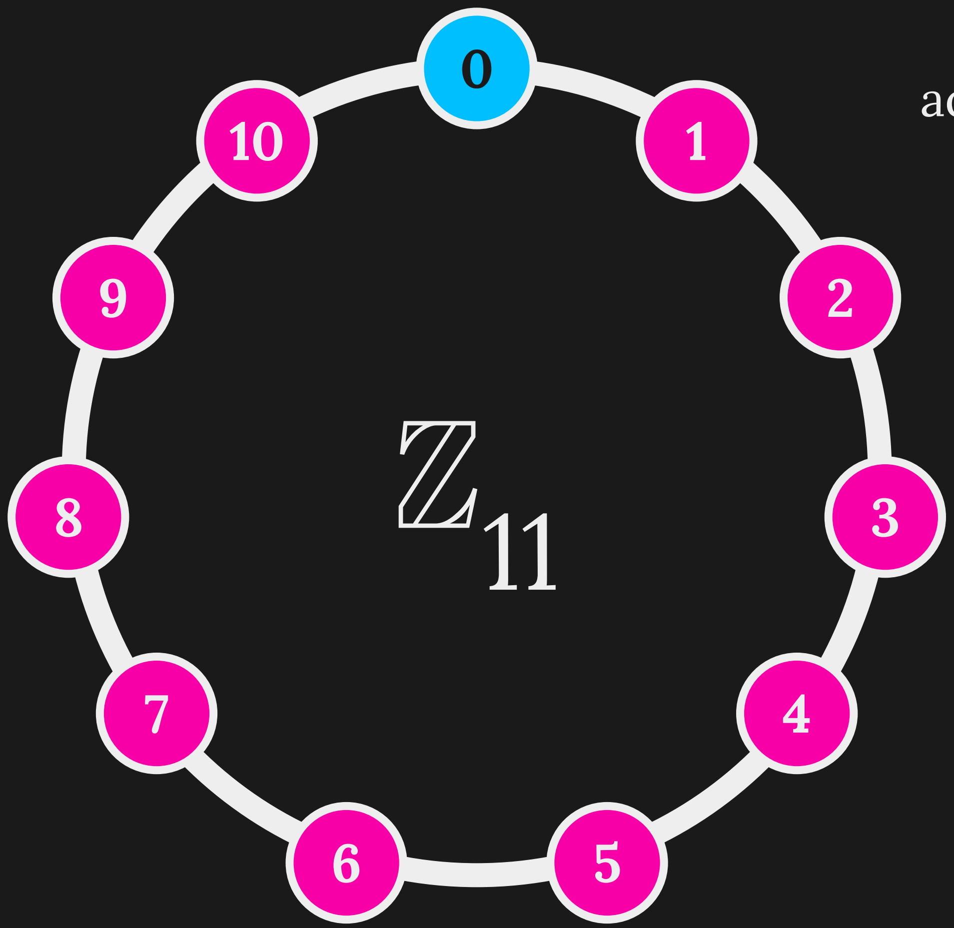
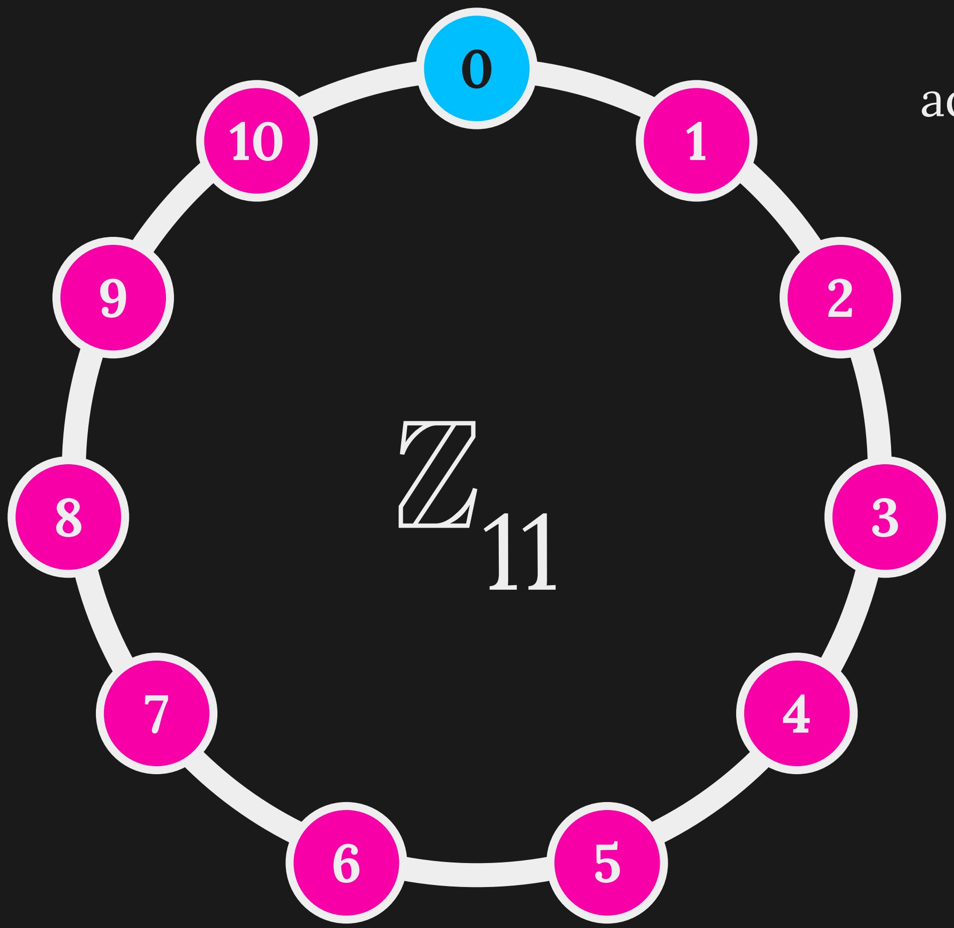
$\mathbb{Z}_{11}$

identity element

adding 0 doesn't change an element





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inverses

for every a in the group, there's a b that makes $a + b = 0$ true

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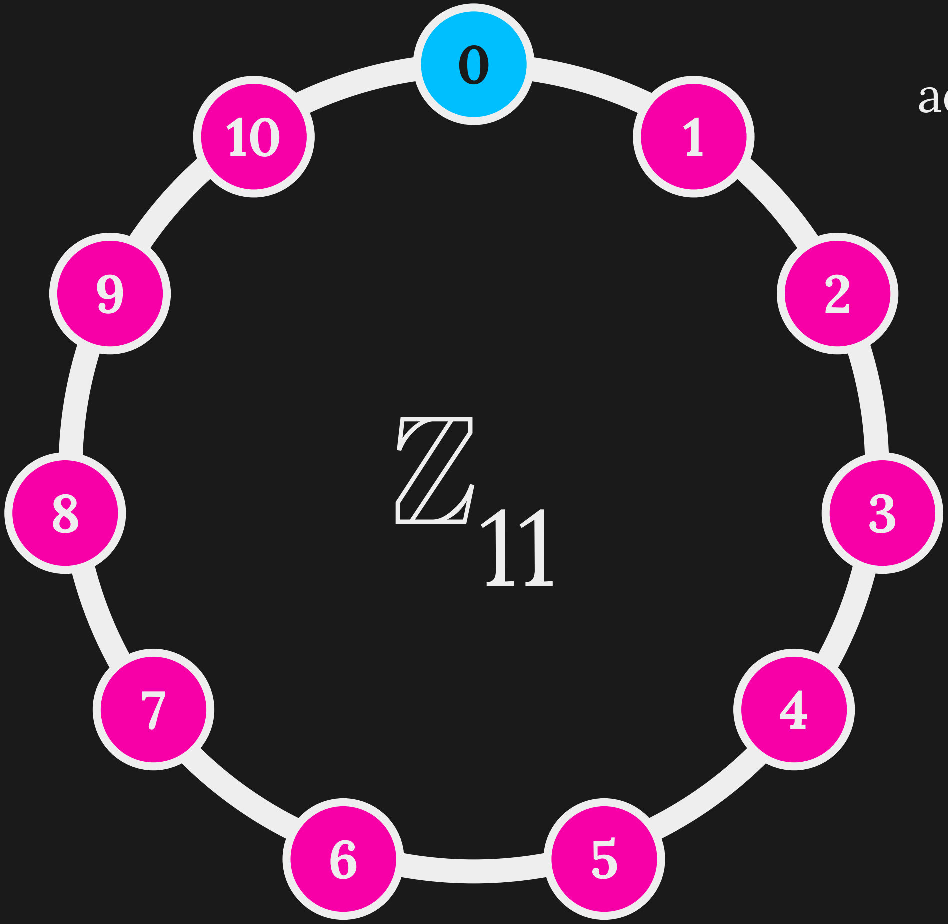

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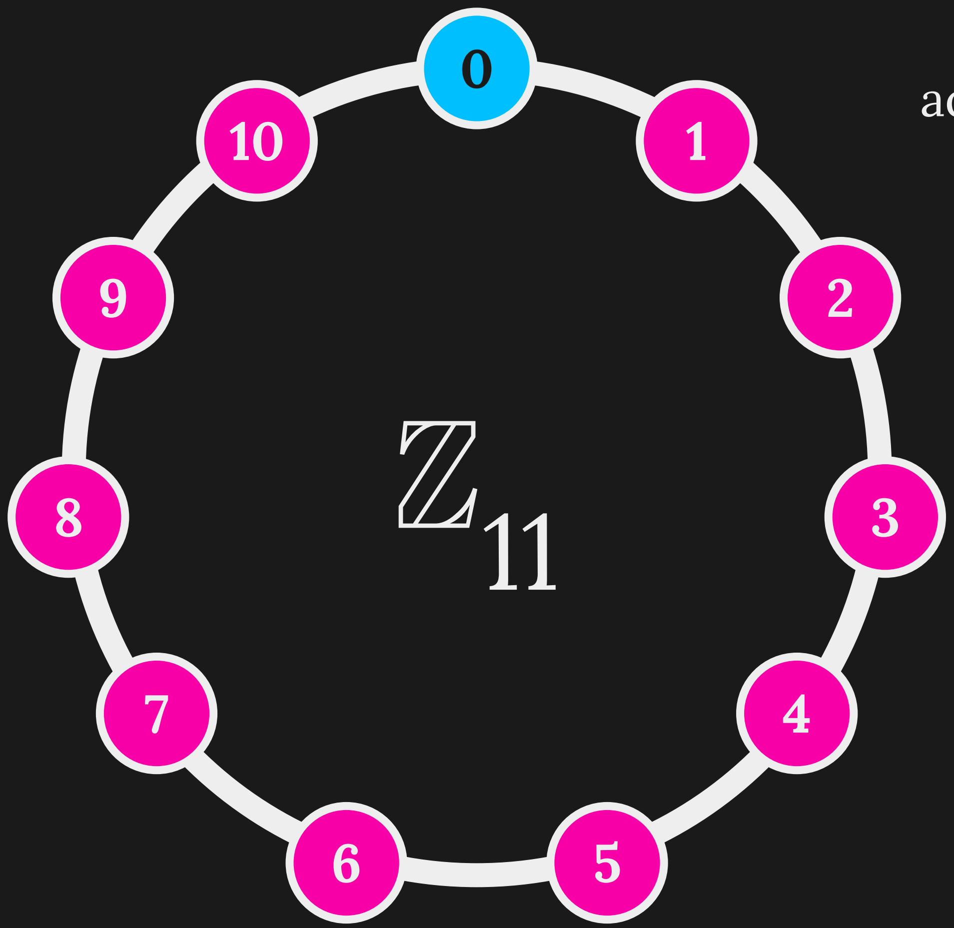
associativity

$1 + (4 + 2) = (1 + 4) + 2$



identity element

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closure

If a and b are in the group and $a + b = c$, then c is in the group

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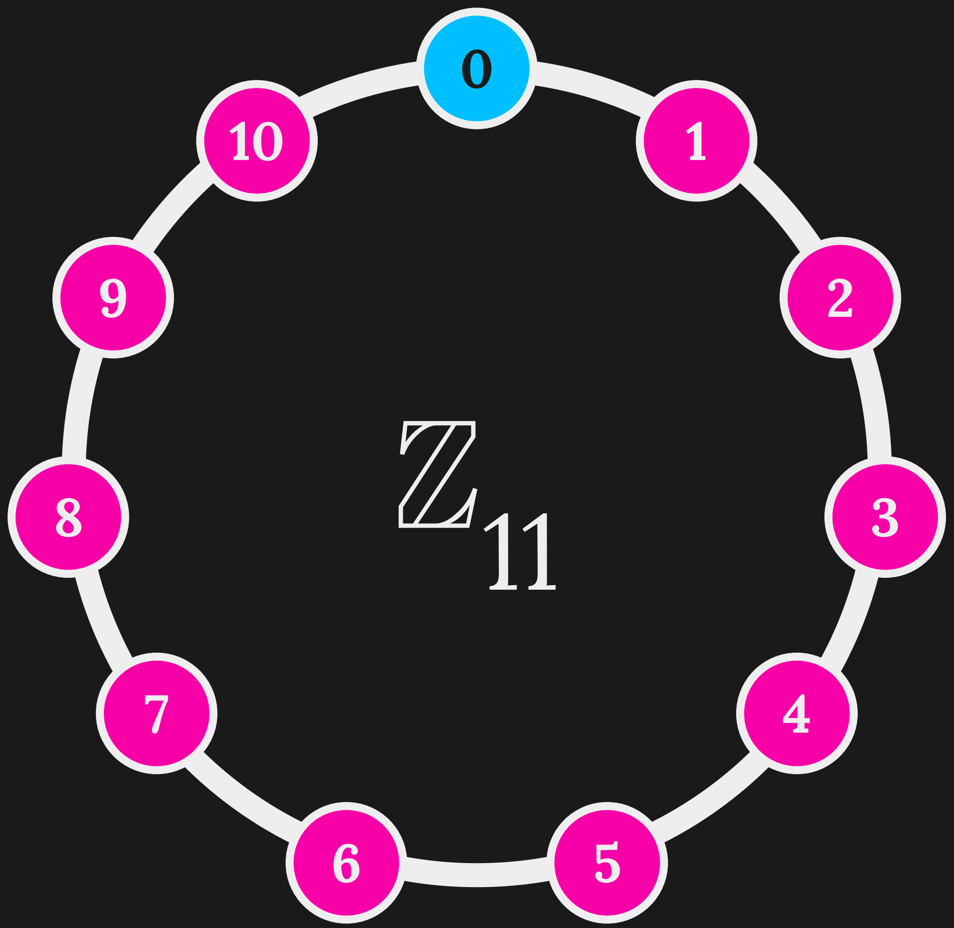
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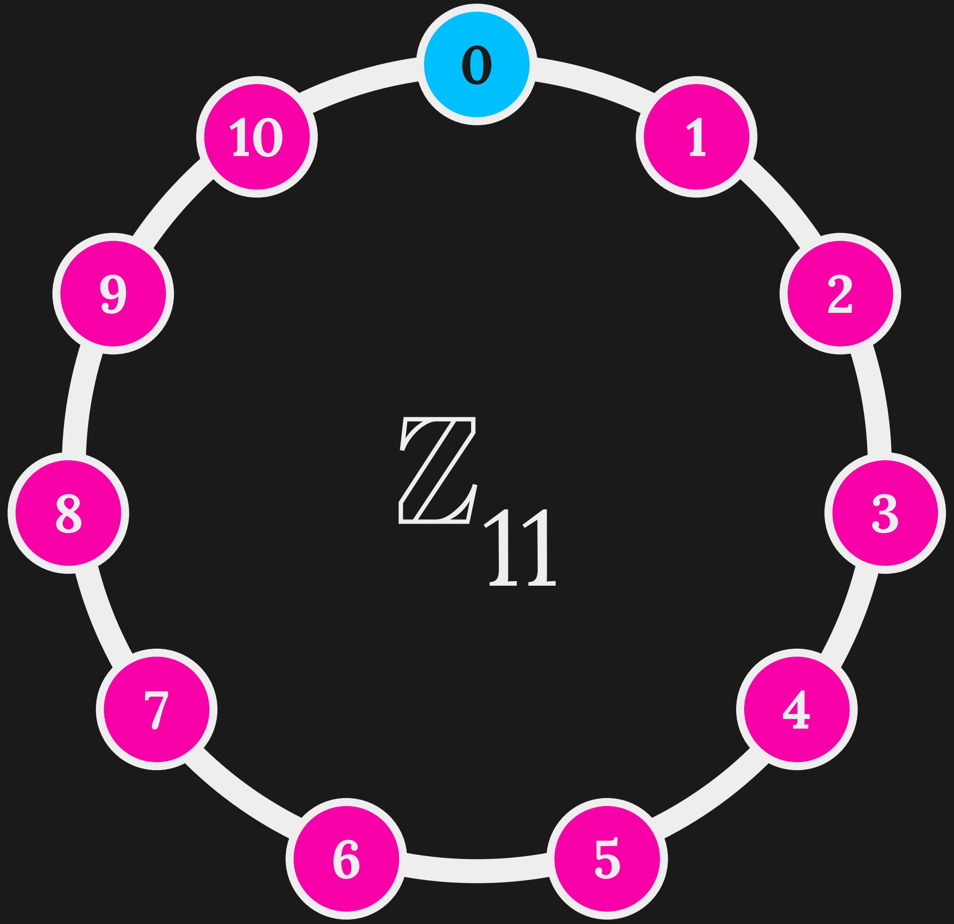

4 x 13 = 52

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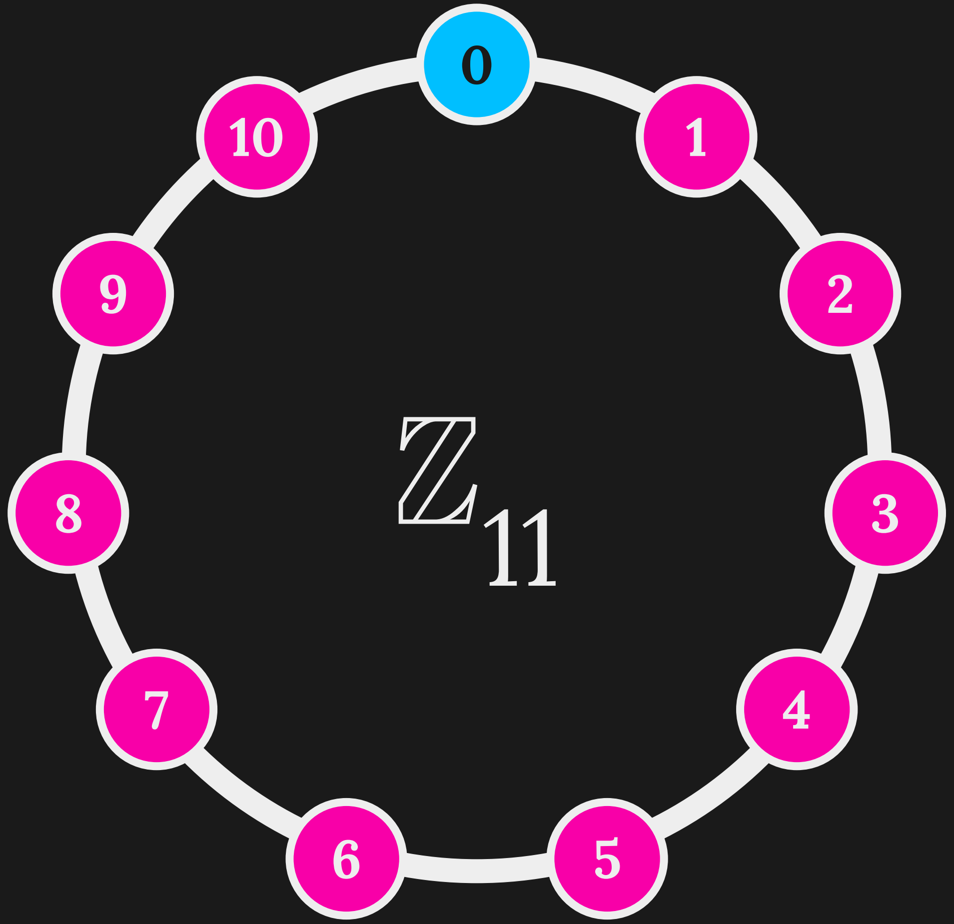

$$4 \times 13 = 52$$
$$= (4 \times 11) + 8$$
$$= 8$$

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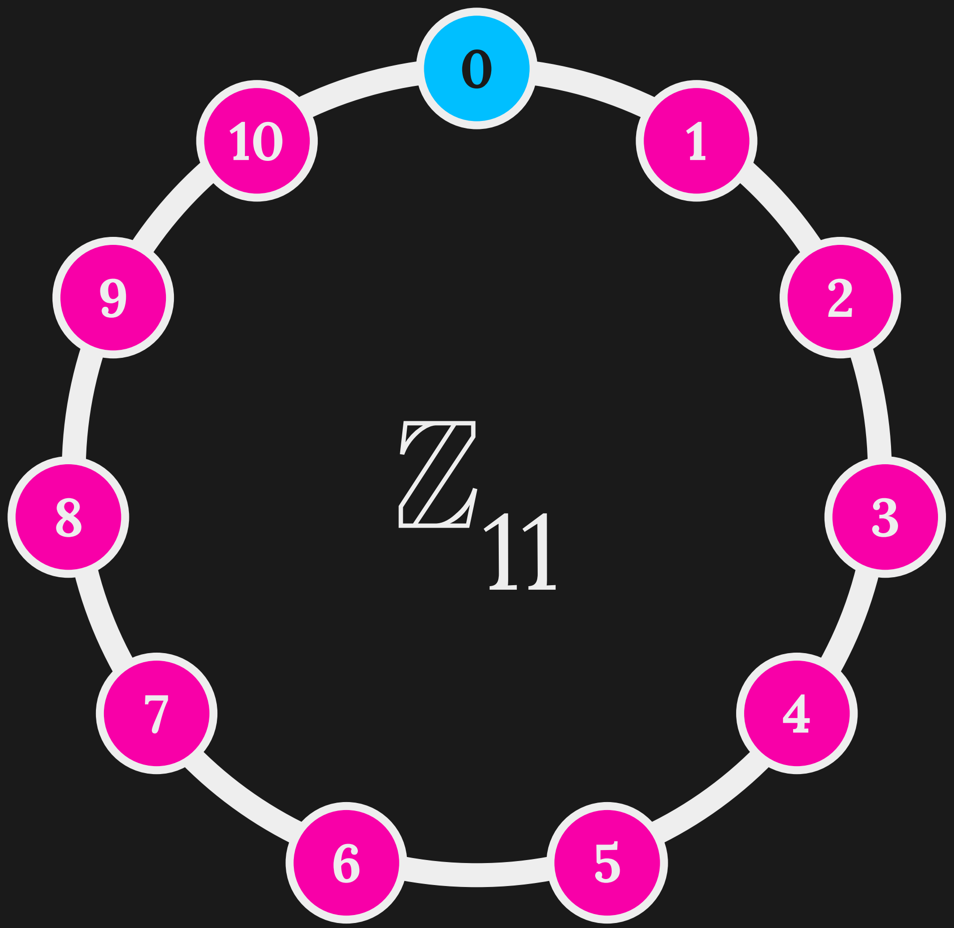

$$4 \times 13 = 52 = (4 \times 11) + 8 = 8$$

you can multiply an element of the group by something that is NOT in the group

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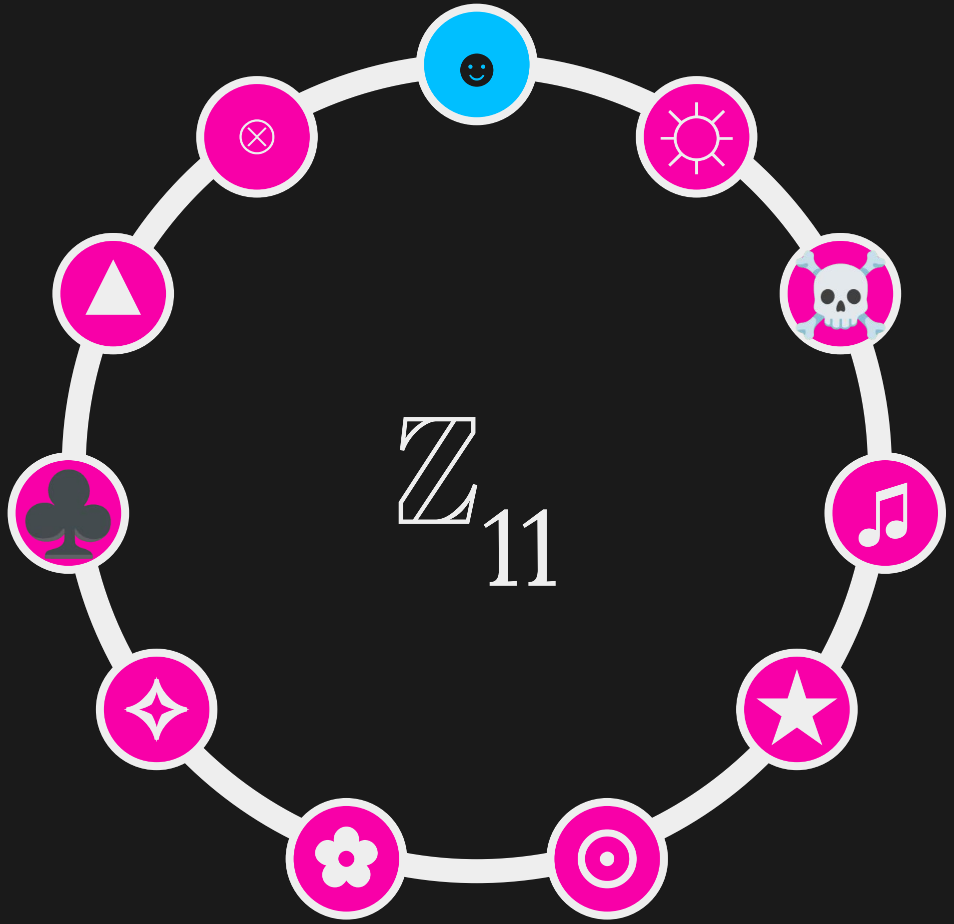

★ x 13 = 52

= (4 x 11) + 8

= ♣

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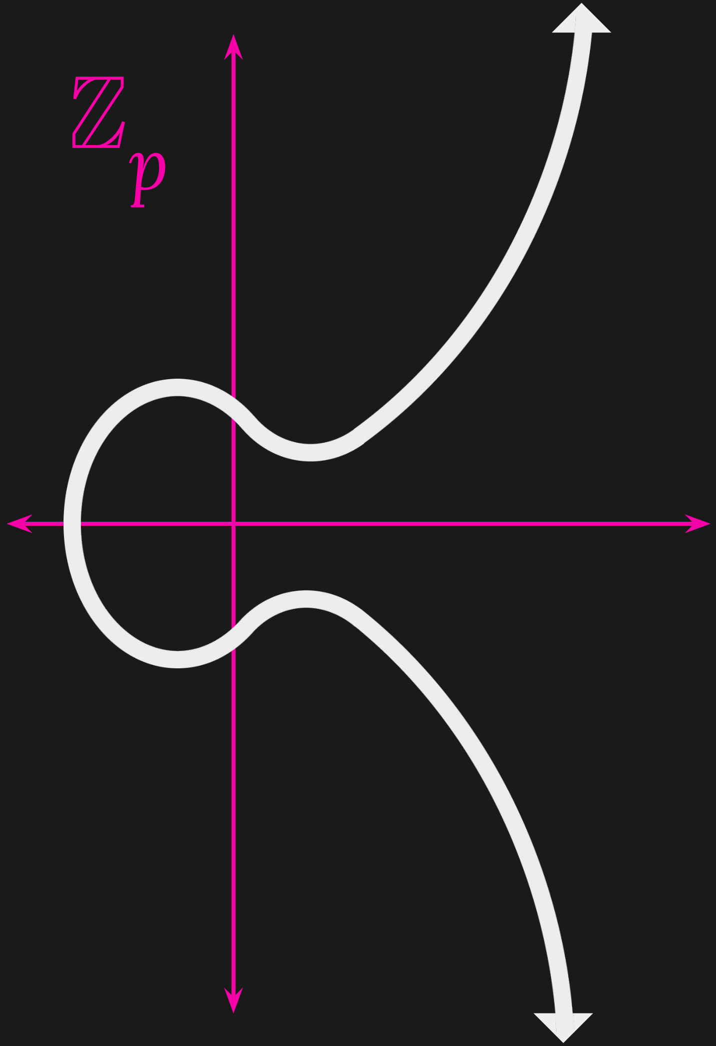
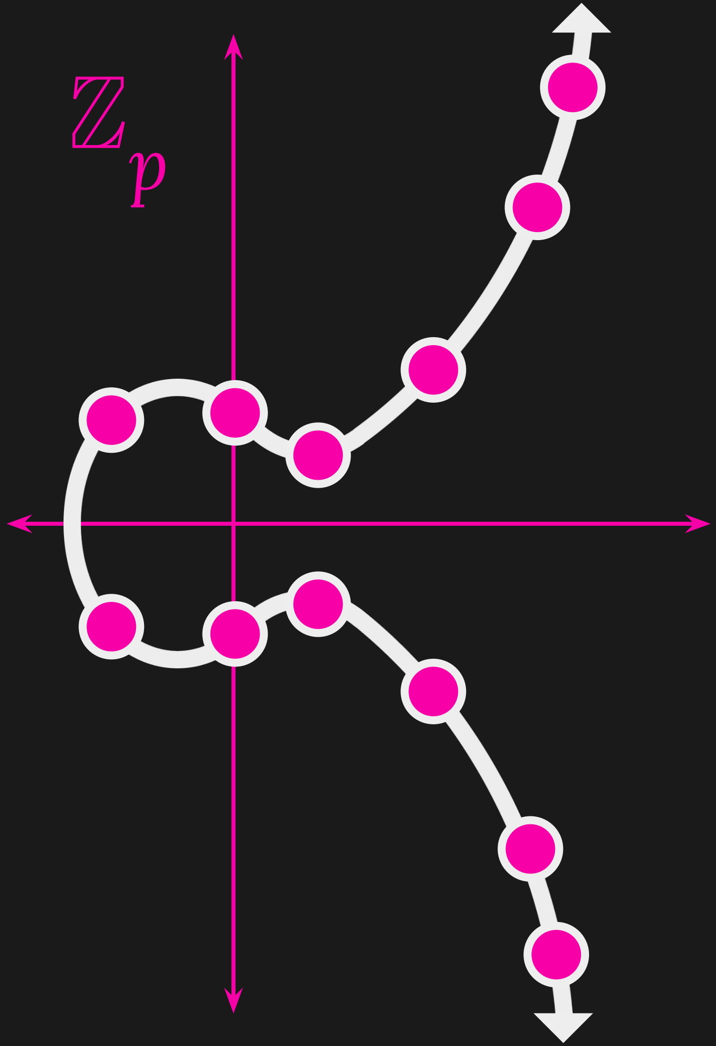
3

Elliptic Curve Cryptography



$$y^2 \equiv x^3 + \mathrm{ax} + \mathrm{b}$$



$$y^2 \equiv x^3 + ax + b$$

where x and y are in $\mathbb{Z}_p$

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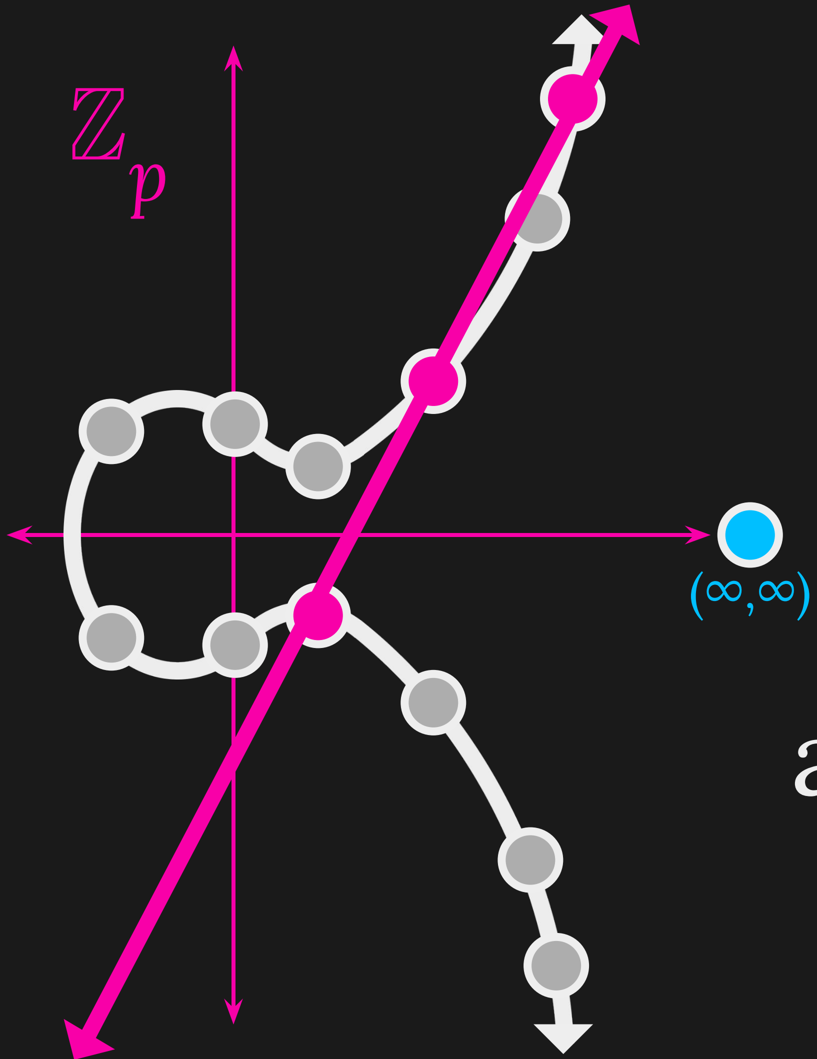

$$y^2 \equiv x^3 + ax + b$$

where x and y are in $\mathbb{Z}_p$

and three collinear points 'sum' to $\mathcal{O}$

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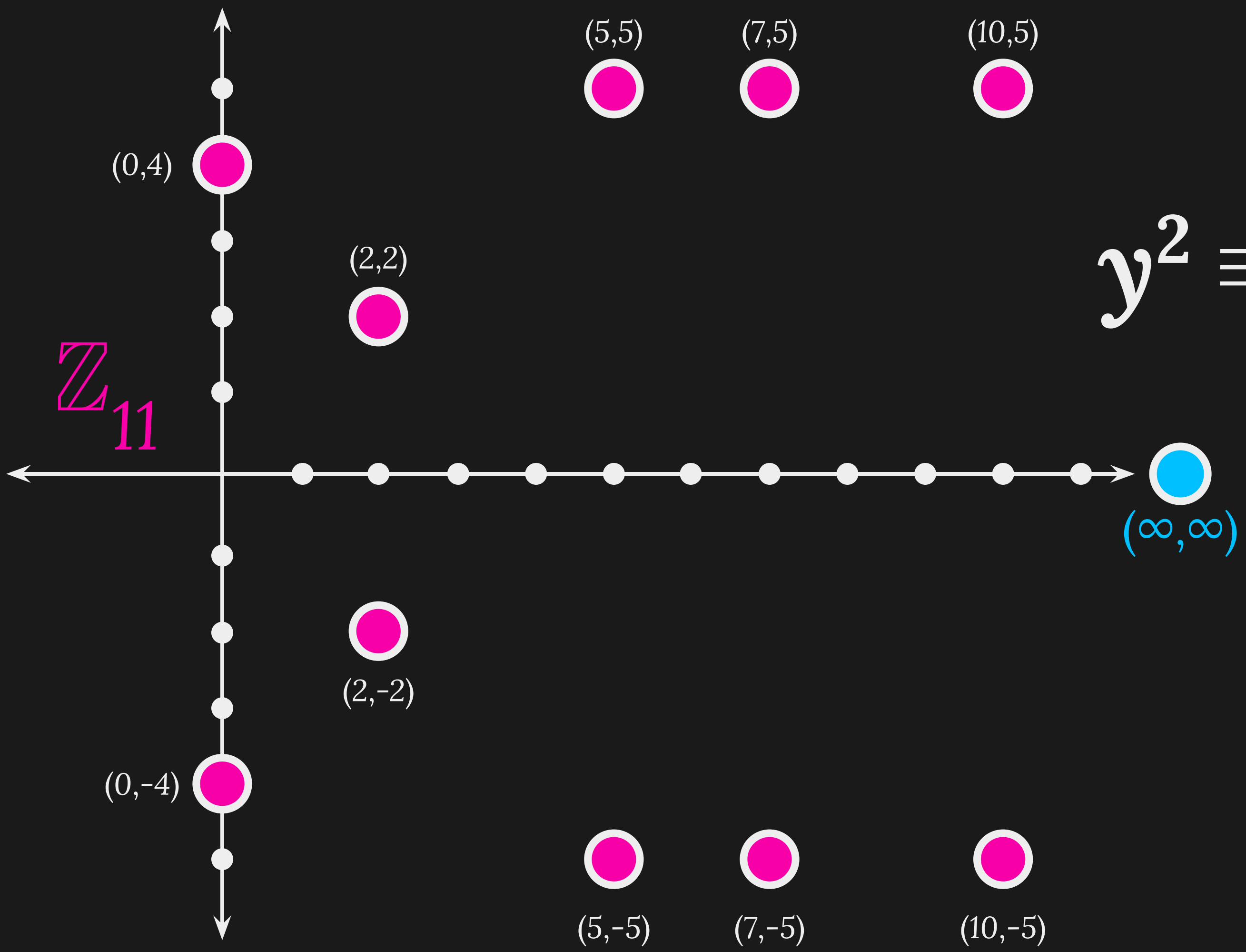

$$y^2 \equiv x^3 + x + 5$$

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elliptic curve domain parameters over F_p

$$T = (p, a, b, G, n, h)$$

Certicom Research: http://www.secg.org/sec2-v2.pdf

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elliptic curve domain parameters over F_p

$$T = (p, a, b, G, n, h)$$

an integer defining
the field F_p

elliptic curve domain parameters over F_p

$$T = (p, a, b, G, n, h)$$

two elements of F_p defining

$$E: y^2 \equiv x^3 + ax + b$$

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elliptic curve domain parameters over F_p

$$T = (p, a, b, G, n, h)$$

a point on $E(F_p)$ written as

$$G = (x_G, y_G)$$

Certicom Research: http://www.secg.org/sec2-v2.pdf

elliptic curve domain parameters over $\boldsymbol{F}_p$

$$T = (p, a, b, \mathrm{G}, n, h)$$

the *order* of G in $E(F_p)$ - i.e.,

$$n \times \mathrm{G} = \mathbf{O}$$

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elliptic curve domain parameters over $\boldsymbol{F}_p$

$$T = (p, a, b, \mathrm{G}, n, h)$$

the *cofactor* of G in $E(F_p)$, which is $|E(F_p)| \,/\, n$

Certicom Research: http://www.secg.org/sec2-v2.pdf

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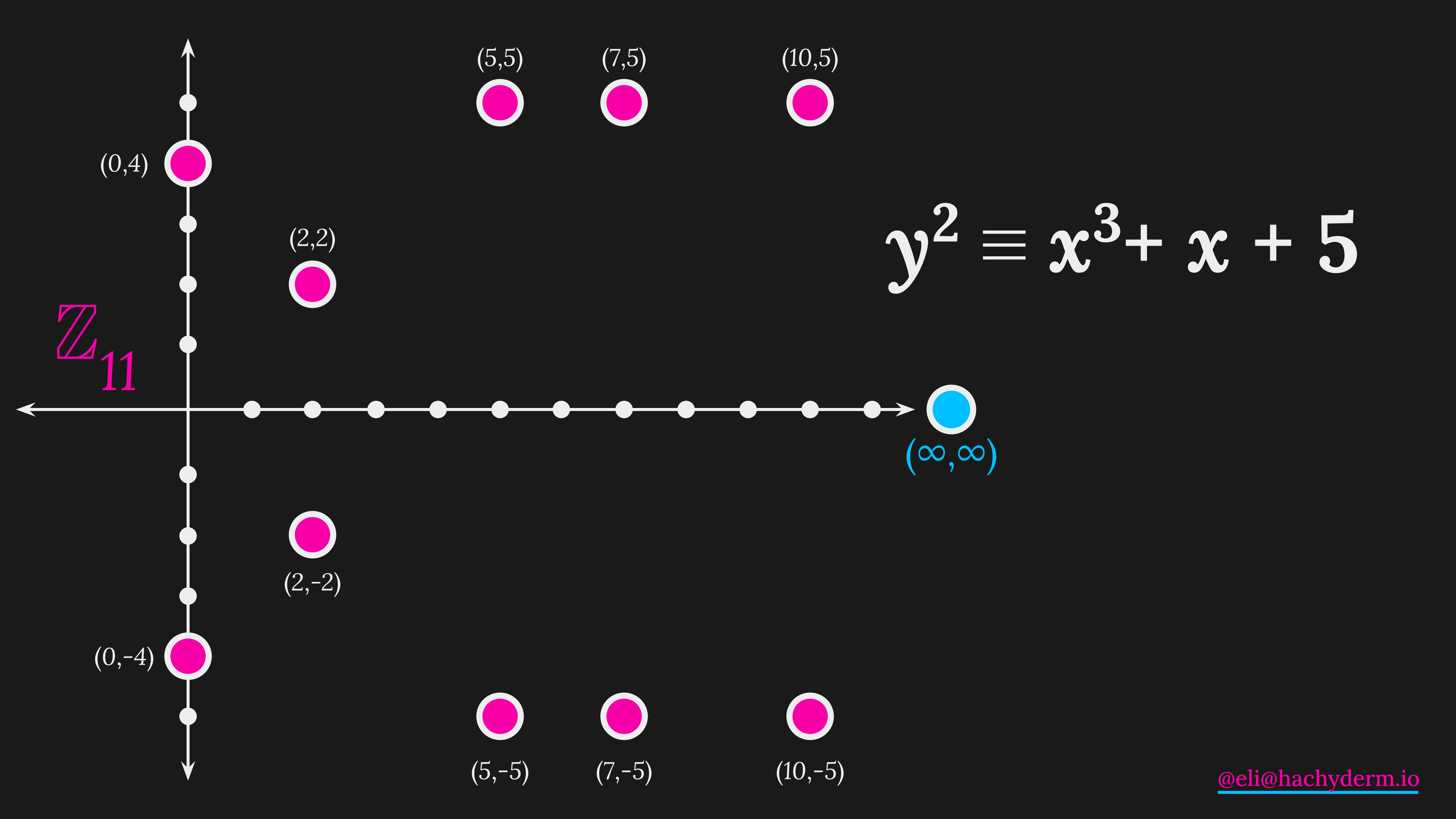
elliptic curve domain parameters over $\boldsymbol{F}_p$

$$T = (p, a, b, G, n, h)$$

or more properly, $orb(G)$

the cofactor of G in $E(F_p)$, which is

$$|E(F_p)| \,/\, n$$



(5,5)
(7,5)
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(2,2)
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$\mathbb{Z}_{11}$
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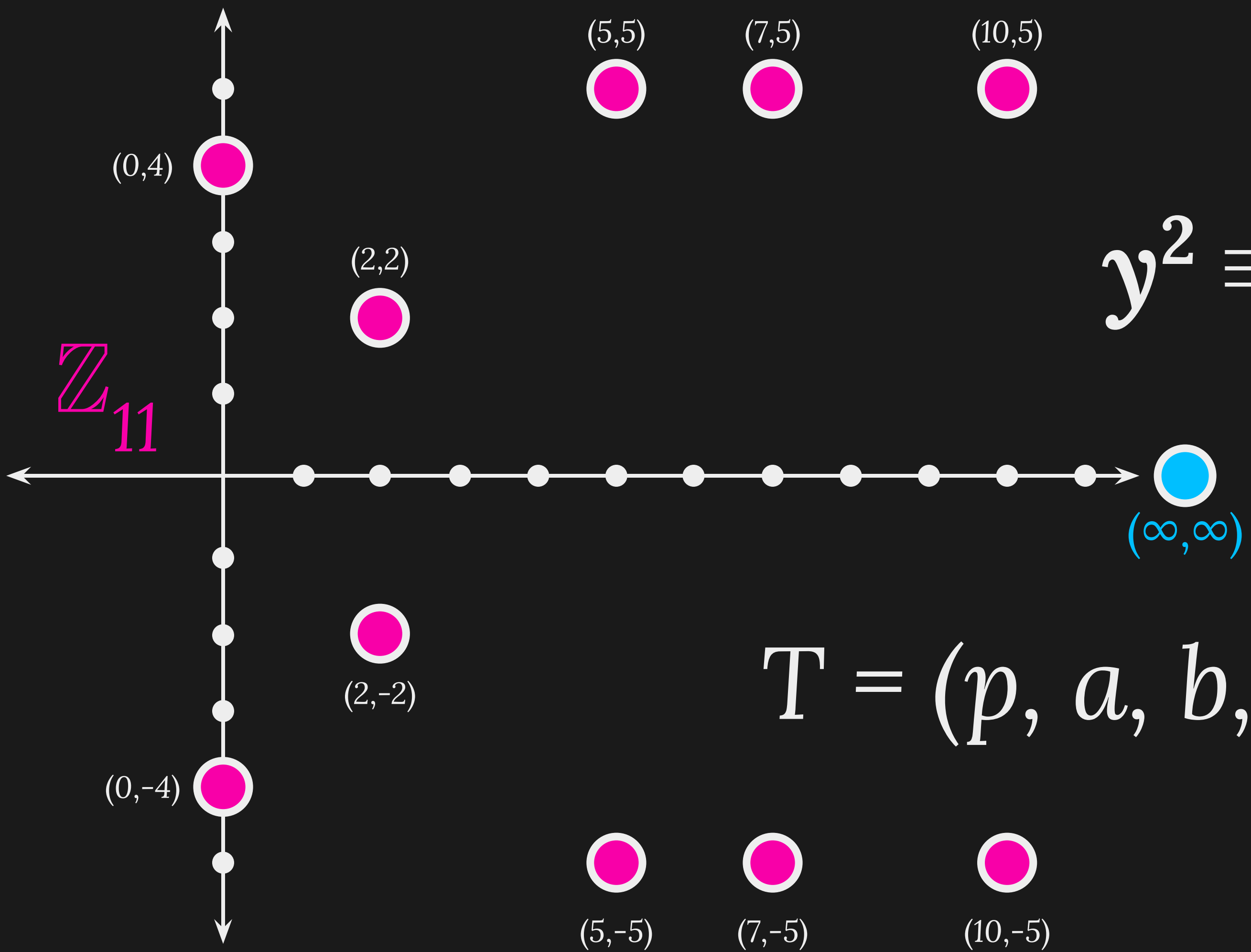

$$y^2 \equiv x^3 + x + 5$$

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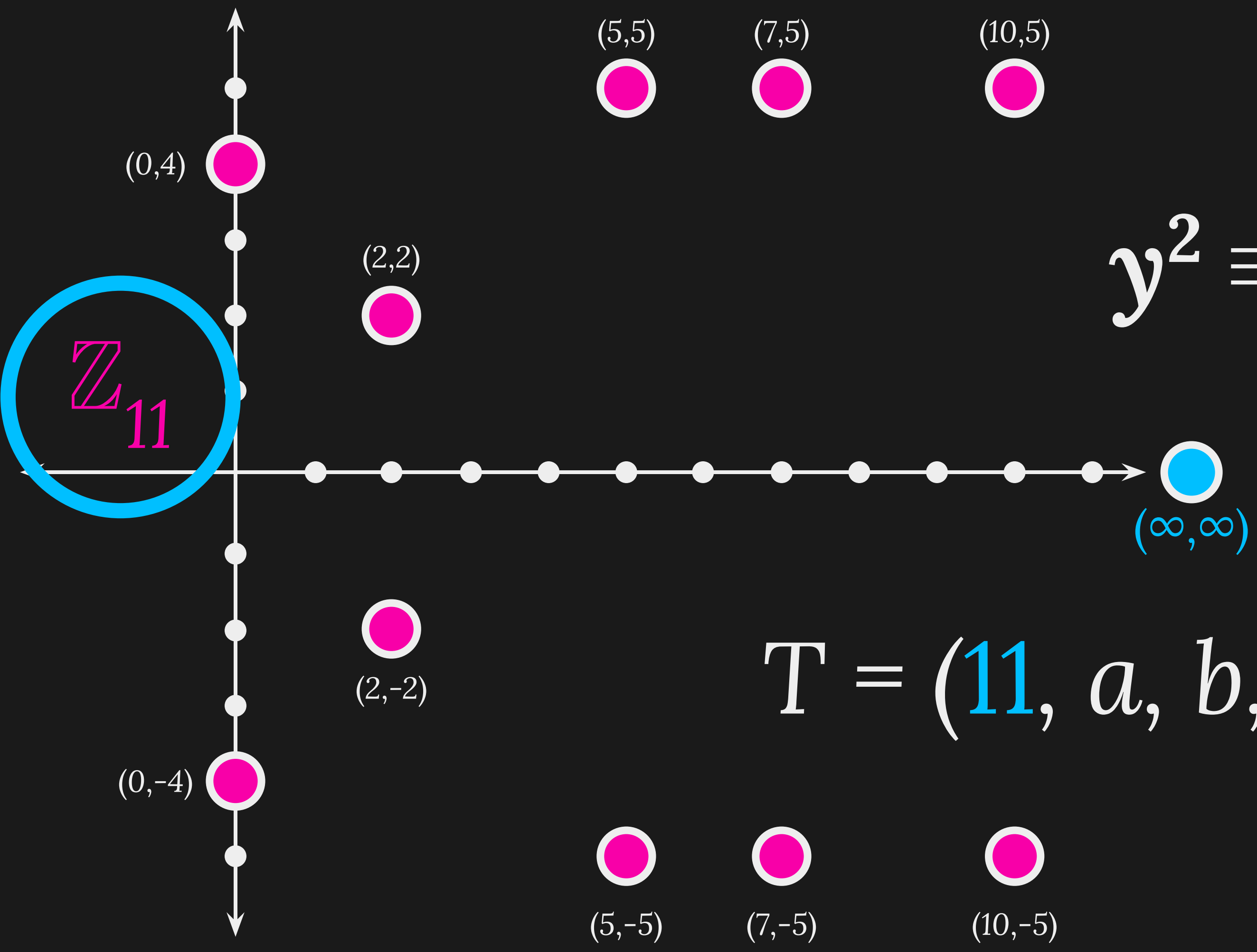

$$y^2 \equiv x^3 + x + 5$$

$$T = (11, a, b, G, n, h)$$

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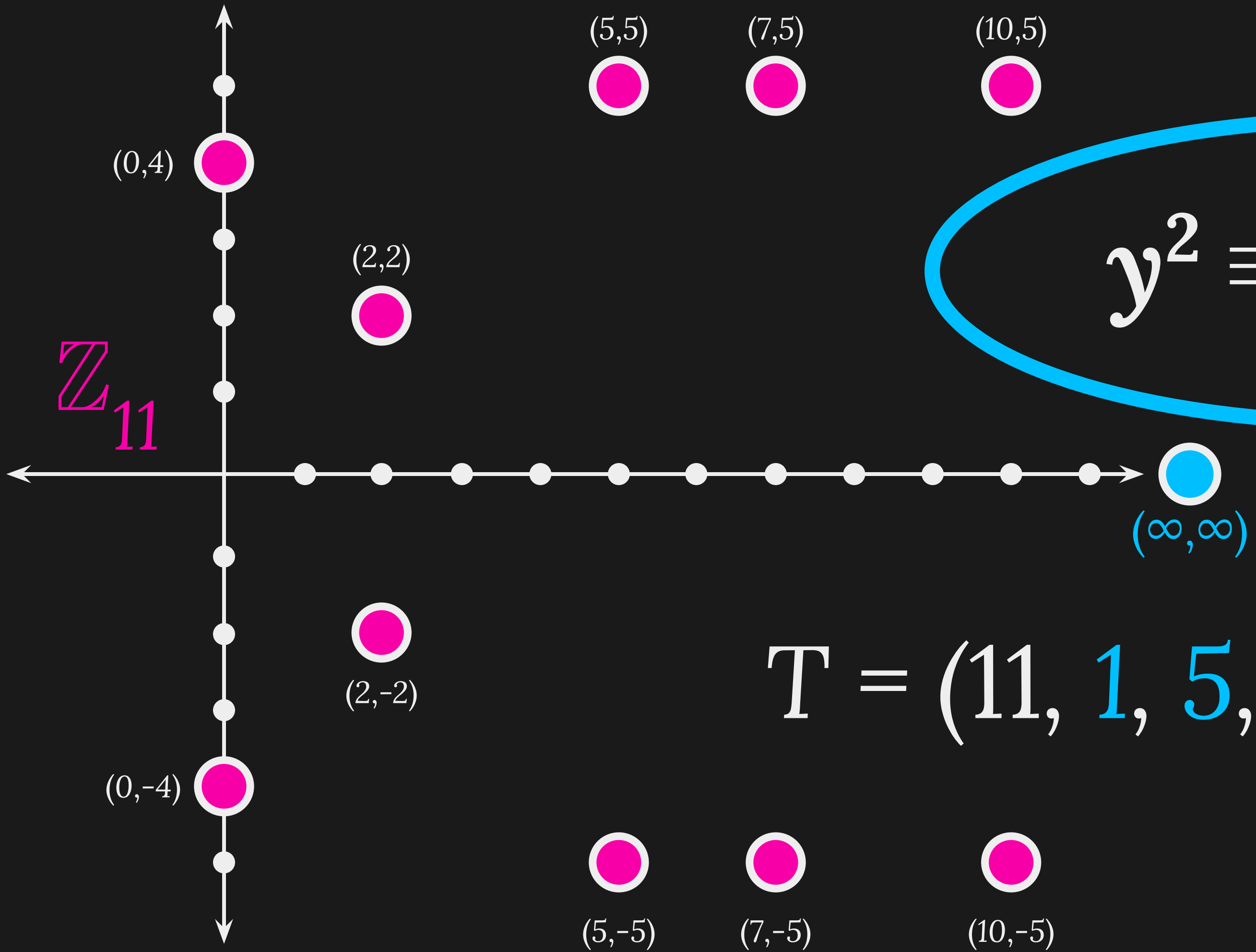
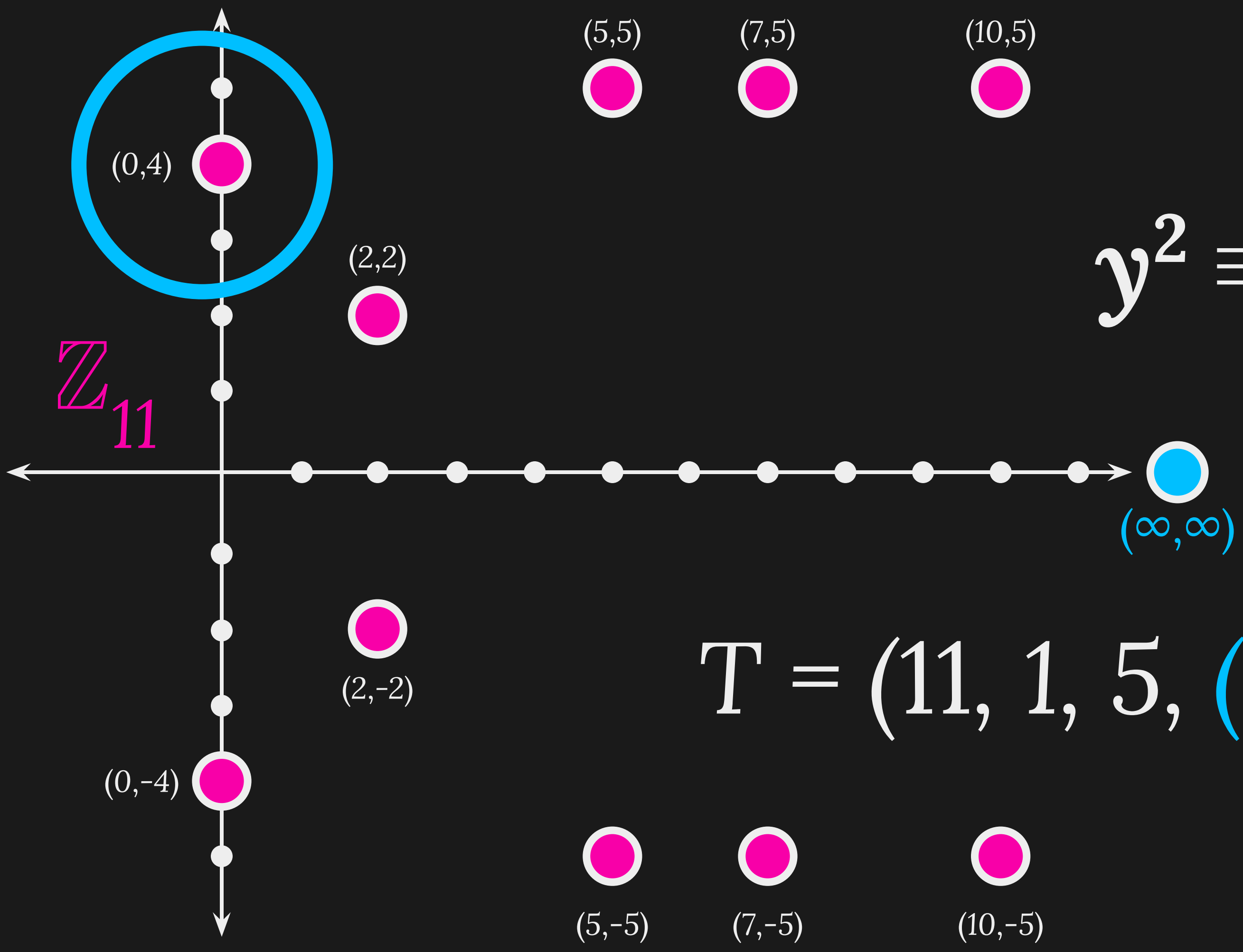


$$y^2 \equiv x^3 + x + 5$$

$$T = (11, 1, 5, G, n, h)$$



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$$T = (11, 1, 5, (0,4), n, h)$$

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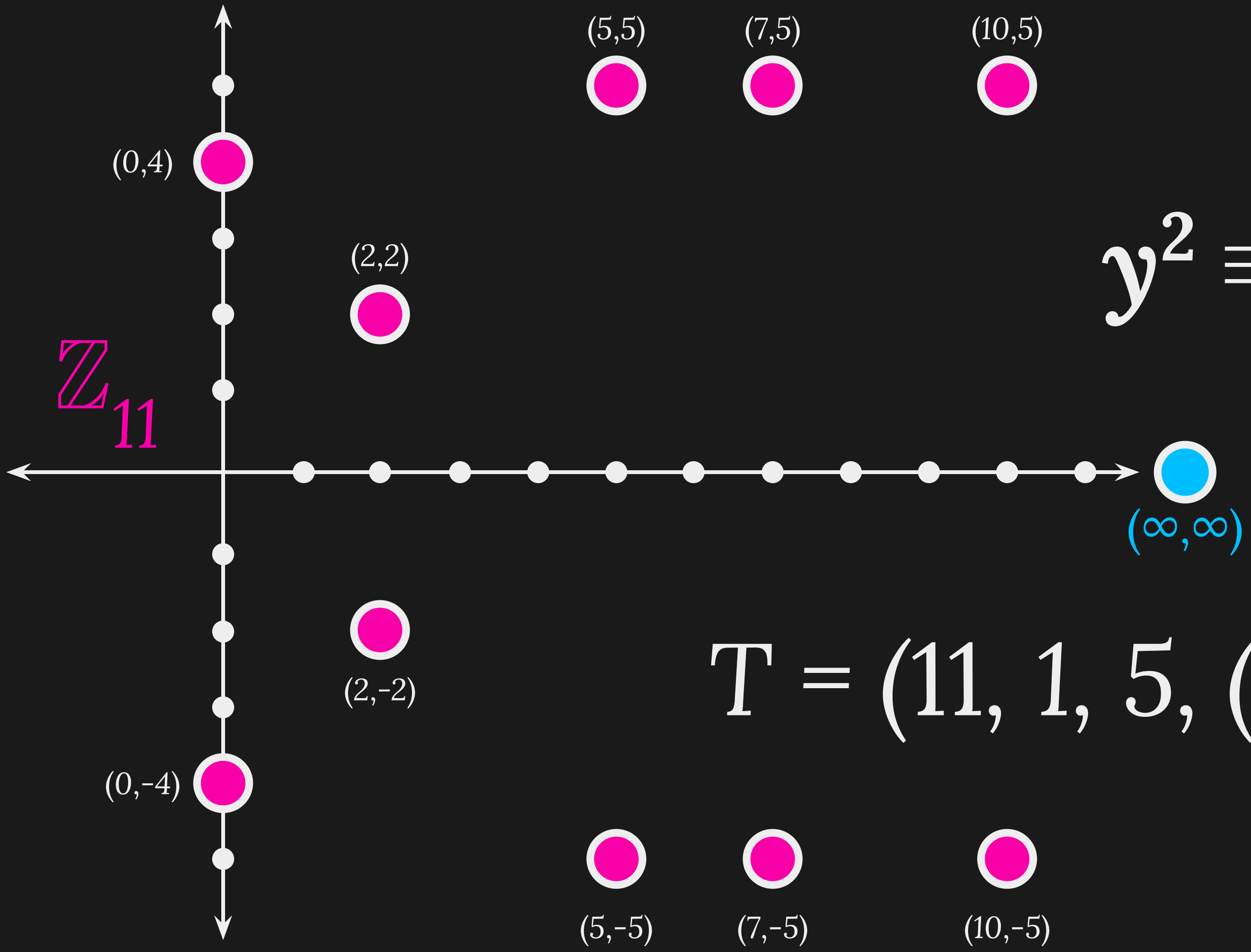

$$y^2 \equiv x^3 + x + 5$$

$$T = (11,\ 1,\ 5,\ (0,4),\ 11,\ 1)$$

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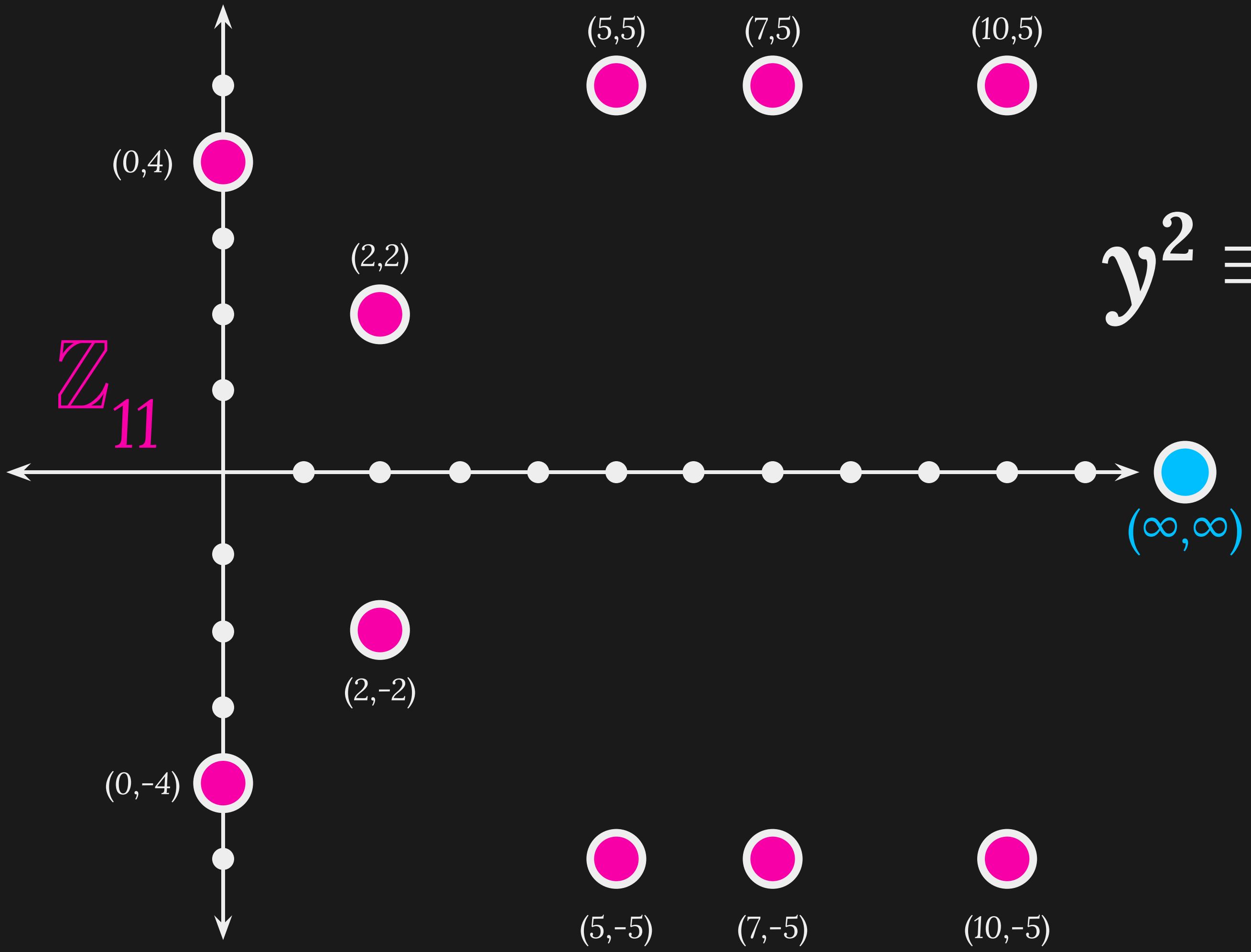
< worked example at the end >



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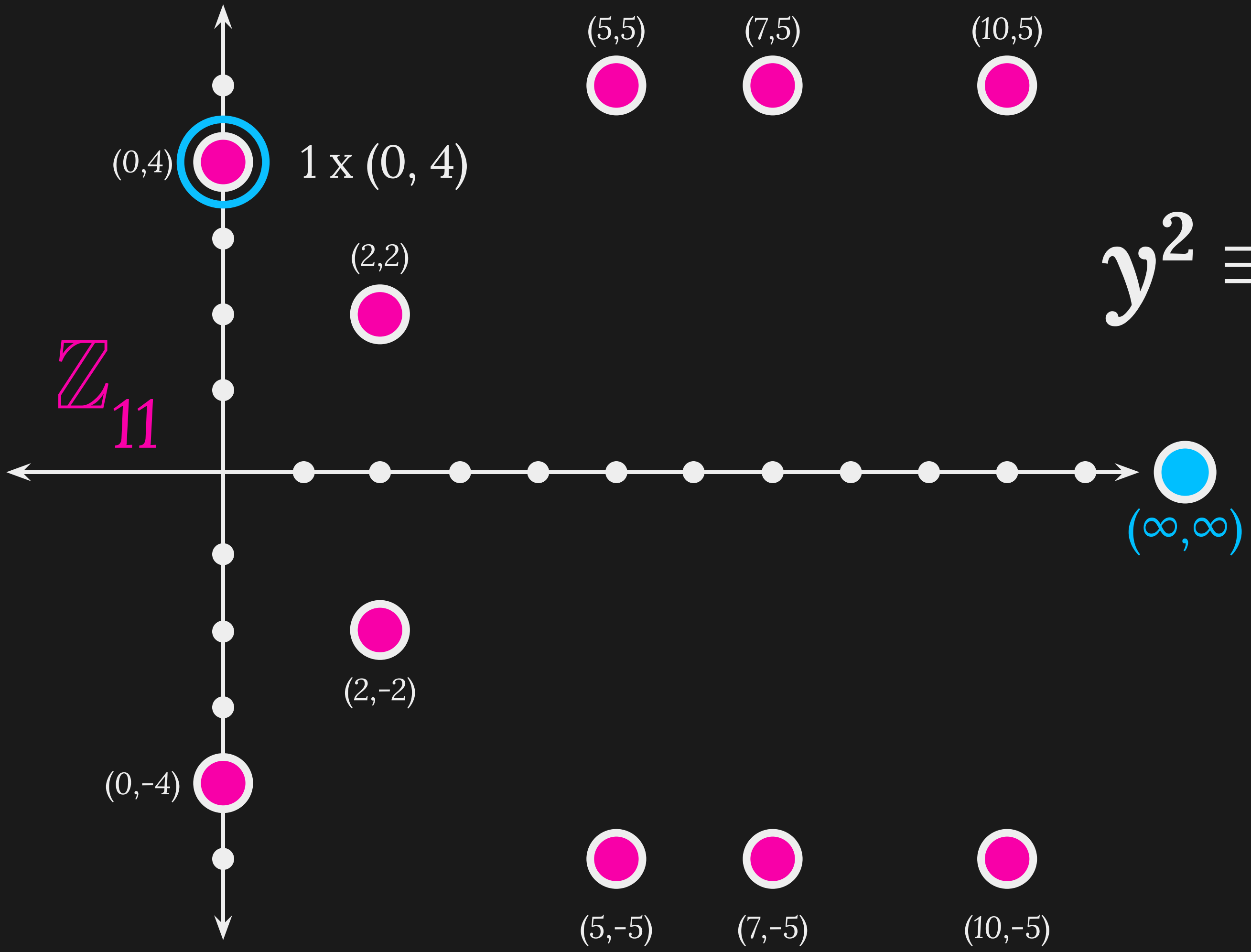

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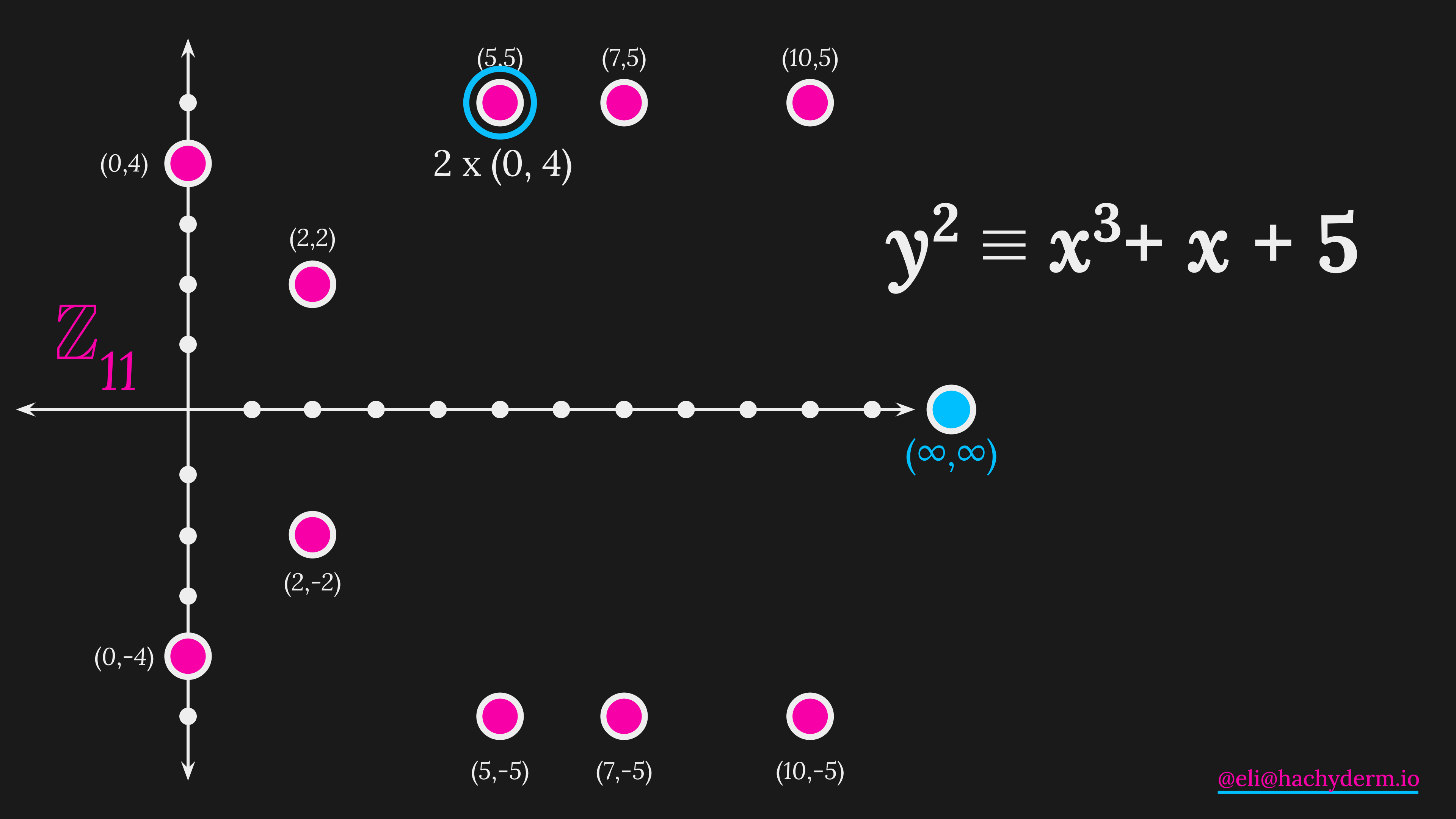
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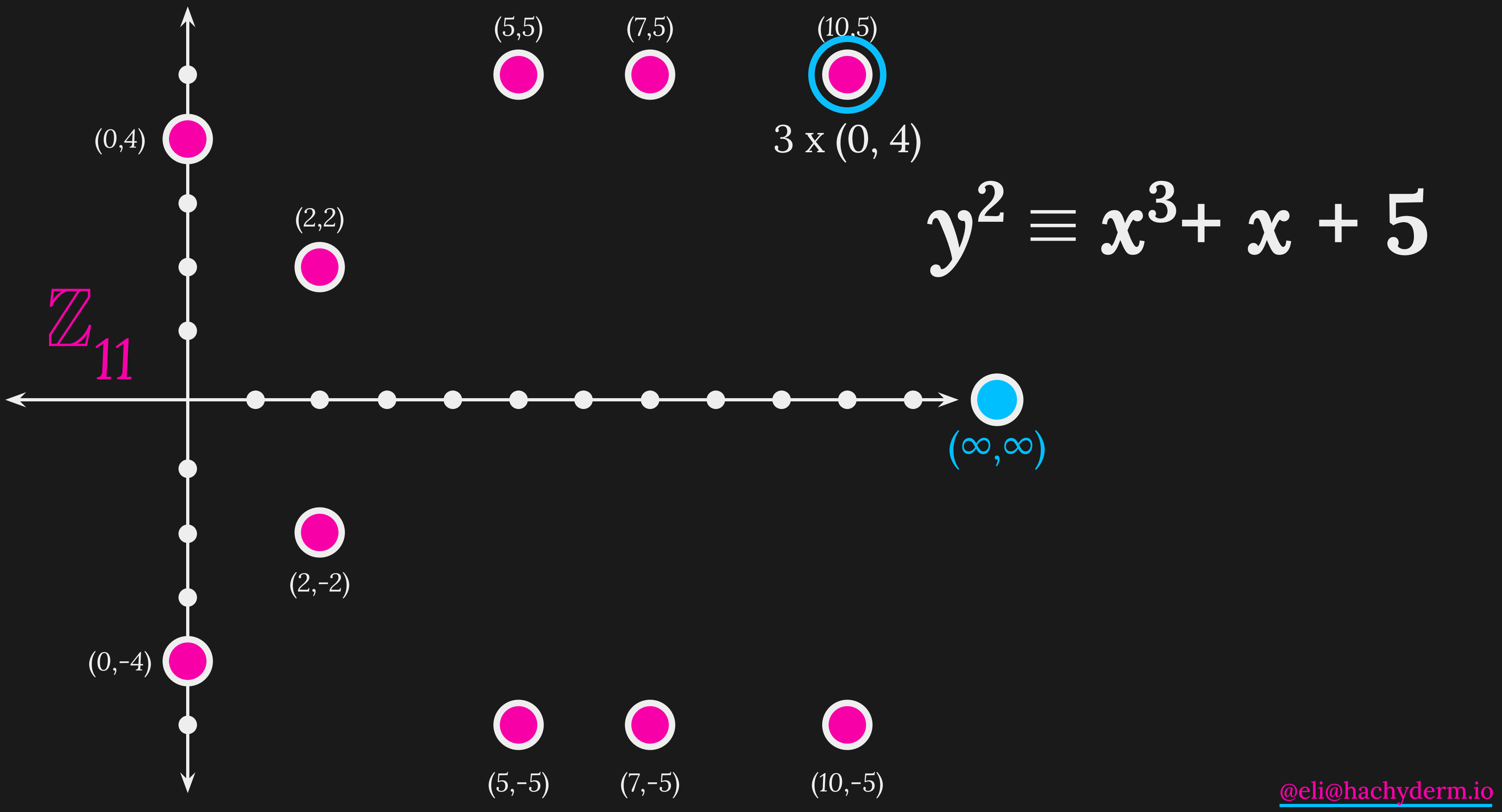
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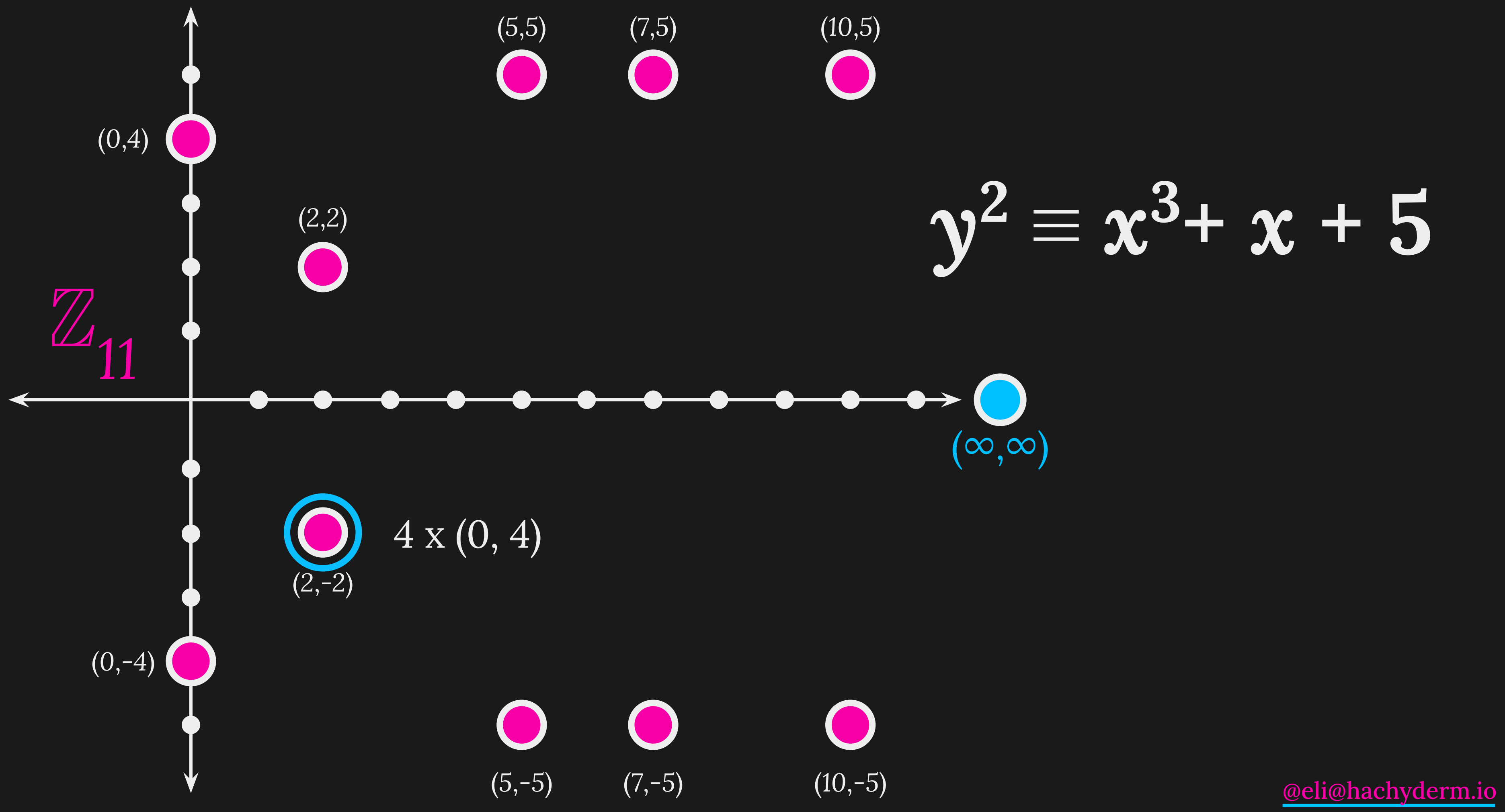
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$$1 \times G = (0, 4)$$
$$2 \times G = (5, 5)$$
$$3 \times G = (10, 5)$$
$$4 \times G = (2, -2)$$
$$5 \times G = (7, -5)$$
$$6 \times G = (7, 5)$$
$$7 \times G = (2, 2)$$
$$8 \times G = (10, -5)$$
$$9 \times G = (5, -5)$$
$$10 \times G = (0, -4)$$

$$11 \times G = (\infty, \infty)$$

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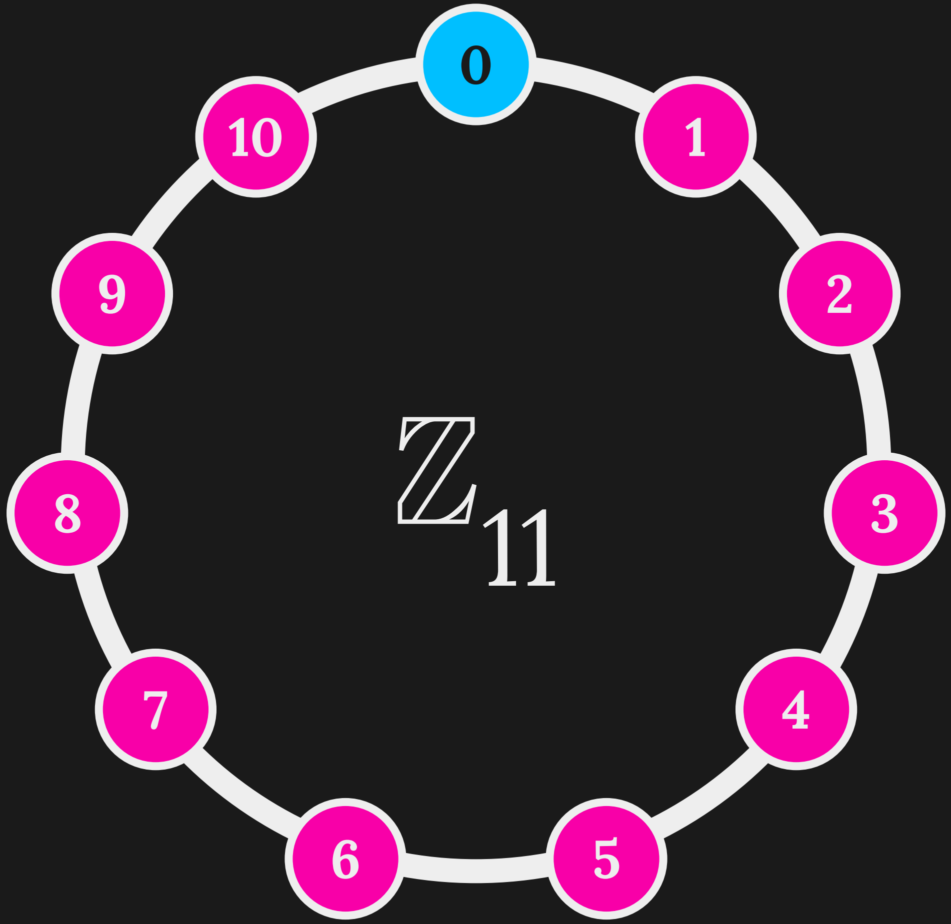

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comparison with RSA

comparison with RSA

smaller key size per security

comparison with RSA

smaller key size per security

smaller payload size

comparison with RSA

smaller key size per security

smaller payload size

faster computation

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4

Quantum Computing & Shor's Algorithms

the Integer Factorisation problem

if pq = N with p & q prime, find p and q given only N

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the Discrete Logarithm problem

if g generates a subgroup of a finite field F, and y is another member of F, find x such that $g^x = y$

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if $pq = N$ with p & q prime, find p and q given only N

the Discrete Logarithm problem

if g generates a subgroup of a finite field F, and y is another member of F, find x such that $g^x = y$

the Elliptic Curve Discrete Logarithm problem

if G generates a subgroup of an elliptic curve over a field F, and P is another member of that elliptic curve, find k such that $P = kG$

Shor's order-finding algorithm

for a given number N, and any number a between 1 and N, we can find the smallest r such that $a^r \equiv 1 \bmod N$, in polynomial time

Shor’s order-finding algorithm

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this breaks RSA!

the Integer Factorisation problem

if pq = N with p & q prime, find p and q given only N

the Discrete Logarithm problem

if g generates a subgroup of a finite field F, and y is another member of F, find x such that $g^x = y$

the Elliptic Curve Discrete Logarithm problem

if G generates a subgroup of an elliptic curve over a field F, and P is another member of that elliptic curve, find k such that $P = kG$

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Post-quantum Cryptography

the isogeny-finding problem

given two elliptic curves between which we know there exists an isogeny, find the mapping that describes it

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SIKE and SIDH, which are considered insecure

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CSIDH

Quantum Security Analysis of CSIDH and Ordinary Isogeny-based Schemes

Xavier Bonnetain[1,2] and André Schrottenloher[2]

[1] Sorbonne Université, Collège Doctoral, F-75005 Paris, France
[2] Inria, France

Abstract. CSIDH is a recent proposal by Castryck, Lange, Martindale, Panny and Renes for post-quantum non-interactive key-exchange. It is similar in design to a scheme by Couveignes, Rostovtsev and Stolbunov

https://who.rocq.inria.fr/Xavier.Bonnetain/pdfs/csidh-attack.pdf

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7 Conclusion

We presented a comprehensive quantum security assessment of CSIDH. In particular, when compared to the cost of a classical key-exchange, we showed that the parameters set in [6] actually seem to provide only around half of the expected security, as summarized in Table 7.

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the isogeny-finding problem

given two elliptic curves between which we know there exists an isogeny, find the mapping that describes it

SIKE and SIDH, which are considered insecure

CSIDH, which should also be considered insecure

the Learning With Errors problem

introducing noise to encodings and using probability to decode

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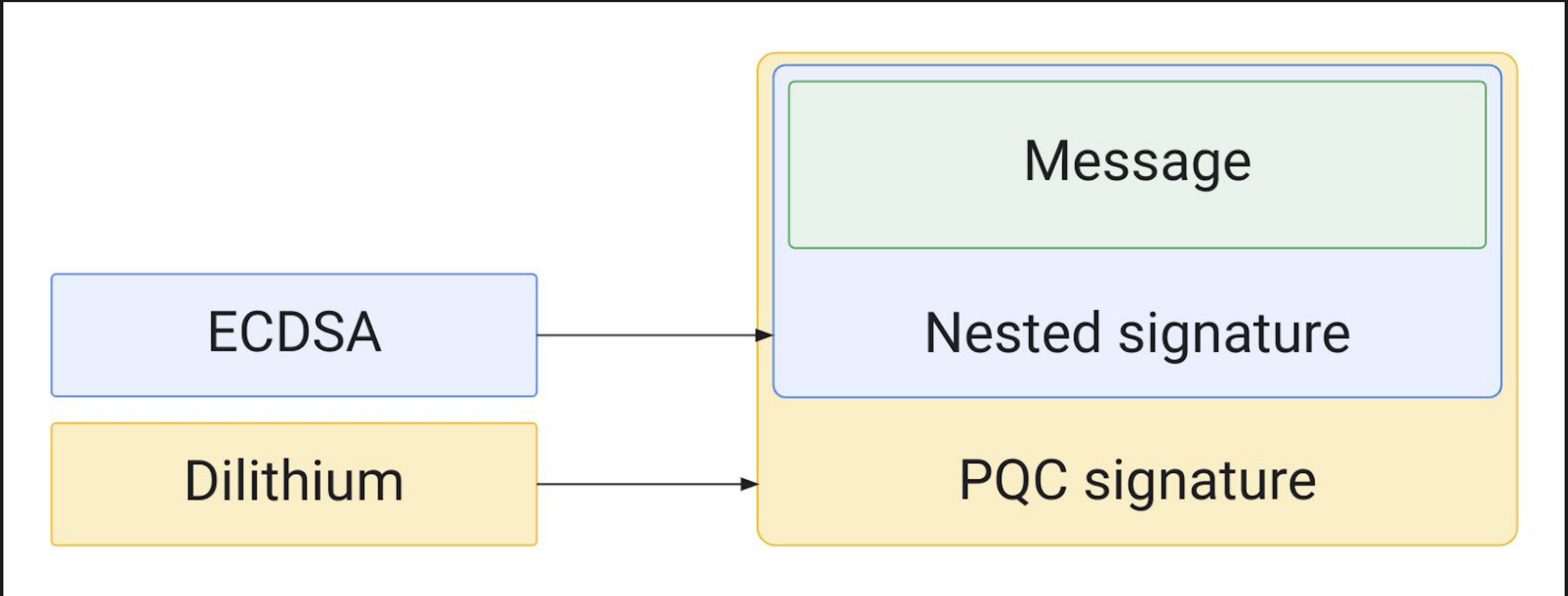
introducing noise to encodings and using probability to decode

CRYSTALS-Kyber (key encapsulation) and
CRYSTALS-Dilithium (signatures)



https://security.googleblog.com/2023/08/toward-quantum-resilient-security-keys.html

In Chrome, you can now enable ‘X25519Kyber768’ for key exchange during TLS

32 bits generated by X25519

32 bits generated by Kyber768

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https://openquantumsafe.org/

what I hope to see

what I hope to see

more diverse quantum-resilient cryptosystems

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more diverse quantum-resilient cryptosystems

quantum-resilient hardware tokens

what I hope to see

more diverse quantum-resilient cryptosystems

quantum-resilient hardware tokens

wider accessibility & rollout

wrapping up

how we got here

how we got here

RSA & ECDSA

how we got here

RSA & ECDSA

...and how quantum breaks them

how we got here

RSA & ECDSA

…and how quantum breaks them

what's next

Asymmetric Cryptography: A Deep Dive

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they/them/theirs

sources: history

https://www.redhat.com/en/blog/brief-history-cryptography

sources: RSA + group theory

https://ee.stanford.edu/~hellman/publications/24.pdf

https://weakdh.org/imperfect-forward-secrecy-ccs15.pdf

https://en.wikipedia.org/wiki/Padding_(cryptography)

sources: ECC

https://scholar.rose-hulman.edu/cgi/viewcontent.cgi?article=1389&context=rhumj

http://koclab.cs.ucsb.edu/teaching/ecc/eccPapers/Washington-ch04.pdf

http://www.secg.org/sec2-v2.pdf

sources: QC & Shor

https://research.kudelskisecurity.com/2021/08/24/quantum-attack-resource-estimate-using-shors-algorithm-to-break-rsa-vs-dh-dsa-vs-ecc/

https://arxiv.org/pdf/quant-ph/9508027.pdf

https://www.omnicalculator.com/math/power-modulo

sources: PQC

https://security.googleblog.com/2023/08/toward-quantum-resilient-security-keys.html

https://csidh.isogeny.org/

https://sike.org/

https://eprint.iacr.org/2019/725

https://blog.chromium.org/2023/08/protecting-chrome-traffic-with-hybrid.html

https://www.ietf.org/archive/id/draft-tls-westerbaan-xyber768d00-02.html

https://openquantumsafe.org/

https://eprint.iacr.org/2022/1225.pdf

https://github.com/signalapp/libsignal/commit/ff09619432e19e96231ebed913fe4433f26ee0d2

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

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$d_{PK} = 3$

Pick a random number d_{PK} from $[1, \dots n\text{-}1] = [1, \dots 10]$.
Let's pick 3. This is our private key.

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$d_{PK} = 3 \quad Q_{PK} = (10, 5)$

Pick a random number d_{PK} from $[1, \ldots n-1] = [1, \ldots 10]$.
Let's pick 3. This is our private key.

Calculate $Q_{PK} = d_{PK} \times G$, which in our case is
$3 \times (0,4) = (10, 5)$. This is our public curve point.

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$d_{PK} = 3 \quad Q_{PK} = (10, 5)$

We have some binary message, e, to sign. Let's say we want to sign the message 01001110 01000100 01000011.

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$z = 3$ $\quad d_{PK} = 3 \quad Q_{PK} = (10, 5)$

We have some binary message, *e*, to sign. Let's say we want to sign the message 01001110 01000100 01000011.

The size of our group is 11, or 1101 in binary - 4 bits long. Take the last 4 bits of our message: 0011. Call it *z*.

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$z = 3$ $\quad d_{PK} = 3$ $\quad Q_{PK} = (10, 5)$

Pick another random number k from $[1,\ldots n\text{-}1]$. This time let's choose 5. This must be random per signature.

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$k^{-1} = 9$ $z = 3$ $d_{PK} = 3$ $Q_{PK} = (10, 5)$

Pick another random number k from $[1,\ldots n-1]$. This time let's choose 5. This must be random per signature.

Find its inverse k^{-1} in $\mathbf{F}_{11}$, which is 9.

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worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$x_k = 7, y_k = -5$ $k^{-1} = 9$ $z = 3$ $d_{PK} = 3$ $Q_{PK} = (10, 5)$

Pick another random number k from $[1,...n-1]$. This time let's choose 5. This must be random per signature.

Find its inverse k^{-1} in $\mathbf{F}_{11}$, which is 9.

Calculate $k \times G = 5 \times (0,4) = (7, -5)$. Take its coordinates, so we have $x_k = 7, y_k = -5$

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$x_k = 7$, $y_k = -5$ $k^{-1} = 9$ $z = 3$ $d_{PK} = 3$ $Q_{PK} = (10, 5)$

Now calculate r and s, where

$$r \equiv x_k \bmod n \text{ and } s \equiv k^{-1}(z + r * d_{PK}) \bmod n$$

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worked example with $T = (11, 1, 5, (0,4), 11, 1)$

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This gives us $r = 7$ and $s = 7$, and this is our signature:

$(r,s) = (7, 7)$.

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If either r or s are 0, we have to go back and pick a different k.

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$x_k = 7, y_k = -5$ $k^{-1} = 9$ $z = 3$ $d_{PK} = 3$ $Q_{PK} = (10, 5)$

We've now generated a signature $(r,s) = (7, 7)$ over the binary message 01001110 01000100 01000011.

Let's verify it!

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$r = 7, s = 7 \qquad Q_{PK} = (10, 5)$

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$z = 3$ $r = 7, s = 7$ $Q_{PK} = (10, 5)$

We have the message, 01001110 01000100 01000011.
Take the last 4 bits as we did before to get $z = 3$.

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worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$u_1 = 2, u_2 = 5$ $z = 3$ $r = 7, s = 7$ $Q_{PK} = (10, 5)$

We have the message, 01001110 01000100 01000011.
Take the last 4 bits as we did before to get $z = 3$.

Calculate $u_1 \equiv zs^{-1} \bmod n$: $u_1 \equiv 3*8 \equiv 2 \bmod 11$
Calculate $u_2 \equiv rs^{-1} \bmod n$: $u_2 \equiv 7*7 \equiv 5 \bmod 11$

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$u_1 = 2, u_2 = 5$ $z = 3$ $r = 7, s = 7$ $Q_{PK} = (10, 5)$

Calculate a new point on the curve, $(x, y) = u_1 \times G + u_2 \times Q_{PK}$

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worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$u_1 = 2, u_2 = 5$ $z = 3$ $r = 7, s = 7$ $Q_{PK} = (10, 5)$

Calculate a new point on the curve, $(x, y) = u_1 \times G + u_2 \times Q_{PK}$

$$u_1 \times G = 2 \times (0,4)$$

$$u_2 \times Q_{PK} = 5 \times (10,5) = 5 \times (3 \times (0,4)) = 4 \times (0,4)$$

worked example with $T = (11, 1, 5, (0,4), 11, 1)$

$u_1 = 2, u_2 = 5$ $z = 3$ $r = 7, s = 7$ $Q_{PK} = (10, 5)$

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so $(x, y) = 2 \times (0,4) + 4 \times (0,4) = 6 \times (0,4) = (7,5)$

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worked example with $T = (11, 1, 5, (0,4), 11, 1)$

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The signature is valid if $x = r \bmod n$, which it is!

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