A Comparison of Some Proofs of Gödel's Completeness Theorem

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Abstract

We present here a comparison of some proofs of Gödel's completeness theorem (1929), with particular emphasis on the proofs given by Henkin (1947), and the one provided by Rasiowa and Sikorski (1950). We provide a description of Henkin's proof, its inherent explanatory power, and also attempt to pinpoint the elements of the Henkin style proof that are able to be extended beyond first-order logic. Similarly we identify the properties that are common to both Henkin's and Rasiowa and Sikorski's proof, in order to identify a common pattern that we find can be employed in more general settings.

1. Introduction

A formulation of the problem of completeness of classical first-order predicate logic was made mathematically precise in 1928, by Hilbert and Ackermann [5], and resolved (positively) by Gödel in 1929 [1]. Gödel's completeness theorem is a fundamental result of first-order logic in that it relates the notions of validity and provability of formulae. It can be stated as follows: any of the standard formal systems for defining provability in first order logic are "complete" in the sense that they contain a derivation for each valid formula — where a formula is valid if and only if it is true for every interpretation of its symbols over any domain of quantification. There are two types of completeness, the *strong* completeness theorem: $\Gamma \models \varphi \implies \Gamma \vdash \varphi$ (where Γ is a set of formulae), and the *weak* completeness theorem: $\models \varphi \implies \vdash \varphi$. We will show the strong completeness theorem, since weak completeness follows immediately from strong completeness by taking $\Gamma = \emptyset$.[2] A great deal of the importance of this theorem comes from the correspondence it establishes between *semantic* truth and *syntactic* provability in first-order logic. Roughly speaking, the semantic notions of models and structures are about sets, together with some operations and relations that satisfy certain sentences; whereas, proofs simply provide rules for the derivation

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[2]Here $\models$ denotes semantic entailment and $\vdash$ denotes syntactic entailment, so the strong completeness theorem can be read as if Γ satisfies φ, then Γ proves φ.

of new sentences. As such, semantics and syntax are traditionally found to live in entirely distinct spheres; however, the completeness theorem tells us that the semantic notion of logical consequence is captured by the syntactic notion of proof, and when taken together with the soundness theorem: $\Gamma \vdash \varphi \implies \Gamma \models \varphi$, these two notions are seen to coincide.

There have been several proofs of the completeness theorem since Gödel's original proof in 1929, most notably the one given by Henkin in 1947 [2], which has become known as the *Henkin method*. Henkin's proof is notable due to its improvement of Gödel's proof (improvement in the sense that the uncountable case is dealt with), along with its ability to be adapted to employ strictly model-theoretic notions. We focus here on the latter aspect since it is among the traits of Henkin's proof that allow it to be generalized. We also seek to pinpoint the elements of the Henkin style proof that lend themselves to such generalizations, since in doing so, we hope to be able to demonstrate that the general ideas behind the Henkin method have utility beyond fist-order logic. Resnik and Kushner provide the following assessment of Henkin's proof:

> The proof is generally regarded as really showing what goes on in the completeness theorem and the proof-idea has been used again and again in obtaining results about other logical systems. Yet again, it is not easy to identify the characterizing property on which it depends. [9 pp. 141]

We will explore this "characterizing property" further in section 5, which we can summarize as the property wherein a systematic extension of some given theory to one that is complete, and whose canonical models satisfy the original theory occurs. We will then look at Henkin's proof that (in ZF) the completeness theorem and the ultrafilter lemma are equivalent [8], this is of note since in later sections we will show that Rasiowa and Sikorski's lemma is also equivalent to the ultrafilter lemma in conjunction with Baire category theorem.

Not long after Henkin's proof, Rasiowa and Sikorski provided yet another proof of the completeness theorem (1951) [3], which we will look at in section 4. The proof provided by Rasiowa and Sikorski is topological in nature (employs the Stone Representation Theorem, and Baire category). In developing this proof, Rasiowa and Sikorski developed their eponymous *Rasiowa-Sikorski Lemma*, which states that for a given Boolean algebra B, a countable collection of subsets of B, (denote $\mathscr{C}$), and a nonzero point $\beta \in B$ there is an ultrafilter U over B such that $\beta \in U$, and U preserves all existing meets (infima) in $\mathscr{C}$. They were then able to prove the completeness theorem by applying this lemma to the Lindenbaum algebra of a countable first-order theory, and constructing a model from the resulting ultrafilter — a model that falsifies some fixed non-theorem. A connection that can be drawn between Henkin's proof, and Rasiowa and Sikorski's proof, is that Henkin's proof implicitly constructs an ultrafilter on the Lindenbaum algebra of formulae. Thus, we note that both Rasiowa and Sikorski's proof and Henkin's proof can be seen as instances of a general pattern. Similar to our analysis of Henkin's proof, along with providing a clear exposition of the Rasiowa-Sikorski proof, we also seek to explore the traits that make it so adaptable.

2. Preliminaries

2.1. ***First-order logic***

In this section we present a brief summary of first-order logic.

Definition 2.1. A first-order *language* $\mathcal{L}$ consists of the following symbols:

1. The logical connectives: the negation sign $\neg$, the disjunction sign $\vee$, the conjunction sign $\wedge$, the implication symbol $\Rightarrow$, and the biconditional symbol $\Leftrightarrow$.
2. The quantifier symbols: the existential quantifier $\exists$, and the universal quantifier symbol $\forall$.[3]
3. Punctuation symbols: left and right parentheses "(" and ")", and the comma ",".
4. A countably infinite set of variables, $x_1, x_2, \ldots$.
5. A countable (possibly empty) set of function symbols.
6. A countable (possibly empty) set of constant symbols.
7. A nonempty set of predicate symbols, including a binary predicate symbol $P_1^2(x, y)$ which we will abbreviate as $x = y$.

Definition 2.2. The set of *terms* of $\mathcal{L}$ is inductively defined by the following clauses:

1. Any variable is a term.
2. Any constant is a term.
3. If $t_1, \ldots, t_n$ are terms, and f is a function symbol, then the expression $f(t_1, \ldots, t_n)$ is also a term.

Definition 2.3. The set of *formulas* is inductively defined by the following clauses:

1. If P_i^n is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $P_i^n(t_1, \ldots, t_n)$ is a formula.
2. if f_i^n is an n-ary function symbol, and $t_1, \ldots, t_n$ are terms, then $f_i^n(t_1, \ldots, t_n)$ is a formula.
3. If t_1 and t_2 are terms then $t_1 = t_2$ is a formula.
4. If φ is a formula, then so is $\neg\varphi$.
5. If φ and ψ are formulas, then $(\varphi \Rightarrow \psi)$ is a formula, and the same holds for the other binary logical connectives.
6. If φ is a formula and x is a variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulae.
   The formulas that are obtained from the first two rules are called *atomic* formulas

Definition 2.4. Let $\mathcal{L}$ be a first-order language. An *interpretation* $\mathfrak{A}$ of $\mathcal{L}$ is comprised of the following:

[3] It suffices to only require $\Rightarrow, \neg, \forall$, in the definition of a language since $\wedge, \vee, \Leftrightarrow, \exists$ can be expressed in terms of these, indeed: $\varphi \wedge \psi$, $\varphi \vee \psi$, $(\exists x_i)\varphi$ are shorthands for $\neg(\varphi \Rightarrow \neg\psi)$, $\neg\varphi \Rightarrow \psi$, $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ and $\neg(\forall x_i)\neg\varphi$ respectively.

1. A nonempty set D, which is called the *domain* of the interpretation.
2. for each n-ary predicate symbol P_i^n of $\mathcal{L}$, an assignment of an n-place relation $(P_i^n)^{\mathfrak{A}}$ in D.
3. For each constant symbol c_i of $\mathcal{L}$, an assignment of a fixed element $c_i^{\mathfrak{A}}$ of D.

We will assume that the reader is familiar with the notions of free and bound variables. For a particular interpretation of a language, a formula of $\mathcal{L}$ with no free variables (called a *sentence*) represents a proposition which is either true or false, on the other hand, a formula that has free variables may be satisfied for some values in the domain and not satisfied for others. We make more explicit (à la Tarski) what is meant by *satisfied* in this context.

2.1.1. A Tarskian definition of satisfiability

Let $\mathfrak{A}$ be an interpretation of a language $\mathcal{L}$, where D is the domain of $\mathfrak{A}$, and let Σ denote the set of all countably infinite sequences of elements of D ($\Sigma = \{s = (s_1, s_2, s_3, \dots) \mid s_i \in D\}$). Before defining what is means for a sequence $s \in \Sigma$ to satisfy a formula φ, it is necessary to define a function $t \mapsto s^*(t)$ which takes a term in $\mathcal{L}$ to an element of D. s^* is defined as follows:

$$s^*(t) = \begin{cases} s_i & \text{if } t \text{ is a variable } x_i \\ c_i^{\mathfrak{A}} & \text{if } t \text{ is a constant } c_i \\ (f_k^n)^{\mathfrak{A}}(s^*(t_1), \dots, s^*(t_n)) & \text{if } t \text{ is a function } f_k^n(t_1, \dots, t_n) \end{cases}$$

The definition of satisfiability, which is an inductive definition can now be stated as follows:

Definition 2.5.

1. If φ is an atomic formula: $P_i^n(t_1, \dots, t_n)$ and $(P_i^n)^{\mathfrak{A}}$ the corresponding relation in D, then a sequence $s = (s_1, s_2, \dots) \in \Sigma$ satisfies φ if and only if $(P_i^n)^{\mathfrak{A}}(s^*(t_1), \dots, s^*(t_n))$ (i.e. the n-tuple $(s^*(t_1), \dots, s^*(t_n))$ is in the relation $(P_i^n)^{\mathfrak{A}}$).[4]
2. A sequence $s \in \Sigma$ satisfies the formula $\neg\varphi$ if and only if s does not satisfy φ
3. A sequence $s \in \Sigma$ satisfies the formula $\varphi \Rightarrow \psi$ if and only if s does not satisfy φ or s satisfies ψ
4. A sequence $s \in \Sigma$ satisfies the formula $(\forall x_i)\varphi$ if and only if every sequence that differs from s only in the i-th component, satisfies φ.

With this definition of satisfiability, we are able to define the notions of truth and falsity of a formula under a given interpretation.

[4] A concrete example makes this more clear, so if we have $D = \mathbb{R}$, and $(P_1^2)^{\mathfrak{A}} = \leq$ and $(f_1^1)^{\mathfrak{A}} = e^x$, then a sequence of real numbers $(s_1, s_2, \dots) \subseteq \mathbb{R}$ satisfies $P_1^2(f_1^1(x_3), x_5)$ if and only if $e^{s_3} \leq s_5$.

Definition 2.6. A formula φ is *true for an interpretation* $\mathfrak{A}$ (denoted $\models_{\mathfrak{A}} \varphi$) if and only if every sequence $s \in \Sigma$ satisfies φ. A formula φ is *false for an interpretation* $\mathfrak{A}$ (denoted $\not\models_{\mathfrak{A}} \varphi$) if and only if there is no sequence $s \in \Sigma$ which satisfies φ. $\mathfrak{A}$ is said to *model* a set of formulae Γ (denoted $\models_{\mathfrak{A}} \Gamma$) if and only if every formula in Γ is true for $\mathfrak{A}$.

Definition 2.7. A formula φ is a *logical consequence* of a set of formulae Γ (denoted $\Gamma \models \varphi$) if and only if under every interpretation, every sequence satisfying every formula in Γ also satisfies φ.

A *first-order theory* in a language $\mathcal{L}$ is a formal theory $\mathcal{T}$ that has the symbols and formulae of $\mathcal{L}$, together with at least the following axioms and rules:
Let φ, ψ, θ be formulae of $\mathcal{L}$, then the axioms of $\mathcal{T}$ are:

$$
\begin{aligned}
&(A1)\ \ \varphi \Rightarrow (\psi \Rightarrow \varphi)\\
&(A2)\ \ (\varphi \Rightarrow (\psi \Rightarrow \theta)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \theta))\\
&(A3)\ \ (\neg\psi \Rightarrow \neg\varphi) \Rightarrow ((\neg\psi \Rightarrow \varphi) \Rightarrow \psi)\\
&(A4)\ \ (\forall x_i)\varphi(x_i) \Rightarrow \varphi(t)\ \text{ if } \varphi(x_i) \text{ is a formula of } \mathcal{L} \text{ and } t\\
&\qquad\quad \text{a term of } \mathcal{L} \text{ which is free for } x_i \text{ in } \varphi(x_i).\\
&(A5)\ \ (\forall x_i)(\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow (\forall x_i)\psi)\ \text{ if there are no free}\\
&\qquad\quad \text{occurrences of } x_i \text{ in } \varphi\\
&(A6)\ \ (\forall x_1)x_1 = x_1\\
&(A7)\ \ x = y \Rightarrow (\varphi(x, x) \Rightarrow \varphi(x, y))
\end{aligned}
$$

The rules of inference for any first-order theory are the following:

1. Modus ponens: φ and $\varphi \Rightarrow \psi$ yield ψ
2. Generalization: $(\forall x_i)\varphi$ follows from φ

Finally we have the following definition:

Definition 2.8. A formula ψ is a *consequence* in $\mathcal{L}$ of a set of formulae Γ, if and only if there is a sequence $\varphi_1, \ldots, \varphi_k$ of formulae, such that φ_k is ψ and for each i it is either the case that φ_i is an axiom, $\varphi_i \in \Gamma$, or φ can be obtained by means of the inference rules applied to some of the preceding formulae in the sequence. This will be denoted $\Gamma \vdash \psi$.

3. A Henkin-style proof

We now present a Henkin-style proof of the strong completeness theorem:

$$\Gamma \models \varphi \implies \Gamma \vdash \varphi \tag{1}$$

This is done in a minimal setting — the language has only one predicate symbol P in addition to the identity predicate, and countably many constant symbols. We will then extend it to the general setting.

We start with some necessary definitions:

Definition 3.1. Let $Cn(\Gamma) = \{\varphi \mid \Gamma \vdash \varphi\}$, this is called the *consequence operator* of Γ.

Definition 3.2. A *theory* is a deductively closed set of sentences, that is a Γ such that $\Gamma = Cn(\Gamma)$.

Definition 3.3. A set Γ is called *consistent* if $\Gamma \nvdash \bot$, and *complete* if there is no proper, consistent extension of Γ in the same language.

We have the following properties:

1. Γ is consistent if and only if not both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, which is the case if and only if there exists a φ such that $\Gamma \nvdash \varphi$.
2. Γ is complete if and only if for all φ, either $\Gamma \vdash \varphi$ or $\Gamma \nvdash \varphi$.

We aim to show the contrapositive of (1), which is: if $\Gamma \nvdash \varphi$ then there is some $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$ but $\mathfrak{A} \not\models \varphi$. Noting that $\Gamma \nvdash \varphi$ is the same as $\Gamma \cup \{\neg\varphi\}$ being consistent, and that the existence of an $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$ but $\mathfrak{A} \not\models \varphi$ is the same as $\Gamma \cup \{\neg\varphi\}$ having a model — we see that it is sufficient to show that:

$$\text{Every consistent set of sentences has a model.} \tag{2}$$

In order to do this, we will also need the notion of a *Henkin theory*.

Definition 3.4. A *Henkin theory* is a theory $\mathcal{T}$ in which there is a witness for each provable existential statement, that is: $\mathcal{T} \vdash (\exists x)\varphi(x) \Longrightarrow \mathcal{T} \vdash \varphi(c)$ for some constant symbol c. We will call such a c a *Henkin witness*.

There are three essential lemmas that go into the proof of the completeness theorem, first we show that every consistent theory has a complete consistent extension (lemma 3.2), next we will show that every consistent set of sentences can be extended to a consistent Henkin theory (lemma 3.3), and finally we show that any complete, consistent Henkin theory has a model (lemma 3.4).

Once we have established these lemmas, the proof of the completeness theorem as stated in (2) becomes straightforward. Indeed, if we apply the lemma 3.2 followed by lemma 3.3, we get that every consistent set of sentences can be extended to a complete, consistent Henkin theory (note that Lemma 3.3 preserves the language — hence the extended theory will still be a Henkin theory). Then applying Lemma 3.4, we will get that the theory, and hence in particular the underlying set, has a model — hence (2) will be established.

The following proposition will enable us to construct a Henkin theory.

Proposition 3.1. *If Γ is a consistent set, then for any fomula $\varphi(x)$, the set $\Gamma \cup \{(\exists x)\varphi(x) \Rightarrow \varphi(c)\}$ is consistent — where c is a constant symbol not occurring in Γ nor in φ (et exigence c antgaureed by expansion of language.*

Proof. Suppose not, then

$$\begin{aligned} \Gamma \cup \{(\exists x)\varphi(x) \Rightarrow \varphi(c)\} \vdash \bot \Longrightarrow &\quad \Gamma \vdash \neg((\exists x)\varphi(x) \Rightarrow \varphi(c)) \\ \Longrightarrow &\quad \Gamma \vdash (\exists x)\varphi(x) \text{ and } \Gamma \vdash \neg\varphi(c) \end{aligned} \tag{3}$$

As c does not occur in Γ nor in φ, we can apply the $(\forall I)$ rule, to get that

$$\begin{aligned} \Gamma \vdash \neg\varphi(c) \Longrightarrow &\quad \Gamma \vdash (\forall x)\neg\varphi(c) \\ \Longrightarrow &\quad \Gamma \vdash (\neg\exists x)\varphi(c) \end{aligned} \tag{4}$$

Then (3) and (4) show that Γ is inconsistent.

□

We now show Lindenbaum's lemma, the proof of which is entirely analogous to the proof for propositional logic.

Lemma 3.2. *(Lindenbaum's Lemma) Every consistent theory has a complete, consistent extension in the same language.*

Proof. Let Γ be a given consistent set, and enumerate the sentences in the language of Γ: $\varphi_0, \varphi_1, \varphi_2, \ldots$. Now recursively define:

$$\begin{aligned} \Gamma_0 &= \Gamma \\ \Gamma_{n+1} &= \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if this set is consistent} \\ \Gamma_n & \text{else.} \end{cases} \\ \Gamma^* &= \bigcup_{n=0}^{\infty} \Gamma_n \end{aligned}$$

Then Γ^* is a complete, consistent extension of Γ in the same language, as desired.

□

Lemma 3.3. *Every consistent set of sentences can be extended into a consistent Henkin theory in an extended language.*

Proof. Suppose that Γ is a consistent set of first-order sentences. Choose a countably infinite set of constant symbols that are not already in $\mathcal{L}$ (the language of $\mathcal{T}$), call it $\mathcal{C}$. Let $(\exists x_0)\varphi(x_0), (\exists x_1)\varphi(x_1), (\exists x_2)\varphi(x_2), \ldots$ be an enumerate all the existential sentences in $\mathcal{L} \cup \mathcal{C}$.

Define Γ_n recursively as follows:

$$\Gamma_0 = \Gamma$$

Let $c_n \in \mathcal{C}$ be the first new constant symbol which does not occur in Γ_n nor in φ_n

$$\Gamma_{n+1} = \Gamma_n \cup \{(\exists x_n)\varphi_n(x_n) \Rightarrow \varphi_n(c_n)\}$$
$$\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$$

Now we claim that $Cn(\Gamma^*)$ is the Henkin theory we want.

First, we note that by induction on n together with Proposition 3.4, we get that each of the Γ_n is consistent. The consistency of the Γ_n, in turn, yield the consistency of Γ^*; indeed, if $\Gamma^* \vdash \perp$, then there is some finite derivation which supports this conclusion, and all the hypotheses of such a derivation would be contained in Γ_n for some n — implying that such a Γ_n is inconsistent.

It is not hard to see that $Cn(\Gamma^*)$ is a Henkin theory. Suppose that $\Gamma^* \vdash (\exists x)\varphi(x)$. Then there is some n such that $(\exists x)\varphi(x)$ is$(\exists x_n)\varphi(x_n)$, and by definition of Γ^* we then get that $\Gamma^* \vdash \varphi(c_n)$.

□

Lemma 3.4. *Every complete, consistent Henkin theory has a model.*

Proof. We will assume that the language, $\mathcal{L}$ of the given Henkin theory $\mathcal{T}$, consists of one binary predicate symbol(in addition to identity symbol), and a countably infinite set of constant symbols which will be enumerated as, $c_0, c_1, \ldots$. We will also assume that $\mathcal{T}$ is a first-order theory with equality. We will construct a model $\mathfrak{A} = \langle A, R, a_0, a_1, \ldots \rangle$ wherein $c_i^{\mathfrak{A}} = a_i$ for $i = 0, 1, \ldots$, and A will consist of equivalence classes of the constant symbols.

Let c_i and c_j be constant symbols, and define the following relation:

$$c_i \sim c_j \iff \mathcal{T} \vdash c_i = c_j$$

From axioms $A6$ and $A7$, it is not hard to see that this is an equivalence relation.

Let $[c]$ denote the equivalence class of c, that is $[c] = \{c \mid \mathcal{T} \vdash c = b\}$, and define A (the universe of $\mathfrak{A}$) as

$$A = \{[c] \mid c \text{ is a constant symbol in } \mathcal{L}\}$$

The interpretation of the predicate symbol P in $\mathfrak{A}$ is the relation R given by:

$$R = \{\langle [c_i], [c_j] \rangle \mid \mathcal{T} \vdash P(c_i, c_j)\}$$

Now we need to show that:

$$R([c_i], [c_j]) \text{ if and only if } \mathcal{T} \vdash P(c_i, c_j)$$

Which is the same as showing that $\sim$ is a congruence with respect to R. The reverse direction is true by definition. On the other hand, suppose that $R([c_i], [c_j])$. Then for $c_{i_1} \sim c_{i_2}$ and $c_{j_1} \sim c_{j_2}$, $\mathcal{T} \vdash P(c_{j_1}, c_{j_2})$. Now, since we know that $\mathcal{T} \vdash c_{i_1} = c_{i_2}$, and that $\mathcal{T} \vdash c_{j_1} = c_{j_2}$, we get that $\mathcal{T} \vdash P(c_{i_2}, c_{j_2})$ by the substitution rule for equality.

The constants of $\mathfrak{A}$ are the equivalence classes $[c_i]$, and for each c_i we define $c_i^{\mathfrak{A}} = [c_i]$, then we have

$$\mathfrak{A} = \langle A, R, [c_0], [c_1], [c_2], \ldots \rangle$$

It remains to show that $\mathfrak{A} \models \mathcal{T}$. We proceed by induction on sentences φ to show that $\mathfrak{A} \models \varphi$ if and only if $\mathcal{T} \vdash \varphi$.

Base case: φ is atomic. We have that

$$\begin{aligned}
\mathfrak{A} \models P([c_i], [c_j]) &\iff R([c_i], [c_j]) \\
&\iff \mathcal{T} \vdash P([c_i], [c_j]), \text{ already shown} \\
\mathfrak{A} \models [c_i] = [c_j] &\iff [c_i] = [c_j] \\
&\iff \mathcal{T} \vdash [c_i] = [c_j], \text{ also shown already}
\end{aligned}$$

Now assume that the result holds for all proper sub-formulas of φ. We show the result for the cases φ is $\psi \Rightarrow \gamma$ and φ is $(\exists x)\psi(x)$, the other cases are shown in the same manner.

$$
\begin{aligned}
\mathfrak{A} \models \psi \Rightarrow \gamma &\Longleftrightarrow \mathfrak{A} \not\models \psi \text{ or } \mathfrak{A} \models \gamma \\
&\Longleftrightarrow \mathcal{T} \not\vdash \psi \text{ or } \mathcal{T} \vdash \gamma, \text{ by induction} \\
&\Longleftrightarrow \mathcal{T} \vdash \neg\psi \text{ or } \mathcal{T} \vdash \gamma, \text{ by completeness of } \mathcal{T} \\
&\Longleftrightarrow \mathcal{T} \vdash \psi \Rightarrow \gamma, \text{ also by completeness of } \mathcal{T} \\
\mathfrak{A} \models \psi \Rightarrow \gamma &\Longleftrightarrow \text{ for some } [c_i] \in |\mathfrak{A}|, \ \mathfrak{A} \models \psi([c_i]) \\
&\Longleftrightarrow \text{ for some } c_j \sim c_i, \ \mathfrak{A} \models \psi(c_j) \\
&\Longleftrightarrow \text{ for some } c_j, \ \mathcal{T} \vdash \psi(c_j), \text{ by induction} \\
&\Longleftrightarrow \mathcal{T} \vdash (\exists x)\psi(x), \text{ since } \mathcal{T} \text{ is a Henkin theory}
\end{aligned}
$$

□

We have shown the completeness theorem in a minimal setting — the language consisted of only one binary predicate symbol (besides the identity predicate), and countably many constant symbols. We now discuss ways to extend our proof to more general settings. In the case where there are multiple predicate symbols (with varying arities), the proof will be almost identical — for each n-ary predicate symbol P_n, we can define its interpretation as $R = P_n^{\mathfrak{A}} = \{\langle [c_1], \ldots, [c_n] \rangle \mid \mathcal{T} \vdash P_n(c_1, \ldots, c_n)\}$, and preform the obvious changes to the remainder of the argument. If, in addition, there are also function symbols, then we can use equivalence classes of arbitrary terms, as opposed to just equivalence classes of constant symbols, where the equivalence relation on terms is defined by $s \sim t \Longleftrightarrow \mathcal{T} \vdash s = t$, and as before — make the rather straightforward changes where necessary. For the further generalized case where there are uncountably many constant symbols, we can make use of the fact that the language has the same cardinality as the set of constant symbols. An application of the axiom of choice allows us to well order the set of constant symbols, along with the formulas of the language, by transfinite ordinals — then arguments that we have provided can be adapted to this new context without much difficulty. An alternate approach which does not make use of transfinite ordinals, relies on applying Zorn's lemma. The proof of lemma 3.2 with Zorn's lemma is still very close to what we have presented, and the proof of lemma 3.3 can be modified as follows:

Given any language, for any formula φ, we can choose a new constant, say c_φ, and if Γ is a consistent set of sentences, then we can recursively define:

$$
\begin{aligned}
\Gamma_0 &= \Gamma \\
\Gamma_{n+1} &= \Gamma_n \cup \{(\exists x)\varphi(x) \Rightarrow \varphi(c_\varphi) \mid \varphi \text{ is a formula in } \mathcal{L}(\Gamma_n)\} \\
\Gamma^* &= \bigcup_{n=0}^{\infty} \Gamma_n
\end{aligned}
$$

Such a construction does not depend on the cardinality of the language — in particular the language may be countable or uncountable.

4. Rasiowa and Sikorski's Proof

In this section we present the proof of the completeness theorem as given by Rasiowa and Sikorski.

4.1. ***Mostowski's functionals*** Φ_α

Definition 4.1. A *Boolean algebra* is a six tuple $(B, \wedge, \vee, ^-, 0, 1)$ that consists of a set B together with two binary operations $\wedge$ and $\vee$, a unary operation $^-$ such that $x \mapsto \bar{x}$ where $x \wedge \bar{x} = 0$ and $x \vee \bar{x} = 1$, least and greatest elements (denoted 0, 1), which is also a distributive lattice.

We will use B to denote a Boolean algebra when the context is clear. B_0 will denote the two-element $(0, 1)$ Boolean algebra.

Definition 4.2. An $(\mathbb{N}, B_0)$-*function* is a map $f^k : \mathbb{N}^k \to B_0$, $\mathfrak{F}_k$ will denote the set of such functions.

Definition 4.3. An $(\mathbb{N}, B_0)$-*functional* is defined as the functional:
$\Phi \mapsto \Phi(x_{j_1}, \ldots, x_{j_n}, f^{k_1}_{j_1}, \ldots, f^{k_m}_{i_m}) \in B_0$, where Φ is a map $\Phi : \mathbb{N}^n \times \mathfrak{F}^n_k \to B_0$.

Any first-order formula: $\alpha = \alpha(x_{j_1}, \ldots, x_{j_n}, f^{k_1}_{j_1}, \ldots, f^{k_m}_{i_m})$ can be interpreted as an $(\mathbb{N}, B_0)$-functional, if the variables x_i have domain $\mathbb{N}$, the function symbols have domain $\mathfrak{F}_k$, and $\vee$, $\neg$, and $\exists$ are interpreted as the Boolean operations of B_0, where $\exists x_n$ is interpreted as $\vee_{n\in\mathbb{N}} x_n$. An $(\mathbb{N}, B_0)$-functional obtained from a first-order formula α will be denoted Φ_α, and called a Mostowski functional, or simply a functional.

4.2. ***Algebraic interpretation of satisfiability***

We have the following lemma which characterizes the relation between satisfiability and the Mostowski functionals:

Lemma 4.1. *For every first-order formula α, the associated functional Φ_α takes the value* 1 (*respectively* 0) *if and only if α* (*respectively* $\neg\alpha$) *is satisfiable.*

Proof. Let $P^m = \{(y_1, \ldots, y_m) \mid y_i \in \mathbb{N}\}$ and let $\chi_{P^m} \in \mathfrak{F}^m$ denote the characteristic function of P^m, then an easy induction shows that a sequence (a_n) satisfies a formula $\alpha = \alpha(x_{j_1}, \ldots, x_{j_n}, f^{k_1}_{j_1}, \ldots, f^{k_m}_{i_m})$ if and only if

$$\Phi_\alpha(a_{j_1}, \ldots, a_{j_n}, \chi_{A^m_{i_1}}, \ldots, \chi_{A^m_{i_m}}) = 1 \in B_0. \tag{5}$$

□

Thus we see that a first-order formula is valid in the set $\mathbb{N}$ if and only if the associated functional is identically $1 \in B_0$.

4.3. ***The Stone spaces of Boolean algebras***

We will now prove a theorem about the existence of certain prime ideals (as specified in Theorem 4.3) in a Boolean algebra.[5]

First we recall that an ideal (filter) $P \subset B$ is called prime if $P \neq B$ and if $a \wedge b \in P$ ($a \vee b \in P$) always implies that $a \in P$ or $b \in P$.

Let B be a Boolean algebra, and let $\mathcal{S}$ denote the set of all prime ideals of B. For each $a \in B$ let $S(a) = \{\, P \in B \mid P \text{ is prime}, a \notin P \,\}$, and let $\mathscr{C}$ denote the class of all such sets. We have by definition, that:

$$P \in S(a) \text{ if and only if } a \notin P \tag{6}$$

Stone showed that $\mathcal{S}$ is a compact Hausdorff space that is totally disconnected[6] (now called a Stone space), where $\mathscr{C}$ is in fact the base of clopen neighborhood, and that the map $S \mapsto S(a)$ is an isomorphism from B into $\mathscr{C}$ [10].

Remark. If $P \in B$ is a prime ideal, then the quotient algebra B/P is B_0, and:

$$[a] = 1 \in B/P \text{ if } a \notin P; \;\; [a] = 0 \in B/P \text{ if } a \in P \tag{7}$$

Definition 4.4. If $a \in B$ can be written as $a = \bigvee_{j \in J} a_j$ in B, then we say that an ideal $I \subseteq B$ *preserves joins* if $[a] = \bigvee_{j \in J} [a_j]$ in B/I

Theorem 4.2. *The set of prime ideals that do not preserve a join is nowhere dense in the space $\mathcal{S}$.*

Proof. Let B be a Boolean algebra, let $\mathcal{V}$ denote the set of prime ideals that do not preserve a join, and let $a = \bigvee_{j \in J} a_j$ in B. Let $P \subset B$ be a prime ideal, then $P \in \mathcal{V}$ if and only if $[a] = 1$ and $[a_j] = 0 \;\; \forall j \in J$, thus by (6) and (7) we have that $P \in \mathcal{V}$ if and only if $P \in S(a) \setminus \bigcup_{j \in J} S(a_j)$, so that $\mathcal{V} = S(a) \setminus \bigcup_{j \in J} S(a_j)$.

Now we claim that $S(a) \setminus \bigcup_{j \in J} S(a_j)$ has empty interior (note that this set is in fact closed so it is equal to its closure). Indeed, suppose not, then we can find some $0 \neq a_0 \in B$ such that $S(a_0) \subsetneq S(a) \setminus \bigcup_{j \in J} S(a_j)$ which is the same as $S(a_j) \subset S(a) \setminus S(a_0) = S(a \wedge a_0)$ for every $j \in J$. Now since $S \mapsto S(b)$ is an isomorphism, we get that $a_j \vee (a \wedge a_0) = a \wedge a_0 \neq a$ but this contradicts the fact that $a = \bigvee_{j \in J} a_j$.

$\square$

[5] This section can be done using the language of filters as opposed to ideals (since they are dual notions) so that upon exchanging $\wedge$ for $\vee$, and requiring that $a_0 \neq 0 \in B$, Theorem 4.3 becomes a fact about the existence of a particular ultrafilter.

[6] Totally disconnected — has a base of clopen neighborhoods, viz., each neighborhood of each point in $\mathcal{S}$ contains a clopen neighborhood of that point.

Remark. The above theorem tells us that the set of prime ideals that do not preserve a join is of first category in $\mathcal{S}$, i.e. it is trivially a countable union of nowhere dense sets.

We now show the Rasiowa-Sikorski Lemma:

Theorem 4.3. *Let $a_0, a_n, a_{n,j}$ ($j \in J_n$ where J_n is some index set), be elements of B such that:*

1. $a_n = \bigvee_{j \in J_n} a_{n,j}$ *in* B $(n = 1, 2, \ldots)$
2. $a_0 \neq 1 \in B$

Then there is a prime ideal $P \subset B$ that preserves all the joins of the form 1. *and contains a_0.*

Proof. Let $\mathscr{P}$ be the set of all prime ideals that preserve all the joins as in 1. Then by the previous theorem, we have that $\mathcal{S} \backslash \mathscr{P}$ is of first category in $\mathcal{S}$. Now since $\mathcal{S}$ is a compact Hausdorff space (in particular locally compact Hausdorff), by Baire category theorem, it is a Baire space. We have that the complement of a first category set is a second category set — hence $\mathscr{P}$ is of second category in $\mathcal{S}$, and since $\mathcal{S}$ is a Baire space, every second category subset of $\mathcal{S}$ is dense. Thus we see that $\mathscr{P}$ is dense in $\mathcal{S}$. We have that $a_0 \neq 1$ by assumption, so $S(a_0') = \mathcal{S} \backslash S(a_0)$ is not empty. Consequently, we have $\mathscr{P} \cdot S(a_0') \neq \emptyset$, and every prime ideal $P \in \mathscr{P} \cdot S(a_0')$ satisfies the conditions of the theorem. $\square$

4.4. ***Lindenbaum's algebra*** B^*

Definition 4.5. The *Lindenbaum algebra* of a first-order theory $\mathcal{T}$ is the algebra that is formed by the equivalence classes of formulae under the relation of provable equivalence in the theory, we will denote it B^*. Denoting provable equivalence in a given theory by $\vdash_{\mathcal{T}} \varphi$, we can define the equivalence class of φ as follows:

$$[\![\varphi]\!]_{\mathcal{T}} = \{\varphi \mid \vdash_{\mathcal{T}} \psi \Rightarrow \varphi \text{ and } \vdash_{\mathcal{T}} \varphi \Rightarrow \psi\}.^7$$

This definition of equivalence classes induces a partially ordered by defining

$$[\![\varphi]\!]_{\mathcal{T}} \leq [\![\psi]\!]_{\mathcal{T}} \text{ if and only if } \vdash_{\mathcal{T}} \varphi \Rightarrow \psi$$

The Lindenbaum algebra is generally not a complete Boolean algebra[8]; however, it always has a what could be called a degree of completeness, as shown in a central result of [3] and [2], which we present in Lemma 4.5.

Theorem 4.4. *The Lindenbaum algebra B^* is a Boolean algebra.*

[7] We will simply use $[\![\cdot]\!]$ when the context is clear

[8] A complete lattice is a partially ordered set where every subset has a least upper bound under $\leq$.

Proof. Under the formulation we have presented, the proof of this fact becomes quite simple. Since it is the case that the axioms of a Boolean algebra exactly mimic the rules of inference, if we set

$$\llbracket\varphi\rrbracket_{\mathcal{T}} \cap \llbracket\psi\rrbracket_{\mathcal{T}} = \llbracket\varphi \wedge \psi\rrbracket_{\mathcal{T}}$$
$$\llbracket\varphi\rrbracket_{\mathcal{T}} \cup \llbracket\psi\rrbracket_{\mathcal{T}} = \llbracket\varphi \vee \psi\rrbracket_{\mathcal{T}}$$

then we obtain a direct translation between algebraic relations and deduction rules. □

The operations on B^* are defined by the equalities:

$$\llbracket\varphi\rrbracket \vee \llbracket\psi\rrbracket = \llbracket\varphi \vee \psi\rrbracket \tag{8}$$

$$\neg\llbracket\varphi\rrbracket = \llbracket\neg\varphi\rrbracket \tag{9}$$

We have the following easy properties that are stated without proof:

$$\llbracket\varphi\rrbracket \subset \llbracket\psi\rrbracket \text{ if and only if } \varphi \Rightarrow \psi \text{ is provable} \tag{10}$$

$$\text{The unit in } B^\star \text{ is the equivalence class of all provable formulae} \tag{11}$$

For each first-order formula φ let $\varphi^{x_i}_{x_j}$ — where x_i is a bound variable in φ, and x_j is free, be the formula obtained from φ in the following manner: Pick $k \in \mathbb{N}$ such that x_k and $\exists x_k$ do not appear in β. Replace every occurrence of the bound variable x_i in φ with x_k, and similarly replace every occurrence of $\exists x_i$ with $\exists x_k$. Finally, replace each occurrence of the free variable x_j with x_k. The formula defined in this manner is not necessarily unique; however, $\llbracket\varphi^{x_i}_{x_j}\rrbracket \in B^\star$ is unique, since it does not depend on the choice of k.

Lemma 4.5. *For every formula φ,*

$$\bigvee_{n\in\mathbb{N}} \llbracket\varphi^{x_n}_{x_m}\rrbracket = \llbracket(\exists x_m)\varphi\rrbracket \tag{12}$$

and the provable formula $\varphi^{x_n}_{x_m} \Rightarrow (\exists x_m)\varphi$ together with (10)*, implies that $\llbracket\varphi^{x_n}_{x_m}\rrbracket \subset \llbracket(\exists x_m)\varphi\rrbracket$ for all $n \in \mathbb{N}$.*

Proof. Let ψ be a first-order formula satisfying $\llbracket\varphi^{x_n}_{x_m}\rrbracket \subset \llbracket\psi\rrbracket$ for all $n \in \mathbb{N}$, by (5) we then have that the formula: $\varphi^{x_n}_{x_m} \Rightarrow \psi$ is provable for all $n \in \mathbb{N}$. Take $k \in \mathbb{N}$ such that x_k is not free in ψ. Then $(\exists x_k)\varphi^{x_n}_{x_m} \Rightarrow \psi$ is provable; so, again by (10) we get $\llbracket(\exists x_k)\varphi\rrbracket = \llbracket\bigvee_{x_k} \varphi\binom{x_n}{x_m}\rrbracket \subset \llbracket\psi\rrbracket$. □

4.5. ***The proof of the completeness theorem***

We recall that a first-order formula, α is valid in the set $\mathbb{N}$ if and only if the associated functional Φ_α is identically $1 \in B_0$. Hence in order to prove the completeness theorem, which can be stated as:

> If a first-order formula α is valid in the domain of positive integers, then α is provable

it is sufficient to show:

Theorem 4.6. *If a first order formula α is not provable, then the $(\mathbb{N}, B_0)$ functional Φ_α assumes the value 0 (the zero element of B_0).*

Proof. Supose that α is a first-order formula that is not provable. Let P^* be a prime ideal of B^* that preserves all joins as in (12), and such that $[\![\alpha]\!] \in P^*$. Note that the existence of such an ideal follows from Theorem 4.3, property (10), and from the fact that the set of all joins of the form (12) is countably infinite (as a remark we note that for B^* the space $\mathcal{S}$ is the Cantor set).

Now $B_0 = B^*/P^*$ is the two element Boolean algebra, and the following hold:

$$[[\![\alpha]\!]] = 0 \qquad \text{Since } \alpha \in P^* \tag{13}$$

$$[[\![\beta]\!]] \vee [[\![\gamma]\!]] = [[\![\beta \vee \gamma]\!]] \qquad \text{by (3)} \tag{14}$$

$$\neg[[\![\beta]\!]] = [[\![\neg\beta]\!]] \qquad \text{by (4)} \tag{15}$$

$$\bigvee_{n\in\mathbb{N}} [[\![\beta^{x_n}_{x_m}]\!]] = [[\![(\exists x_m)\beta]\!]] \tag{16}$$

where the last equality follows from Lemma 4.5 and since P^* preserves all the joins in (14).

Now for $k, j = 1, 2, \ldots$, let $g^k_j \in \mathfrak{F}^k$ be an $(\mathbb{N}, B_0)$ function given by,

$$g^k_j(a_1, a_2, \ldots, a_k) = [[\![f^k_j(x_{a_1}, x_{a_2}, \ldots, x_{a_k})]\!]] \tag{17}$$

where $a_i \in \mathbb{N}$. For each formula β let Φ^0_β denote the value of the $(\mathbb{N}, B_0)$ functional Φ_β with the following values of its arguments: $x_i = i$ and $f^k_j = g^k_j$. Then, from the equalities (14-17), we get that $\Phi^0_\beta = [[\![\alpha]\!]]$ for ever formula β. The straightforward proof (by induction on the length of β) is omitted. In particular, we have that $\Phi^0_\beta = [[\![\alpha]\!]] = 0 \in B_0$ by (14), and hence the claim follows.

$\square$

4.6. ***Generalizations***

We can use the same method of proof to show that the completeness theorem holds for first order logic with equality. The axioms of this system are A1-7.

The algebraic interpretation of the formula $x_k = x_l$ is $\varphi(x_k, x_l)$ where $f \in \mathfrak{F}^2$ is an $(\mathbb{N}, B_0)$ function defined by:

$$f(m,n) = \begin{cases} 1 \in B_0 & m = n \\ 0 \in B_0 & m \neq n \end{cases}$$

5. Comparison of the Proofs and Generalizations

The "characterizing property" of Henkin's proof that we mention in the introduction is best illustrated by means of some examples of generalizations of the Henkin method. The *omitting types theorem* is the simplest such example: all that is required to obtain this result is to, in the construction of a Henkin theory, add the requirement that for each $\varphi(a,x)$, and for each non-principal p, there is a p-omitting witness (viz. the consistency of $(\exists x)\varphi(a,x) \wedge \neg\psi(x)$ for some $\psi(x) \in p$).[9] It is not difficult to enforce such a requirement — each term (in the language with new constants) must omit the type. Another, more exotic, use of Henkin's method was by Baldwin and Lachlan in 1971 [19] to show that an $\aleph_1$-categorical theory which is not $\aleph_0$-categorical cannot have finitely many countable models. The fundamental idea of the Henkin method (or the "characterizing property"), namely — the systemic extension of a given theory to a complete one whose canonical model satisfies a desired property can be seen in a more rigorous context; Abraham Robinson's concept of finite and infinite forcing [20]. This concept has been recently employed (and extended) to construct certain counterexamples in functional analysis, for example of specific C^*-algebras. The authors of [21] write, "We describe a way of constructing C^*-algebras (and metric structures in general) by Robinson forcing (also known as the Henkin construction)." It is interesting to note that both Rasiowa and Sikorski's proof method and Henkin's method have become useful techniques in the area of forcing (The Rasiowa-Sikorski lemma is one of the fundamental facts presently used in forcing).

Turning now to Rasiowa and Sikorski's proof of the completeness theorem, we first seek to understand some of the motivation behind their approach. In order to do this we focus on the preface of their text: *The Mathematics of the Metamathematics.* Therein, Rasiowa and Sikorski suggest that in metamathematical investigations, infinitistic methods should be employed — by this they mean using whatever mathematical

[9] The omitting types theorem: Let $\mathcal{L}$ be a countable first-order language, and $\mathcal{T}$ a theory of $\mathcal{L}$, further let $\{p_i \mid i \in \mathbb{N}\}$ be a countable set of non-isolated n-types of $\mathcal{T}$ then there is a countable structure $\mathcal{M}$ such that $\mathcal{M} \models \mathcal{T}$ and $\mathcal{M}$ omits each p_i.

tool is available (in particular those from abstract algebra, lattice theory, set theory, and topology). They advocate a deviation from the proof-theoretic approach found in the formalist trend in metamathematics, which they assert is an artificial limitation that serves to obscure the understanding of metamathematical notions. In their words:

> The title of the book is inexact since not all mathematical methods used in metamatemathics are exposed in it. [...] The exact title of the book should be: Algebraic, lattice-theoretical, set-theoretical and topological methods in metamathematics.
>
> The *finitistic* approach of Hilbert's school is completely abandoned in this book. On the contrary, the *infinitistic* methods, making use of the more profound ideas of mathematics, are distinctly favoured. This brings out clearly the mathematical structure of metamathematics. It also permits a greater simplicity and clarity in the proofs of the basic metamathematical theorems and emphasizes the mathematical contents of these theorems. [...]
>
> The theorem on the completeness of the propositional calculus is seen to be exactly the same as Stone's theorem on the representation of Boolean algebras. [...] It is surprising that the Gödel completeness theorem [of the predicate calculus] can be obtained, for example, as a result of the Baire theorem on sets of the first category in topological spaces, etc.
>
> [12 pp. 5-6]

It seems reasonable to assume that this view was somewhat controversial in certain academic contexts; there were in fact some reviewers such as Kreisel [13] and Beth [14] that were critical of this approach. On the other hand, reviewers such as Feferman [15] and Robinson [16] praised such an approach for the simplicity of the proofs that can be obtained. We come now to an important similarity between Rasiowa and Sikorski and Henkin's proofs of the completeness theorem — both proofs could not avoid infinitistic methods, as Feferman highlights in his review of [3]:

> In the opinion of the reviewer, this paper represents a distinct advance over all preceding proofs; for on the one hand, much less formal development from the axioms is required than in the proofs similar to Gödel's, [...] Moreover, the present derivation [...] has the special advantage of bringing out the essentially algebraic character to the method first used by Henkin.
>
> [16 pp.131]

We end by proving some results that tie together many of the aspects of Henkin and Rasiowa-Sikorski proofs. The first main theorem that we show is that the completeness theorem, the compactness theorem, and the ultrafilter lemma are equivalent. We are careful to avoid any implicit applications of the axiom of choice in showing this result. We will then show that the Rasiowa-Sikorski lemma is equivalent to the ultrafilter lemma in conjunction with Baire category. The importance of the (strong) ultrafilter lemma is that it indirectly plays a key role in the Rasiowa-Sikorski proof of completeness, since it is required to show that the stone space is compact.

Theorem 5.1. *The following are equivalent:*

1. *Every Boolean algebra contains an ultrafilter (weak ultrafilter lemma).*
2. *Each filter in a Boolean algebra can be extended to an ultrafilter (strong ultrafilter lemma).*
3. *Every consistent set of first-order sentences has a model (completeness theorem).*
4. *If Γ is a set of first order sentences such that every finite subset has a model, then Γ has a model (compactness theorem).*

Proof. [10] (1) $\Rightarrow$ (2) Suppose that (1) holds and let B be a Boolean algebra containing a filter F. Consider the quotient algebra B/F. Since the quotient algebra is in fact a Boolean algebra, by (1) it contains an ultrafilter U. Let $\pi : B \to B/F$ be the canonical projection homomorphism from B into the quotient. Then we have that $\pi^{-1}(U)$ is an ultrafilter in B extending F.
(2) $\Rightarrow$ (3) Assume that (2) holds, let $\mathcal{L}$ be a first order language, and let Γ be a consistent set of sentences, which are taken to be in prenex normal form. For each formula φ of $\mathcal{L}$ which is in prenex normal form, let φ^* be the *Skolem transform* of φ — that is, φ^* is the formula obtained from φ by first taking the closure of φ, introducing Skolem functions and constants in order to eliminate the existential quantifiers, and then suppressing any remaining universal quantifiers (So for example if we have the prenex formula: $(\exists x)(\forall y)(\exists z)\varphi(x,y,z)$ then its Skolem transform would be $\varphi(c,y,f(y))$, where c is a new constant, and f a new function symbol). Without loss of generality, we may assume that new functions symbols and new constants introduced for each formula are distinct. Let $\mathcal{L}^*$ be the language that is formed from $\mathcal{L}$ by appending all its symbols except for the quantifiers, and also the new function symbols and constants that arise from taking the Skolem transforms of the formulas of $\mathcal{L}$. The inference rules of $\mathcal{L}^*$ are modus ponens and substitution of terms for variables, and the axioms are A1-3 as defined in section 2, together with

$$\tau_1 = \tau_1$$
$$\tau_1 = \tau_2 \Rightarrow (\varphi(\tau_1) \Rightarrow \varphi(\tau_2))$$

where τ_1, τ_2 are terms and φ a formula.

It is the case Γ is consistent in $\mathcal{L}$ if and only if $\Gamma^* = \{\varphi^* \mid \varphi \in \Gamma\}$ is consistent in $\mathcal{L}^*$.[11] Let B^* be the Lindenbaum algebra of $\mathcal{L}^*$. Then as Γ is consistent, so is Γ^*, and hence $[\![\Gamma^*]\!]_{\mathcal{L}^*} = \{[\![\varphi^*]\!]_{\mathcal{L}^*} \mid \varphi^* \in \Gamma^*\}$ has the finite intersection property. Thus $[\![\varphi^*]\!]_{\mathcal{L}^*}$

[10] This proof is due to Bell J.L., and Slomson A.B. [22]

[11] This result is known as the *second ε-theorem* which we state without proof, for a proof consult [23]. Note in particular that the proof does not make any use of the axiom of choice.

can be extended to a filter in B^*, and hence — by the assumption (2), can in turn be extended to an ultrafilter F in B^*. With this ultrafilter F, we can construct an interpretation $\mathfrak{A}$ of $\mathcal{L}^*$ in the following manner. The domain of $\mathfrak{A}$ consists of the set of all terms of $\mathcal{L}^*$, $c_i^{\mathfrak{A}} = c_i$ for each constant c_i of $\mathcal{L}^*$, and if f^n is an n-ary function symbol, then its interpretation is $F(\tau_1, \dots, \tau_n) = f(\tau_1, \dots, \tau_n)$. For each n-ary predicate symbol P^n, we define $P^n(\tau_1, \dots, \tau_n)$ to be true in $\mathfrak{A}$ if and only if $[\![P^n(\tau_1, \dots \tau_n)]\!] \in F$. For the sake of this argument, we do not assume that "=" is the identity relation on $\mathcal{A}$, but rather take it to be a non-logical predicate symbol. A straightforward induction shows that any formula ψ of $\mathcal{L}^*$ is valid in $\mathfrak{A}$ if and only if $[\![\psi]\!] \in F$, and since we have that $[\![\Gamma^*]\!] \subseteq F$, each member of Γ^* is valid in $\mathfrak{A}$. By construction of Γ^* (from Γ) it follows that each element of Γ is also valid in $\mathfrak{A}$. So we see that $\mathfrak{A}$ will be a model of Γ in the event that the equivalence relation that corresponds with the equality symbol was the identity relation; however, this is not generally the case. For the case where this does not occur, we can "contract" $\mathfrak{A}$ to a model $\mathfrak{A}'$ such that $\mathfrak{A}'$ does satisfy this condition in a canonical manner.[12] Thus $\mathfrak{A}'$ is a model of Γ and so we have (3).
(3) $\Rightarrow$ (4) Suppose that (3) holds, and let Γ be a set of first-order sentences which is such that every finite subset has a model. Then each finite subset of Γ is consistent, which in turn gives us that Γ itself is consistent. Hence by assumption (3), Γ has a model.
(4) $\Rightarrow$ (1) Suppose that (4) holds, let B be a Boolean algebra, let $\mathcal{L}$ be the appropriate language for a Boolean algebra, for each element of B fix a constant and let $\mathcal{L}(B)$ be the language obtained from $\mathcal{L}$ by adding all the aforementioned constants. Take Γ_1 to be the set of all open sentences of $\mathcal{L}(B)$ that hold in B when the constants are interpreted as the corresponding elements of B, and take Γ_2 to be some set of axioms for a Boolean algebra. Add a unary predicate symbol U to $\mathcal{L}(B)$ and call this language $\mathcal{L}^+$. Finally take Γ_3 to be the set of sentences which say that the elements of $\mathcal{L}^+$ which satisfy U form an ultrafilter. We claim that every subset of $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ has a model.

Let $\Gamma_0 \subset \Gamma$ be a finite subset, and A_0 be the set of elements of B which correspond to the constants of Γ_0. Now let $\mathfrak{A}$ be the (finite) subalgebra of B that is generated by A_0. As $\mathfrak{A}$ is finite, we know (without invoking the axiom of choice) that $\mathfrak{A}$ contains an ultrafilter, call it F. Now if a is an enumeration of $\mathfrak{A}$, then we get that $(\mathfrak{A}, F, a)$ is a model of Γ_0. Thus by assumption (4), Γ has a model, say B', wherein $U_{B'}$ — the value of U, is an ultrafilter. As B' is a model of Γ_1 it is an extension of B (up to isomorphism). Therefore $U_{B'} \cap B$ is an ultrafilter contained in B.

□

In order to highlight the dependence that the Rasiowa-Sikorski lemma has on the ultrafilter lemma, we present here another proof (the standard topological proof) of the lemma.

Let B be a Boolean algebra, and $X \subseteq B$ such that $\bigwedge X \in B$ — that is X has a

[12]See [24 p. 98]

meet in B. A filter F of B *preserves meets* if $X \subseteq F$ implies $\bigwedge X \in F$. We will assume that filters are proper ($0 \notin F$), unless stated otherwise.

We restate the Rasiowa-Sikorski lemma as follows:

Theorem 5.2. *Let $\{X_n \mid n \in \omega\}$ be a countable collection of subsets of B, where each contains a meet in B. Then for any nonzero $a \in B$, there is an ultrafilter F of B that preserves the meets $\bigwedge X_n$ for each n, and which contains a.*

Proof. Let S_B be the set of ultrafilters on B endowed with the topology generated by the basic open sets $S_b = \{F \in S_b \mid b \in F\}$ for all $b \in B$. Then S_B is the stone space of B and so is compact and Hausdorff, so by Baire Category S_B is a Baire space. Now, if we set $S_n = \{F \in S_B \mid F \text{ preserves } \bigwedge X_n\}$, then we get that:

$$S_n = \left(\bigcup_{x \in X_n} S_{\overline{x}} \right) \cup S_{\bigwedge x_n}$$

and thus S_n is open (as a union of opens). Further, S_n is dense in S_B. Indeed, we recall that a set A is dense if and only if for every nonempty open set U we have $A \cap U \neq \emptyset$. It suffices to check this on a basis, so let S_b be a nonempty basic open set, then we either have that:

1. $b \leq \bigwedge X_n$, so that $S_b \cap S_n = S_b \neq \emptyset$; or
2. there is some $x \in X_n$ such that $b \nleq x$, hence $b \wedge \overline{x} \neq 0$, so for any $F \in S_B$ such that $b \wedge \overline{x} \in F$ we have that $F \in S_b \cap S_n$.

Thus we see that in either case $S_b \cap S_n \neq \emptyset$. Now since S_B is a Baire space, any intersection of countably many dense open sets remains dense, and so $\bigcap_{n \in \omega} S_n$ is dense in S_B. Note that $\bigcap_{n \in \omega} S_n$ is the set of ultrafilters which preserve all meets $\bigwedge X_n$. If $a \neq 0$ then S_a is nonempty and open, and so intersects $\bigwedge X_n$ — from which we see that any element of this intersection will meet the requirements of the theorem.

$\square$

We now show that the Rasiow-Sikorski lemma is equivalent to Tarski's Lemma and the Ultrafilter lemma.

Theorem 5.3. *Tarski's Lemma. Let $\{X_n \mid n \in \omega\}$ be a countable collection of subsets of B, where each contains a meet in B. Then for any nonzero $a \in B$, there is a filter F of B that decides $\bigwedge X_n$ for each n, and which contains a. Where a filter decides a meet $\bigwedge X$ if $\bigwedge X \in F$ or $\overline{x} \in F$ for some $x \in X$.*

Proof. Let $F_0 \subseteq F_1 \subseteq \cdots$ be an increasing sequence of finite sets which are defined as follows:

1. $F_0 = a$
2. If F_n has been defined, then:
   (a) if $\bigwedge F_n \leq \bigwedge X_n$, set $F_{n+1} = F_n \cup \{\bigwedge X_n\}$

(b) if $\bigwedge F_n \not\leq \bigwedge X_n$, then there is some $x \in X_n$ such that $\bigwedge F_n \not\leq x$. Pick such an x and put $F_{n+1} = F_n \cup \{\overline{x}\}$.

Now if $\bigwedge F_n \neq 0$ then by the above construction, $\bigwedge F_{n+1} \neq 0$ Thus we have that $\overline{F} = \bigcup_{n\in\omega} F_n$ has the finite meet property, as each finite subset has non trivial meet — hence $\overline{F}$ generates the filter we seek. □

If a filter decides $\bigwedge X$, then any extension of it will also decide $\bigwedge X$. Now a filter deciding $\bigwedge X$ also preserves $\bigwedge X$, and the converse holds for ultrafilters. Thus we see that Tarski's lemma together with the ultrafilter lemma imply the Rasiowa-Sikorski lemma, and the Rasiowa-Sikorski lemma in turn implies Tarski's lemma. On the other hand, the weak ultrafilter lemma is implicit in the Rasiowa-Sikorski lemma, indeed take $X_n = \emptyset$ and $a = 1$. Hence we have that the Rasiowa and Sikorski lemma is equivalent to the ultrafilter lemma, together with Tarski's lemma. Goldblatt in [11] then showed that Tarski's lemma is equivalent to Baire category.

As a final remark, we note that Rasiowa [26] provided a completeness theorem for intuitionistic first-order logic with respect to algebraic models, specifically Heyting-algebra-valued models. Using results due to McKinsey and Tarski [27] along with the MacNeille completion, Rasiowa was able to show that the characterization could be restricted to models on complete Heyting algebras. Additionally, such an approach could be used in order to show the completeness of classical first-order logic with respect to complete-Boolean-algebra-valued models. This being said, the main contribution of the Rasiowa-Sikorski lemma was in showing "...that a Boolean-valued model can be factored through a join-preserving ultrafilter to obtain a 2-valued model, hence a model in the sense of Tarski, and that this procedure provides sufficiently many such 2-valued models to yield a proof of Gödel's completeness theorem." [6 pp.175]

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