Hausdorff Dimension and Pro-P Groups

Eli Vigneron

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Abstract. We look at the concept of Hausdorff dimension in particular how it relates to profinite groups. The main focus is what can be said of the relation between Hausdorff spectra and p-adic analytic pro-p groups. In particular we want to study the conditions under which Hausdorff spectra can be said to characterize p-adic analytic pro-p groups. In the process of examining this question, we also consider when the Hausdorff dimension, respectively spectrum, agree under some of the 'standard' filtration's.

1 Introduction

Part of the motivation for looking at Hausdorff dimension comes from considering the question of how 'big' a subgroup $H \leq G$ is. An initial guess might be $\frac{1}{|G:H|} = \frac{|H|}{|G|}$. In the case of p-groups we might consider $\frac{\log|H|}{\log|G|}$. When G is a finite group, this seems to be a decent choice, however in the infinite case we encounter some issues — subgroups of infinite index are not distinguished, and we want a subgroup of finite index of an infinite group to have dimension 1.

These issues can be avoided in the case of profinite groups by considering the Hausdorff dimension which is a concept that was originally introduced in the study of metric spaces. Unfortunately there is no canonical choice of metric on profinite groups, but rather each filtration series induces a translation invariant metric on G.

Definition 1. *If G is a profinite group then a* ***filtration series*** *$\mathcal{S}$ of G is a chain*

$$G = G_0 \supseteq G_1 \supseteq \dots$$

where each of the G_i are open normal subgroups and form a base for the neighbourhoods of the identity. Note that we also have $\bigcap_i G_i = 1$.

Definition 2. *Let G be countably based and profinite, let $\mathcal{S}$ be a filtration series of G. Note that $\mathcal{S}$ forms a neighbourhood base of the identity. In the event that*

G is infinite the filtration series $\mathcal{S}$ induces the following metric on G

$$d_{\mathcal{S}}(x,y) = \inf\left\{|G:G_i|^{-1} \mid x \equiv y \pmod{G_i}\right\}$$

Using this metric we can define the notion of the ***Hausdorff dimension*** $\mathrm{hdim}_G^{\mathcal{S}}(U) \in [0,1]$ *for a subset $U \subseteq G$.*

We also have the alternative characterization of the Hausdorff dimension that is more 'algebraic' in flavour and is due to Barnea and Shalev [4].

Theorem 1. *Let $H \leq_c G$ be a closed subgroup of G, $\mathcal{S}$ a filtration of G. Then we can define:*

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = \liminf_{i\to\infty} \frac{\log|HG_i : G_i|}{\log|G:G_i|}; \tag{1}$$

We can interpret this as a sort of density function, and as alluded to previously, we can define $\mathrm{hdim}_G^{\mathcal{S}}(H) = {}^{\log|H|}/_{\log|G|}$ when G is finite. This characterization makes it clearer that although the same topology arises from the various metrics on G, the Hausdorff dimension is liable to change.

For every finitely generated pro-p group G there is a rather natural choice of filtration, namely the **p-power series**, which we will denote by $\mathcal{P}$ throughout

$$\pi_i(G) = G^{p^i} = \langle x^{p^i} \mid x \in G\rangle, \quad i \in \mathbb{N}.$$

Another useful index we will need is the **Hausdorff spectrum** of G, with respect to $\mathcal{S}$, which is defined as:

$$\mathrm{hspec}^{\mathcal{S}}(G) = \{\mathrm{hdim}_G^{\mathcal{S}}(H) \mid H \leq_c G\} \subseteq [0,1].$$

The Hausdorff spectrum gives us a loose measure of the 'complexity' of the subgroup structure of G. We briefly summarize some of the straightforward properties for a countably based profinite group G.

1. If $H \leq_c G$ then $\mathrm{hdim}_G^{\mathcal{S}}(H) \in [0,1]$
2. If $H, K \leq_c G$ and $K \leq H$ then $\mathrm{hdim}_G^{\mathcal{S}}(K) \leq \mathrm{hdim}_G^{\mathcal{S}}(H)$ and further, if $|H:K| < \infty$ then we have equality $\mathrm{hdim}_G^{\mathcal{S}}(K) = \mathrm{hdim}_G^{\mathcal{S}}(H)$
3. If G is infinite then open subgroups have Hausdorff dimension 1 and finite subgroups have Hausdorff dimension 0. So we have $\{0,1\} \subseteq \mathrm{hspec}^{\mathcal{S}}(G) \subseteq [0,1]$.

2 Hausdorff Dimension

In this section we will work towards establishing Theorem 1 by building up the required prerequisite material. We also look at some of the fractal dimensions that are related to the Hausdorff dimension.

Definition 3. *Let (X, d) be a metric space, Y a subset of X and $\{U_i\}_i$ a cover of Y where each of the U_i have diameter no more than some δ and let $s \geq 0$.*

$$\mathcal{H}^s_\delta(Y) = \inf \sum_i (\operatorname{diam} U_i)^s$$

where the infimum is taken over all covers.

Note that as δ becomes small, the class of covers of Y reduces and in turn the infimum $\mathcal{H}^s_\delta$ does not increase as δ goes to zero, so approaches a limit. We put

$$\mathcal{H}^s(Y) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(Y)$$

and call this the s-dimensional Hausdorff measure of Y.

$\mathcal{H}^s$ can easily be seen to be an outer measure on X.

Lemma 1. *If $\mathcal{H}^s(Y) < \infty$ and $s < t$, then $\mathcal{H}^s(Y) = 0$*

Proof. We have,

$$\sum_i (\operatorname{diam} U_i)^t \leq \sum_i (\operatorname{diam} U_i)^{t-s} (\operatorname{diam} U_i)^s \leq \delta^{t-s} \sum_i (\operatorname{diam} U_i)^s$$

and letting $\delta \to 0$ yields the result. □

So if we were to plot $\mathcal{H}^s(Y)$ against s, we would see that there is a critical value of s where $\mathcal{H}^s(Y)$ jumps from ∞ to 0. This critical value is the Hausdorff dimension of Y. More precisely, the Hausdorff dimension of a set $Y \subseteq X$, is

$$\operatorname{hdim} Y = \sup\{s \mid \mathcal{H}^s(Y) = \infty\} = \inf\{s \mid \mathcal{H}^s(Y) = 0\}$$

and we have

$$\mathcal{H}^s(Y) = \begin{cases} \infty & 0 \leq s < \operatorname{hdim} Y \\ 0 & s > \operatorname{hdim} Y \end{cases}$$

Another fractal dimension that we are interested in is the so called **box dimension**.

Definition 4. *Fix some δ positive and take $N_\delta(Y)$ to be the least number of sets of diameter δ which can cover Y, then*

$$\underline{\operatorname{bdim}}\ Y = \liminf_{\delta \to 0} \frac{\log N_\delta(Y)}{-\log \delta}, \qquad \overline{\operatorname{bdim}}\ Y = \limsup_{\delta \to 0} \frac{\log N_\delta(Y)}{-\log \delta}$$

are the lower and upper box dimensions respectively.

These are related to Hausdorff as expressed by the following claim

Lemma 2. *For each $Y \subseteq X$,*

$$\operatorname{hdim} Y \leq \underline{\operatorname{bdim}}\ Y \leq \overline{\operatorname{bdim}}\ Y.$$

Proof. Let $1 < \mathcal{H}^s(Y) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(Y)$. Then for δ small enough,

$$1 < \mathcal{H}^s_\delta(Y) \leq N_\delta(Y)\delta^s$$

Taking logarithms, we have

$$0 < \log N_\delta(Y) + s \log \delta$$

and we see that $s \leq \liminf_{\delta \to 0} \frac{\log N_\delta(Y)}{-\log \delta}$. ☐

It is important to note that equality does not generally hold here, Hausdorff dimension and box dimension agree on 'nice' sets, but there are many cases where the inequality is strict. Fortunately, for our purposes i.e. when G is profinite, we will see that $\operatorname{hdim}^{\mathcal{S}}_G H = \underline{\operatorname{bdim}}\ H$.

We start by defining the box dimension for profinite groups as well. If G is a profinite group and $\mathcal{S} = G_i$ a filtration of G, then as before we have the metric

$$d_{\mathcal{S}}(x, y) = \inf \left\{ |G : G_i|^{-1} \mid x \equiv y \pmod{G_i} \right\}$$

and the balls that this metric defines are the left (resp. right) cosets of G_i, meanwhile the diameter is found to be $|G : G_i|^{-1}$. In fact sets of diameter $|G : G_i|^{-1}$ will be contained in some coset of G_i. If we have $H \leq_c G$, and if $\delta = |G : G_i|^{-1}$ then we see that $N_\delta(H) = |HG_i : G_i| = |H : H \cap G_i|$. Applying the definition of the box dimension

$$\underline{\operatorname{bdim}}\ H = \liminf_{i \to \infty} \frac{\log|HG_i : G_i|}{\log|G : G_i|}, \qquad \overline{\operatorname{bdim}}\ H = \limsup_{i \to \infty} \frac{\log|HG_i : G_i|}{\log|G : G_i|}$$

Abercrombie showed, in [2, Prop. 2.6], that

$$\operatorname{hdim}^{\mathcal{S}}_G H \geq \liminf_{i \to \infty} \frac{\log|HG_i : G_i|}{\log|G : G_i|} = \underline{\operatorname{bdim}}\ H$$

This combined with Lemma 2 establishes Theorem 1.

Summarizing this as a theorem, we have

Theorem 2. *For a profinite group G with filtration $\mathcal{S}$ and a closed subgroup $H \leq_c G$, we have the following chain of equalities:*

$$\operatorname{hdim}^{\mathcal{S}}_G H = \underline{\operatorname{bdim}}\ H = \liminf_{i \to \infty} \frac{\log|HG_i : G_i|}{\log|G : G_i|} = \liminf_{i \to \infty} \frac{\log|H : H \cap G_i|}{\log|G : G_i|}$$

Now we can see that indeed finite subgroups of G will have Hausdorff dimension zero, for G infinite; whereas, G and its open subgroups will have Hausdorff dimension 1 as we initially wanted.

3 Hausdorff Spectrum

In what follows we will need the following definition:

Definition 5. *A p-**adic analytic group** is a p-adic analytic manifold over $\mathbb{Q}_p$ which is also a group, the group operations being analytic functions.*

We note that if G is a p-adic analytic pro-p group, then its closed subgoups remain p-adic analytic. We will use $\dim G$ to denote the dimension of G viewed as a p-adic analytic group.

Returning to the Hausdorff spectrum, we have the following results due to Barnea and Shalev [4, Th. 1.1].

Theorem 3. *Let G be a p-adic analytic pro-p group, $H \leq_c G$ a closed subgroup, then*

$$\operatorname{hdim}_G^{\mathcal{P}}(H) = \frac{\dim H}{\dim G}$$

Proof. Recall that we are computing the Hausdorff dimension with respect to the filtration $\mathcal{P}$ i.e. $G_n = G^{p^n}$. A result due to Lazard [11, P. 95] tells us that

$$\dim G = \lim_{n\to\infty} \frac{\log|G : G^{p^n}|}{n}$$
$$\dim H = \lim_{n\to\infty} \frac{\log|H : H^{p^n}|}{n}$$

Then $H^{p^n} \leq H \cap G_n$ and so we can conclude that

$$\frac{\log|H : H \cap G_n|}{\log|G : G_n|} \leq \frac{\log|H : H^{p^n}|/n}{\log|G : G_n|/n} \xrightarrow{n\to\infty} \frac{\dim H}{\dim G}.$$

Then by Theorem 2,

$$\operatorname{hdim}_G^{\mathcal{P}}(H) = \liminf_{n\to\infty} \frac{\log|H : H \cap G_n|}{\log|G : G_n|} \leq \frac{\dim H}{\dim G}$$

For the reverse inequality, we make use of [4, Cor. 3.2] as well as the fact that $H \cap G_{n+c} \leq H^{p^n} = H_n$ for each n, to find that

$$\operatorname{hdim}_G^{\mathcal{P}}(H) = \liminf_{n\to\infty} \frac{\log|H : H \cap G_{n+c}|}{\log|G : G_{n+c}|} \geq \frac{\log|H : H_n|/n}{\log|G : G_{n+c}|/n} = \frac{\dim H}{\dim G}$$

□

Corollary 1. *With G as in Theorem 3 and letting d denote $\dim G$, we have*

$$\operatorname{hspec}^{\mathcal{P}} G \subseteq \{0, 1/d, 2/d, \ldots, (d-1)/d, 1\}$$

is finite and consists of rationals.

An obvious question that now arises is whether or not the converse of Corollary 1 holds, that is — does $\operatorname{hspec}^{\mathcal{P}} G$ being finite imply that G is p-adic analytic. If it does, then we can extend the following characterization.

Let G be a pro-p group, then the following are equivalent, (for a more comprehensive list see [5, Pg. 97]).

1. G is p-adic analytic [5, Cor. 8.34]
2. G has finite rank
3. G is finitely generated and virtually uniform [5, Cor. 4.3]
4. G has polynomial subgroup growth [5, Th. 3.19]
5. G is isomorphic to a closed subgroup of $GL_d(\mathbb{Z}_p)$ for some d [5, Th. 7.19 and Th. 5.2]
6. G is finitely generated, and there is no infinite closed subgroup of G with Hausdorff dimension zero [5, Ex. 4.16]

Before digging deeper into this question, we consider a related question — if G is a finitely generated pro-p group and $\mathcal{S}_1, \mathcal{S}_2$ are filtration series of G, can we say that $\operatorname{hspec}^{\mathcal{S}_1} G$ is finite precisely whenever $\operatorname{hspec}^{\mathcal{S}_2} G$ is?

It turns out the answer to this question is negative as illustrated by the following example.

Example 1. *Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$.*

1. $\operatorname{hspec}^{\mathcal{P}} G = \{0, 1/2, 1\}$ *via Theorem 3*
2. *There is a filtration series $\mathcal{S}$ such that* $\left[\frac{1}{p+1}, \frac{p-1}{p+1}\right] \subseteq \operatorname{hspec}^{\mathcal{S}} G$ *[1, Prop. 3.3]*

Motivated by this somewhat surprising result, we turn to an examination of how the Hausdorff spectrum behaves with respect to some of the standard filtrations.

4 Filtration Dependence

The next question we will turn to is what happens when we change the filtration, does Theorem 3 remain true for any of the other standard filtrations. We can consider this question in the context of some of the filtrations that typically arise.

First we illustrate by way of concrete example the dependence Hausdorff dimension has on the chosen filtration.

Example 2. *Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and consider the subgroup $H = 0 \oplus \mathbb{Z}_p$. Then for the filtration $\mathcal{S} : G_n = p^n\mathbb{Z}_p \oplus p^n\mathbb{Z}_p$ we find that $\operatorname{hdim}_G^{\mathcal{S}}(H) = \frac{1}{2}$.*

On the other hand, with respect to the filtration $\mathcal{S} : G_n = p^{2n}\mathbb{Z}_p \oplus p^n\mathbb{Z}_p$, $\operatorname{hdim}_G^{\mathcal{S}}(H) = \frac{1}{3}$.

The 'standard' filtrations that we will look at are the following: The **lower p-series** of a finitely generated pro-p group G which we will denote by $\mathcal{L}$. We recall that this series is defined as

$$P_1(G) = G, \quad \text{and} \quad P_i(G) = P_{i-1}(G)^p\,[P_{i-1}(G), G] \quad \text{for } i \geq 2,$$

the next series in question is the **Frattini series** of G (we will denote by $\mathcal{F}$), is given by

$$\Phi_0(G) = G, \quad \text{and} \quad \Phi_i(G) = \Phi_{i-1}(G)^p\,[\Phi_{i-1}(G), \Phi_{i-1}(G)] \quad \text{for } i \geq 1.$$

Lastly, the **dimension subgroup series** (also referred to as the Jennings or Zassenhaus series) of G (denoted $\mathcal{D}$) is defined as

$$D_1(G) = G, \quad \text{and} \quad D_i(G) = D_{\lceil i/p \rceil}(G)^p \prod_{1 \leq j < i} [D_j(G), D_{i-j}(G)] \quad \text{for } i \geq 2.$$

We are working towards showing which conditions are necessary in establishing equality of the Hausdorff dimension with respect to the above mentioned filtration series. Beforehand, however, we need the following results.

Lemma 3. *[10, Lem. 7.1] Suppose that G is countably based profinite group and suppose further that we have the following two filtration series of G*

$$\mathcal{S}\colon G = G_0 \supseteq G_1 \supseteq \dots, \qquad \mathcal{S}^*\colon G = G_0^* \supseteq G_1^* \supseteq \dots$$

If it is known that

$$\lim_{i\to\infty} \frac{\log|G_iG_i^* : G_i|}{\log|G : G_iG_i^*|} = \lim_{i\to\infty} \frac{\log|G_iG_i^* : G_i^*|}{\log|G : G_iG_i^*|} = 0.$$

Then every closed subgroup $H \leq G$ is such that $\operatorname{hdim}_G^{\mathcal{S}}(H) = \operatorname{hdim}_G^{\mathcal{S}^*}(H)$.

Corollary 2. *[1, Cor. 2.3] Let G be a countably based profinite group, and let*

$$\mathcal{X}\colon G = X_0 \supseteq X_1 \supseteq \dots, \qquad \mathcal{Y}\colon G = Y_0 \supseteq Y_1 \supseteq \dots$$

be filtration series of G. Suppose that

$$\mathbb{N} \to \mathbb{N}, \quad i \mapsto i^* \qquad \textit{and} \qquad \mathbb{N} \to \mathbb{N}, \quad j \mapsto j'$$

are such that

$$\lim_{i\to\infty} \frac{\log|X_iY_{i^*} : X_i|}{\log|G : X_iY_{i^*}|} = \lim_{i\to\infty} \frac{\log|X_iY_{i^*} : Y_{i^*}|}{\log|G : X_iY_{i^*}|} = 0$$

and

$$\lim_{j\to\infty} \frac{\log|X_{j'}Y_j : X_{j'}|}{\log|G : X_{j'}Y_j|} = \lim_{j\to\infty} \frac{\log|X_{j'}Y_j : Y_j|}{\log|G : X_{j'}Y_j|} = 0.$$

Then every closed subgroup $H \le G$ satisfies $\mathrm{hdim}_G^{\mathcal{X}}(H) = \mathrm{hdim}_G^{\mathcal{Y}}(H)$*.*

Proof. Note that

$$\mathcal{X}' \colon G = X_{0'} \supseteq X_{1'} \supseteq \dots, \qquad \mathcal{Y}^* \colon G = Y_{0^*} \supseteq Y_{1^*} \supseteq \dots$$

are filtration series of G, which are produced by a thinning out of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Thus we can apply the previous lemma, namely Lemma 3, to conclude that for every closed subgroup $H \le G$,

$$\mathrm{hdim}_G^{\mathcal{X}}(H) = \mathrm{hdim}_G^{\mathcal{Y}^*}(H) \ge \mathrm{hdim}_G^{\mathcal{Y}}(H) = \mathrm{hdim}_G^{\mathcal{X}'}(H) \ge \mathrm{hdim}_G^{\mathcal{X}}(H). \qquad \square$$

Proposition 1. *Let G be a p-adic analytic pro-p group, and let $H \le_c G$ be a closed subgroup. Then under the series $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{D}$, the Hausdorff dimension function agrees on closed subgroups. In other words*

$$\mathrm{hdim}_G^{\mathcal{P}}(H) = \mathrm{hdim}_G^{\mathcal{F}}(H) = \mathrm{hdim}_G^{\mathcal{D}}(H).$$

As a direct consequence, we find that

$$\mathrm{hspec}^{\mathcal{P}}(G) = \mathrm{hspec}^{\mathcal{F}}(G) = \mathrm{hspec}^{\mathcal{D}}(G).$$

Proof. The conditions on G (namely being p-adic analytic) ensure that it will have finite rank. Let $H \le_c G$ be a closed subgroup, and as usual, let $\mathcal{P} \colon G_i = \pi_i(G) = G^{p^i}$, $i \in \mathbb{N}$, denote the p-power series of G.

We begin by comparing the p-power series $\mathcal{P}$ with the Frattini series $\mathcal{F} \colon \Phi_i(G)$, $i \in \mathbb{N}$ $G_i \subseteq \Phi_i(G)$ for all $i \in \mathbb{N}$ is immediate. Conversely, [5, Prop. 3.9 and Th. 4.5] guarantees the existence of a $j \in \mathbb{N}$ such that both G_i and $\Phi_i(G)$ are uniformly powerful for each of the $i \ge j$.

In particular we can conclude that for $i \ge j$ there are $x_1, \dots, x_d \in G$ such that $G_i = \langle x_1^{p^i}, \dots, x_d^{p^i} \rangle$, and consequently $G_{i+1} = \langle x_1^{p^{i+1}}, \dots, x_d^{p^{i+1}} \rangle = G_i^p$ (Where $d = \dim G$).

So we find that $\log_p|G_i : G_{i+1}| = \dim(G) = \log_p|\Phi_i(G) : \Phi_{i+1}(G)|$ for all $i \ge j$, and hence

$$\log_p|\Phi_i(G) : G_i| = \log_p|\Phi_j(G) : G_j|$$

is constant for $i \ge j$. And finally Lemma 3 yields $\mathrm{hdim}_G^{\mathcal{P}}(H) = \mathrm{hdim}_G^{\mathcal{F}}(H)$.

Now we will look at $\mathcal{P}$ in comparison with the dimension subgroup series $\mathcal{D} \colon D_i(G)$, $i \in \mathbb{N}$.

$G_i \subseteq D_i(G)$ for all $i \in \mathbb{N}$ again is immediate. Repeating the above argument, augmented by [5, Th. 11.4, 11.5;Lem. 11.22], we can find a $j \in \mathbb{N}$ such that: G_i and $D_i(G)$ are uniformly powerful, and we also have $D_{pi}(G) = D_i(G)^p$ for each $i \geq j$.

Next we make use of the following constructed filtration series. Put $\mathcal{D}^* \colon D_i^*(G) = D_{p^i j}(G)$, $i \in \mathbb{N}$.

Then $G_i \subseteq D_{i-j}^*$ for all $i \geq j$, where $\log_p |G_i : G_{i+1}| = \dim(G) = \log_p |D_{i-j}^*(G) : D_{i-j+1}^*(G)|$. As a result — we find

$$\log_p |D_{i-j}^*(G) : G_i| = \log_p |D_0^*(G) : G_j|$$

to be constant.

So we can employ Lemma 3 to conclude that

$$\mathrm{hdim}_G^{\mathcal{P}}(H) = \mathrm{hdim}_G^{\mathcal{D}^*}(H).$$

Then, because we also have $\log_p |D_i^*(G) : D_k(G)| \leq \dim(G)$ for $p^i j \leq k \leq p^{i+1} j$, we can match the terms of $\mathcal{D}^*$ to the terms $D_{p^i j}(G)$, $D_{p^i j+1}(G)$, $\ldots$, $D_{p^{i+1} j-1}(G)$, and lastly make use of Corollary 2 to find $\mathrm{hdim}_G^{\mathcal{D}^*}(H) = \mathrm{hdim}_G^{\mathcal{D}}(H)$.

$\square$

Unfortunately, the case for the lower p-series is not as nice. In fact there are p-adic analytic pro-p groups for which $\mathrm{hspec}^{\mathcal{P}} G \neq \mathrm{hspec}^{\mathcal{L}} G$. To see that this is the case, we need to construct a somewhat elaborate example.

Example 3. *We start by considering the cyclotomic extension of the p-adic integers* $\mathbb{Z}_p[\zeta_{p^n}]$. *The extension* $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ *is totally ramified. Indeed,* $\pi = \zeta_{p^n} - 1$ *is a root of the polynomial* $f(x) = (1+x)^{(p-1)p^{n-1}} + (1+x)^{(p-1)p^{n-2}} + \cdots + 1$ *which is Eisenstein over* $\mathbb{Q}_p$ *and we have* $\mathbb{Q}_p(\pi) = \mathbb{Q}_p(\zeta_{p^n})$ *hence the degree of the extension is* $\phi(p^n) = (p-1)p^{n-1}$.

Further, $f(0) = p = \prod_{\sigma \in Gal(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p)} \sigma(\pi)$. *And we also have that*

$$|\pi|_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p} = |\sigma(\pi)|_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}$$

hence $|p|_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p} = |\pi^{\phi(p^n)}|_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}$. *So we see that* $p \in \pi^{\phi(p^n)} \mathcal{O}^{\times}_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}$. *Then since the* $\phi(p^n)$ *is also the degree of the extension* $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$, *and* π *is a uniformizer, the extension is indeed totally ramified. All said, we see that*

$$\mathbb{Z}_p[\zeta_{p^n}]_+ = \mathbb{Z}_p \oplus \pi\mathbb{Z}_p \oplus \ldots \oplus \pi^{\phi(p^n)-1}\mathbb{Z}_p \cong \mathbb{Z}_p^{\phi(p^n)}.$$

and we have $\pi^{\phi(p^n)} \mathbb{Z}_p[\zeta_{p^n}] = p\mathbb{Z}_p[\zeta_{p^n}]$.

The G *in question is constructed as the semidirect product of*

$$S = \langle s_0, s_1, \ldots, s_{d-1} \rangle \cong \mathbb{Z}_p^{\,d} \tag{2}$$

$$T = \langle t_0, \ldots, t_{\phi(p^n)-1} \rangle \cong \mathbb{Z}_p[\zeta_{p^n}]_+ \tag{3}$$

Where the isomorphism in (2) is by way of the map $\varphi : T \to \mathbb{Z}_p[\zeta_{p^n}]$, $\prod t_i^{\lambda_i} \mapsto \sum \lambda_i \pi^i$ *and* d *is just some natural. Note that the action* $T \curvearrowright S$ *is by way of*

$$(v^{s_0})\varphi = v\varphi \cdot \zeta_{p^n} \quad \textit{and} \quad [v, s_1] = \ldots = [v, s_{d-1}] = 1 \quad \textit{for } v \in T.$$

Under this construction, the dimension of G *is seen to be* $\dim(G) = d + \phi(p^n)$ *and computation of the lower* p*-series* $\mathcal{L}$ *produces*

$$P_i(G) = \langle s_0^{p^{i-1}}, \ldots, s_d^{p^{i-1}} \rangle \ltimes T_i, \quad \textit{where } T_i\varphi = \pi^{i-1}\mathbb{Z}_p[\zeta_{p^n}].$$

We find that

$$\log_p |P_i(G) : P_{i+1}(G)| = d + 1$$

and as a result, we have

$$\mathrm{hdim}_G^{\mathcal{L}}(\langle s_0 \rangle) = \frac{1}{d+1} \quad \textit{and} \quad \mathrm{hdim}_G^{\mathcal{L}}(\langle t_0 \rangle) = \frac{1}{(d+1)\phi(p^n)}.$$

The above, in turn, allows us to compute the Hausdorff spectrum to be

$$\mathrm{hspec}^{\mathcal{L}}(G) = \big\{ (d+1)^{-1} \phi(p^n)^{-1} j \mid j \in \{0, 1, \ldots, (d+1)\phi(p^n)\} \big\}.$$

For an arbitrary subgroup $H \leq G$ *we have*

$$\mathrm{hdim}_G^{\mathcal{L}}(H) = \frac{1}{d+1} \left(\dim(HT/T) + \frac{1}{\phi(p^n)} \dim(H \cap T) \right).$$

and this is seen to be in contradistinction with Theorem 3.

Note that in particular, whenever $0 \leq j \leq d$ *and* $0 \leq k \leq \phi(p^n)$

$$\langle s_0^{p^n}, \ldots, s_{j-1}^{p^n}, t_0, \ldots, t_{k-1} \rangle$$

has Hausdorff dimension

$$\frac{1}{d+1} \left(j + \frac{k}{\phi(p^n)} \right).$$

Returning again to the question of characterization, we have the following result due to Klopsch

Theorem 4. *If* G *is a finitely generated soluble pro-*p *group, and* $\mathcal{S}$ *is one of either* $\mathcal{P}$, $\mathcal{F}$, *or* $\mathcal{D}$ *then whenever* G *is not* p*-adic analytic — the Hausdorff spectrum* $\mathrm{hspec}^{\mathcal{S}} G$ *contains a non-trivial interval.*

A direct corollary of this result is the following

Corollary 3. *If G is a finitely generated soluble pro-p group and $\mathcal{S}$ is as above, then G is p-adic analytic if and only if* $\operatorname{hspec}^{\mathcal{S}} G$ *is finite.*

We will provide only a very loose proof sketch of Theorem 4. For full details, see [1, Th. 5.4].

Proof sketch of Theorem 4. Take $\{G^{(m)}\}_{m\geq 1}$ to be the derived series of G and k to be the maximum so that $\operatorname{hdim}_G^{\mathcal{S}} G^{(k)} = 1$.

Writing $\operatorname{hdim}_G^{\mathcal{S}} G^{(k+1)} = \eta$ for each $K \leq G^{(k)}$ finitely generated, another result due to Shalev entails that

$$\operatorname{hdim}_G^{\mathcal{S}} K \leq \operatorname{hdim}_G^{\mathcal{S}} G^{(k+1)} = \eta < 1$$

Now, fixing some $\xi \in [\eta, 1]$ we can build a series $H_0 \leq H_1 \leq \dots$ of finitely generated groups such that

$$\operatorname{hdim}_G^{\mathcal{S}} \langle H_i \rangle = \xi$$

□

It is unclear whether or not a statement similar to Theorem 4 holds for the lower p-series.

References

[1] B. Klopsch and A. Thillaisundaram and A. Zugadi-Reizabal, Hausdorff dimensions in p-adic analytic groups, 1702.06789, (2017), arXiv, math.GR

[2] A. G. Abercrombie, Subgroups and subrings of profinite rings, *Math. Proc. Camb. Phil. Soc.* **116 (2)** (1994), 209–222.

[3] Y. Barnea and B. Klopsch, Index-subgroups of the Nottingham group, *Adv. Math.* **180** (2003), 187–221.

[4] Y. Barnea and A. Shalev, Hausdorff dimension, pro-p groups, and Kac-Moody algebras, *Trans. Amer. Math. Soc.* **349** (1997), 5073–5091.

[5] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, Second edition, Cambridge Studies in Advanced Mathematics, **61**, Cambridge University Press, Cambridge, 1999.

[6] K. J. Falconer, *Fractal geometry. Mathematical foundations and applications*, John Wiley & Sons, Ltd., Chicester, 1990.

[7] G. A. Fernández-Alcober, E. Giannelli, and J. González-Sánchez, Hausdorff dimension in R-analytic profinite groups, *J. Group Theory* **20** (2017), 579–587.

[8] G. A. Fernández-Alcober, A. Zugadi-Reizabal, GGS-groups: order of congruence quotients and Hausdorff dimension, *Trans. Amer. Math. Soc.* **366** (2014), 1993–2017.

[9] A. Jaikin-Zapirain and B. Klopsch, Analytic groups over general pro-p domains, *J. London Math. Soc.* **76** (2007), 365–383.

[10] B. Klopsch, Substitution Groups, Subgroup Growth and Other Topics, D.Phil. Thesis, University of Oxford, 1999.

[11] M. Lazard, Groupes analytiques p-adiques. *Publ. Math. IHÉS* **26** (1965), 389–603.