Finite Presentation of Chow Groups for a Toric Variety

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The result we will be looking at concerns the finite presentation of the Chow group $A_k(X)$ for a toric variety toric variety X. More precisely, we are looking to show that there is a finite presentation of the Chow group $A_k(X)$ in terms of torus-invariant subvarieties and torus-invariant divisors. We will follow the notation used in [Ful93] concerning toric varieties.

We recall that for a given variety X, the Chow group $A_k(X)$ is the free abelian group on the k-dimensional irreducible closed subvarieties of X modulo the subgroup generated by cycles $[\mathrm{div}(f)]$ with f a non-zero rational function on a $(k+1)$-dimensional subvariety of X.

It is known that for a given toric variety X, the toric divisors generate the group $A_{n-1}(X)$ of Weil divisors modulo rational equivalences. This can be generalized in the following manner.

The Chow group $A_k(X)$ of a given variety X is generated by all k-dimensional closed subvarieties of X, with relations generated by divisors of rational functions on $(k+1)$-dimensional subvarieties. In particular, as mentioned, $A_k(X)$ can be finitely presented in terms of torus-invariant subvarieties and torus invariant divisors when X is a toric variety.

Taking $X = X(\Delta)$ to be the toric variety corresponding to some fan Δ in a lattice N of dimension n, we can say that the torus-invariant closed subvarieties of X take the form $V(\sigma)$ where σ ranges over the cones in Δ and $\dim(V(\sigma)) = \mathrm{codim}(\sigma) = n - \dim(\sigma)$. The cones τ determine sublattices of the form $M(\tau) = \tau^{\perp} \cap M$ with M the dual lattice of N. The non-trivial elements $u \in M(\tau)$ determine rational functions x^u on $V(\tau)$ with the divisor of such rational functions being of the form

$$[\mathrm{div}(x^u)] = \sum_{\sigma} \langle u, n_{\sigma}, \tau \rangle [V(\sigma)] \tag{1}$$

the sum is taken over cones σ containing τ with $\dim(\tau) + 1$, and $n_{\sigma,\tau}$ is a lattice point in σ such that its image generates the one-dimensional lattice N_σ/N_τ. N_σ, and N_τ, are the sublattices of N generated by $\sigma \cap N$ and $\tau \cap N$ in that order.

Restating this as a theorem we have

Theorem 1. *The Chow group $A_k(X)$ of a toric variety $X = X(\Delta)$ is generated by the classes of the orbit closures $V(\sigma)$ of the cones σ of dimension $n-k$ of Δ.*

Proof. Take $X_i \subseteq X$ to be the union of all the $V(\sigma)$ with σ of dimension at least $n-i$. From this we obtain a filtration $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_{-1} = \emptyset$ by closed subschemes (with reduced structure). $X_i \setminus X_{i-1} = \coprod_\sigma O_\sigma$ where σ ranges over the cones of dimension $n-i$. Making use of the exact sequence reltating a closed subscheme and its complement, we obtain

$$A_k(X_{i-1}) \longrightarrow A_k(X_i) \longrightarrow \bigoplus_{\dim \sigma = n-i} A_k(O_\sigma) \longrightarrow 0$$

Since a torus O_σ is open in the affine space $\mathbb{A}^i$, we can similarly determine that $A_i(O_\sigma) = \mathbb{Z} \cdot [O_\sigma]$ and $A_k(O_\sigma) = 0$ when $k \neq i$. Then, as the restriction from $A_k(X_i)$ to $A_k(O_\sigma)$ takes $[V(\sigma)]$ to $[O_\sigma]$, straightforward induction then establishes that $[V(\sigma)]$, and the class $\dim(\sigma) = n-k$ generate $A_k(X_i)$ □

We also have the following:

Theorem 2. *The group of relations on the generators mentioned in Theorem in 1 is generated by all relations as in (1) where τ ranges over cones of codimension $k+1$ in Δ, u runs over some generating set of $M(\tau)$.*

This result is a direct consequence of the following theorem.

Theorem 3. *If a connected solvable linear algebraic group Γ acts on a scheme X, then the homomorphism $A_k^\Gamma(X) \to A_k(X)$ is an isomorphism.*

Here $A_k^\Gamma(X)$ is described as follows. First letting $Z_k(X)$ denote the free abelian group generated by all k-dimensional closed subvarieties of X and $R_k(X)$ the subgroup generated by cycles $[\mathrm{div}(f)]$ with f a non-zero rational function on a $(k+1)$-dimensional subvarieties W of X — then if an algebraic group Γ acts on X, we can form a group $A_k^\Gamma(x) = Z_k^\Gamma(X)/R_k^\Gamma(X)$, where $Z_k^\Gamma(X)$ is the free abelian group generated by all Γ-stable closed subvarieties of X and $R_k^\Gamma(X)$ is the subgroup generated by all divisors of eigenfunctions on Γ-stable $(k+1)$-dimensional subvarieties. Where a function in $R(W)^*$ is an eigenfunction if $g \cdot f = \chi(g)f$ for all $g \in \Gamma$, $\chi = \chi_f$ some character on Γ.

If Γ acts on X with only finitely many orbits, we obtain a finite presentation of the Chow groups of X. Indeed this is a consequence of two properties: when Γ has a dense orbit in W, a given eigenspace $R(W)_\chi$ can have dimension of at most one; the group of characters is finitely generated. It is worth noting that it is not always easy to make this presentation explicit. For the case of toric varieties however, we can obtain a finite presentation of the Chow groups via the combinatorics of the fan. Indeed, the invariant subvarieties correspond to cones in the defining fan, and the eigenfuntions on a corresponding variety are determined by points in the lattice dual to the cone.

We end by looking at an explicit example.

Example 1. *Take Δ to be the fan over the faces of the cube which has vertices at the points $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{Z}^3$. Take N to be the lattice $N = \{(x, y, z) \in (Z)^3 : x \equiv y \equiv z \mod 2\}$. Let Δ_k be the complete nonprojective fan obtained from Δ by changing the generator $(1, 1, 1)$ to $(1, 1, 2k+1)$. Then we can use the above results to compute $A_1(X(\Delta_k)) = \mathbb{Z}/k\mathbb{Z}$. In particular we find that*

$$A_1(X(\Delta)) = \mathbb{Z}, \qquad A_1(X(\Delta_1)) = 0, \qquad A_1(X(\Delta_2)) = \mathbb{Z}/2\mathbb{Z}.$$

When Δ is complete, results 1 and 2 imply that $A_0(X(\Delta)) = \mathbb{Z}$, where the isomorphism from $A_0(X(\Delta))$ to $\mathbb{Z}$ is the degree map which satisfies $\deg([V(\sigma)]) = 1$ for each n-dimensional cone $\sigma \in \Delta$. On the other hand, when Δ is not complete we can conclude that $A_0(X(\Delta)) = 0$.

References

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