Internal Languages

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Abstract. In this paper we will attempt to summarize some of the progress on the internal language conjectures looking in particular at the equivalence between CxlCat and CwA. We will proceed by building up the required material to precisely state the internal language conjectures, and then turn our attention to a small part of the first conjecture.

1 Introduction

One of the questions that arises when considering internal languages is what the correct notion of a weak equivalences between type theories is. There are at least two ways to motive an answer to this question. From the more logically oriented view, it would seem desirable to have things like conservativity and bi-interpretability. Such properties suggest, for example, Definition 9 where weak equivalences of theories are translations that permit lifting types up to equivalence and terms up to propositional equality. Another motivation comes from a homotopy-theoretic perspective. It may be noted that to each intensional type theory, there is an associated classifying fibration category. Thus it would seem desirable to require a weak equivalence between theories to induce an equivalence of the underlying fibration categories.

A nice consequence of working in HoTT is that the above described notions happen to coincide.

Any contextual functor is a type-theoretic equivalence if and only if it is a homotopy-theoretic equivalence.

One of the takeaways then is that since a category with weak equivalences is a presentation of an ∞-category (or a homotopy theory), we end up getting several ∞-categories of type theories.

For example we have ∞-categories of type theories with Id-types and dependent sums, $\mathrm{Th}_{\mathrm{Id},\Sigma}$, we also have ∞-categories of type theories: $\mathrm{Th}_{Id,\Sigma,\Pi}$ which have dependent products as well. What we would like to have is an ∞-category corresponding to $\mathrm{Th}_{\mathrm{HoTT}}$. Working just with the type theories that we can define and their ∞-counterparts, we proceed by constructing functors from these ∞-categories into ∞-categories of ∞-categories. In particular we obtain maps $\mathrm{Th}_{Id,\Sigma} \to \mathrm{Lex}_\infty$,

$\mathrm{Th}_{Id,\Sigma,\Pi} \to \mathrm{LCCC}_\infty$ and we conjecture that there is a map $\mathrm{Th}_{\mathrm{HoTT}} \to \mathrm{ElTop}_\infty$. Where Lex_∞ denotes ∞-categories with finite limits, LCCC_∞ locally cartesian closed ∞-categories, and ElTop_∞ elementary ∞-toposes.

Then the conjecture is that these functors are ∞-equivalences. To avoid any reliance on the initiality conjecture the formulations are done using contextual categories, but the desired goal is to have the conjectures hold for syntactically-presented type theories.

The next step in the process is equipping $\mathrm{Th}_{\mathrm{Id},\Sigma}$, and $\mathrm{Th}_{\mathrm{Id},\Sigma,\Pi}$ with a left semi-model structre. This happens to be a very natural construction, and for example we find that cofibrant type theories (those thought of as *free*) are exactly the theories that are extensions by types and terms without equations.

Proving the first conjecture, that is $\mathrm{Th}_{\mathrm{Id},\Sigma} \simeq \mathrm{Lex}_\infty$, follows by observing that there is a decomposition:

$$\mathrm{Th}_{\mathrm{Id},\Sigma} \to \mathrm{Trb} \to \mathrm{FibCat} \to \mathrm{Lex}_\infty$$

Here Trb is the category of (Joyal's) tribes, and FibCat the category of fibration categories.

The reason that this decomposition is useful is that tribes are related to comprehension categories, in fact the ∞-category of comprehension categories with Id and Σ types is equivalent to the of tribes. Further, there is a proof that $\mathrm{FibCat} \to \mathrm{Lex}_\infty$ is an equivalence (cf. Kapulkin, Szlumo). So the only missing piece is an equivalence from $\mathrm{Trb} \simeq \mathrm{FibCat}$. We begin by briefly introducing the necessary foundation to state and examine the internal language conjectutes.

2 Contextual Categories

Definition 1. *A* ***contextual category*** $\mathbf{C}$ *is comprised of the following:*

1. *a grading* $\mathrm{Ob}\mathbf{C} = \coprod_{n:\mathbb{N}} \mathrm{Ob}_n\mathbf{C}$
2. *an object* $1 \in \mathrm{Ob}_0\mathbf{C}$
3. *father operations* $\mathrm{ft}_n : \mathrm{Ob}_{n+1}\mathbf{C} \longrightarrow \mathrm{Ob}_n\mathbf{C}$
4. *for each* $\Gamma \in \mathrm{Ob}_{n+1}\mathbf{C}$*, a map* $p_\Gamma : \Gamma \longrightarrow \mathrm{ft}\Gamma$ *(**canonical projections**)*
5. *for each* $\Gamma \in \mathrm{Ob}_{n+1}\mathbf{C}$ *and* $f : \Delta \longrightarrow \mathrm{ft}\Gamma$*, an object* $f^*\Gamma$ *together with a* ***connecting map*** $f.\Gamma : f^*\Gamma \longrightarrow \Gamma$

   such that

6. 1 *is the unique object in* $\mathrm{Ob}_0\mathbf{C}$*, it is also terminal in* $\mathbf{C}$
7. *for each* $\Gamma \in \mathrm{Ob}_{n+1}\mathbf{C}$ *and* $f : \Delta \longrightarrow \mathrm{ft}\Gamma$*, we have* $\mathrm{ft}(f^*\Gamma) = \Delta$ *and the square*

$$\begin{tikzcd}
f^*\Gamma \arrow[r, "f.\Gamma"] \arrow[d, "p_{f^*\Gamma}"'] & \Gamma \arrow[d, "p_\Gamma"] \\
\Delta \arrow[r, "f"] & \mathrm{ft}\ \Gamma
\end{tikzcd}$$

is a pullback (the ***canonical pullback*** *of* Γ *along* f*)*

8. *these canonical pullbacks are functorial:* $\Gamma \in \mathrm{Ob}_{n+1}\mathbf{C}$, $\mathrm{id}^*_{\mathrm{ft}\Gamma} = \Gamma$ *and* $\mathrm{id}_{\mathrm{ft}\Gamma}.\Gamma = \mathrm{id}_\Gamma$*; and for each* $\Gamma \in \mathrm{Ob}_{n+1}\mathbf{C}$ *and* $f : \Delta \longrightarrow \mathrm{ft}\Gamma$ *and* $g : \Theta \longrightarrow \Delta$*, we have* $(fg)^*\Gamma = g^*f^*\Gamma$ *and* $fg.\Gamma = f.\Gamma \circ g.f^*\Gamma$

Definition 2. *A* ***contextual functor*** $F : \mathbf{C} \longrightarrow \mathbf{D}$ *is a functor which preserves all the above listed structure of a contextual category. That is – the grading, the terminal object, the father maps, the dependent projections, the canonical pullbacks, and the connecting maps are all preserved.*

Definition 3. *For an object* $\Gamma \in \mathrm{Ob}_n\mathbf{C}$*, by* $\mathrm{Ty}_{\mathbf{C}}(\Gamma)$ *we will mean the set of objects* $\{\Gamma' \in \mathrm{Ob}_{n+1}\mathbf{C} \mid \mathrm{ft}(\Gamma') = \Gamma\}$ *and we will call such a set the* ***types in context*** Γ*. To each morphism of contexts* $f : \Gamma' \longrightarrow \Gamma$*, there is an associated map* $f^* : \mathrm{Ty}_{\mathbf{C}}(\Gamma) \to \mathrm{Ty}_{\mathbf{C}}(\Gamma')$ *which is defined by the pullback operation of* $\mathbf{C}$*. We remark that the axioms of a contextual category ensure that* $\mathrm{Ty}_{\mathbf{C}}$ *is a presheaf,* $\mathrm{Ty}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$*.*

Definition 4. *A* ***context extension*** *of* $\Gamma \in \mathrm{Ob}_n\mathbf{C}$ *will be an object* $\Gamma' \in \mathrm{Ob}_{n+m}\mathbf{C}$ *such that* $\mathrm{ft}^m\Gamma' = \Gamma$*. Such an extension will be written as* $\Gamma.\Delta$ *where* Δ *is a context extension of* Γ *of degree* m*.*

Remark 1. *Since we are concerned with the connection between Martin-Löf type theory and category theory, we need to equip our contextual categories with operations corresponding to the type-constructors of Martin-Löf type theory. For the time being it will suffice to consider operations corresponding to identity types (*Id*), unit types (*1*), dependent sum/sigma types (*Σ*), dependent function types and their functional extensionality rules (*Π_{ext}*). We will denote the* ***category of contextual categories*** *as* $\mathrm{CxlCat}_{\mathrm{Id},1,\Sigma}$ *and* $\mathrm{CxlCat}_{\mathrm{Id},1,\Sigma,\Pi_{\mathrm{ext}}}$ *respectively, where the subscripts denote the additional structure.*

Definition 5. *From* Id*-types on a contextual category, we can construct* ***identity contexts****. If we start with an object* $\Gamma \in \mathrm{Ob}_n\mathbf{C}$ *and a context extension* $\Gamma.\Delta \in \mathrm{Ob}_{n+m}\mathbf{C}$*, then we can form a further context extension* $\Gamma.\Delta.p^*_\Delta.\mathrm{Id}_\Delta \in \mathrm{Ob}_{n+3m}$*, as well as a reflexively map and an elimination operation. Such operations serve to generalize the counterparts corresponding to an identity type* $\Gamma.A.p^*_A A.\mathrm{Id}_A$ *of a single type in context* Γ

Definition 6. *Using the above definition, we can define a* ***homotopy*** *in a contextual category. Let* $f, g : \Gamma \longrightarrow \Delta$ *be morphisms in a context category* $\mathbf{C}$*, then a homotopy* $H : f \sim g$ *is a factorization of* $(f, g) : \Gamma \longrightarrow \Delta \times \Delta = \Delta.p^*_\Delta\Delta$ *through the identity context* $\Delta.p^*_\Delta\Delta.\mathrm{Id}_\Delta \xrightarrow{p_{\mathrm{Id}_\Delta}} \Delta.p^*_\Delta\Delta$*.*

Definition 7. *Now we can define the notion of equivalence in a context category with identity types (in fact there are several equivalent definitions).*

*A **structured equivalence** $w : \Gamma \simeq \Delta$ is given by a map $f : \Gamma \longrightarrow \Delta$, along with two maps $g_1, g_2 : \Delta \longrightarrow \Gamma$ such that there are homotopies $\eta : fg_1 \sim 1_\Delta$ and $\varepsilon : g_2 f \sim 1_\Gamma$*

An **eqivalence** $\Gamma \xrightarrow{\sim} \Delta$ in $\mathbf{C}$ is a map $f : \Gamma \longrightarrow \Delta$ for which we can find some $g_1, g_2, \eta, \varepsilon$ which turn it into a structured equivalence.

Definition 8. *A functor between fibration categories is **exact** if it preserves fibrations, acyclic finrations, pullbacks along fibrations, and terminal objects.*

*An exact functor is a **weak equivalence** of fibration categories if it iduces an equivalence of homotopy categories.*

Part of the purpose of bringing fibration categories into the picture is that they can be useful for looking at the homotopy theoretic aspects of type theory. If we start with a contextual category $\mathbf{C}$ that has identity types, we can turn it into a fibration category by taking the class of weak equivalences ($\mathcal{W}$) to be the equivalences in contextual categories that we just defined, and taking the class of fibrations ($\mathcal{F}$) to be maps which can be decomposed as a series of canonical projections.

We state this as a theorem

Theorem 1. [KL16, Theorem 2.13]

1. *A contextual category $\mathbf{C}$ with $\mathrm{Id}, 1$ and Σ can be made into a fibration category via the classes $\mathcal{F}$ and $\mathcal{W}$.*

2. *There is a full and faithful functor $\mathrm{CxlCat}_{\mathrm{Id},1,\Sigma} \longrightarrow \mathrm{FibCat}$ which sends contextual categories to fibration categories in light of 1.*

We can use the above fact to describe the homotopy category of a contextual category. An important theorem which we will state at the end of this section is that a contextual functor is a type-theoretic equivalence if and only if it is a homotopy-theoretic equivalence, and the following lemma is an essential ingredient in establishing this fact.

Lemma 1. [KL16, Lemma 2.14], [AKL15, Theorem 3.2.5] *Let $\mathbf{C} \in \mathrm{CxlCat}_{\mathrm{Id},1,\Sigma}$ be a contextual category. The **homotopy category** of $\mathbf{C}$ (when considered as a fibration category) is given by*

1. *objects of $\mathrm{Ho}(\mathbf{C})$ are just the objects of $\mathbf{C}$*

2. *morphism in $\mathrm{Ho}(\mathbf{C})$ are homotopy classes of maps in $\mathbf{C}$ (where homotopy is as defined above)*

Proof. We start by describing the path objects in $\mathbf{C}$. Let $A \in \mathbf{C}$ then $\mathsf{P}A \equiv \sum_{x,y:A} x = y$, i.e. it is constructed by its identity types. It comes equipped with a fibration $\pi : \mathsf{P}A \to A \times A$, and there is also a natural map $r : A \to \mathsf{P}A$ which takes $a \in A$ to the pair $((a, a), r(a))$. The map from $\mathsf{P}A$ to A which sends $((a, a'), p) \mapsto a$ (or a') provides a quasi-inverse to r. All together this provides a factorization of the diagonal of A as a weak equivalence followed by a fibration, pictorally:

$$\begin{tikzcd}
 & \mathsf{P}A \arrow[dr, "\pi"] & \\
A \arrow[ur, "r"] \arrow[rr, "\Delta"'] & & A \times A
\end{tikzcd}$$

From this we see that homotopy as in Definition 6 coincides with right homotopy as defined for a fibration category. Then, as the fibration categories which arise from contextual categories have the convenient property that all of their objects are also cofibrant (all acyclic fibrations have a section), this notion also corresponds to weak right homotopy. We complete this proof by verifying our claim that objects of a contextual category are cofibrant. Let $\mathsf{hfib}(f, y) \equiv \sum_{x:X}(f(x) = y)$ denote the **homotopy fibre** of $f : X \to Y$ over an element $y \in Y$. For a fibration $\pi_1 : \sum_{x:A} B(x) \to A$ and an $a : A$ we claim that $B(a) \simeq \mathsf{hfib}(\pi_1, a)$. The map from $B(a)$ to $\mathsf{hfib}(\pi_1, a)$ is given by sending b to $((a, b), \mathsf{refl}(a))$, and the map in the other direction is found by sending $(a', b), p) : \mathsf{hfib}(\pi_1, a)$ $(b : B(a'),\ p : a' = a)$ to $p_*(b) : B(a)$. Now, if π_1 is an acylic fibration, then it admits a section as follows: for a family consisting of contractions of the fibres of $\mathsf{hfib}(\pi_1, x)$, let $x : A$ get sent to the image of the centre of contraction via the equivalence $\mathsf{hfib}(\pi_1, x) \simeq B(x)$. Then for $C \in \mathbf{C}$ and $\bar{f}$ as below:

$$\begin{tikzcd}
 & \sum_{x:A} B(x) \arrow[d, "\pi_1"] \\
C \arrow[ur, dotted, "\bar{f}"] \arrow[r, "f"'] & A
\end{tikzcd}$$

define $\bar{f}$ as the composite of f and this section. □

Now we seek to examine the two notions of equivalence between contextual categories namely, type theoretic (logical equivalences) equivalences and homotopy theoretic equivalences.

Definition 9. *A functor of contextual categories, $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a* ***type-theoretic equivalence*** *if the following conditions are satisfied:*

1. **weak type lifting**: For all $\Gamma \in \mathbf{C}$ and $A \in \mathrm{Ty}(F\Gamma)$, there is an $\bar{A} \in \mathrm{Ty}(\Gamma)$ and an equivalence $F\bar{A} \xrightarrow{\sim} A$ over $F\Gamma$:

$$\begin{tikzcd}
\bar{A} \arrow[r, dotted] \arrow[d, two heads] & F\bar{A} \arrow[rr, "\sim"] \arrow[dr, two heads] & & A \arrow[dl, two heads] \\
\Gamma \arrow[rr, dotted] & & F\Gamma &
\end{tikzcd}$$

2. **weak term lifting**: For all $\Gamma \in \mathbf{C}$, $A \in \mathrm{Ty}(\Gamma)$ and $a \in \mathrm{Tm}(FA)$, there is a term $\bar{a} \in \mathrm{Tm}(A)$ as well as an element of the identity type $e \in \mathrm{Tm}(\mathsf{Id}_{FA}(F\bar{a}, a))$:

$$\begin{tikzcd}
A \arrow[r, dotted] & FA \\
\Gamma \arrow[u, "\bar{a}"] \arrow[r, dotted] & F\Gamma \arrow[u, bend left, "F\bar{a}"] \arrow[u, bend right, "a"']
\end{tikzcd}$$

We can strengthen the lifting properties in the following sense:

Lemma 2. [KL16, Lemma 3.2] *All type-theoretic equivalences* $F : \mathbf{C} \xrightarrow{\sim} \mathbf{D}$ *satisfies:*

1. ***weak context lifting**: for all contexts* $\Gamma \in \mathbf{C}$ *and context extensions* $F\Gamma.\Delta$, *there is a context extension* $\Gamma.\bar{\Delta}$ *as well as an equivalence* $F(\Gamma.\bar{\Delta}) \xrightarrow{\sim} F\Gamma.\Delta$

2. ***weak section lifting**: for any context extension* $\Gamma.\Delta \in \boldsymbol{C}$ *and section* $s : F\Gamma \longrightarrow F\Gamma.F\Delta$ *of the projection* $p_F\Delta$, *there is a section* $\bar{s} : \Gamma \longrightarrow \Gamma.\Delta$ *of* p_Δ *and a homotopy* $e : F\bar{s} \sim s$ *over* $F\Gamma$

Note that we can also define a **homotopy theoretic equivalence** $F : \mathbf{C} \longrightarrow \mathbf{D}$ if the induced functor $\mathrm{Ho}F : \mathrm{Ho}\mathbf{C} \longrightarrow \mathrm{Ho}\mathbf{D}$ is an equivalence of categories. Fortunately, both of these definitions are equivalent.

Theorem 2. [KL16, Proposition 3.3] *A contextual functor is a type theoretic equivalence if and only if it is a homotopy-theoretic equivalence.*

3 Comprehension Categories

One of the problems encountered when modelling type theory is that most categories do not have the structure of a contextual category. The strategy is thus to consider other 'structured' categories which are more ubiquitous, and then provide a means by which to turn these 'structured' categories into contextual categories. The obvious question that then arises is why we even bother with contextual categories, why not just use these as yet undefined 'structured' categories? Part of the reason is that we want the canonical pullbakcs to be strictly functorial, and the mere existence of pullbacks is not enough to ensure canonicity. A further issue is the logical structure required to interpret the logical constructors. As an illustrative example, consider the coproduct type former $+$, and a substitution f^*. In the syntactic category of a type theory $\mathrm{Cl}(\mathbb{T})$ we have $f^*(A + B) = f^*(A) + f^*(B)$ viz. substitution is strictly functorial with respect to the logical structure. So, since this happens to be a property of the syntax, it must be a property of each model. From the categorical point of view (in categorical semantics) type formers are interpreted using universal properties, or some even weaker notion. Due to this, substitution is guaranteed to be functorial (in the best cases) up to canonical isomorphism. Models that only weakly satisfy such stability conditions are said to be weak models, and the problem of strictifying weak models is known as the **coherence problem**. One example of weak models comes from **comprehension categories**. Strictifying a weak model will be done by constructing a functor from the category of comprehension categories to the category of contextual categories. This is usually done by introducing a third type of model, which lies between comprehension categories and contexutal categories, and are called **categories with attributes**. So we have something like

$$\text{Comprehension Categories} \to \text{Categories with Attributes} \to \text{Contextual Categories}$$

$$\begin{tikzcd} \text{weak stability} \arrow[r, leftrightarrow, dashed] & \text{strict stability} \end{tikzcd}$$

To construct a strictification functor from comprehension categories to categories with attributes, we can make use of Voevodsky's universe method. Indeed, given any category with a universe the ensuing construction will yield a contextual category.

We begin by defining some of the terms mentioned above.

Definition 10. *A* ***cleaving*** *for a (Grothendieck) fibration $p : \mathcal{T} \to \mathcal{C}$ is a choice function which takes a given $t \in \mathcal{T}$ and some $u : b \to p(t)$ in $\mathcal{C}$ and returns a P-cartesian lift $f : t' \to t$ of u. The cleaving is said to be* ***normal*** *if it lifts identities to identites, and said to be* ***split*** *if it is both normal, and it preserves compositions (i.e. lifts of compositions are compositions of lifts – the lifts are functorial).*

Definition 11. *A* ***comprehension category*** *consists of the following data:*

1. a category $\mathcal{C}$;
2. a functor $\chi : \mathcal{T} \to \mathcal{C}^{\to}$ (the **comprehension**);

such that:

1. $P :=$ cod $\circ\, \mathcal{T} \to \mathcal{C}$ is a Grothendieck fibration;
2. for each P-cartesian morphism f, χf is a pullback in $\mathcal{C}$.

Definition 12. *A comprehension category is* ***full*** *if χ is fully faithful and* ***cloven*** *if P is.*

Going forward, we will assume that all our comprehension categories are full and cloven.

As with contextual categories, we can interpret contexts as objects in the category $\mathcal{C}$ and substitutions as morphisms. Using P to denote the composition cod$\circ\chi : \mathcal{T} \to \mathcal{C}$ we can view a comprehension category as a commutative diagram of the form:

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\chi"] \arrow[dr, "P"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

Then, types in some context can be interpreted via the fibration $P : \mathcal{T} \to \mathcal{C}$ – a type X in context Γ is an object $X \in \mathcal{T}$ such that $P(X) = \Gamma$. The display maps are defined as follows:

Definition 13. *Let $(\mathcal{C}, \chi)$ be a comprehension category and Γ an object of $\mathcal{C}$, denote the fibre $P^{-1}(\Gamma)$ by $\mathcal{T}_\Gamma$. For some $\sigma \in \mathcal{T}_\Gamma$ the object $\chi(\sigma) \in \mathcal{C}^{\to}$ is a map $p_\sigma : \Gamma.\sigma \to \Gamma$ in $\mathcal{C}$ which is called the* ***display map****.*

Actions of substitutions of the form $f : \Gamma \to \Delta$ on a type X in context Γ are determined by the source of the P-cartesian lift of f which arises from the cleaving.

Definition 14. *A **category with attributes** is a full split comprehension category. Morphisms between comprehension categories are functors among the underlying categories and a functor between the total space which commutes with the cleaving. We will use* CwA *to represent the category of categories with attributes.*

We will use CompCat to denote category of comprehension categories which are not required to be split.

4 Internal Language Conjectures

Our goal in this section is to make precise the conjectures that dependent type theories are internal languages of appropriate higher categories.

We start by introducing some necessary terms and definitions so that we can give such precise statements.

A functor between categories with weak equivalences is called **homotopical**, if it preserves the weak equivalences. So we can consider a category, call it weCat, which has categories with weak equivalence as objects, and homotopical functors as morphisms.

Relating this to something we already know, we note that for a fibration category say $(\mathcal{C}, \mathcal{F}, \mathcal{W})$, we trivially have a category with weak equivalences given by $(\mathcal{C}, \mathcal{W})$. Further, we can apply Ken Brown's Lemma to get that all exact functors are homotopical.[1]

So, since we can show that any contextual category can be made into a fibration category, in particular, the contextual category has an underlying category with weak equivalences. Using this construction, we get a functor $\mathrm{CxlCat}_{\mathrm{id},1,\Sigma} \to \mathrm{weCat}$.

In fact, we can turn weCat into a category with weak equivalences by letting the set $\mathcal{W}$ be comprised of the Dwyer-Kan equivalences. DK-equivalences are maps which induce weak equivalences between simplicial sets and and an equivalence of categories between the homotopy categories.

If we let Cat_∞ be the full subcategory of sSet, with objects taken as quasicategories, then Cat_∞ can be viewed as a category with weak equivalences where the weak equivalences are categorical equivalences. We have a DK-equivalence:

$$\mathrm{weCat} \xrightarrow{\mathrm{Ho}_\infty} \mathrm{Cat}_\infty$$

Where Ho_∞ is taken as the hammock localization composed with the fibrant replacement of the homotopy coherent nerve. Then we will let $\mathrm{Cat}_\infty^{\mathrm{Id},1,\Sigma}$ denote the functor

$$\mathrm{CxlCat}_{\mathrm{Id},1,\Sigma} \longrightarrow \mathrm{weCat} \xrightarrow{\mathrm{Ho}_\infty} \mathrm{Cat}_\infty$$

Similarly, we write $\mathrm{Cat}_\infty^{\mathrm{Id},1,\Sigma,\Pi_{\mathrm{ext}}}$ for

$$\mathrm{CxlCat}_{\mathrm{Id},1,\Sigma,\Pi_{\mathrm{ext}}} \longrightarrow \mathrm{CxlCat}_{\mathrm{Id},1,\Sigma} \longrightarrow \mathrm{weCat} \xrightarrow{\mathrm{Ho}_\infty} \mathrm{Cat}_\infty$$

[1] In short Ken Brown's lemma says that a functor from a fibration category is homotopical if it takes acyclic fibrations to weak equivalences.

In practice, we will use the construction given by Szuiło, namely that of quasicategory of frames, denoted $\mathrm{N_f}\mathcal{C}$ where $\mathcal{C}$ is a fibration category, to work with the functors just described. Essentially we make use of the equivalence $\mathrm{N_f}\mathcal{C} \simeq Ho_\infty\mathcal{C}$. This equivalence is useful since $\mathrm{N_f}\mathcal{C}$ gives us a more practical description of $Ho_\infty\mathcal{C}$ than those which are garnered from general constructions.

Theorem (1) The functor $\mathrm{Cl}_\infty^{\mathrm{Id},1,\Sigma}$ takes values in the category Lex_∞ of quasicategories with finite limits and finite limit preserving functors. Moreover, it takes equivalences of contextual categories to categorical equivalences of quasicategories.

(2) The functor $\mathrm{Cl}_\infty^{\mathrm{Id},1,\Sigma,\Pi_{\mathrm{ext}}}$ takes values in the category LCCC_∞ of locally cartesian closed quasicategories and locally cartesian closed functors. As before, it takes equivalences of contextual categories to categorical equivalences of quasicategories.

We are now in a position to state the first conjecture.

The functors:

$$\begin{aligned}
\mathrm{Cl}_\infty^{\mathrm{Id},1,\Sigma} &: \mathrm{CxlCat}_{\mathrm{Id},1,\Sigma} \longrightarrow \mathrm{Lex}_\infty \\
\mathrm{Cl}_\infty^{\mathrm{Id},1,\Sigma,\Pi_{\mathrm{ext}}} &: \mathrm{CxlCat}_{\mathrm{Id},1,\Sigma,\Pi_{\mathrm{ext}}} \longrightarrow \mathrm{LCCC}_\infty
\end{aligned}$$

are DK-equivalences of categories with weak equivalences.

The ultimate goal is to have horizontal DK-equivalence's in the following:

$$\begin{tikzcd}
\mathrm{CxlCat}_{\mathrm{HoTT}} \arrow[r, dashed, "\mathrm{Cl}_\infty^{\mathrm{HoTT}}", "\sim"'] \arrow[d, dashed] & \mathrm{ElTop}_\infty \arrow[d, dashed] \\
\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathrm{ext}}} \arrow[r, "\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma,\Pi_{\mathrm{ext}}}", "\sim"'] \arrow[d] & \mathrm{LCCC}_\infty \arrow[d] \\
\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \arrow[r, "\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma}", "\sim"'] & \mathrm{Lex}_\infty
\end{tikzcd}$$

The break-down of the first conjecture can be represented as:

$$\begin{tikzcd}
& & \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \arrow[rr, "\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma}", "\sim"'] \arrow[dl, dotted, no head] & & \mathrm{Lex}_\infty \arrow[dr, dotted, no head] & \\
& \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \arrow[r] \arrow[dl, dotted, no head] & \mathrm{Trb} \arrow[rr, "\sim"] \arrow[dr, dotted, no head] & & \mathrm{FibCat} \arrow[r, "\sim"] & \mathrm{Lex}_\infty \\
\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \arrow[r] & \mathrm{CwA}_{\mathsf{Id},1,\Sigma} \arrow[r] & \mathrm{CompCat}_{\mathsf{Id},1,\Sigma} \arrow[r, "\sim"] & \mathrm{Trb} & &
\end{tikzcd}$$

It is not within the scope of this paper to present proof's of each of the equivalences along the bottom string of functors. We will, however, construct a functor $F : \mathrm{CxlCat} \longrightarrow \mathrm{CwA}$, and then discuss a few strictification theorems.

The distinguished terminal object as well as the underlying category are the same. The objects in the total space $\mathcal{T}$ are given by pairs (Γ, σ) such that $ft(\sigma) = \Gamma$; and morphisms $(\Delta, \delta) \to (\Gamma, \sigma)$ are maps $\Delta.\delta \to \Gamma.\sigma$ in $\mathbf{C}$. The comprehension functor χ takes (Γ, σ) to p_σ and the split cleaving will be the given by the canonical pullbacks of $\mathbf{C}$ (note that this implies that χ maps cartesian morphisms to pullbacks.

If we start with a category with attributes, it is reasonable to ask if there is a grading which turns it into a contextual category, and still yet, if there are many such gradings. It turns out that having a grading is in fact a property of a category with attributes and doesn't require any extra structure. In other words, the functor F is fully faithful. We state this as a proposition:

Proposition 1. *The functor* $F : \mathrm{CxlCat} \longrightarrow \mathrm{CwA}$ *is fully faithful.*

Proof. Let $\mathcal{C} \in \mathrm{CwA}$. Then a grading of the objects in $\mathcal{C}$ means that for each $\Gamma \in \mathcal{C}$ either Γ is a terminal object or there is a choice of $\Delta \in \mathcal{C}$ and a $\sigma \in \mathcal{T}_\Delta$ such that $\Delta.\sigma = \Gamma$. Further, the length of each Γ must be finite. In order to see that having a grading is an inherent property of a category with attributes, we can proceed by induction on the length of Γ. The base case is immediate. For the inductive case, suppose that for some Γ we have choices of $\Delta \in \mathcal{C}$, $\sigma \in \mathcal{T}_\Delta$ and $\Delta' \in \mathcal{C}$, $\sigma \in \mathcal{T}_{\Delta'}$. Then $\Delta.\sigma = \Gamma = \Delta'.\sigma'$ so that $\Delta = p(\Delta.\sigma) = p(\gamma) = p(\Delta'.\sigma') = \Delta$. So the inductive hypothesis guarantees uniqueness of the grading. $\square$

In the other direction, we want to construct a functor $\mathrm{core} : \mathrm{CwA} \longrightarrow \mathrm{CxlCat}$. We can do this as follows: Let $(\mathcal{C}, \chi) \in \mathrm{CwA}$, and let 1 be the distinguished terminal object. The objects of the underlying category of $\mathrm{core}(\mathcal{C})$ are given by lists $(A_0, A_1, \ldots, A_n)$ where $A_0 \in \mathcal{T}_1$ (1 here representing the distinguished terminal object of $\mathcal{C}$) and $A_{i+1} \in \mathcal{T}_{1.A_0.\cdots.A_i}$. The morphisms, $(A_0, \ldots, A_n) \longrightarrow (B_0, \ldots, B_m)$ are just the morphisms $1.A_0.A_1.\cdots.A_n \to 1.B_0.B_1.\cdots.B_m$ in $\mathcal{C}$. The terminal object is the empty list. The grading comes from the length of the lists and the father maps just forget the last element. The canonical projections are the display maps $p_\sigma : \Gamma.\sigma \to \Gamma$. The canonical pullbacks come from the cleaving since χ takes cartesian maps to pullbacks. Further, the cleaving being split guarantees that the pullbacks are functorial.

We note that the functor core as defined might add in the same object many times. In order to avoid this we need to ensure that each object is either the distinguished terminal object, or that there is a unique composite $1.\cdots.\sigma$ from it to the terminal object.

Now we seek to demonstrate that

Proposition 2. *The functor* core *is right adjoint to the functor* $F : \mathrm{CxlCat} \longrightarrow \mathrm{CwA}$.

Proof. Let $\mathbf{C}$ be a fixed contextual category, and let $(\mathcal{D}, \chi : \mathcal{T} \to \mathcal{D}^{\to})$ be a category with attributes. Let $m : F(\mathbf{C}) \to \mathcal{D}$ be a functor of categories with attributes. We will represent m by a functor G between the underlying categories, and a functor T between the total spaces. Then define the contextual functor $\hat{m} : \mathbf{C} \to \mathrm{core}(\mathcal{D})$ first on objects, by means of induction on the degree of $\Gamma \in \mathbf{C}$ as follows: When Γ is the unique object of degree 0, send it to the empty list in $core(\mathcal{D})$. If Γ has degree $n+1$, take the display map $\Gamma \to ft(\Gamma)$.

Applying G to the aforementioned display map we get a map $G(\Gamma) \to G(ft(\Gamma))$, and using the inductive hypothesis, we have an object $\hat{m}(ft(\Gamma)$ represented by a list $(D_0, \ldots, D_n)$. Definition of the functor F implies that the display map is represented by an object t in the total space of $F(\mathbf{C})$. Next, applying T to t, we get an object D_{n+1} over $G(ft(c))$. We then define $\hat{m}(\Gamma) := (D_0, \ldots, D_n, D_{n+1})$.

$\hat{m}$ acts on morphism in the obvious way, as the morphisms between $(D_0, \ldots, D_n)$ and $(E_0, \ldots, E_\ell)$ in core($\mathcal{D}$) correspond to morphisms between $1.D_0.\cdots.D_n$ and $1.E_0.\ldots.E_\ell$ in $\mathcal{D}$.

Suppose that $\hat{m}$ is a contextual morphism from $\mathbf{C}$ to core($\mathcal{D}$). We start by defining a functor G which goes from the underlying category of $F(\mathbf{C})$ (which is also the underlying category of $\mathbf{C}$) to the underlying category of $\mathcal{D}$.

Let $\Gamma \in \mathbf{C}$, $\hat{m}(c) = (D_0, \ldots, D_n)$, and define $G(\Gamma) := D_0.\cdots.D_n$. Then again, since morphisms between $(D_0, \ldots, D_n)$ and $(E_0, \ldots, E_\ell)$ in core($\mathcal{D}$) correspond to morphisms between $1 : D_0, \cdots, D_n$ and $1.E_0.\cdots.E_\ell$ in $\mathcal{D}$, we see how G acts on morphisms.

Now we need to define a functor T which runs between the total spaces. The objects of the total space of $F(\mathbf{C})$ are pairs (Γ, σ) where $\Gamma, \sigma \in \mathbf{C}$ and $ft(\sigma) = \Gamma$.

The map $\hat{m}(p_\sigma)$ in core($\mathcal{D}$) is a display map, so it can be written as $\hat{m}(p_\sigma) : (D_0, \ldots, D_n, D_{n+1}) \to (D_0, \ldots, D_n)$. Let $T(\Gamma, \sigma) := D_{n+1} \in \mathcal{T}$.

Then since maps of the form $f : (\Delta.\delta) \to (\Gamma.\sigma)$ in the total space of $F(\mathbf{C})$ correspond to morphisms $f : \Delta.\delta \to \Gamma.\sigma$ in $\mathbf{C}$, we can apply $\hat{m}$ and then by definition of morphisms in core($\mathcal{D}$), we find that we have a morphism $1 : D_0.\cdots.D_n \to 1.E_0.\cdots.E_\ell$. Finally, since χ is fully faithful, we end up with a unique morphism $D_n \to E_\ell$ in $\mathcal{T}$. $\square$

If Th is a dependent type theory that only posses the structural rules, then the category with attributes $F(\mathrm{Cl}(\mathrm{Th}))$ is initial. Indeed, this follows from the fact that the functor $F : \mathrm{CxlCat} \longrightarrow \mathrm{CwA}$ is left adjoint and so preserves initial objects. The upshot is that categories with attributes model type theory just as contextual categories do, but categories with attributes are slightly more natural in that we don't have to worry about the grading.

We would like to have some additional logical structure on comprehension categories, and the first step in this direction is to consider strictification theorems.

5 Strictification

There is a strictification theorem due to Hofmann [Hof95] which gives us a functor from $(-)_* : \mathrm{CompCat} \to \mathrm{CwA}$ and works well when modelling *extensional* type theory; however, it does not handle non-trivial identity types. To model *intensional* type theory then, we need to look for another strictification theorem. Lumsdaine and Warren [LW15] have done exactly this. The methods that they used were inspired by Voevodsky's universe construction which we will now breifly outline.

Definition 15. *For some category $\mathcal{C}$, a* ***universe*** *in $\mathcal{C}$ is a morphism $p : \tilde{U} \to U$ in $\mathcal{C}$ such that for every morphism $f : X \to U$ there is a choice of pullback square:*

$$\begin{tikzcd} (X; f) \arrow[r, "Q(f)"] \arrow[d, "P_{(X,f)}"'] & \tilde{U} \arrow[d, "p"] \\ X \arrow[r, "f"] & U \end{tikzcd}$$

The first thing to note is that we can iteratively perform pullbacks, i.e. given some morphisms $f_1 : X \to U, f_2 : (X; f_1) \to U$ we can then consider an object $((X; f_1); f_2))$ and so on. For clarity, we will denote such an iterated pullback as $(X : f_1, \ldots, f_n)$. Let us now consider the contextual category that arises from a universe U.

Definition 16. *Starting with a category C, a universe U and a terminal object 1 the contextual category arising from U (denoted C_U) is comprised of the following data:*

1. $\mathrm{Ob}_n C_U \coloneqq \{(f_1, \ldots, f_n) \mid f_i : (1; f_1, \ldots f_{i-1}) \to U; 1 \leq i \leq n\}$;
2. Morphisms $(f_1, \ldots, f_n) \to (g_1, \ldots, g_m)$ in C_U are just the morphisms $(1; f_1, \ldots, f_n) \to (1; g_1, \ldots, g_m)$ in C;
3. The distinguished terminal object of C_U is the empty sequence ();
4. The father maps are $ft(f_1, \ldots, f_{n+1}) \coloneqq (f_1, \ldots, f_n)$;
5. Projections $p_{(f_1,\ldots,f_{n+1})} : (f_1, \ldots, f_{n+1}) \to (f_1, \ldots, f_n)$ are given by the map $P_{(1;f_1,\ldots,f_{n+1})}$ arising from the universe construction on U;
6. for an object $(f_1, \ldots, f_{n+1})$ as well as a morphism $\alpha : (g_1, \ldots, g_m) \to (f_1, \ldots, f_n)$ in C_U, the canonical pullback $\alpha^*(f_1, \ldots, f_{n+1})$ is $(g_1, \ldots, g_m, f_{n+1} \circ \alpha)$ which corresponds to $(1; g_1, \ldots, g_m, f_{n+1} \circ \alpha)$ in the following diagram:

$$\begin{tikzcd} (1; g_1, \ldots, g_m, f_{n+1} \circ \alpha) \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "Q(f_{n+1} \circ \alpha)"] & (1; f_1, \ldots, f_{n+1}) \arrow[r, "Q(f_{n+1})"'] \arrow[d] & \tilde{U} \arrow[d, "p"] \\ (1; g_1, \ldots, g_m) \arrow[r, "\alpha"] & (1; f_1, \ldots, f_n) \arrow[r, "f_{n+1}"] & U \end{tikzcd}$$

   where the dashed arrow denotes the projection which is induced by the pullback on the right

One of the nice properties of this construction is that the contextual category C_U only depends on the map $p : \tilde{U} \to U$ and not on the choice of pullbacks or terminal object. Fruther, the functoriality of the data is a result of using the composition in C, which is strictly associative. An additional pleasing property is the following:

Proposition 3. *For a small contextual category $\mathcal{C}$, we can consider the universe U in the prefsheaf category $Set^{C^{\mathrm{op}}}$ which is given by:*

$$U = \{Y \mid ft(Y) = X\}$$
$$\tilde{U}(X) = \{(Y, s) \mid ft(Y) = X, s \text{ is a section of } p_Y\}$$

together with the evident projection map. Then $(Set^{C^{\mathrm{op}}})_U$ are isomorphic as contexutal categories.

This is especially useful since structure on the morphism $p : \tilde{U} \to U$ corresponds to logical structure on C_U.

Now we can describe a functor $(-)_! : \mathrm{CompCat} \to \mathrm{CwA}$. For the ensuing construction, assume that $\mathcal{C}$ is locally cartesian closed. Let $(\mathcal{C}, \mathcal{T}, \chi)$ be a comprehension category, and define $(\mathcal{C}_!, \mathcal{T}_!, \chi_!)$ such that $\mathcal{C}_! := \mathcal{C}$, objects in $\mathcal{T}_!$ over some $\Gamma \in \mathcal{C}$ are tripples of the form $(V_A, E_A, \{A\})$ where $V_A \in \mathrm{Ob}\mathcal{C}$, $E_A \in \mathrm{Ob}(\mathcal{T}(V_A))$ and $\{A\}$ is a morphism $\Gamma \to V_A$ in $\mathcal{C}$. Let $[A]$ denote the reindexing of E_A along the map $\{A\}$. A morphism $(V_B, E_B, \{B\}) \to (V_A, E_A, \{A\})$ in $\mathcal{T}_!$ over a map $m : \Delta \to \Gamma$ consists of maps $[B] \to [A]$ in $\mathcal{T}$ over m. We immediately get the projection $p_! = \mathcal{T}_! \to \mathcal{C}_!$, taking $(V_B, E_B, \{B\})$ to the domain of $\{B\}$.

If $m : \Delta \to \Gamma$ is a morphism in $\mathcal{C}$, $(V_A, E_A, \{A\})$ is an object in $\mathcal{T}_!$ over Γ, set $A[m] := (V_A, E_A, \{A\} \circ m)$ and take $A_m : A[m] \to A$ to be the canonical map $[A[m]] \to [A]$ over m in $\mathcal{T}$ which comes from $(E_A)_{\{A\}}$ being P-cartesian. Then, such choices make $p_!$ a split fibration. Lastly, we can send $(V_A, E_A, \{A\})$ to $\chi(\{A\})$ yielding $\chi_! : \mathcal{T}_! \to \mathcal{C}_!^{\to}$, and since χ sends P-cartesians to pullbacks, so does $\chi_!$.

The above construction gives a general method for producing models of intensional dependent type theories. We state this as a theorem:

Theorem 3. *Let $\mathcal{C}$ be a comprehension category that satisfies the LF-condition, then $\mathcal{C}$ is equivalent to a split comprehension category $\mathcal{C}_!$. Moreover, if $\mathcal{C}$ has weak Σ-types (respectively Π-types, Id-types, W-types, and so forth), then $C_!$ has strictly stable analogues.*

Thus we can rewrite our rough sketch from the start of the previous section as follows.

$$\begin{tikzcd} \mathrm{CxlCat} \arrow[r, "\top", "F"'] & \mathrm{CwA} \arrow[l, bend right, "\mathrm{core}"'] \arrow[r, hook, "\top", "i"'] & \mathrm{CompCat} \arrow[l, bend right, "(-)_*"'] \arrow[l, bend left, "(-)_!"] \arrow[r] & \mathrm{Trb} \end{tikzcd}$$

There is not sufficient space to detail the proof that $\mathrm{Trb} \simeq \mathrm{FibCat}$, so we end by giving a very brief summary. This equivalence is shown by first considering the semi-simplicially enriched versions of fibration categories and tribes, and then showing that they both have the structure of a fibration category. From here it is then possible to explicitly construct inverses to show that they are equivalent to their unenriched counterparts (cf prop 3.12). The last step is to show that the functor running from semi-simplicial tribes to semi-simplicial fibration categories satisfies the Approximation Properties. Once this is established, the result follows from the 2-out-of-3 property.

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6 Appendix

Type theories that are more interesting in terms of their utility usually have some additional logical structure on top of their structural rules. Examples of this include Σ-types, Π-types, Id-types, W-types and so forth. The logical structure is then provided by further rules which usually consist of a formation rule, an introduction rule, an eliminations rule, and a computation rule. We will illustrate the case of Σ-types.

Definition 17. *A type theory with Σ-types includes the followng rules:*

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B(x)}{\Gamma \vdash \Sigma_{x:A} B(x)} \; \Sigma\text{-Form}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B(x)}{\Gamma, x : A, y : B(x) \vdash \langle x, y \rangle_{A,B} : \Sigma_{x:A} B(x)} \; \Sigma\text{-Intro}$$

$$\frac{\Gamma, z : \Sigma_{x:A} B(x) \vdash C(z) \qquad \Gamma, x : A, y : B(x) \vdash m(x, y) : C(\langle x, y \rangle_{A,B})}{\Gamma, z : \Sigma_{x:A} B(x) \vdash \mathsf{ind}_m(z) : C(z)} \; \Sigma\text{-Elim}$$

$$\frac{\Gamma, z : \Sigma_{x:A} B(x) \vdash C(z) \qquad \Gamma, x : A, y : B(x) \vdash m(x, y) : C(\langle x, y \rangle_{A,B})}{\Gamma, x : A, y : B(x) \vdash \mathsf{ind}_m(\langle x, y \rangle_{A,B}) \equiv m(x, y) : C(\langle x, y \rangle_{A,B})} \; \Sigma\text{-Comp}$$

In order to find a model for such a type theory, we need to interpret Σ-types. Looking at the induced structure on the classifying category, we see that the following definition is in order.

Definition 18. *A Σ-type structure on a contextual category* $\mathbf{C}$ *is comprised of the following data:*

1. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\mathbf{C}$, there is an object $(\Gamma, \Sigma(A, B)) \in \mathrm{Ob}_{n+1}$ (the formation rule);
2. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\mathbf{C}$, there is a morphism $\langle , \rangle_{A,B} : (\Gamma, A, B) \longrightarrow (\Gamma, \Sigma(A, B))$ which commutes with the canonical projections to Γ (the introduction rule);
3. for all (Γ, A, B), $(\Gamma, \Sigma(A, B), X)$ and each morphism $m : (\Gamma, A, B) \longrightarrow (\Gamma, \Sigma(A, B), X)$ such that $p_X \circ m = \langle , \rangle_{A,B}$, there is a morphism $\mathsf{ind}_m(\Gamma, \Sigma(A, B)) \longrightarrow (\Gamma, \Sigma(A, B), X)$;

such that for each context morphism $f : \Delta \longrightarrow \Gamma$:

$$f^*(\Gamma, \Sigma(A, B)) = (\Delta, \Sigma(f^*A, f^*B))$$
$$f^*\langle , \rangle_{A,B} = \langle , \rangle_{f^*A, f^*B}$$
$$f^*\mathsf{ind}_m = \mathsf{ind}_{f^*m}$$

We then have the following correctness result:

Theorem 4. *Given a dependent type theory $\mathbb{T}$ with the typical structural rules, plus some combination of the logical rules for Σ-,Γ-,W- or Id-types, its classifying category $C(\mathbb{T})$ is initial among contextual categories with the corresponding structure.*