Problem 1.

(a)

To evaluate the definite integral

$$\int_{-1}^{1} x\,e^{x}dx$$

We can use integration by parts:

$$\int_{-1}^{1} x\,e^{x}dx = (xe^{x} - e^{x})|_{-1}^{1} = \frac{2}{e} \approx 0.735758882343$$

Using the composite trapezoidal method, we approximate this answer numerically:

```
>> P1_Trapezoid
k = 0;  h =    1;  T(h) = 1.175201193643801; |I - T(h)|  = 4.4e-01;
 k = 1;  h = 5.0e-01;  T(h) = 0.848148249568774; |I - T(h)|  = 1.1e-01; |I - T(2h)|\|I - T(h)| = 3.9
 k = 2;  h = 2.5e-01;  T(h) = 0.764019438861344; |I - T(h)|  = 2.8e-02; |I - T(2h)|\|I - T(h)| = 4.0
 k = 3;  h = 1.2e-01;  T(h) = 0.742834305720519; |I - T(h)|  = 7.1e-03; |I - T(2h)|\|I - T(h)| = 4.0
 k = 4;  h = 6.2e-02;  T(h) = 0.737528382265355; |I - T(h)|  = 1.8e-03; |I - T(2h)|\|I - T(h)| = 4.0
 k = 5;  h = 3.1e-02;  T(h) = 0.736201297598931; |I - T(h)|  = 4.4e-04; |I - T(2h)|\|I - T(h)| = 4.0
 k = 6;  h = 1.6e-02;  T(h) = 0.735869488674432; |I - T(h)|  = 1.1e-04; |I - T(2h)|\|I - T(h)| = 4.0
 k = 7;  h = 7.8e-03;  T(h) = 0.735786534083122; |I - T(h)|  = 2.8e-05; |I - T(2h)|\|I - T(h)| = 4.0
 k = 8;  h = 3.9e-03;  T(h) = 0.735765795287779; |I - T(h)|  = 6.9e-06; |I - T(2h)|\|I - T(h)| = 4.0
 k = 9;  h = 2.0e-03;  T(h) = 0.735760610579723; |I - T(h)|  = 1.7e-06; |I - T(2h)|\|I - T(h)| = 4.0
 k = 10;  h = 9.8e-04;  T(h) = 0.735759314402132; |I - T(h)|  = 4.3e-07; |I - T(2h)|\|I - T(h)| = 4.0
```

(b) See print out above.

(c) According to the theorem of precision of trapezoid rule, since f(x) has a continuous second derivative on the interval [-1,1], the error

$$I - T(h) = O(h^{2})$$

In the program above, each time the step size is made smaller by a factor of 2, the error decreases by a factor of 4, which is consistent with an error of O(h^2).

(d) Programming the Romberg Algorithm, we get the approximations:

```
n = 1;  h = 5.0e-01; R(n,1) = 0.739130601543765
n = 2;  h = 2.5e-01; R(n,1) = 0.735976501958868; |I - R(k-1,0)|\|I - R(k,0)| = 15.5
n = 3;  h = 1.2e-01; R(n,1) = 0.735772594673577; |I - R(k-1,0)|\|I - R(k,0)| = 15.9
n = 4;  h = 6.2e-02; R(n,1) = 0.735759741113634; |I - R(k-1,0)|\|I - R(k,0)| = 16.0
n = 5;  h = 3.1e-02; R(n,1) = 0.735758936043457; |I - R(k-1,0)|\|I - R(k,0)| = 16.0
n = 6;  h = 1.6e-02; R(n,1) = 0.735758885699598; |I - R(k-1,0)|\|I - R(k,0)| = 16.0
n = 7;  h = 7.8e-03; R(n,1) = 0.735758882552686; |I - R(k-1,0)|\|I - R(k,0)| = 16.0
n = 8;  h = 3.9e-03; R(n,1) = 0.735758882355998; |I - R(k-1,0)|\|I - R(k,0)| = 16.0
n = 9;  h = 2.0e-03; R(n,1) = 0.735758882343704; |I - R(k-1,0)|\|I - R(k,0)| = 16.0
n = 10;  h = 9.8e-04; R(n,1) = 0.735758882342935; |I - R(k-1,0)|\|I - R(k,0)| = 16.4
```

(e) Above we have the ratios between the previous and new Romberg approximation of the integral at each step. Theoretically, as f has continuous derivatives of all orders, the theory predicts R(n,1) should have an error of $O(h^4)$. This agrees with our program, in which halving the step size makes the error 16 times smaller.

Code Printout:

P1_Trapezoid.m

```
a = -1;
b = 1;
I = 2 / exp(1);          % Correct value of the integral
prev_error = 0;
for k = 0:10
    h = 2^(-k);                  % Length of each subinterval
    n = (b-a) / h;       % Number of subintervals

    %% Triangular Rule
    % (1) Add the average of f(x) at the endpts, with the f(x) at the
    % rest of points
    sum = 1/2 * (f(a) + f(b));

    for i = 1:n-1
        x_i = a + h*i;
        sum = sum + f(x_i);
    end
    % (2) multiply by step size
    sum = sum*h;
    if k == 0
        fprintf("n = %i;  h = %3.1d;  T(h) = %.15f; |I - T(h)|  = %3.1d;  \n", k, h, sum, abs(I - sum))  % |I - T(h)| /|I - T(h\2)|
    else
        romberg = 4/3 * sum - 1/3 * prev_sum;
        if k ~= 1
            fprintf("n = %i;  h = %3.1d;   T(h) = %.15f; |I - T(h)|  = %3.1d; |I - T(2h)|\\|I - T(h)| = %.1f; R(n,1) = %.15f; |I - R(k-1,0)|\\|I - R(k,0)| = %.1f\n", ...
                k, h, sum, abs(I - sum), abs(I - prev_sum)/abs(I - sum), romberg,  abs(I - prev_romberg)/abs(I - romberg))

        else
            fprintf("n = %i;  h = %3.1d;   T(h) = %.15f; |I - T(h)|  = %3.1d; |I - T(2h)|\\|I - T(h)| = %.1f; R(n,1) = %.15f\n", ...
                k, h, sum, abs(I - sum), abs(I - prev_sum)/abs(I - sum), romberg)

        end
        prev_romberg = romberg;
    end
    % (3) Keep track of previous approximations
    prev_sum = sum;



end


function y = f(x)
y = x * exp(double(x));
end
```

Problem 2.

Suppose for a function $f(x)$, the error of our trapezoidal approximation takes the form:

$$\int_a^b f(x)dx = R(n,0) + a_3h^3 + a_6h^6 + \ldots \qquad \textbf{(I)}$$

Then note that:

$$\int_a^b f(x)dx = R(n-1,0) + a_3(2h)^3 + a_6(2h)^6 + \ldots \qquad \textbf{(II)}$$

Then taking 8 times **(I)** minus **(II)** divided by 8, we get:

$$\int_a^b f(x)dx = \frac{8}{7}R(n,0) - \frac{1}{7}R(n-1,0) + O(h^6)$$

Therefore we should define R(n,1) as:

$$R(n,1) = \frac{8}{7}R(n,0) - \frac{1}{7}R(n-1,0)$$

.

Problem 3

Given the nodes:

i	x	$f(x)$
0	1	10
1	1.25	8
2	1.5	7
3	1.75	6
4	2	5

The composite Simpson rule give us:

$$\int_1^2 f(x)dx \approx (.25/3)[(10+5)+4(8+6)+2(7)] = 85/12$$

$$\approx 7.083$$

Problem 4(a)

4. (a) Verify $\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

is exact for $p \in \Pi_3$

- $f(x) = 1 \Rightarrow \int_{-1}^{1} f(x)\,dx = 2$

  and $f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 1 + 1 = 2$

- $f(x) = x \Rightarrow \int_{-1}^{1} f(x)\,dx = 0$

  and $f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = 0$

- $f(x) = x^2 \Rightarrow \int_{-1}^{1} f(x)\,dx = \frac{2}{3}$

  and $f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

- $f(x) = x^3 \Rightarrow \int_{-1}^{1} f(x)\,dx = 0$

  and $f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} = 0$

(b)

(b) Verify $\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

is exact for $f(x) \in \Pi_5$

$f(x) = 1$ LHS: $\int_{-1}^{1} 1\,dx = 2$

RHS: $\frac{5}{9}(1) + \frac{8}{9}(1) + \frac{5}{9}(1) = 2$

$f(x) = x$ LHS: $\int_{-1}^{1} x\,dx = 0$

RHS: $\frac{5}{9}\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + \frac{5}{9}\left(\frac{\sqrt{3}}{\sqrt{5}}\right) = 0$

$f(x) = x^2$ LHS: $\int_{-1}^{1} x^2\,dx = 2/3$

RHS: $\frac{5}{9}\left(\frac{3}{5}\right) + 0 + \frac{5}{9}\left(\frac{3}{5}\right) = 2/3$

$f(x) = x^3$ LHS: $\int_{-1}^{1} x^3\,dx = 0$

RHS: $-\frac{5}{9}\left(\frac{3}{5}\right)^{3/2} + \frac{5}{9}\left(\frac{3}{\sqrt{5}}\right)^{3/2} = 0$

$f(x) = x^4$ LHS: $\int_{-1}^{1} x^4\,dx = 2/5$

RHS: $\frac{5}{9}\left(\frac{9}{25}\right) + \frac{5}{9}\left(\frac{9}{25}\right) = 2/5$

$f(x) = x^5$ LHS: $\int_{-1}^{1} f(x)\,dx = 0$

RHS: $\frac{5}{9}\left(-\frac{3}{5}\right)^{5/2} + \frac{5}{9}\left(\frac{3}{5}\right)^{5/2} = 0$

(c) The Gaussian Quadrature formulas with 2 and 3 interpolating nodes give an approximation of the integral

$$\int_{-1}^{1} x\, e^{x}\, dx$$

Accurate to 2 and 4 decimal places respectively.

```
>> P4_Gauss_Quadrature
Integral: 0.735758882342885

Guassian quadrature approximation (n = 1): 0.704325909483092
Accuracy: 3.14e-02

Guassian quadrature approximation (n = 2): 0.735362144160853
Accuracy: 3.97e-04
```

Code:

```
% Goal: Test accuracy of Gaussian Curvature Formulas

I = 2 / exp(1);         % Correct value of the integral
fprintf("Integral: %.15f\n", I)

approx_1 = f(1/sqrt(3)) + f(-1/sqrt(3));
fprintf("\nGuassian quadrature approximation (n = 1): %.15f\n", approx_1)
fprintf("Accuracy: %.2d\n\n", abs(I - approx_1))

approx_2 = (5/9)*f(-sqrt(3)/sqrt(5)) + (5/9)*f(sqrt(3)/sqrt(5)) + (8/9)*f(0);
fprintf("Guassian quadrature approximation (n = 2): %.15f\n", approx_2)
fprintf("Accuracy: %.2d\n\n", abs(I - approx_2))




function y = f(x)
y = x * exp(double(x));
end
```