2. Determine the natural cubic spline that interpolates $f(x) = x^{-6}$ over $[0, 2]$ using knots $0, 1, 2$

- We know the spline $S \in S_3^2(\{0,1,2\})$
- This spline space has a basis of

$$\{1, x, x^2, x^3, (x-1)_+^3\}$$

- The spline then is of the form

$$S(x) = A + Bx + Cx^2 + Dx^3 + E(x-1)_+^3$$

- Fitting the interpolation and continuity equations...

$$\begin{aligned}
&(1) & S(0) &= A = 0 \\
&(2) & S(1) &= A + B + C + D = 1 \\
&(3) & S(2) &= A + 2B + 4C + 8D + E = 64 \\
&(4) & S'(1) &= B + 2C + 3D = 6 \\
&(5) & S''(1) &= 2C + 6D = 30
\end{aligned}$$

$(1) \Rightarrow A = 0$, $(4) - (2) \Rightarrow C + 2D = 5$ (6)

$(5) - 2 \times (6) \Rightarrow 2D = 20 \Rightarrow D = 10$

$\Rightarrow C = -15 \Rightarrow B = 6$

$(3) \Rightarrow 0 + 12 + (-60) + 80 + E = 64$

$\Rightarrow 32 + E = 64 \Rightarrow E = 32$

- So an equation for the cubic spline is

$$\boxed{S(x) = 6x - 15x^2 + 10x^3 + 32(x-1)_+^3}$$

expressed in truncated power series basis

Problem 3

$$S(x) = \begin{cases} \overbrace{x^3 - 1}^{S_0(x)} & -9 \le x \le 0 \\ \underbrace{a + bx + cx^2 + dx^3}_{S_1(x)}, & 0 \le x \le 5 \end{cases}$$

Imposing continuity equations

- $S_1(0) = S_0(0) \Rightarrow a = -1$
- $S_1'(0) = S_0'(0) \Rightarrow b = 0$
- $S_1''(0) = S_0''(0) \Rightarrow 2c = 0$

- $S(1) = 2 \Rightarrow a + b + c + d = 2$

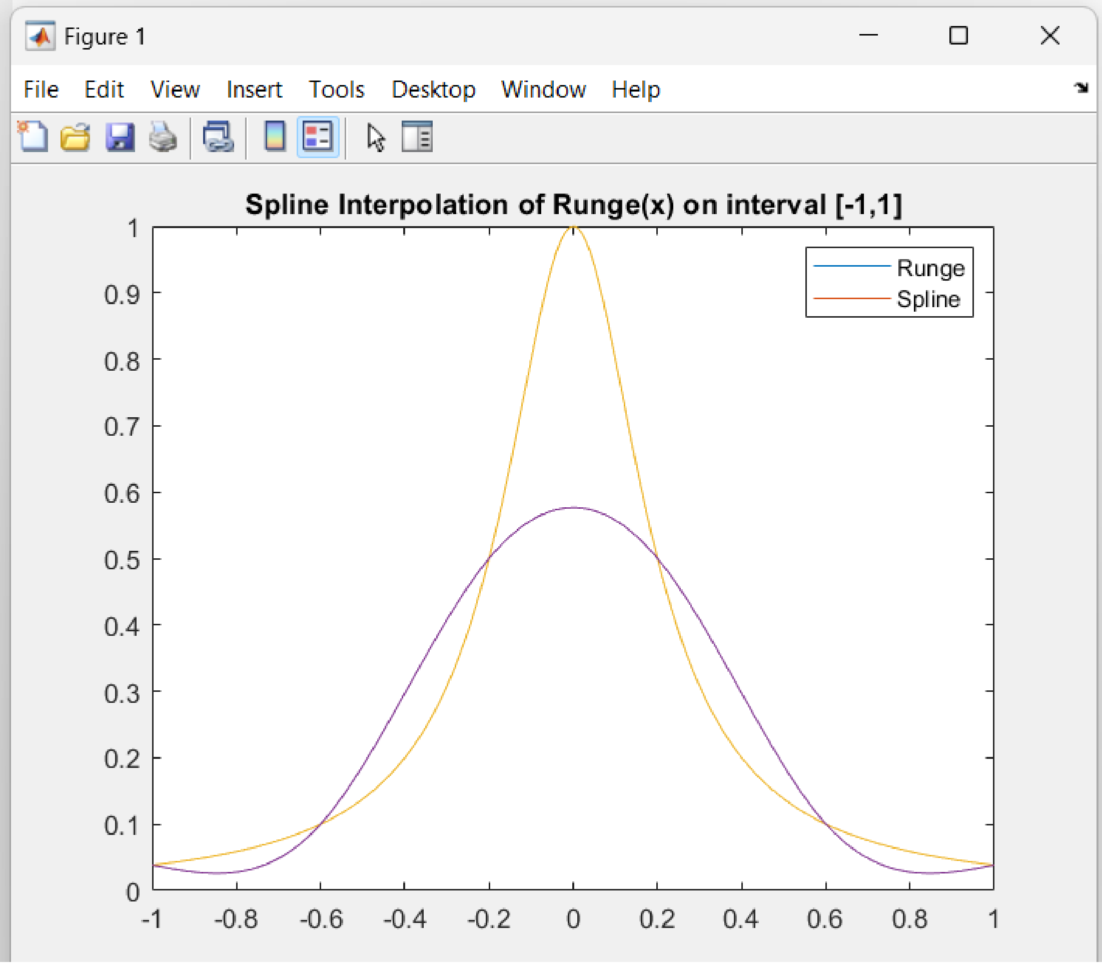
$\Rightarrow d = 3$

$$\boxed{\Rightarrow (a, b, c, d) = (-1, 0, 0, 3)}$$

Problem 1.

(a – f) Initially using 5 knots to generate the spline, the maximum error of the spline approximation of the Runge function was .42

```
The Maximum Error: 0.423481781376518>>
```



Increasing the number of knots to n=4,8,16,32,64…, we observe:

```
>> P1_Natural_Cubic_Spline
For n = 4, error = 0.279311381158094
For n = 8, error = 0.056073644925168, convergence rate = 5.0
For n = 16, error = 0.003710967584725, convergence rate = 15.1
For n = 32, error = 0.000637335632232, convergence rate = 5.8
For n = 64, error = 0.000036066324368, convergence rate = 17.7
For n = 128, error = 0.000002053431999, convergence rate = 17.6
For n = 256, error = 0.000000159303183, convergence rate = 12.9
For n = 512, error = 0.000000009676896, convergence rate = 16.5
For n = 1024, error = 0.000000000331312, convergence rate = 29.2
For n = 2048, error = 0.000000000030329, convergence rate = 10.9
For n = 4096, error = 0.000000000002131, convergence rate = 14.2
```

The spline appears to converge quadratically to the runge function, as each time the size of the intervals are halfed, the error is atleast 4x smaller. The spline s appears to converge to f, and this agrees with the theoretical result that splines can converge to f.

Code:

P1_Natural_Cubic_Spline.m

```
% Goal: Create a cubic spline that interpolate a function (Runge)

% (0) Set up the interpolation problem
% (0.1) Sample the Runge function at (t_0,y_0)...(t_n,y_n)
for n=2.^(2:14)
    t = linspace(-1, 1, n+1);
    y = arrayfun(@Runge, t);

    % (1) Compute spline coeffients
    z = Spline_Coef(n, t, y);

    % (2) Evaluate the spline
    m = 200;
    X = linspace(-1, 1, m+1);
    Y_Runge = arrayfun(@Runge, X);
    Y_Spline = arrayfun(@(x) Spline_Eval(x, n, t, y, z), X);

    % (3) Printout
    plot(X, Y_Runge)
    hold on
    plot(X, Y_Spline)
    hold on
    legend('Runge','Spline')
    title("Spline Interpolation of Runge(x) on interval [-1,1]")

    error = max(abs(Y_Runge-Y_Spline));
    if n == 4
        fprintf("For n = %i, error = %.15f\n", n, error)
    else
        fprintf("For n = %i, error = %.15f, convergence rate = %3.1f\n", n,error, prev_error/error)
    end
    prev_error = error;
end
```

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Function Definitions
function sum = Spline_Eval(x, n, t, y, z)
for i=n:-1:1
    if x - t(i) >= 0
        break
    end
end
h = t(i+1) - t(i);

sum = z(i+1)/(6*h) * (x - t(i))^3 ...
   +  z(i)/(6*h) * (t(i+1) - x)^3 + ...
           (y(i+1) / h - h/6 *z(i+1))*(x-t(i)) + (y(i) / h - h/6 * z(i)) *(t(i+1) - x);
end


function z = Spline_Coef(n, t, y)
    for i=1:n
        h(i) = t(i+1) - t(i);           % Length of ith interval
        b(i) = (y(i+1) - y(i))/h(i);    % Average slope in ith interval
    end

    u(2) = 2 * (h(1) + h(2));
    v(2) = 6 * (b(2)-b(1));

    for i=3:n
        u(i) = 2 * (h(i) + h(i-1)) - h(i-1)^2/u(i-1);
        v(i) = 6 * (b(i) - b(i-1)) - h(i-1)*v(i-1) / u(i-1);
    end

    z(n+1) = 0;
    z(1) = 0;
    for i = n:-1:2
        z(i) = (v(i) - h(i) * z(i+1)) / u(i);
    end
end

function y = Runge(x)
y = 1/(1 + 25*x^2);
end
```

Problem 2