

Lab and Exercises IV

Exercise I.

(i) Implement the finite difference scheme that we discussed for solving the equation (for simplicity $q = 0$ and $r = 0$):

$$-p\partial_x^2 u = f \quad \text{for } a < x < b,$$
$$\text{with } u(a) = g_a,\ u(b) = g_b.$$

(i) Test your code on the problem $[a, b] = [0, 1]$, $f(x) = \sin(\pi x)$, and $g_a = g_b = 0$. What solution $u(x)$ do you expect? Does your computed solution $\{U_i\}_{i=0}^{M+1}$ approximate the expected function well?

(ii) With part (i) at hand you have already almost implemented the Crank-Nicolson scheme for solving the time-dependent heat equation

$$\partial_t u(t,x) - p\partial_x^2 u(t,x) = f(t,x) \quad \text{for } a < x < b,$$
$$\text{with } u(a) = g_a,\ u(b) = g_b.$$

Modify your code accordingly and implement the Crank-Nicolson scheme. What happens when you run your code with $[a, b] = [0, 1]$, $f(t, x) = 0$, $g_a = g_b = 0$, and initial value $u(0, x) = \sin(\pi x)$. What do you observe? What happens if you change the right hand side to a time-dependent function $f(t, x) = \sin(\pi x)\sin(\pi t)$ instead?

Exercise II.

Implement the second-order explicit time-stepping scheme that we discussed for the wave equation: Given initial values U_i^0 at time t_0, and U_i^1 at time t_1 for $i = 1, \ldots, M$, create a sequence of approximations at time t_{n+1} by setting

$$U_i^{n+1} = 2\,U_i^n - U_i^{n-1} + c^2\,\frac{\tau^2}{h^2}\big(U_{i+1}^n - 2\,U_i^n + U_{i-1}^n\big),$$

for $i = 0, \ldots, M$, and homogeneous Dirichlet conditions: $U_0^{n+1} = U_{M+1}^{n+1} = 0$.

Warning: it is vital to choose the time step size τ small enough so that the CFL condition holds.

(i) Experiment with different initial values for $\{U_i^0\}$ and $\{V_i^0\}$. Describe what you observe. What happen when a wave hits the boundary? What happens if you choose the time step size too large?

Exercise III.

En plus. Implement the explicit first-order time-stepping scheme that we discussed for the wave equation: Given initial values U_i^0, V_i^0 for $i = 1, \dots, M$, create a sequence of approximations at time t_{n+1} by setting

$$\begin{aligned} U_i^{n+1} &= U_i^n - \frac{\tau}{2h}\big(V_{i+1}^n - V_{i-1}^n\big), \\ V_i^{n+1} &= V_i^n - c^2\frac{\tau}{2h}\big(U_{i+1}^n - U_{i-1}^n\big), \end{aligned}$$

for $i = 0, \dots, M$, where we assume *periodic* boundary conditions throughout. Concretely, for the equation with $i = 0$ set

$$U_{-1}^n = U_M^n, \quad \text{and} \quad V_{-1}^n = V_M^n,$$

and for $M+1$ simply set $U_{M+1}^{n+1} = U_0^{n+1}$ and $V_{M+1}^{n+1} = V_0^{n+1}$.

Warning: it is vital to choose the time step size τ small enough so that the CFL condition holds.

(i) Experiment with different initial values for $\{U_i^0\}$ and $\{V_i^0\}$. Describe what you observe. What happens if you choose the time step size too large?