Proof of the stability theorem

Let us start with the definition of the error:

$$\vec{e}^{\,n+1} = \vec{x}(t_{n+1}) - \vec{y}^{\,n+1}$$

$$= \vec{x}(t_n) + \tau \vec{F}(t_n, \vec{x}(t_n)) + \tau \vec{\pi}^{\,n} - \vec{y}^{\,n+1}$$

def. of π^n

$$= \vec{x}(t_n) + \tau \vec{F}(t_n, \vec{x}(t_n)) + \tau \vec{\pi}^{\,n} - \vec{y}^{\,n} - \tau \vec{F}(t_n, \vec{y}^{\,n})$$

construction of $\vec{y}^{\,n}$

$$= \vec{e}^{\,n} + \tau \left\{ \vec{F}(t_n, \vec{x}(t_n)) - \vec{F}(t_n, \vec{y}^{\,n}) \right\} + \tau \vec{\pi}^{\,n}$$

Taking the norm on both sides and using triangle inequality:

$$\|\vec{e}^{\,n+1}\| \le \|\vec{e}^{\,n}\| + \tau \|\vec{F}(t_n, \vec{x}(t_n)) - \vec{F}(t_n, \vec{y}^{\,n})\| + \tau \|\vec{\pi}^{\,n}\|$$

$$\le \|\vec{e}^{\,n}\| + \tau L \|\vec{x}(t_n) - \vec{y}^{\,n}\| + \tau \|\vec{\pi}^{\,n}\|$$

$\vec{F}$ is Lipschitz-continuous

and we arrive at $\boxed{\|\vec{e}^{\,n+1}\| \le \|\vec{e}^{\,n}\| + \tau L \|\vec{e}^{\,n}\| + \tau \|\vec{\pi}^{\,n}\|}$

Substitute recursively:

$$\|\vec{e}^{\,n+1}\| \le \underbrace{\|\vec{e}^{\,0}\|}_{=0} + \tau L \sum_{\nu=0}^{n} \|\vec{e}^{\,\nu}\| + \tau \sum_{\nu=0}^{n} \|\vec{\pi}^{\,\nu}\| \qquad (*)$$

Lemma (Grönwall):

Suppose we have nonnegative sequences $\{w_n\}$ and $\{b_n\}$, where b_n are increasing, and a constant $a > 0$ such that

$$w_0 \leq b_0 \quad \text{and} \quad w_{n+1} \leq a \sum_{\nu=0}^{n} w_\nu + b_{n+1} \quad \text{for } n \geq 0.$$

Then, $w_{n+1} \leq \exp((n+1)a)\, b_{n+1}$ for $n \geq 0$ (and $w_0 \leq b_0$).

Applying the lemma to our inequality (*) with

$$a = \tau L, \quad b_{n+1} = \tau \sum_{\nu=0}^{n} \|\vec{\tau}^{\,\nu}\| \quad \text{and} \quad w_n = \|\vec{e}^{\,n}\|$$

gives
$$\|\vec{e}^{\,n+1}\| \leq e^{(n+1)L\tau} \left\{ \tau \sum_{\nu=0}^{n} \|\vec{\tau}^{\,\nu}\| \right\}.$$

Proof of the Grönwall Lemma:

Let us set $S_{n+1} := a \sum_{\nu=0}^{n} w_\nu + b_{n+1}$. We now show $S_{n+1} \leq e^{(n+1)a} b_{n+1}$.

First of all, we have that $S_0 \leq b_0$. Now assume the statement holds true for n, meaning

$$S_n \leq e^{na} b_n \quad \text{and by assumption } w_n \leq S_n.$$

We then have $S_{n+1} - S_n = a\,w_n + b_{n+1} - b_n$.

$$\leadsto S_{n+1} \leq (1+a) S_n + b_{n+1} - b_n$$

$$\leq \underbrace{(1+a)}_{\downarrow} e^{na} b_n + \underbrace{b_{n+1} - b_n}_{>0}$$

$$\leq e^{a} e^{na} b_n + \underbrace{e^{(n+1)a}}_{\geq 1} (b_{n+1} - b_n)$$

$$\leq e^{(n+1)a} b_{n+1}$$

This implies that $w_{n+1} \leq S_{n+1} \leq e^{(n+1)a} b_{n+1}$ for all $n \geq 0$

$\square$