Finite difference schemes for time-dependent problems, part I

In the previous lecture we have seen how to approximate the solution to $-p\,\partial_x^2 u + q\,\partial_x u + ru = f$.

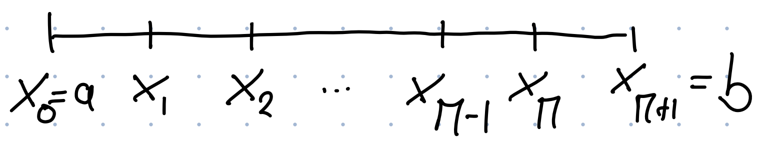
Let us now discuss how to solve the time-dependent problem

$$\partial_t u - p\,\partial_x^2 u + q\,\partial_x u + ru = f.$$

For simplicity we will set $q=0$ and $r=0$ and only discuss the "diffusion" equation $\partial_t u - p\,\partial_x^2 u = f$. (You can simply add back the missing terms from the previous lecture...)

Recall that for the spatial discretization we chose

- <u>Spatial discretization</u>: Subdivide interval $\Omega = [a,b]$ into $M+1$ subintervals of equal size: $x_i = a + ih$, $h = \dfrac{b-a}{M+1}$



$\boxed{\text{index } i}$

We will now augment this by discretizing in time:

Temporal discretization: Subdivide interval $I = [t_0, T]$ into N subintervals of equal size: $t_n = t_0 + \tau n$, $\tau = \frac{T - t_0}{N}$

$t_0 \quad t_1 \quad t_2 \quad \cdots \quad t_{N-2} \quad t_{N-1} \quad t_N = T$

$\boxed{\text{index } n}$

And we approximate now: $U_i^n \approx u(t_n, x_i)$

$$\partial_t u - p\,\partial_x^2 u = f$$

forward Euler:

$$\frac{U_i^{n+1} - U_i^n}{\tau} \underbrace{- \frac{p}{h^2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)}_{\text{at time } t_n} = \underbrace{f(t_n, x_i)}_{\text{at time } t_n}$$

Combining this with boundary and initial conditions:

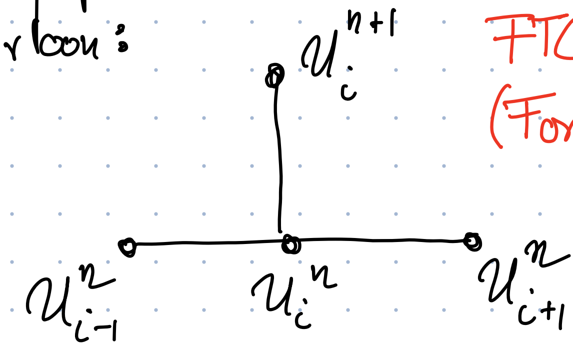
Task (Forward Euler)

Given $\{U_i^n\}_{i=0}^{M+1}$ at time t_n and boundary values g_a, g_b construct $\{U_i^{n+1}\}_{i=0}^{M+1}$ at time t_{n+1} by setting

$$(*) \quad U_i^{n+1} = U_i^n + p\frac{\tau}{h^2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) + \tau f(t_n, x_i)$$

and $U_0^{n+1} = g_a$ and $U_{M+1}^{n+1} = g_b$

This is again an explicit time-marching scheme. We can simply compute u_i^{n+1} without the need to solve a linear system! Cartoon:



FTCS
(Forward time centered space)

But, it is only first-order in time (in contrast to second order in space). Worse, there is another problem: For this method to work we need a time step size restriction:

τ has to be chosen such that $\tau \leq \frac{1}{2p} h^2$.
"parabolic CFL condition"

named after Richard Courant, Kurt Friedrichs, Hans Lewy

<u>Remark</u>: The CFL condition can be derived by analyzing carefully how (*) propagates and "amplifies" error: Say, the approximation $\{u_i^n\}$ at time t_n has the error $e_i^n = \underbrace{u(t_n, x_i)}_{\text{true solution}} - u_i^n$, then how large is the error of $\{u_i^{n+1}\}$ at time t_{n+1}, $e_i^{n+1} = u(t_{n+1}, x_i) - u_i^{n+1}$?

Such an analysis is known as "von Neumann stability analysis".
Cutting a lot of corners, let's assume that the error is given by a combination of "error modes"

$$\hat{e}_m^n = a^n e^{ikm}$$

a = amplification factor
k = wavenumber

($i \to$ writing m instead of i) (i = imaginary unit)

that each solve (*): $\hat{e}_m^{n+1} = (1-2p\tau/h^2)\hat{e}_m^n + p\tau/h^2(\hat{e}_{m-1}^n + \hat{e}_{m+1}^n)$.

Substituting:

$$a^{n+1}e^{ikm} \overset{!}{=} a^n e^{ikm} + p\tau/h^2 \underbrace{(e^{-ik}-2+e^{ik})}_{=-4\sin(k/2)} a^n e^{ikn}$$

and dividing by $a^n e^{ikn}$:

$$a = 1 - 4p\tau/h^2 \underbrace{\sin(k/2)}_{\geq 0}$$

≥ 0 (which we have not discussed)

We now want that the amplification factor a has an absolute value not greater than one: $|a| \overset{!}{\leq} 1$.

This implies that

$$1 \geq a \geq 1 - 4p\tau/h^2 \overset{!}{\geq} -1$$

always true $\quad$ sin is at most one

Rearranging: $4p\tau/h^2 \leq 2$. ∎

If we want to avoid this time-step size restriction we need to use an implicit time stepping method:

$$\partial_t u - p\,\partial_x^2 u = f$$

$$\frac{U_i^{n+1} - U_i^{n}}{\tau} - \cancel{\ast} = \cancel{\ast}$$

trapezoidal rule: half at time t_n and half at time t_{n+1} !!

$$\frac{U_i^{n+1} - U_i^{n}}{\tau} - \frac{p}{2h^2}\Big(\underbrace{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}_{\text{at time } t_{n+1}} + \underbrace{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}_{\text{at time } t_n} \Big)$$

$$= \frac{1}{2}\big(f(t_n, x_i) + f(t_{n+1}, x_i) \big)$$

Writing this out again:

Task (Crank-Nicolson scheme)

Given $\{U_i^n\}_{i=0}^{M+1}$ at time t_n and boundary values g_a, g_b construct $\{U_i^{n+1}\}_{i=0}^{M+1}$ at time t_{n+1} by solving

$$U_i^{n+1} - p\frac{\tau}{2h^2}\left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} \right)$$
$$= U_i^{n} + p\frac{\tau}{2h^2}\left(U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n} \right)$$
$$+ \frac{\tau}{2}\left\{ f(t_n, x_i) + f(t_{n+1}, x_i) \right)$$

for $i = 1, \dots, M$

and $U_0^{n+1} = g_a$ and $U_{M+1}^{n+1} = g_b$

Remark: In this case the stability analysis results in

$$a\left(1+2p\,\tau/h^2\sin(k/2)\right) = \left(1-2p\,\tau/h^2\sin(k/2)\right)$$

$$\Rightarrow \quad |a| = \left|\frac{1-2p\,\tau/h^2\sin(k/2)}{1+2p\,\tau/h^2\sin(k/2)}\right| \le 1 \quad \text{unconditionally.}$$

Cartoon for Crank-Nicolson scheme: