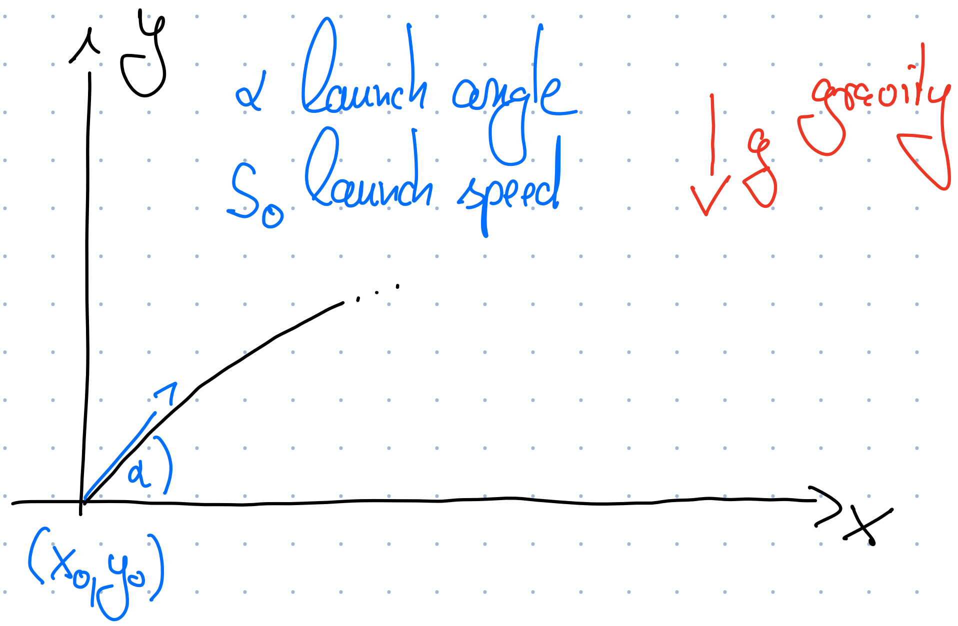
Introduction & Ordinary Differential Equations

Warmup: Trajectory of a projectile, revisited



[a] Newton's laws of motion

Projectile at time t: horizontal/vertical position: $x(t)/y(t)$

horizontal/vertical velocity: $v_x(t)/v_y(t)$

$$\begin{cases} \frac{d}{dt}x(t) = v_x(t), & \frac{d}{dt}v_x(t) = 0, \\ \frac{d}{dt}y(t) = v_y(t), & \frac{d}{dt}v_y(t) = -g, \\ x(0) = x_0, & v_x(0) = v_{x0} \\ y(0) = y_0, & v_y(0) = v_{y0} \end{cases}$$

"first principles"

modeling assumption: constant gravity

... of course this system can be solved by hand:

$$x(t) = x_0 + v_{x,0}\, t, \quad y(t) = y_0 + v_{y,0}\, t - \frac{1}{2} g t^2$$

<u>Typical optimization problem:</u> by varying α, maximize distance travelled.

- With $v_{x,0} = S_0 \cos\alpha$ and $v_{y,0} = S_0 \sin\alpha$ we can find $t^* > 0$ with $y(t^*) = 0$ $\Rightarrow$ $0 \stackrel{!}{=} y_0 + S_0 \sin\alpha\, t - \frac{1}{2} g t^2$

$$\Rightarrow \quad t^* = \frac{S_0 \sin\alpha + \sqrt{S_0^2 \sin^2\alpha + 2 g y_0}}{g}$$

$$D(\alpha) := x(t^*) = x_0 + S_0 \cos\alpha \, \frac{S_0 \sin\alpha + \sqrt{S_0^2 \sin^2\alpha + 2 g y_0}}{g}$$

- Necessary condition for optimum $\frac{d}{d\alpha} D(\alpha) \stackrel{!}{=} 0$:

For $y_0 = 0$: $\quad 0 \stackrel{!}{=} \frac{2 S_0^2}{g} \left(-\sin^2\alpha + \cos^2\alpha \right)$

which has solution $\alpha = 45°$

5 ... and subject to air resistance

Model: Newtonian drag

Air resistance leads to a force acting on the projectile opposing the movement:

$$\vec{F} = -\frac{1}{2} m \mu \|\vec{v}\| \vec{v}$$

mass

drag coefficient

force

Here, $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ and $\|\vec{v}\| = \sqrt{|v_x|^2 + |v_y|^2}$

This leads to

$$\begin{cases} \frac{d}{dt} x(t) = v_x(t), & \frac{d}{dt} v_x(t) = -\frac{1}{2}\mu \|\vec{v}\| v_x \\ \frac{d}{dt} y(t) = v_y(t), & \frac{d}{dt} v_y(t) = -g - \frac{1}{2}\mu \|\vec{v}\| v_y \\ x(0) = x_0, & v_x(0) = v_{x0} \\ y(0) = y_0, & v_y(0) = v_{y0} \end{cases}$$

Key difference: This ODE does not admit a »closed-form« solution.

↳ For all practical purposes we have to approximate solutions to (*)

Background: Initial value problems

Definition: An initial value problem is the task to find a differentiable function $\vec{x}(t): I \to \mathbb{R}^d$ such that

$(t_0 \in I)$ $$\frac{d}{dt}\vec{x}(t) = \vec{F}(t, \vec{x}(t)), \quad \vec{x}(t_0) = \vec{x}_0,$$ for given

$(*)$ initial value $\vec{x}_0 \in \mathbb{R}^d$ and RHS $\vec{F}: I \times \mathbb{R}^d \to \mathbb{R}^d$.

Theorem (Cauchy-Peano): Let $\vec{F}$ be a continuous function. Then, provided I is small enough, there exists a solution to the IVP $(*)$

Definition: The function $\vec{F}: I \times \mathbb{R}^d \to \mathbb{R}^d$ is said to satisfy a uniform Lipschitz condition if there exists $L > 0$ such that

$$\|\vec{F}(t, \vec{x}) - \vec{F}(t, \vec{y})\| \leq L \|\vec{x} - \vec{y}\| \quad \text{for all } \vec{x}, \vec{y} \in \mathbb{R}^d$$

Theorem (Picard-Lindelöf):

If $\vec{F}$ also satisfies a uniform Lipschitz condition then the solution to IVP (*) is unique.

Important:

(a) The interval might be very small!

Example: $\frac{d}{dt}x(t) = (x(t))^2$, $x(1) = 1$ has solution

$$y(t) = \frac{1}{2-t} \text{ with } I = (-\infty, 2)$$

"finite time blowup"

(b) These results are strict: Non-uniqueness!

Example: $\frac{d}{dt}x(t) = 2\sqrt{x(t)}$, $x(0) = 0$ has infinitely many solutions: Fix $c \geq 0$ and set

$$x(t) = \begin{cases} 0 & \text{for } t \leq c, \\ (t-c)^2 & t > c. \end{cases}$$

(c) And even when a unique solution exists for all times the system might be incredibly hard to solve:

Example: <u>The Lorenz system</u>

Named after Edward Norton Lorenz 1917-2008, pioneer of the mathematical discipline of chaos theory.

"A simple weather model"

$$\begin{cases} \text{Find } x(t), y(t), z(t) \text{ solving} \\ \frac{d}{dt} x = \sigma(y-x), \quad \frac{d}{dt} y = x(\rho - z), \quad \frac{d}{dt} z = xy - \beta z \\ \text{with } \sigma = 10, \ \beta = 8/3 \text{ and } \rho = 28 \end{cases}$$