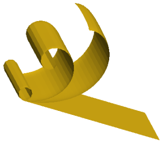
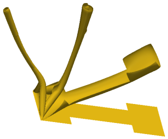
Curved Origami

Andrea Bonito



Department of Mathematics
Texas A&M University



AMUSE Seminar • Texas A&M University • Sept. 13, 2023

OUTLINE

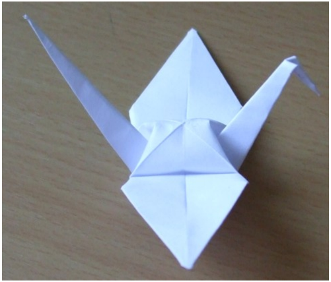
OUTLINE

Traditional Origami

Origami: means "paper to fold" and is the Japanese are of folding paper.

Dates: First known book in 1797, Black forest Cuckoo clock in 1987.

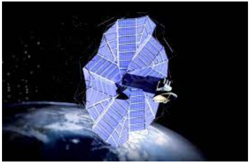
Mathematical Theory: Possible folds, see Robert Lang TED's talk.



Japanese crane
(folded by Anita Raj)

Black Forest Cuckoo Clock
(folded and designed by Robert J. Lang)

Applications



Space Solar Panels
[NSF.GOV]

Biomedical Stents
[MDDIONLINE.COM]

Bullet-proof shields
[BYU]

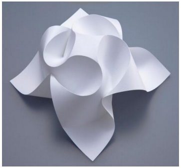
OUTLINE

Applications

Nature & Art:

Venus flytrap
[Wikipedia]

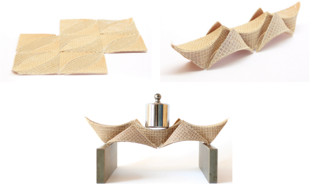
Ladybird's wing
[University of Tokyo]



Curved Origami
[Jun Mitani]

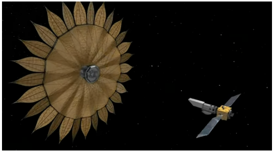
Applications II

Curved folding for flat packed product design



[A. Menges et al. (2020)]

Space Origami: NASA exoplanet program - Starshade



[exoplanets.nasa.gov]

Applications III

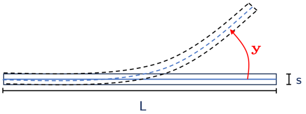
What about this?

i'm lovin'it

OUTLINE

2D Bending Model - Thin Structures

Domain: $\Omega_s = [0, L] \times (-s/2, s/2) \subset \mathbb{R}^2$ with thickness s and midline $[0, L]$;



2d deformation: $\mathbf{u} : \Omega_s \to \mathbb{R}^2$, $\nabla \mathbf{u} : \Omega_s \to \mathbb{R}^{2\times 2}$.

Simplified St-Venant Scaled hyperelastic energy:

$$E^s[\mathbf{u}] = \frac{1}{4s^3} \int_{\Omega_s} |\nabla \mathbf{u}^T \nabla \mathbf{u} - I_2|^2 dx_1 dx_2$$

1d deformation: $\mathbf{y} : [0, L] \subset \mathbb{R} \to \mathbb{R}^2$; We are interested in the limit as $s \to 0^+$.

Modified Kirchhoff-Love assumption:

$$\mathbf{u}(x_1, x_2) = \mathbf{y}(x_1) + x_2 \boldsymbol{\nu}(x_1) \qquad \boldsymbol{\nu} := \frac{(\mathbf{y}')^\perp}{|(\mathbf{y}')^\perp|}.$$

Plate Model: Simplified Asymptotics

Gradients: $\nabla \mathbf{u}(x_1, x_2) = [\mathbf{y}'(x_1) + x_2 \boldsymbol{\nu}'(x_1) \quad | \quad \boldsymbol{\nu}(x_1)]$

2d energy:

$$E^s[\mathbf{u}] = \frac{1}{4s^3} \int_0^L \int_{-s/2}^{s/2} \left| \begin{bmatrix} |\mathbf{y}'|^2 - 1 & 0 \\ 0 & \cancel{|\boldsymbol{\nu}|^2 - 1} \end{bmatrix} \right.$$

$$\left. + x_2 \begin{bmatrix} 2\boldsymbol{\nu}' \cdot \mathbf{y}' & \cancel{\boldsymbol{\nu}' \cdot \boldsymbol{\nu}} \\ \cancel{\boldsymbol{\nu} \cdot \boldsymbol{\nu}'} & 0 \end{bmatrix} + x_2^2 \begin{bmatrix} |\boldsymbol{\nu}'|^2 & 0 \\ 0 & 0 \end{bmatrix} \right|^2 dx_1 dx_2.$$

First and second fundamental forms:

$$I[\mathbf{y}] := |\mathbf{y}'|^2 \quad II[\mathbf{y}] := -\boldsymbol{\nu}' \cdot \mathbf{y}' = \mathbf{y}'' \cdot \boldsymbol{\nu}.$$

Reduced energy: As $s \to 0^+$, $\mathbf{y}$ satisfies $I\mathbf{y}' = 1$ (isometry constraint)

$$E[\mathbf{y}] := \frac{1}{2} \int_0^L |\mathbf{y}''|^2 dx_1.$$

Equilibrium Shapes: Minimize bending!

$$\min_{\mathbf{y}\,:\,|\mathbf{y}'|^2 = 1} E[\mathbf{y}].$$

Minimization Process:

Energie: $E[\mathbf{y}] = \frac{1}{2}\int_0^L |\mathbf{y}''|^2$.

Gâteau Derivative:

$$\delta E(\mathbf{y})(\mathbf{z}) := \lim_{t\to 0} \frac{E(\mathbf{y}+t\mathbf{z}) - E(\mathbf{y})}{t} = \int_0^L \mathbf{y}'' \cdot \mathbf{z}'' dx_1 = \int_0^L \mathbf{y}^{(iv)} \cdot \mathbf{z} dx_1.$$

with appropriate boundary conditions on $\mathbf{y}$ and $\mathbf{z}$.

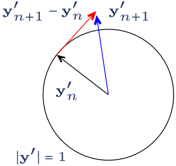
Gradient:

$$\nabla E(\mathbf{y}) := \mathbf{y}^{(iv)}.$$

Gradient Descent: For $\tau > 0$, if one has $\mathbf{y}_n$, find $\mathbf{y}_{n+1}$ so that

$$\mathbf{y}_{n+1} = \mathbf{y}^n - \tau \nabla E(\mathbf{y}_{n+1}), \qquad \text{i.e.} \qquad \mathbf{y}_{n+1} + \tau \mathbf{y}_{n+1}^{(iv)} = \mathbf{y}^n.$$

Isometry Constraint: $|\mathbf{y}_{n+1}|^2 = 0$ is too difficult, instead we impose



$$(\mathbf{y}'_{n+1} - \mathbf{y}'_n) \cdot \mathbf{y}'_n = 0$$

Taylor expansion:

$$f(\mathbf{y}^{n+1}) \approx f(\mathbf{y}^n) + Df(\mathbf{y}^n)(\mathbf{y}^{n+1} - \mathbf{y}^n).$$

What can we prove?

Modeling: $E = \Gamma - \lim_{s \to 0} E^s$. If x_k is a global minimizer of $E^{1/k}$, then there is a subsequence (x_{n_k}) converging to a minimizer of E.

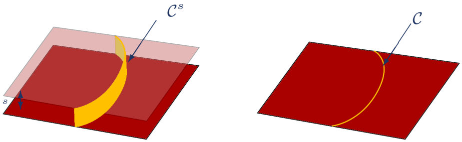
Finite Element Approximation Convergence: Use piecewise quadratics to approximate $\mathbf{y}$.

Isometry defect:

$$\left| |\mathbf{y}_n'|^2 - 1 \right| \leq C\tau, \qquad \forall n \geq 0.$$

Folds in $\mathbb{R}^3$

3D/2D models:



$\Omega \times (-s/2, s/2)$ Ω

Reduced Energy: $\mathbf{y} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ satisfies $\nabla \mathbf{y}^T \nabla \mathbf{y} = I_2$, and

$$E_{\mathcal{C}}[\mathbf{y}] = \frac{1}{2} \int_{\Omega \setminus \mathcal{C}} \left| D^2 \mathbf{y} \right|^2 \, d\mathbf{x}'.$$

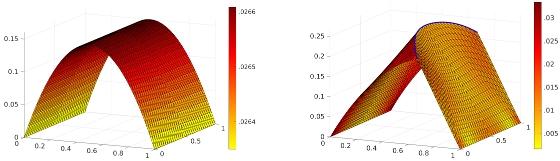
Γ**-convergence:** This limit is rigorous [Bartels, Bonito, Hornung (2022)].

Discretization

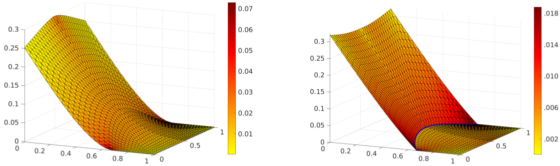
Need Math 437 and Math 610.

Experiments: Effect of crease (linearized case)

Test 1: Two sides fixed, no vs. quadratic interface



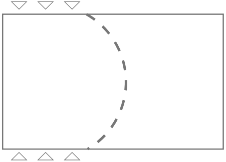
Test 2: One side clamped, one boundary point fixed, no vs. quadratic interface

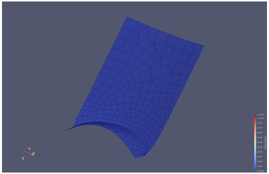


Observation: Crease significantly affects bending behavior [Bartels, Bonito, and Tscherner (2023)].

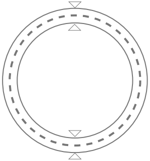
Experiments (nonlinear case)

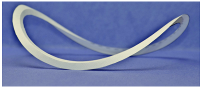
Test 1: Curved arc on rectangular plate with compressive BCs





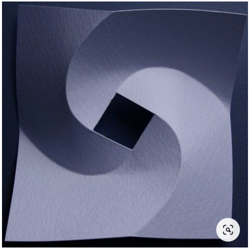
Test 2: Ring with central folding curve compressed at two opposite points



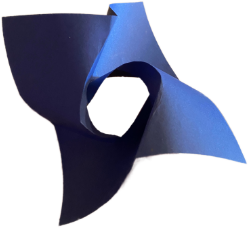


[Dias et al. (2012)]

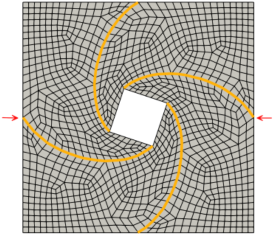
Origami Flower

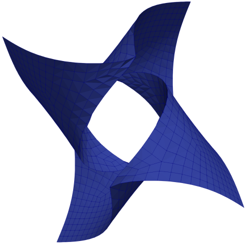


[Pinterest, Curved Fold Study, Prof. YM]



[Bonito's lab]





[Bartels, Bonito, and Hornung (2022)]

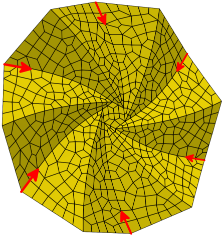
Space origami

Starshade project from NASA/JPL-Caltech:

→ search for life on exoplanets;

→ use a starshade between the star and the telescope to block the light;

→ the starshade need to be folded when launched into space to fit in the rocket.



Source: exoplanets.nasa.gov

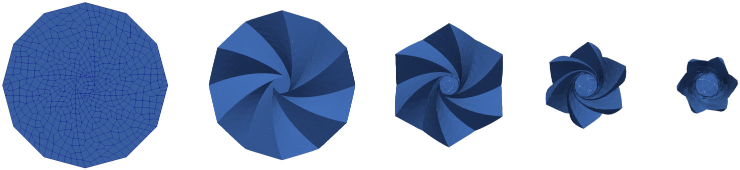
Space origami

Starshade project from NASA/JPL-Caltech:

→ search for life on exoplanets;

→ use a starshade between the star and the telescope to block the light;

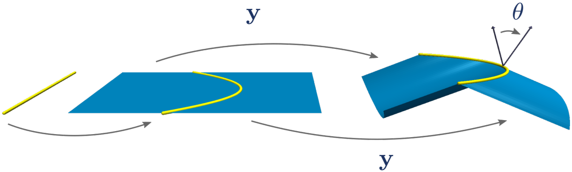
→ the starshade need to be folded when launched into space to fit in the rocket.



[BONITO, GUIGNARD, MORVANT (2023)]

Curvatures: *Geodesic curvature* k_g (maintained under isometric deformations) and *normal curvature* μ^1, μ^2 curvatures are related via

$$\boxed{k_g \sin\left(\frac{\theta}{2}\right) = \pm\mu^\ell \cos\left(\frac{\theta}{2}\right)}$$



McFries Box

$$\boxed{k_g \sin\left(\frac{\theta}{2}\right) = \pm\mu^{\ell} \cos\left(\frac{\theta}{2}\right)}$$

Implications: For deformed plate along crease $\mathcal{C}$

- if $k_g = 0$ then either unfolded or folded back or θ constant and $\mu^{\ell} = 0$
- if $k_g \neq 0$ then either unfolded or $\mu^{\ell} \neq 0$ and θ uniquely defined

Related: Pogorelov theorem?

OUTLINE

Origami

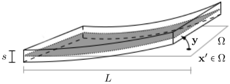
Curved Origami

Plate Formulations

Extension to Bilayer Plates

Conclusions

Bilayer Plates: 3d Bending Model - no Crease



Bilayer: $\{x \in \Omega_s : \pm x_3 > 0\}$ react differently to external actuation.

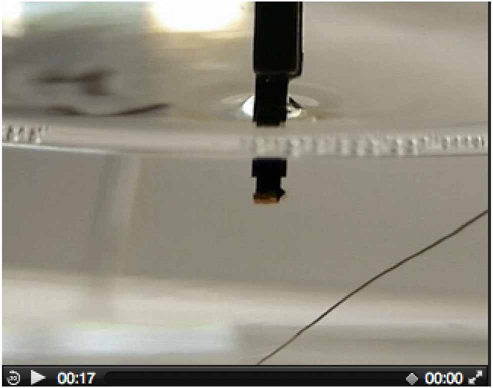
Reduced bending energy: $\mathbf{y}$ satisfies $(\nabla \mathbf{y})^T \nabla \mathbf{y} = I_2$ and

$$\lim_{s \to 0} E^s[\mathbf{u}] \approx \frac{1}{2} \int_\Omega \left| D^2 \mathbf{y} \cdot \boldsymbol{\nu} - Z \right|^2 d\mathbf{x}',$$

with Z reflects the material mismatch (spontaneous curvature).

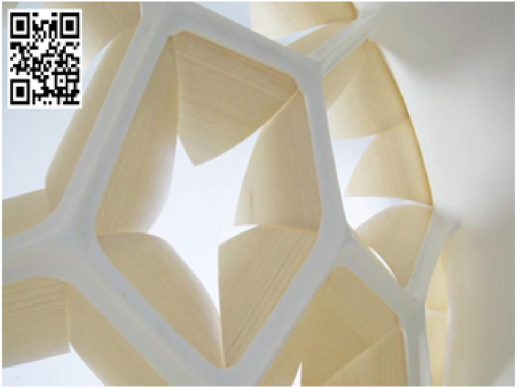
Experiment 4: Microhand and Hair

The actuators move from completely flat to fully curled and back (to/from fully oxidized to/from fully reduced) in about 1 second (the bilayer is 0.5 μm thick).



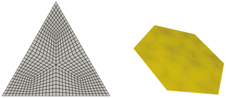
E. Smela, O. Inganäs, and I. Lundström, *Controlled folding of micrometer-size structures*, Science, 268 (1995), 1735–1738.

FRAC Centre Orléans[1,2,3]

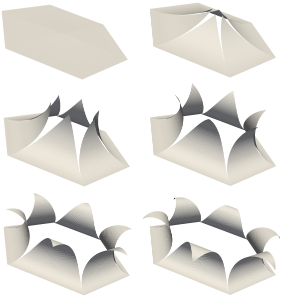


[1] http://icd.uni-stuttgart.de ; Achim Menges
[2] [Reichert, Menges, and Correa (2015)]
[3] [Menges and Reichert (2015)]

Hygroskin

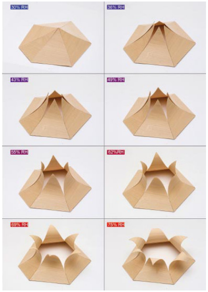


$$Z = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \qquad \text{with } i = 0, 1, 2, 3, 4, 5.$$



Numerical simulations

[Bonito, Nochetto, Yang (2023+)]

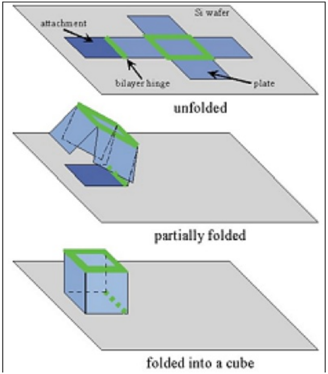


Laboratory experiments

[Menges and Reichert (2012)]

Laboratory Experiments of E. Smela (Mech. Eng., UMD)

Conducting layers of polypyrrole (polymer) and gold (Au) were used as hinges to connect ridig plates to each other and to a Si substrate. The bending of the hinges was electrically controlled. [4]



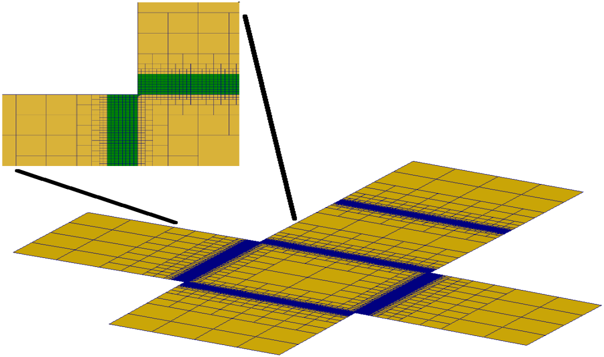
[4] [Jager, Smela, and Inganäs (2000)]

Self-Assembling Composite-Material Box

Domain Ω: 6 squares of size 1x1;

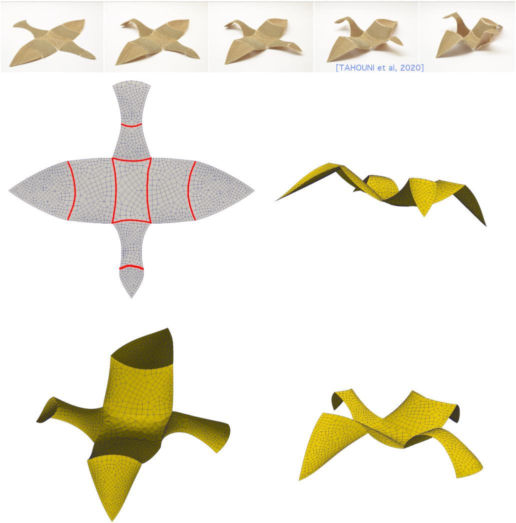
Bilayer Hinges: width $\pi/24$ and activated by heat diffusing in the device

Spontaneous curvature: $Z \propto$ temperature. [5]



[5] [Bartels, Bonito , Muliana, Nochetto (2018)]

Mixing it up: Bilayer Plates with Folding[6]



[6] Experiment: [Tahouni et al (2020)]; Simu: [Bonito, Guignard, Morvant (2023)]

OUTLINE

Conclusions

Motivations:

- **Properties:** Rigidity, Robustness and ability to deploy rigid and wrapping structures with relatively small energy
- **Applications:** Airspace engineering, mechanical engineering, microengineering, material science and biomedical science.

Mathematical Skills:

- **Modeling**
- **Analysis:** calculus of variations, geometric analysis.
- **Numerical Analysis:** design of algorithm, analysis, implementations.

Software: Deal.ii

Support: NSF DMS-2110811

More info: `people.tamu.edu/~bonito`

Pine Cone (quora)

Almond leave with fungus
(Sharon - Efrati 2010)

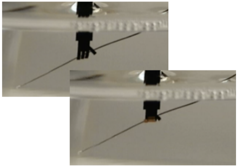
Hygroskin (FRAC Centre Orléan, A. Menges)

Venus Flytrap (Wiki)

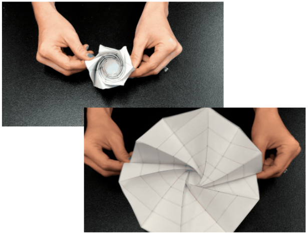
Rigid Box (McBonito)

Shield (BYU Photo)

Bandaid Kirigami (Ruike Zhao @ MIT)



Micro-gripper with human hair
(Smella Website @UMD)



Starshade (NASA-CalTech)

Support: DMS-2110811

Softwares: deal.ii and paraview

Thank you for your attention!

Venus Flytrap (Wiki)

Pine Cone (quora)

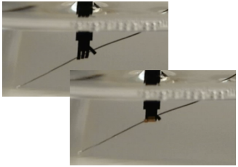
Rigid Box (McBonito)

Almond leave with fungus
(Sharon - Efrati 2010)

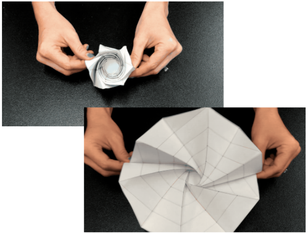
Shield (BYU Photo)

Hygroskin (FRAC Centre Orléan, A. Menges)

Bandaid Kirigami (Ruike Zhao @ MIT)



Micro-gripper with human hair
(Smella Website @UMD)



Starshade (NASA-CalTech)

Support: DMS-2110811

Softwares: deal.ii and paraview