FINITE ELEMENT METHOD - EXERCISES

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(1) For $N \in \mathbb{N}$, let $0 \leq x_0 < x_1 < ... < ...x_N = 1$ be a partition of $[0,1]$. For $i = 1, ..., N-1$, we define the hat function

$$\phi_i(x) := \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & x \in (x_{i-1}, x_i), \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x \in [x_i, x_{i+1}), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $\phi_i \in H_0^1(0,1)$ for $i = 1, ..., N-1$;

(b) Show that $\text{span}(\phi_1, ..., \phi_{N-1})$ contains all continuous piecewise linear functions subordinate to the partition $\{x_i\}_{i=0}^N$ and that vanish at $x = 0$ and $x = 1$.

(c) Compute $\int_0^1 \phi_i'(x)\phi_j'(x)\, dx$ for $i, j = 1, ..., N-1$.

(d) Consider now uniform partitions of $[0,1]$, i.e., $x_i = i/N$. Implement a FEM code to approximate the solution u of the ODE

$$-(p(x)u'(x))' = 1, \qquad 0 < x < 1 \qquad \text{and} \qquad u(0) = u(1) = 0,$$

where $p(x)$ is given by

$$p(x) = \begin{cases} 1 & x \in (0, 1/2) \\ \alpha & x \in [1/2, 1) \end{cases}$$

for some $\alpha > 0$. Plot your FEM approximations for $\alpha \in \{1, 2, 5, 10\}$ and N large enough.

(e) Show that the weak solution $u \in H_0^1(0,1)$ to the above ODE satisfies

$$J(u) \leq J(v) \qquad \forall v \in H_0^1(0,1),$$

where for $v \in H_0^1(0,1)$

$$J(v) := \frac{1}{2}\int_0^1 p(x)|v'(x)|^2 dx - \int_0^1 f(x)v(x)dx.$$

(2) Consider the same ODE but supplemented with mixed boundary conditions

$$-(p(x)u'(x))' = 1, \qquad 0 < x < 1 \qquad \text{and} \qquad u(0) = 0, \quad u'(1) = 1.$$

(a) Determine the corresponding weak formulation. Think first what are reasonable test functions to multiply the ODE with.

(b) Show that $\text{span}(\phi_1, ..., \phi_N)$, where

$$\phi_i(x) := \begin{cases} \frac{x-x_{N-1}}{x_N-x_{N-1}} & x \in (x_{N-1}, x_N], \\ 0 & \text{otherwise,} \end{cases}$$

contains all continuous piecewise linear functions subordinate to the partition $\{x_i\}_{i=0}^N$ and that vanish at $x = 0$.

(c) Modify your code to approximate the solution to the ODE with mixed boundary conditions.

Date: May 24, 2024.