Lecture 1: Parabolic equations

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1 Objectives

The goal of the lecture is to investigate the structure of the following scalar-valued initial boundary value problem: Find $u : \mathbb{R}\times\mathbb{R}_{\geq 0} \to \mathbb{R}$ so that

$$\partial_t u(x,t) - p\partial_{xx}u(x,t) + q\partial_x u(x,t) + ru(x,t) = f(x,t), \quad x \in \mathbb{R}, t \in \mathbb{R}_{>0} \tag{1a}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1b}$$

$$\lim_{x\to\pm\infty} u(x,t) = 0, \qquad \forall t \in \mathbb{R}_{>0}. \tag{1c}$$

Here p, q, and r are constants, f is a function that is square integrable in time and space. u_0 is a square integrable in space. We also assume that $p > 0$.

Remark 1.1 (The Devil is at infinity)**.** The statements $\lim_{x\to\pm\infty} u(x,t) = 0$ mean that the following holds $\lim_{N\to-\infty} u(N,t) = 0$ and $\lim_{N'\to+\infty} u(N',t) = 0$.

Remark 1.2 (Terminology)**.** We say that f is the *source* of the problem (1) and u_0 is the *initial data*. The conditions $\lim_{x\to-\infty} u(x,t) = 0$ and $\lim_{x\to+\infty} u(x,t) = 0$ are called *Dirichlet boundary conditions* at infinity.

Remark 1.3 (Terminology)**.** If one would formally replace ∂_t by t, ∂_x by x and ∂_{xx} by x^2, one would obtain $(t - px^2 + qx + r)u = f$ (the reason for doing this is rooted in Fourier analysis . . .). As we assumed $p > 0$, the two-variate polynomial $t - px^2 + qx + r$ generates a parabola. It is then customary to say that the problem (1) is *parabolic*.

Remark 1.4 (Domain and boundary conditions)**.** The problem (1) can be formulated over a finite interval in space, say $(-L, L)$, $L > 0$. The *Dirichlet boundary conditions* consist of setting $u(-L,t) = 0$ and $u(+L,t) = 0$. More will be said later on boundary conditions.

Remark 1.5 (Existence, uniqueness, stability)**.** The first three natural questions that come to mind when considering (1) are the following:

(i) Does the problem (1) has a solution? This is an *existence* question.

(ii) If a solution exists, is it unique? This is a *uniqueness* question.

(iii) If a solution exists, is the solution operator $(f, u_0) \mapsto u$ continuous in some sense? In other words, if two source/initial data pairs (f^1, u_0^1), (f^2, u_0^2) are close, are the corresponding solutions u^1, u^2 close? This is a *stability/continuity* question.

It turns out that the *existence* question is the most difficult one as it requires some knowledge in real analysis (Lebesgue integration, Hilbert spaces, Bochner spaces). The easiest question is that concerning the *stability/continuity*. Hence, we are going investigate first, the stability, then the uniqueness, and finally say a few words regarding existence.

2 Energy method

The *energy method* is a principle that is useful to derive *a priori estimate* from partial differential equations (PDEs). The idea is to multiply the PDE by the solution itself or some other quantity (it could be the time derivative, a space derivative, a power of the solution, or any other functional involving the solution), integrate over time and space, integrate by parts so as to obtain only terms with precise signs (positive or negative), and conclude something useful facts from this manipulation.

Here are some techniques that we are going to use repeatedly.

2.1 Leibniz's rule

$$\partial_t(vw) = w\partial_t v + v\partial_t w, \tag{2a}$$

$$\partial_x(vw) = w\partial_x v + v\partial_x w. \tag{2b}$$

As an application we have

$$v\partial_t v = \frac{1}{2}\partial_t v^2, \tag{3a}$$

$$v\partial_x v = \frac{1}{2}\partial_x v^2. \tag{3b}$$

2.2 Fundamental theorem of calculus

$$\int_a^b \partial_x v(x,t)\,\mathrm{d}x = v(b,t) - v(a,t). \tag{4}$$

2.3 Integration by parts

Combining Leibniz's rule and the fundamental theorem of calculus, we obtain

$$\int_a^b v\partial_t w\,\mathrm{d}t = -\int_a^b w\partial_t v\,\mathrm{d}t + v(b)w(b) - v(a)w(a). \tag{5a}$$

$$\int_a^b v\partial_x w\,\mathrm{d}x = -\int_a^b w\partial_x v\,\mathrm{d}x + v(b)w(b) - v(a)w(a). \tag{5b}$$

2.4 Commutation

$$\int_{\mathbb{R}>0} \partial_x v(x,t)\,\mathrm{d}t = \partial_x \int_{\mathbb{R}>0} v(x,t)\,\mathrm{d}t \tag{6a}$$

$$\int_{\mathbb{R}} \partial_t v(x,t)\,\mathrm{d}x = \partial_t \int_{\mathbb{R}} v(x,t)\,\mathrm{d}x. \tag{6b}$$

2.5 Cauchy-Schwarz inequality

Let us set $\|v\|_{L^2(\mathbb{R})} := \left(\int_{\mathbb{R}} v^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}$. Then

$$\left|\int_{\mathbb{R}} v(x)w(x)\,\mathrm{d}x\right| \leq \|v\|_{L^2(\mathbb{R})}\|w\|_{L^2(\mathbb{R})}, \tag{7a}$$

$$\left|\int_{\mathbb{R}} v(x,t)w(x,t)\,\mathrm{d}x\right| \leq \|v(\cdot,t)\|_{L^2(\mathbb{R})}\|w(\cdot,t)\|_{L^2(\mathbb{R})}. \tag{7b}$$

2.6 Young's inequality

For all $a \geq 0$, $b \geq 0$, $\gamma > 0$

$$ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2. \tag{8}$$

3 Stability

Let us assume that a solution to (1) exists. This statement is quite vague at the moment, since we have to state what the smoothness of the solution that we claim exists is. We are going to discover a smoothness class for which existence is ensured as we investigate stability and uniqueness. We are going to use the *energy method* to do that.

3.1 L^2-estimates

Let us apply the *energy method* (1a). We multiply (1a) by u and integrate over space:

$$\int_{\mathbb{R}} \big(u(x,t)\partial_t u(x,t) - pu(x,t)\partial_{xx}u(x,t) + qu(x,t)\partial_x u(x,t) + ru^2(x,t)\big)\,\mathrm{d}x = \int_{\mathbb{R}} u(x,t)f(x,t)\,\mathrm{d}x,$$

Using the techniques describes above, we have

$$\int_{\mathbb{R}} \left(\partial_t(\frac{1}{2}u^2(x,t)) - pu(x,t)\partial_{xx}u(x,t) + q\partial_x(\frac{1}{2}u^2)(x,t) + ru^2(x,t)\right)\,\mathrm{d}x = \int_{\mathbb{R}} u(x,t)f(x,t)\,\mathrm{d}x,$$
$$\partial_t \int_{\mathbb{R}} \frac{1}{2}u^2(x,t)\,\mathrm{d}x + \int_{\mathbb{R}} \left(-pu(x,t)\partial_{xx}u(x,t) + q\partial_x(\frac{1}{2}u^2)(x,t) + ru^2(x,t)\right)\,\mathrm{d}x = \int_{\mathbb{R}} u(x,t)f(x,t)\,\mathrm{d}x.$$

Using integration by parts we have

$$\begin{aligned}\lim_{N,N'\to+\infty} \int_{-N'}^{N} -u(x,t)\partial_{xx}u(x,t)\,\mathrm{d}x &= \lim_{N,N'\to+\infty} \int_{-N'}^{N} \partial_x u(x,t)\partial_x u(x,t)\,\mathrm{d}x \\ &\quad - \lim_{N,N'\to+\infty} u(N,t)\partial_x u(N,t) - u(N',t)\partial_x u(N',t).\end{aligned}$$

Now we invoke the boundary conditions and get

$$\lim_{N,N'\to+\infty} \int_{-N'}^{N} -u(x,t)\partial_{xx}u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} (\partial_x u(x,t))^2\,\mathrm{d}x.$$

(Note in passing that for this to make sense we have to assume that the solution is smooth enough so that the product $u(N,t)\partial_x u(N,t)$ does converge to zero as $N \to +\infty \ldots$) Similarly we have

$$\lim_{N,N'\to+\infty} \int_{-N'}^{N} \partial_x(\frac{1}{2}u^2)(x,t)\,\mathrm{d}x = \lim_{N,N'\to+\infty} \frac{1}{2}u^2(N,t) - \frac{1}{2}u^2(N',t) = 0.$$

We are left with

$$\partial_t \int_{\mathbb{R}} \frac{1}{2} u^2(x,t)\,\mathrm{d}x + \int_{\mathbb{R}} p(\partial_x u(x,t))^2 + r u^2(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} u(x,t) f(x,t)\,\mathrm{d}x.$$

Let us assume that $r \geq 0$. The above identity together with the Cauchy-Schwarz inequality implies

$$\frac{1}{2}\partial_t \|u(\cdot,t)\|^2_{L^2(\mathbb{R})} + r\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} u(x,t) f(x,t)\,\mathrm{d}x \leq \|u(\cdot,t)\|_{L^2(\mathbb{R})}\|f(\cdot,t)\|_{L^2(\mathbb{R})}.$$

Observing that $\frac{1}{2}\partial_t \|u(\cdot,t)\|^2_{L^2(\mathbb{R})} = \|u(\cdot,t)\|_{L^2(\mathbb{R})}\partial_t \|u(\cdot,t)\|_{L^2(\mathbb{R})}$ (by Leibniz's rule), we infer that

$$\partial_t \|u(\cdot,t)\|_{L^2(\mathbb{R})} + r\|u(\cdot,t)\|_{L^2(\mathbb{R})} \leq \|f(\cdot,t)\|_{L^2(\mathbb{R})}.$$

Dropping the term $r\|u(\cdot,t)\|_{L^2(\mathbb{R})}$ and integrating in time over the interval $(0,T)$, this gives, the following *a priori* L^2-estimate

$$\boxed{\|u(\cdot,T)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})} + \int_0^T \|f(\cdot,t)\|_{L^2(\mathbb{R})}\,\mathrm{d}t.} \tag{9}$$

where we used $\|u(\cdot,0)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R}}$.

3.2 Refined L^2-estimates

Let us assume that $r > 0$. Then the above computations gives

$$\partial_t(e^{rt}\|u(\cdot,t)\|_{L^2(\mathbb{R})}) = e^{rt}\partial_t \|u(\cdot,t)\|_{L^2(\mathbb{R})} + re^{rt}\|u(\cdot,t)\|_{L^2(\mathbb{R})} \leq e^{rt}\|f(\cdot,t)\|_{L^2(\mathbb{R})}.$$

Integrating in time over the interval $(0,T)$, this gives the following refined *a priori* L^2-estimate

$$\boxed{\|u(\cdot,T)\|_{L^2(\mathbb{R})} \leq e^{-rT}\|u_0\|_{L^2(\mathbb{R})} + \int_0^T e^{r(t-T)}\|f(\cdot,t)\|_{L^2(\mathbb{R})}\,\mathrm{d}t.} \tag{10}$$

3.3 Another L^2-estimates

Let us assume again that $r > 0$. Let us use Young's inequality,

$$\int_{\infty} u(x,t) f(x,t)\,\mathrm{d}x \leq \|u(\cdot,t)\|_{L^2(\mathbb{R})}\|f(\cdot,t)\|_{L^2(\mathbb{R})} \leq \frac{r}{2}\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2r}\|f(\cdot,t)\|^2_{L^2(\mathbb{R})}.$$

Then we obtain

$$\frac{1}{2}\partial_t\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} + p\|\partial_x u(\cdot,t)\|^2_{L^2(\mathbb{R})} + r\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} \le \frac{r}{2}\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2r}\|f(\cdot,t)\|^2_{L^2(\mathbb{R})},$$

and we arrive at

$$\frac{1}{2}\partial_t\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} + p\|\partial_x u(\cdot,t)\|^2_{L^2(\mathbb{R})} + r\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} \le \frac{1}{2r}\|f(\cdot,t)\|^2_{L^2(\mathbb{R})}.$$

Integrating over the time interval $(0,T)$, we obtain another *a priori estimate*:

$$\boxed{\begin{aligned}&\frac{1}{2}\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} + \int_0^T \left(p\|\partial_x u(\cdot,t)\|^2_{L^2(\mathbb{R}} + r\|u(\cdot,t)\|^2_{L^2(\mathbb{R}}\right)\,\mathrm{d}t\\ &\qquad\le \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R})} + \frac{1}{2r}\int_0^T \|f(\cdot,t)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t.\end{aligned}} \tag{11}$$

3.4 Stability/Continuity

Let us consider two pairs of sources and initial data (f^1, u_0^1), (f^2, u_0^2). Let u^1 and u^2 be the corresponding solutions to (1). Let $\phi := u^1 - u^2$, then linearity implies that

(12a)
$$\partial_t\phi(x,t) - p\partial_{xx}\phi(x,t) + q\partial_x\phi(x,t) + r\phi(x,t) = f^1(x,t) - f^2(x,t), \quad x\in\mathbb{R}, t\in\mathbb{R}_{>0}$$

$$\phi(x,0) = u_0^1(x) - u_0^2, \qquad x\in\mathbb{R}, \tag{12b}$$

$$\lim_{x\to\pm\infty}\phi(x,t) = 0, \qquad \forall t\in\mathbb{R}_{>0}. \tag{12c}$$

We can then use the *a priori* estimate (9) to infer that

(13)
$$\boxed{\|(u^1-u^2)(\cdot,T)\|_{L^2(\mathbb{R})} = \|\phi(\cdot,t)\|_{L^2(\mathbb{R})} \le \|u_0^1 - u_0^2\|_{L^2(\mathbb{R})} + \int_0^T \|(f^1-f^2)(\cdot,t)\|_{L^2(\mathbb{R})}\,\mathrm{d}t.}$$

We have discovered a notion of continuity for the solution operator. The inequality (13) shows that if $\|u_0^1 - u_0^2\|_{L^2(\mathbb{R})}$ and $\int_0^T \|(f^1-f^2)(\cdot,t)\|_{L^2(\mathbb{R})}\,\mathrm{d}t$ are both small, then the difference $\|(u_1-u_2)(\cdot,T)\|_{L^2(\mathbb{R})}$ is small. Moreover, let us consider the source and initial data (f,u_0), and consider a sequence of pairs $(f^n,u_0^n)_{n\in\mathbb{N}}$. Then if $\lim_{n\to+\infty}\int_0^T \|(f^n-f)(\cdot,t)\|_{L^2(\mathbb{R})}\,\mathrm{d}t = 0$ and $\lim_{n\to+\infty}\|u_0^n - u_0\|_{L^2(\mathbb{R})} = 0$, then $\lim_{n\to+\infty}\|(u^n-u)(\cdot,T)\|_{L^2(\mathbb{R})} = 0$. A similar argument holds with the estimate (10).

Many more *a priori* estimates can be deduced for the solution to (13).

Remark 3.1 (Smoothness class for existence)**.** The estimate (11) shows that a good candidate for a smoothness class where the existence of a solution could be established is the space composed of the functions $v : \mathbb{R}\times\mathbb{R}_{>0} \to \mathbb{R}$ for which $\mathbb{R}_{>0} \ni t \mapsto \|v(\cdot,t)\|_{L^2(\mathbb{R})} \in \mathbb{R}$ is continuous and the following quantity is bounded: $\int_0^T \left(p\|\partial_x u\|^2_{L^2(\mathbb{R})} + r\|u\|^2_{L^2(\mathbb{R})}\right) \mathrm{d}t < \infty$ for all $T > 0$. We define the space

$$
\begin{aligned}
X := \{v : \mathbb{R}\times\mathbb{R}_{>0} \to \mathbb{R} \mid \|v(\cdot,t)\|_{L^2(\mathbb{R})} \in C^0((0,+\infty);\mathbb{R}),\\
\int_0^T \left(p\|\partial_x u\|^2_{L^2(\mathbb{R})} + r\|u\|^2_{L^2(\mathbb{R})}\right) \mathrm{d}t < \infty, \forall T > 0\}.
\end{aligned}
\tag{14}
$$

(We equip X with the natural norm and corresponding topology.) The estimate (11) also shows that it is likely that a sufficient condition on the data (f, u_0) for the existence of a solution to (1) are that $\|u_0\|_{L^2(\mathbb{R})}$ is bounded and $\int_0^T \|f(\cdot,t)\|_{L^2(\mathbb{R})} \mathrm{d}t$ is bounded for all $T > 0$. Accordingly, we define

$$
Y := \{(f, u_0) \mid \|u_0\|_{L^2(\mathbb{R})} < \infty, \int_0^T \|f(\cdot,t)\|_{L^2(\mathbb{R})} \mathrm{d}t < \infty, \forall T > 0\}.
\tag{15}
$$

(We equip Y with the natural norm and corresponding topology.)

4 Uniqueness

Let (f, u_0) be a data pair in the normed space Y (source and initial data). Let us assume that there exists at least one solution u in the space X. We now show that, this solution must be unique. Assume that this is not the case, and let u^1, u^2 be two solutions corresponding to the same data pair (f, u_0). Then the estimate (13) immediately shows that

$$
\|(u^1 - u^2)(\cdot,T)\|_{L^2(\mathbb{R})} = 0, \quad \forall T > 0.
\tag{16}
$$

Hence $u^1 = u^2$, whence the assertion.

5 Existence

Proving existence of a solution to (1) in X with data in Y is quite technical. They are many ways to do so. This can be done by using the Fourier transform technique in space. Another method that is closer to numerical analysis consists of constructing finite-dimensional approximations that are uniformly bounded in X and passing to the limit. This second method works particularly well when the space domain is a bounded interval $(-L, L)$. A general existence result for parabolic equations is known in the literature as Lions' theorem [1, Thm. 2.1,p. 219].

6 More on boundary conditions

We assume in this section that the model problem (1) is set over the finite interval $(-L, L)$. In this case, many boundary conditions can be enforced. As we have seen above, deriving *a priori estimates* is essential to define a smoothness class where one can prove existence, uniqueness, and stability of a solution. All the arguments invoked above using the energy method can be applied. The key point is the integration by parts in

$$
\begin{aligned}
-\int_{-L}^{L} & pu(x,t)\partial_{xx}u(x,t)\,\mathrm{d}x + \int_{-L}^{L} qu(x,t)\partial_x u(x,t)\,\mathrm{d}x \\
&= \int_{-L}^{L} p(\partial_x u(x,t))^2\,\mathrm{d}x + \int_{-L}^{L} q\partial_x \frac{1}{2} u(x,t)\,\mathrm{d}x - pu(x)\partial_x u(x)|_{-L}^{L} \\
&= \int_{-L}^{L} p(\partial_x u(x,t))^2\,\mathrm{d}x + \left(q\tfrac{1}{2}(u(x))^2 - pu(x)\partial_x u(x)\right)\big|_{-L}^{L}
\end{aligned}
$$

Then, admissible boundary conditions are obtained by ensuring that the boundary terms appearing above produce non-negative terms. For instance, we could try to enforce

$$
\begin{aligned}
u(L)(q\tfrac{1}{2}u(L) - p\partial_x u(L)) \geq 0, \\
u(-L)(q\tfrac{1}{2}u(-L) - p\partial_x u(-L)) \leq 0.
\end{aligned}
$$

This can be achieved by enforcing *Dirichlet* boundary conditions:

$$u(-L) = 0, \tag{17}$$
$$u(L) = 0. \tag{18}$$

If $q \geq 0$, one can also enforce a *Dirichlet* boundary condition at $-L$ and a *Neumann* boundary condition at $+L$:

$$u(-L) = 0, \tag{19}$$
$$\partial_x u(L) = 0, \tag{20}$$

and the other way around if $q \leq 0$. One can also enforce *Robin* boundary conditions

$$-\partial_x u(L) = Hu(L), \tag{21}$$

$$\partial_x u(-L) = Hu(-L). \tag{22}$$

where H is such that $H > -\frac{1}{2}q$.

References

[1] J.-L. Lions. Quelques remarques sur les équations différentielles opérationnelles du **1^er^** ordre. *Rend. Sem. Mat. Univ. Padova*, 33:213–225, 1963.