~~Lecture XII~~ Finite Element Method

→ Alternative to the finite difference method

→ more general and applicable.

→ but conceptually a bit more complicated

Lecture XII : Finite Element I

We consider the ODE for $u:(0,1)\to\mathbb{R}$

$$-(p(x)u'(x))' = f(x) \qquad x\in(0,1)$$

here $p:(0,1)\to\mathbb{R}$ and $f:(0,1)\to\mathbb{R}$ are given
$p>0$

Example: u: equilibrium heat on domain $(0,1)$

p: heat conductivity

f: heat source

BC: (D) $\begin{cases} u(0)=\alpha \\ u(1)=\beta \end{cases}$ impose temperature at 0 and 1

(N) $\begin{cases} u'(0)=\alpha \\ u'(1)=\beta \end{cases}$ impose temperature flux at 0 and 1

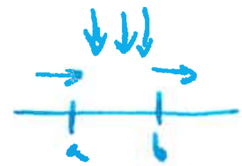
or combination of both. We focus on (D)

Derivation:

Fix law heat diffusion flux $J = -p\,u'$

for $0 < a < b < 1$

$$J(b) - J(a) = \int_a^b f(x)\,dx$$



$$\int_a^b J'(x)\,dx = \int_a^b f(x)\,dx$$

$$-\int_a^b \left(p(x)u'(x)\right)'\,dx = \int_a^b f(x)\,dx$$

this holds for all $a < b$

$$\Rightarrow \quad -\left(p(x)u'(x)\right)' = f(x)$$

Energy Estimate:

$$\begin{cases} -\left(p(x)u'\right)' = f & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

Multiply ~~by u~~ and integrate

$$-\int_0^1 (p(x)u'(x))' \, u(x)\,dx = \int_0^1 f(x)u(x)\,dx$$

integrate by parts

$$\int_0^1 p(x)\, u'(x)\, u'(x) - \underbrace{p(x)u'(x)\,u(x)\Big|_0^1}_{\to 0} = \int_0^1 f(x)u(x)\,dx$$

$$\underbrace{\int_0^1 p(x)\,|u'(x)|^2}_{\text{heat energy}} = \underbrace{\int_0^1 f(x)u(x)\,dx}_{\text{heat work}}$$

Heat Energy

$$\frac{1}{2}\int_0^1 p(x)\,|u'(x)|^2\,dx$$

~~This relation~~ The energy does not require $\begin{cases} u \text{ to be } C^2 \\ p \text{ to be } C^1 \\ f \text{ to be } C^0 \end{cases}$

As we shall see, it does not even require

$u \in C^1 \implies$ $\boxed{\text{weak derivatives}}$

Can we construct a numerical scheme taking advantage of this? i.e not assuming "smooth" solutions, rather solutions with bounded energies $\implies$ $\boxed{\text{Finite Element Method}}$

⚠ $\boxed{\text{life is rough}}$

Weak Derivatives

We say that a function $f:(0,1)\to\mathbb{R}$ is square integrable on $(0,1)$ if integrable and $\int_0^1 |f(x)|^2\,dx < \infty$

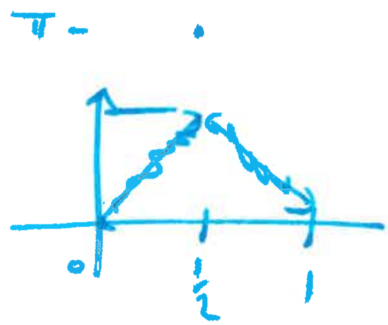
All these functions are gathered in the set

$$L^2(0,1) := \left\{ f:(0,1)\to\mathbb{R} \;\middle|\; f \text{ integrable } \int_0^1 f^2 < \infty \right\}$$

$$\|f\|_{L^2(0,1)} = \left(\int_0^1 f^2\right)^{\frac{1}{2}}$$

Note that this set contains discontinuous functions. For example

$$f(x) = \begin{cases} 1 & x\in(0,\tfrac{1}{2}) \\ -1 & x\in(\tfrac{1}{2},1) \\ \pi & x=\tfrac{1}{2} \end{cases}$$



$$\int_0^1 f(x)\,dx = \int_0^{\frac{1}{2}} f(x)\,dx + \int_{\frac{1}{2}}^1 f(x)\,dx = \frac{1}{2} - \frac{1}{2} = 0$$

and $$\int_0^1 |f(x)|^2\,dx = \int_0^{1/2} 1^2 + \int_{1/2}^1 1 = 1 < \infty$$

Also, note that the value at one point does not matter for the integral.

Suppose $v \in C^1(0,1)$ continuous, with continuous derivative

Take $w \in C^1(0,1)$ with $w(0) = w(1) = 0$ and compute

$$\boxed{\int_0^1 v'w = -\int_0^1 v w' + \cancel{v w \Big|_0^1}}$$

Given v, the above equation must hold for every w!

Definition (weak derivative in L^2)

Let $v \in L^2(0,1)$, we say that v has a weak derivative in L^2 if there exists $\varphi \in L^2(0,1)$ st

$$-\int_0^1 v w' = \int_0^1 \varphi w \quad \text{for all } w \in C^1(0,1) \text{ st } w(0)=w(1)=0.$$

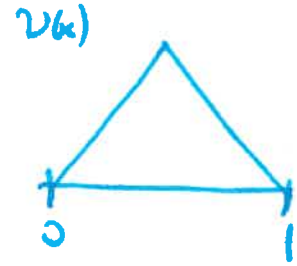
We write $v' = \varphi$

Clearly if $v \in C^1(0,1)$ then v has a weak derivative in $L_2(0,1)$

In fact, the standard (strong) derivative is a weak derivative.

Example

$$v(x)=\begin{cases} x & x<\frac{1}{2} \\ 1-x & x>\frac{1}{2}\end{cases}$$



This function is not C^1!

However for any $w\in C^1(0,1)$ with $w(0)=w(1)=0$

$$\int_0^1 v w'\,dx = \int_0^{1/2} v w'\,dx + \int_{1/2}^1 v w'\,dx$$

$$= \int_0^{1/2} x\, w'(x)\,dx + \int_{1/2}^1 (1-x)\, w'(x)\,dx$$

$$= -\int_0^{1/2} 1\, w(x)\,dx + \cancel{\frac{1}{2} w(\tfrac{1}{2})}$$

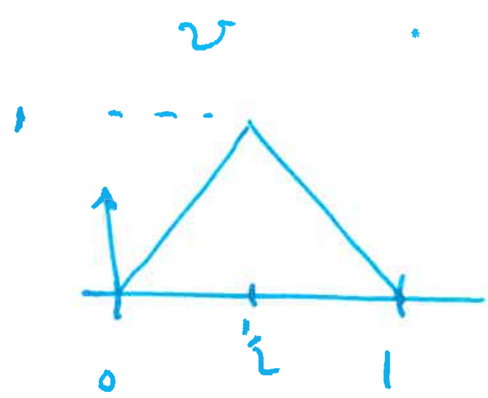
$$+ \int_{1/2}^1 1\, w(x)\,dx + \cancel{(1-x)\,w(x)\Big|_{1/2}^{1}}$$

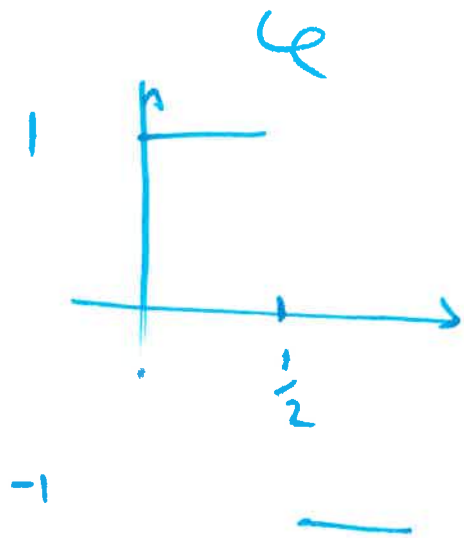
$$\cancel{-\frac{1}{2} w(\tfrac{1}{2})}$$

$$= -\int_0^{1/2} (+1)\, w(x)\,dx - \int_{1/2}^1 (-1)\, w(x)\,dx$$

Set $\varphi(x)=\begin{cases} 1 & x<\frac{1}{2} \\ -1 & x>\frac{1}{2}\end{cases}$ $\Rightarrow \displaystyle\int_0^1 v w'\,dx = -\int_0^1 \varphi w\,dx$

$\Rightarrow \varphi$ is the weak derivative of v!





We denote by

$$H^1(0,1) = \{ f:(0,1)\to\mathbb{R} \mid f\in L^2(0,1) \text{ and } f \text{ has a weak derivative in } L^2(0,1) \}$$

$$\|f\|_{H^1(0,1)} = \left(\|f\|^2_{L^2(0,1)} + \|f'\|^2_{L^2(0,1)} \right)^{\frac{1}{2}}$$

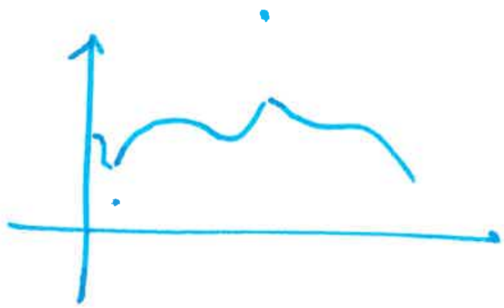
~~weak derivatives, if exist, are unique~~

Some Remarks:

- Changing the value of a function at one point will not change its weak derivative
- Weak derivatives, if exists, are unique up to the value at a finite ~~constant~~ number of points

  $\to$ equivalence class
- If $f\in H^1(0,1)$ $\to$ there is an equivalent function that is continuous!

We will always associate $f \in H^1(0,1)$ with its continuous representation!



$$H_0^1(0,1) = \{ f \in H^1(0,1) \mid f(0) = f(1) = 0 \}$$

Weak Formulation

~~Given~~ We consider the ODE

$$\begin{cases} -(p(x)u'(x))' = f(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

with $0 < p(x) \leq P \quad \forall x \in (0,1) \qquad f \in L^2(0,1)$

not necessarily continuous!

and look for a solution in a weak sense TBD.

To define a solution to the ODE but requiring less regularity, we multiply the ODE by $v \in H_0^1(0,1)$ and integrate

$$\int_0^1 -(pu')'v = \int_0^1 fv$$

integrate by parts

$$\int_0^1 pu'v'\,dx - \cancel{pu'v\Big|_0^1} = \int_0^1 fv$$

So u must satisfy

$$\int_0^1 pu'v'\,dx = \int_0^1 fv \quad \text{for every } v \in H_0^1(0,1)$$

We say that $u \in H_0^1(0,1)$ is a weak solution if

$$\int_0^1 pu'v' = \int_0^1 fv \quad \forall\, v \in H_0^1(0,1)$$

Weak Formulation

Note that ~~all~~ the term make sense

thanks to the Cauchy-Schwartz inequality

$f, g \in L^2(0,1)$

$$\int_0 |f \cdot g| \leq \left(\int f^2\right)^{\frac{1}{2}} \left(\int g^2\right)^{\frac{1}{2}} = \|f\|_{L^2(0,1)} \|g\|_{L^2(0,1)}$$

Therefore

$$\int_0^1 f v \leq \|f\|_{L^2(0,1)} \|v\|_{L^2(0,1)} < \infty$$

$$\int_0^1 p(x) u' v' \leq P \int_0^1 |u'||v| \leq P \|u'\|_{L^2(0,1)} \|v'\|_{L^2(0,1)} < \infty$$

~~**Result** There exists a unique solution~~

~~$u \in H^1_0(0,1)$, satisfying the weak formulation~~

~~and $\int_0^1 p(x)|u'|^2 = \int_0 f u$~~

~~if u happens to be C^2 $\Rightarrow$ u solves the ODE (strong)~~

Result : The weak ~~sol~~ formulation has one and only one solution $u \in H_0^1(0,1)$.

It satisfies $\int_0^1 p|u'|^2 = \int_0^1 fu$

and $\frac{1}{2}\int_0^1 p|u'|^2 - \int_0^1 fu \leq \frac{1}{2}\int_0^1 p|v'|^2 - \int_0^1 fv \quad \forall v \in H_0^1(0,1)$

- If u happens to be C^2

$\Rightarrow$ u solves the ODE

One last thing about $H_0^1(0,1)$:

v smooth $v(0)=0$

$$v(x)^2 - \cancel{v(0)^2} = \int_0^x (v(s)^2)' ds$$

$$v(x)^2 = 2\int_0^x v(s)v'(s)\,ds$$

$$\leq 2 \int_0^1 |v(s)\, v'(s)|\,ds$$

$$\leq 2\left(\int_0^1 v^2\right)^{\frac{1}{2}}\left(\int_0^1 v'\right)^{\frac{1}{2}}$$

$$\Rightarrow \int_0^1 v^2(x)\,dx \leq 2\,\|v\|_{L^2(0,1)}\,\|v'\|_{L^2(0,1)}$$

$$\Rightarrow \|v\|_{L^2(0,1)} \leq 2\,\|v'\|_{L^2(0,1)}$$

$\boxed{\text{Poincaré Estimate}}$

Note that it is important that $v(0)=0$, otherwise not true.

Stability Estimate:

$$\int_0^1 p(x)|u'|^2 = \int_0^1 f u \qquad v = u$$

$$\leq \|f\|_{L^2(0,1)}\,\|u\|_{L^2(0,1)}$$

$$\leq 2\,\|f\|_{L^2(0,1)}\,\|u'\|_{L^2(0,1)}$$

$$p_{min} = \min_{x \in (0,1)} p(x) > 0$$

$$p_{min} \underbrace{\int_0^1 |u'|^2}_{= \|u'\|^2_{L^2(0,1)}} \leq \int_0^1 p(x)|u'|^2 \leq 2\,\|f\|_{L^2(0,1)}\,\|u'\|_{L^2(0,1)}$$

$$\Rightarrow \boxed{\|u'\|_{L^2(0,1)} \leq \frac{2}{p_{min}}\,\|f\|_{L^2(0,1)}}$$

II Finite Element Method I

Weak Formulation

$$u \in H_0^1(0,1): \quad \int_0^1 p(x) u'(x) v'(x)\, dx = \int_0^1 f(x) v(x)\, dx$$

ODE $\quad -\left(p(x) u'(x)\right)' = f(x)$

FD: discretize the derivatives, i.e operator

FEM: discretize the function

We want to replace $H_0^1(0,1)$ with a finite dimentional space $\Rightarrow$ computable

For $N>0$ let $\quad x_0 = 0 < x_1 < x_2 < \dots < x_N = 1$

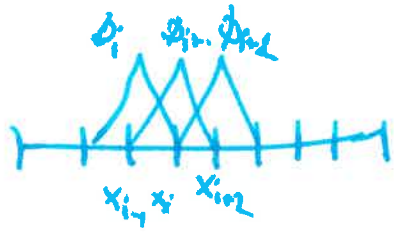
partition of $[0,1]$ ~~not~~, $\triangle$ not necessarily $x_i = \frac{i}{N}$

For each x_i, $i = 1, \dots, N-1$, we define

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ & \text{otherwise} \end{cases}$$

II

if $x_i = \frac{i}{N}$



$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h} & x \in (x_i, x_{i+1}) \\ 0 & \text{otherwise} \end{cases} \qquad h = \frac{1}{N}$$

$$= x_i - x_{i-1}$$

$$= x_{i+1} - x_i$$

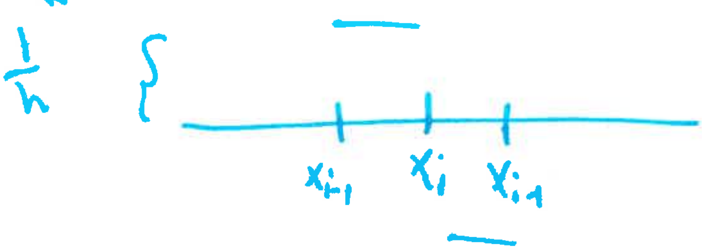
Properties of ϕ_i

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

ϕ_i are continuous but not differentiable

$$\phi_i'(x) = \begin{cases} \frac{1}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{1}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & 0 \end{cases}$$

when $x_i = \frac{i}{N}$



$\phi_i \in H_0^1(0,1)$

Define $\mathbb{V}_N = \mathrm{span}(\phi_1, \dots, \phi_{N-1}) \subset H^1_0(0,1)$

ϕ_i are linearly independent:

$$\sum_{i=1}^{N-1} \alpha_i \phi_i(x) = 0 \qquad \forall \; x \in [0,1]$$

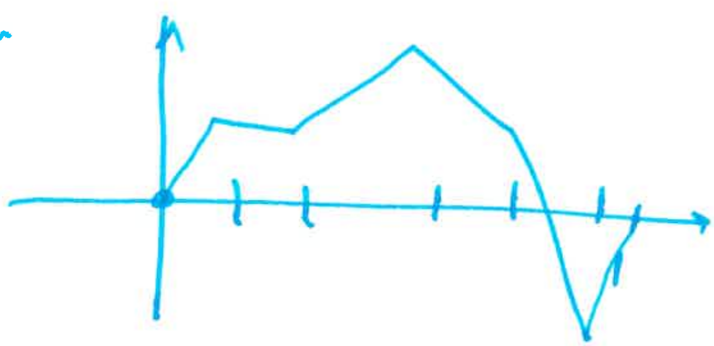
$$\sum_{i=1}^{N-1} \alpha_i \underbrace{\phi_i(x_j)}_{\delta_{ij}} = 0 \qquad j = 1, \dots, N-1$$

$$\alpha_j = 0 \qquad j = 1, \dots, N-1$$

$\Rightarrow \dim \mathbb{V}_N = N-1$
and $\{\phi_i\}$ basis of $\mathbb{V}_N$

Moreover, ϕ_i are continuous functions vanishing at 0 and 1 that are linear on each interval $[x_j, x_{j+1}]$. We say ϕ_i is continuous pw linear

How many parameters are needed to represent all continuous pw linear vanishing at 0 and 1



$\Rightarrow N-1$ parameters (values at x_i)

$\Rightarrow \mathbb{V}_N = \{$all continuous pw linear functions vanishing at 0 and at 1$\}$

IV

We approximate / replace $H^1_0(0,1)$ by V_N

$$v \in V_N \iff v(x) = \sum_{i=1}^{N-1} v_i \,\phi_i(x) \qquad v_i \in \mathbb{R}$$

FEM formulation:

Find $u_N \in V_N$ st

$$\int_0^1 p(x)\, u_N'(x)\, v_N'(x)\, dx = \int_0^1 f(x)\, v_N(x)\, dx$$

$$\forall v_N \in V_N$$

$$\iff \int_0^1 p(x)\, u_N'(x)\, \phi_j'(x)\, dx = \int_0^1 f(x)\, \phi_j(x)\, dx$$

$$j = 1, \dots, N-1$$

$$u_N = \sum_{i=1}^{N-1} u_i\, \phi_i(x)$$

$$\iff \sum_{i=1}^{N-1} u_i \int_0^1 p(x)\, \phi_i'(x)\, \phi_j'(x)\, dx = \int_0^1 f(x)\, \phi_j(x)\, dx$$

$$i = 1 \dots, N-1$$

Matrix version

Define $\quad A = (a_{ij})_{i,j=1}^{N-1}$

$$a_{ij} = \int_0^1 p(x)\,\phi_j'\,\phi_i'$$

$$F = (F_i)_{i=1}^{N-1} \qquad F_i = \int_0^1 f(x)\,\phi_i(x)$$

$$U = (u_i)_{i=1}^{N-1}$$

$$AU = F$$

check!

Linear system for the coefficients of u_N in the basis $\{\phi_i\}_{i=1}^{N-1}$

Remark $\int_0^1 p(x)\,\phi_i'\,\phi_j'$ is mostly zero except when $|i-j|\leq 1$

$$\begin{pmatrix} \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

. when $p(x)\equiv 1$ and $x_j = \frac{j}{N}$ $\Rightarrow$ ~~FD~~ FD matrix

Does $AU=F$ has a unique solution $U \in \mathbb{R}^{N-1}$?

A invertible ? $A \in \mathbb{R}^{N \times N}$

$\operatorname{Ker} A = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$?

Assume $AU = \underset{\sim}{0}$

$$U^T A U = 0$$

$$\sum_{i,j=1}^{N-1} u_i a_{ij} u_j = 0$$

$$\sum_{i,j=1}^{N-1} u_i u_j \int_0^1 p(x) \phi_i' \phi_j' = 0$$

$$\int_0^1 p(x) \cancel{\phi} |u_N'|^2 = 0$$

$$p_{min} \int_0^1 |u_N'|^2 = 0$$

$$\Rightarrow \int_0^1 |u_N'|^2 = 0 \Rightarrow \int_0^1 |u_N|^2 = 0$$

$$u_N = 0 \Rightarrow U_i = 0, \quad i=1,\dots,N-1$$

Energy and Stability

$$\int_0^1 p(x)\, u_N'\, v_N' = \int_0^1 f v_N \qquad \forall v_N \in V_N$$

$v_N = u_N$

$$\int_0^1 p(x)|u_N'|^2 = \int_0^1 f u_N \qquad \text{energy}$$

$$\leq \|f\|_{L^2(0,1)} \|u_N\|_{L^2(0,1)}$$

$$\leq 2\|f\|_{L^2(0,1)} \|u_N'\|_{L^2(0,1)}$$

$$p_{min} \|u_N'\|_{L^2(0,1)} \leq 2 \|f\|_{L^2(0,1)}$$

$$\|u_N'\|_{L^2(0,1)} \leq \frac{2}{p_{min}} \|f\|_{L^2(0,1)}$$

Result: $u_N \in V_N$ satisfies

$$\frac{1}{2}\int_0^1 p(x)|u_N'|^2 - \int_0^1 f u_N \leq \frac{1}{2}\int_0^1 p(x)|v_N'|^2 - \int_0^1 f v_N$$

$$\forall v_N \in V_N$$

Some Analysis

$$\int_0^1 p(x)\,u'(x)\,v'(x)\,dx = \int_0^1 f v \qquad \forall v \in H_0^1(0,1)$$

$$\int_0^1 p(x)\,u_N'(x)\,v_N'(x)\,dx = \int_0^1 f v_N \qquad \forall v_N \in V_N$$

$$V_N \subset H_0^1(0,1)$$

$$v \Leftarrow v_N$$

$$\int_0^1 p(x)\big(u'(x) - u_N'(x)\big)v_N' = 0 \qquad \forall v_N \in V_N$$

$$p_{min}\,\|u'-u_N'\|^2_{L^2(0,1)} = p_{min}\int_0^1 (u'-u_N')^2 \le \int_0^1 p(x)\,|u'-u_N'|^2$$

$$= \int_0^1 p(x)(u'-u_N')(u'-u_N')$$

$$= \int_0^1 p(x)(u'-u_N')u' - \underbrace{\int_0^1 p(x)(u'-u_N')u_N'}_{=\,0\,=\,\int_0^1 p(x)(u'-u_N')v_N' \quad \forall v_N \in V_N}$$

$$p_{min}\|u'-u_N'\|^2_{L^2(0,1)} \leq \int_0^1 p(x)(u'-u_N')(u'-v_N')$$

$$\leq p_{max}\|u'-u_N'\|_{L^2(0,1)}\|u'-v_N'\|_{L^2(0,1)}.$$

$$\Rightarrow \|u'-u_N'\|_{L^2(0,1)} \leq \frac{p_{max}}{p_{min}}\|u'-v_N'\|_{L^2(0,1)} \quad \forall v_N \in V_N$$

Quasi-best approximation!

How small can $\|u'-v_N'\|_{L^2(0,1)}$ be?

For now we assume $u \in C^2$

$\cancel{\text{let } e = u - u_N}$ and we define

$$I_N u(x) = \sum_{i=1}^{N-1} u(x_i)\phi_i(x) \in V_N$$

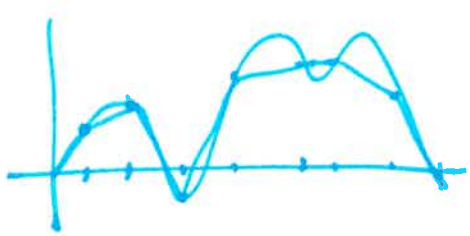
$\cancel{x \in (x_i, x_{i+1})}$

interpolant of u.

$\cancel{e'(x) = \int}$

$$I_N u(x_j) = \sum_{i=1}^{N-1} u(x_i)\underbrace{\phi_i(x_j)}_{\delta_{ij}}$$

$$= u(x_j)$$



Note that $e(x) = u(x) - I_N u(x)$

satisfies $e(x_i) = 0 \quad i = 0, \dots, N$

$\Rightarrow \exists\ \xi_j \in (x_j, x_{j+1})$ st $e'(\xi_j) = 0 \quad j = 0, \dots, N-1$

Rolles

$x \in (x_j, x_{j+1})$

$$e'(x) = \int_{\xi_j}^{x} e''(s)\,ds = \int_{\xi_j}^{x} u''(s)\,ds$$

$$e'(x)^2 = \left(\int_{\xi_j}^{x} u''(s)\,ds \right)^2 \le |x - \xi_j| \int_{\xi_j}^{x} (u''(s))^2$$

$x \in (x_j, x_{j+1})$

$$e'(x)^2 \le |x_{j+1} - x_j| \int_{x_j}^{x_{j+1}} |u''(s)|^2\,ds$$

$$\int_{x_j}^{x_{j+1}} e'(x)^2\,dx \le |x_{j+1} - x_j|^2 \int_{x_j}^{x_{j+1}} |u''(s)|^2\,ds$$

Σ_j

$$\int_0^1 e'(x)^2\,dx \le \max_{j=0,\dots,N-1} |x_{j+1} - x_j|^2 \int_0^1 |u''(s)|^2\,ds$$

$$\| u' - I_N u' \|_{L^2(0,1)} \le \max_{j=0,\dots,N-1} |x_{j+1} - x_j| \left(\int_0^1 |u''(x)|^2\,dx \right)^{\frac{1}{2}}$$

$$\leq \max_{j=0,\dots,N-1} |x_{j+1} - x_j| \max_{x \in [0,1]} |u''(x)|$$ $\boxtimes$

Returning to $u' - u_N'$

$$\|u' - u_N'\|_{L^2(0,1)} \leq \frac{p_{max}}{p_{min}} \max_{x \in [0,1]} |u''(x)| \max_{j=1,\dots,N-1} |x_{j+1} - x_j|$$

~~We need to~~

For instance $x_i = \frac{i}{N}$

$$\leq \frac{p_{max}}{p_{min}} \max_{x \in [0,1]} |u''(x)| \frac{1}{N} \xrightarrow[N \to \infty]{} 0$$