Exercises

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1 Lecture 1

Question 1: Let $y(x,t) = x\cos(2t + \log(|x|))$. Compute $\partial_{tt}y + x^2\partial_{xx}y - x\partial_x y + 6y$.

Solution: This exercise is meant to check whether you understand the notion of partial derivatives and the chain rule

$$\begin{aligned}
\partial_{tt}y(x,t) &= -4x\cos(2t + \log(|x|)) = -4y(x,t),\\
\partial_x y(x,t) &= \cos(2t + \log(|x|)) - x\sin(2t + \log(|x|))\frac{1}{x} = \cos(2t + \log(|x|)) - \sin(2t + \log(|x|)),\\
\partial_{xx}y(x,t) &= -\frac{1}{x}\sin(2t + \log(|x|)) - \frac{1}{x}\cos(2t + \log(|x|))
\end{aligned}$$

In conclusion

$$\begin{aligned}
\partial_{tt}y + x^2\partial_{xx}y - x\partial_x y + 6y &= -4x\cos(2t + \log(|x|)) - x\sin(2t + \log(|x|))\\
&\quad - x\cos(2t + \log(|x|)) - x\cos(2t + \log(|x|)) + x\sin(2t + \log(|x|)) + 6x\cos(2t + \log(|x|))\\
&= 0,
\end{aligned}$$

that is to say, $y(x,t)$ solve the PDE $\partial_{tt}y + x^2\partial y - x\partial_x y + 6y = 0$.

Question 2: Let $u, f : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions of class C^1. (a) Compute $\partial_x f(u(x))$.

Solution: Using the chain rule we obtain

$$\partial_x f(u(x)) = f'(u(x))\partial_x u.$$

where f' denotes the derivative of f.

(b) Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be functions of class C^1. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $F(v) = \int_0^v f'(t)\psi'(t)\,\mathrm{d}t$. Use (a) to compute $\partial_x(F(u(x)) - \partial_x(f(u(x)))\psi'(u(x))$.

Solution: Using the chain rule we obtain

$$\partial_x(F(u(x)) = F'(u(x))\partial_x u(x) = f'(u(x))\psi'(u(x))\partial_x u(x) = \partial_x(f(u(x)))\psi'(u(x)).$$

This means that $\partial_x(F(u(x)) = \partial_x(f(u(x)))\psi'(u(x))$.

(c) Using the notation of (a) and (b), assume that $u(\pm\infty) = 0$ and compute $\int_{-\infty}^{+\infty} \partial_x(f(u(x)))\psi'(u(x))\,\mathrm{d}x$.

Solution: Using (b) and $u(\pm\infty) = 0$ we have

$$\int_{-\infty}^{+\infty} \partial_x(f(u(x)))\psi'(u(x))\,\mathrm{d}x = \int_{-\infty}^{+\infty} \partial_x(F(u(x)))\,\mathrm{d}x = F(u(x))|_{-\infty}^{+\infty} = F(0) - F(0) = 0.$$

Question 3: Let u solve $\partial_t u - \partial_x((3x+1)\partial_x u) = -3$, $x \in (0, L)$, with $\partial_x u(0,t) = 1$, $\partial_x u(L,t) = \alpha$, $u(x,0) = f(x)$.

(a) Compute $\int_0^L u(x,t)dx$ as a function of t.

Solution: Integrate the equation over the domain $(0, L)$:

$$\begin{aligned} d_t \int_0^L u(x,t)dx &= \int_0^L \partial_t u(x,t)dx = \int_0^L \partial_x((3x+1)\partial_x u)dx - 3L \\ &= (3L+1)\partial_x u(L,t) - \partial_x u(0,t) - 3L = (3L+1)\alpha - 1 - 3L \\ &= (3L+1)(\alpha - 1) \end{aligned}$$

That is $d_t \int_0^L u(x,t)dx = (3L+1)(\alpha-1)$. This implies $\int_0^L u(x,t)dx = (3L+1)(\alpha-1)t + \int_0^L f(x)dx$.

(b) For which value of α the quantity $\int_0^L u(x,t)dx$ does not depend on t?

Solution: The above computation yields $\int_0^L u(x,t)dx = (3L+1)(\alpha-1)t + \int_0^L f(x)dx$. This is independent of t if $(3L+1)(\alpha-1) = 0$, meaning $\alpha = 1$.

Question 4: Let u solve $\partial_t u + \partial_x\big(v(x,t)u - \mu(x,t)\partial_x u\big) = g(x)e^{-t}$, $x \in (0, L)$, $t > 0$, with $\mu(0,t)\partial_x u(0,t) = 1$, $\mu(L,t)\partial_x u(L,t) = 1 + 2e^{-t}$, $u(x, 0) = f(x)$, where v, $\mu > 0$, f and g are four smooth functions and $v(0) = v(L) = 0$.

(a) Compute $\frac{d}{dt}\int_0^L u(x,t)dx$ as a function of t.

Solution: Integrate the equation over the domain $(0, L)$ and apply the Fundamental Theorem of calculus:

$$\begin{aligned}\frac{d}{dt}\int_0^L u(x,t)\,\mathrm{d}x = \int_0^L \partial_t u(x,t)\,\mathrm{d}x &= \int_0^L \partial_x(-v(x,t)u + \mu(x,t)\partial_x u)\,\mathrm{d}x + e^{-t}\int_0^L g(x)\,\mathrm{d}x \\ &= \mu(L,t)\partial_x u(L) - \mu(0,t)\partial_x u(0) + e^{-t}\int_0^L g(x)\,\mathrm{d}x \\ &= 1 + 2e^{-t} - 1 + e^{-t}\int_0^L g(x)\,\mathrm{d}x.\end{aligned}$$

That is

$$\frac{d}{dt}\int_0^L u(x,t)\,\mathrm{d}x = e^{-t}(\int_0^L g(x)\,\mathrm{d}x + 2).$$

(b) Use (a) to compute $\int_0^L u(x,t)dx$ as a function of t.

Solution: Applying the Fundamental Theorem of calculus again (with respect to time this time) gives

$$\begin{aligned}\int_0^L u(x,T)\,\mathrm{d}x &= \int_0^L u(x,0)\,\mathrm{d}x + \int_0^T e^{-t}\,\mathrm{d}t(\int_0^L g(x)\,\mathrm{d}x + 2). \\ &= \int_0^L f(x)dx + (1 - e^{-T})(\int_0^L g(x)dx + 2).\end{aligned}$$

(c) What is the limit of $\int_0^L u(x,t)dx$ as $t \to +\infty$?

Solution: The above formula gives

$$\lim_{T\to+\infty}\int_0^L u(x,T)dx = \int_0^L f(x)dx + \int_0^L g(x)dx + 2.$$

Question 5: Consider the differential equation $-\frac{d^2\phi}{dt^2} = \lambda\phi$, $t \in (0, \pi)$, supplemented with the boundary conditions $\phi(0) = 0$, $3\phi(\pi) = -\phi'(\pi)$.
(a) Prove that it is necessary that λ be positive for a non-zero solution to exist.

Solution: (i) Let ϕ be a non-zero solution to the problem. Multiply the equation by ϕ and integrate over the domain.

$$\int_0^{\pi} (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) = \lambda \int_0^{\pi} \phi^2(t)dt.$$

Using the BCs, we infer

$$\int_0^{\pi} (\phi'(t))^2 dt + 3\phi(\pi)^2 = \lambda \int_0^{\pi} \phi^2(t)dt,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^{\pi} (\phi'(t))^2 dt = 0$ and $\phi(\pi)^2 = 0$, which implies that $\phi'(t) = 0$ and $\phi(\pi) = 0$. The fundamental theorem of calculus implies $\phi(t) = \phi(\pi) + \int_{\pi}^{t} \phi'(\tau)d\tau = 0$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero solution to exist.

(b) Find the equation that λ must solve for the above problem to have a nonzero solution.

Solution: Since λ is positive, ϕ is of the following form

$$\phi(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition $\phi'(\pi) = -3\phi(\pi)$ implies $\sqrt{\lambda}c_2 \cos(\sqrt{\lambda}\pi) = -3c_2 \sin(\sqrt{\lambda}\pi)$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, λ must solve the following equation

$$\sqrt{\lambda} \cos(\sqrt{\lambda}\pi) + 3 \sin(\sqrt{\lambda}\pi) = 0,$$

for a nonzero solution ϕ to exist.

2 Lecture 2

Question 6: Consider the following conservation equation $\partial_t \rho + \partial_x(q(\rho)) = 0$, $x \in (-\infty, +\infty)$, $t > 0$, $\rho(x, 0) = \frac{1}{2}$ if $x < 0$, and $\rho(x, 0) = 1$ if $x > 0$. where $q(\rho) = 2\rho + 3\rho^3 - \sin(\rho^2)$ (and $\rho(x, t)$ is the conserved quantity). What is the wave speed for this problem?

Solution: The wave speed is the quantity $q'(\rho) = 2 + 9\rho^2 - 2\rho\cos(\rho^2)$.

Question 7: Consider the following conservation equation $\partial_t \rho + \partial_x(q(\rho)) = 0$, $x \in (-\infty, +\infty)$, $t > 0$, $\rho(x, 0) = \frac{1}{2}$ if $x < 0$, and $\rho(x, 0) = 1$ if $x > 0$. where $q(\rho) = 2\rho + 3\rho^3 - \sin(\rho^2)$ (and $\rho(x, t)$ is the conserved quantity). What is the wave speed for this problem?

Solution: The wave speed is the quantity $q'(\rho) = 2 + 9\rho^2 - 2\rho\cos(\rho^2)$.

Question 8: Consider the conservation equation with flux $q(\rho) = \rho^4$. Assume that the initial data is $\rho_0(x) = 2$, if $x < 0$, $\rho_0(x) = 1$, if $0 < x < 1$, and $\rho_0(x) = 0$, if $1 < x$. (i) Draw the characteristics

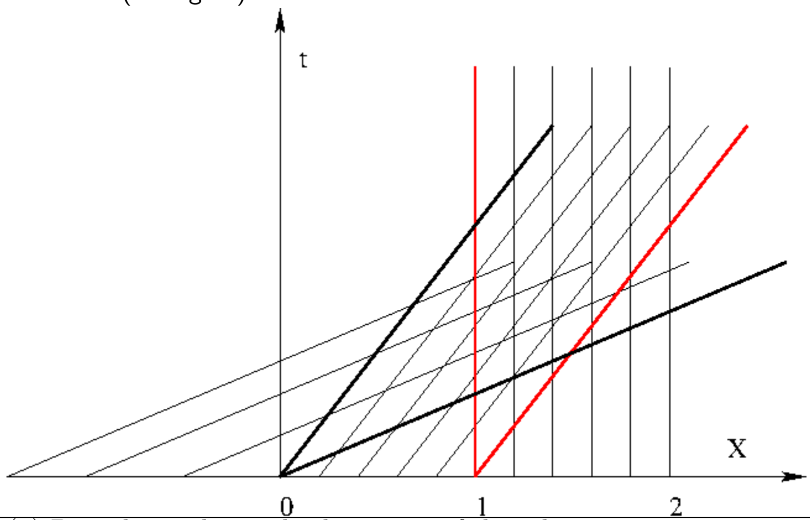
Solution: There are three families of characteristics.

Case 1: $s < 0$, $X(s,t) = 32t + s$. In the x-t plane, these are lines with slope $\frac{1}{32}$.

Case 2: $0 < s < 1$, $X(s,t) = 4t + s$. In the x-t plane, these are lines with slope $\frac{1}{4}$.

Case 3: $1 < s$, $X(s,t) = s$. In the x-t plane, these are vertical lines.

One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).



(ii) Describe qualitatively the nature of the solution.

Solution: We have two shocks moving to the right. One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).

(iii) When does the left shock catch up with the right one?

Solution: The speeds of the shocks are

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{2^4 - 1}{2 - 1} = 15, \quad \text{and} \quad \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \frac{1 - 0}{1 - 0} = 1.$$

The location of the left shock at time t is $x_1(t) = 15t$ and that of the right shock is $x_2(t) = t + 1$. The two shocks are at the same location when $15t = t + 1$, i.e., $t = \frac{1}{14}$.

Question 9: Consider the conservation equation $\partial_t \rho + \partial_x(\sin(\frac{\pi}{2}\rho)) = 0$, $x \in \mathbb{R}$, $t > 0$, with initial data $\rho_0(x) = 0$ if $x < 0$ and $\rho_0(x) = 1$ if $x > 0$. Draw the characteristics and give the explicit representation of the solution.

Solution: The implicit representation of the solution to the equation $\partial_t\rho + \partial_x q(\rho) = 0$, $\rho(x, 0) = \rho_0(x)$, is

$$X(s,t) = q'(\rho_0(s))t + s; \quad \rho(X(s,t),t) = \rho_0(s). \tag{1}$$

The explicit representation is obtained by expressing s in terms of X and t.

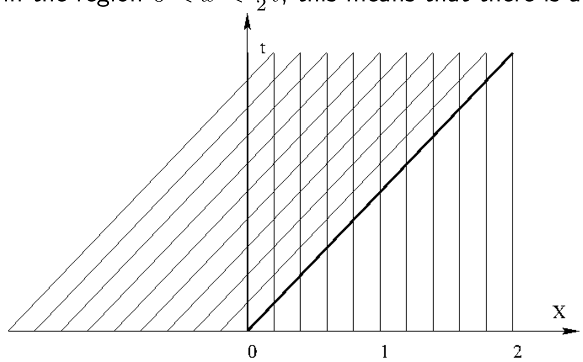
Case 1: $s < 0$, we have $\rho_0(s) = 0$, $q'(\rho_0(s)) = \frac{\pi}{2}\cos(0) = \frac{\pi}{2}$, $X = \frac{\pi}{2}t + s$, which means $s = X - \frac{\pi}{2}t$. Then

$$\rho(x,t) = 0 \text{ if } x < \frac{\pi}{2}t.$$

Case 2: $0 < s$, we have $\rho_0(s) = 1$, $q'(\rho_0(s)) = \frac{\pi}{2}\cos(\frac{\pi}{2}) = 0$, $X = s$. Then

$$\rho(x,t) = 1 \text{ if } 0 < x.$$

The characteristics cross in the region $0 < x < \frac{\pi}{2}t$; this means that there is a shock in this region.

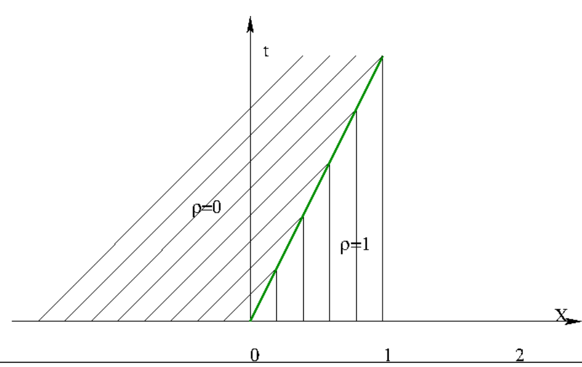


The Rankin-Hugoniot formula gives the speed of the shock

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{\sin(\frac{\pi}{2}) - \sin(0)}{1-0} = 1.$$

The equation of the trajectory of the shock is $x_s(t) = t$. Finally

$$\rho(x,t) = \begin{cases} 0 & \text{if } x < t, \\ 1 & \text{if } t < x. \end{cases} \tag{2}$$



Question 10: For all $k \in \mathbb{R}$, consider the entropy $\eta(v, k) := |v - k|$. Compute the entropy flux associated with this entropy, $q(v)$, with the normalization $q(k) := 0$.

Solution: By definition, $q(u) = \int_k^u \mathrm{sign}(v - k) f'(v) \, \mathrm{d}v$, where $\mathrm{sign}(z) = -1$ if $z < 0$ and $\mathrm{sign}(z) = 1$ if $z > 0$. If $u < k$, then $q(u) = -\int_k^u f'(v) \, \mathrm{d}v = \int_u^k f'(v) \, \mathrm{d}v = f(k) - f(u) = \mathrm{sign}(u - k)(f(u) - f(k))$. We obtain the same result if $k < u$. Hence, $q(u) = \mathrm{sign}(u - k)(f(u) - f(k))$.

Question 11: Consider Burgers' equation with $D := \mathbb{R}$ and $u_0(x) := H(x)$, where H is the Heaviside function. (a) Verify that $u_1(x,t) := H(x - \frac{1}{2}t)$ and $u_2(x,t) := 0$ if $x < 0$, $u_2(x,t) := \frac{x}{t}$, if $0 < x < t$, $u_2(x,t) := 1$ if $x > t$, are both weak solutions.

Solution: Let us look at u_1 first. In the distribution sense, we have $\partial_t u_1 = -\frac{1}{2}\delta(x - \frac{1}{2}t)$, where δ is the Dirac measure, and upon observing that $u_1^2 = u_1$, we also have $\partial_x(\frac{1}{2}u_1^2) = -\frac{1}{2}\delta(x - \frac{1}{2}t)$. Hence, $\partial_t u_1 + \partial_x(\frac{1}{2}u_1^2) = 0$. Let us now look at u_2. Upon observing that u_2 is a continuous function in $\mathbb{R}\times\mathbb{R}_+$, we have $\partial_t u_2(x,t) = 0$ if $x < 0$, $\partial_t u_2(x,t) = -\frac{x}{t^2}$, if $0 < x < t$, $\partial_t u_2(x,0) = 0$ if $x > t$, and $\partial_x(\frac{1}{2}u_2^2(x,t)) = 0$ if $x < 0$, $\partial_x(\frac{1}{2}u_2^2(x,t)) = \frac{x}{t^2}$, if $0 < x < t$, $\partial_t(\frac{1}{2}u_2^2(x,0)) = 0$ if $x > t$. This proves that $\partial_t u_2 + \partial_x(\frac{1}{2}u_2^2) = 0$.

(ii) Verify that u_1 does not satisfy the entropy inequalities, whereas u_2 does.

Solution: Let $k \in (0,1)$. Let us consider the Kružkov entropy pair $\eta(u) = |u - k|$ and $q(u) = \text{sign}(u-k)(f(u) - f(k)) = \text{sign}(u-k)\frac{1}{2}(u^2 - k^2)$ (i.e., $\boldsymbol{q}(u) = q(u)\boldsymbol{e}_x$). Then, for u_1, we have $\eta(u_1) = |H(x - \frac{1}{2}t) - k| = k$ if $x < \frac{1}{2}t$ and $\eta(u_1) = |H(x - \frac{1}{2}t) - k| = 1 - k$ if $x > \frac{1}{2}t$. This means that $\eta(u_1) = H(x - \frac{1}{2}t) - (2H(x - \frac{1}{2}t) - 1)k$. This shows that $\partial_t \eta(u_1) = (-\frac{1}{2} + k)\delta(x - \frac{1}{2}t)$. Similarly, we have

$$
\begin{aligned}
q(u_1) &= \text{sign}(u_1 - k)\frac{1}{2}(u_1^2 - k^2) = \tfrac{1}{2}\text{sign}(H(x - \tfrac{1}{2}t) - k)(H^2(x - \tfrac{1}{2}t) - k^2) \\
&= \begin{cases} \frac{1}{2}k^2 & \text{if } x < \frac{1}{2}t, \\ \frac{1}{2}(1 - k^2) & \text{if } x > \frac{1}{2}t. \end{cases}
\end{aligned}
$$

This means that $\partial_x q(u_1) = (\frac{1}{2} - k^2)\delta(x - \frac{1}{2}t)\boldsymbol{e}_x$. Hence, we have

$$
\partial_t \eta(u_1) + \partial_x q(u_1) = \big(-\tfrac{1}{2} + k + \tfrac{1}{2} - k^2\big)\delta(x - \tfrac{1}{2}t) = k(1-k)\delta(x - \tfrac{1}{2}t),
$$

which proves that u_1 is not the entropy solution since $k(1-k)\delta(x - \frac{1}{2}t)$ is a positive measure for all $k \in (0,1)$.

We now do the computation for u_2. Clearly, $\partial_t \eta(u_2) + \partial_x q(u_2) = 0$ if $x < 0$ or $t < x$. Let us now assume that $0 < x < t$. Then $\eta(u_2) = |\frac{x}{t} - k| = k - \frac{x}{t}$ if $x < kt$ and $\eta(u_2) = |\frac{x}{t} - k| = \frac{x}{t} - k$ if $x > kt$, meaning that $\partial_t \eta(u_2) = +\frac{x}{t^2}$ if $x < kt$ and $\eta(u_2) = -\frac{x}{t^2}$ if $x > kt$. Similarly, $q(u_2) = \text{sign}(u_2 - k)\frac{1}{2}(u_2^2 - k^2) = -\frac{1}{2}(\frac{x^2}{t^2} - k^2)$ if $x < kt$, and $q(u_2) = \frac{1}{2}(\frac{x^2}{t^2} - k^2)$ if $x > kt$, meaning that $\partial_x q(u_2) = -\frac{x}{t^2}$ if $x < kt$, and $q(u_2) = \frac{x}{t^2}$ if $x > kt$. Hence, $\partial_t \eta(u_2) + \partial_x q(u_2) = 0$ a.e. in x and t. In conclusion, u_2 is the entropy solution.

3 Lecture 3

Question 12: Consider the quasilinear Klein-Gordon equation: $\partial_{tt}\phi(x,t)-c^2\partial_{xx}\phi(x,t)+m^2\phi(x,t)+\beta^2\phi^3(x,t)=0$, $x\in\mathbb{R}$, $t>0$, with $\phi(x,0)=f(x)$, $\partial_t\phi(x,0)=g(x)$ and $\phi(\pm\infty,t)=0$, $\partial_t\phi(\pm\infty,t)=0$, $\partial_x\phi(\pm\infty,t)=0$. Find an energy $E(t)$ which is invariant with respect to time (Hint: test with $\partial_t\phi(x,t)$ and use $\phi^p\phi'=(\frac{1}{p+1}\phi^{p+1})'$.)

Solution: Testing with $\partial_t\phi(x,t)$ and integrating over $\mathbb{R}$ and using the property $\partial_t\phi(\pm\infty,t)=0$, $\partial_x\phi(\pm\infty,t)=0$, we obtain

$$\begin{aligned}0&=\int_{-\infty}^{+\infty}\partial_t(\frac{1}{2}(\partial_t\phi)^2)\,\mathrm{d}x-c^2\int_{-\infty}^{+\infty}\partial_{xx}\phi\partial_t\phi\,\mathrm{d}x+m^2\int_{-\infty}^{+\infty}\partial_t(\frac{1}{2}\phi^2)\,\mathrm{d}x+\beta^2\int_{-\infty}^{+\infty}\partial_t(\frac{1}{4}\phi^4)\,\mathrm{d}x\\&=d_t\int_{-\infty}^{+\infty}\frac{1}{2}(\partial_t\phi)^2\,\mathrm{d}x+c^2\int_{-\infty}^{+\infty}\partial_x\phi\partial_t\partial_x\phi\,\mathrm{d}x+d_t\int_{-\infty}^{+\infty}(\frac{m^2}{2}\phi^2+\frac{\beta^2}{4}\phi^4)\,\mathrm{d}x\\&=d_t\int_{-\infty}^{+\infty}\frac{1}{2}(\partial_t\phi)^2\,\mathrm{d}x+d_t\int_{-\infty}^{+\infty}\frac{c^2}{2}(\partial_x\phi)^2\,\mathrm{d}x+d_t\int_{-\infty}^{+\infty}(\frac{m^2}{2}\phi^2+\frac{\beta^2}{4}\phi^4)\,\mathrm{d}x\\&=d_t\int_{-\infty}^{+\infty}\left(\frac{1}{2}(\partial_t\phi)^2+\frac{c^2}{2}(\partial_x\phi)^2+\frac{m^2}{2}\phi^2+\frac{\beta^2}{4}\phi^4\right)\mathrm{d}x.\end{aligned}$$

Introduce

$$E(t)=\int_{-\infty}^{+\infty}\left(\frac{1}{2}(\partial_t\phi)^2+\frac{c^2}{2}(\partial_x\phi)^2+\frac{m^2}{2}\phi^2+\frac{\beta^2}{4}\phi^4\right))\,\mathrm{d}x.$$

Then

$$d_tE(t)=0.$$

The fundamental Theorem of calculus gives

$$E(t)=E(0).$$

In conclusion the quantity $E(t)$ is invariant with respect to time, as requested.

Question 13: (a) Compute the Fourier transform of the function $f(x)$ defined by

$$f(x)=\begin{cases}1 & \text{if } |x|\le 1\\ 0 & \text{otherwise}\end{cases}$$

Solution: By definition

$$\begin{aligned}\mathcal{F}(f)(\omega)=\frac{1}{2\pi}\int_{-1}^{1}e^{i\xi\omega}&==\frac{1}{2\pi}\frac{1}{i\omega}(e^{i\omega}-e^{-i\omega})\\&=\frac{1}{2\pi}\frac{2\sin(\omega)}{\omega}.\end{aligned}$$

Hence

$$\mathcal{F}(f)(\omega)=\frac{1}{\pi}\frac{\sin(\omega)}{\omega}.$$

(b) Find the inverse Fourier transform of $g(\omega)=\frac{\sin(\omega)}{\omega}$.

Solution: Using (a) we deduce that $g(\omega) = \pi\mathcal{F}(f)(\omega)$, that is to say, $\mathcal{F}^{-1}(g)(x) = \pi\mathcal{F}^{-1}(\mathcal{F}(f))(x)$. Now, using the inverse Fourier transform, we deduce that $\mathcal{F}^{-1}(g)(x) = \pi f(x)$ at every point x where $f(x)$ is of class C^1 and $\mathcal{F}^{-1}(g)(x) = \frac{\pi}{2}(f(x^-) + f(x^+))$ at discontinuity points of f. As a result:

$$\mathcal{F}^{-1}(g)(x) = \begin{cases} \pi & \text{if } |x| < 1 \\ \frac{\pi}{2} & \text{at } |x| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Question 14: Use the Fourier transform technique to solve the following ODE $y''(x) - y(x) = f(x)$ for $x \in (-\infty, +\infty)$, with $y(\pm\infty) = 0$, where f is a function such that $|f|$ is integrable over $\mathbb{R}$.

Solution: By taking the Fourier transform of the ODE, one obtains

$$-\omega^2 \mathcal{F}(y) - \mathcal{F}(y) = \mathcal{F}(f).$$

That is

$$\mathcal{F}(y) = -\mathcal{F}(f)\frac{1}{1+\omega^2}.$$

and the convolution Theorem gives

$$\mathcal{F}(y) = -\pi \mathcal{F}(f)\mathcal{F}(e^{-|x|}) = -\frac{1}{2}\mathcal{F}(f * e^{-|x|}).$$

Applying $\mathcal{F}^{-1}$ on both sides we obtain

$$y(x) = -\frac{1}{2} f * e^{-|x|} = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|} f(z) dz$$

That is

$$y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|} f(z) dz.$$

Question 15: Solve the integral equation: $f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1}dy = \frac{1}{x^2+4} + \frac{1}{x^2+1}$, for all $x \in (-\infty, +\infty)$.

Solution: The equation can be re-written

$$f(x) + \frac{1}{2\pi} f * \frac{1}{x^2+1} = \frac{1}{x^2+4} + \frac{1}{x^2+1}.$$

We take the Fourier transform of the equation and we apply the Convolution Theorem (see (**??**))

$$\mathcal{F}(f) + \frac{1}{2\pi} 2\pi \mathcal{F}(\frac{1}{x^2+1})\mathcal{F}(f) = \mathcal{F}(\frac{1}{x^2+4}) + \mathcal{F}(\frac{1}{x^2+1}).$$

Using (**??**), we obtain

$$\mathcal{F}(f) + \frac{1}{2}e^{-|\omega|}\mathcal{F}(f) = \frac{1}{4}e^{-2|\omega|} + \frac{1}{2}e^{-|\omega|},$$

which gives

$$\mathcal{F}(f)(1 + \frac{1}{2}e^{-|\omega|}) = \frac{1}{2}e^{-|\omega|}(\frac{1}{2}e^{-|\omega|} + 1).$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{2}e^{-|\omega|}.$$

Taking the inverse Fourier transform, we finally obtain $f(x) = \frac{1}{x^2+1}$.

Question 16: Use the Fourier transform method to solve the equation $\partial_t u + \frac{2t}{1+t^2}\partial_x u = 0$, $u(x,0) = u_0(x)$, in the domain $x \in (-\infty, +\infty)$ and $t > 0$.

Solution: We take the Fourier transform of the equation with respect to x

$$\begin{aligned} 0 &= \partial_t \mathcal{F}(u) + \mathcal{F}(\frac{2t}{1+t^2}\partial_x u) \\ &= \partial_t \mathcal{F}(u) + \frac{2t}{1+t^2}\mathcal{F}(\partial_x u) \\ &= \partial_t \mathcal{F}(u) - i\omega \frac{2t}{1+t^2}\mathcal{F}(u). \end{aligned}$$

This is a first-order linear ODE:

$$\frac{\partial_t \mathcal{F}(u)}{\mathcal{F}(u)} = i\omega \frac{2t}{1+t^2} = i\omega \frac{\mathrm{d}}{\mathrm{d}t}(\log(1+t^2))$$

The solution is

$$\mathcal{F}(u)(\omega, t) = K(\omega)e^{i\omega \log(1+t^2)}.$$

Using the initial condition, we obtain

$$\mathcal{F}(u_0)(\omega) = \mathcal{F}(u)(\omega, 0) = K(\omega).$$

The shift lemma (see formula (**??**)) implies

$$\mathcal{F}(u)(\omega, t) = \mathcal{F}(u_0)(\omega)e^{i\omega \log(1+t^2)} = \mathcal{F}(u_0(x - \log(1+t^2))),$$

Applying the inverse Fourier transform finally gives

$$u(x,t) = u_0(x - \log(1+t^2)).$$

Question 17: Use the Fourier transform technique to solve $\partial_t u(x,t) + \sin(t)\partial_x u(x,t) + (2+3t^2)u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$. (Use the shift lemma: $\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta}$ and the definition $\mathcal{F}(f)(\omega) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx$)

Solution: Applying the Fourier transform to the equation gives

$$\partial_t \mathcal{F}(u)(\omega,t) + \sin(t)(-i\omega)\mathcal{F}(u)(\omega,t) + (2+3t^2)\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t \mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = i\omega\sin(t) - (2+3t^2).$$

Then applying the fundamental theorem of calculus between 0 and t, we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = -i\omega(\cos(t)-1) - (2t+t^3).$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{-i\omega(\cos(t)-1)}e^{-(2t+t^3)}.$$

Then the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0(x+\cos(t)-1)(\omega)e^{-(2t+t^3)}.$$

This finally gives

$$u(x,t) = u_0(x+\cos(t)-1)e^{-(2t+t^3)}.$$

Question 18: Solve the PDE

$$
\begin{aligned}
&u_{tt} - a^2 u_{xx} = 0, && -\infty < x < +\infty, \quad 0 \le t, \\
&u(x,0) = \cos(x), \quad u_t(x,0) = -a\sin(x), && -\infty < x < +\infty.
\end{aligned}
$$

Solution: Apply D'Alembert's Formula.

$$
\begin{aligned}
u(x,t) &= \tfrac{1}{2}(\cos(x-at) + \cos(x+at)) - \tfrac{1}{2a}\int_{x-at}^{x+at} a\sin(\xi)d\xi \\
&= \tfrac{1}{2}(\cos(x-at) + \cos(x+at)) + \tfrac{1}{2}(\cos(x+at) - \cos(x-at)) \\
&= \cos(x+at).
\end{aligned}
$$

Hence $u(x,t) = \cos(x+at)$.

Question 19: Solve the wave equation on the semi-infinite domain $(0, +\infty)$,

$$\partial_{tt} w - 4\partial_{xx} w = 0, \quad x \in (0, +\infty),\ t > 0$$
$$w(x, 0) = (1 + x^2)^{-1}, \quad x \in (0, +\infty); \qquad \partial_t w(x, 0) = 0, \quad x \in (0, +\infty); \quad \text{and} \quad \partial_x w(0, t) = 0, \quad t > 0.$$

(Hint: Consider a particular extension of w over $\mathbb{R}$)

Solution: We define $f(x) = (1 + x^2)^{-1}$ and its even extension $f_e(x)$ on $-\infty, +\infty$. Let w_e be the solution to the wave equation over the entire real line with f_e as initial data:

$$\begin{aligned}
&\partial_{tt} w_e - 4\partial_{xx} w_e = 0, \quad x \in \mathbb{R},\ t > 0 \\
&w_e(x, 0) = f_e(x), \quad x \in \mathbb{R}, \\
&\partial_t w_e(x, 0) = 0, \quad x \in \mathbb{R}.
\end{aligned}$$

The solution to this problem is given by the D'Alembert formula

$$w_e(x, t) = \frac{1}{2}(f_e(x - 2t) + f_e(x + 2t)), \qquad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$$

Then $w(x, t) = w_e(x, t)$ for all $x \in (0, +\infty)$, since by construction $\partial_x w_e(0, t) = 0$ for all times.

Case 1: If $x - 2t > 0$, $f_e(x - 2t) = f(x - 2t)$; as a result

$$w(x, t) = \frac{1}{2}(f(x - 2t) + f(x + 2t), \quad \text{If } x - 2t > 0.$$

Case 2: If $x - 2t < 0$, $f_e(x - 2t) = f(-x + 2t)$; as a result

$$w(x, t) = \frac{1}{2}(f(-x + 2t) + f(x + 2t)), \quad \text{If } x - 2t < 0.$$

Note that $f(x) = (1 + x^2)^{-1}$; as a result, the solution can also be re-written as follows:

$$w(x, t) = \frac{1}{2}\left(\frac{1}{1 + (x - 2t)^2} + \frac{1}{1 + (x + 2t)^2}\right).$$