Scalar Conservation Equations

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Introduction

Goal

Implement a solver for scalar conservation equations that is (a) conservative, (b) preserves local maximum principle, (c) converges to the physically relevant solution.

Application

- Linear transport equation.
- Burgers' equation.

Conservation Equations:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where f is called the flux and u is the conserved quantity. Here f must be at least Lipschitz.

Method of Characteristics

If $X(t)$ is function that satisfies $\begin{cases} \frac{dX}{dt} = f'(u(X(t),t)) \\ X(0) = x_0 \end{cases}$ then $u(X(t),t) = u_0(x_0)$ for all time $t \geq 0$

An Example

Consider the flux $f(x) = \frac{u^2}{2}$, and $u_0 = \begin{cases} x+1 & -1 < x \leq 0 \\ 1-x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$. Solving the PDE with the given initial data gives

$$u(x,t) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{t+1} & -1 \leq x \leq t \\ \frac{x-1}{t-1} & t \leq x \leq 1 \\ 0 & 1 < x \end{cases}$$

However, this only makes sense for times $t \leq 1$, where at $t = 1$, the function gains a discontinuity. So we must find a weak solution.

Weak Solution

If we multiply the PDE by a smooth test function with compact support $\Phi(x)$ and apply integration by parts we get

$$\int_0^\infty \int_{\mathbb{R}} (u\partial_t\Phi + f(u)\partial_x\phi)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \Phi(x,0)u_0(x)\mathrm{d}x = 0$$

and we say u is a weak solution if this statement is true for all possible Φ.

In this example, solving this when $t >= 1$ gives

$$u(x,t) = \begin{cases} \frac{x+1}{t+1} & -1 < x < \sqrt{2t+2} - 1 \\ 0 & \text{o.w.} \end{cases}$$

Maximum Principle

Maximum Principle

Let u be a solution to the conservation equations. Then for all time $t \geq 0$, we have that $\max_x |u(x,t)| \leq \max_x |u_0|$

This gives stability to our solution and ensures that the solution staying within the bounds of the initial values.

Numerical Solution

Goal

Our goal is to find a numerical solution with analogs of:

1. Uniqueness
2. Conservation
3. Maximum principle

Creating a mesh

- We create a uniform mesh of x-points: $x_j = j \cdot \Delta x$
- We also discretize time t_n up to the final T of prediction (Δt is not constant)
- $\left\{ V_j^n \right\}$ = Value of numerical solution at (x_j, t_n)

Numerical Solution

Construction

- Define,

$$V(x, t_n) = \sum_j V_j^n \chi_{[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]}(x)$$

  where χ is the characteristic function.
- Define for $t_n < t < t_{n+1}$

$$V(x, t) = V(x, t_n)$$

- Our initial function is given by,

$$V_j^0 = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x) dx \tag{2}$$

Adding Diffusion

Diffusion

$$\text{Conservation Equation:} \quad u_t + f(u)_x = 0$$
$$\text{With Non-uniform Diffusion:} \quad u_t + f(u)_x = (\epsilon(u) \cdot u_x)_x$$

It turns out by using finite differences we get a time-stepping algorithm:

$$V_i^{n+1} = V_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)$$

Where $F_{i+\frac{1}{2}}$ depends on our chosen $\epsilon_{i+\frac{1}{2}}$

Stability

This time-stepping scheme gives us discrete conservation (and maximum principle):

$$\sum_i V_i^{n+1} = \sum_i V_i^n$$

Riemann Problem

Assume that f is Lipshitz continous: $\left|\frac{f(x)-f(y)}{x-y}\right| \leq L \in \mathbb{R}$

The Riemann Problem consists of solving the Cauchy problem (1) but where our initial data u_0:

$$u_0 := u(x,0) := \begin{cases} u_i & \text{if x} < 0 \\ u_{i+1} & \text{if x} > 0 \end{cases}$$

A change of variable (x,t)$\rightarrow$(βx,βt) does not change our PDE. So our solution can be written $U(x,t) = U(\frac{x}{t})$.

Riemann Solution

Assume that f is convex. Then f' is monotonically increasing.

1. If $u_i < u_{i+1}$, the left state (u_i) moves with speed $f'(u_i)$ and the right state (u_{i+1}) moves with speed $f'(u_{i+1})$. The two states are connected by monotone increasing profile called an expansion wave.
2. If $u_i > u_{i+1}$, the solution is a discontinuity that travels with speed s defined by the Rankine-Hugoniot condition:

$$s = \frac{f(u_i) - f(u_{i+1})}{u_i - u_{i+1}}$$

The max wave speed $\lambda_{i+\frac{1}{2}}$ is given by

$$\lambda_{i+\frac{1}{2}} = \begin{cases} \max(|f'(u_i)|, |f'(u_{i+1})|) & u_i < u_{i+1} \\ |s| & u_i > u_{i+1} \end{cases}$$

which allows us to define the diffusion: $\epsilon_{i+\frac{1}{2}} = \lambda_{i+\frac{1}{2}} \Delta x$

Algorithm

1. Create the mesh: $h = \frac{b-a}{N}$ where b is the right end point and a is the left end point.
2. Create the initial data $u_0(x)$, eg. the hat function
3. Compute $\lambda^n_{i+\frac{1}{2}}$ (Speed)

Case Study

Case 1: $f(u) = \beta u$. $\lambda_{i+\frac{1}{2}} = |\beta|$.

Case 2: Burgers $f(u) = \frac{u^2}{2}$.

- $u_i \geq u_{i+1}$: $\lambda_{i+\frac{1}{2}} = |\frac{u_i + u_{i+1}}{2}|$
- $u_i < u_{i+1}$: $\lambda_{i+\frac{1}{2}} = \max(|u_i|, |u_{i+1}|)$

Algorithm (Cont'd)

4. Compute Δt to satisfy CFL Condition

CFL Condition

$\forall i \in \mathbb{Z}$, find Δt such that

$$1 - 2\frac{\Delta t}{h}(\lambda_{i+\frac{1}{2}} + \lambda_{i-\frac{1}{2}}) \geq 0. \tag{3}$$

As u^n steps forward together, we can find

$$\lambda_{max} = \max\{\lambda_{1+\frac{1}{2}} + \lambda_{1-\frac{1}{2}}, ..., \lambda_{N+\frac{1}{2}} + \lambda_{N-\frac{1}{2}}\}.$$

That is, in one time step, we solve (2) and get that $\forall i \in \mathbb{Z}$, $\Delta t = \frac{h}{2\lambda_{max}}$.

- Note: If $t_n + \Delta t \geq T$, then reset $\Delta t = T - t_n$.

5. Advance in time $\{u_i^n\} \to \{u_i^{n+1}\}$

Discrete Version of True PDE: $u_t + f(u)_x = (\epsilon(u) \cdot u_x)_x$

- Forward Euler on u_t
- Central Difference on $f(u)_x$ and $(\epsilon(u) \cdot u_x)_x$
- Define $\lambda_{i+\frac{1}{2}} = \frac{\epsilon^n_{i+1/2}}{\Delta x}$.
- After the algebraic manipulation, we can get an explicit 3-point scheme approximation.

$$
\begin{aligned}
u_i^{n+1} = u_i^n &- \frac{\Delta t}{2h}(f(u_{i+1}^n) - f(u_{i-1}^n)) \\
&+ \frac{\Delta t}{h}(\lambda_{i+\frac{1}{2}} + (u_{i+1}^n - u_i^n) - \lambda_{i-\frac{1}{2}}(u_{i-1}^n - u_i^n))
\end{aligned}
$$

Error Tables

The error rate α is calculated using the following formula:

$$\alpha = \frac{log(\frac{e^{N-1}}{e^{N}})}{log(2)}$$

Refinement	Number of sub-intervals	L_1 Error	α
0	100	0.216306	-
1	200	0.129453	0.740
2	400	0.070520	0.876
3	800	0.035997	0.970
4	1600	0.018119	0.990
5	3200	0.009674	0.993

Table: Error rate α for Linear Transport on initial data: Hat function. $t_f = 1$.

Refinement	Number of sub-intervals	Error	α
0	100	0.098155	-
1	200	0.060255	0.703
2	400	0.035863	0.748
3	800	0.021263	0.754
4	1600	0.012357	0.782
5	3200	0.007074	0.804

Table: Error rate α for Burger's flux on initial data: Hat function. $t_f = 1$.

Animations

Play animations

Conclusion

In conclusion, when working with the nonlinear PDE (1).

1. Due to a lack of regularity, a weak solution is needed.
2. There is no unique weak solution unless some other restrictions are enforced.

In this presentation, we used the Forward Euler method on two cases.

1. Linear transport with initial data being the hat function.
2. Burger's Equation with initial data being the hat function.

To show the accuracy of the method described today:

1. The exact solution in both cases was compared against an approximated solution.
2. Lastly, the rate of error was calculated when a finer discretization is taken.