Finite Difference Methods

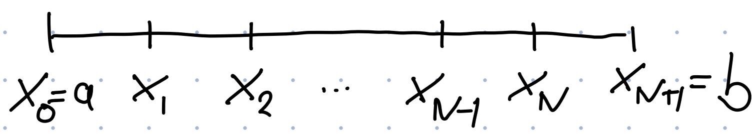
- How to approximate the solution $u(t,x)$ to

$$\underbrace{\cancel{\partial_t u}}_{=0} - p\partial_x^2 u + q\partial_x u + ru = f \quad \text{for } a<x<b$$

(for now)

with $u(a) = g_a$ and $u(b) = g_b$

- Spatial discretization: Subdivide interval $\Omega = [a,b]$ into $N+1$ subintervals of equal size: $x_i = a + ih$, $h = \dfrac{b-a}{N+1}$



Approximate a differential with a difference operator:

$$f'(x) \approx \delta_h^+ f(x) := h^{-1}\{f(x+h) - f(x)\} \quad \text{forward difference}$$

$$f'(x) \approx \delta_h^- f(x) := h^{-1}\{f(x) - f(x-h)\} \quad \text{backward difference}$$

$$f'(x) \approx \delta_h f(x) := (2h)^{-1}\{f(x+h) - f(x-h)\} \quad \text{central difference}$$

Remark: How good are these discretization approaches?

Forward difference: use Taylor series expansion of u at around x:

$$\delta_h^+ f(x) = h^{-1}\left(f(x+h) - f(x)\right)$$

$$= h^{-1}\left(f(x) + h f'(x) + \frac{1}{2}h^2 f''(\xi) - f(x)\right) \quad \text{for some } \xi_i \in (x, x+h)$$

$$= f'(x) + \frac{1}{2} h f''(\xi).$$

The forward difference operator is a first order approximation:

$$\left| \delta_h^+ f(x) - f'(x) \right| \leq \frac{1}{2} h \max_{\xi \in \Omega} f''(\xi)$$

"half the mesh size half the error"

Backward difference: same (verify!)

Central difference: apply Taylor series expansion to $f(x_i - h)$ as well:

$$\delta_h f(x) = (2h)^{-1}\left(f(x+h) - f(x-h)\right)$$

$$= (2h)^{-1}\Big(f(x) + h f'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(\xi^+) - \left\{f(x) - h f'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(\xi^-)\right)\Big)$$

$$= (2h)^{-1}\Big(2h f'(x) + \frac{1}{6}h^3 \underbrace{\{f'''(\xi^-) + f'''(\xi^+)\}}_{= 2 f'''(\xi)}\Big)$$

choose ξ such that $\rightarrow$ (intermediate value theorem)

$$= f'(x) + \frac{1}{6}h^2 f'''(\xi)$$

The central difference operator is a second order approx.:

$$|\delta_h f(x) - f'(x)| \leq \frac{1}{6}h^2 \max_{\xi\in\Omega} f'''(\xi)$$

Applying this to the second derivative:

$$\partial_x^2 u(x) \approx \delta_{h/2}(\delta_{h/2} u)$$

$$= \delta_{h/2}\left(h^{-1}\left(u(x+h/2) - u(x-h/2)\right)\right)$$

$$= h^{-1}\left(h^{-1}\{u(x+h) - u(x)\} - h^{-1}\{u(x) - u(x-h)\}\right)$$

$$= h^{-2}\left(u(x+h) - 2u(x) + u(x-h)\right) = \Delta_h^{(3)} u(x)$$

(also called the "three point stencil")

This is again a second order approximation:

$$|\partial_x^2 u(x) - \Delta_h^{(3)} u(x)| \leq \frac{1}{12} h^2 \max_{\xi\in\Omega} |u''''(\xi)|$$

(Lab)

Applying this to $-p\partial_x^2 u + q\partial_x u + ru = f$:

not equal

$$-p\,\Delta_h^{(3)} u(x_i) + q\,\delta_h u(x_i) + r u(x_i) \approx f(x_i)$$

$$\Rightarrow \quad -p\big(u(x_{i+1}) - 2u(x_i) + u(x_{i-1})\big) + \frac{qh}{2}\big(u(x_{i+1}) - u(x_i)\big) + r h^2 u(x_i) \approx h^2 f(x_i)$$

Rearranging:

$$\left\{-p + \tfrac{1}{2}qh\right\} u(x_{i+1}) + \left\{2p + rh^2\right\} u(x_i) + \left\{-p - \tfrac{1}{2}qh\right\} u(x_{i-1}) \approx h^2 f(x_i)$$

Idea: Use this to construct a discrete approximation

Task (Finite difference approximation)

Construct $\{U_i\}_{i=0}^{N+1}$ with $U_i \approx u(x_i)$ by requiring:

$$\begin{cases} \left(-p + \frac{qh}{2}\right) U_{i+1} + (2p + rh^2) U_i + \left(-p - \frac{qh}{2}\right) U_{i-1} = h^2 f(x_i) \quad \text{for } i = 1, \dots, N \\ U_0 = g_a \text{ and } U_{N+1} = g_b \end{cases}$$

Note: This is a linear system of equations!

To make this clear let's device a matrix-vector form:

Solution vector: $\mathcal{U} := [U_1, \dots, U_N]^T \in \mathbb{R}^N$

and matrix $A \in \mathbb{R}^{N \times N}$:

$$A = \frac{1}{h^2} \begin{bmatrix} 2p+rh^2 & -p+\frac{1}{2}qh & & & \\ -p-\frac{1}{2}qh & 2p+rh^2 & -p+\frac{1}{2}qh & & \\ & \ddots & \ddots & \ddots & \\ & & -p-\frac{1}{2}qh & 2p+rh^2 & -p+\frac{1}{2}qh \\ & & & -p-\frac{1}{2}qh & 2p+rh^2 \end{bmatrix}$$

and a right hand side $B \in \mathbb{R}^N$ with

$$B = \left[f(x_1) + \frac{(p+\frac{1}{2}qh)}{h^2} g_a, f(x_2), \dots, f(x_{N-1}), f(x_N) + \frac{(p-\frac{1}{2}qh)}{h^2} g_b \right]^T$$

And we can write

$$\boxed{\text{Find } \mathcal{U} \in \mathbb{R}^N \text{ such that } A\mathcal{U} = B \quad *}$$