Lecture 3: 1D wave equation

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1 Objectives

The goal of this lecture is to investigate the following linear partial differential equation: Find $u : \mathbb{R}\times\mathbb{R}_{>0} \to \mathbb{R}$ so that

$$\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}_{>0} \tag{1a}$$
$$u(x,0) = f(x), \qquad x \in \mathbb{R}, \tag{1b}$$
$$\partial_t u(x,0) = g(x), \qquad x \in \mathbb{R}, \tag{1c}$$
$$\lim_{x\to-\infty}(u(x,t) - f(x)) = 0, \quad \lim_{x\to+\infty}(u(x,t) - f(x)) = 0 \qquad \forall t \in \mathbb{R}_{>0}. \tag{1d}$$

Here $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are initial data. Above equation is called wave equation. The constant is called wave speed.

In this lecture we construct a solution to this problem using the Fourier transform technique.

2 Fourier transform

2.1 Definitions and properties

We denote by $L^1(\mathbb{R};\mathbb{C})$ the space composed of those complex-valued functions that are integrable over the entire real line. That is

$$L^1(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f(x)| \,\mathrm{d}x < \infty\}, \tag{2}$$

and we set $\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| \,\mathrm{d}x$.

Definition 2.1 (Fourier transform)**.** *Let $f \in L^1(\mathbb{R};\mathbb{C})$. We define the Fourier transform of f, the function $\mathcal{F}(f) : \mathbb{R} \to \mathbb{C}$ so that*

$$\mathcal{F}(f)(\omega) := \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{\mathrm{i}\omega x} \, \mathrm{d}x. \tag{3}$$

Notice that this definition makes sense since $|\int_{\mathbb{R}} f(x) e^{\mathrm{i}\omega x} \, \mathrm{d}x| \leq \int_{\mathbb{R}} |f(x)||e^{\mathrm{i}\omega x}| \, \mathrm{d}x = \|f\|_{L^1(\mathbb{R})} < \infty$.

Definition 2.2 (Inverse Fourier transform)**.** *Let $f \in L^1(\mathbb{R};\mathbb{C})$. We define the Fourier transform of f, the function $\mathcal{F}^{-1}(f) : \mathbb{R} \to \mathbb{C}$ so that*

$$\mathcal{F}^{-1}(f)(\omega) := \int_{\mathbb{R}} f(x) e^{-\mathrm{i}\omega x} \, \mathrm{d}x. \tag{4}$$

Remark 2.1 (Conventions)**.** There are many ways to define the Fourier transform. Some authors use (4) to define the Fourier transform and use (3) to define the inverse Fourier transform. Some other authors define the Fourier transform with the factor $\frac{1}{\sqrt{(2\pi)}}$ instead of $\frac{1}{2\pi}$ and modify the definition of the inverse Fourier transform accordingly. All these and just conventions.

Example 2.2. Here are examples of Fourier transform of some standard functions.

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi} \frac{\alpha}{\omega^2 + \alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2 + \alpha^2})(\omega) = e^{-\alpha|\omega|}, \tag{5}$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}} \tag{6}$$

$$\mathcal{F}(H(x)e^{-ax})(\omega) = \frac{1}{2\pi} \frac{1}{a - \mathrm{i}\omega}, \qquad H \text{ is the Heaviside function} \tag{7}$$

$$\mathcal{F}(\mathrm{pv}(\frac{1}{x}))(\omega) = \frac{\mathrm{i}}{2} \mathrm{sign}(\omega) = \frac{\mathrm{i}}{2}(H(\omega) - H(-\omega)) \tag{8}$$

$$\mathcal{F}(\mathrm{sech}(ax)) = \frac{1}{2a} \mathrm{sech}(\frac{\pi}{2a}\omega), \qquad \mathrm{sech}(ax) := \frac{1}{\mathrm{ch}(x)} = \frac{2}{e^x + e^{-x}}. \tag{9}$$

The two important results of this section are the following.

Theorem 2.3 (Inverse Fourier transform)**.** *Let $f \in L^1(\mathbb{R};\mathbb{C})$. Assume also that f is of class C^1, then*

$$\mathcal{F}^{-\infty}(\mathcal{F}(\{))(\S) = \{(\S), \quad \forall \S \in \mathbb{R}. \tag{10}$$

If f is discontinuous at x_0 but is of class C^1 on the left and on the right of x_0, then

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \frac{1}{2}(f(x_0^-) + f(x_0^+)). \tag{11}$$

Theorem 2.4 (Derivation)**.** *Let $f \in L^1(\mathbb{R};\mathbb{C})$ and assume also that $\partial_x f \in L^1(\mathbb{R};\mathbb{C})$. Then*

$$\boxed{\mathcal{F}(\partial_x f)(\omega) = -\mathrm{i}\omega\mathcal{F}(f)(\omega)}, \quad \forall\omega \in \mathbb{R}. \tag{12}$$

Notice that the above result can be applied again to $\partial_x f$ if $\partial_{xx} f \in L^1(\mathbb{R};\mathbb{C})$, and in this case we obtain $\mathcal{F}(\partial_{xx} f)(\omega) = (-\mathrm{i}\omega)^2\mathcal{F}(f)(\omega)$.

Remark 2.3 (The importance of the Fourier transform)**.** Theorem 2.4 shows that the Fourier transform changes derivations into products by $-\mathrm{i}\omega$

Remark 2.4 (Time and space Commutation)**.** Let $f : \mathbb{R}\times\mathbb{R}_{\geq 0}$. Assume that for all $t \in \mathbb{R}_{\geq 0}$, $f(\cdot,t) \in L^1(\mathbb{R};\mathbb{C})$. Assume also that $\partial_t f(\cdot,t) \in L^1(\mathbb{R};\mathbb{C})$. Then the definition of the Fourier transform implies that

$$\boxed{\mathcal{F}(\partial_t f(\cdot,t))(\omega) = \partial_t\mathcal{F}(f)(\omega,t)}. \tag{13}$$

We finish with a result called shift lemma in the literature

Lemma 2.5 (Shift Lemma)**.** *Let $f \in L^1(\mathbb{R};\mathbb{C})$. Let $\beta \in \mathbb{R}$. Then*

$$\boxed{\mathcal{F}(f(x-\beta))(\omega) = e^{\mathrm{i}\beta\omega}\mathcal{F}(f)(\omega)}, \tag{14}$$

2.2 Convolution

We now introduce the notion of convolution product.

Definition 2.6 (Convolution product)**.** *Let $f \in L^1(\mathbb{R};\mathbb{C})$ and $g \in L^1(\mathbb{R};\mathbb{C})$. We define a new function denoted $f * g$ and called convolution product of f and g the function defined by*

$$f * g(x) := \int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y. \tag{15}$$

Lemma 2.7 (Commutation)**.** *For all $f \in L^1(\mathbb{R};\mathbb{C})$ and all $g \in L^1(\mathbb{R};\mathbb{C})$,*

$$f * g = g * f. \tag{16}$$

The following result is essential for the rest of the lecture.

Theorem 2.8 (Convolution theorem)**.** *$f \in L^1(\mathbb{R};\mathbb{C})$ and $g \in L^1(\mathbb{R};\mathbb{C})$. Then*

$$\boxed{\mathcal{F}(f * g) = 2\pi\mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega)}. \tag{17}$$

Remark 2.5 (The importance of the convolution theorem)**.** Theorem 2.8 shows that the product of two Fourier transform, is actually a Fourier transform. If particular, if f and g are both of class C^1, this theorem shows that

$$\mathcal{F}^{-1}(\mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega)) = \frac{1}{2\pi} f * g. \tag{18}$$

3 The d'Alembert Formula

We use the Fourier transform technique to solve the wave equation in this section.

3.1 Fourier transform technique

Let us take the Fourier transform of (1).

$$\mathcal{F}(\partial_{tt}u)(\omega,t) - c^2\mathcal{F}(\partial_{xx}u)(\omega,t) = 0$$

Now we use the commutation result (13) and the (12).

$$\partial_{tt}\mathcal{F}(u)(\omega,t) + c^2\omega^2\mathcal{F}(u)(\omega,t) = 0.$$

The above equation is a linear second-order ordinary differential equation. The solution to this equation can be written in the following form:

$$\mathcal{F}(u)(\omega,t) = A(\omega)e^{\mathsf{i}\omega ct} + B(\omega)e^{-\mathsf{i}\omega ct}.$$

We now compute the function $A(\omega)$ and $B(\omega)$ by matching the initial data. We must have

$$\begin{aligned}\mathcal{F}(u)(\omega,0) &= A(\omega) + B(\omega),\\ \partial_t\mathcal{F}(u)(\omega,0) = \mathcal{F}(\partial_t u)(\omega,0) &= \mathsf{i}\omega cA(\omega)e^{\mathsf{i}\omega ct} - \mathsf{i}\omega c\mathsf{B}(\omega)\mathsf{e}^{-\mathsf{i}\omega\mathsf{ct}}.\end{aligned}$$

But $\mathcal{F}(u)(\omega,0) := \mathcal{F}(u(\cdot,0))(\omega) = \mathcal{F}(f)(\omega)$ and $\mathcal{F}(\partial_t u)(\omega,0) := \mathcal{F}(\partial_t u(\cdot,0))(\omega) = \mathcal{F}(g)(\omega)$. Hence,

$$\begin{aligned}\mathcal{F}(f)(\omega) &= A(\omega) + B(\omega),\\ \mathcal{F}(g)(\omega,0) &= \mathsf{i}\omega cA(\omega)e^{\mathsf{i}\omega ct} - \mathsf{i}\omega cB(\omega)e^{-\mathsf{i}\omega ct}.\end{aligned}$$

Solving the above linear system for $A(\omega)$ and $B(\omega)$ gives

$$\begin{aligned}A(\omega) &= \frac{1}{2}\mathcal{F}(f)(\omega) + \frac{1}{2\mathsf{i}\omega c}\mathcal{F}(g)(\omega)\\ B(\omega) &= \frac{1}{2}\mathcal{F}(f)(\omega) - \frac{1}{2\mathsf{i}\omega c}\mathcal{F}(g)(\omega).\end{aligned}$$

Finally,

$$\begin{aligned}\mathcal{F}(u)(\omega,t) &= \frac{1}{2}\left(\mathcal{F}(f)(\omega) + \frac{1}{\mathsf{i}\omega c}\mathcal{F}(g)(\omega)\right)e^{\mathsf{i}\omega ct} + \frac{1}{2}\left(\mathcal{F}(f)(\omega) - \frac{1}{\mathsf{i}\omega c}\mathcal{F}(g)(\omega)\right)e^{\mathsf{i}\omega ct}\\ &= \frac{1}{2}\left(\mathcal{F}(f)(\omega)e^{\mathsf{i}\omega ct} + \mathcal{F}(f)(\omega)e^{-\mathsf{i}\omega ct} + \frac{1}{\mathsf{i}\omega c}\mathcal{F}(g)(\omega)e^{\mathsf{i}\omega ct} - \frac{1}{\mathsf{i}\omega c}\mathcal{F}(g)(\omega)e^{-\mathsf{i}\omega ct}\right)\end{aligned}$$

The shift lemma 2.5 gives

$$\mathcal{F}(f)(\omega)e^{\mathrm{i}\omega ct} + \mathcal{F}(f)(\omega)e^{-\mathrm{i}\omega ct} = \mathcal{F}(f(x-ct) + f(x+ct))(\omega).$$

Now let us define $G(x) = \int_0^x g(\xi)\,\mathrm{d}\xi$. Then $\partial_x G(x) = g(x)$ and $-\mathrm{i}\omega\mathcal{F}(G)(\omega) = \mathcal{F}(g)(\omega)$. This shows that

$$\frac{1}{\mathrm{i}\omega}\mathcal{F}(g)(\omega)e^{\mathrm{i}\omega ct} - \frac{1}{\mathrm{i}\omega}\mathcal{F}(g)(\omega)e^{-\mathrm{i}\omega ct} = -\mathcal{F}(G)(\omega)e^{\mathrm{i}\omega ct} + \mathcal{F}(G)(\omega)e^{-\mathrm{i}\omega ct}.$$

And the shift lemma gives

$$\frac{1}{\mathrm{i}\omega}\mathcal{F}(g)(\omega)e^{\mathrm{i}\omega ct} - \frac{1}{\mathrm{i}\omega}\mathcal{F}(g)(\omega)e^{-\mathrm{i}\omega ct} = -\mathcal{F}(G(x-ct) + G(x+ct))(\omega).$$

Putting everything together, we obtain

$$\begin{aligned}\mathcal{F}(u)(\omega,t) &= \mathcal{F}\left(\frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}(G(x+ct) - G(x-ct))\right) \\ &= \mathcal{F}\left(\frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,\mathrm{d}\xi\right).\end{aligned}$$

Using the inverse Fourier transform lemma 2.3, we have established the following result.

Theorem 3.1 (D'alembert formula)**.** *The unique weak solution to* (1) *is*

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,\mathrm{d}\xi. \tag{19}$$

3.2 Uniqueness

Let us convince ourselves that if f and g are smooth enough, then the solution to (1) given by (3.1) is unique. We proceed by using the energy method. We multiply (1a) by $\partial_t u$ and integrate over space.

$$\begin{aligned}0 = \int_{\mathbb{R}} \left(\partial_t u \partial_{tt} u - c^2 \partial_t u \partial_{xx} u\right) \mathrm{d}x &= \int_{\mathbb{R}} \left(\partial_t \frac{1}{2}(\partial_t u)^2 + c^2 \partial_t \partial_x u \partial_x u\right) \mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\partial_t \frac{1}{2}(\partial_t u)^2 + c^2 \partial_t \frac{1}{2}(\partial_x u)^2\right) \mathrm{d}x \\ &= \partial_t \left(\int_{\mathbb{R}} \left(\frac{1}{2}(\partial_t u)^2 + c^2 \frac{1}{2}(\partial_x u)^2\right) \mathrm{d}x\right).\end{aligned}$$

After integrating over the time interval $(0,T)$, we obtain the a priori estimate

$$\|\partial_t u\|^2_{L^2(\mathbb{R})} + c^2\|\partial_x u\|^2_{L^2(\mathbb{R})} = \|g\|^2_{L^2(\mathbb{R})} + c^2\|\partial_x f\|^2_{L^2(\mathbb{R})}.$$

This estimate implies uniqueness in the class of solutions for which $[0,\infty) \ni t \mapsto \|\partial_t u\|^2_{L^2(\mathbb{R})} + c^2\|\partial_x u\|^2_{L^2(\mathbb{R})}$ is continuous and bounded, which is the case for the solution given by the d'Alembert formula if f and g are smooth enough.

References