Exercises

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1 Lecture 1

Question 1: Let $y(x,t) = x\cos(2t + \log(|x|))$. Compute $\partial_{tt} y + x^2 \partial_{xx} y - x\partial_x y + 6y$.

Question 2: Let $u, f : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions of class C^1. (a) Compute $\partial_x f(u(x))$.

(b) Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be functions of class C^1. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $F(v) = \int_0^v f'(t)\psi'(t)\,\mathrm{d}t$. Use (a) to compute $\partial_x(F(u(x)) - \partial_x(f(u(x)))\psi'(u(x))$.

(c) Using the notation of (a) and (b), assume that $u(\pm\infty) = 0$ and compute $\int_{-\infty}^{+\infty} \partial_x(f(u(x)))\psi'(u(x))\,\mathrm{d}x$.

Question 3: Let u solve $\partial_t u - \partial_x((3x+1)\partial_x u) = -3$, $x \in (0, L)$, with $\partial_x u(0,t) = 1$, $\partial_x u(L,t) = \alpha$, $u(x,0) = f(x)$.

(a) Compute $\int_0^L u(x,t)dx$ as a function of t.

(b) For which value of α the quantity $\int_0^L u(x,t)dx$ does not depend on t?

Question 4: Let u solve $\partial_t u + \partial_x\big(v(x,t)u - \mu(x,t)\partial_x u\big) = g(x)e^{-t}$, $x \in (0,L)$, $t > 0$, with $\mu(0,t)\partial_x u(0,t) = 1$, $\mu(L,t)\partial_x u(L,t) = 1 + 2e^{-t}$, $u(x,0) = f(x)$, where v, $\mu > 0$, f and g are four smooth functions and $v(0) = v(L) = 0$.

(a) Compute $\frac{d}{dt}\int_0^L u(x,t)dx$ as a function of t.

(b) Use (a) to compute $\int_0^L u(x,t)dx$ as a function of t.

(c) What is the limit of $\int_0^L u(x,t)dx$ as $t \to +\infty$?

Question 5: Consider the differential equation $-\frac{d^2\phi}{dt^2} = \lambda\phi,\ t \in (0, \pi)$, supplemented with the boundary conditions $\phi(0) = 0,\ 3\phi(\pi) = -\phi'(\pi)$.

(a) Prove that it is necessary that λ be positive for a non-zero solution to exist.

(b) Find the equation that λ must solve for the above problem to have a nonzero solution.

2 Lecture 2

Question 6: Consider the following conservation equation $\partial_t \rho + \partial_x(q(\rho)) = 0$, $x \in (-\infty, +\infty)$, $t > 0$, $\rho(x, 0) = \frac{1}{2}$ if $x < 0$, and $\rho(x, 0) = 1$ if $x > 0$. where $q(\rho) = 2\rho + 3\rho^3 - \sin(\rho^2)$ (and $\rho(x, t)$ is the conserved quantity). What is the wave speed for this problem?

Question 7: Consider the following conservation equation $\partial_t \rho + \partial_x(q(\rho)) = 0$, $x \in (-\infty, +\infty)$, $t > 0$, $\rho(x, 0) = \frac{1}{2}$ if $x < 0$, and $\rho(x, 0) = 1$ if $x > 0$. where $q(\rho) = 2\rho + 3\rho^3 - \sin(\rho^2)$ (and $\rho(x, t)$ is the conserved quantity). What is the wave speed for this problem?

Question 8: Consider the conservation equation with flux $q(\rho) = \rho^4$. Assume that the initial data is $\rho_0(x) = 2$, if $x < 0$, $\rho_0(x) = 1$, if $0 < x < 1$, and $\rho_0(x) = 0$, if $1 < x$. (i) Draw the characteristics

(ii) Describe qualitatively the nature of the solution.

(iii) When does the left shock catch up with the right one?

Question 9: Consider the conservation equation $\partial_t \rho + \partial_x(\sin(\frac{\pi}{2}\rho)) = 0$, $x \in \mathbb{R}$, $t > 0$, with initial data $\rho_0(x) = 0$ if $x < 0$ and $\rho_0(x) = 1$ if $x > 0$. Draw the characteristics and give the explicit representation of the solution.

Question 10: For all $k \in \mathbb{R}$, consider the entropy $\eta(v,k) := |v-k|$. Compute the entropy flux associated with this entropy, $q(v)$, with the normalization $q(k) := 0$.

Question 11: Consider Burgers' equation with $D := \mathbb{R}$ and $u_0(x) := H(x)$, where H is the Heaviside function. (a) Verify that $u_1(x,t) := H(x - \frac{1}{2}t)$ and $u_2(x,t) := 0$ if $x < 0$, $u_2(x,t) := \frac{x}{t}$, if $0 < x < t$, $u_2(x,t) := 1$ if $x > t$, are both weak solutions.

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(ii) Verify that u_1 does not satisfy the entropy inequalities, whereas u_2 does.

3 Lecture 3

Question 12: Consider the quasilinear Klein-Gordon equation: $\partial_{tt}\phi(x,t) - c^2\partial_{xx}\phi(x,t) + m^2\phi(x,t) + \beta^2\phi^3(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $\phi(x,0) = f(x)$, $\partial_t\phi(x,0) = g(x)$ and $\phi(\pm\infty,t) = 0$, $\partial_t\phi(\pm\infty,t) = 0$, $\partial_x\phi(\pm\infty,t) = 0$. Find an energy $E(t)$ which is invariant with respect to time (Hint: test with $\partial_t\phi(x,t)$ and use $\phi^p\phi' = (\frac{1}{p+1}\phi^{p+1})'$.)

Question 13: (a) Compute the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Find the inverse Fourier transform of $g(\omega) = \frac{\sin(\omega)}{\omega}$.

Question 14: Use the Fourier transform technique to solve the following ODE $y''(x) - y(x) = f(x)$ for $x \in (-\infty, +\infty)$, with $y(\pm\infty) = 0$, where f is a function such that $|f|$ is integrable over $\mathbb{R}$.

Question 15: Solve the integral equation: $f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1} dy = \frac{1}{x^2+4} + \frac{1}{x^2+1}$, for all $x \in (-\infty, +\infty)$.

Question 16: Use the Fourier transform method to solve the equation $\partial_t u + \frac{2t}{1+t^2} \partial_x u = 0$, $u(x, 0) = u_0(x)$, in the domain $x \in (-\infty, +\infty)$ and $t > 0$.

Question 17: Use the Fourier transform technique to solve $\partial_t u(x,t) + \sin(t)\partial_x u(x,t) + (2 + 3t^2)u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$. (Use the shift lemma: $\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta}$ and the definition $\mathcal{F}(f)(\omega) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx$)

Question 18: Solve the PDE

$$
\begin{aligned}
&u_{tt} - a^2 u_{xx} = 0, && -\infty < x < +\infty, \quad 0 \leq t,\\
&u(x,0) = \cos(x), \quad u_t(x,0) = -a\sin(x), && -\infty < x < +\infty.
\end{aligned}
$$

Question 19: Solve the wave equation on the semi-infinite domain $(0, +\infty)$,

$\partial_{tt} w - 4\partial_{xx} w = 0, \quad x \in (0, +\infty),\ t > 0$
$w(x, 0) = (1 + x^2)^{-1}, \quad x \in (0, +\infty); \qquad \partial_t w(x, 0) = 0, \quad x \in (0, +\infty);$ and $\partial_x w(0, t) = 0, \quad t > 0.$

(Hint: Consider a particular extension of w over $\mathbb{R}$)