Lecture 2: 1D Scalar conservation equations

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1 Objectives

The goal of this lecture is to investigate the following nonlinear partial differential equation: Find $u : \mathbb{R}\times\mathbb{R}_{>0} \to \mathbb{R}$ so that

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}_{>0} \tag{1a}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1b}$$

$$\lim_{x\to-\infty}(u(x,t) - u_0(x)) = 0, \quad \lim_{x\to+\infty}(u(x,t) - u_0(x)) = 0 \qquad \forall t \in \mathbb{R}_{>0}. \tag{1c}$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function.

Remark 1.1 (Terminology)**.** We say that f is the *flux* of the problem (1) and u_0 is the *initial data*. Typical examples are

(i) Linear transport: $f(v) = \beta v$, $\beta \neq 0$.

(ii) Burgers equations: $f(v) = \frac{1}{2}v^2$,

(iii) Traffic flow equation: $f(\rho) = u_{\max}\rho(1 - \frac{\rho}{\rho_{\max}})$,

(iv) Buckley-Leverett equation: $f(v) = \frac{v^2}{v^2+(1-v)^2}$.

Remark 1.2 (Existence, uniqueness, stability)**.** Three natural questions come to mind when considering (1):

(i) Does the problem (1) has a solution? This is an *existence* question.

(ii) If a solution exists, is it unique? This is a *uniqueness* question.

(iii) If a unique solution exists, is the solution operator $(f, u_0) \mapsto u$ continuous in some sense? In other words, if two source/initial data pairs (f^1, u_0^1), (f^2, u_0^2) are close, are the corresponding solutions u^1, u^2 close? This is a *stability/continuity* question.

2 Method of characteristics

In this section we describe a method to construct a solution to (1) assuming that the solution is continuous in space at all times. The method is based on the notion of characteritic curves (or just characteristics).

2.1 The characteristic curves

Let us assume that (1) has a unique solution $u(x,t)$ and let us assume that this solution is locally Lipschitz with respect to x and continuous with respect to t, at least over some time interval $t \in (0,T^*)$. The idea is to introduce a change a variable based on u. We introduce a new function $X : \mathbb{R}\times\mathbb{R}_{>0}$ which we define to be the solution to the nonlinear ordinary differential equation

$$\partial_t X(s,t) = f'(u(X(s,t),t)), \quad s \in \mathbb{R},\ t \in (0,T^*) \tag{2a}$$

$$X(s,0) = s, \quad s \in \mathbb{R}. \tag{2b}$$

Notice that, owing to the assumption we made on the solution u, the Cauchy-Lipchitz theorem (a.k.a. Picard–Lindelöf theorem) implies that $X(s,t)$ is well defined for all $s \in \mathbb{R}$ and all $t \in (0,T^*)$. For the time being the situation looks desperate since $X(s,t)$ is defined by invoking u which is still unknown, but a little miracle will happen and will solve this conondrum.

Remark 2.1 (Terminology)**.** For each $s \in \mathbb{R}$ the curve $(X(s,t),t)$ in the half plane $\mathbb{R}\times\mathbb{R}_{>0}$ is called characteristic curve of (1) passing through s at $t = 0$.

Remark 2.2 ($f'(v)$)**.** Be aware that $f'(v)$ means *derivative* of f at v; that is, $f'(v)$ is another way of writting $\partial_v f(v)$. For instance, if $f(v) = \frac{1}{2}v^2$, then $f'(v) = v$, and if $f(\rho) = u_{\max}\rho(1-\frac{\rho}{\rho_{\max}})$, then $f'(\rho) = u_{\max}(1-2\frac{\rho}{\rho_{\max}})$. Do not confuse $f'(v(x,t))$ with either $\partial_t f(v(x,t))$ or $\partial_x f(v(x,t))$.

2.2 The change of variable

We now define a new function $\phi : \mathbb{R}\times\mathbb{R}_{t>0} \to \mathbb{R}$ by setting

$$\phi(s,t) := u(X(s,t),t). \tag{3}$$

Let us compute $\partial_t \phi(s,t)$. Using the chain rule (9), we have

$$\partial_t \phi(s,t) = (\partial_x u)(X(s,t),t)\partial_x X(s,t) + (\partial_t u)(X(s,t),t)\partial_t t \tag{4}$$

$$= (\partial_x u)(X(s,t),t) f'(u(X(s,t))) + (\partial_t u)(X(s,t),t). \tag{5}$$

Recall that the chain rule applied to univariate functions gives $\partial_x f(u(y,z)) := \partial_x(f \circ u)(y,z)) = f'(u(y,z))(\partial_x u)(y,z)$. Hence, $(\partial_x u)(X(s,t),t)f'(u(X(s,t))) = \partial_x(f \circ u)(X(s,t),t)$. In conclusion,

$$\partial_t \phi(s,t) = (\partial_t u + \partial_x(f \circ u))(X(s,t),t) = 0, \tag{6}$$

because u solves (1). This means that

$$u(X(s,t),t) := \phi(s,t) = \phi(s,0) = u(X(s,0),0) = u(s,0) = u_0(s). \tag{7}$$

In conclusion, we have established that

$$\boxed{u(X(s,t),t) = u_0(s).} \tag{8}$$

In other words, $u(X(s,t),t)$ does not depend on t.

Remark 2.3 (Chain rule for two-variate functions)**.** Consider the two-variate functions $v(\alpha,\beta)$, $Y(s,t)$, $Z(s,t)$. We recall that the chain rule for two-variate functions is as follows

$$\partial_t(v(Y(p,q),Z(p,q))) = (\partial_\alpha v)(Y(p,q),Z(p,q))(\partial_t Y)(p,q) \tag{9}$$
$$+ (\partial_\beta v)(Y(p,q),Z(p,q))(\partial_t Z)(p,q). \tag{10}$$

2.3 Computing the characteristics

One can now solve the ordinary differential equation (2) since the right-hand side of (2a) does not depend on t. Integrating (2a) over $(0,t)$, we obtain, for all $s \in \mathbb{R}$,

$$X(s,t) - X(s,0) = tu_0(s). \tag{11}$$

Hence, for all $s \in \mathbb{R}$,

$$\boxed{X(s,t) = s + tf'(u_0(s)).} \tag{12}$$

2.4 Implicit representation of the solution

The solution to (1) over the time interval $(0,T^*)$ is given by

$$\boxed{\begin{aligned} u(X(s,t),t) &= u_0(s), \\ X(s,t) &= s + tf'(u_0(s)). \end{aligned}} \tag{13}$$

Given $s \in \mathbb{R}$ and $t > 0$, the value of u at the location $X(s,t)$ and time t is $u_0(s)$.

2.5 Explicit representation of the solution

Obtaining an explicit representation of the solution to (1) using the methods of characteristics is in general nontrivial. This is done by expressing s as a function of x and t.

Let x in $\mathbb{R}$, t in $\mathbb{R}_{>0}$, and let us compute $u(x,t)$. To do so we must find $s \in \mathbb{R}$ so that

$$s + tf'(u_0(s)). = x. \tag{14}$$

This equation is nonlinear due to the presence of $f'(u_0(s))$. Let us apply the implicit function theorem to the equation $G(s) = 0$ with $G(s) := s + tf'(u_0(s)) - x$. Let us assume that $f \in C^2(\mathbb{R})$. If $1 + tf''(u_0(s))\partial_x u_0(s) \neq 0$, then there is $S(x,t)$ so that $G(S(x,t)) = 0$. With the function $S(x,t)$ this defined, we have

$$u(x,t) = u_0(S(x,t)). \tag{15}$$

We have proved the following result.

Theorem 2.1 (Existence time for a strong solution)**.** *Assume that f is of class C^2, u_0 is of class C^1, and $\inf_{s\in\mathbb{R}} \min(f''(u_0(s))u_0'(s), 0) > -\infty$. Then* (1) *has a unique strong solution over the time interval $[0, T^*)$, where $T^* := \infty$ if $\inf_{s\in\mathbb{R}} f''(u_0(s))u_0'(s) \geq 0$ and otherwise we have*

$$T^* := \inf_{s\in\mathbb{R}} \frac{-1}{\min(f''(u_0(s))u_0'(s), 0)} < \infty. \tag{16}$$

2.6 Examples

Let us assume that u_0 is of class C^1. If u_0 and f are such that $1+tf''(u_0(s))\partial_x u_0(s) \neq 0$ for all $s \in \mathbb{R}$ and all $t > 0$, then $S(x,t)$ is always well defined. In this case $T^* = \infty$.

This above situation occurs if f is convex (i.e., $f''(v) \geq 0$ for all $v \in \mathbb{R}$) and u_0 is monotone increasing function (i.e., $\partial_x u_0(s) \geq 0$ for all $s \in \mathbb{R}$. The same conclusion holds if f is concave (i.e., $f''(v) \leq 0$ for all $v \in \mathbb{R}$) and u_0 is monotone decreasing function (i.e., $\partial_x u_0(s) \leq 0$ for all $s \in \mathbb{R}$.

2.6.1 Linear transport

Let us illustrate this situtation for the linear transport case $f(v) = \beta v$. In this case we have $f'(v) = \beta$ and $f''(v) = 0$. Then $S(x,t) = x - \beta t$. In conclusion, for all $x \in \mathbb{R}$ and all $t > 0$,

$$u(x,t) = u_0(x - \beta t). \tag{17}$$

2.6.2 Burgers

Consider Burgers' equation, $f(v) = \frac{1}{2}v^2$. Assume that

$$u_0(s) = \begin{cases} -1 & s \leq -1, \\ x & -1 < s \leq 1, \\ 1 & 1 < s. \end{cases} \tag{18}$$

Notice that u_0 is not strictly speaking of class C^1, but u_0 is continuous and piecewise of class C^1. One can prove that the solution constructed by the method of characteristics is still valid.

Case 1: $s \leq -1$, then $u_0(s) = -1$ and $f'(u_0(s)) = u_0(s) = -1$. The characteristics are given by $X(s,t) = s - t$, i.e., $s = X(s,t) + t$. This means $S(x,t) = x + t$ and $u(x,t) = u_0(S(x,t)) = -1$. But this holds true only when $s \leq -1$, i.e., $x + t \leq -1$. Hence

$$u(x,t) = -1, \quad \text{if } x \leq -1 - t. \tag{19}$$

Case 2: $-1 < s \leq 1$, then $u_0(s) = s$ and $f'(u_0(s)) = u_0(s) = s$. The characteristics are given by $X(s,t) = s + ts$, i.e., $s = \frac{X(s,t)}{1+t}$. This means $S(x,t) = \frac{x}{1+t}$ and $u(x,t) = u_0(S(x,t)) = S(x,t) = \frac{x}{1+t}$. But this holds true only when $-1 < s \leq 1$, i.e., $-1 < \frac{x}{1+t} \leq 1$. Hence

$$u(x,t) = \frac{x}{1+t}, \quad \text{if } -1 - t < x \leq 1 + t. \tag{20}$$

Case 3: $1 < s$, then $u_0(s) = 1$ and $f'(u_0(s)) = 1$. The characteristics are given by $X(s,t) = s + t$, i.e., $s = X(s,t) - t$. This means $S(x,t) = x - t$ and $u(x,t) = 1$. But this holds true only when $1 < s$, i.e., $1 < x - t$. Hence,

$$u(x,t) = 1, \quad \text{if } 1 + t < x. \tag{21}$$

2.6.3 Solution developing a shock

Consider again Burgers' equation, $f(v) = \frac{1}{2}v^2$. Assume now that

$$u_0(s) = \begin{cases} 1 & s \leq 0, \\ 1 - x & 0 < s \leq 1, \\ 0 & 1 < s. \end{cases} \tag{22}$$

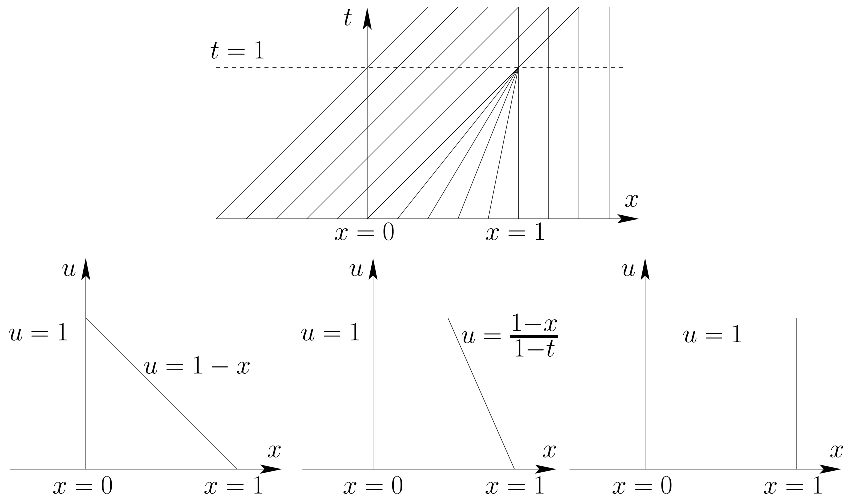

Figure 1: Top: characteristics for Burgers' equation. Bottom, from left to right: (i) solution at $t = 0$; (ii) solution at $t \in (0, 1)$; (iii) solution at $t = 1$.

Case 1: $s \leq 0$, then $u_0(s) = 1$ and $f'(u_0(s)) = 1$. The characteristics are given by $X(s,t) = s + t$, i.e., $s = X(s,t) - t$. This means $S(x,t) = x - t$ and $u(x,t) = 1$. But this holds true only when $s \leq 0$, i.e., $x - t \leq 0$. Hence,

$$u(x,t) = 1, \quad \text{if } x \leq t. \tag{23}$$

Case 2: $0 < s \leq 1$, then $u_0(s) = 1 - s$ and $f'(u_0(s)) = u_0(s) = 1 - s$. The characteristics are given by $X(s,t) = s + t(1-s)$, i.e., $s = \frac{X(s,t)-t}{1-t}$. This means $S(x,t) = \frac{x-t}{1-t}$ and $u(x,t) = u_0(S(x,t)) = 1 - S(x,t) = 1 - \frac{x-t}{1+t} = \frac{1-x}{1-t}$. Notice that the characteristics are well defined only for $t \in [0,1)$; i.e.,, here we have $T^* = 1$. But the argument holds true only when $0 < s \leq 1$, i.e., $0 < \frac{x-t}{1-t} \leq 1$. Hence

$$u(x,t) = \frac{x-t}{1+t}, \quad \text{if } t < x \leq 1 \text{ and } 0 \leq t < 1. \tag{24}$$

Case 3: $1 < s$, then $u_0(s) = 0$ and $f'(u_0(s)) = 0$. The characteristics are given by $X(s,t) = s$, i.e., $s = X(s,t)$. This means $S(x,t) = x$ and $u(x,t) = 0$. But this holds true only when $1 < s$, i.e., $1 < x$. Hence,

$$u(x,t) = 0, \quad \text{if } 1 < x. \tag{25}$$

The characteristics at time $t = 1$ and the graph of the solution at $t = 0$, $t \in (0,1)$ and $t = 1$ are shown in Figure 1 The striking property here is that the solution become s discontinuous at $t = 1$. A key feature of nonlinear conservation equations is that smoothness of the initial data can be lost in finite time.

3 Weak solutions, existence, uniqueness

3.1 Weak solutions

In order to make sense of solutions to (1) that are not of class C^1, because either the initial data is not of class C^1 or smoothness is lost at some time T^*, we now introduce the notion of weak solutions. A weak formulation of (1) is obtained by testing the equation with smooth test functions that are compactly supported in $\mathbb{R}\times\mathbb{R}_{\geq 0}$, say $\phi \in C^1_0(\mathbb{R}\times\mathbb{R}_{\geq 0})$, integrating over the space-time domain $\mathbb{R}\times\mathbb{R}_{\geq 0}$, and integrating by parts as follows:

$$\int_0^\infty \int_{\mathbb{R}} (u\partial_t\phi + f(u)\partial_x\phi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \phi(x,0)u_0(x)\,\mathrm{d}x = 0. \tag{26}$$

Since $\mathbb{R}_{\geq 0} := [0,\infty)$, the function $\phi(0,\cdot)$ can be nonzero over a compact subset of the line $\mathbb{R}\times\{t{=}0\}$. Let us denote by $L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$, the space composed of the functions that are bounded on each compact subset of $\mathbb{R}\times\mathbb{R}_+$.

Definition 3.1 (Weak solution)**.** *We say that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$ is a* weak solution *to* (1) *if u satisfies* (26) *for all $\phi \in C^1_0(\mathbb{R}\times\mathbb{R}_{\geq 0})$.*

If u is smooth and is a weak solution to (26), then restricting the test functions in (26) to $C^\infty_0(\mathbb{R}\times(0,\infty))$ shows that u solves $\partial_t u + \partial_x f(u) = 0$.

Example 3.1 (Linear transport)**.** Assuming that $u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, let us show that $u(x,t) = u_0(x-\beta t)$ is indeed a weak solution to the linear transport equation $\partial_t u + \partial_x(\beta u) = 0$, where $\beta \in \mathbb{R}$ is constant. Let us denote $\mathfrak{I} := \int_0^\infty \int_{\mathbb{R}}(u\partial_t\phi(x,t) + u\beta\partial_x\phi(x,t))\,\mathrm{d}x\,\mathrm{d}t$, and let us make the change of variable $x' = x - \beta t$. We obtain

$$\mathfrak{I} = \int_0^\infty \int_{\mathbb{R}} \big(u_0(x')\partial_t\phi(x'+\beta t,t) + u_0(x')\beta\partial_x\phi(x'+\beta t,t)\big)\,\mathrm{d}x'\,\mathrm{d}t.$$

For all $x' \in \mathbb{R}$, let us set $\psi(x',t) := \phi(x'+\beta t,t)$. Then $\partial_t\psi(x',t) = \beta\partial_x\phi(x'+\beta t,t) + \partial_t\phi(x'+\beta t,t)$. Applying Fubini's theorem gives $\mathfrak{I} = \int_{\mathbb{R}} u_0(x')\int_0^\infty \partial_t\psi(x',t)\,\mathrm{d}t\,\mathrm{d}x' = \int_{\mathbb{R}} -u_0(x)\psi(x,0)\,\mathrm{d}x = \int_{\mathbb{R}} -u_0(x)\phi(x,0)\,\mathrm{d}x$. In conclusion, the identity (26) holds true.

In general, there are infinitely many weak solutions to (1). Consider for instance Burgers' equation in dimension one with $u_0(x) := H(x)$, where H is the Heaviside function (i.e., $H(x) := 1$ if $x \geq 0$ and $H(x) := 0$ if $x < 0$). Let us verify that the following two functions:

$$u_1(x,t) := H(x - \tfrac{1}{2}t) \quad \text{and} \quad u_2(x,t) := \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{t} & \text{if } 0 < x < \text{t}, \\ 1 & \text{if } x > t, \end{cases} \tag{27}$$

are weak solutions, that is, let us show that (26) holds true with $\mathbb{R} := \mathbb{R}$ in both cases for every test function $\phi \in C_0^1(\mathbb{R}\times\mathbb{R}_{\geq 0})$. Let us denote by $\mathfrak{I}_1$ the double integral on the left-hand side of (26) with $u := u_1$. Using Fubini's theorem for the double integral involving $\partial_t \phi$, we obtain

$$\begin{aligned}\mathfrak{I}_1 &= \int_{\mathbb{R}} \int_0^\infty H(x - \tfrac{1}{2}t)\partial_t \phi \,\mathrm{d}t \,\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \frac{1}{2} H^2(x - \tfrac{1}{2}t)\partial_x \phi \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_0^\infty \int_0^{2x} \partial_t \phi \,\mathrm{d}t \,\mathrm{d}x + \int_0^\infty \int_{\frac{t}{2}}^\infty \frac{1}{2} \partial_x \phi \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_0^\infty (\phi(x, 2x) - \phi(x, 0)) \,\mathrm{d}x - \frac{1}{2} \int_0^\infty \phi(\frac{t}{2}, t) \,\mathrm{d}t = -\mathfrak{I}_0,\end{aligned}$$

with $\mathfrak{I}_0 := \int_0^\infty \phi(x, 0) \,\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\phi(x, 0) \,\mathrm{d}x$ since $u_0(x) = 0$ if $x < 0$. Let us denote by $\mathfrak{I}_2$ the double integral on the left-hand side of (26) with $u := u_2$. Then $\mathfrak{I}_2 := \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2}$ with

$$\begin{aligned}\mathfrak{I}_{2,1} &:= \int_0^\infty \Big(\int_0^x \partial_t \phi \,\mathrm{d}t + \int_x^\infty \frac{x}{t} \partial_t \phi \,\mathrm{d}t \Big) \,\mathrm{d}x \\ &= \int_0^\infty -\phi(x, 0) \,\mathrm{d}x + \int_0^\infty \int_x^\infty \frac{x}{t^2} \phi \,\mathrm{d}t \,\mathrm{d}x,\end{aligned}$$

where we used Fubini's theorem and integrated by parts in time, and

$$\mathfrak{I}_{2,2} := \int_0^\infty \Big(\int_0^t \frac{1}{2} \frac{x^2}{t^2} \partial_x \phi \,\mathrm{d}x + \int_t^\infty \frac{1}{2} \partial_x \phi \,\mathrm{d}x \Big) \,\mathrm{d}t = \int_0^\infty \int_0^t -\frac{x}{t^2} \phi \,\mathrm{d}x \,\mathrm{d}t,$$

where we integrated by parts in space. Invoking once again Fubini's theorem and observing that $\{x \in \mathbb{R}_{\geq 0},\, t \geq x\} = \{t \in \mathbb{R}_{\geq 0},\, x \in (0, t)\}$ leads to $\mathfrak{I}_2 = \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2} = -\mathfrak{I}_0$.

3.2 Existence and uniqueness

The nonuniqueness problem can be solved by invoking additional considerations on viscous dissipation. We say that u is a physically relevant solution to (1) if it is a weak solution and if it is the limit in some appropriate topology of the unique solution to the following perturbed problem as $\epsilon \to 0$:

$$(28) \qquad \partial_t u_\epsilon + \partial_x f(u_\epsilon) - \epsilon \Delta u_\epsilon = 0, \qquad u_\epsilon(x, 0) = u_0(x), \quad (x, t) \in \mathbb{R}\times\mathbb{R}_{\geq 0}.$$

We say that u_ϵ is the *viscous regularization* of u or the viscous approximation to (1). The limiting process has been studied in detail in Oleĭnik [4, 5], Kružkov [3], where it is proved that requesting that a weak solution to (26) be such that $\lim_{\epsilon\to 0}\|u_\epsilon - u\|_{L^1(\mathbb{R}\times(0,T);\mathbb{R})} = 0$ is equivalent to requiring that u satisfy the additional entropy inequalities $\partial_t\eta(u)+\partial_x q(u)\le 0$ (in the distribution sense) for any convex function $\eta\in\mathrm{Lip}(\mathbb{R};\mathbb{R})$ with associated flux $q\in\mathrm{Lip}(\mathbb{R};\mathbb{R})$ s.t. $q_l(u) := \int_0^u \eta'(v)f_l'(v)\,\mathrm{d}v$ for all $l\in\{1{:}d\}$. The functions η and q are called *entropy* and *entropy flux*.

Theorem 3.2 (Entropy solution)**.** *Let $f\in\mathrm{Lip}(\mathbb{R};\mathbb{R})$ and $u_0\in L^\infty(\mathbb{R})$. Let the assumptions on the boundary conditions stated in §**??** hold true. There is a unique entropy solution to* (1)*, i.e., there is a unique function $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$ that is a weak solution and that satisfies the following* entropy inequalities*:*

$$-\int_0^\infty\int_{\mathbb{R}}(\eta(u)\partial_t\phi+q(u)\partial_x\phi)\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathbb{R}}\phi(x,0)\eta(u_0)\,\mathrm{d}x\le 0, \tag{29}$$

for all the entropy pairs (η,q) and all $\phi\in C^1_0(\mathbb{R}\times\mathbb{R}_{\ge 0};\mathbb{R}_+)$ (note that ϕ here takes nonnegative values). In other words, we have $\partial_t\eta(u)+\partial_x q(u)\le 0$ in the distribution sense in $\mathbb{R}\times(0,\infty)$.

Theorem 3.3 (Maximum principle)**.** *Let us set* $u_{\min} := \operatorname{ess\,inf}_{x\in\mathbb{R}} u_0(x)$ *and* $u_{\max} := \operatorname{ess\,sup}_{x\in\mathbb{R}} u_0(x)$*. The entropy solution satisfies the following maximum principle:*

$$u(x,t)\in[u_{\min},u_{\max}],\qquad \textit{for a.e. } (x,t)\in\mathbb{R}\times\mathbb{R}_{\ge 0}. \tag{30}$$

Remark 3.2 (Kružkov entropies)**.** It can be shown that Theorem 3.2 holds true if the inequality (29) is satisfied only for the Kružkov entropies $\eta_k(u) := |u-k|$, with flux $q_k(u) := \mathrm{sign}(u-k)(f(u)-f(k))$ for all $k\in[u_{\min},u_{\max}]$.

Remark 3.3 (Strong solution)**.** Strong solutions are also weak solutions and they satisfy all the entropy inequalities. This follows from the definition of the entropy flux and the chain rule.

4 Riemann problem

In this section, we introduce the notion of Riemann problem and give a brief overview of the construction of its solution. Understanding the structure of the solution to the Riemann problem is important to understand the numerical techniques invoking the maximum wave speed Riemann problems. In the entire section, we assume that f is at least Lipschitz, i.e., $f\in\mathrm{Lip}(\mathbb{R};\mathbb{R})$.

4.1 One-dimensional Riemann problem

The Riemann problem is a particular instance of the Cauchy problem (1), where the initial data consists of two constant states. More precisely, the *Riemann problem* consists of solving the following Cauchy problem:

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,0) := \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases} \tag{31}$$

with $u_L, u_R \in \mathbb{R}$. Since the solution to (31) is trivial if $u_L = u_R$, we focus on the case $u_L \neq u_R$. The key idea is that the solution to (31) is self-similar, i.e., it only depends on the ratio $\frac{x}{t}$. In other words, there is a function $w : \mathbb{R} \to \mathbb{R}$ such that $u(x,t) := w(\frac{x}{t})$. The motivation for looking for a solution of this form is the observation that if $u(x,t)$ solves (31), then $u_\lambda(x,t) = u(\lambda x, \lambda t)$ also solves (31) for all $\lambda > 0$. After setting $\xi := \frac{x}{t}$ and inserting the ansatz $u(x,t) = w(\xi)$ into (31), one obtains $-\frac{x}{t^2} w'(\xi) + \frac{1}{t} f'(w(\xi)) w'(\xi) = 0$, so that $u(x,t) = w(x/t)$ solves (31) iff w satisfies the identity

$$\xi = f'(w(\xi)). \tag{32}$$

Solving this nonlinear equation requires that we investigate the monotonicity properties of f'.

4.2 Convex or concave flux

If f' is strictly monotone, then $f' : \mathbb{R} \to \mathbb{R}$ is invertible and the solution to $\xi = f'(w(\xi))$ is $w(\xi) = (f')^{-1}(\xi)$. Let us now make sense of this argument.

Let us assume that $u_L < u_R$. Assume that f is of class C^2 and strictly convex in the interval $[u_L, u_R]$. Then both $f' : [u_L, u_R] \to \mathbb{R}$ and $(f')^{-1} : [f'(u_L), f'(u_R)] \to \mathbb{R}$ are monotonically increasing. Since for every $t \geq 0$ the viscous solution to (31) is monotone in x (see Holden and Risebro [2, §2.1]), we connect u_L to u_R with a monotone increasing profile by setting

$$u(x,t) := \begin{cases} u_L & \text{if } \frac{x}{t} \leq f'(u_L), \\ (f')^{-1}(\frac{x}{t}) & \text{if } f'(u_L) < \frac{x}{t} \leq f'(u_R), \\ u_R & \text{if } f'(u_R) < \frac{x}{t}. \end{cases} \tag{33}$$

It can be proved that this is indeed the entropy solution to (31) (see Holden and Risebro [2, §2.2]). This solution is called *expansion wave*. The above argument does not make sense if f is strictly concave, since in this case $f'(u_L) > f'(u_R)$. It can then be shown that the correct solution is a discontinuity moving with the velocity $s := \frac{f(u_L)-f(u_R)}{u_L-u_R}$, i.e.,

$$u(x,t) := \begin{cases} u_L & \text{if } \frac{x}{t} \le s, \\ u_R & \text{if } s < \frac{x}{t}, \end{cases} \qquad s := \frac{f(u_L)-f(u_R)}{u_L-u_R}. \tag{34}$$

This solution is called *shock wave* or simply *shock*. Graphical representations of the expansion wave and the shock wave are shown in Figure 2.

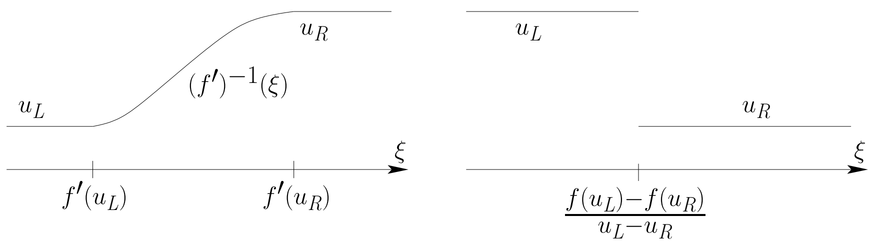

Figure 2: Solution to the Riemann problem $u(x,t) = w(\xi)$ when f is strictly convex. From left to right: (i) expansion wave; (ii) shock.

Recalling that for the time being we have assumed that $u_L < u_R$, the expansion wave (33) and the shock wave (34) can be recast into a single formalism by introducing the lower convex envelope of f over $[u_L, u_R]$:

$$\underset{\smile}{f}(v) := \sup\{g(v) \mid g(z) \le f(z), \forall z \in [u_L, u_R],\ g \text{ convex}\}. \tag{35}$$

To visualize the graph of $\underset{\smile}{f}$, think of a rubber band in $\mathbb{R}^2$ fixed at $(u_L, f(u_L))$ and $(u_R, f(u_R))$ and passing underneath the graph of f. This definition implies that $\underset{\smile}{f}(v) = f(u_L)\frac{v-u_R}{u_L-u_R} + f(u_R)\frac{v-u_L}{u_R-u_L}$ if f is concave and $\underset{\smile}{f}(v) = f(v)$ if f is convex. The two expressions (33) and (34) can be recast into a single formalism as follows:

$$u(x,t) := \begin{cases} u_L & \text{if } \frac{x}{t} \le \underset{\smile}{f}'(u_L), \\ (\underset{\smile}{f}')^{-1}(\frac{x}{t}) & \text{if } \underset{\smile}{f}'(u_L) < \frac{x}{t} \le \underset{\smile}{f}'(u_R), \\ u_R & \text{if } \underset{\smile}{f}'(u_R) < \frac{x}{t}. \end{cases} \tag{36}$$

Note that if f is concave, $\underset{\smile}{f}'(u_L) = \underset{\smile}{f}'(u_R) = s$ and the measure of the set $(\underset{\smile}{f}'(u_L), \underset{\smile}{f}'(u_R)]$ is zero, i.e., one does not have to bother to define $(\underset{\smile}{f}')^{-1}(s)$.

One treats the situation $u_L > u_R$ similarly by invoking the change of variable $x \to -x$ and $f \to -f$, but in this case the lower convex envelope of $-f$ is the upper concave envelope of f over $[u_R, u_L]$ defined by

$$\overset{\frown}{f}(v) := \inf\{g(v) \mid f(z) \le g(z), \forall z \in [u_R, u_L],\ g \text{ concave}\}. \tag{37}$$

To visualize the graph of $\overset{\frown}{f}$, think of a rubber band in $\mathbb{R}^2$ fixed at $(u_L, f(u_L))$ and $(u_R, f(u_R))$ and passing above the graph of f. The solution is defined by setting $u(x,t) := u_L$ if $\frac{x}{t} \leq \overset{\frown}{f}'(u_L)$, $u(x,t) := (\overset{\frown}{f}')^{-1}(\frac{x}{t})$ if $\overset{\frown}{f}'(u_L) < \frac{x}{t} \leq \overset{\frown}{f}'(u_R)$, and $u(x,t) := u_R$ if $\overset{\frown}{f}'(u_R) < \frac{x}{t}$.

Remark 4.1 (Rankine–Hugoniot)**.** When the solution to (31) is a shock wave, the identity $s = \frac{f(u_L)-f(u_R)}{u_L-u_R}$ is called *Rankine–Hugoniot condition* and s is called *shock speed*.

4.3 General case

It turns out that the above argumentation can be generalized to any Lipschitz flux with finitely many inflection points.

Theorem 4.1 (Riemann solution)**.** *Assume that the interval $[u_L, u_R]$ can be divided into finitely many subintervals where f has a continuous and bounded second derivative, and where f is either strictly convex or strictly concave. The entropy solution to* (31) *is given by*

$$u(x,t) := \begin{cases} u_L & \text{if } \frac{x}{t} \leq \underset{\smile}{f}'(u_L), \\ (\underset{\smile}{f}')^{-1}(\frac{x}{t}) & \text{if } \underset{\smile}{f}'(u_L) < \frac{x}{t} \leq \underset{\smile}{f}'(u_R), \\ u_R & \text{if } \underset{\smile}{f}'(u_R) < \frac{x}{t}, \end{cases} \tag{38}$$

if $u_L < u_R$, and $\underset{\smile}{f}$ must be replaced by $\overset{\frown}{f}$ in (38) *if $u_L > u_R$.*

Proof. See Dafermos [1, Lem. 3.1] for the construction of the solution to the Riemann problem assuming that the flux is piecewise linear. See Holden and Risebro [2, §2.2] for a detailed proof. We refer to Osher [6, Thm. 1] for another interesting representation of the solution. □

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