Finite difference schemes for time dependent problems, part II

As a second type of time-dependent PDE let us discuss the acoustic wave equation:

$$(*)\quad \begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0 \quad \text{with } u(a) = g_a,\ u(b) = g_b \\ \text{and initial values } u(0,x) = u_0(x) \text{ and } \partial_t u(0,x) = v_0(x) \end{cases}$$

With the same discretization that we introduced previously, we now approximate $\partial_t^2 u - c^2 \partial_x^2 u = 0$

$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\tau^2} - c^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0$$

Combining this with boundary and initial conditions:

Task ("Centered time centered space")

Given $\{U_i^n\}_{i=0}^{M+1}$ at time t_n and $\{U_i^{n-1}\}_{i=0}^{M+1}$ at time t_{n-1} construct $\{U_i^{n+1}\}_{i=0}^{M+1}$ at time t_{n+1} by setting $U_0^{n+1} = g_a$, $U_{M+1}^{n+1} = g_b$,

$$U_i^{n+1} = 2U_i^n - U_i^{n-1} + c^2 \frac{\tau^2}{h^2} \left\{ U_{i+1}^n - 2U_i^n + U_{i-1}^n \right\}_{i=1,\dots,M}$$

(*)

Similarly to what we did for the parabolic problem in the previous lecture we can again perform a von Neumann stability analysis, that will reveal that (*) is subject to a

$$\boxed{\begin{array}{c}\tau \text{ has to be chosen such that } c\tau < h \\ \text{"hyperbolic CFL condition"}\end{array}}.$$

But there is another issue with this approach. We need to know values for U_i^n at time t_0 and t_1 in order to "start" the procedure.

Remark: This is typically avoided by removing the second time derivative: Introduce $v(t,x) = \partial_t u(t,x)$ and approximate

$$\begin{cases} \partial_t u - v = 0, \\ \partial_t v - c^2 \partial_x^2 u = 0, \end{cases}$$

instead.

But we will now try something slightly different instead: "The wave equation in conservative form"

Let us introduce a function $v(t,x)$ by setting

$$\partial_t v = -c^2 \partial_x u$$

and substituting into the wave equation:

$$0 = \partial_t^2 u - c^2 \partial_x^2 u$$

substituting

$$= \partial_t^2 u + \partial_x(\partial_t v)$$

rearranging

$$= \partial_t \underbrace{(\partial_t u + \partial_x v)}_{=0}$$

certainly sufficient, but v can be chosen such that we have 0 here...

And we get the wave equation in conservative form:

Find $u(t,x)$ and $v(t,x)$ solving

$$\partial_t u + \partial_x v = 0,$$

$$\partial_t v + c^2 \partial_x u = 0,$$

... boundary cond. ...

with initial condition $u(t_0,\cdot) = u_0(\cdot)$ and $v(t_0,\cdot) = v_0(\cdot)$.

Remark: Why is this a "conservation equation"?

Suppose $u(t,x)$ and $v(t,x)$ are given on the entire real number line, $x \in \Omega = \mathbb{R}$. But far away from the origin

u and v are identically 0. Then first integrate from $-a$ to a:

$$\partial_t u + \partial_x v = 0$$

$$\Rightarrow \partial_t \int_{-a}^{a} u(t,x)\,dx + \int_{-a}^{a} \partial_x v(t,x)\,dx = 0$$

$$\Rightarrow \partial_t \int_{-a}^{a} u(t,x)\,dx + \Big[v(t,x)\Big]_{-a}^{a} = 0$$

fundamental theorem of calculus

and now let $a \to \infty$

$$\Rightarrow \partial_t \int_{-\infty}^{\infty} u(t,x)\,dx = 0$$

This means that the total integrals $\int_{-\infty}^{\infty} u(t,x)\,dx$ and $\int_{-\infty}^{\infty} v(t,x)\,dx$ are constant in time, they are conserved.

Let us try to translate this property to our discretization:

$$\partial_t u + \partial_x v = 0$$

forward Euler

central difference

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{V_{i+1}^n - V_{i-1}^n}{2h} = 0$$

and similarly: $$\frac{V_i^{n+1} - V_i^n}{\tau} + c^2 \frac{u_{i+1}^n - u_{i-1}^n}{2h} = 0$$

Task: ("Inviscid approximation")

Given $\{U_i^n\}$ and $\{V_i^n\}$ at time t_n compute an update $\{U_i^{n+1}\}$ and $\{V_i^{n+1}\}$ by setting

$$U_i^{n+1} = U_i^n - \tfrac{\tau}{2h}\left(V_{i+1}^n - V_{i-1}^n\right) \quad \text{for } i=0,\dots,M,$$

$$V_i^{n+1} = V_i^n - c^2 \tau/2h\left(U_{i+1}^n - U_{i-1}^n\right) \quad \text{for } i=0,\dots,M.$$

for simplicity with periodic boundary conditions

- for the equation with $\boxed{i=0}$ set $U_{-1}^n = U_M^n$, $V_{-1}^n = V_M^n$,
- and set $U_{M+1}^{n+1} = U_0^{n+1}$ and $V_{M+1}^{n+1} = V_0^{n+1}$.

Technicality: How to integrate point values? Idea:

$$\int_a^b u(t_n,x)\,dx \quad \leftrightarrow \quad \sum_{i=0}^{M} h\, U_i^n =: I(U^n)$$

periodic, $U_0^n = U_{M+1}^n$

trapezoidal rule

Observation: The scheme is conservative, meaning

$$I(U^{n+1}) = I(U^n) \text{ and } I(V^{n+1}) = I(V^n)$$

Proof:

$$I(u^{n+1}) = \sum_{i=0}^{M} h u_i^{n+1}$$

$$= \sum_{i=0}^{M} h\left(u_i^n - \frac{\tau}{2h}\left(V_{i+1}^n - V_{i-1}^n\right)\right)$$

method

$$= \sum_{i=0}^{M} h u_i^n - \frac{\tau}{2h}\left\{\sum_{i=0}^{M} V_{i+1}^n - \sum_{i=0}^{M} V_{i-1}^n\right\}$$

and using $V_{-1}^n = V_M$ we have $\sum_{i=0}^{M} V_{i-1}^n = \sum_{i=0}^{M} V_i^n$

$$= I(u_i^n) - \tau/2h\left\{\sum_{i=0}^{M} V_{i+1}^n - \sum_{i=0}^{M} V_i^n\right\}$$

telescopic sum!

$$= I(u_i^n) - \tau/2h\underbrace{\left\{V_{M+1}^n - V_0^n\right\}}_{=0 \text{ due to periodicity!}}$$

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