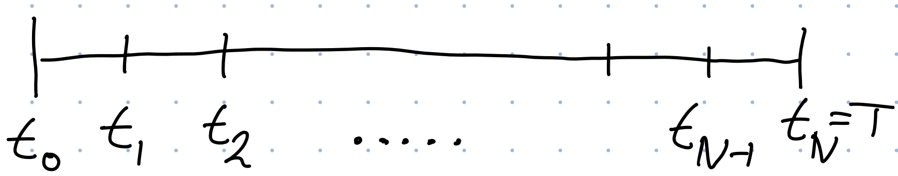
Timestepping Methods

- Let's try to find a procedure for approximating the solution $\vec{x}(t)$ of

$$\text{IVP: } \begin{cases} \frac{d}{dt}\vec{x}(t) = \vec{f}(t, \vec{x}(t)), \\ \vec{x}(t_0) = \vec{x}_0 \end{cases} \quad \text{on } I = [t_0, T]$$

↳ Divide interval into subintervals: ($\text{with } \tau := t_{n+1} - t_n$)



↳ Find a discrete approximation $\{\vec{y}^n\}_0^N$ of $\vec{x}(t)$ with $\vec{y}^n \approx \vec{x}(t_n)$ for $n = 0, \dots, N$.

Idea: approximate differential operator by difference operator:

$$\frac{d}{dt}\vec{x}(t_n) \approx \frac{\vec{x}(t_{n+1}) - \vec{x}(t_n)}{t_{n+1} - t_n}$$

Thus: $$\frac{\vec{y}^{n+1} - \vec{y}^n}{\tau} = \vec{f}(t_n, \vec{y}^n)$$

Definition (Euler's Method):

Construct a sequence of approximations $\{\vec{y}^{\,n}\}_0^N$ as follows

$$\vec{y}^{\,0} = \vec{x}_0$$
$$\vec{y}^{\,n+1} = \vec{y}^{\,n} + \tau \vec{f}(t_n, \vec{y}^{\,n}) \quad (*)$$

This is an explicit time-marching procedure

RHS $(*)$ depends only on t_n and y^n

" $\vec{y}^{\,n} \curvearrowright \vec{y}^{\,n+1}$ "

Question: How good is this procedure ?

To gain some insight we first answer the question "How well does $\vec{x}(t)$ fit into $(*)$?"

Definition: (Truncation error of Euler's Method)

$$\vec{\tau}^{\,n} := \frac{\vec{x}(t_{n+1}) - \vec{x}(t_n)}{\tau} - \vec{f}(t_n, \vec{x}(t_n))$$

What can we say about $\vec{\tau}^{\,n+1}$?

- We know that $\vec{F}(t_n, \vec{x}(t_n)) = \frac{d}{dt}\vec{x}(t_n)$
- For $\vec{x}(t_{n+1})$ use a Taylor series expansion:

$$\vec{x}(t_{n+1}) = \vec{x}(t_n) + \frac{d}{dt}\vec{x}(t_n)(t_{n+1} - t_n) + \frac{1}{2}\frac{d^2}{dt^2}\vec{x}(\xi_n)(t_{n+1} - t_n)^2$$

with some $\xi_n \in (t_n, t_{n+1})$

— Lagrange remainder —

Substituting:

$$\vec{\tau}^n = \frac{\cancel{\vec{x}(t_n)} + \cancel{\frac{d}{dt}\vec{x}(t_n)\cdot\tau} + \frac{1}{2}\frac{d^2}{dt^2}\vec{x}(\xi)\,\tau^2 - \cancel{\vec{x}(t_n)}}{\tau} - \cancel{\frac{d}{dt}\vec{x}(t_n)}$$

$$= \frac{1}{2}\tau\frac{d^2}{dt^2}\vec{x}(\xi)$$

This implies:

Lemma: (Truncation error of Euler's Method)

$$\boxed{\max_n \|\vec{\tau}^n\| \leq \frac{1}{2}\tau \max_{\xi \in \bar{I}} \left\|\frac{d^2}{dt^2}\vec{x}(\xi)\right\|}$$

This is a first order approximation: if we half the stepsize τ we half the truncation error $\|\vec{\tau}^n\|$!

Definition:

- a one-step method is consistent with order K if

$$\max_n \|\vec{\tau}^n\| \leq C \tau^K$$

constant c depending on $\vec{x}(t)$

- a one-step method is convergent with order K if for the error $\vec{e}^n = \vec{x}(t_n) - \vec{y}^n$:

$$\max_n \|\vec{e}^n\| \leq C \tau^K$$

(constant c depending on $\vec{x}(t)$)

Theorem (Discrete stability of Euler's method):

Let $\vec{f}$ be Lipschitz continuous with Lipschitz-constant L. Let

- $\vec{x}(t)$ be the solution to $\frac{d}{dt}\vec{x}(t) = \vec{f}(t, \vec{x}(t))$,
- $\{\vec{y}^n\}$ be constructed with Euler's method, $\vec{y}^{n+1} = \vec{y}^n + h\vec{f}(t_n, \vec{y}^n)$

Then,

$$\max_n \|e^n\| \leq e^{LT} T \max_n \|\tau^n\|$$

Important principle: $\boxed{\text{consistency}} \xrightarrow{\text{stability}} \boxed{\text{convergence}}$

Proof: Lab exercise!

Explicit Runge-Kutta methods

It is often necessary to construct time-stepping schemes that are more than first order convergent. A huge class of such schemes fit into the framework of a Runge-Kutta method. These are defined as follows.

$$\begin{cases} \vec{y}^{\,n+1} = \vec{y}^{\,n} + \tau \sum_{r=1}^{R} b_r \vec{k}_r, \text{ where} \\ \vec{k}_1 = \vec{f}(t_n, y^n) \quad \text{and for } r \geq 2: \\ \vec{k}_r = \vec{f}\left(t_n + \tau c_r, \vec{y}^{\,n} + \tau \sum_{s=1}^{r-1} a_{rs} \vec{k}_s\right) \end{cases}$$

Task: Find sensible coefficients a_{rs}, b_r, c_r ...

The coefficients are typically arranged in a Butcher tableau:

$$\begin{array}{c|cccccc} 0 & 0 & & & & & \\ c_2 & a_{21} & 0 & & & & \\ c_3 & a_{31} & a_{32} & 0 & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ & & & & & 0 & \\ c_R & a_{R1} & a_{R2} & \cdots & & a_{R,R-1} & 0 \\ \hline & b_1 & b_2 & \cdots & & b_{R-1} & b_R \end{array}$$

Examples:

$\boxed{R=1}$ and the (only possible) choice $b_1 = 1$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \vec{y}^{\,n+1} = \vec{y}^{\,n} + \tau \vec{F}(t_n, \vec{y}^{\,n}) \qquad \text{forward Euler}$$

$\boxed{R=2}$ and we have some choices: $\begin{array}{c|cc} 0 & 0 & \\ c_2 & a_{21} & 0 \\ \hline & b_1 & b_2 \end{array}$

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \vec{y}^{\,n+1} = \vec{y}^{\,n} + \tau/2 \left\{ \vec{F}(t_n, \vec{y}^{\,n}) + \vec{F}(t_n + \tau, \vec{y}^{\,n} + \tau \vec{F}(t_n, \vec{y}^{\,n})) \right\}$$

Heun's method (second order)

$$\begin{array}{c|cc} 0 & 0 & \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array} \qquad \vec{y}^{\,n+1} = \vec{y}^{\,n} + \tau \vec{F}\left(t_n + \tfrac{1}{2}\tau, \vec{y}^{\,n} + \tfrac{1}{2}\tau \vec{F}(t_n, \vec{y}^{\,n})\right)$$

Modified Euler (second order)

$\boxed{R=4}$ Classical Runge-Kutta scheme (fourth order)

$$\begin{array}{c|cccc} 0 & 0 & & & \\ 1/2 & 1/2 & 0 & & \\ 1/2 & 0 & 1/2 & 0 & \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 2/6 & 2/6 & 1/6 \end{array}$$

$$\vec{y}^{\,n+1} = \vec{y}^{\,n} + \tau/6 \left\{ \vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4 \right\},$$

where
$$\vec{k}_1 = \vec{F}(t_n, \vec{y}^{\,n})$$
$$\vec{k}_2 = \vec{F}(t_n + \tfrac{1}{2}\tau, \vec{y}^{\,n} + \tfrac{1}{2}\tau \vec{k}_1)$$
$$\vec{k}_3 = \vec{F}(t_n + \tfrac{1}{2}\tau, \vec{y}^{\,n} + \tfrac{1}{2}\tau \vec{k}_2)$$
$$\vec{k}_4 = \vec{F}(t_n + \tau, \vec{y}^{\,n} + \tau \vec{k}_3)$$