# Exercises

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## 1 Lecture 1

Question 1: Let  $y(x,t) = x\cos(2t + \log(|x|))$ . Compute  $\partial_{tt}y + x^2\partial_{xx}y - x\partial_xy + 6y$ .

(b) Let  $\psi: \mathbb{R} \to \mathbb{R}$  be functions of class  $C^1$ . Let  $F: \mathbb{R} \to \mathbb{R}$  be defined by  $F(v) = \int_0^v f'(t)\psi'(t) dt$ . Use (a) to compute  $\partial_x(F(u(x)) - \partial_x(f(u(x)))\psi'(u(x))$ .

(c) Using the notation of (a) and (b), assume that  $u(\pm \infty) = 0$  and compute  $\int_{-\infty}^{+\infty} \partial_x (f(u(x))) \psi'(u(x)) dx$ .

Question 3: Let u solve  $\partial_t u - \partial_x ((3x+1)\partial_x u) = -3$ ,  $x \in (0,L)$ , with  $\partial_x u(0,t) = 1$ ,  $\partial_x u(L,t) = \alpha$ , u(x,0) = f(x).

(a) Compute  $\int_0^L u(x,t)dx$  as a function of t.

(b) For which value of  $\alpha$  the quantity  $\int_0^L u(x,t)dx$  does not depend on t?

Question 4: Let u solve  $\partial_t u + \partial_x \left(v(x,t)u - \mu(x,t)\partial_x u\right) = g(x)e^{-t}, \ x \in (0,L), \ t > 0$ , with  $\mu(0,t)\partial_x u(0,t) = 1, \ \mu(L,t)\partial_x u(L,t) = 1 + 2e^{-t}, \ u(x,0) = f(x)$ , where  $v, \ \mu > 0$ , f and g are four smooth functions and v(0) = v(L) = 0.

(a) Compute  $\frac{d}{dt} \int_0^L u(x,t) dx$  as a function of t.

(b) Use (a) to compute  $\int_0^L u(x,t)dx$  as a function of t.

(c) What is the limit of  $\int_0^L u(x,t)dx$  as  $t \to +\infty$ ?

Question 5: Consider the differential equation  $-\frac{d^2\phi}{dt^2} = \lambda\phi$ ,  $t \in (0,\pi)$ , supplemented with the boundary conditions  $\phi(0) = 0$ ,  $3\phi(\pi) = -\phi'(\pi)$ .

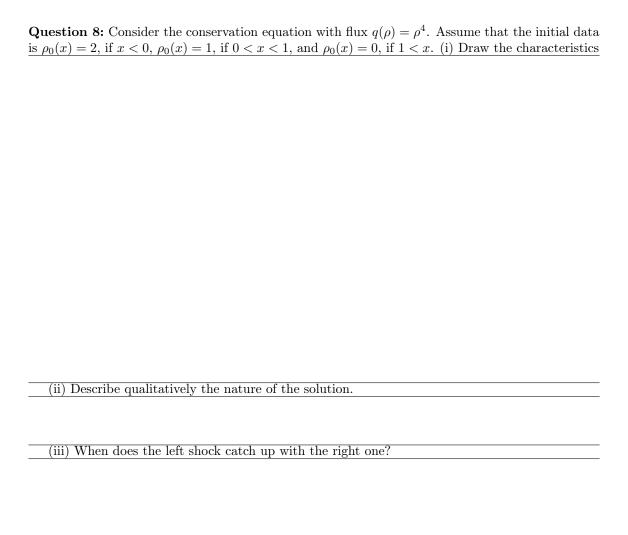
(a) Prove that it is necessary that  $\lambda$  be positive for a non-zero solution to exist.

(b) Find the equation that  $\lambda$  must solve for the above problem to have a nonzero solution.

## 2 Lecture 2

**Question 6:** Consider the following conservation equation  $\partial_t \rho + \partial_x (q(\rho)) = 0$ ,  $x \in (-\infty, +\infty)$ , t > 0,  $\rho(x,0) = \frac{1}{2}$  if x < 0, and  $\rho(x,0) = 1$  if x > 0. where  $q(\rho) = 2\rho + 3\rho^3 - \sin(\rho^2)$  (and  $\rho(x,t)$  is the conserved quantity). What is the wave speed for this problem?

**Question 7:** Consider the following conservation equation  $\partial_t \rho + \partial_x (q(\rho)) = 0$ ,  $x \in (-\infty, +\infty)$ , t > 0,  $\rho(x,0) = \frac{1}{2}$  if x < 0, and  $\rho(x,0) = 1$  if x > 0. where  $q(\rho) = 2\rho + 3\rho^3 - \sin(\rho^2)$  (and  $\rho(x,t)$  is the conserved quantity). What is the wave speed for this problem?





Luestion	10: Fo	or all $k \in$	$\in \mathbb{R}, \text{ con}$	sider tl	he entro	py $\eta(v, k)$	v(t) :=  v	-k .	Compute	the entr	opy flu
ssociated	with ti	nis entro	py, q(v),	with th	ne norm	anzation	$q(\kappa) :=$	0.			

Question 11: Consider Burgers' equation with  $D := \mathbb{R}$  and  $u_0(x) := H(x)$ , where H is the Heaviside function. (a) Verify that  $u_1(x,t) := H(x - \frac{1}{2}t)$  and  $u_2(x,t) := 0$  if x < 0,  $u_2(x,t) := \frac{x}{t}$ , if 0 < x < t,  $u_2(x,t) := 1$  if x > t, are both weak solutions.

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(ii) Verify that  $u_1$  does not satisfy the entropy inequalities, whereas  $u_2$  does.

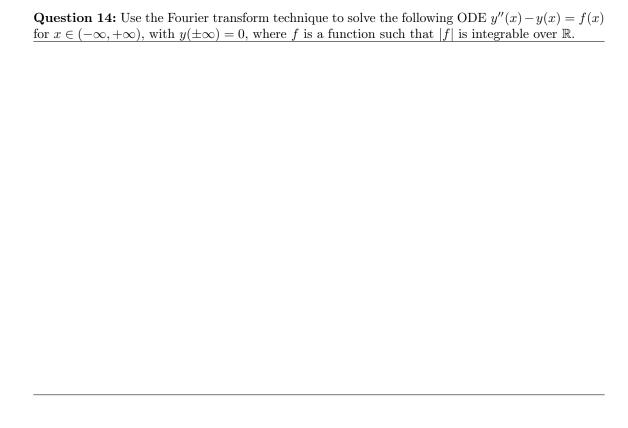
#### 3 Lecture 3

Question 12: Consider the quasilinear Klein-Gordon equation:  $\partial_{tt}\phi(x,t)-c^2\partial_{xx}\phi(x,t)+m^2\phi(x,t)+\beta^2\phi^3(x,t)=0, x\in\mathbb{R}, t>0$ , with  $\phi(x,0)=f(x), \partial_t\phi(x,0)=g(x)$  and  $\phi(\pm\infty,t)=0, \partial_t\phi(\pm\infty,t)=0$ ,  $\partial_x\phi(\pm\infty,t)=0$ . Find an energy E(t) which is invariant with respect to time (Hint: test with  $\partial_t\phi(x,t)$  and use  $\phi^p\phi'=(\frac{1}{p+1}\phi^{p+1})'$ .)

Question 13: (a) Compute the Fourier transform of the function f(x) defined by

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1\\ 0 & \text{otherwise} \end{cases}$$

(b) Find the inverse Fourier transform of  $g(\omega) = \frac{\sin(\omega)}{\omega}$ .



Question 15: Solve the integral equation:  $f(x) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1} dy = \frac{1}{x^2+4} + \frac{1}{x^2+1}$ , for all  $\underline{x} \in (-\infty, +\infty)$ .

<b>Question 16:</b> Use the Fourier transform method to solve the equation $\partial_t u + \frac{2t}{1+t^2} \partial_x u = 0$ , $u(x,0) = u_0(x)$ , in the domain $x \in (-\infty, +\infty)$ and $t > 0$ .								
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Question 17: Use the Fourier transform technique to solve  $\partial_t u(x,t) + \sin(t)\partial_x u(x,t) + (2+3t^2)u(x,t) = 0, x \in \mathbb{R}, t > 0$ , with  $u(x,0) = u_0(x)$ . (Use the shift lemma:  $\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta}$  and the definition  $\mathcal{F}(f)(\omega) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)e^{i\omega x} dx$ )

#### Question 18: Solve the PDE

$$u_{tt} - a^2 u_{xx} = 0,$$
  $-\infty < x < +\infty, \quad 0 \le t,$   $u(x,0) = \cos(x), \quad u_t(x,0) = -a\sin(x),$   $-\infty < x < +\infty.$ 

**Question 19:** Solve the wave equation on the semi-infinite domain  $(0, +\infty)$ ,

$$\begin{split} &\partial_{tt} w - 4 \partial_{xx} w = 0, \quad x \in (0, +\infty), \ t > 0 \\ & w(x, 0) = (1 + x^2)^{-1}, \quad x \in (0, +\infty); \qquad \partial_t w(x, 0) = 0, \quad x \in (0, +\infty); \quad \text{and} \quad \partial_x w(0, t) = 0, \quad t > 0. \end{split}$$

(Hint: Consider a particular extension of w over  $\mathbb{R}$ )