MATH 240 – Lecture 10

Enlai Li

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1 Relations

$R \subseteq A \times A$ intention of pairing together elemtns that are "equivalent" from a certain point of view ...
ex:

$$\frac{a}{b} \sim \frac{c}{d} \iff ad = bc$$

What does it mean for 2 things to be "equivalent" from a mathematical POV?
Definition: A relation $R \subseteq A \times A$ is <u>reflexive</u> if $\forall x \in A, xRx$
ex:

- $=$: because $x = x$
- $\sim$: $\frac{a}{b} \sim \frac{a}{b} \iff ab = ba$
- $|$: $x|y \iff y = kx (k \in \mathbb{Z})$
  Reflexive: $x|x \iff \exists k.x = kx$ True with $k = 1$

non-examples:

- Unit cercle: $xCy \iff x^2 + y^2 = 1$
  We do not necessarily have $xCx \ldots$, that would mean
  $$\begin{aligned} x^2 + x^2 &= 1 \\ 2x^2 &= 1 \\ x &= + - \frac{1}{\sqrt{2}} \end{aligned}$$
- Strict order: $<$ (on $\mathbb{R}$, on $\mathbb{Z}, \ldots$)
  Counterexample: $42 \not< 42$, in fact $x < x$ is never true. *but <= is reflexive.*
  $x <= x$

1.1 Transitive

Def: $R \subseteq A \times A$ is transitive of $\forall x, y, z \in A, xRyhatyRz \Rightarrow xRz$

Assume: $x \sim y$ and $y \sim z$ i.e. $ad = bc$ and $cf = de$ Goal: show $x \sim z$, i.e. $af = be$?

$$\begin{aligned} df &= bc \\ adf &= bcf \\ a\cancel{d}f &= b\cancel{d}e \\ ad &= be \end{aligned}$$

1.2 Divisibility

$$x|ycapy|z? \Rightarrow x|z$$

Assume $x|y$ and $y|z$ where $y = kx$ and $z = ly$. Then $z = ly = l(kx) = (lk)x \Rightarrow x|z$

Note: $<$ and $<=$ are also transitive

Non example:

Unit circle:

$$\begin{aligned} 0 \subset 1 &\text{ because } 0^2 + 1^2 = 1 \\ 1 \subset 0 &\text{ because } 0^2 + 1^2 = 1 \\ \text{But } 0 \not\subset 0 &\text{ because } 0^2 + 0^2 \neq 1 \end{aligned}$$

Non-equality($\neq$):

$$0 \neq 1 \text{ and } 1 \neq 0$$
$$\text{but it is not true that } 0 \neq 0$$

1.3 Symmetric

Def: $R \subseteq A \times A$ is symmetric if $xRy \Rightarrow yRx$

ex:

$$\begin{aligned} x = y \Rightarrow y = x \\ \text{Unit cercle} \\ xCy \Rightarrow x^2 + y^2 = 1 \\ \Rightarrow y^2 + x^2 = 1 \Rightarrow yCx \end{aligned}$$

Fraction:

$$\begin{aligned} \frac{a}{b} \sim \frac{c}{d} &\iff ad = bc \\ &\iff bc = ad \\ &\iff \frac{c}{d} = \frac{a}{b} \end{aligned}$$

Non-ex:
$<$ and $<=$ are not symmetric
Divisibility:

$$2|6 \quad 6 = 3 \times 2$$
$$but 6 \not| 2 \text{ because } 2 = k6$$
$$k = \frac{2}{6} \notin \mathbb{Z}$$

1.4 equivalence relation

Def: if R is reflective, transitive and symmetric
ex:
$=$ and $\sim$
$A \sim B if |A| = |B| \iff \exists f : A \to B$ Bijective
Show equivalence relation:

1. Reflexive: $A \sim A$
   consider identity function

$$id_A : A \to A \text{ invertible}$$
$$id_A(x) = x \text{ bijective}$$

2. Symmetric: $A \sim B \Rightarrow \exists f : a \to$ bijective. Then f is invertible and $f^{-1} i B \to A$ and f^{-1} is also invertible (hence bijective)

$$(f^{-1})^{-1} = f \Rightarrow B \sim A$$

$\frac{2}{3}$ and $\frac{4}{6}$ are equivalent because $\frac{2}{3} \sim \frac{4}{6}$

1.5 equivalent relation

Def: Given an equivalent relation on a set A and an elemtn $a \subseteq A$, the equivalence class of a is the set

$$[a]\tilde{} = \{x \in A \mid x \sim a\}$$

ex:

$$= \text{ Then } [a]_= = \{a\}$$
$$\text{on } \mathbb{F} : [\frac{a}{b}]_\sim = \{\frac{c}{d} \mid ad = bc\}$$
$$\text{ex: } [\frac{1}{2}]_\sim = \{\frac{1}{2}, \frac{2}{4}, \frac{42}{84}, \dots\}$$
$$[\mathbb{N}] = \{\text{comptable infinite sets}\}$$

Remark:

1. $[a]_\sim \neq \emptyset$ because $a \in [a]$ be reflexivity $a \sim a$
2. $a \sim b \iff [a]_\sim = [b]_\sim$
   Assume $a \sim b$ NTS $[a]_\sim = [b]_\sim$
   Double inclusion: $[a]_\sim \subseteq [b]_\sim$
3. if $a \not\sim b$, then $[a]_\sim \sim [b]_\sim = \emptyset$
   proof by contrapositive: if $[a]_\sim \sim [b]_\sim \neq \emptyset$

In the case of fractions we can ocnsider that teh rational number $\frac{1}{2}$ is the class $[\frac{1}{2}]_\sim$
Def: equivelence relation on A, then the quotient set of $Az \sim$

$$A/\sim = \{[x]_\sim \mid x \in A\}$$

is the set of equivalence classes. We could define $\mathbb{Q} = \mathbb{F}/\sim$
so when we write

$$\frac{1}{2} = \frac{2}{4} \qquad \text{(as in } \mathbb{Q}\text{)}$$

it means $[\frac{1}{2}{}_\sim] = [\frac{2}{4}{}_\sim]$