# MATH240 – Lecture 3

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## 1 Mathematical logic

$$\begin{aligned}
\text{Premise} &\rightarrow \text{All men are } \underbrace{mortal}_{M} \\
&\rightarrow \underbrace{Socrates}_{S} \text{ is a } \underbrace{man}_{H(human)} \\
\text{Conclusion} &\rightarrow \text{Therefore Socrates is mortal}
\end{aligned}$$

All x such that x is H is M.
S is H
Therefore S is M

$$\begin{array}{l}
\forall x, H(x) \Rightarrow M(x) \\
H(S) \\
\hline
M(S)
\end{array}$$

## 2 Propositional logic (True/False)

### 2.1 Atomic propositions

The building block of propositional logic are propositions: statements that are either true or false.
ex:

p: 21 is a multiple of 7 (True proposition)

q: $2 + 2 = 5$ (False proposition)

r: there exists an extraterrestrial life form (proposition)

## 2.2 Compound propositions

Atomic propositions combined with logical connectors
ex:

$$2+2=5 \text{ and "there exists and extraterrestial lifeform}$$
$$= q \wedge r$$

## 2.3 Logical connectors

### 2.3.1 conjunction ($\wedge$) and

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

### 2.3.2 Negation ($\neg$) not

$$\neg p = \text{ Not } p = \overline{p}$$
$$q = \text{"}2+2=5\text{"}$$
$$\neg q = \text{"}2+2 \neq 5\text{"}$$

| p | $\neg p$ |
|---|---|
| T | F |
| F | T |

### 2.3.3 Disjunction ($\vee$) or

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

### 2.3.4 More complex propositions

$$\neg(\neg p \wedge \neg q) = ?$$

| p | q | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ | $\neg(\neg p \wedge \neg q)$ |
|---|---|---|---|---|---|
| T | T | F | F | F | T |
| T | F | F | T | F | T |
| F | T | T | F | F | T |
| F | F | T | T | T | F |

### 2.3.5 Set laws and logical equivalence

Set laws translate into logical equivalence:

| Set | Logic |
|---|---|
| $U$ | True |
| $\emptyset$ | False |
| $\cup$ | $\wedge$ |
| $\cap$ | $\vee$ |
| $\overline{A}$ | $\neg A$ |

Since $p \vee q$ has the same truth table as $\neg(\neg p \wedge \neg q)$, they are logically equivalent:

$$p \vee q \equiv \neg(\neg p \wedge \neg q)$$

### 2.3.6 Exclusive or ($\oplus$) xor

Def: $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$

| p | q | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

### 2.3.7 Conditional ($\Rightarrow$)

Def: $p \Rightarrow q \equiv \neg(p \wedge \neg q) \equiv \neg p \vee q$
$p \Rightarrow q$: "p implies q": "if p then q"
ex:

p: It rains

q: It's cloudy

$p \Rightarrow q$: If it rains outside, then it is cloudy

$p \Rightarrow q$ is always true when p is false

| p | q | $p \Rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

### 2.3.8 Biconditional ( $\iff$ )

$p \iff q$: p if and only if q

$$\begin{aligned} \text{Def: } p \iff q &\equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \\ &\equiv (p \wedge q) \vee (\overline{p} \wedge \overline{q}) \\ &\equiv \overline{p \oplus q} \end{aligned}$$

| p | q | $p \Rightarrow q$ | $q \Rightarrow p$ | $p \iff q \equiv ((p \Rightarrow q) \wedge (q \Rightarrow p))$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

## 2.4 Set law in logic

- double negation:

$$\neg\neg p \equiv p$$

- idempotent:

$$p \wedge p \equiv p$$
$$p \vee p \equiv p$$

- commutative:

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$

- absorption:

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

- association:

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$
$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

- distribution:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

- De Morgan:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

- complement:

$$p \vee (\neg p) \equiv T$$
$$p \wedge (\neg p) \equiv F$$

- identity:

$$p \wedge T = p$$
$$p \vee F = p$$

- domination:

$$p \vee T = T$$
$$p \wedge F = F$$

- negation:

$$\neg F = T$$
$$\neg T = F$$

## 2.5 converse, contrapositive and inverse

- statement: $p \Rightarrow q$
- converse: $q \Rightarrow p$
- contrapositive: $\neg q \Rightarrow \neg p$
- inverse: $\neg p \Rightarrow \neg q$