Combinatorics (counting)

Product principle

$$|A \times B| = |A| \times |B|$$

Sum principle

$$|A \cup B| = |A| + |B| \text{ if } A \cap B = \emptyset$$

ex :

Usernames on some site must be between 6 and 8 characters.
among a-z, A-Z, 0-9
How many admissible usernames are there?

The number of characters sets three disjoint scenarios:

$$U_n = \{\text{Username with } n \text{ characters}\}$$

$$U = \{\text{Usernames}\}$$

$$\Rightarrow U = U_6 + U_7 + U_8$$

by sum principle: $|U| = |U_6| + |U_7| + |U_8|$

Let's count U_n

$$\underset{\substack{\uparrow \\ \text{username}}}{x} = \underset{c_1}{\_}\ \underset{c_2}{\_}\ \cdots\ \underset{c_n}{\_}$$

Alphabet - $A = \{a..z, A...Z, 0...9\}$

$\Rightarrow U_n = A \times A \times \dots A$ (n times)

Product principle $|U_n| = |A|^n$

$A = \{\text{lowercase}\} \cup \{\text{uppercase}\} \cup \{\text{digits}\}$

Sum principle: $|A| = 26 + 26 + 10$
$= 62$

Putting this together: $|U| = |U_6| + |U_7| + |U_8|$
$= 62^6 + 62^7 + 62^8$
$\approx 2.22 \cdot 10^{14}$

Complement principle

If $B \subseteq A$, then $|A \setminus B| = |A| - |B|$

Proof: $A = B \cup (A \setminus B)$ disjoint

Sum $\Rightarrow |A| = |B| + |A \setminus B|$

$|A \setminus B| = |A| - |B|$

Ex: how many usernames (same as example above) contain at least one digit

Sol: instead of counting this directly we can count usernaes with no digits... (same as in example above but only 52 char in alphabet) $|A| = 52$

$|U| = 52^6 + 52^7 + 52^8$

Usernames with at least one digit are the usernames that do not have 0 digits...
By the complement principle, that's

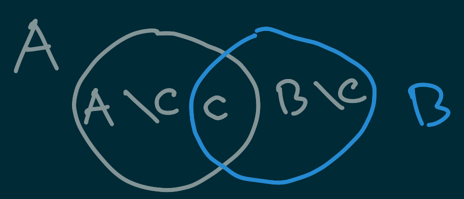
$$\underbrace{(62^6 + 62^7 + 62^8)}_{\text{All username}} - \underbrace{(52^6 + 52^7 + 52^8)}_{\text{No digits}}$$

Inclusion-Exclusion Principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof: Let $C = A \cap B$

Then $A \cup B = (A \setminus C) \cup C \cup (B \setminus C)$



Disjoint

Sum principle $\Rightarrow |A \cup B| = |A \setminus C| + |C| + |B \setminus C|$

Complement principle

$$|A \cup B| = |A| - \cancel{|C|} + \cancel{|C|} + |B| - |C|$$

$$= |A| + |B| - |A \cap B|$$

Ex: How many bit strings of length 8 start with 1 or end with 00?

union (not disjoint)

Sol: Let

$A = \{$Bit strings, length 8, start with 1$\}$

$B = \{$Bit strings, length 8, end with 00$\}$

Want $|A \cup B| = |A| + |B| - |A \cap B|$

Count A

$$x = \underline{1}\ \overbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}^{7}$$

$|A| = 2^7$ (product principle)

Count B

$$x = \overbrace{\_\ \_\ \_\ \_\ \_\ \_}^{6}\ \underline{0}\ \underline{0}$$

$|B| = 2^6$

$A \cap B$:

$$x = \underline{1}\ \overbrace{\_\ \_\ \_\ \_\ \_}^{5}\ \underline{0}\ \underline{0}$$

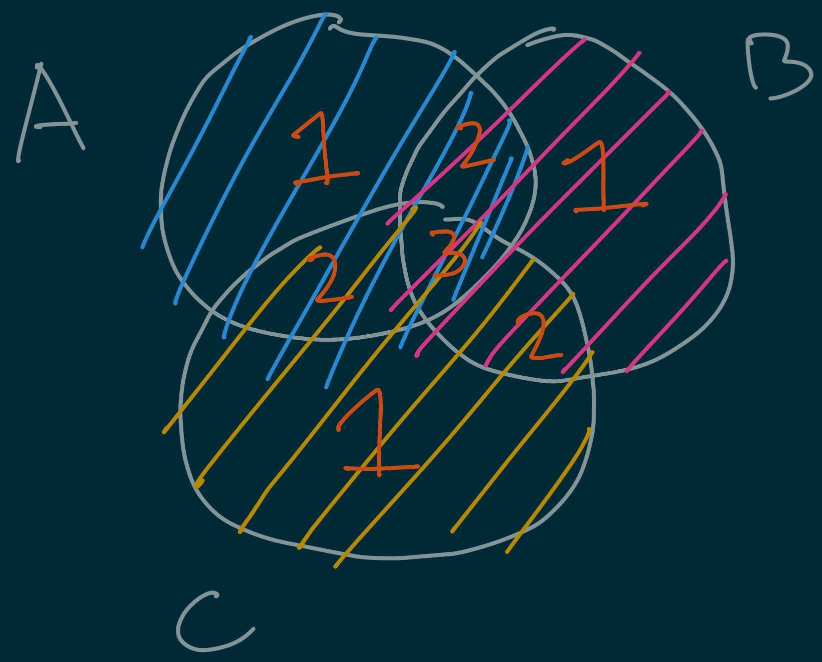
$$|A \cap B| = 2^5$$

Conclusion:

$$|A \cup B| = 2^7 + 2^6 - 2^5$$
$$= 2^5(4 + 2 - 1)$$
$$= 5 \cdot 2^5$$

General inclusion exclusion

What if there are 3 sets?



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B|$$
$$- |A \cap C|$$
$$- |B \cap C|$$
(middle sub 3 times) $+ |A \cap B \cap C|$

General pattern (n sets)

$$|A_1 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n|$$
$$- |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n|$$
$$+ |A_1 \cap A_2 \cap A_3| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n|$$
$$- |A_1 \cap A_2 \cap A_3 \cap A_4| - \ldots - |A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n|$$
$$+ \ldots + (-1)^n |A_1 \cap A_2 \ldots \cap A_n|$$

Oh the k-th step above, we write all intersections of k sets alternate the + or - sign as k increases ex :

How many integers 0 < n < 10000 are neither squares, cubes, nor multiples of 7

$X = \{1, 2, 3, \ldots, 9999\}$ (universe)

$S = \{ n \in X \mid n = k^2$ for some $k \}$

$C = \{ n \in X \mid n = k^3$ for some $k \}$

$M = \{ n \in X \mid n = 7k$ for some $k \}$

Want $\bar{S} \cap \bar{C} \cap \bar{M}$

$= \overline{(S \cup C \cup M)}$

$= X \setminus (S \cup C \cup M)$

Count $S \cup C \cup M$ (inclusion-exclusion) and then subtract from X (complement principle)

$$|S \cup C \cup M| = |S| + |C| + |M| - |S \cap C| - |S \cap M| - |C \cap M| + |S \cap C \cap M|$$

S : $S = \{1, 4, 9, \ldots, k^2\}$

$k^2 \leq 9999 \Rightarrow k \leq \sqrt{9999}$ $\quad k = \lfloor \sqrt{9999} \rfloor = 99$

$\sqrt{10000} = 100$

$\Rightarrow k = 99 \Rightarrow |S| = 99$

Notation (floor function)

$\lfloor x \rfloor$ = the largest integer <= x

$C = \{1, 8, 27, \ldots, k^3\}$

$k^3 \leq 9999$

$k \leq \sqrt[3]{9999}$ (as large as possible

$k = \lfloor \sqrt{9999} \rfloor$

$\lfloor \sqrt[3]{9999} \rfloor = 21, \cancel{54\ldots} = 21 = |C|$

$M = \{1, 7, 14, \ldots 7k\}$

$K = \left\lfloor \frac{9999}{7} \right\rfloor = 1428$

$S \cap C$? n is square AND cube

$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_l^{\alpha_l}$

where α_i is a multiple of 2 and a multiple of 3

Since $GCD(2,3) = 1$, then α is multiple of 6

$\Rightarrow n = k^6$ for some k

$\Rightarrow S \cap C = \{1, 2^6, 3^6, \ldots, k^6\}$ where $K = \lfloor \sqrt[6]{9999} \rfloor$

$\Rightarrow |S \cap C| = 4$

$S \cap M = \{7^2, 14^2, 21^2, \ldots, (7k^2)\}$

Since 7 is prime $\qquad (7K)^2 \leq 9999$

$k \leq \frac{\sqrt{9999}}{7}$

$\Rightarrow S \cap M = \left\lfloor \frac{\sqrt{9999}}{7} \right\rfloor = 14$

$C \cap M = \{7^3, 14^3, \dots, (7k)^3\}$ where $k = \left\lfloor \frac{\sqrt[7]{9999}}{7} \right\rfloor$

$|C \cap M| = 3$

$S \cap C \cap M = \{7^7, 14^6, \dots, (7k)^6\} = \varnothing$

$|S \cap C \cap M| = \left\lfloor \frac{\sqrt[6]{9999}}{7} \right\rfloor = 0$

A assume $|X| - |S \cup C \cup M|$

$= |X| - |S| - |C| - |M|$

$+ |S \cap C| + |S \cap M| + |C \cap M|$

$- |S \cap C \cap M|$

$= 9999 - 99 - 21 - 1428 + 4 + 14 + 3 - 0$

$= 8472$