MATH240 – Lecture 3

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1 Mathematical logic

$$\begin{aligned}
\text{Premise} \rightarrow\ & \text{All men are } \underbrace{\textit{mortal}}_{M} \\
\rightarrow\ & \underbrace{\textit{Socrates}}_{S} \text{ is a } \underbrace{\textit{man}}_{H(human)} \\
\text{Conclusion} \rightarrow\ & \text{Therefore Socrates is mortal}
\end{aligned}$$

All x such that x is H is M.
S is H
Therefore S is M

$$\begin{array}{l}
\forall x, H(x) \Rightarrow M(x) \\
H(S) \\
\hline
M(S)
\end{array}$$

2 Propositional logic (True/False)

2.1 Atomic propositions

The building block of propositional logic are propositions: statements that are either true or false.
ex:

p: 21 is a multiple of 7 (True proposition)

q: $2 + 2 = 5$ (False proposition)

r: there exists an extraterrestrial life form (proposition)

2.2 Compound propositions

Atomic propositions combined with logical connectors
ex:

$$2 + 2 = 5 \text{ and "there exists and extraterrestial lifeform} \\ = q \wedge r$$

2.3 Logical connectors

2.3.1 conjunction ($\wedge$) and

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2.3.2 Negation ($\neg$) not

$$\neg p = \text{ Not } p = \overline{p}$$
$$q = \text{"}2 + 2 = 5\text{"}$$
$$\neg q = \text{"}2 + 2 \neq 5\text{"}$$

p	$\neg p$
T	F
F	T

2.3.3 Disjunction ($\vee$) or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

2.3.4 More complex propositions

$$\neg(\neg p \wedge \neg q) = ?$$

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

2.3.5 Set laws and logical equivalence

Set laws translate into logical equivalence:

Set	Logic
U	True
$\emptyset$	False
$\cup$	$\wedge$
$\cap$	$\vee$
$\overline{A}$	$\neg A$

Since $p \vee q$ has the same truth table as $\neg(\neg p \wedge \neg q)$, they are logically equivalent:

$$p \vee q \equiv \neg(\neg p \wedge \neg q)$$

2.3.6 Exclusive or ($\oplus$) xor

Def: $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

2.3.7 Conditional ($\Rightarrow$)

Def: $p \Rightarrow q \equiv \neg(p \wedge \neg q) \equiv \neg p \vee q$
$p \Rightarrow q$: "p implies q": "if p then q"
ex:

p: It rains

q: It's cloudy

$p \Rightarrow q$: If it rains outside, then it is cloudy

$p \Rightarrow q$ is always true when p is false

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

2.3.8 Biconditional ($\iff$)

$p \iff q$: p if and only if q

$$\begin{aligned} \text{Def: } p \iff q &\equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \\ &\equiv (p \wedge q) \vee (\overline{p} \wedge \overline{q}) \\ &\equiv \overline{p \oplus q} \end{aligned}$$

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \iff q \equiv ((p \Rightarrow q) \wedge (q \Rightarrow p))$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

2.4 Set law in logic

- double negation:

$$\neg\neg p \equiv p$$

- idempotent:

$$p \wedge p \equiv p$$
$$p \vee p \equiv p$$

- commutative:

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$

- absorption:

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

- association:

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$
$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

- distribution:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

- De Morgan:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

- negation:

$$p \vee (\neg p) \equiv T$$
$$p \wedge (\neg p) \equiv F$$

- identity:

$$p \wedge T = p$$
$$p \vee F = p$$

- domination:

$$p \vee T = T$$
$$p \wedge F = F$$

2.5 converse, contrapositive and inverse

- statement: $p \Rightarrow q$
- converse: $q \Rightarrow p$
- contrapositive: $\neg q \Rightarrow \neg p$
- inverse: $\neg p \Rightarrow \neg q$