

De Morgan still applies

$$\begin{aligned}\neg (\exists x \in A. P(x)) &= \neg (\exists x. ((x \in A) \wedge P(x))) \\ &\equiv \forall x, \neg ((x \in A) \wedge P(x)) \\ &\equiv \forall x, \neg (x \in A) \vee \neg P(x) \\ &\equiv \forall x, (x \in A) \Rightarrow \neg P(x) \\ &\equiv \forall x \in A. \neg P(x)\end{aligned}$$

Similarly: $\neg (\forall x \in A. P(x)) \equiv \exists x \in A. \neg P(x)$

Nested Quantifiers

$$\models x : \forall x \in \mathbb{R}. \exists y \in \mathbb{R}. x + y = 0$$

This is true. For any given x , you can simply choose $y = -x$ as an example, which makes $x + y = 0$ true

$$\models x \not\models : \exists y \in \mathbb{R}. \forall x \in \mathbb{R}. x + y = 0$$

This is false. No matter the y that you choose, there are plenty of examples of x (namely, all real numbers except $-y$) which make $x + y \neq 0$

Generally: $\forall x. \exists y. P(x, y) \neq \exists y. \forall x. P(x, y)$

The order in which the variables are quantified matters. One way to think about this is in terms of a two-player game

Game rules:

- Your opponent chooses a value for the \forall -quantified variable; you choose a value for the \exists -quantified ones
- Take turns according to the order in which the variables are quantified (left to right)
- You win if the resulting proposition is true, you lose if it is false

With this in mind, a "proof" of a (true) formula is just a strategy for winning at that game, no matter the choices that the opponent makes. Disproving a formula (i.e. showing that it is false) is finding a strategy for the opponent to win

What about $\forall \forall$ and $\exists \exists$?

$$\exists x : \forall y \in \mathbb{R}, \forall z \in \mathbb{R} . x^2 + y^2 \geq 0$$

Here the opponent just chooses two numbers, x and y , before you have any say about them. So it does not affect your strategy in what order he chooses them

The result would be the same with:

$$\forall y \in \mathbb{R}, \forall x \in \mathbb{R} . x^2 + y^2 \geq 0$$

Or equivalently, these two formulas are equivalent to:

$$\forall (x, y) \in \mathbb{R}^2, x^2 + y^2 \geq 0$$

so x and y may very well be chosen together

Generally : $\forall x . \forall y . P(x, y) \equiv \forall y . \forall x . P(x, y)$

So when it is the same quantifier that is involved, the order does not matter

Similarly : $\exists x . \exists y . P(x, y) \equiv \exists y . \exists x . P(x, y)$