MATH240 – Lecture 4

Enlai Li

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1 Logic continued

1.1 Negating a sentence

Ex:

p = you can access the internet on campus only if you are a student or an employee

I = you can access the internet on campus

S = you are a student

E = you are an employee

$$p = I \Rightarrow (S \vee E)$$

Let's find a sentence for $\overline{p}$

$$\begin{aligned} \neg p &= \neg(I \Rightarrow (S \vee E)) \\ &= \neg(\neg I \vee (S \vee E)) \text{ (Def of } \Rightarrow) \\ &= \neg\neg I \wedge \neg(S \vee E) \text{ (De Morgan)} \\ &= I \wedge (\neg S \wedge \neg E) \text{ (De Morgan + Double negation)} \end{aligned}$$

$\neg p$ = You can access the internet on camput and you are not a student and you are not an employee

1.2 Four types of logical formulas

A logical formula is:

- satisfiable if its truth table contains at least one T
- a contradiction if not satisfiable (always false)
- falsifiable if its truth table contains at least one F
- a tautology if not falsifiable (always true)

ex1: $p \wedge \neg p$: contradiction

contradictions are always falsifiable

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

ex2: $p \vee \neg p$: tautology

tautology are always satisfiable

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

ex3: $p \iff q$: satisfiable and falsifiable

At least one T and one F

p	q	$p \iff q$
T	T	T
T	F	F
F	T	T
F	F	T

1.3 constant logical formulas

Formula A is a tautology iff A = 1, and a contradiction iff A = 0

1: formula always true

0: formula always false

ex: show that the following is a tautology

$$
\begin{aligned}
&((p \Rightarrow q) \wedge p) \Rightarrow q \\
&\equiv ((\neg p \vee q) \wedge p) \Rightarrow q \text{ (Def of } p \Rightarrow q) \\
&\equiv \neg((\neg p \vee q) \wedge p) \vee q \text{ (Def of } p \Rightarrow q) \\
&\equiv (\neg(\neg p \vee q) \vee \neg p) \vee q \text{ (De Morgan)} \\
&\equiv ((p \wedge \neg q) \vee \neg p) \vee q \text{ (De Morgan)} \\
&\equiv (p \wedge \neg q) \vee (\neg p \vee q) \text{ (Associative)} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee q) \text{ (De Morgan)} \\
&\equiv 1 \text{ (Complement)}
\end{aligned}
$$

1.4 rules of inference for propositional logic

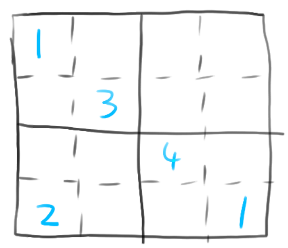
Tautologies can be used for logical reasoning (as inference rules)

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

1.5 application of satisfiability (the SAT problem)

Given a logical formula, find if it's satisfiable or not. This general problem is much more difficult than it looks. We could always compute the truth table to find out, but with n propositional variable, the table would have 2^n rows! There are softwares called "sat solvers" specialized in solving this problem efficiently (in some cases), but none of them defeats the exponential completely, and it is believed that no such program is possible (this is called the $P \neq NP$ conjecture, 1 million \$ prize attached!)

SAT encodes a lot of practical problem in CS. Ex: Sudoku



Fill the grid with numbers {1,2,3,4} such that

1. Each row contains all 4 numbers
2. Each column contains all 4 numbers
3. Each 2x2 contains all 4 numbers

We can solve this using a SAT solver. Variables:

There are 4x4x4 = 64 variables

$$p_{i,j,n} = \text{"The cell in row i, column j, has value n"}$$

Then the three conditions translate into formulas:

1. $p_{i1n} \vee p_{i2n} \vee p_{i3n} \vee p_{i4n}$ for all i an n in {1,2,3,4}
2. and 3. : similar
   add some formulas for consistency
4. No cell contains two different numbers:

$$p_{ijn} \Rightarrow \neg p_{ijk}, \text{ for all } i, j, n, k, , \text{ where } n \neq k$$

5. initial conditions:

$$p_{111} \vee p_{223} \vee p_{334} \vee p_{441} \vee p_{412}$$

The conjunction of all the above formulas form a big formula, but a solution which satisfies it is a solution to the sudoku problem