Predicate logic

p = "All men are mortal"
is an atomic proposition from the point of view of propositional logic (cannot be decomposed into simpler propositions)

Still has some structure:

"men" and "mortal" are predicates
"All" is a quantifier
"are" is a copula

A predicate is something that is true or false, about a subject (which may vary)

Ex: Mortal
Socrates is mortal: True
Zeus is mortal: False

Mathematically, a predicate is a function

$$P : x \in U \longrightarrow Bool = \{T, F\}$$

U : universe of discourse

Subject $\longmapsto$ Truth value

In predicate logic, instead of dealing with propositional variables, we deal with predicates, P(s) and our variables x range over any universe U

$P(x) = $ "$x \geq 0$" $\to$ When you give a value to x you find a proposition

$P(2) = $ "$2 \geq 0$" : True $P(-1)$: "$-1 \geq 0$" False

Quantifiers:

$\forall$:"For all" (Universal quantifier)

$\exists$:"There exists" (Existential quantifier)

We can bind a variable (in a proposition) to a quantifier, so that it can no longer be freely set

Ex : $M(x) =$ " x is mortal "

$\forall x, M(x) =$ " For all x, x is mortal "

Proposition (True or False)

Ex2 : $H(x) \Rightarrow M(x) =$ " If x is a man, then x is mortal "

$\forall x (H(x) \Rightarrow M(x)) =$ " For all x, if x is a man, then x is mortal "

$=$ " All men are mortal "

Ex3 : $E(x) =$ " x is even "

$\exists x, E(x) =$ "There exists x such that x is even"

$=$ "There exists an even x" True

$\rightarrow$ ex : $E(2)$ True, hence $\exists x, E(x)$ True

When is a quantified proposition true or false?

$\forall x, P(x)$ is <u>true</u> is P(x) is true at any input x (p is a constant "True" predicate)

$\exists x, P(x)$ is <u>true</u> if there is at least one example of an x value that makes it true

$\forall x. P(x)$ is <u>false</u> if there is an x value that makes P(x)= F

$\exists x, P(x)$ is <u>false</u> if P(x) is constantly false (false for all x)

In short

$$\left.\begin{array}{l} \neg(\forall x . P(x)) \equiv \exists x . \neg P(x) \\ \neg(\exists x . P(x)) \equiv \forall x . \neg P(x) \end{array}\right\} \text{De Morgan for quantifiers}$$

Ex: $P \equiv \forall x . (P(x) \vee \exists y . Q(x,y))$

$$\neg P \equiv \neg \forall x . (P(x) \vee \exists y . Q(x,y))$$

$$\equiv \exists x . (\neg P(x) \wedge \neg \exists y . Q(x,y))$$

$$\equiv \exists x . (P(x) \wedge \forall y \neg Q(x,y))$$

Restricted quantifiers

Ex: $P(x) =$ "$x > 9000$"

$P(\pi) =$ "$\pi > 9000$" False (but makes sense)

$P(\text{Goku}) =$ "$\underline{\text{Goku}} > 9000$" Sentence does not make

We may want to restrict the domain of a predicate to a specific set...

$$\underbrace{(\forall x \in \mathbb{R})}_{\text{quantifier restricted to } \mathbb{R}} . P(x) \stackrel{def}{=} \forall x ((x \in \mathbb{R}) \Rightarrow P(x))$$

For the existential quantifier

$$(\exists x \in A) . P(x) \stackrel{def}{=} \exists x ((x \in A) \wedge P(x))$$

De Morgan still applies

$$\neg(\exists x \in A . P(x)) = \neg(\exists x . ((x \in A) \wedge P(x)))$$
$$\equiv \forall x, \neg((x \in A) \wedge P(x))$$
$$\equiv \forall x, \neg(x \in A) \vee \neg P(x)$$
$$\equiv \forall x, (x \in A) \Rightarrow \neg P(x)$$
$$\equiv \forall x \in A . \neg P(x)$$

Similarly: $\neg(\forall x \in A . P(x)) \equiv \exists x \in A . \neg P(x)$

Nested Quantifiers

Ex: $\forall x \in \mathbb{R} . \exists y \in \mathbb{R} . x + y = 0$

This is true. For any given x, you can siply choose y = -x as an example, which makes x + y = 0 true

Ex2: $\exists y \in \mathbb{R} . \forall x \in \mathbb{R} . x + y = 0$

This is false. No matter the y that you choose, there are plenty of examples of x (namely, all real numbers except -y) which make x + y != 0

Generally: $\forall x . \exists y . P(x, y) \not\equiv \exists y . \forall x . P(x, y)$

The order in which the variable are quantified matters. One way to think about this is in terms of a two-player game

Game rules:

- Your opponent chooses a value for the $\forall$-quantified variable; you choose a value to the $\exists$-quantified ones

- Take turns according to the order in which the variables are quantified (left to right)

- You win if the resulting proposition if true, you lose if it is false

With this in mind, a "proof" of a (true) formula is just a strategy for winning at that game, no matter the choices that the opponent makes. Disproving a formula (i.e. showig that it is false) is finding a strategy for the opponent to win

What about $\forall\forall$ and $\exists\exists$?

E x : $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} . x^2 + y^2 \geq 0$

Here the opponent just chooses two numbers, x and y, before you have any say about them. So it does not affect your strategy in what order he chooses them

The result would be the same with:

$$\forall y \in \mathbb{R}, \forall x \in \mathbb{R} . x^2 + y^2 \geq 0$$

Or equivalently, these two formulas are equivalent to:

$$\forall (x, y) \in \mathbb{R}^2, x^2 + y^2 \geqslant 0$$

so x and y may very well be chosen together

Generally : $\forall x . \forall y . P(x, y) \equiv \forall y . \forall x . P(x, y)$

So when it is the same quantifier that is involved, the order does not matter

Similarly : $\exists x , \exists y P(x, y) \equiv \exists y . \exists x . P(x, y)$