Permutations and combinations

Ex: From a class of 20 students, form a committee with 3 positions: president, VP, secretary

In how many ways can we possibly do that?

→ Sol: 3 chairs

20 options	19 options	18 options
P	VP	S

Product principle: $20 \times 19 \times 18$ committees

Def: a k-permutation of a set with n elmenets is an ordered arrangement (without repetition) of k elements from that set

By the same logic as in the last example, the number of k-permutations is

$$_nP_k = P(n,k) = n\cdot(n-1)\times\cdots\times(n-k+1)$$

ex: $P(20,3) = 20\times19\times18 = 6840$

Special case $k = n$, we simply call that a permutation

$$\Rightarrow P(n,n) = n(n-1)\cdots(n-n+1) = n!$$

Ex: The number of permutations of S = {1,2,3,4,5} is 5!

Ex2: Form a committee of 3 students from the same class, without positions (everyone's "equal"). In how many ways we can do that?
In the solution of the previous problem, the same three people are counted several times...

$$\left.\begin{array}{ccc} \underline{A} & \underline{B} & \underline{C} \\ P & VP & S \\ B & A & C \\ B & C & A \\ \vdots & & \end{array}\right\}$$

they are now the same

6 possible arrangements

If x is the number of committees with different 3 people

$$6x = P(n, k)$$

$$x = \frac{P(n,k)}{6} = 1140$$

Def: a k-combination of a set of n elements is a subset with k elements. The number of k combinations is:

$${}_nC_k = C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!}$$

n choose k

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

$$= \frac{n!}{k!(n-k)!}$$

This is also called a binomial coefficient

Ex1: Number of poker hands from a regular deck...

$$\binom{52}{5} = \frac{52!}{5!\,47!} = 2598960 \text{ hands}$$

Ex2: Count the number of binary strings of length 10 that have exactly four "zeros" in them

$$\underset{1}{\underline{1}}\ \underset{2}{\underline{0}}\ \underset{3}{\underline{0}}\ \underset{4}{\underline{1}}\ \underset{5}{\underline{1}}\ \underset{6}{\underline{0}}\ \underset{7}{\underline{1}}\ \underset{8}{\underline{1}}\ \underset{9}{\underline{0}}\ \underset{10}{\underline{1}}$$

Choose the four positions that will contain a 0

That's $\binom{10}{4}$ possibilities

The rest of the string is filled with 1

$$\Rightarrow \text{Answer } \binom{10}{4} = \frac{10!}{6!\,4!}$$

Ex3: a chocolate box contains chocolates of 7 different flavors: A,B,C,D,E,F,G

Assume that there are at least four of each

Want: choose four chocolates
You can tell the difference between two chocolates of a different flavor, but not between two chocolates of the same flavor

Also: order does not matter
AABC = ABAC

How many options are there?

Answer is not $\binom{7}{4}$, we want to choose chocolates, not flavors!

Trick: stars and bars
Imagine a shelf with separators...

This display bijectively represents the things that we want to count! That's a string with 2 types of characters ⋆ and |

How many stars?
The number of chocolates = 4
Hoe many bars?
The number of flavors − 1 = 6

String length = 6+4 = 10

So we want the number of binary strings of length 10 that contain exactly four ⋆

Thats ex2! Answer: $\binom{10}{4}$

Combinatorial identities

ex: $\binom{n}{k} = \binom{n}{k-n}$

Proof 1 : Algebraic

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$$

Proof 2 : Combinatorial proof

Show that the two sides of the identity solve the same counting problem

LHS : $\binom{n}{k}$ = The nmber of binary strings of length n with exactly k zeros

RHS : $\binom{n}{n-k}$ = The number of binary strings of length n with exactly (n−k) ones

Same set of strings!

Therefore : LHS = RHS

Ex 2 : Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Combinatorial proof

LHS : Number of committees with k students from a class of n+1 students

RHS : Goal: Solve the same problem in a way that leads to the LHS formula. The + suggests that we split the problem in two disjoint cases

Class = {0,1,2,3,4,...,n}

Case 1: 0 is on the committee. Then there are k−1 other students to choose from n possibilities

$$\Rightarrow \binom{n}{k-1} \text{ committees}$$

Case 2: 0 is not on the committee. Then we must choose the k students among {1,2,3,4,...,n}

$$\Rightarrow \binom{n}{k}$$

Since these two cases are disjoint and cover all possibilities, then by the sum rule, the total number of committees is

$$\binom{n}{k-1} + \binom{n}{k}$$

Conclusion: LHS = RHS

Application: Pascal's Triangle
A recursive way of calculating binomial coefficients...

Uses! $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

$$\binom{n}{0} = \binom{n}{n} = 1$$