MATH240 – Lecture 2

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1 Set algebra

When are sets equal? For instance:

$$A = \{x \in \mathbb{Z} \mid x = 2k - 1 \text{ for some } k \in \mathbb{Z}\}$$
$$B = \{x \in \mathbb{Z} \mid x = 2n + 1 \text{ for some } n \in \mathbb{Z}\}$$

We need to prove:

1. $A \in B$
2. $B \in A$

1. NTS (need to show): if $x \in A$ then $x \in B$

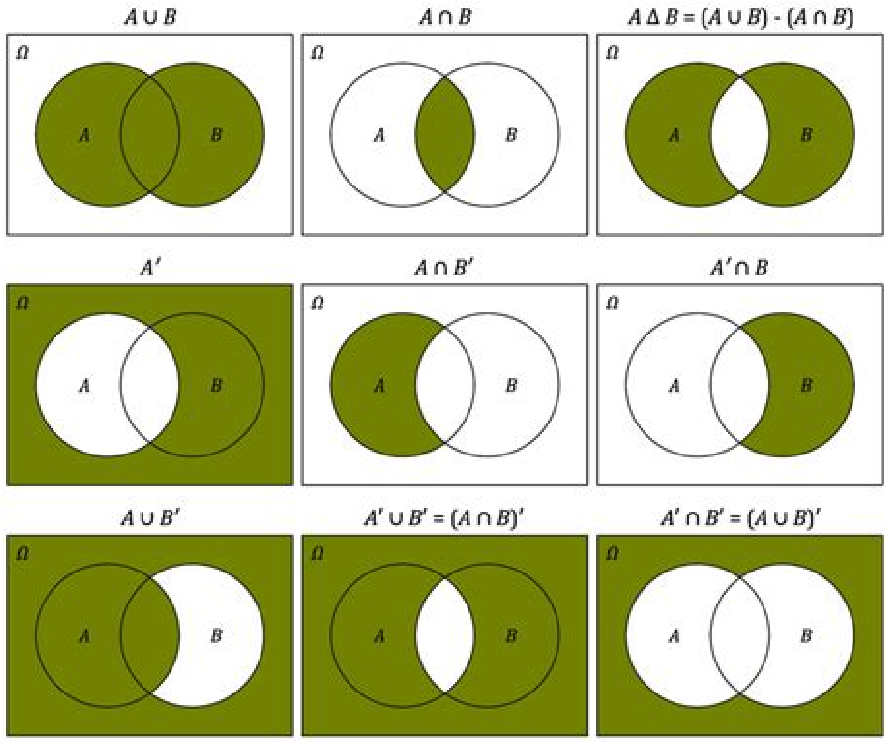
$$\text{Assume } x \in A \text{ so } x = 2k - 1 \text{ for some } k \in \mathbb{Z} = 2k - 2 + 2 - 1 = 2(k-1) + 1$$
$$\text{With } n = k - 1 \text{ we have } x = 2n + 1$$
$$\text{therefore } A = B$$

2. $x \in B \Rightarrow x \in A$ Let $x = 2n + 1$ where $n \in \mathbb{Z}$, then

$$\begin{aligned} x &= 2n + 2 - 2 + 1 \\ &= 2(n+1) - 1 \end{aligned}$$

If $k = n + 1$ then $x = 2k - 1 \Rightarrow x \in A$

2 Set operations



$U =$ The universe of objects

- Union
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

- Intersection
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

- Difference
$$A \backslash B = A - B = \{x \in A \mid x \notin B\}$$

- Complement
$$\overline{A} = A' = \{x \in U \mid x \notin A\} = U \backslash A$$

  ex:

$$\begin{aligned} A &= \{1, 2, 3\} \\ U &= \mathbb{N} \\ \overline{A} &= \{0, 4, 5, \dots\} \end{aligned}$$

- Symmetric Difference

$$\begin{aligned} A \oplus B &= \{x \in U \mid x \in A \text{ or } x \in B \text{ , but not both } \} \\ &= (A \backslash B) \cup (B \backslash A) \\ &= (A \cup B) \backslash (A \cap B) \\ &= (A \cup B) \cap (A \cup B) \end{aligned}$$

ex:

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{3, 4, 5\} \\ A \oplus B &= \{1, 2, 4, 5\} \end{aligned}$$

3 Set identities

Theorem: If two sets A and B have the same Venn diagram, then $A = B$
Proof: We need to show $A \subseteq B$ and $B \subseteq A$ (They have similar proof)

1. $A \subseteq B$
   Let $x \in A$, then x is in the shaded region of A's Venn diagram. So it's also in shaded version of B's Venn diagram, so $x \in B$

2. $B \subseteq A$
   Let $x \in B$, then x is in A's region. So it's also in B's region, so $x \in A$

$\square$

3.1 Laws of boolean algebra

Laws of boolean algebra are complete set of rules

- Identify law:

$$\begin{aligned} A \cup \emptyset &= A \\ A \cap U &= A \end{aligned}$$

- Idempotent laws:

$$\begin{aligned} A \cup A &= A \\ A \cap A &= A \end{aligned}$$

- Domination laws:

$$\begin{aligned} A \cup U &= U \\ A \cap \emptyset &= \emptyset \end{aligned}$$

- Double complement laws:

$$\overline{\overline{A}} = A$$

- Commutative laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- Complement laws:

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

- Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- De Morgan's law:

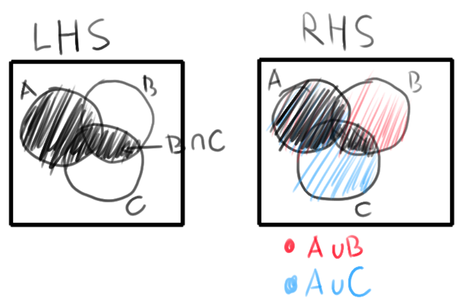
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption law:

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

Note: Past 3 sets or more than 2 levels of set equations, Venn diagrams become impossible to read But we can use the laws of boolean algebra to derive new set identities algebraically.

ex: Check $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



ex: use set laws to show

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cup \overline{(B \cup C)} \text{ (De Morgan)} \\
&= \overline{A} \cup (\overline{B} \cap \overline{C}) \text{ (De Morgan)} \\
&= \overline{A} \cup (\overline{C} \cap \overline{B}) \text{ (Commutativity)} \\
&= (\overline{C} \cap \overline{B}) \cup \overline{A} \text{ (Commutativity)}
\end{aligned}
$$

3.2 External set operations

The following set operations change the universe of the set

- Cartesian product: $A \times B = \{(a, b) \mid a \in A, b \in B\}$ (set of ordered pairs)
  ex:

  Cardinality: $|A \times B| = |A| \times |B|$

  $$A = \{1, 2, 3\} \quad B = \{0, 1\}$$
  $$A \times B = \{(1, 0), (2, 0), (3, 0), (1, 1)(2, 1)(3, 1)\}$$

  ex2:

  $$\{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \text{ (The set of 2-dimensional vectors)}$$

- Power set: $\{x \mid x \subseteq A\}$ (set of all subsets) ex:

  Cardinality: $|P(A)| = 2^{|A|}$

  $$
  \begin{aligned}
  A &= \{1, 2, 3\} \\
  P(A) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\} \text{ (8 elements)}
  \end{aligned}
  $$

  ex2:

  $$P(\emptyset) = \{\emptyset\} \text{ (1 element)}$$
  $$|P(\emptyset)| = 1$$