MATH 240 — Lecture 9

Enlai Li

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1 Bijection Principle

If A, B are finite sets:

$$|A| = |B| \iff thereexistsabijectivefunction f : A \to B$$

Ex: Show that for a finite set A

$$|P(A)| = 2^{|A|}$$

$$B_n = \{0,1\}x\{0,1\} - x\{0,1\} = \{0,1\}^n$$

So we just need to find a bijection

$$f : P(A) \to B_n$$

in order to conclude (with the bijection Principle) that:

$$|P(A)| = 2^n$$

Let $x \in P(A)$, then $x \subseteq A$
Let $A = \{a_1, a_2, \ldots, a_n\}$ Define: $f(x) = (b_1, b_2, \ldots, b_n)$

$$\text{where } b_i = \begin{cases} 1 & \text{if } a_i \in X \\ 0 & \text{if } a_i \notin X \end{cases} \tag{1}$$

Ex:

$$A = \{1,2,3,4\}, X = \{1,4\} \Rightarrow f(x) = (1,0,0,1)$$

This f is <u>clearly</u> invertible

$$f^{-1}(b_1, b_2, \ldots, b_n) = \{a_i \in A | b_i = 1\}$$

invertible $\Rightarrow$ bijective

$$f^{-1}(f(X)) = Xf(f^{-1}(b)) = b$$

2 Infinite Cardinalities

Let's extend the bijection principle to Infinite sets Def: Two sets A and B have the same cardinality if there is a bijection $f : A \leftarrow B$

We then write

$$|A| = |B|$$

Ex:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

$$\mathbb{E} = \{0, 2, 4, 6, 8, \ldots\}$$

Theorem: There is no bijection between $\mathbb{N}$ and $\mathbb{R}$ for $(0, 1)$

That $\mathbb{R}$ is strictly larger than $\mathbb{R}$

$$\mathbb{N} < \mathbb{R}$$

There are at least two different infinities! (In fact, there are infinitely many)

Proof: (Canton's Diagonal Argument)

By contradiction. Assume there is a bijection

$$f : \mathbb{N} \leftarrow (0, 1)$$

Note: any $x \in (0, 1)$ can be written in decimal notation:

$$x = 0, a_1, a_2, a_3, \ldots (a_i \in 0, 1, \ldots, 9)$$

$$Ex : \frac{1}{3} = 0.33333 \ldots a_i = 3, \forall_i$$

$$f(0) = 0, a_0$$

Now consiter the following nunber:

$$C = 0.C_0C_1C_2C_3 \ldots$$

where

$$\text{where } c_i = \begin{cases} 4 & \text{if} a_{ii} \neq 4 \\ 2 & \text{if} a_{ii} = 4 \end{cases} \tag{2}$$

What matters is $c_i \neq a_i i(\forall_i) \Rightarrow C \neq f(n), \forall n \in \mathbb{N}$

That's a contradiction! $c \in (0, 1)$ but it is not in the "list" (which should have been complete) $\square$

2.1 Some remarks

1. Sets in bijection with $\mathbb{N}$ are called countable sets, ex:

$$\mathbb{N}, \mathbb{E}, \mathbb{O}, \mathbb{Z}, \mathbb{Q}$$

They are the "smallest" kind of infinite sets

2. Sets in bijection with $\mathbb{R}$ are called "continuous"
ex: $\mathbb{R}, (0,1)$, any real interval, ...

3. There are sets that are larger than $\mathbb{R}$ ex:

$$\mathbb{R} \subset |P(\mathbb{R})| \subset |P(P(\mathbb{R})) \subset \ldots$$

The proof of that is analogous to the one we just did

4. Is there a set X with $|\mathbb{N}| < |\mathbb{X}| < |\mathbb{R}|$?
This problem is called the continuum hypothesis and is known to be undecidable!

2.2 Relations

Ex: Unit circle

$$C = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$$

Def: A <u>relation</u> on a set A is a subset $R \subseteq A \times A$
Ex: Function $f : A \to A$

$$\text{Graph } (f) = \{(x,y) \in A \times A | y = f(x)\}$$

That is a relation which satisfies the vertical line test ...
Ex: Equality on a set A:

$$A = \{(x,x) | x \in A\}$$

Ex:

$$(5,5) \in =_{\mathbb{Z}}: 5 =_{\mathbb{Z}} 5 \tag{3}$$

$$(2,5) \in \neq_{\mathbb{Z}}: 2 \neq_{\mathbb{Z}} 5 \tag{4}$$

2.3 Infix Notation

$$s \tag{5}$$

Ex4: U: some universe of sets ...

$$A \sim B \Rightarrow |A| = |B| \text{ meaning } \exists f : A \to B \text{ bijective}$$

$\sim$ is a relation on U:

$$\mathbb{N} \sim \mathbb{Z}$$

$$3 \tag{6}$$