MATHR323 – Lecture 10

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1 HyperGeometric

N: size of population

M: size of subpopulation of interest

n: sample size

Y: # of tagged fish in the sample of size n

$$H\mathbb{G}(N, M, n)$$
$$\mathbb{E}(Y) = \frac{nM}{N}$$
$$V(Y) = n(\frac{M}{N})(1 - \frac{M}{N})(\frac{N-n}{N-1})$$

$\boxed{R \quad B}$

M: # of red balls in the bag

N: # of blue balls in the bag

$$x_i = \begin{cases} 1 & R \\ 0 & B \end{cases}$$

$$Y = \sum_{i=1}^{n} X_i$$

$$x_1, \dots, x_n$$

$$P(x_i = 1) = \frac{M}{N} = P$$

$$Z = \sum_{i=1}^{n} X_i \sim Bin(n, p)$$

$$Y = \sum_{i=1}^{n} X_i \sim H\mathbb{G}(N, M, n)$$

$$\mathbb{E}(Y) = \mathbb{E}[\sum_{1}^{n} X_i] = \sum_{i=1}^{n} X_i \mathbb{E}(x_i)$$

$$\mathbb{E}[Y] = \frac{nM}{N}$$

$$P(Y = k) = \frac{\binom{M}{K}\binom{N-M}{n-k}}{\binom{N}{n}}, \text{ where } k = 0, 1, \dots n$$

$$\sum_{k=0}^{n} P(Y = k) \stackrel{?}{=} 1$$

$$\sum_{k=0}^{n} \frac{\binom{M}{K}\binom{N-M}{n-k}}{\binom{N}{n}} \stackrel{?}{=} 1$$

$$\sum_{k=0}^{n} \binom{M}{K}\binom{N-M}{n-k} = \binom{N}{n}$$

$$P(x_1 = 1) = \frac{M}{N}$$

We take a second ball without looking at the color:

$$P(x_2 = 1) = P(x_2 = 1 \mid x_1 = 1)P(x_1 = 1) + P(x_2 = 1|x_1 = 0)P(x_1 = 0)$$

$$= \frac{M-1}{N-1} \times \frac{M}{N} + \frac{M}{N-1}(1 - \frac{M}{N})$$

$$= \frac{M(M-1)}{N(N-1)} + \frac{M(N-M)}{N(N-1)}$$

$$= \frac{-M + MN}{N(N-1)}$$

$$= \frac{M}{N}$$

The probability does not change, what changes is the independence:

$$
\begin{aligned}
P(x_i = 1) &= \frac{M}{N}, i = 1, 2, \ldots, n \\
p(X_1 = 1, x_2 = 1) &= P(x_2 = 1 \mid x_1 = 1)p(X_1 = 1) \\
&= \frac{M-1}{N-1} \times \frac{M}{N} \\
&= \frac{M(M-1)}{N(N-1)}
\end{aligned}
$$

1.1 Question 3.33, p130

$$(1+x)^N = (1+x)^M(1+x)^{N-M}$$

A group of 20 peoples (N), 8 black peoples (M), a random sample of 6 people(n), only 1 black person (Y) among the jury. Is there a reason to doubt the randomness?

$$
\begin{aligned}
N &= 20 \\
M &= 8 \\
n &= 6 \\
Y &= 1 \\
\mathbb{E}(Y) &= \frac{nM}{N} = \frac{6 \times 8}{20} = 2.4 \\
P(Y = 1) &= \frac{\binom{8}{1}\binom{12}{5}}{\binom{20}{6}} \\
&= 0.1634
\end{aligned}
$$

When sample is very big, it does not matter if we return the ball

$$n(\frac{M}{N})(1 - \frac{M}{N})(\frac{N-n}{N-1})$$

$$\lim_{N \to \infty} \frac{\binom{M}{Z}\binom{N-M}{n-k}}{\binom{N}{n}}$$

$$\to \binom{n}{k} \frac{M}{N}^K (1 - \frac{M}{N})^{n-k}$$

Radioactive decay: for large n, p is not fixed and vary with n

$$\binom{n}{k} p_n^k (1 - p_n)^{n-k}$$

$$np_n \to \lambda, \text{ as } n \to \infty$$

1.

$$\lim_{N\to\infty} \frac{n(n-1)\dots(n-k+1)}{n\times n\cdots\times n} = lim \prod_{g=0}^{k-1}(\frac{n-j}{n})$$
$$= lim_{N\to\infty} \prod_{j=0}^{k-1}(\frac{1-j}{n})$$
$$= \prod_{j=0}^{k-1} \underbrace{lim_{N\to\infty}(\frac{1-j}{n})}_{1}$$
$$= \prod_{j=0}^{k} a_j = a_0 \times a_1 \dots a_k$$

2.

$$lim_{N\to\infty}(1-\frac{\lambda}{n})^n = e^{-\lambda}$$
$$lim_{N\to\infty}(1-\frac{\lambda}{n})^{-k} = e^{-\lambda}$$
$$lim_{N\to\infty}\binom{n}{k}p_n^k(1-p)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!} \text{ If } np_n \approx \lambda$$

$$\lim_{N\to\infty}\binom{n}{k}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k}$$
$$= \lim_{N\to\infty}\frac{n!}{k!(n-k)!} \times \frac{\lambda^k}{n^k}(1-\frac{\lambda}{n})^{n-k}$$
$$= \frac{\lambda^k}{k!}\lim_{N\to\infty}\frac{n(n-1)\dots(n-k+1)}{n\times n\cdots\times n}(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-k}$$

1.2 Approximation of binomial

$$np_n \approx \lambda \text{ and } n\to\infty$$
$$P(x=k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, \dots$$

Poisson p.m.f

$$\sum_{k=0}^{\infty} P(x=k) \stackrel{?}{=} 1$$

$$\begin{aligned}
\sum_{k=0}^{\infty} P(x=k) &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1 \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}
\end{aligned}$$

if $x \sim P_0(\lambda)$