

# MATH323 – Lecture

Enlai Li

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# 1 Geometric probability mass function (p.m.f.)

$$P(Y = k) = (1 - p)^{k-1} p \text{ where } k = 1, 2, \dots$$

$$\sum_{k=1}^{\infty} P(Y = k) = 1$$

$$\begin{aligned} \mathbb{E}(Y) &= \sum_{k=1}^{\infty} k P(Y = k) \\ &= \sum_{k=1}^{\infty} (1 - p)^{k-1} p \\ &= p \sum_{k=1}^{\infty} k q^{k-1} \\ &= p \sum_{k=1}^{\infty} \frac{d}{dq} (q^k) \end{aligned}$$

$$\sum_{n=i}^{\infty} a_n x^n \quad \text{note: } \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\begin{aligned} &= p \frac{d}{dq} \left[ \sum_{k=1}^{\infty} q^k \right] \\ &= p \frac{d}{dq} \left[ \frac{q}{1 - q} \right] \\ &= p \frac{1 \times (1 - q) + q}{(1 - q)^2} \\ &= p \frac{1}{(1 - q)^2} \\ &= \cancel{p} \frac{1}{p^x} \\ &= \frac{1}{p} \end{aligned}$$

$$\mathbb{E}(Y) = \frac{1}{p}$$

$$\begin{aligned}
& \binom{49}{6} \\
p &= \frac{1}{\binom{49}{6}} \\
\mathbb{E}(Y) &= \binom{49}{6} \\
2 \times 52.5 &= 105 \\
\binom{49}{6} &= 13,983,816 \\
\frac{\binom{49}{6}}{105} &= \frac{13,983,816}{105} \\
&= 133,180 \text{ games}
\end{aligned}$$

$$\begin{aligned}
V(Y) &= \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 \\
\mathbb{E}(Y^2) &= \mathbb{E}[Y(Y-1)] + \mathbb{E}[Y] \\
\mathbb{E}[Y(Y-1)] &= \sum_{k=1}^{\infty} k(k-1)P(y=k) \\
&= \sum_{k=1}^{\infty} k(k-1)q^{k-1}p \\
&= pq \sum_{k=1}^{\infty} k(k-1)q^{k-2} \\
&= pq \frac{d^2}{dq^2} \left[ \sum_{k=1}^{\infty} q^k \right] \\
V(Y) &= \frac{(1-p)}{p^2}
\end{aligned}$$

## 1.1 Negative Binomial Distribution

Notation:  $Bin(n, p)$ ,  $Geo(p)$ ,  $NB(r, p)$

$$X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

$\underline{r}$  successes:

$$\begin{aligned}
& r \\
Y &= K
\end{aligned}$$

There is at least 2 success. Consider  $K$  boxes filled with 0, there is a success box with 1 at position  $K$ . There is  $r - 1$  boxes, corresponding to  $k - 1$  boxes, it then becomes

$$P(Y = k) = \binom{k-1}{r-1} p^r q^{k-r} \text{ where } k = r, r+1, \dots$$

$$q = 1 - p$$

$$\left(\frac{1}{1-x}\right)^r = \sum_{k=r}^{\infty} \binom{k-1}{r-1} x^{k-r} \text{ where } 0 < x < 1$$

$$\begin{aligned} \sum_{k=r}^{\infty} P(Y = K) &= p^r \sum_{k=r}^{\infty} \binom{k-1}{r-1} q^{k-r} \\ &= p^r \left(\frac{1}{1-q}\right)^r \\ &= 1 \end{aligned}$$

If  $Y \sim NB(r, p)$  then

$$\mathbb{E}(Y) = \frac{r}{p}$$

$$V(Y) = \frac{r(1-p)}{p^2}$$

$$Y \sim NB(r, p)$$

$$Y = \sum_{i=1}^r X_i \text{ where } X_i \overset{ind}{\sim} Geo(p)$$

$$\text{If } Y \sim Bin(n, p) \text{ then } Y = \sum_{i=1}^n X_i, X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$