MATH323 – Lecture 9

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1 Geometric probability mass function (p.m.f.)

$$P(Y=k) = (1-p)^{k-1}p \text{ where } k = 1, 2, \ldots$$

$$\sum_{k=1}^{\infty} P(Y=k) = 1$$

$$\begin{aligned}
\mathbb{E}(Y) &= \sum_{k=1}^{\infty} kP(Y=k) \\
&= \sum_{k=1}^{\infty} (1-p)^{k-1}p \\
&= p\sum_{k=1}^{\infty} kq^{k-1} \\
&= p\sum_{k=1}^{\infty} \frac{d}{dq}(q^k)
\end{aligned}$$

$$\sum_{n=i}^{\infty} a_n x^n \qquad \text{note: } \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\begin{aligned}
&= p\frac{d}{dq}[\sum_{k=1}^{\infty} q^k] \\
&= p\frac{d}{dq}[\frac{q}{1-q}] \\
&= p\frac{1 \times (1-q) + q}{(1-q)^2} \\
&= p\frac{1}{(1-q)^2} \\
&= \cancel{p}\frac{1}{p^{\cancel{2}}} \\
&= \frac{1}{p}
\end{aligned}$$

$$\mathbb{E}(Y) = \frac{1}{p}$$

$$\binom{49}{6}$$

$$p = \frac{1}{\binom{49}{6}}$$

$$\mathbb{E}(Y) = \binom{49}{6}$$

$$2 \times 52.5 = 105$$

$$\binom{49}{6} = 13,983,816$$

$$\frac{\binom{49}{6}}{105} = \frac{13,983,816}{105}$$

$$= 133,180 \text{ games}$$

$$
\begin{aligned}
V(Y) &= \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2 \\
\mathbb{E}(Y^2) &= \mathbb{E}[Y(Y-1)] + \mathbb{E}[Y] \\
\mathbb{E}[Y(Y-1)] &= \sum_{k=1}^{\infty} k(k-1)P(y=k) \\
&= \sum_{k=1}^{\infty} k(k-1)q^{k-1}p \\
&= pq\sum_{k=1}^{\infty} k(k-1)q^{k-2} \\
&= pq\frac{d^2}{dq^2}[\sum_{k=1}^{\infty} q^k]
\end{aligned}
$$

$$V(Y) = \frac{(1-p)}{p^2}$$

1.1 Negative Binomial Distribution

Notation: $Bin(n,p)$, $Geo(P)$, $NB(r,p)$

$$X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

r successes:

$$r$$

$$Y = K$$

There is at least 2 success. Consider K boxes filled with 0, there is a success box with 1 at position K. There is $r-1$ boxes, corresponding to $k-1$ boxes, it then becomes

$$\binom{k-1}{r-1}p^r q^{k-r}$$

$$P(Y=k) = \binom{k-1}{r-1}p^r q^{k-r} \text{ where } k = r, r+1, \ldots$$

$$q = 1-p$$

$$\boxed{(\frac{1}{1-x})^r = \sum_{k=r}^{\infty}\binom{k-1}{r-1}x^{k-r} \text{ where } 0 < x < 1}$$

$$\begin{aligned}\sum_{k=r}^{\infty}P(Y=K) &= p^r\sum_{k=r}^{\infty}\binom{k-1}{r-1}q^{k-r}\\ &= p^r(\frac{1}{1-q})^r\\ &= 1\end{aligned}$$

If $Y \sim NB(r,p)$ then

$$\mathbb{E}(Y) = \frac{r}{p}$$

$$V(Y) = \frac{r(1-p)}{p^2}$$

$$Y \sim NB(r,p)$$

$$Y = \sum_{i=1}^{r}X_i \text{ where } X_i \overset{ind}{\sim} Geo(P)$$

If $Y \sim Bin(n,p)$ then $Y = \sum_{i=1}^{n}X_i$, $X_i = \begin{cases}1 & p\\ 0 & 1-p\end{cases}$

1.2 HyperGeometric Distribution

$R + B = N$:

$$\boxed{R \quad B}$$

Y = # of rectangles in a sample of size n

$$S \qquad \binom{N}{n}$$

$$P(Y = r)$$

$$A = \binom{R}{r}\binom{B}{n-r}$$

$$P(Y = r) = \frac{\binom{R}{r}\binom{B}{n-r}}{\binom{N}{n}}, \text{ where } r = 0, 1, 2, \ldots, n$$

$$Y \sim H\mathbb{G}(N, M, n)$$

$$P(Y = r) = \frac{\binom{M}{r}\binom{N-M}{n-r}}{\binom{N}{n}}, \text{ where } r = 0, 1, 2, \ldots, n$$

Take n balls:

$$Y = \sum_{i=1}^{n} X_i$$

$$x_i = \begin{cases} 0 & 1 - p_n \\ 1 & p_n \end{cases}$$