2.2 Bayesian Optimisation

The core contributions of this thesis relate to the use of optimisation techniques in the context of the data-driven clutch control tuning during gearshifts. The selected optimisation algorithm is the *Bayesian Optimisation* and in this regard, this section is devoted to explain its principles.

Bayesian optimisation is a very versatile algorithm, that is designed for black-box derivative-free global optimisation, where the analytical expression of the objective function $f(x)$ is unknown and so are its derivatives. It belongs in the class of *reinforcement learning* algorithms, in which a software agent is taught to maximise its acquisition of rewards in a given environment through observations of it. In principle, the evaluation of the function is restricted to sampling at a point x and getting a, possibly noisy, response, which is assessed by the agent in order to make decisions about the next actions. Ultimately, the agent tries to maximise the reward (or minimise the cost) which corresponds to converging to the global maximum (or minimum) of the unknown objective function $f(x)$.

In mathematical terms, suppose there is a function $f\colon \mathcal{X} \to \mathbb{R}$ and the domain of interest is $X \subseteq \mathcal{X}$. Then, the solution of the optimisation problem is the value

$$x^* = \arg\max_{x \in X} f(x) \tag{2.1}$$

Although there are also other algorithms that follow similar principles, the innovation of Bayesian optimisation lies on the way that the next sampling point is selected and on its attempt to converge to the global optima in the minimum number of iterations. This is particularly useful for problems where the sampling process is expensive (either in terms of time or resources), like the ones that this thesis tackles.

A Bayesian optimisation algorithm consists of two main components; a *surrogate model* and an *acquisition function*

Surrogate Model

The surrogate model is used to approximate the unknown objective function $f(x)$. There are several ways to model the function $f(x)$ and its uncertainty, but the most popular approach is to use Gaussian processes (GPs).

A GP is a random process where any point $\boldsymbol{x} \in \mathbb{R}^d$ is assigned a random variable $f(\boldsymbol{x})$ and where the joint distribution of a finite number of these variables $p(f(\boldsymbol{x}_1), , f(\boldsymbol{x}_N))$ is itself a Gaussian.

This statistical model provides a posterior probability distribution that describes potential values for $f(x)$ at a candidate point x. Each time f is sampled at a new point, this posterior distribution is updated.

From a mathematical perspective, given observations $\boldsymbol{f} = (f(\boldsymbol{x}_1), f(\boldsymbol{x}_2), ..., f(\boldsymbol{x}_t))$ at t points, the posterior distribution should be found or equivalently an estimation about the function value at a new point $\boldsymbol{x}^*$ should be made, which can be denoted as $f^* = f(\boldsymbol{x}^*)$ and in the case of a GP, it is equal to

$$Pr(f^*|\boldsymbol{f}) = \mathcal{N}(\mu(\boldsymbol{x}^*), \sigma^2(\boldsymbol{x}^*)) \tag{2.2}$$

where $\mu(\boldsymbol{x})$ corresponds to the mean function and $\sigma^2(\boldsymbol{x}^*)$ to the kernel (or covariance) function. Using this formula, the distribution of the function at any new point x^* can be estimated. The best estimate of $f(x)$ is given by the mean $\mu(\boldsymbol{x})$ and the uncertainty is given by the variance $\sigma^2(\boldsymbol{x}^*)$.

In Fig 2.3 an example of the above procedure is showed, when an additional sample is added to the current sampling set. The blue line represents the approximation of the objective function that the GP realises and the light blue area corresponds to the uncertainty of this approximation. For simplicity, we assume that the underlying objective function is deterministic and therefore each sample is surrounded by negligible uncertainty. When the new sample is added, the approximation is updated and the uncertainty is reduced.

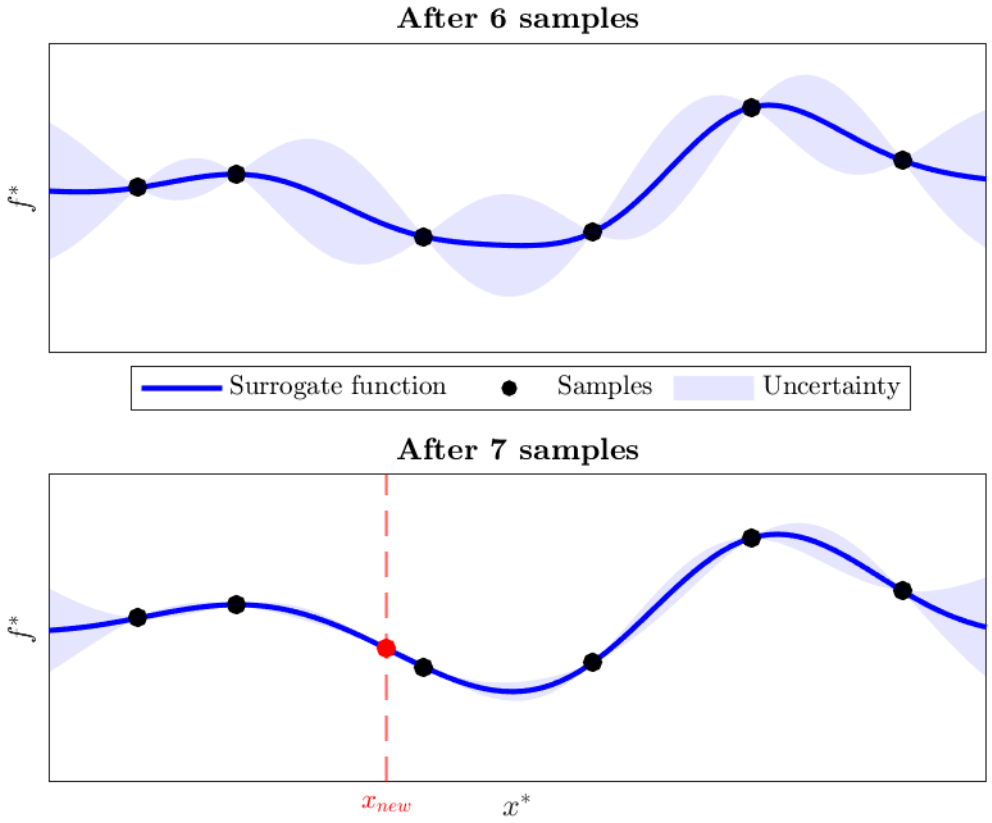

Figure 2.3: Update of a Gaussian process model

Acquisition Function

The acquisition function proposes sampling points in the search space. It takes the mean and variance at each point $\boldsymbol{x}$ on the function and computes a value that indicates how desirable it is to sample next at this position. A good acquisition function should trade off exploration and exploitation. Exploitation means sampling where the surrogate model predicts a high objective and exploration means sampling at locations where the prediction uncertainty is high. Both correspond to high acquisition function values and the goal is to maximise the acquisition function to determine the next sampling point.

There are a lot of possible acquisition functions available, with the most commonly used to be the *maximum probability of improvement (MPI)*, *expected im-*

provement (EI) and *upper confidence bound (UCB)*. In the following, the expected improvement (EI) will be further analysed, since it is the acquisition function that was used during the thesis.

Expected improvement computes the expected amount of improvement and it is given by

$$EI(\boldsymbol{x}) = \mathbb{E}max(f(\boldsymbol{x}) - f(\boldsymbol{x}^+), 0) \tag{2.3}$$

where $f(\boldsymbol{x}^+)$ is the value of the best sample so far and $\boldsymbol{x}^+$ is the location of that sample. Under the assumption of GP model , the EI can be expressed in an analytical way as follows [1]

$$EI(\boldsymbol{x}) = \begin{cases} (\mu(\boldsymbol{x}) - f(\boldsymbol{x}^+) - \xi)\Phi(Z) + \sigma(\boldsymbol{x})\phi(Z) & \text{if } \sigma(\boldsymbol{x}) > 0 \\ 0 & \text{if } \sigma(\boldsymbol{x}) = 0 \end{cases} \tag{2.4}$$

where

$$Z = \begin{cases} \frac{\mu(\boldsymbol{x}) - f(\boldsymbol{x}^+) - \xi}{\sigma(\boldsymbol{x})} & \text{if } \sigma(\boldsymbol{x}) > 0 \\ 0 & \text{if } \sigma(\boldsymbol{x}) = 0 \end{cases} \tag{2.5}$$

where $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are the mean and the standard deviation of the GP posterior predictive at x, respectively. Φ and ϕ are the CDF and PDF of the standard normal distribution, respectively.

The first summation term in Equation 3.2 is the exploitation term and second summation term is the exploration term. Parameter ξ determines the amount of exploration during optimisation and higher ξ values lead to more exploration. In other words, a high value of ξ increases the importance of potential improvement in regions of high prediction uncertainty (large $\sigma(\boldsymbol{x})$).

Combining the surrogate model with the acquisition function, the algorithm of the Bayesian optimisation can be summarised as described in Algorithm 1 [2]. In Figure 2.4, an illustrative 1D example for three consecutive iterations of a Bayesian optimisation process is showed. The green plot illustrates the EI values for each possible new sample given the current posterior probability distribution

Algorithm 1: Pseudo-code for Bayesian optimisation

```
Place a Gaussian process prior on f
Observe f at n_0 points according to an initial space-filling design.
Set n = n_0.
while n < N do
    Update the posterior probability distribution on f
    Let x_n be a maximiser of the acquisition function over x
    Observe y(x_n) = f(x_n)
    Increment n
end while
Return a solution: either the point evaluated with the largest f(x), or
  the point with the largest posterior mean.
```

on f. The acquisition is high where the GP predicts a high objective (exploitation) and where the prediction uncertainty is high (exploration); areas with both attributes are sampled first. Note that the area on the far left remains unsampled, as while it has high uncertainty, it is (correctly) predicted to offer little improvement over the highest observation.

Note that the algorithm does not try to model the unknown objective function but only to find its global optimum.

2.3 Literature Analysis

In the literature, the clutch control in vehicles has been a matter of several studies [3, 4, 5]. However, regarding the development of such systems in motorcycles, the research is in its initial stages, since the great majority of the literature is applied to four-wheel vehicles, both for manual and automated transmissions.

The work in [6] signifies the first contributions in the area of control-oriented analysis for the gear shifting manoeuvre of motorcycles. In the paper, the detailed