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Errata for *Quantitative Robust Uncertainty Principles and Optimally Sparse Decompositions* (DOI: 10.1007/s10208-004-0162-x)

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In the proof of Theorem 4.1, $\Phi = \begin{pmatrix} I & F^* \end{pmatrix}$ is the dictionary constructed by concatenating the Dirac and Fourier orthobases, and Φ_Γ, $\Phi_{\Gamma'}$ are subdictionaries constructed by extracting columns from Φ corresponding to the index sets Γ, Γ'. The assertion is made that if $|\Gamma| = |\Gamma'|$, and if both Φ_Γ, $\Phi_{\Gamma'}$ are full rank, then it must follow that $\mathrm{Range}(\Phi_{\Gamma\setminus\Gamma'}) = \mathrm{Range}(\Phi_{\Gamma'\setminus\Gamma})$. This is true if Φ_Γ and $\Phi_{\Gamma'}$ are both orthogonal matrices, but is false in general (including the context of the Theorem).

A correct proof of Theorem 4.1 requires a different tack. The statement is the same, except with a very minor change in the constant. We will also not require Lemma 4.2.

Theorem 4.1. *Let $f = \Phi\alpha$ be a signal of length $N \geq 512$ with support set $\Gamma = T \cup \Omega$ sampled uniformly at random with*

$$|T| + |\Omega| \leq \frac{.2681\ N}{\sqrt{(\beta+1)\log N}},$$

and with coefficients α sampled as in Section 2. Then the solution to (P_0) is unique and equal to α with probability at least $1 - O((\log N)^{1/2} \cdot N^{-\beta})$.

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Proof. Theorem 3.1 is easily generalized so that if Γ is chosen uniformly at random with

$$|T| + |\Omega| \leq \frac{.5583\, q\, N}{\sqrt{(\beta+1)\log N}},$$

for any $0 < q \leq 1/2$, then

$$\|F_{\Omega T}{}^* F_{\Omega T}\| \leq q \tag{4.1}$$

with probability $1 - O((\log N)^{1/2} \cdot N^{-\beta})$. We will show that taking q just less than $1/2$ ($q \approx .4802$) will guarantee (with probability 1) that a random coefficient sequence on a Γ which satisfies (4.1) can be recovered by solving (P_0).

Given a Γ obeying (4.1), the (continuous) probability distribution on the $\{\alpha(\gamma), \gamma \in \Gamma\}$ induces a continuous probability distribution on $\mathrm{Range}(\Phi_\Gamma)$. We will show that for every $\Gamma' \neq \Gamma$ with $|\Gamma'| \leq |\Gamma|$

$$\mathrm{Range}(\Phi_{\Gamma'}) \neq \mathrm{Range}(\Phi_\Gamma). \tag{4.2}$$

As such, the set of signals in $\mathrm{Range}(\Phi_\Gamma)$ that have expansions on alternate supports Γ' that are *at least* as sparse as their expansions on Γ is at most a finite union of subspaces of dimension strictly smaller than $|\Gamma|$. This set has measure zero as a subset of $\mathrm{Range}(\Phi_\Gamma)$, and hence the probability of observing such a signal is zero.

Consider any $\Gamma' = T' \cup \Omega'$ different than Γ with $|\Gamma'| \leq |\Gamma|$. The range of Φ_Γ will equal the range of $\Phi_{\Gamma'}$ only if each column φ_γ for $\gamma \in \Gamma' \backslash \Gamma$ is in the range of Φ_Γ. Without loss of generality, suppose $T' \backslash T \neq \emptyset$ (the same argument, with the roles of time and frequency reversed, also applies to the case where $\Omega' \backslash \Omega \neq \emptyset$). Take $\varphi_\gamma = \delta_{t_0}$ to be a spike at location $t_0 \in T' \backslash T$. Using the uncertainty principle, we will show that δ_{t_0} cannot be in $\mathrm{Range}(\Phi_\Gamma)$.

Arguing by contradiction, suppose that $\delta_{t_0} \in \mathrm{Range}(\Phi_\Gamma)$. Then there must be a linear combination of the sinusoids in Φ_Ω that is zero everywhere except on $T \cup \{t_0\}$. Expressed differently, there exists α_0 supported on Ω such that $f = F^* \alpha_0$ vanishes outside of $T \cup \{t_0\}$. Let f_T be the values of f on T, and $f_{\{t_0\}}$ the value at t_0. Since $\hat{f}$ is supported on Ω and the pair (T, Ω) obeys (4.1), it follows that

$$\|f_T\|_2^2 \;=\; \|F^*_{\Omega T} R_\Omega \hat{f}\|^2 \;\leq\; q\|f\|_2^2,$$

which gives $|f_{\{t_0\}}|^2 \geq (1-q)\|f\|_2^2$. By construction, $1_{\Omega^c} \cdot \hat{f} = 0$ or, equivalently, $F R_T^* f_T = f_{\{t_0\}}\, F\delta_{t_0}$ on Ω^c implying that

$$\|1_{\Omega^c} \cdot F R_T^* f_T\|_2^2 = |f_{\{t_0\}}|^2 \|1_{\Omega^c} \cdot F\delta_{t_0}\|_2^2 = |f_{\{t_0\}}|^2 \cdot \left(1 - \frac{|\Omega|}{N}\right). \tag{4.3}$$

On the one hand, we have

$$\|1_{\Omega^c} \cdot F R_T^* f_T\|_2^2 \leq \|f_T\|_2^2 \leq q\|f\|_2^2,$$

and on the other,

$$
\begin{aligned}
|f_{\{t_0\}}|^2 \|1_{\Omega^c} \cdot F\delta_{t'}\|_2^2 &\geq (1-q) \cdot \left(1 - \frac{|\Omega|}{N}\right) \cdot \|f\|_2^2 \\
&\geq (1-q) \cdot \left(1 - \frac{.5583\, q}{\sqrt{(\beta+1)\log N}}\right) \cdot \|f\|_2^2 \\
&\geq (1-q) \cdot (1 - .1581q) \cdot \|f\|_2^2,
\end{aligned}
$$

where the last inequality holds for $\beta \geq 1$ and $N \geq 512$. Therefore, (4.3) can hold only if

$$q \geq (1-q) \cdot (1 - .1581q)$$

which is not true for $q \leq .48026$. As a result, (4.2) holds, and α is ℓ_0-unique with probability 1 (conditioned on Γ obeying (4.1)). □

The generalization to Theorem 5.2 (whose statement does not change) is also an easy change. We simply apply Theorem 5.1 with $C'_\beta = C_\beta/2$, using the same reasoning about support sizes as in Corollary 4.1.

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