1.1

$$u = \begin{pmatrix} 1 \\ 2i \\ 3-3i \end{pmatrix} \quad v = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

1. $\langle u, u \rangle = (1 \cdot 1) + -2i \cdot 2i + (3-3i) \cdot (3+3i)$ $\quad A \cdot B = \sum_i \bar{a}_i \cdot b_i$ dot product

$$= 1 - 4i^2 + 9 - 9i - 9i^2$$

$$= 1 + 4 + 9 + 9$$

$$= 23$$

$$u^T = [1 \quad 2i \quad 3-3i]$$

2. $u^T u = 1 \times 1 + 2i \times 2i + (3-3i) \times (3-3i) = \underline{-3i - 18i}$

3. $u^* = \begin{pmatrix} 1 \\ -2i \\ 3+3i \end{pmatrix} \qquad u^* u = 1 \times 1 + 2i \times -2i + (3+3i) \times (3-3i) = \underline{23}$

4. $v^T = [3 \; 2 \; 1] \qquad u v^T = \begin{bmatrix} 3 & 2 & 1 \\ 6i & 4i & 2i \\ 9-9i & 6-6i & 3-3i \end{bmatrix}$

5. Not possible since u^T and v^T are both of size 1×3

6.

$$u \times v = \begin{pmatrix} 1 \\ 2i \\ 3-3i \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{matrix} 2i & 3-3i & 1 & 2i \\ 2 & 1 & 3 & 2 \end{matrix}$$

$$= (2i - 2(3-3i)), (3(3-3i - 1), (2 - 6i)$$

$$\begin{bmatrix} -6 + 8i \\ 8 - 9i \\ 2 - 6i \end{bmatrix}$$

1.2

1. $A^T = \begin{bmatrix} 4 & 1 \\ 5 & 0 \\ 6 & 2 \end{bmatrix}$ $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 0 & 2 \end{bmatrix}$ $A^T A = \begin{bmatrix} 17 & 20 & 26 \\ 20 & 25 & 30 \\ 26 & 30 & 40 \end{bmatrix}$ Symmetric

2. $A^T = \begin{bmatrix} 4 & 1 \\ 5 & 0 \\ 6 & 2 \end{bmatrix}$ $(BA) = \begin{bmatrix} 3 & 5 & 4 \\ -1 & 0 & -2 \end{bmatrix}$ $A^T(BA) = \begin{bmatrix} 11 & 20 & 14 \\ 15 & 25 & 20 \\ 16 & 30 & 20 \end{bmatrix}$

3. $B^T = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ AB^T is not possible, incorrect dimensions

4. A^2 is not possible, not quadratic

5. $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. $B^T B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ $B^T B + A$ is not possible $2 \times 2 \neq 2 \times 3$

7. $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 0 & 2 \end{bmatrix}$ $A^T = \begin{bmatrix} 4 & 1 \\ 5 & 0 \\ 6 & 2 \end{bmatrix}$ $A A^T = \begin{bmatrix} 77 & 16 \\ 16 & 5 \end{bmatrix}$

1.3

1. Normalized : $|\vec{v}| = 1$

$$|n| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \qquad \hat{n} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$|l| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5 \qquad \hat{l} = \begin{pmatrix} 0 \\ 3/5 \\ 4/5 \end{pmatrix}$$

length of projection

2. $$P_{\hat{n}}(\hat{l}) = \underbrace{\left(\frac{\hat{l} \cdot \hat{n}}{\|\hat{n}\|^2}\right)}_{\text{length of projection}} \hat{n} = \left(\frac{\frac{1}{\sqrt{3}}\left(\frac{3}{5}+\frac{4}{5}\right)}{1^2}\right)\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\left(\frac{3}{4}+\frac{4}{5}\right) \\ \text{—} \sim \text{—} \\ \text{—} \sim \text{—} \end{pmatrix} = \begin{pmatrix} 7/15 \\ 7/15 \\ 7/15 \end{pmatrix}$$

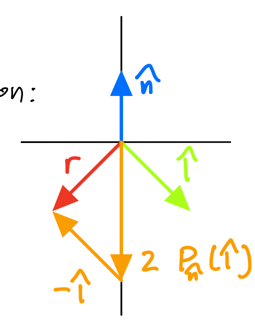
$$|P_{\hat{n}}(\hat{l})| = \sqrt{\left(\tfrac{7}{15}\right)^2 + \left(\tfrac{7}{15}\right)^2 + \left(\tfrac{7}{15}\right)^2} = 0{,}808$$

To get the projection $P_{\hat{l}}\,\hat{n}$ we have to change wich vector we multiply with the length of the projection. That does <u>not</u> influence $|P_{\hat{l}}(\hat{n})|$

3 $$r = 2(\hat{n} \cdot \hat{l})\hat{n} - \hat{l} = 2\left(\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3/5 \\ 4/5 \end{pmatrix}\right)\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 3/5 \\ 4/5 \end{pmatrix} = 2 \cdot \frac{7}{5\sqrt{3}} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 3/5 \\ 4/5 \end{pmatrix}$$

$$= \begin{pmatrix} 14/15 \\ 14/15 \\ 14/15 \end{pmatrix} - \begin{pmatrix} 0 \\ 9/15 \\ 12/15 \end{pmatrix} = \begin{pmatrix} 14/15 \\ 1/3 \\ 2/15 \end{pmatrix}$$

Illustration:



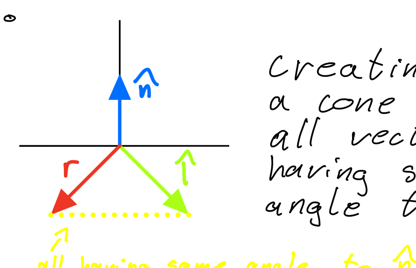
Using the projection $P_{\hat{n}}(\hat{l})$

$$ref_{\hat{n}}(\hat{l}) = 2 \cdot P_{\hat{n}}(\hat{l}) - \hat{l} = 2\begin{pmatrix} 7/15 \\ 7/15 \\ 7/15 \end{pmatrix} - \begin{pmatrix} 0 \\ 3/5 \\ 4/5 \end{pmatrix} = \begin{matrix} 14/15 \\ 1/3 \\ 2/15 \end{matrix}$$

$$|r| = \sqrt{\left(\tfrac{14}{15}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{2}{15}\right)^2} = 1$$, since its normalized : $\hat{r} = r$

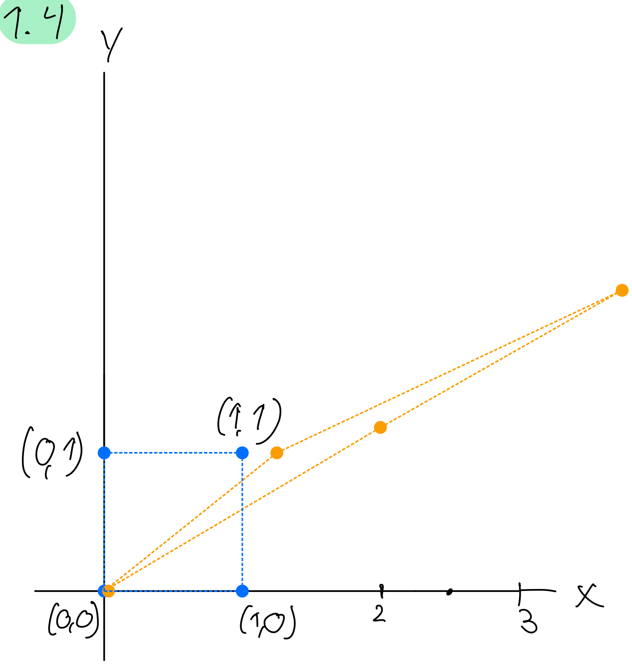
4. $$\theta_{\hat{n}\hat{l}} = \theta_{\hat{r}\hat{n}} = \cos^{-1}\left(\frac{7}{5\sqrt{3}}\right) \approx 0{,}63 \text{ rad} \approx 36°$$

This is not sufficient proof that r is a reflection vector since we operate in R^3 space and all vectors between I and r will have the same angle



Creating a a cone with all vectors having same angle to $\hat{n}$

1.4



Shear matrixes:

Vertical $S_1 : \begin{bmatrix} 1 & 0 \\ x_1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{matrix} a(1+x_1) \\ b \end{matrix}$

Horizontal $S_2 : \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{matrix} a \\ b(1+x_2) \end{matrix}$

Looking at the transformation from the original quad we can assume that $S_1 = \begin{bmatrix} 1 & 0 \\ 1{,}2 & 1 \end{bmatrix}$ to shif the point $(0,1)$ vertically to $(1.2, 1)$. We can also see that the point $(1,0)$ shifts horizontally by $1{,}25$. We assume $S_2 = \begin{bmatrix} 1 & 1.25 \\ 0 & 1 \end{bmatrix}$. $T = S_2 S_1 = \begin{bmatrix} 1 & 1{,}25 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1{,}2 & 1 \end{bmatrix} = \begin{bmatrix} 2.5 & 1.25 \\ 1.2 & 1 \end{bmatrix}$

1. Size of T: 2×2
2. Determinant of $T = 2.5 \cdot 1 - 1{,}25 \cdot 1.2 = 1$
3. Trace of $T = 2{,}5 + 1 = 3{,}5$
4. Rank of T: $\text{rref}(T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ rank $= 2$

1.5

$$M = \frac{1}{3}\begin{pmatrix} 1+i & -2+i & 1+i \\ 1+i & 1+i & -2+i \\ -2+i & 1+i & 1+i \end{pmatrix}$$

1 $$M^T = \frac{1}{3}\begin{pmatrix} 1+i & 1+i & -2+i \\ -2+i & 1+i & 1+i \\ 1+i & -2+i & 1+i \end{pmatrix}$$

2 $$M^* = \frac{1}{3}\begin{pmatrix} 1-i & 1-i & -2-i \\ -2-i & 1-i & 1-i \\ 1-i & -2-i & 1-i \end{pmatrix}$$

3. $$M = \frac{1}{3}\begin{pmatrix} 1+i & -2+i & 1+i \\ 1+i & 1+i & -2+i \\ -2+i & 1+i & 1+i \end{pmatrix} \quad M^T = \frac{1}{3}\begin{pmatrix} 1+i & 1+i & -2+i \\ -2+i & 1+i & 1+i \\ 1+i & -2+i & 1+i \end{pmatrix} \quad M M^T = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_{11} = (1+i)\cdot(1-i) + (-2+i)(-2-i) + (1+i)(1-i) = 1 - i^2 + 1 - i^2 + 4 - i^2 = 6 - 3i^2 = 9$$

$$x_{12} = (1+i)(1-i) + (-2+i)(1-i) + (1+i)(1-i) = 1 - i^2 - 2 - i^2 - 2 - i^2 = -3 - 3i^2 = 0$$

4 $$M^{-1} = \frac{1}{3}\begin{pmatrix} 1-i & 1-i & -2-i \\ -2-i & 1-i & 1-i \\ 1-i & -2-i & 1-i \end{pmatrix} = M^* \neq \frac{1}{3}\begin{pmatrix} 1+i & 1+i & -2+i \\ -2+i & 1+i & 1+i \\ 1+i & -2+i & 1+i \end{pmatrix}$$

1.7

A matix is positive-definite if it's symmetric and eigen values are positive. The signs of the pivots are signs of the eigenvalues.

$M(v) = Av + b \quad A = \begin{pmatrix} 1 & 2 \\ 2 & a_{22} \end{pmatrix} \in \mathbb{R}^2 \qquad b \in \mathbb{R}^2$

$\begin{pmatrix} 1 & 2 \\ 2 & a_{22} \end{pmatrix} R_2 = R_2 - 2R_1 \begin{pmatrix} 1 & 2 \\ 0 & a_{22}-4 \end{pmatrix} \qquad a_{22} - 4 > 0$

$\det(1) = 1 \quad , \det(A) = a_{2.2} - 4$

$1, a_{2.2} - 4 > 0 \Rightarrow$ Positive definite

which means $\underline{a_{2.2} > 4}$