Vector Spaces

Helwig Hauser *et al.*,
UiB Dept. of Informatics

Looking Back & Forth

Last time: linear systems

- linear operations, linear equation systems
- Gauss elimination, LU decomposition

Repetition: left- vs. right-multiplication

Today: vector spaces

- basis of a vector space
- standard basis
- basis of a sub-space
- coordinates wrt. a basis
- changing the basis
- changing the basis of a mapping
- projections

Repetition: Left-, Right-Multiplication

Matrix multiplication in order to execute an operation:

- left-multiplication of **A** with **M**, i.e., **B** = **M A**:
  - matrix **M** represents the operation executed on **A**
  - **M** tells, how **A**'s rows are linearly recombined into **B**'s rows
  - **B**'s row #i is a linear comb. of **A**'s rows, coefficients: **M**'s row #i
- right multiplication of **A** with **M**, i.e., **A M** = **B**:
  - matrix **M** represents the operation executed on **A** (as above)
  - **M** tells, how **A**'s columns are linearly recombined into **B**'s cols.
  - **B**'s col. #i is a linear comb. of **A**'s cols., coefficients: **M**'s col. #i

Illustration

Left-multiplication of M with C, i.e., C M:

– the new row i =

$$
\begin{aligned}
& c_{i,1} \cdot \text{old row } 1 \\
+\ & c_{i,2} \cdot \text{old row } 2 \\
\\
+\ & c_{i,m} \cdot \text{old row } m
\end{aligned}
$$

$$
\begin{bmatrix} \vdots \\ \text{new row } i \\ \vdots \end{bmatrix}
=
\begin{bmatrix} & \vdots & & \\ c_{i,1} & c_{i,2} & \cdots & c_{i,m} \\ & \vdots & & \end{bmatrix}
\begin{bmatrix} \text{old row } 1 \\ \text{old row } 2 \\ \vdots \\ \text{old row } m \end{bmatrix}
$$

Right-multiplication of M with C, i.e., M C:

– the new col. i =

$$
\begin{aligned}
& c_{1,i} \cdot \text{old col. } 1 \\
+\ & c_{2,i} \cdot \text{old col. } 2 \\
\\
+\ & c_{m,i} \cdot \text{old col. } m
\end{aligned}
$$

$$
\begin{bmatrix} \text{old col. } 1 & \text{old col. } 2 & \cdots & \text{old col. } m \end{bmatrix}
\begin{bmatrix} & c_{1,i} & \\ \cdots & c_{2,i} & \cdots \\ & \vdots & \\ & c_{m,i} & \end{bmatrix}
=
\begin{bmatrix} \cdots & \text{new col. } i & \cdots \end{bmatrix}
$$

Operations: Examples (left-mult.)

Swapping equations (permutation operation)

– operation **Π**: swapping rows 1 and 4 $\Rightarrow$ left-multiplication with **P**

$$\Pi\,[\ \mathbf{A}\,\mathbf{x} = \mathbf{b}\ ]$$

$$\Pi\,[\ \mathbf{A}\,\mathbf{x}\ ] = \Pi\,[\ \mathbf{b}\ ]$$

$$\mathbf{P}\,\mathbf{A}\,\mathbf{x} = \mathbf{P}\,\mathbf{b}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \rightarrow \text{“bring row \#4 here”} \\ \rightarrow \text{“keep row \#2 in place”} \\ \rightarrow \text{“keep row \#3 in place”} \\ \rightarrow \text{“bring row \#1 here”} \end{array}$$

Eliminate a variable (elimination operation)

– operation **Λ**: use pivot row 2 to … $\Rightarrow$ left-multiplication with **E**

$$\mathbf{A} = \begin{pmatrix} -2 & -1 & 3 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & -4 & 3 & -1 \\ 0 & -6 & -2 & 4 \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{array}{l} \rightarrow \text{“keep row \#1 in place”} \\ \rightarrow \text{“keep the pivot row \#2”} \\ \rightarrow \text{“add twice the pivot row”} \\ \rightarrow \text{“add 3* the pivot row”} \end{array}$$

Operations: Examples (left-mult.)

Eliminate a variable (elimination operation)

– operation **Λ**: use pivot row 2 to … ⇒ left-multiplication with **E**

$$\mathbf{A} = \begin{pmatrix} -2 & -1 & 3 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & \boxed{-4} & 3 & -1 \\ 0 & \boxed{-6} & -2 & 4 \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{array}{l} \rightarrow \text{“keep row \#1 in place”} \\ \rightarrow \text{“keep the pivot row \#2”} \\ \rightarrow \text{“add twice the pivot row”} \\ \rightarrow \text{“add 3* the pivot row”} \end{array}$$

Update (“undo” elimination operation)

– operation **Λ’**: compensate **Λ** ⇒ left-multiplication with **E’**

$$\mathbf{E}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix} \begin{array}{l} \rightarrow \text{“keep row \#1 in place”} \\ \rightarrow \text{“keep the pivot row \#2”} \\ \rightarrow \text{“subtract twice the pivot row”} \\ \rightarrow \text{“subtract 3* the pivot row”} \end{array}$$

Operations: Examples (right-mult.)

Do a linear combination of vectors $\mathbf{v}_i$

– operation $\boldsymbol{\Lambda}_{\mathbf{c}}$ (coeffs. $\mathbf{c}$) $\Rightarrow$ right-multiplication with $\mathbf{c}$

$$\boldsymbol{\Lambda}_{\mathbf{c}}\,\mathbf{V} = \mathbf{V}\,\mathbf{c} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \\ 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

"add col. #1 and twice col. #2"

Introduction

Vector spaces…

… are **omnipresent in visual computing** (and elsewhere!)

- the 2D / 3D “world” in which objects “live”
- the camera space in computer graphics
- etc.

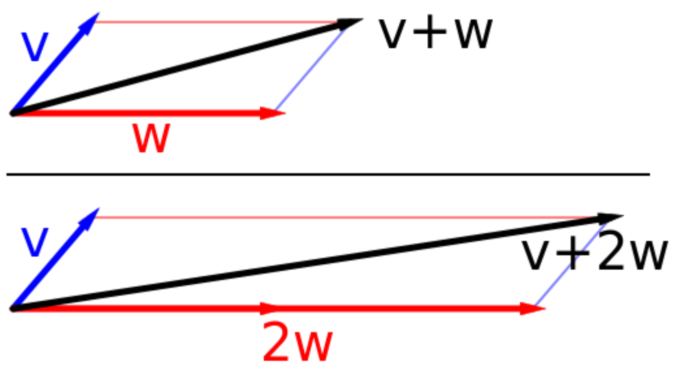
… **transformations** (in particular linear transformations) map between spaces

- affine transformations like rotation, scaling, …
- viewing transformation, projection
- etc.

Vector Spaces

A vector space (a space) **is**

– a set of vectors
  such that every possible, specific lin. comb. of them is also in the same space



Properties:

Axiom	Meaning
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Vectors Spanning a Vector Space

Given some vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

– they ***span a vector space***,
encompassing all possible (specific) linear combinations of them: $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_n \mathbf{v}_n$

Basis of a Vector Space

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$:

- form a *basis* of a vector space $\boldsymbol{\Omega}$,
  - if they are linearly independent
  - *and* if they span $\boldsymbol{\Omega}$
- such a basis
  - is *orthogonal*, if $\mathbf{v}_i^\mathsf{T} \mathbf{v}_j = 0 \quad \forall\, i \neq j$
  - is *orthonormal*, if $\mathbf{v}_i^\mathsf{T} \mathbf{v}_j = \delta_{i,j}$ (Kronecker delta $\delta_{i,j}$),
    for ex., the *standard basis* $(\mathbf{e}_1\ \mathbf{e}_2 \ldots \mathbf{e}_n)$ – more in the following
- any vector $\mathbf{v}$ (in vector space $\boldsymbol{\Omega}$, spanned by $\mathbf{v}_i$, $i=1 \ldots n$)
  - can be written as $\mathbf{v} = (\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n)\, \mathbf{c}$
  - with the $\mathbf{c}$ being called the *coordinates* of $\mathbf{v}$
    with respect to basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$
- basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ (set of vectors) vs. basis-matrix $\mathbf{V} = (\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n)$

Basis – example

Vectors $\mathbf{v}_1 = (1\ 2)^\top$ and $\mathbf{v}_2 = (2\ 1)^\top$ form a basis of $\mathbb{R}^2$ –

they are linearly independent,
since equation $(\mathbf{v}_1\ \ \mathbf{v}_2)\ \mathbf{c} = \mathbf{0}$ can only be solved for $\mathbf{c} = \mathbf{0}$,

and they span all of $\mathbb{R}^2$,
i.e., equation $(\mathbf{v}_1\ \ \mathbf{v}_2)\ \mathbf{c} = \mathbf{d}$ can be solved (for $\mathbf{c}$) for any $\mathbf{d}$

Not a Basis – example 1

Vectors $\mathbf{v}_1 = (1\ 2)^\top$, $\mathbf{v}_2 = (2\ 1)^\top$, and $\mathbf{v}_3 = (1\ 1)^\top$
do *not* form a basis of $\mathbb{R}^2$,

since the first condition (linear independency of all $\mathbf{v}_i$)
is violated:

it is possible to find a non-trivial $\mathbf{c} \neq \mathbf{0}$
such that $(\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3)\,\mathbf{c} = \mathbf{0}$, for ex. $\mathbf{c}^* = (1\ 1\ \ -3)^\top$:

$$\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} - 3\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1+2-3\\2+1-3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Not a Basis – example 2

Vector $\mathbf{v}_1 = (1\ 2)^\top$ does *not* form a basis of $\mathbb{R}^2$,
since the second condition (all $\mathbf{v}_i$ must span $\mathbf{\Omega} = \mathbb{R}^2$)
is violated:

we can find a vector $\mathbf{d} \in \mathbf{\Omega} = \mathbb{R}^2$
that cannot be written as a linear combination of all $\mathbf{v}_i$,
for ex. $\mathbf{d}^* = (2\ 1)^\top$:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} c = \mathbf{d}^* = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{cannot be solved for any } c$$

In other words:

a basis of an *n*-dimensional vector space must consist of *n* vectors

Standard Basis of $\mathbb{R}^n$

Often implicitly,
we "per default" assume the standard basis *E* (of $\mathbb{R}^n$):

- the basis vectors $\mathbf{e}_i$ of ***E*** are given as:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

- matrix **E**, representing ***E***, is then, of course, the identity matrix **I**
- accordingly, the standard basis ***E*** is then also orthonormal, in particular is $\mathbf{E}^{-1} = \mathbf{E}^{*} = \mathbf{E}$

Basis of a Sub-Space

A basis $\boldsymbol{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ **with all** $\mathbf{v}_i \in \mathbb{R}^n$, $i=1\ldots m$, **can span**

- either all of $\mathbb{R}^n$ – then $m=n$
- or a sub-space of $\mathbb{R}^n$ which is m-dimensional ($m<n$), including the origin ($\mathbf{0}$)

Examples

- standard basis $\boldsymbol{E}_n = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ spans all of $\mathbb{R}^n$
- the basis $\boldsymbol{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ with

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

spans a two-dimensional sub-space of $\mathbb{R}^3$, i.e., a plane in 3D

Coordinates wrt. a Basis

Given a basis $\boldsymbol{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ **of a vector space** $\boldsymbol{\Omega}$

- and its according matrix $\mathbf{B} = (\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n)$,
- we can write any $\mathbf{v} \in \boldsymbol{\Omega}$ as $\mathbf{v} = \mathbf{B}\,\mathbf{c}$
- with $\mathbf{c}$
  being $\mathbf{v}$'s **coordinates** *wrt. basis* $\boldsymbol{B}$

[see MATLAB example 2b_A]

Coordinates – example 1

Whenever we write a vector $\mathbf{v} \in \mathbb{R}^n$ in coordinate form, i.e., as $\mathbf{v} = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}^\top$,

we implicitly assume that the v_i are the n coordinates of $\mathbf{v}$ wrt. the standard basis $\mathcal{E}$ with $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\mathbf{E} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{pmatrix} = \mathbf{I}_n$.

Then, clearly, $\mathbf{v}$ can be written as

$$\mathbf{v} = \mathbf{I}\,\mathbf{v} = \mathbf{E}\,\mathbf{v} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$= v_1\,\mathbf{e}_1 + v_2\,\mathbf{e}_2 + \cdots + v_n\,\mathbf{e}_n$$

Coordinates – example 2 (a)

Given two linearly independent, three-dimensional vectors $\mathbf{v}_1 = (1\ 1\ 1)^\top$ and $\mathbf{v}_2 = (2\ 1\ 0)^\top$,
they form a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of a plane $\mathbf{\Pi}$ in 3D
$(\mathbf{B} = (\mathbf{v}_1\ \mathbf{v}_2))$.

Any vector $\mathbf{v}$ in plane $\mathbf{\Pi}$ can then be written as

$$\mathbf{v} = \mathbf{B}\,\mathbf{c} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2$$

with $\mathbf{c} = (c_1\ c_2)^\top$ being the (2D) coordinates of $\mathbf{v}$ wrt. basis $\mathcal{B}$.

Coordinates – example 2 (b)

Any vector $\mathbf{v}$ in plane $\mathbf{\Pi}$ can then be written as

$$\mathbf{v} = \mathbf{B}\,\mathbf{c} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2$$

with $\mathbf{c} = (c_1 \; c_2)^\top$ being the (2D) coordinates of $\mathbf{v}$ wrt. basis $\mathcal{B}$.

Considering vector $\mathbf{v} = (4 \; 3 \; 2)^\top \in \mathbf{\Pi}$,

$\mathbf{v}$ can be written

- either wrt. the standard basis $\mathcal{E}_3 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
- or wrt. basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$:

$$\mathbf{v} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \mathbf{I}_3 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Notation for "wrt. to basis *B*"

When working with a particular basis (*B*)

- it seems appropriate to indicate this explicitly
  (otherwise, it can be *very* confusing very easily!)
- one notation to do so, is to use subscripts to indicate
  with respect to which basis the coordinates are to be interpreted:

$$\mathbf{v} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}_{\mathcal{E}_3} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathcal{B}}$$

! Note that we are only considering one vector **v** here –
just with *different coordinates* wrt. two *different bases*

Changing the Basis (1)

We can change the basis of a vector v:

– we need to know, how to get
  - from **v**'s coordinates wrt. basis ***B***, i.e., $(c_i)_{\boldsymbol{B}}$
  - to **v**'s coordinates wrt. basis ***C***, i.e. , $(d_i)_{\boldsymbol{C}}$, so that $(c_i)_{\boldsymbol{B}} = (d_i)_{\boldsymbol{C}}$

– by now,
  we know, how to get from basis ***B*** to the standard basis ***E***:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_{\mathcal{E}_n} = \mathrm{B} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}_{\mathcal{B}}$$

– this means:
  if we know **v**'s coordinates wrt. ***B*** (and matrix **B**, of course),
  we can get **v**'s standard coordinates

Changing the Basis (2a)

Of course, we also need to get back (from ***E*** to ***B***), if possible:

- let's start with an example:

  Again, we consider the plane $\mathbf{\Pi} \in \mathbb{R}^3$,
  spanned by $\mathbf{v}_1 = (1\ 1\ 1)^\top$ and $\mathbf{v}_2 = (2\ 1\ 0)^\top$, as above.

  We now consider a vector $\mathbf{v}$ in this plane with the following standard coordinates: $\mathbf{v} = (5\ 3\ 1)^\top$.

  What are $\mathbf{v}$'s coordinates $(c_j)_\mathcal{B}$ wrt. basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$?

Changing the Basis (2b)

Of course, we also need to get back (from ***E*** to ***B***), if possible:

– let's start with an example:

What are $\mathbf{v}$'s coordinates $(c_j)_{\mathcal{B}}$ wrt. basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$?

We know that $(v_i)_{\mathcal{E}} = \mathbf{B}\,(c_j)_{\mathcal{B}}$.

Accordingly, we can solve $(v_i)_{\mathcal{E}} = \mathbf{B}\,(c_j)_{\mathcal{B}}$ for the c_j:

(it's a system of linear equations, after all)

Changing the Basis (2c)

Of course, we also need to get back (from ***E*** to ***B***), if possible:

- let's start with an example:

Accordingly, we can solve $(v_i)_\mathcal{E} = \mathbf{B}\,(c_j)_\mathcal{B}$ for the c_j:

$$\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}_\mathcal{E} = \mathbf{B} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_\mathcal{B} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_\mathcal{B}$$

leading to the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & 1 & 3 \\ 1 & 0 & 1 \end{array}\right)$$

Changing the Basis (2d)

Of course, we also need to get back (from ***E*** to ***B***), if possible:

– let's start with an example:

Gaussian elimination leads to

$$\left(\begin{array}{rr|r} 1 & 2 & 5 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{array}\right)$$

first, and then to

$$\left(\begin{array}{rr|r} 1 & 2 & 5 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

Changing the Basis (2e)

Of course, we also need to get back (from ***E*** to ***B***), if possible:

– let's start with an example:

first, and then to

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

Accordingly,
$-c_2 = -2 \ \ (\Rightarrow \ \ c_2 = 2)$ and $c_1 + 2c_2 = 5 \ \ (\Rightarrow \ \ c_1 = 1)$,
giving $(c_j)_{\mathcal{B}}$ from $(v_i)_{\mathcal{E}}$.

double-check and compute **B** $(1\ 2)^T$!

Changing the Basis (3)

If $n = m$, i.e., if the n-dim. basis vectors $\mathbf{v}_i$ span all of $\mathbb{R}^n$, we can use the inverse of $\mathbf{B} = (\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n)$, i.e., $\mathbf{B}^{-1}$, to convert standard coordinates to $\mathcal{B}$-coordinates:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_{\mathcal{E}} = \mathbf{B} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{\mathcal{B}} \qquad \Rightarrow \qquad \mathbf{B}^{-1} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_{\mathcal{E}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{\mathcal{B}}$$

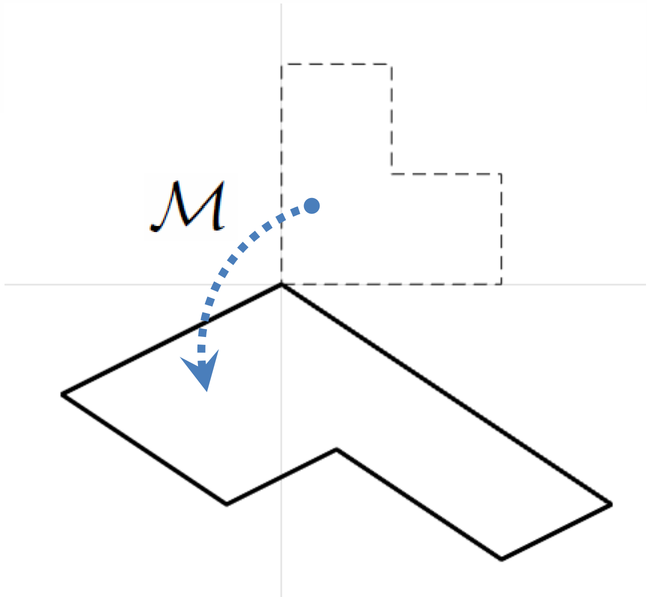
If basis $\mathcal{B}$ is orthonormal (as $\mathcal{E}$), then $\mathbf{B}^{-1} = \mathbf{B}^*$, which makes going back and forth even easier:

$(v_i)_{\mathcal{E}} = \mathbf{B}\,(c_j)_{\mathcal{B}}$ and $(c_j)_{\mathcal{B}} = \mathbf{B}^*\,(v_i)_{\mathcal{E}}$

Changing the Basis of Mappings (1)

Assume we have a linear mapping $\mathcal{M}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e., $\mathcal{M}: \mathbb{R}^n \to \mathbb{R}^m$ and $\mathcal{M}(\mathbf{v}) = \mathbf{A}\,\mathbf{v}$ (in standard coordinates).

The left-multiplication with which matrix $\mathbf{D}$ performs the same transformation $\mathcal{M}: \mathbb{R}^n \to \mathbb{R}^m$, if we consider $\mathbf{v}$ in terms of $\mathcal{B}$-coordinates instead of standard coordinates?



note that the mapping stays the same irrespective of which coordinates are used

Changing the Basis of Mappings (2)

Using the above solution for a change of the basis,
we can achieve the same mapping $\mathcal{M}$ by
first changing the basis (back) to the standard basis,
before then using $\mathbf{A}$ to do the transformation:

$$(v_i)_{\mathcal{E}} = \mathbf{B}\,(c_j)_{\mathcal{B}}$$

leading to

$$\mathcal{M}(\mathbf{v}) = \mathbf{A}\,(v_i)_{\mathcal{E}} = \mathbf{A}\,\mathbf{B}\,(c_j)_{\mathcal{B}} = \mathbf{D}\,(c_j)_{\mathcal{B}}$$

Changing the Basis of Mappings (3)

If we wish to consider $\mathcal{M}(\mathbf{v})$ in terms of another basis $\mathcal{G}$, we can use the other solution to convert $\mathcal{M}(\mathbf{v})_{\mathcal{E}}$ into $\mathcal{M}(\mathbf{v})_{\mathcal{G}}$:

$$(c_j)_{\mathcal{G}} = \mathbf{G}^{-1} \, (v_i)_{\mathcal{E}}$$

leading to

$$\mathcal{M}(\mathbf{v})_{\mathcal{G}} = \mathbf{G}^{-1} \, \mathcal{M}(\mathbf{v})_{\mathcal{E}} = \mathbf{G}^{-1} \, \mathbf{A} \, \mathbf{B} \, (c_j)_{\mathcal{B}} = \mathbf{D} \, (c_j)_{\mathcal{B}}$$

Realizing $\mathcal{M}(\mathbf{v})$, based on considering $\mathbf{v}$ in $\mathcal{B}$-coordinates and the result, $\mathcal{M}(\mathbf{v})$, in $\mathcal{G}$-coords., $\mathcal{M}(\,(c_j)_{\mathcal{B}}\,)_{\mathcal{G}}$ amounts to the left-multiplication of $(c_j)_{\mathcal{B}}$ with $\mathbf{D} = \mathbf{G}^{-1}\mathbf{A}\mathbf{B}$.

Changing the Basis of Mappings (4)

Accordingly, we can also do it the other way 'round:

We could first consider $\mathbf{v}$ in $\mathcal{B}$-coordinates,
then applying a linear mapping (by left-multiplying with $\mathbf{D}$)
to end up at a result in $\mathcal{G}$-coordinates,
before reverting to standard coordinates from there.

This means that we can then write

$$\mathcal{M}(\,(v_i)_{\mathcal{E}}\,)_{\mathcal{E}} = \mathbf{A}\,(v_i)_{\mathcal{E}} = \mathbf{G}\,\mathbf{D}\,\mathbf{B}^{-1}\,(v_i)_{\mathcal{E}}$$

and we read it in the following way:

first, we change the basis of $\mathbb{R}^n$ (to $\mathcal{B}$-coordinates),
then, we left-multiply with $\mathbf{D}$,
giving a result in $\mathbb{R}^m$ (in $\mathcal{G}$-coordinates),
before we eventually change back to standard coordinates.

Illustration (1)

T maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ … T(v) = A v in std. coords. E_n, E_m

… $\boldsymbol{B}_n$: alternative basis of $\mathbb{R}^n$, ergo $[\mathbf{v}]_{\boldsymbol{E}n} = \mathbf{B}_n\,[\mathbf{v}]_{\boldsymbol{B}n}$

$\boldsymbol{B}_m$: alternative basis of $\mathbb{R}^m$, ergo $[\mathbf{v}]_{\boldsymbol{E}m} = \mathbf{B}_m\,[\mathbf{v}]_{\boldsymbol{B}m}$

$\Rightarrow [\mathbf{T}(\mathbf{v})]_{\boldsymbol{E}m} = \mathbf{B}_m\,\mathbf{D}\,\mathbf{B}_n^{-1}\,[\mathbf{v}]_{\boldsymbol{E}n}$

Illustration (2)

T maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ … $\mathbf{T}(\mathbf{v}) = \mathbf{A}\,\mathbf{v}$ in std. coords. E_n, E_m

… B_n: alternative basis of $\mathbb{R}^n$, ergo $[\mathbf{v}]_{E_n} = \mathbf{B}_n\,[\mathbf{v}]_{B_n}$
B_m: alternative basis of $\mathbb{R}^m$, ergo $[\mathbf{v}]_{E_m} = \mathbf{B}_m\,[\mathbf{v}]_{B_m}$

$\Rightarrow [\mathbf{T}(\mathbf{v})]_{E_m} = \mathbf{B}_m\,\mathbf{D}\,\mathbf{B}_n^{-1}\,[\mathbf{v}]_{E_n}$

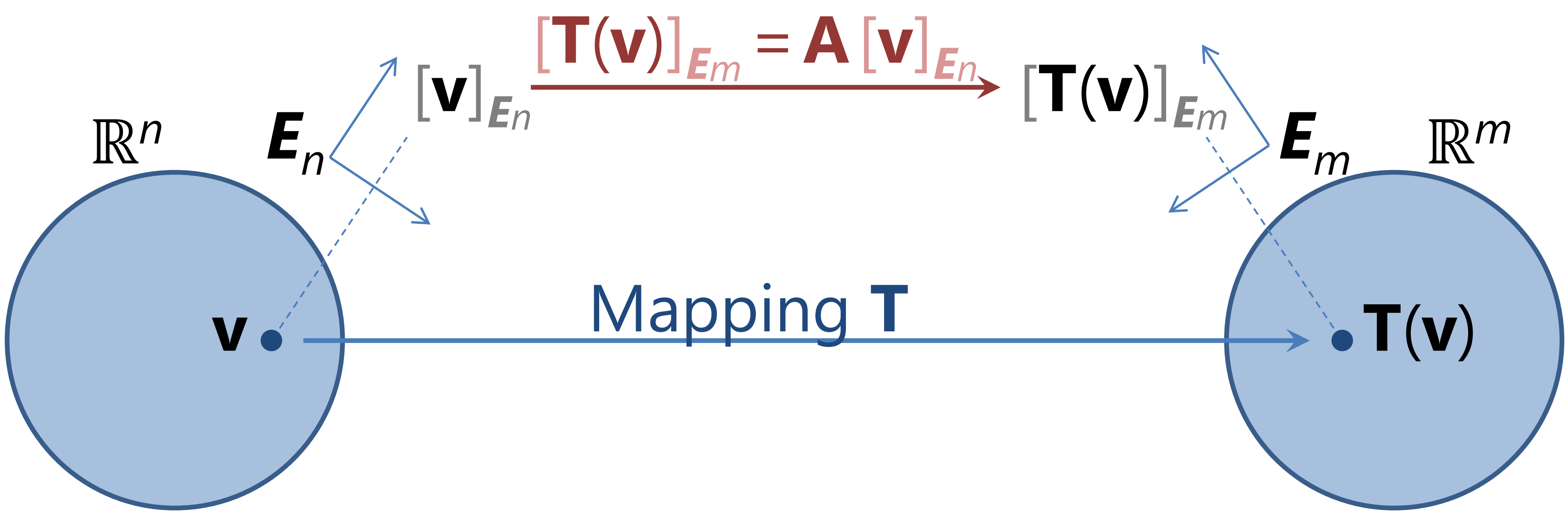


Illustration (3)

T maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ … T(v) = A v in std. coords. $\boldsymbol{E}_n$, $\boldsymbol{E}_m$

… $\boldsymbol{B}_n$: alternative basis of $\mathbb{R}^n$, ergo $[\mathbf{v}]_{\boldsymbol{E}_n} = \mathbf{B}_n\,[\mathbf{v}]_{\boldsymbol{B}_n}$

$\boldsymbol{B}_m$: alternative basis of $\mathbb{R}^m$, ergo $[\mathbf{v}]_{\boldsymbol{E}_m} = \mathbf{B}_m\,[\mathbf{v}]_{\boldsymbol{B}_m}$

$\Rightarrow [\mathbf{T}(\mathbf{v})]_{\boldsymbol{E}_m} = \mathbf{B}_m\,\mathbf{D}\,\mathbf{B}_n^{-1}\,[\mathbf{v}]_{\boldsymbol{E}_n}$

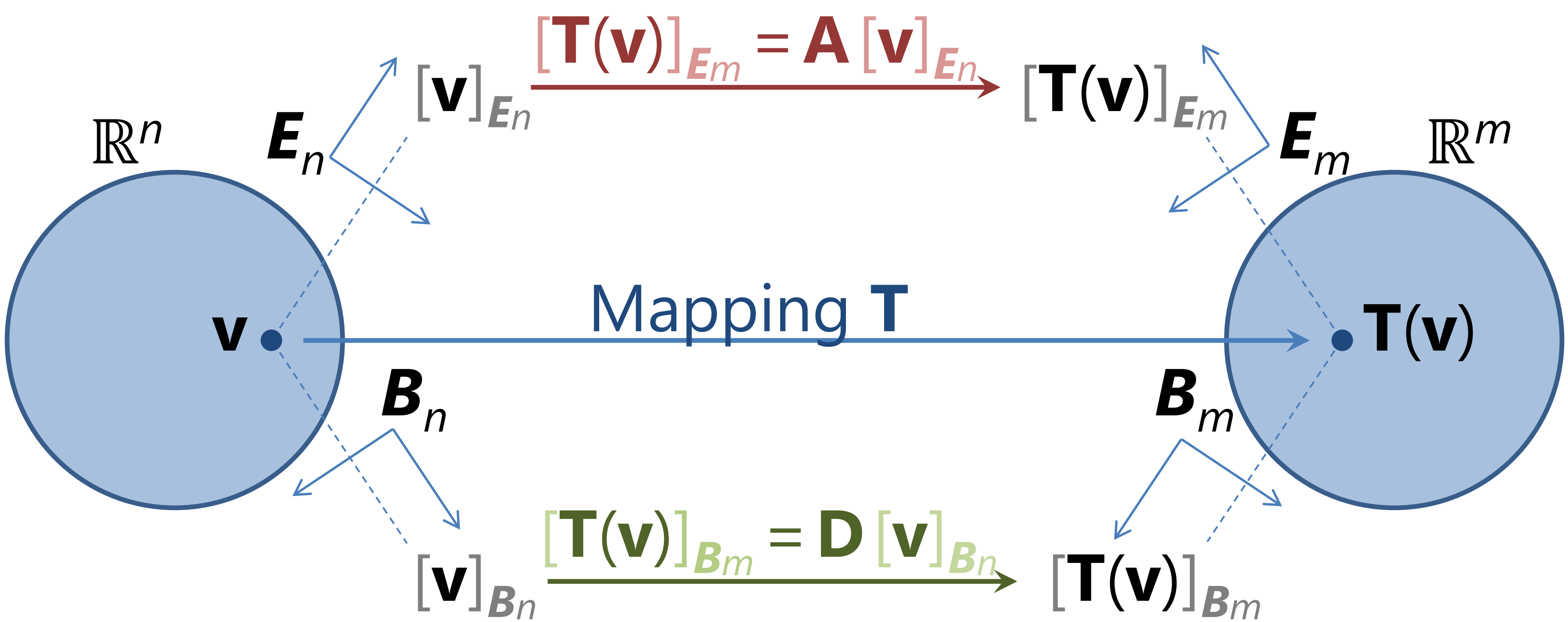


Illustration (4)

T maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ … T(v) = A v in std. coords. $\boldsymbol{E}_n$, $\boldsymbol{E}_m$

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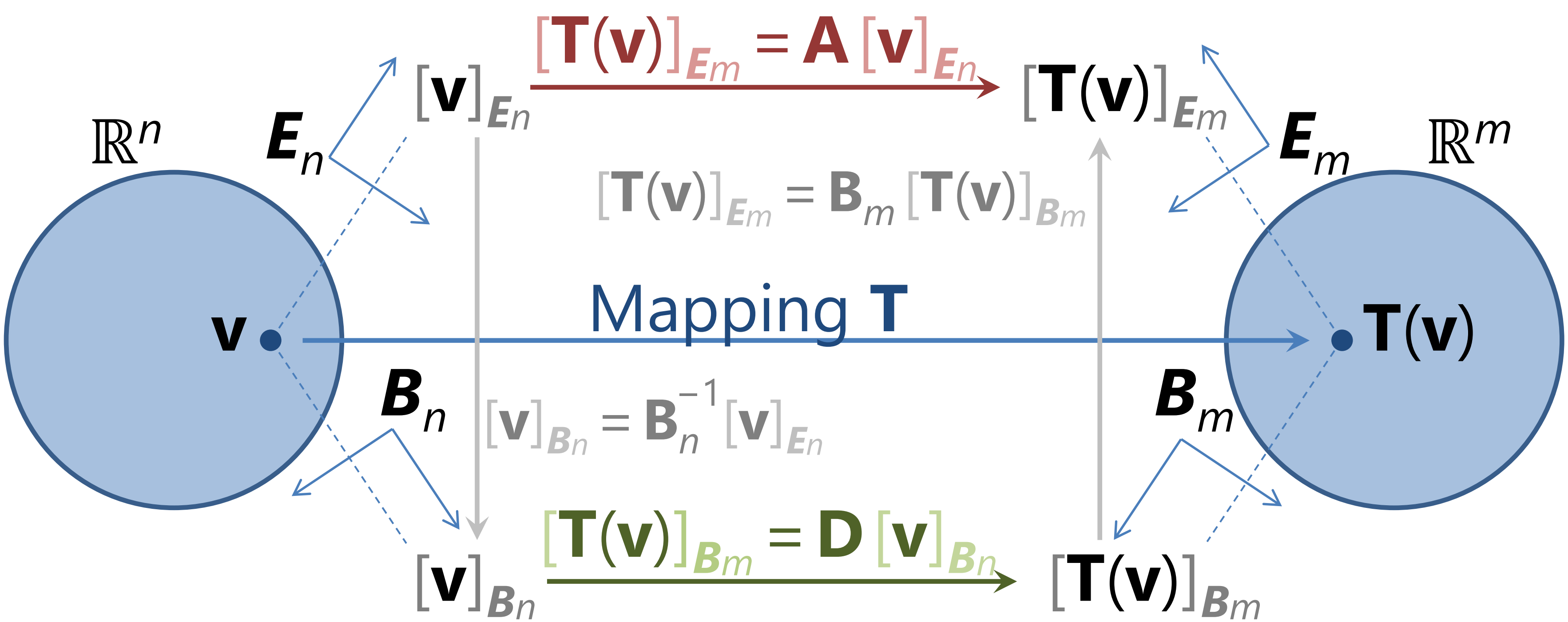
$\boldsymbol{B}_m$: alternative basis of $\mathbb{R}^m$, ergo $[\mathbf{v}]_{\boldsymbol{E}_m} = \mathbf{B}_m\,[\mathbf{v}]_{\boldsymbol{B}_m}$

$\Rightarrow [\mathbf{T}(\mathbf{v})]_{\boldsymbol{E}_m} = \mathbf{B}_m\,\mathbf{D}\,\mathbf{B}_n^{-1}\,[\mathbf{v}]_{\boldsymbol{E}_n}$



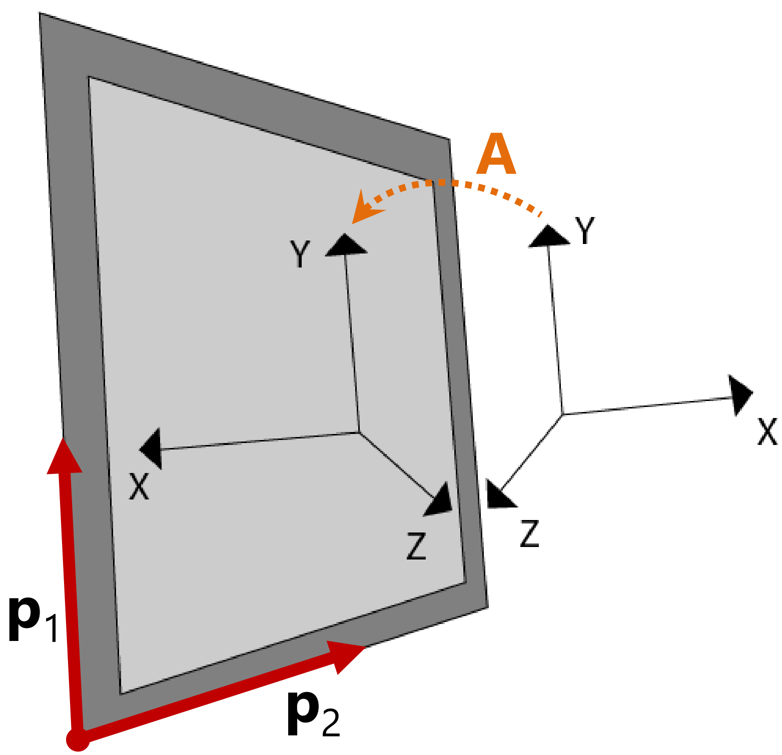
Mapping Example (1)

Assume a plane p with basis $\mathbf{p}_1$ and $\mathbf{p}_2$ in 3D:

– assume further that the left-multiplication with **A** (in std. coords.) should amount to mirroring 3D-vectors **v** about plane **p**:

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{p}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

– now find **A**



es this straight-forward!

3 (with matrix **B**) that is "aligned" with **p**,

$\mathbf{b}_3$ orthogonal to $\mathbf{b}_1$ and $\mathbf{b}_2$

ɔf *B*,

$$\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$$

'd *B*-coordinate!

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathrm{B} = \begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & -1 \\ 3 & 1 & -1 \end{pmatrix}$$

Mapping Example (1)

Assume a plane p with basis $\mathbf{p}_1$ and $\mathbf{p}_2$ in 3D:

- assume further that the left-multiplication with **A** (in std. coords.) should amount to mirroring 3D-vectors **v** about plane **p**:
- now find **A**

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \; \mathbf{p}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

A change of the basis makes this straight-forward!

- let's assume a basis ***B*** of $\mathbb{R}^3$ (with matrix **B**) that is "aligned" with **p**, i.e., $\mathbf{b}_1 = \mathbf{p}_1$, $\mathbf{b}_2 = \mathbf{p}_2$, and $\mathbf{b}_3$ orthogonal to $\mathbf{b}_1$ and $\mathbf{b}_2$
- given vectors **v** in terms of ***B***, mirroring about plane **p** is equal to negating the 3rd ***B***-coordinate!

$$\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$$

- thus $\mathbf{A} = \mathbf{B}\,\mathbf{M}\,\mathbf{B}^{-1}$ with

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Mapping Example (2, computing $\mathbf{B}^{-1}$)

$$\mathbf{M}_{\text{aug}} = \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \qquad \mathbf{B}^{-1} = \frac{1}{27}\begin{pmatrix} 2 & 5 & 5 \\ -1 & -16 & 11 \\ 5 & -1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & -1 & -11 & -2 & 1 & 0 \\ 0 & 1 & -16 & -3 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 11 & 2 & -1 & 0 \\ 0 & 0 & -27 & -5 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 11 & 2 & -1 & 0 \\ 0 & 0 & 1 & 5/27 & -1/27 & -1/27 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/27 & 5/27 & 5/27 \\ 0 & 1 & 0 & -1/27 & -16/27 & 11/27 \\ 0 & 0 & 1 & 5/27 & -1/27 & -1/27 \end{array}\right)$$

Mapping Example (**3**, computing **A**, etc.)

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & -1 \\ 3 & 1 & -1 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathbf{B}^{-1} = \frac{1}{27} \begin{pmatrix} 2 & 5 & 5 \\ -1 & -16 & 11 \\ 5 & -1 & -1 \end{pmatrix}$$

$$\mathbf{M}\,\mathbf{B}^{-1} = \frac{1}{27} \begin{pmatrix} 2 & 5 & 5 \\ -1 & -16 & 11 \\ -5 & 1 & 1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{B}\,\mathbf{M}\,\mathbf{B}^{-1} = \frac{1}{27} \begin{pmatrix} -23 & 10 & 10 \\ 10 & 25 & -2 \\ 10 & -2 & 25 \end{pmatrix}$$

Finding A was easier via *B*

we "test" **A** with $\mathbf{v} = 2\,\mathbf{b}_1 + \mathbf{b}_2$:

$$\mathbf{v} = 2\mathbf{b}_1 + \mathbf{b}_2 = \begin{pmatrix} 2+0 \\ 4-1 \\ 6+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$$

think first – what to expect?

$$\mathbf{T}(\mathbf{v}) = \mathbf{A}\,\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$$

next, we "check" $\mathbf{w} = \begin{pmatrix} 17 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}_B \qquad \mathbf{T}(\mathbf{w}) = \begin{pmatrix} -13 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}_B$

Mapping Example (3, computing A, etc.)

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & -1 \\ 3 & 1 & -1 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathbf{B}^{-1} = \frac{1}{27} \begin{pmatrix} 2 & 5 & 5 \\ -1 & -16 & 11 \\ 5 & -1 & -1 \end{pmatrix}$$

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think again – what to expect?

Mapping Example (3, computing A, etc.)

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & -1 \\ 3 & 1 & -1 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathbf{B}^{-1} = \frac{1}{27}\begin{pmatrix} 2 & 5 & 5 \\ -1 & -16 & 11 \\ 5 & -1 & -1 \end{pmatrix}$$

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Finding A was easier via *B*

we “test” **A** with $\mathbf{v} = 2\,\mathbf{b}_1 + \mathbf{b}_2$:

$$\mathbf{v} = 2\mathbf{b}_1 + \mathbf{b}_2 = \begin{pmatrix} 2+0 \\ 4-1 \\ 6+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{B}\,\mathbf{M}\,\mathbf{B}^{-1} = \frac{1}{27}\begin{pmatrix} -23 & 10 & 10 \\ 10 & 25 & -2 \\ 10 & -2 & 25 \end{pmatrix}$$

$$\mathbf{T}(\mathbf{v}) = \mathbf{A}\,\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$$

next, we “check” $\mathbf{w} = \begin{pmatrix} 17 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}_B \qquad \mathbf{T}(\mathbf{w}) = \begin{pmatrix} -13 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}_B$

Projection

Projecting some data D onto some (sub)space Ω …

… means to "talk about **D** in terms (coordinates) of **Ω** (only)"—
obviously, this reminds us of bases and the change of a basis!

Formal definition of projection

– a projection **P** is a linear operation that fulfills $\mathbf{P}^2 = \mathbf{P}$,
i.e., if applied a 2nd time, nothing changes anymore

outer product

Example:

– projection onto a line with **u** as a unit vector: $\mathbf{P_u} = \mathbf{u}\,\mathbf{u}^\top$

$$\underbrace{\mathbf{v} = (\mathbf{u}\ \mathbf{u}_\perp)\ (c_1\ c_2)^\top}_{\text{express } \mathbf{v} \text{ in terms of } \mathbf{u} \text{ and } \mathbf{u}_\perp} \quad \mathrm{P_u}\,\mathbf{v} = \mathrm{P_u}\,(c_1\mathbf{u}+c_2\mathbf{u}_\perp) = \mathrm{P_u}\,c_1\mathbf{u}+\mathrm{P_u}\,c_2\mathbf{u}_\perp =$$

$$c_1\mathrm{P_u}\,\mathbf{u}+c_2\mathrm{P_u}\,\mathbf{u}_\perp =$$

$$c_1\mathbf{u}\,\underbrace{\mathbf{u}^\top\mathbf{u}}_{1} + c_2\mathbf{u}\,\underbrace{\mathbf{u}^\top\mathbf{u}_\perp}_{0} =$$

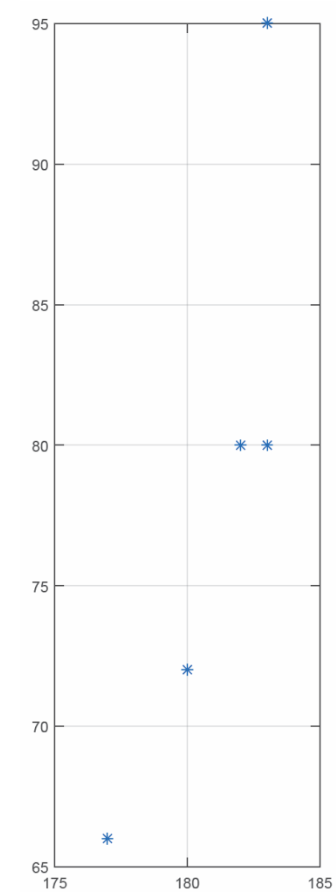
$$\boxed{c_1\mathbf{u}}$$

– generalizes, accordingly:
given orthonormal **A** = ($\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k$)
defines a subspace of $\mathbb{R}^n$, then
$\mathbf{P_A} = \mathbf{A}\,\mathbf{A}^\top$ projects from $\mathbb{R}^n$ onto **A**

Projection Example (1)

Assume, we are in $\mathbb{R}^2$:

– assume further, we have vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^2$, given also a matrix $\mathbf{V} = (\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n)$ – ex.: $\mathbf{V} = \begin{pmatrix} 180 & 183 & 183 & 182 & 177 \\ 72 & 95 & 80 & 80 & 66 \end{pmatrix}$



[see MATLAB example 2b_B]

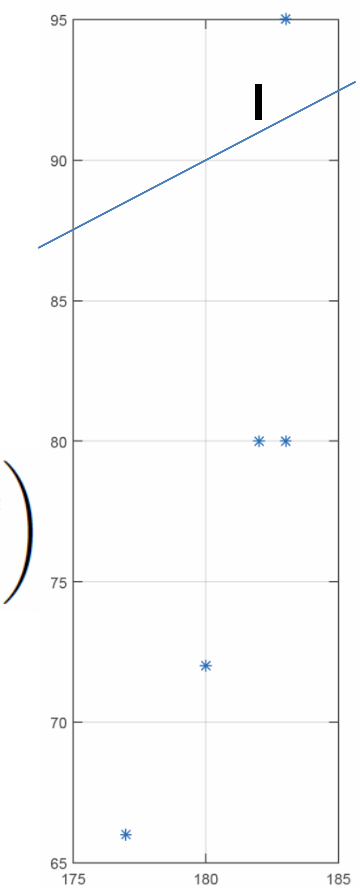
Projection Example (2)

Assume, we are in $\mathbb{R}^2$:

– assume further, we have vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^2$,
given also a matrix $\mathbf{V} = (\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n)$ – ex.: $\mathbf{V} = \begin{pmatrix} 180 & 183 & 183 & 182 & 177 \\ 72 & 95 & 80 & 80 & 66 \end{pmatrix}$

– further, we assume $\mathbf{d} = (2\ 1)^\top$
a 1D subspace (a line $\mathbf{l}$), defined by $s\,\mathbf{d},\ s \in \mathbb{R}$

– $L = \{\mathbf{d}'\}$ is an orthonormal basis of $\mathbf{l}$ with $\mathbf{d}' = \mathbf{d}/|\mathbf{d}|$
and $\mathbf{L} = \mathbf{d}'$ the corresponding "matrix"



Assume

– the projection onto $\mathbf{l}$ is defined by $\mathbf{P_l} = \mathbf{d}'\,\mathbf{d}'^\top$

– $\mathbf{P_l} = \mathbf{d}'\,\mathbf{d}'^\top = \begin{pmatrix} 0.8944 \\ 0.4472 \end{pmatrix} \begin{pmatrix} 0.8944 & 0.4472 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{pmatrix}$

– projecting $\mathbf{V}$
then means to compute $\mathbf{P_l}\,\mathbf{V}$ as

$$\mathbf{P_l}\mathbf{V} = \begin{pmatrix} 172.8 & 184.4 & 178.4 & 177.6 & 168.0 \\ 86.4 & 92.2 & 89.2 & 88.8 & 84.0 \end{pmatrix}$$

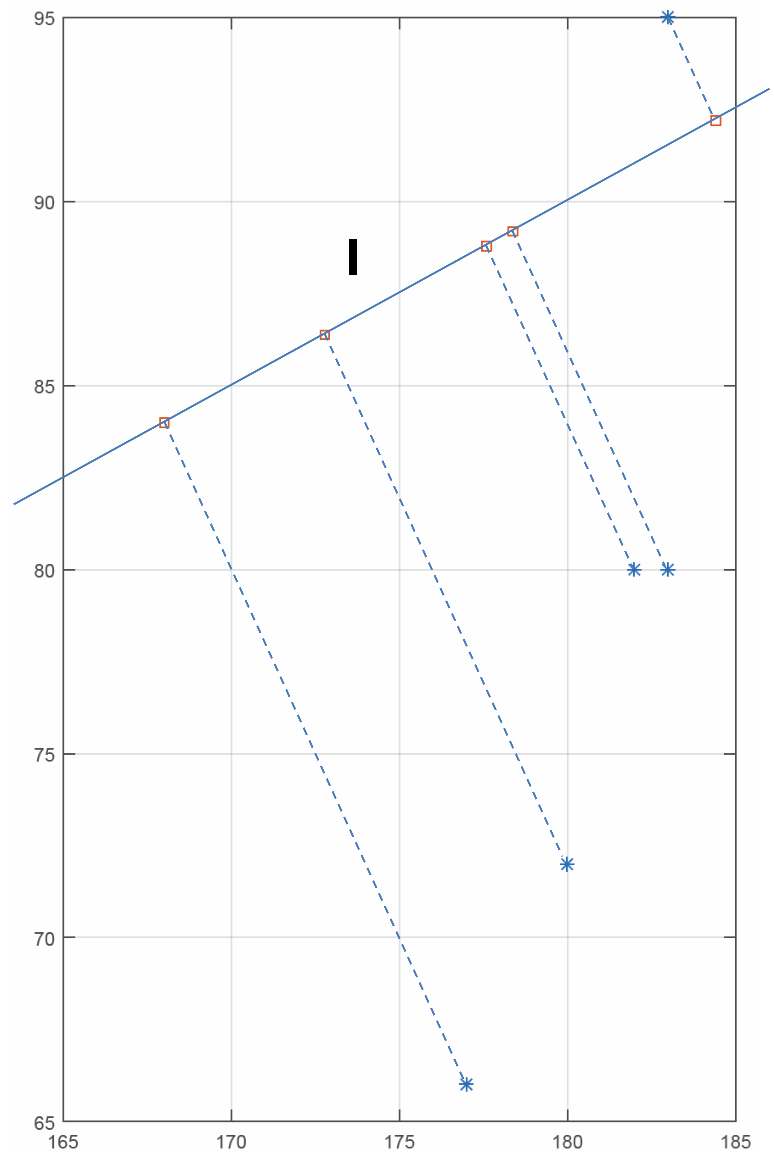
Projection Example (3)

Illustration:

– assume further, we have vectors $\mathbf{v}_1$, $\mathbf{v}_2$, given also a matrix $\mathbf{V} = (\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n)$

$$\mathbf{V} = \begin{pmatrix} 180 & 183 & 183 & 182 & 177 \\ 72 & 95 & 80 & 80 & 66 \end{pmatrix}$$

$$\mathbf{P}_\mathbf{l}\mathbf{V} = \begin{pmatrix} 172.8 & 184.4 & 178.4 & 177.6 & 168.0 \\ 86.4 & 92.2 & 89.2 & 88.8 & 84.0 \end{pmatrix}$$



Outlook

Next:

- singular-value decomposition, SVD
- eigenanalysis
- principal component analysis, PCA
- more examples

Reading and Related Material

In the book:

- chapter 2 (on linear systems), mostly section 2.1, and related parts
- chapter 15 (on SVD, etc.): coming soon! :–)

Course notes:

- section 4

An interactive online-book:

- http://ImmersiveMath.com/,
  chapters (1–4 &) 5 (on Gaussian elimination)
  and 6 (on The Matrix)

On Wikipedia:

- many good pages