3.1 Quadratic Fit

Professor H. Incognito got the impression that the number of students, who watch lecture videos instead of attending the live-streamed lectures, follows a parabola $f(x) = ax^2 + bx + c$ with x denoting the lecture-number and $f(x)$ modeling how many students watch the recorded lecture. His hypothesis is that the students prefer live-streamed lectures initially, but later "rearrange" their preferences/schedule and rather watch videos. As the exam approaches, it gets important to ask questions, reducing the number of students watching videos again. This semester, he already has the number of students, who watched the videos of the first four lectures. Given a best-possible quadratic fit (in the least-squares sense), he wonders which lecture video will be watched by most students.

According to his quadratic model, which lecture video will be watched by the largest number of students? And how many students will that be? Provide also the equation of the parabola that fits the data.

Lecture 1	Lecture 2	Lecture 3	Lecture 4
40	50	66	68

Table 1: Number of students watching the recorded lecture.

Given formula:

$$\begin{bmatrix} \sum_k x_k^4 & \sum_k x_k^3 & \sum_k x_k^2 \\ \sum_k x_k^3 & \sum_k x_k^2 & \sum_k x_k \\ \sum_k x_k^2 & \sum_k x_k & k \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \sum_k x_k^2 y_k \\ \sum_k x_k y_k \\ \sum_k y_k \end{bmatrix}$$

$\sum_k x_k^4 = 352$ $\qquad \sum_k x_k^2 + y_k = 1922$

$\sum_k x_k^3 = 100$ $\qquad \sum_k x_k + y_k = 610$

$\sum_k x_k^2 = 30$ $\qquad \sum_k y_k = 224$

$\sum_k x_k = 10$ $\qquad k = 4$

$$\begin{bmatrix} 352 & 100 & 30 \\ 100 & 30 & 10 \\ 30 & 10 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1922 \\ 610 \\ 224 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 352A & 100B & 30C & 1922 \\ 100A & 30B & 10C & 610 \\ 30A & 10B & 4C & 224 \end{array}\right]$$

Guassian elimination

$$\left[\begin{array}{ccc|c} 352A & 100B & 30C & 1922 \\ 0A & \frac{310}{177}B & \frac{90}{59}C & \frac{11870}{177} \\ 0A & \frac{90}{59}B & \frac{86}{59}C & \frac{3606}{59} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 352A & 100B & 30C & 1922 \\ 0A & \frac{310}{177}B & \frac{90}{59}C & \frac{11870}{177} \\ 0A & 0B & \frac{4}{31}C & \frac{84}{31} \end{array}\right]$$

3. $\to 354A = 1922 - 100B - 30C \Rightarrow A = -2$

2. $\to \frac{310}{177}B = \frac{11870}{177} - \frac{90}{59} \cdot C \Rightarrow B = 20$

1. $\to \frac{4}{31}C = \frac{84}{31} \Rightarrow C = 21$

$f(x) = -2x^2 + 20x + 21$

Finding max $f'(x) = 0$, since $f(x)$ is concave (negated sign), there must be a max, not min.

$f'(x) = -4x + 20 = 0 \Rightarrow x = 5$

$f(5) = -2 \cdot 5^2 + 20 \cdot 5 + 21 = 71$

Max point $(5, 71)$

3.3 Hermite Spline Curve

Assume a Hermite spline curve $\mathbf{c}(t)$, $t \in [0,1]$, starting in $\mathbf{a} = \mathbf{c}(0)$ and ending in $\mathbf{b} = \mathbf{c}(1)$. Given $\mathbf{a}$, $\mathbf{b}$, and the tangent vectors in $\mathbf{a}$ and $\mathbf{b}$ as follows, find the three standard coordinates of $\mathbf{c}(\frac{1}{2})$, i.e., the mid-point of that curve.

$$\mathbf{a} = \begin{pmatrix} 2 \\ -8 \\ -8 \end{pmatrix}_{\mathcal{E}}, \ \mathbf{b} = \begin{pmatrix} 7 \\ -2 \\ 3 \end{pmatrix}_{\mathcal{E}}, \ \mathbf{t_a} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}_{\mathcal{E}}, \ \mathbf{t_b} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{E}}$$

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$q(t) = (t^3 \ t^2 \ t \ 1) \ M^H \ (a \ b \ t_a \ t_b)$$

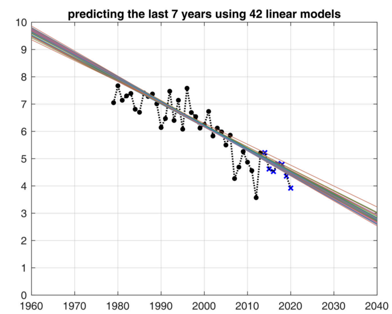
$$q(\tfrac{1}{2}) = \left[\tfrac{1}{2}^3 \ \ \tfrac{1}{2}^2 \ \ \tfrac{1}{2} \ \ 1\right] \cdot M^H \cdot \begin{bmatrix} 2 & 7 & 0 & 4 \\ -8 & -2 & 1 & 1 \\ -8 & 3 & -4 & 0 \end{bmatrix} = [4 \ \ -5 \ \ -3]$$

$$q(\tfrac{1}{2}) = \begin{pmatrix} 4 \\ -5 \\ -3 \end{pmatrix}$$

3.2 Predicting the future

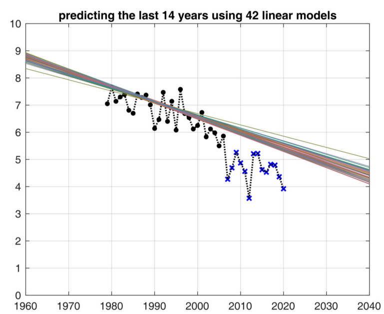
1. For each prediction scenario, comment on how good the prediction was for the following years up to 2020 (blue crosses), choosing from "completely off", "quite bad", "kind of OK", "actually quite good", and "excellent". Consider also, how consistent the prediction was, given all 42 variants. In your assessment, hypothesize also about the prediction for 2040 – how many Mkm^2 of sea ice to expect? If you think that a prediction was "quite bad" or "completely off", include also a bit of explanation.

Linear models:



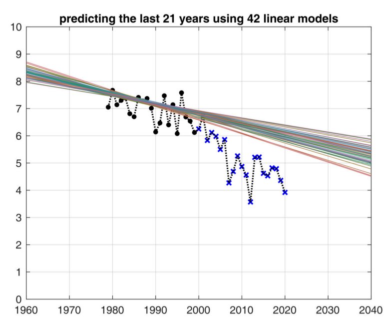
Excellent

Consistent



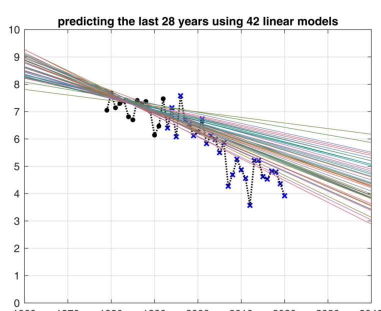
Off

Consistent



Off

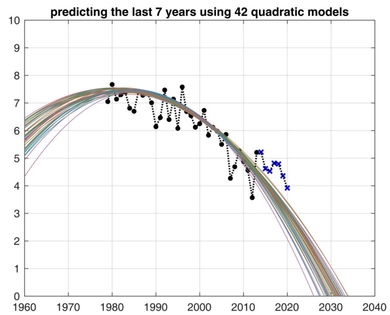
Inconsistent



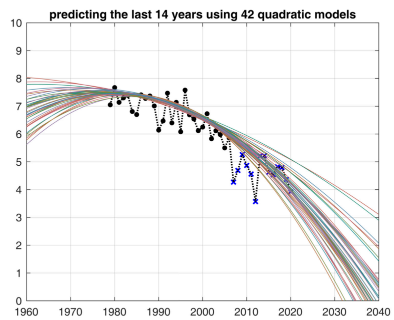
Ok

Inconsistent

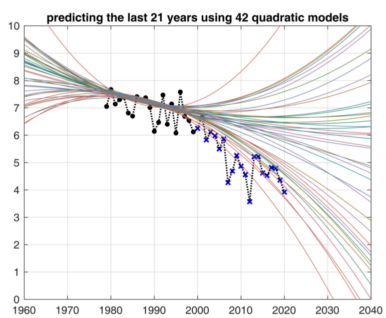
Quadratic models:



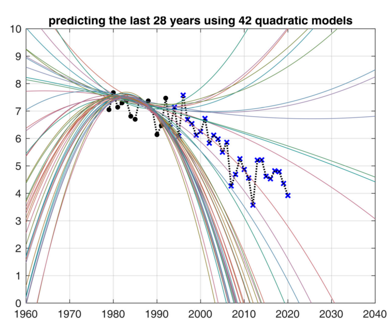
Ok
Consistent



Excellent
Inconsistent

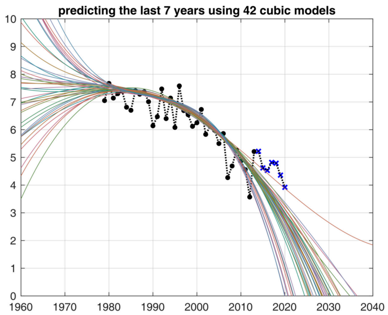


Ok
Inconsistent



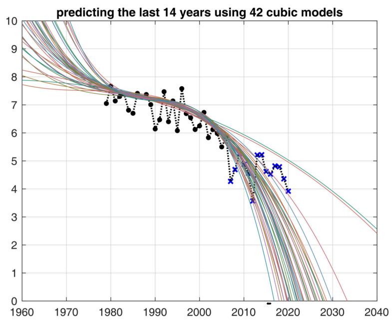
Off
Inconsistent

Qubic models:



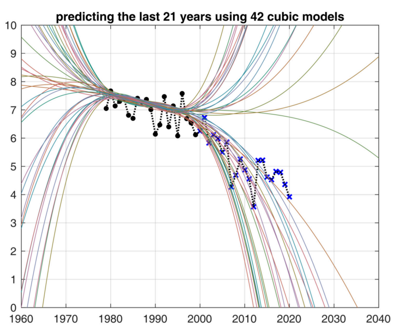
Ok

Consistent



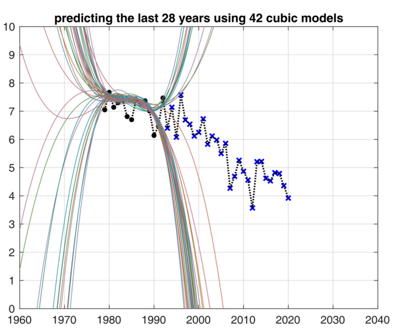
Bad

Inconsistent



Bad

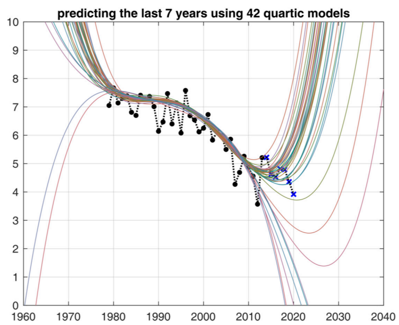
Inconsistent



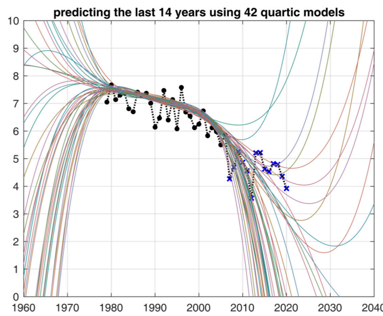
Off

Inconsistent

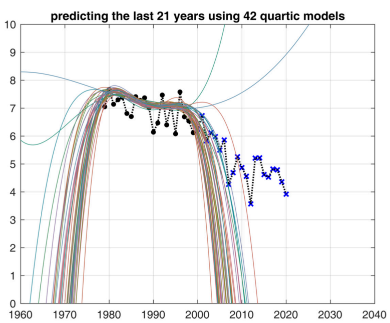
Quartic models:



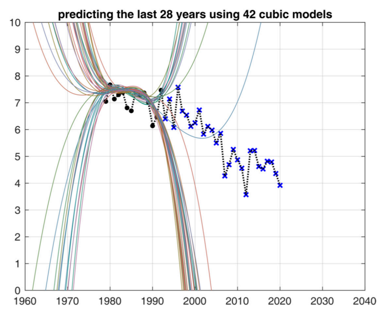
Good
Inconsistent



Good
Inconsistent

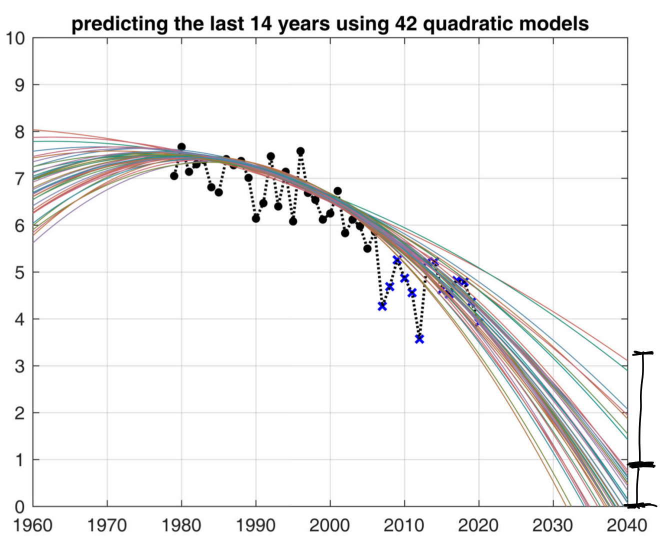


Bad
Inconsistent



Off
Inconsistent

My prediction for 2040: 1 Mkm^2



Optimistic

3.4 Converting between splines

Consider a single, cubic Hermite spline curve $\mathbf{p}_H(t),\ t \in [0,1]$, in 2D with geometry $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{t}_0, \mathbf{t}_1)$, a cubic Catmull-Rom spline curve $\mathbf{p}_{CR}(t),\ t \in [0,1]$, with geometry $(\mathbf{c}_{-1}, \mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2)$, and a cubic Bézier spline curve $\mathbf{p}_B(t),\ t \in [0,1]$, with control points $(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, and assume that all three spline curves are identical, i.e., $\mathbf{p}_H(t) = \mathbf{p}_{CR}(t) = \mathbf{p}_B(t)\ \forall t \in [0,1]$.

1. Given $\mathbf{G}_H = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{t}_0, \mathbf{t}_1) = \begin{pmatrix} 0\,1\,4\,3 \\ 0\,5\,4\,7 \end{pmatrix}$, find the control points $\mathbf{C}_{CR} = (\mathbf{c}_{-1}, \mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2)$ for the Catmull-Rom spline curve. Specify the transformation matrix $\mathbf{T}_{H\to CR}$ such that $\mathbf{C}_{CR}^\top = \mathbf{T}_{H\to CR}\,\mathbf{G}_H^\top$. Could you also go the other way and obtain the Hermite spline geometry from the Catmull-Rom control points (explain why/why not)?

2. Given $\mathbf{G}_H = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{t}_0, \mathbf{t}_1) = \begin{pmatrix} 1\,2\,2\,3 \\ 1\,5\,9\,3 \end{pmatrix}$, find the control points $\mathbf{Q}_B = (\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ for the Bézier spline curve. Specify the transformation matrix $\mathbf{T}_{H\to B}$ such that $\mathbf{Q}_B^\top = \mathbf{T}_{H\to B}\,\mathbf{G}_H^\top$. Also here: Could you also go the other way and obtain the Hermite spline geometry from the Bézier control points (explain why/why not)?

   Explain, also, how $\mathbf{t}_0$ relates to $\mathbf{q}_1 - \mathbf{q}_0$, and how $\mathbf{t}_1$ relates to $\mathbf{q}_3 - \mathbf{q}_2$.

1.

Hermit $q(t) = (t^3\ t^2\ t\ 1)\ M_H\ (q_0\ q_1\ t_0\ t_1)$

Catmull-Rom $p^{CR}(u) = (u^3\ u^2\ u\ 1) \cdot \frac{1}{2} M^{CR} \cdot (P_{k-1}\ P_k\ P_{k+1}\ P_{k+2})$

our result

$t = u,\quad q(t) = p^{CR}(t)$

$$\cancel{(t^3\ t^2\ t\ 1)}\ M_H\ (q_0\ q_1\ t_0\ t_1)^T = \cancel{(t^3\ t^2\ t\ 1)} \cdot \frac{1}{2} M^{CR} \cdot \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \\ P_{k+2} \end{bmatrix}$$

$$M_H \begin{bmatrix} 0 & 1 & 4 & 3 \\ 0 & 5 & 4 & 7 \end{bmatrix}^T = \frac{1}{2} M^{CR} \cdot \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \\ P_{k+2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 5 \\ 4 & 4 \\ 3 & 7 \end{bmatrix} = \frac{1}{2} M^{CR} \cdot \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \\ P_{k+2} \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ -8 & 0 \\ 4 & 4 \\ 0 & 0 \end{bmatrix} = \frac{1}{2} M^{CR} \cdot \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \\ P_{k+2} \end{bmatrix} \quad \Big| \left(\frac{1}{2} M^{CR}\right)^{-1}$$

$$\boxed{M^{CR} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix},\quad \left(\frac{1}{2} M^{CR}\right)^{-1} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -8 & 0 \\ 4 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \\ P_{k+2} \end{bmatrix} \to \begin{bmatrix} -7 & 3 \\ 0 & 0 \\ 1 & 5 \\ 6 & 14 \end{bmatrix} = \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \\ P_{k+2} \end{bmatrix}$$

$P_{k-1} = (-7 \;\; 3) = C_{-1}$

$P_k = (0, 0) = C_0$

$P_{k+1} = (1, 5) = C_1$

$P_{k+2} = (6, 14) = C_2$

The transformation matrix is the steps we just did.

$$T_{H \to CR} = \left(\tfrac{1}{2} M^{CR}\right)^{-1} M^{H} = \begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$T_{H \to CR} \begin{bmatrix} 0 & 1 & 4 & 3 \\ 0 & 5 & 4 & 7 \end{bmatrix}^T = \begin{bmatrix} -7 & 3 \\ 0 & 0 \\ 1 & 5 \\ 6 & 14 \end{bmatrix}$$

The transformation $T_{CR \to H}$ is simply the inverse of $T_{H \to CR}$

$$(T_{H \to CR})^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = T_{CR \to H}$$

$$T_{CR \to H} \begin{bmatrix} -7 & 3 \\ 0 & 0 \\ 1 & 5 \\ 6 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & 3 \\ 0 & 5 & 4 & 7 \end{bmatrix}^T$$

2. Hermit $q(t) = (t^3\ t^2\ t\ 1)\ M_H\ (q_0\ q_1\ t_0\ t_1)$

Bezier $q(t) = (t^3\ t^2\ t\ 1)\ M_{Bez}\ (p_0\ p_1\ p_2\ p_3)$

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad M_{Bez} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

for $n=3$

$$M_H (q_0\ q_1\ t_0\ t_1) = M_{Bez} (p_0\ p_1\ p_2\ p_3)$$

$$M_H \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 5 & 9 & 3 \end{bmatrix}^T = M_{Bez} (p_0\ p_1\ p_2\ p_3)$$

$$(M_{Bez})^{-1}\ M_H \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 5 & 9 & 3 \end{bmatrix}^T = (p_0\ p_1\ p_2\ p_3)$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 1/3 & 1/3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 5 \\ 2 & 9 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5/3 & 4 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}$$

$p_0 = (1\ \ 1) = q_0$

$p_1 = (5/3\ \ 4) = q_1$

$p_2 = (1\ \ 4) = q_2$

$p_3 = (2\ \ 5) = q_3$

$$T_{H \to Bez} = (M_{Bez})^{-1} M_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & -1/3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Also invertible

$$T_{Bez \to H} = (T_{H \to Bez})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$t_0 = \begin{pmatrix} 2 \\ 9 \end{pmatrix} \quad q_1 - q_0 = \begin{pmatrix} 5/3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 9/3 \end{pmatrix} = 3t_0$

$t_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad q_3 - q_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix} = 3t_1$

Cubic Bezier curve with $n = 3$

3.5 De Casteljau algorithm

Consider the four control points $\mathbf{p}_0 = (0,2)^\top$, $\mathbf{p}_1 = (2,2)^\top$, $\mathbf{p}_2 = (1,4)^\top$, and $\mathbf{p}_3 = (4,1)^\top$. Your job is to calculate four points of the Bézier curve, $\mathbf{b}(t)$, corresponding to parameter values $t \in \{0, 0.25, 0.5, 0.75, 1\}$. Present the answer as a decimal number rounded to 10^{-1} precision. Then plot your results and draw your best approximation of the Bézier curve defined by the four control points.

$t = 0$

$\mathrm{lerp}(p_0, p_1, 0) = (1-0)\cdot(0\ 2)^T + 0(2\ 2)^T = (0\ 2)^T$

$\mathrm{lerp}(p_1, p_2, 0) = (1-0)\cdot(2\ 2)^T + 0(1\ 4)^T = (2\ 2)^T$

$\mathrm{lerp}(p_2, p_3, 0) = (1-0)\cdot(1\ 4)^T + 0(4\ 1)^T = (1\ 4)^T$

$\mathrm{lerp}((0\ 2)^T, (2\ 2)^T, 0) = (0\ 2)^T$

$\mathrm{lerp}((2\ 2)^T, (1\ 4)^T, 0) = (2\ 2)^T$

$\rightarrow \mathrm{lerp}((0\ 2)^T, (2\ 2)^T, 0) = (0\ 2)^T \qquad b(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$t = 0.25$

$\mathrm{lerp}(p_0, p_1, 0.25) = (1-0.25)\cdot(0\ 2)^T + 0.25(2\ 2)^T = (0.5\ \ 2)^T$

$\mathrm{lerp}(p_1, p_2, 0.25) = (1-0.25)\cdot(2\ 2)^T + 0.25(1\ 4)^T = (1.8\ \ 2.5)^T$

$\mathrm{lerp}(p_2, p_3, 0.25) = (1-0.25)\cdot(1\ 4)^T + 0.25(4\ 1)^T = (1.8\ \ 3.3)^T$

$\mathrm{lerp}((0.5\ 2)^T, (1.8\ 2.5)^T, 0.25) = (0.8\ \ 2.1)^T$

$\mathrm{lerp}((1.8\ 2.5)^T, (1.8\ \ 3.3)^T, 0.25) = (1.8\ \ 2.7)^T$

$\Rightarrow \mathrm{lerp}((0.8\ 2.1)^T, (1.8\ \ 2.7)^T, 0.25) = (1\ \ \ 2.3)^T \qquad b(0.25) = \begin{bmatrix} 1 \\ 2.3 \end{bmatrix}$

$t = 0.5$

$\mathrm{lerp}(p_0, p_1, 0.5) = (1\ \ 2)^T$

$\mathrm{lerp}(p_1, p_2, 0.5) = (1.5\ \ 3)^T$

$\mathrm{lerp}(p_2, p_3, 0.5) = (2.5\ \ 2.5)^T$

$\mathrm{lerp}((1\ 2)^T, (1.5\ 3)^T, 0.5) = (1.3\ \ 2.5)^T$

$\mathrm{lerp}((1.5\ 3)^T, (2.5\ 2.5)^T, 0.5) = (2\ \ 2.8)^T$

$\Rightarrow \mathrm{lerp}((1.3\ 2.5)^T, (2\ 2.8)^T, 0.5) = (1.7\ \ 2.7)^T \qquad b(0.5) = \begin{bmatrix} 1.7 \\ 2.7 \end{bmatrix}$

$t = 0.75$

$\mathrm{lerp}(p_0, p_1, 0.75) = (1.5\ \ 2)^T$

$\mathrm{lerp}(p_1, p_2, 0.75) = (1.3\ \ 3.5)^T$

$\mathrm{lerp}(p_2, p_3, 0.75) = (3.3\ \ 1.8)^T$

$\mathrm{lerp}((1.5\ 2)^T, (1.3\ 3.5)^T, 0.75) = (1.4\ \ 3.1)^T$

$\mathrm{lerp}((1.3\ 3.5)^T, (3.3\ \ 1.8)^T, 0.75) = (2.8\ \ 2.2)^T$

$\Rightarrow \mathrm{lerp}((1.4\ 3.1)^T, (2.8\ 2.2)^T, 0.75) = (2.5\ \ 2.4)^T \qquad b(0.75) = \begin{bmatrix} 2.5 \\ 2.4 \end{bmatrix}$

$\mathrm{lerp}(P_0, P_1, 1) = (2\ \ 2)^T$

$\mathrm{lerp}(P_1, P_2, 1) = (1\ \ 4)^T$

$\mathrm{lerp}(P_2, P_3, 1) = (4\ \ 1)^T$

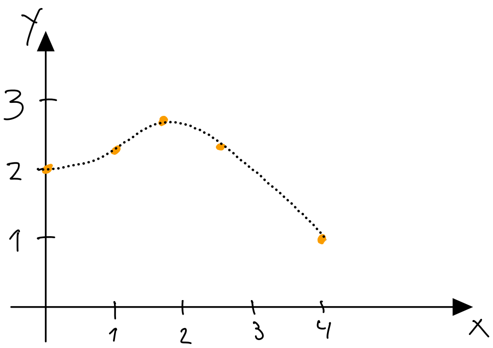
$\mathrm{lerp}((2\ 2)^T, (1\ \ 4)^T, 1) = (1\ \ 4)^T$

$\mathrm{lerp}((1\ \ 4)^T, (4\ \ 1)^T, 1) = (4\ \ 1)^T$

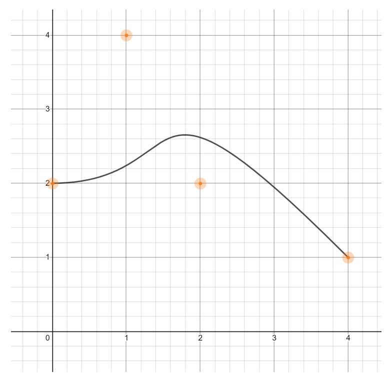
$t = 1$

$\Rightarrow \mathrm{lerp}((1\ \ 4)^T, (4\ \ 1)^T, 1) = (4\ \ 1)^T \qquad b(1) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Plot



On PC



3.6 Iterate to success

In this exercise, we compare the bisection method to the secant method. Assume that function $f(x) : \mathbb{R} \to \mathbb{R}$ is given by samples in tabulated form:

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0	-1.5	1	-2.25	2	-0.5	3	-2.5	4	1	5	2.45	6	2.3	7	2.05
0.125	-1.55	1.125	-2.25	2.125	0.05	3.125	-2.5	4.125	1.65	5.125	2.5	6.125	2.25	7.125	2.1
0.25	-1.65	1.25	-2.2	2.25	0.25	3.25	-2.1	4.25	1.875	5.25	2.5	6.25	2.2	7.25	2.15
0.375	-1.75	1.375	-2.1	2.375	0.05	3.375	-1.6	4.375	2	5.375	2.5	6.375	2.15	7.375	2.15
0.5	-1.875	1.5	-2	2.5	-0.5	3.5	-1	4.5	2.15	5.5	2.5	6.5	2.1	7.5	2.2
0.625	-2	1.625	-1.75	2.625	-1	3.625	-0.35	4.625	2.25	5.625	2.45	6.625	2.05	7.625	2.25
0.75	-2.1	1.75	-1.5	2.75	-1.65	3.75	-0.05	4.75	2.35	5.75	2.4	6.75	2	7.75	2.3
0.875	-2.2	1.875	-1	2.875	-2.2	3.875	0.05	4.875	2.4	5.875	2.35	6.875	2	7.875	2.4
														8	2.5

Find a root x_r of $f(x)$ both with the bisection and with the secant method. Accept x_r as root, if $|f(x_r)| < 0.1$. Provide your answers to the following questions:

1. Starting with $a_1 = 0$ and $b_1 = 8$, which root does the bisection method report? Provide all intervals $[a_i,\, b_i]$ until convergence.
2. Starting with $x_0 = 0$ and $x_1 = 8$, which root does the secant method report? Provide all values x_i until convergence.
3. Assuming that $f(x)$ is continuous on $[0,\, 8]$ (and that there is a way, besides the provided table, to retrieve $f(x)$ for any $x \in [0,\, 8]$), what is the condition (in terms of the starting interval $[a_1,\, b_1]$) for ensuring that the bisection method in fact converges?
4. Knowing that the secant method can also diverge, find specific starting values x_0 and x_1 that lead to such a situation? Explain why the secant method does not converge for these starting values.

1. Bisection

$\varepsilon = 0{,}1$

$a_1 = 0 \quad b_1 = 8$

$\rightarrow c = (0+8)/2 = 4 \quad f(4) = 1 \quad f(a_1) \cdot f(4) = -1.5 \cdot 1 = -1.5 < 0$

$a_2 = 0 \quad b_2 = 4$

$\rightarrow c = (0+4)/2 = 2 \quad f(2) = -0.5 \quad f(a_2) \cdot f(2) = -1.5 \cdot -0.5 = 0.75 > 0$

$a_3 = 2 \quad b_3 = 4$

$\rightarrow c = (2+4)/2 = 3 \quad f(3) = -2.5 \quad f(a_3) \cdot f(3) = -0.5 \cdot -2.5 = 1.25 > 0$

$a_4 = 3 \quad b_4 = 4$

$\rightarrow c = (3+4)/2 = 3.5 \quad f(3.5) = -1 \quad f(a_3) \cdot f(3.5) = -2.5 \cdot -1 = 2.5 > 0$

$a_5 = 3.5 \quad b_5 = 4$

$\rightarrow c = (3.5+4)/2 = 3.75 \quad f(3.75) = -0.05 \quad |f(3.75)| < \varepsilon$

$c = 3.75$ is root

2. Sacant

$a_0 = 0 \quad b_0 = 8$

$\rightarrow x_0 = 0 - f(0)\dfrac{8-0}{f(8)-f(0)} = 0 + 1.5\,\dfrac{8}{2.5+1.5} = 3$

$\rightarrow f(0)\,f(x_0) = -1.5 \cdot -2.5 = 3.75 > 0$
$f(8)\,f(x_0) = 2.5 \cdot -2.5 = -6.25 < 0 \rightarrow a_1 = x_0 \,,\; b_1 = b_0$

$a_1 = 3 \quad b_1 = 8$

$\rightarrow x_1 = 3 - f(3)\dfrac{8-3}{f(8)-f(3)} = 5.5$

$\rightarrow f(3)\,f(x_1) = -6.25 < 0 \rightarrow a_2 = a_1 \,,\; b_2 = x_1$
$f(8)f(x_1) = 6.25 > 0$

$a_2 = 3 \quad b_2 = 5.5$

$\rightarrow x_2 = 3 - f(3)\dfrac{5.5-3}{f(5.5)-f(3)} = 4.25$

$\rightarrow f(3)\cdot f(x_2) = -4.6875 < 0 \rightarrow a_3 = a_2 \,,\; b_3 = x_2$
$f(5.5)\cdot f(x_2) = 4.6875$

$a_3 = 3 \quad b_3 = 4.25$

$\rightarrow x_3 = 3 - f(3)\dfrac{4.25-3}{f(4.25)-f(3)} = 3.7143$

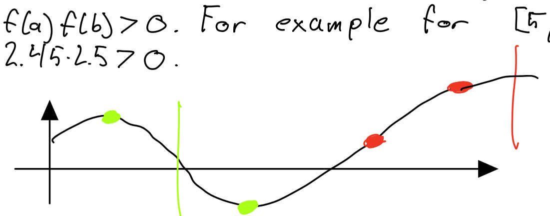
$\rightarrow f(3)\cdot f(x_3) = 0.125 > 0$
$f(4.25)\cdot f(x_3) = \cdot 0.0938 < 0 \rightarrow a_4 = x_3 \quad b_4 = b_3$

$a_4 = 3.7143 \quad b_4 = 4.25$

$\rightarrow |f(a_4)| < 0.01$ so 3.7143 is root

3. The condition for starting interval $[a_1, b_1]$ is that $f(a_1)$ and $f(b_1)$ must have opposite signs. $f(a_1)\cdot f(b_1) < 0$. With this condition there must be a root in interval $[a_1, b_1]$

4. The Sacant method diverges for intervals $[a, b]$ where $f(a)f(b) > 0$. For example for $[5, 8]$ where $f(5) = 2.45$ and $f(8) = 2.5$ $2.45 \cdot 2.5 > 0$.



3.7 Critical point characterization

For the following iteration scenarios, i.e., $x_{i+1} = f(x_i)$, involving a particular function $f(x)$ and a fixed point (critical point) x_c with $f(x_c) = x_c$, answer the following questions: 1., Confirm that x_c is a critical point. 2., Given $x_0 = x_c - \varepsilon$, $\varepsilon \in \left(0, \frac{1}{2}\right)$, does series $\{x_i\}$ converge to x_c for increasing i? If so, why? 3., Does series $\{x_i\}$ exhibit monotone or oscillatory behavior near x_c? What's the key argument for explaining this?

1. $f(x) = \frac{(x-6)^2+3}{4}$, $x_c = 3$
2. $f(x) = \frac{x^2+3}{4}$, $x_c = 3$
3. $f(x) = \frac{x^2+3}{4}$, $x_c = 1$
4. $f(x) = \frac{(x-2)^2+3}{4}$, $x_c = 1$

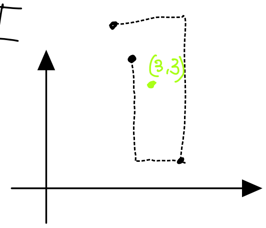
1

I. $f(x_c) = \frac{(3-6)^2+3}{4} = 3 = x_c$

II $f(x_c - \frac{1}{2}) = \frac{(2.5-6)^2+3}{4} = 3.8$

$f(3.8) = 1.9 \quad f(1.9) = 4.9$

III



Diverges oscillatory

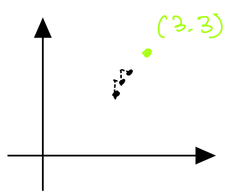
Oscillates around critical point

2.

I $f(x_c) = \frac{3^2+3}{4} = 3 = x_c$

II $f(x_c - \frac{1}{2}) = \frac{2.5^2+3}{4} = 2.3$

III $f(2.3) = 2.1 \quad f(2.1) = 1.8$



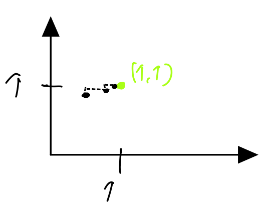
Diverges monotone

3.

I $f(x_c) = \frac{1^2+3}{4} = 1 = x_c$

II $f(x_c - \frac{1}{2}) = \frac{0.5^2+3}{3} = 0.8$

III $f(0.8) = 0.9 \quad f(0.9) = 0.96$



Converges monotone

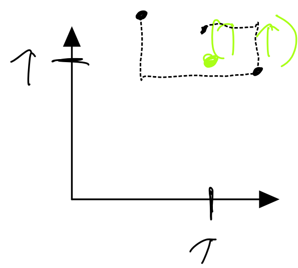
4.

I $f(x_c) = \frac{(1-2)^2+3}{4} = 1 = x_c$

II $f(x_c - \frac{1}{2}) = \frac{(0.5-2)^2+3}{4} = 1.3$

III $f(1.3) = 0.9 \qquad f(0.9) = 1.1$

Converging oscillatory



3.8 Do a Jacobi!

Assume a solvable, 3-by-3 equation system $\mathbf{A}(a_{2,1})\,\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A}(a_{2,1}) = \begin{pmatrix} 4 & -1 & 2 \\ a_{2,1} & -3 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Now do the following (in this sequence, also):

1. Find the largest-possible $a_{2,1} \in \mathbb{N}$ such that $\mathbf{A}(a_{2,1})$ is diagonally dominant.
2. Find $\mathbf{b}$ such that $\mathbf{x} = (1\ 1\ 1)^\top$ is a solution, i.e., the only one solution of $\mathbf{A}(a_{2,1})\,\mathbf{x} = \mathbf{b}$ (use $a_{2,1}$ as found in the first step).
3. Write down the Jacobi iteration formula, $\mathbf{x}_{i+1} = \mathbf{J}(\mathbf{x}_i)$, in as-simple-as-possible terms of $\mathbf{x}_i$ and numbers (in vector/matrix organization).
4. Compute the first Jacobi iteration, i.e., $\mathbf{x}_1$, starting from $\mathbf{x}_0 = (2\ 2\ 2)^\top$.
5. Determine the distances of both $\mathbf{x}_0$ and $\mathbf{x}_1$ from the solution. How much closer to the solution did you get (in terms of a factor/percent)?

1. $|a_{2,2}| \geq |a_{21}| + |a_{23}|$

$3 \geq 2 + 1$

$a_{2,2} = 2$

2. $\begin{bmatrix} 4 & -1 & 2 \\ 2 & -3 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - 1 + 2 \\ 2 - 3 + 1 \\ -1 + 0 + 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

3.

$x^{(k+1)} = D^{-1}(b - Ax^k)$

$x_i^{(k)} = \frac{1}{a_{ii}} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} (-a_{ij}\, x_j^{(k-1)}) + b_i \right)$

by hand

Conditions: Square matrix, Linear and Ideally Diagonally dominant to ensure convergence

① Assume a guess for $x^{(0)}, y^{(0)}, z^{(0)}$

② Rewrite system of equations to solve for x, y, z

③ Iterate and replace $x^{(k)}, y^{(k)}, z^{(k)}$ with $x^{(k-1)}, y^{(k-1)}, z^{(k-1)}$ in equations to get new values for x, y, z.

4.

$$4x - y + 2z = 5$$
$$2x - 3y + z = 0$$
$$-x + z = 0$$

$$x = \frac{5+y-2z}{4}$$

$$y = \frac{-2x-z}{-3}$$

$$z = x$$

	x_0	x_1
x	2	$\frac{5+2-2\cdot 2}{4} = \frac{3}{4}$
y	2	$\frac{-2\cdot 2-2}{-3} = 2$
z	2	2

Distance from $(1,1,1)^T$ to $x_0 = (2,2,2)^T = \sqrt{(2-1)^2+(2-1)^2+(2-1)^2} = \sqrt{3}$

Distance from $(1,1,1)^T$ to $x_1 = (\frac{3}{4},2,2)^T = 1.436$

Difference in distance: 17% decrease
20%