Computational Fluid Dynamics

Introduction to numerical methods in CFD I

Lecture 16

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SUMMARY OF LECTURE:
Introduction to numerical methods in CFD I

- ❑ Understand how total stress can be found using RANS
- ❑ Be aware of how “Hairpin vortices” affect the BL
- ❑ Be aware of Boussinesq hypothesis restrictions
- ❑ Be aware of the definition of CFD and how it is used in scientific work
- ❑ Be aware of numerical methods used in CFD software

Lecture 14: Questions to be discussed next

- ❑ How do we physically interpret the Reynold stress term in RANS?
- ❑ How do we “explain” the flat velocity profile in a channel flow?
- ❑ Why does the viscous sublayer end at y+=5?

This lecture

- ❑ How do we obtain the stress distribution in a channel flow?

$$\boxed{T_{tot}(y) = \tau_w \left(1 - \frac{y}{\delta}\right)}$$

- ❑ How is the stress related to the vortex dynamics near walls?
- ❑ When is Boussinesq hypothesis valid?

How do we obtain the stress distribution in a channel flow?

$$\boxed{T_{tot}(y) = \tau_w \left(1 - \frac{y}{\delta}\right)}$$

Assume fully developed turbulent flow where the mean flow is in x-direction, but changes in y-direction (sheared flow). It also mean that all statistical properties are independent of x (except P which drives the flow).

$$\rho \underbrace{\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right)}_{=0} = -\frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}}{\partial x_j} + \frac{\partial R_{ij}}{\partial x_j}$$

$$T_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \qquad R_{ij} \equiv -\rho \langle u_i' u_j' \rangle$$

$$\boxed{x: \frac{\partial T_{1j}}{\partial x_j} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = \cancel{\frac{\partial T_{xx}}{\partial x}} + \frac{\partial T_{xy}}{\partial y} + \cancel{\frac{\partial T_{xz}}{\partial z}}}$$

$$\boxed{y: \frac{\partial T_{2j}}{\partial x_j} = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = \cancel{\frac{\partial T_{yx}}{\partial x}} + \cancel{\frac{\partial T_{yy}}{\partial y}} + \cancel{\frac{\partial T_{yz}}{\partial z}}}$$

$$\boxed{\left(\frac{\partial T_{xy}}{\partial y} = \mu \frac{\partial^2 U(y)}{\partial y^2}\right)}$$

$$\boxed{x: \frac{\partial R_{1j}}{\partial x_j} = \cancel{\frac{\partial R_{xx}}{\partial x}} + \frac{\partial R_{xy}}{\partial y} + \cancel{\frac{\partial R_{xz}}{\partial z}}}$$

$$\boxed{y: \frac{\partial R_{2j}}{\partial x_j} = \cancel{\frac{\partial R_{yx}}{\partial x}} + \frac{\partial R_{yy}}{\partial y} + \cancel{\frac{\partial R_{yz}}{\partial z}}}$$

$$\Rightarrow \quad x: \; 0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(T_{xy} + R_{xy}\right) \qquad y: \; 0 = -\frac{\partial P}{\partial y} + \frac{\partial R_{xy}}{\partial y}$$

$$\boxed{x:\ 0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(T_{xy} + R_{xy}\right)}$$

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 U(y)}{\partial y^2} - \rho \frac{\partial}{\partial y}\langle u'v'\rangle \quad \Rightarrow \quad \frac{\partial P}{\partial x} = \frac{\partial}{\partial y}\Bigg[\underbrace{\mu \frac{\partial U(y)}{\partial y} - \rho \langle u'v'\rangle}_{T_{tot}}\Bigg]$$

Integrate over y (LHS independent of y)

$$\frac{\partial P}{\partial x} y + C = T_{tot}$$

Determine C from BC.
No-slip condition makes Reynold stress to be zero at wall.
Total stress has to be zero at center line (by symmetry, see previous lecture).

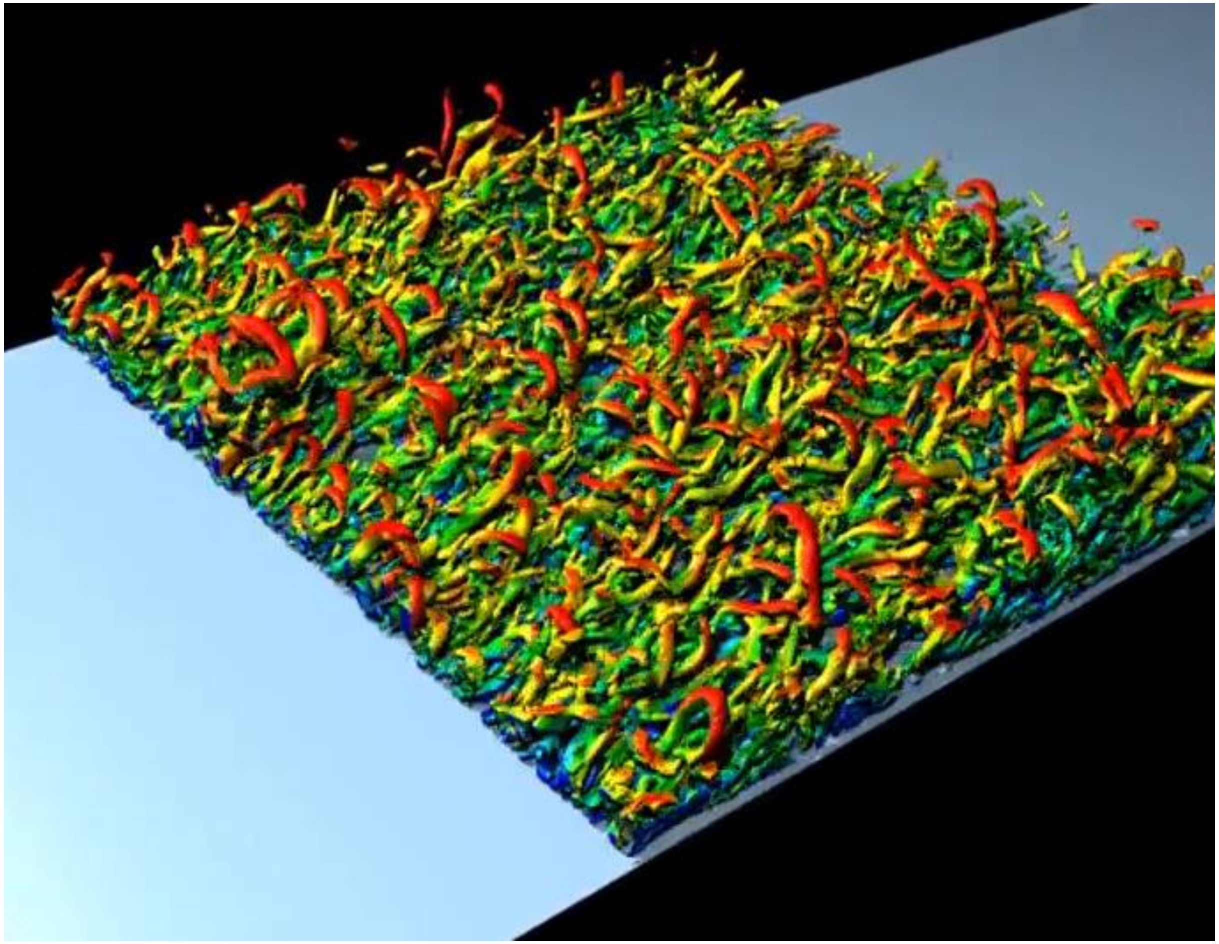
$$\boxed{y=0} \qquad T_{tot}(0) = \mu \frac{\partial U(0)}{\partial y} - \rho \langle u'v'\rangle_{y=0} = \mu \frac{\partial U(0)}{\partial y} \equiv \tau_w \quad \Rightarrow \quad C = \tau_w \quad \Rightarrow \quad T_{tot} = \frac{\partial P}{\partial x} y + \tau_w$$

$$\boxed{y=\delta} \qquad 0 = \frac{\partial P}{\partial x}\delta + \tau_w \quad \Rightarrow \quad \frac{\partial P}{\partial x} = -\frac{\tau_w}{\delta}$$

$$T_{tot} = \frac{\partial P}{\partial x} y + \tau_w \quad \Rightarrow \quad \boxed{T_{tot} = -\frac{\tau_w}{\delta} y + \tau_w = \tau_w\left(1 - \frac{y}{\delta}\right)}$$

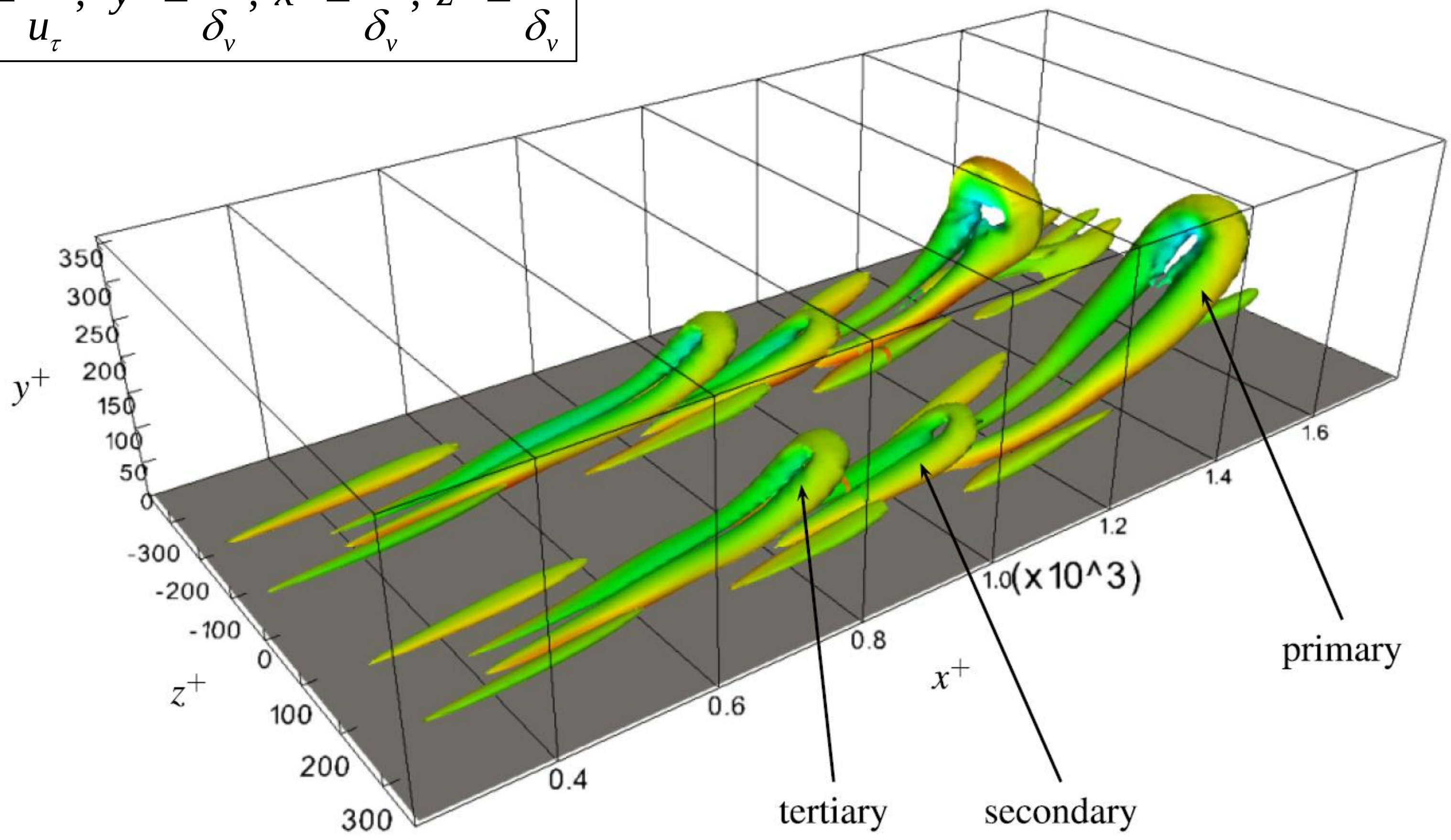
How is the stress related to the vortex dynamics near walls?

As we have discussed before the near wall stress can be viewed as the momentum transfer in perpendicular direction to the wall.



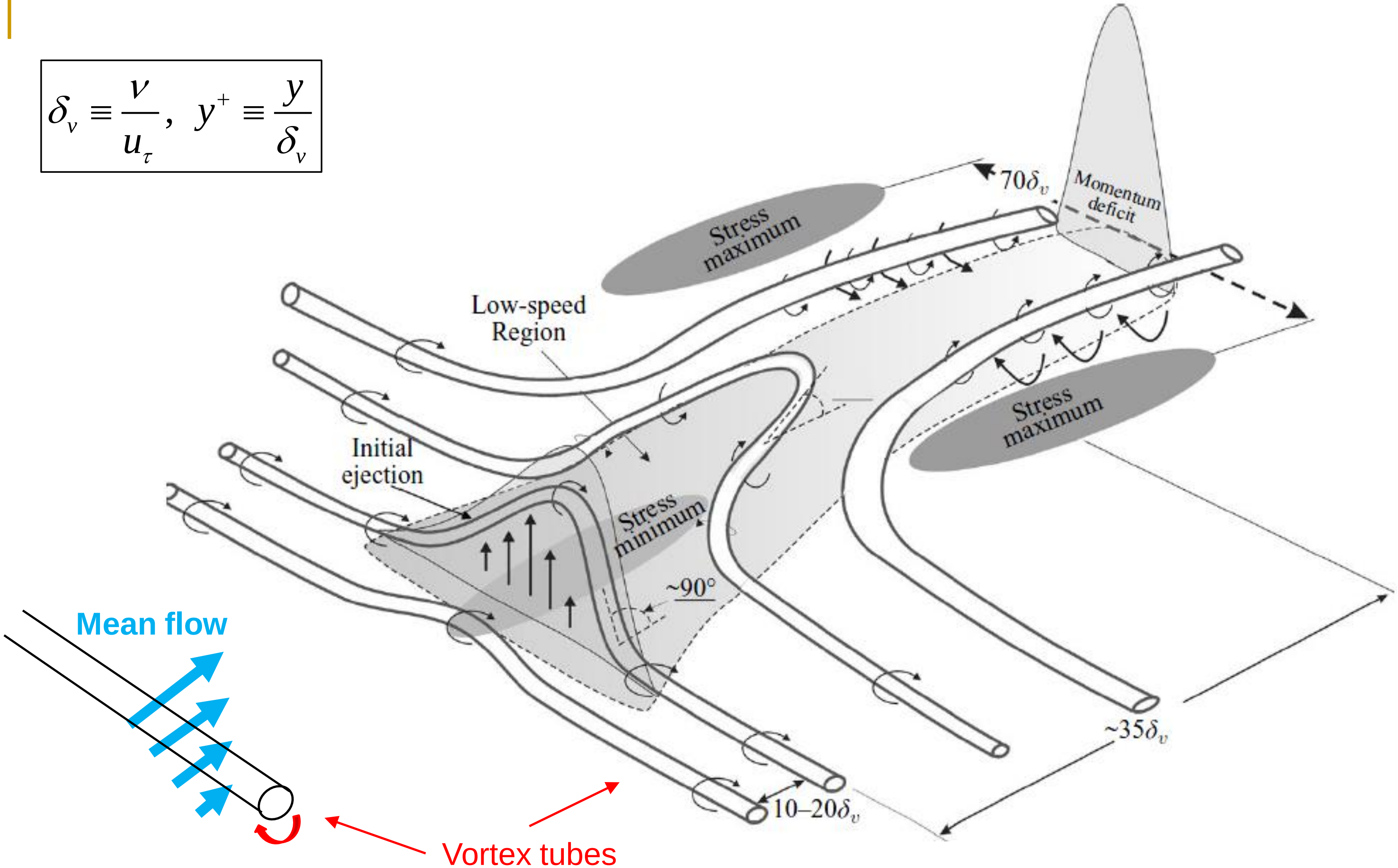
"Hairpin vortices in turbulent boundary layers"
by G. Eitel-Amor etal. 2015

$$\delta_v \equiv \frac{\nu}{u_\tau}, \quad y^+ \equiv \frac{y}{\delta_v}, \; x^+ \equiv \frac{x}{\delta_v}, \; z^+ \equiv \frac{z}{\delta_v}$$



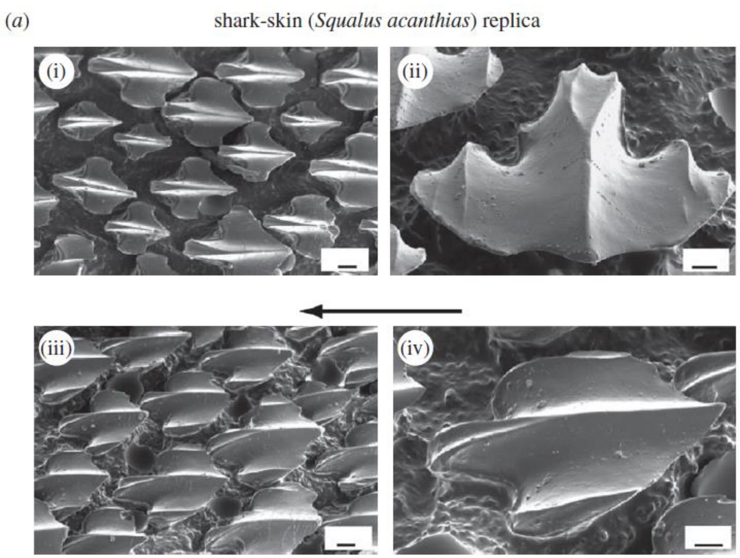
DNS results: Creation of Hairpin-vortex

$$\delta_\nu \equiv \frac{\nu}{u_\tau}, \quad y^+ \equiv \frac{y}{\delta_\nu}$$



Creation of a Hairpin-vortex (Sheng, Malkiel and Katz, J.Fluid Mech, vol 633, 17-60, 2009)

Example: Shark skin effects on streamwise vortices



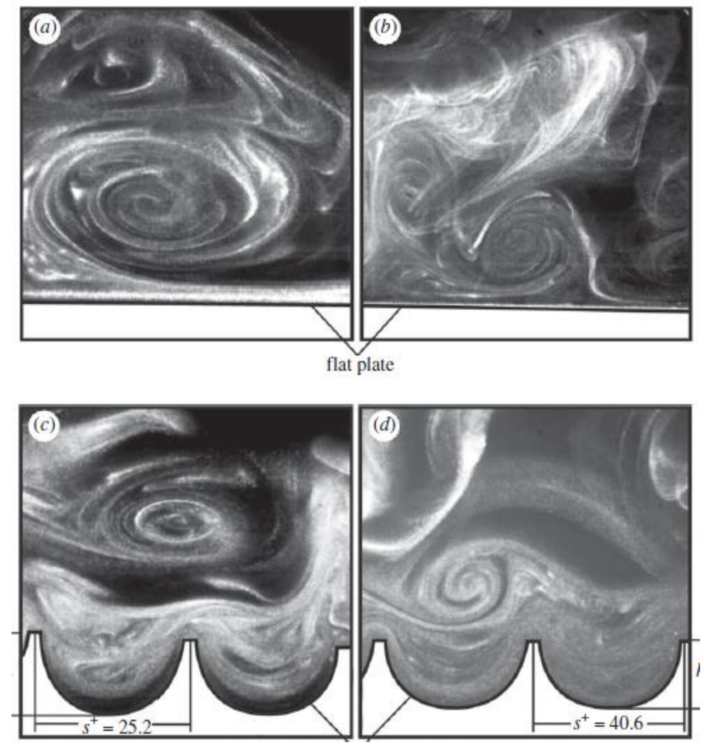


Figure. Shark skin effects on turbulence (Dean and Bhushan, Phil. Trans. R. Soc. A 368, 4775-4806, 2010)

When is Boussinesq hypothesis valid?

Hypothesis

$$R_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij}$$

$$R_{ij} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

Definitions

$$R_{ij} \equiv -\rho \left\langle u_i' u_j' \right\rangle$$

$$k \equiv \frac{1}{2} \left\langle u_i' u_i' \right\rangle$$

$$S_{ij} \equiv \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

From Davidson Ch. 4.6.1

1) The Reynolds stress and the strain rate is related by a scalar (the turbulent viscosity), and not a tensor, which is unlikely to be valid in strongly anisotrophic turbulence

2) If we have a flow with zero strain rate, then the hypothesis predict that

$$\left\langle u_x' u_x' \right\rangle = \left\langle u_y' u_y' \right\rangle = \left\langle u_z' u_z' \right\rangle,$$

which is not in general found to be the case.

$$-\rho \left\langle u_x' u_x' \right\rangle = -\tfrac{2}{3}\rho k$$

$$-\rho \left\langle u_y' u_y' \right\rangle = -\tfrac{2}{3}\rho k$$

$$-\rho \left\langle u_z' u_z' \right\rangle = -\tfrac{2}{3}\rho k$$

3) There is an impicit assumption that the Reynolds stress is controlled by the **local** strain in the mean flow, and not by the history of straining of the turbulence.This leads to errorneous results in those cases where the turbulence is subject to rapid straining by the mean flow, so that the shape of the eddies, and hence the magnitude of the Reynolds stresses, depend upon the immediately history of the turbulence.

When is Boussinesq hypothesis valid? (cont)

Hypothesis

$$R_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij}$$

Definitions

$$R_{ij} \equiv -\rho \left\langle u_i' u_j' \right\rangle$$

$$k \equiv \frac{1}{2} \left\langle u_i' u_i' \right\rangle$$

$$S_{ij} \equiv \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

Realizable k-ε model (one correction)

Due to flaws in Boussinesq physically non-realizable states of negative kinetic energy can be obtained

$$R_{11} = -\rho \left\langle u' u' \right\rangle < 0$$

$$R_{11} = \mu_t \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial x} \right) - \frac{2}{3} \rho k \Rightarrow \mu_t \frac{\partial U}{\partial x} - \frac{1}{3} \rho k < 0$$

or

$$\boxed{C_\mu \frac{k}{\varepsilon} \frac{\partial U}{\partial x} - \frac{1}{3} < 0}$$

How do we avoid breaking this condition during a simulation?

Solution: Let C_μ be a function depending on the flow and adjust its value according to the condition.

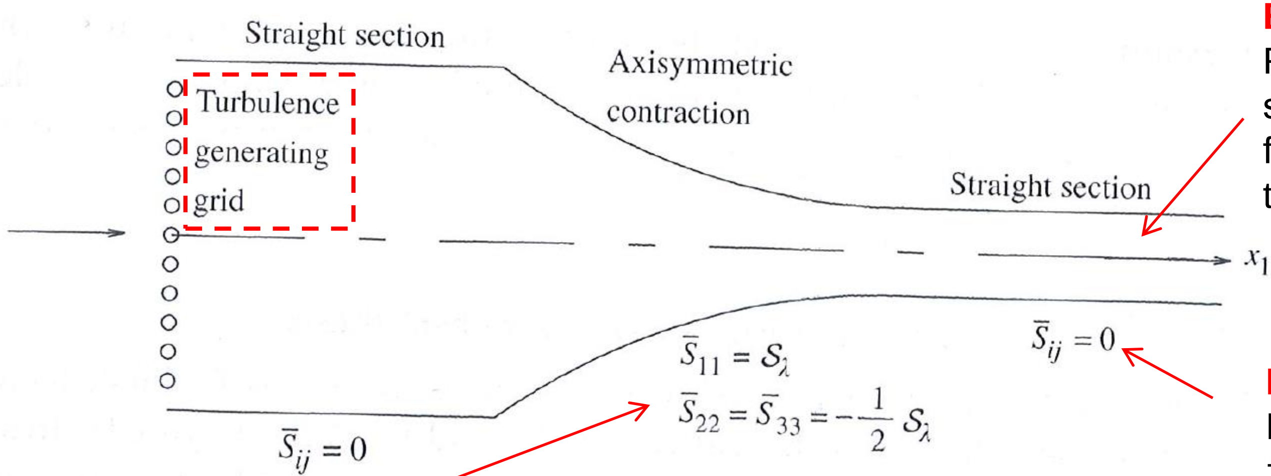
Inspection of Boussinesq hypothesis: Classical example of failure

Boussinesq hypothesis: $R_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij}$

Assumption: Valid in all points and for all times

1) The hypothesis work reasonable well in simple shear flows where the mean flow gradients and turbulence change relatively slowly along the mean flow, e.g. Channel flow, Mixing layer, Round jet, Boundary layer (Pope Ch10).

2) An example where the hypothesis fails is the axi-symmetric contraction (Pope Ch10, Fig. 10.1):



Experiments:
Reynolds stress not zero since there is a “memory” from the contraction left in the flow

Boussinesq:
Reynolds stress must be zero since no gradients

By design of the channel

Incompressible fluid

$$\bar{S}_{11} + \bar{S}_{22} + \bar{S}_{33} = S_k - \tfrac{1}{2} S_k - \tfrac{1}{2} S_k = 0$$

RANS Turbulence: Model Descriptions (ANSYS 2006)

Model	Description
Spalart – Allmaras	A single transport equation model solving directly for a modified turbulent viscosity. Designed specifically for aerospace applications involving wall-bounded flows on a fine near-wall mesh. FLUENT's implementation allows the use of coarser meshes. Option to include strain rate in k production term improves predictions of vortical flows.
Standard k–ε	The baseline two-transport-equation model solving for k and ε. This is the default k–ε model. Coefficients are empirically derived; valid for fully turbulent flows only. Options to account for viscous heating, buoyancy, and compressibility are shared with other k–ε models.
RNG k–ε	A variant of the standard k–ε model. Equations and coefficients are analytically derived. Significant changes in the ε equation improves the ability to model highly strained flows. Additional options aid in predicting swirling and low Reynolds number flows.
Realizable k–ε	A variant of the standard k–ε model. Its "realizability" stems from changes that allow certain mathematical constraints to be obeyed which ultimately improves the performance of this model.
Standard k–ω	A two-transport-equation model solving for k and ω, the specific dissipation rate (ε / k) based on Wilcox (1998). This is the default k–ω model. Demonstrates superior performance for wall-bounded and low Reynolds number flows. Shows potential for predicting transition. Options account for transitional, free shear, and compressible flows.
SST k–ω	A variant of the standard k–ω model. Combines the original Wilcox model for use near walls and the standard k–ε model away from walls using a blending function. Also limits turbulent viscosity to guarantee that $\tau_T \sim k$. The transition and shearing options are borrowed from standard k–ω. No option to include compressibility.
Reynolds Stress	Reynolds stresses are solved directly using transport equations, avoiding isotropic viscosity assumption of other models. Use for highly swirling flows. Quadratic pressure-strain option improves performance for many basic shear flows.

From ANSYS Introductory FLUENT Notes (FLUENT v6.3 December 2006)

RANS Turbulence: Model Behavior and Usage (ANSYS 2006)

Model	Behavior and Usage
Spalart-Allmaras	Economical for large meshes. Performs poorly for 3D flows, free shear flows, flows with strong separation. Suitable for mildly complex (quasi-2D) external/internal flows and boundary layer flows under pressure gradient (e.g. airfoils, wings, airplane fuselages, missiles, ship hulls).
Standard k–ε	Robust. Widely used despite the known limitations of the model. Performs poorly for complex flows involving severe pressure gradient, separation, strong streamline curvature. Suitable for initial iterations, initial screening of alternative designs, and parametric studies.
RNG k–ε	Suitable for complex shear flows involving rapid strain, moderate swirl, vortices, and locally transitional flows (e.g. boundary layer separation, massive separation, and vortex shedding behind bluff bodies, stall in wide-angle diffusers, room ventilation).
Realizable k–ε	Offers largely the same benefits and has similar applications as RNG. Possibly more accurate and easier to converge than RNG.
Standard k–ω	Superior performance for wall-bounded boundary layer, free shear, and low Reynolds number flows. Suitable for complex boundary layer flows under adverse pressure gradient and separation (external aerodynamics and turbomachinery). Can be used for transitional flows (though tends to predict early transition). Separation is typically predicted to be excessive and early.
SST k–ω	Offers similar benefits as standard k–ω. Dependency on wall distance makes this less suitable for free shear flows.
Reynolds Stress	Physically the most sound RANS model. Avoids isotropic eddy viscosity assumption. More CPU time and memory required. Tougher to converge due to close coupling of equations. Suitable for complex 3D flows with strong streamline curvature, strong swirl/rotation (e.g. curved duct, rotating flow passages, swirl combustors with very large inlet swirl, cyclones).

Overview of numerical approaches in CFD

Lab 1:

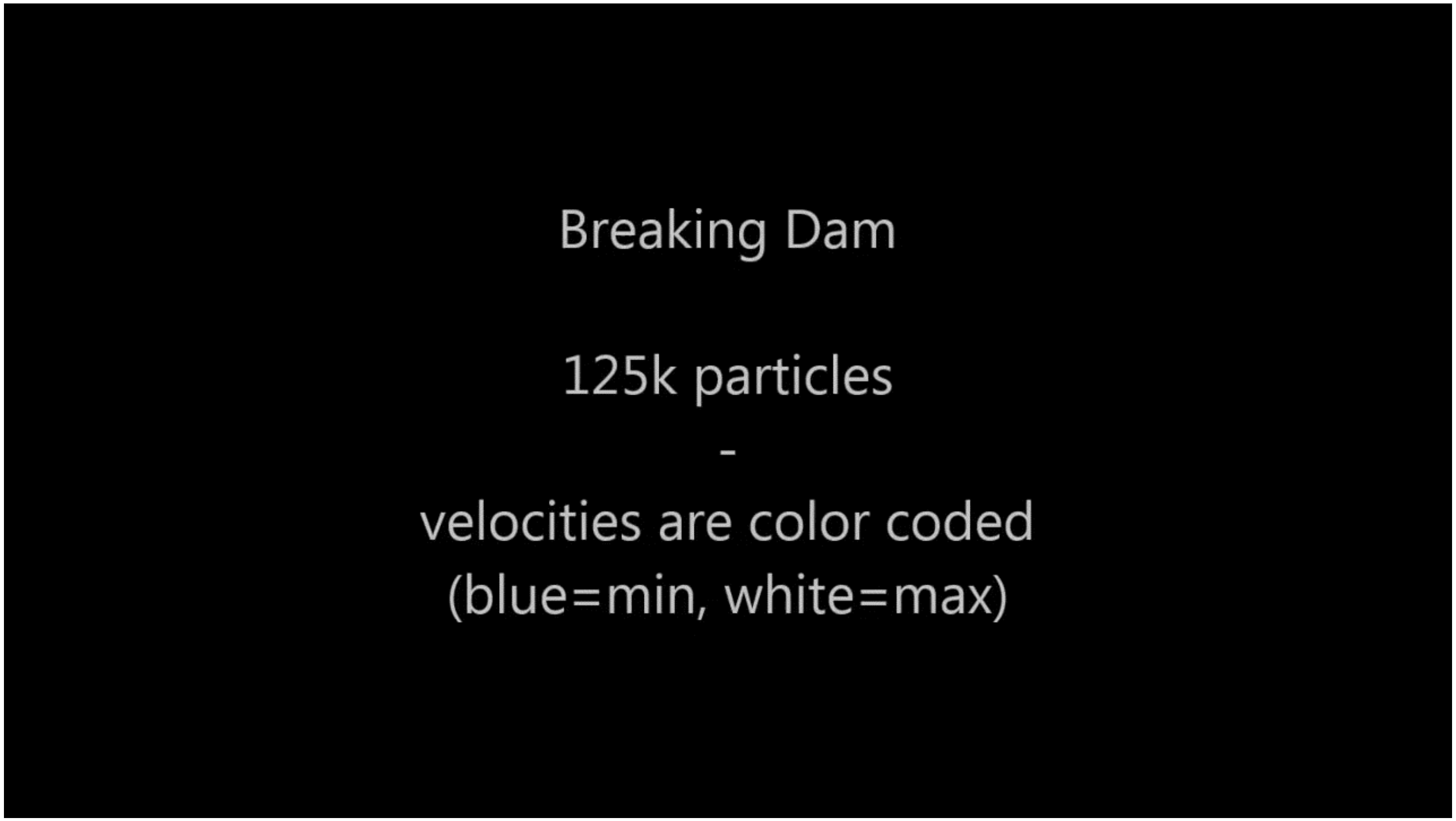
- Get used to Comsol
- Learn how to modify Heat Transfer Module to describe a Levelset approach
- Learn how to compare simulation with experimental data
- Learn how to extract and visualize simulated data

Recap: Using the levelset approach we were able to simulate two different fluids simultaneously with only one set of equations. We were able to reproduce the dynamics of the moving interface between the fluids.

In general complicated dynamics of moving surfaces, like for example breaking waves etc, are very hard to do in numerical approaches that use static meshes.

However, using meshless approaches such dynamics often are much easier. One example of such method is “Smoothed Particle Hydrodynamic” (SPH).

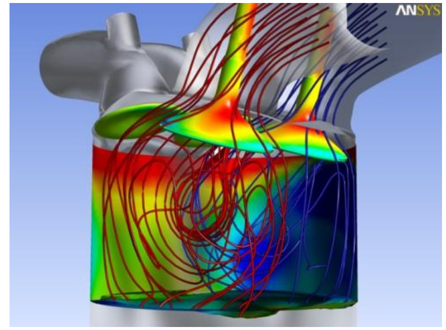
Smoothed Hydrodynamic Particle (SPH) Method
-Mesh free numerical method

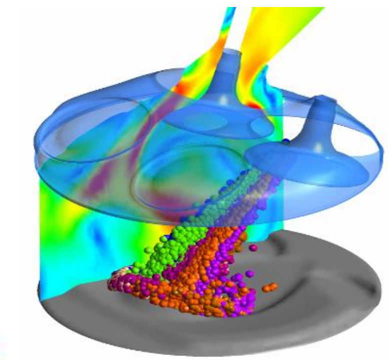


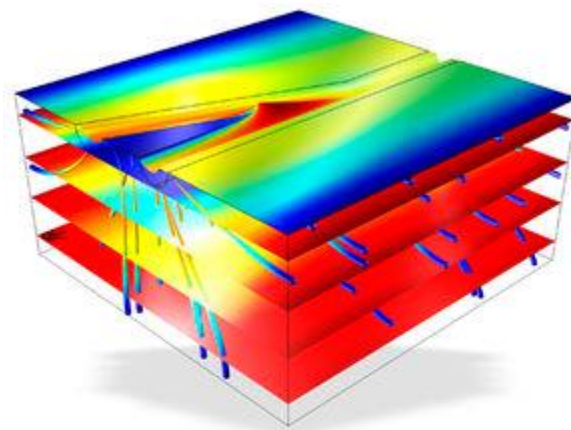
CFD-software

Different software:

- ANSYS CFX
- ANSYS FLUENT
- STAR-CCM+
- Open FOAM
- COMSOL
- Many others…

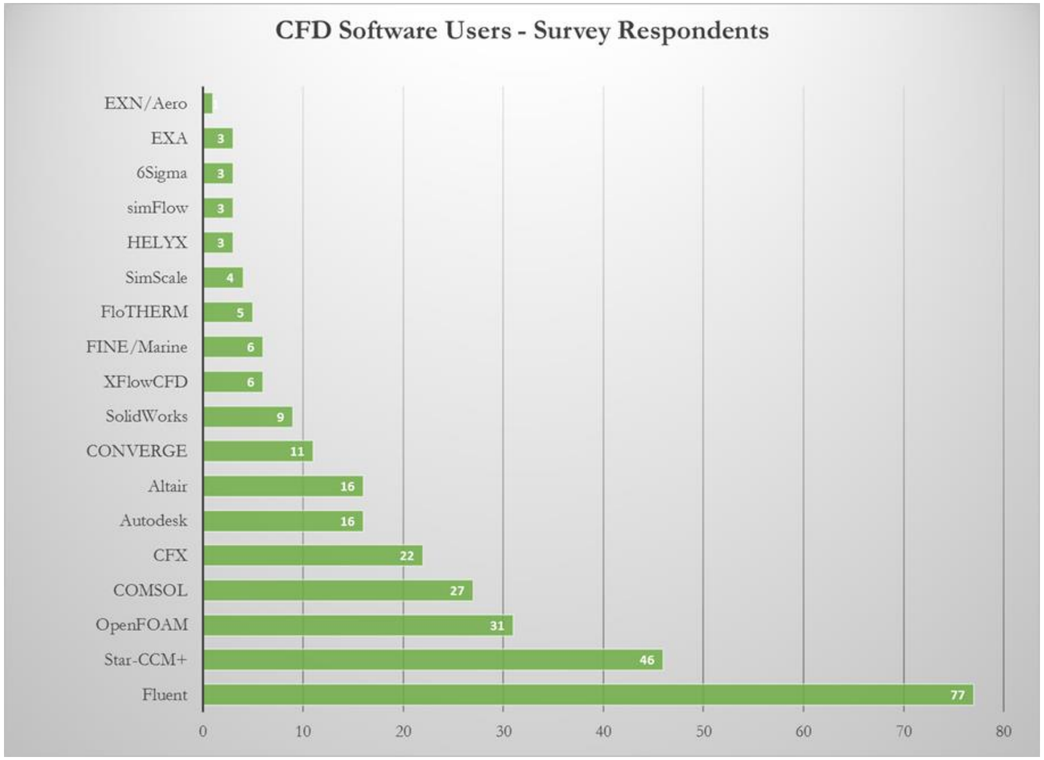






CFD Software Survey 2016 (only 242 responders…):

1) Fluent = 32%
2) Star CCM+ = 19%
3) OpenFoam = 13%
4) Comsol = 11%
5) CFX = 9%



www.resolvedanalytics.com/theflux/comparing-cfd-software

Numerical methods in CFD

Spatial discretization

Most common approaches:

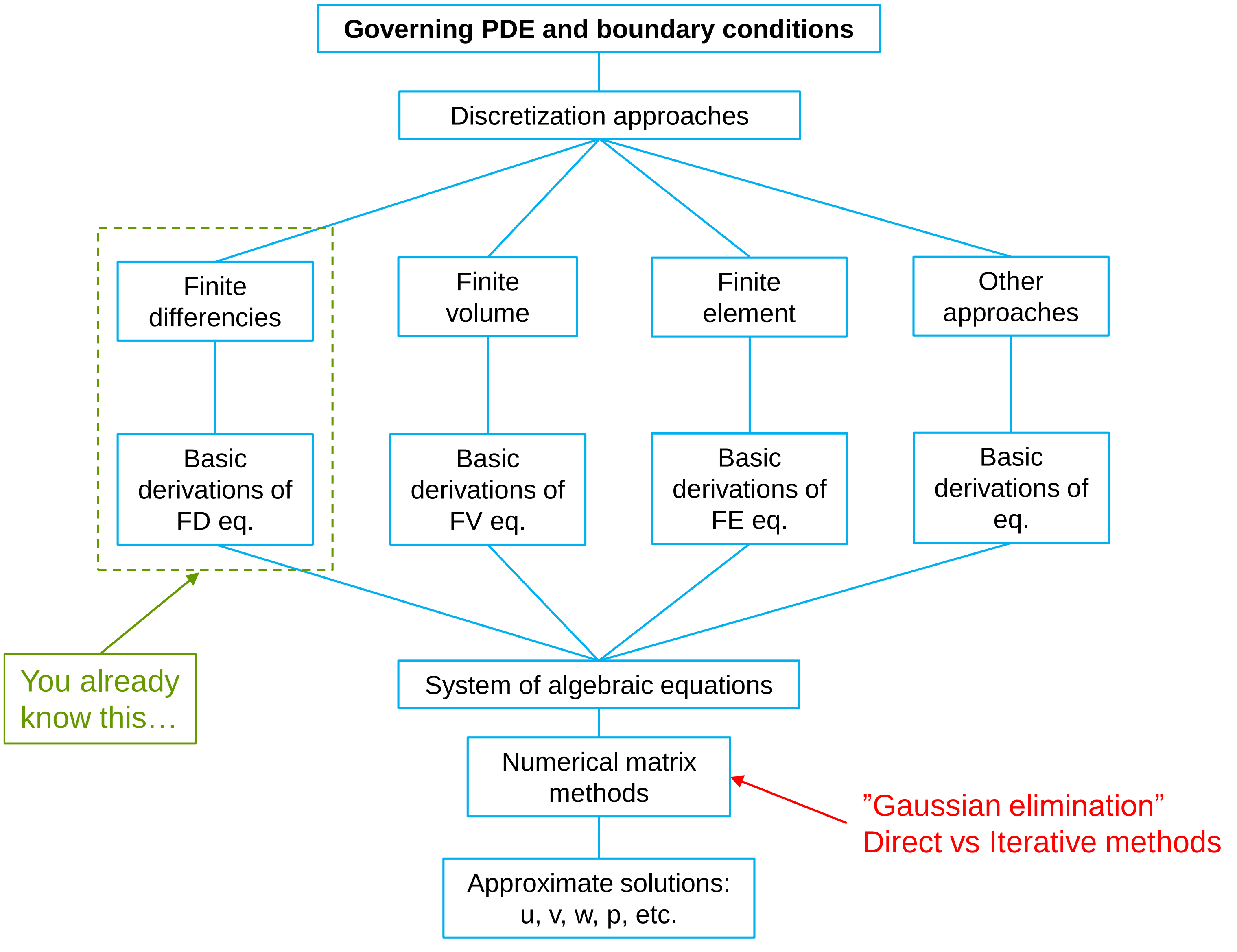
- ❑ Finite Diffrence Method (FDM)
- ❑ Finite Volume Method (FVM)
- ❑ Finite Element Method (FEM)

Other approaches:

- ❑ Smoothed Particle Hydrodynamics (SPH)
- ❑ Lattice Boltzman Method (LBM)
- ❑ Vorticity based methods
- ❑ …

1) Fluent (FVM)
2) Star CCM+ (FVM)
3) OpenFoam (FVM)
4) Comsol (FEM)
5) CFX (FVM)

Most approaches use FDM to handle time-stepping


Governing PDE and boundary conditions
Discretization approaches
Finite differencies
Finite volume
Finite element
Other approaches
Basic derivations of FD eq.
Basic derivations of FV eq.
Basic derivations of FE eq.
Basic derivations of eq.
You already know this…
System of algebraic equations
Numerical matrix methods
”Gaussian elimination”
Direct vs Iterative methods
Approximate solutions: u, v, w, p, etc.

Examples of governing equations

Example 1: Steady 1D diffusion

$$\frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) + q_{\phi} = 0$$

Diffusion coefficient

Source

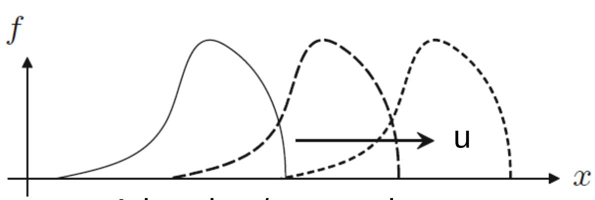
Example 2: Steady 1D advection-diffusion

$$\rho u \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) + q_{\phi}$$

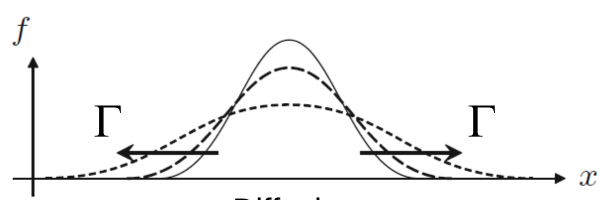
Fluid density

Fluid velocity

In CFD Φ could correspond to temperature, concentration, turbulence properties,…



Advection / convection



Diffusion

Example 3: Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial x^2}$$

A simplified version of Navier-Stokes equation

Often used to validate numerical methods since exact solution can be found by a Hopf-Cole transformation which converts the equation to a linear diffusion equation.
However, this also shows that Burgers equation lack some of the main characteristics of Navier-Stokes / turbulence, the chaotic behavior due to extreme sensitivity to initial conditions.

End of lecture