Computational Fluid Dynamics
Turbulence Models II (cont)
Near wall flows
Lecture 11

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SUMMARY OF LECTURE: TURBULENCE MODELS II (CONT)

- Be able to describe how a boundary layer near a wall is divided into sublayers
- Be able to mathematically describe the velocity profiles of Viscous sublayer and Log-layer
- Understand how y+ is used to describe boundary layers
- Understand how viscous stresses and turbulent stresses are related in viscous sublayer and log-layer
- Understand the origin of damping functions
- Be aware of near wall improvements of zero-equation models
- Be able to use and understand the Low Re k-epsilon models
- Understand how to analyze the near wall behavior of turbulent quantities
- Be aware of the Wall-function-approach and its strengths and weaknesses
- Be able to perform near wall analysis, using a Taylor-expansion-approach, on terms in the k-epsilon equation
- Be able to use log-layer properties to analyze turbulence in the log-layer
- Be aware of some turbulence models: Spalart-Allmaras, Wilcox k-omega, Menter SST k-omega and two-layer k-epsilon

Recap: Standard k-ε model + Near wall modifications

The Low Re k-ε model

Model k-equation

$$\frac{Dk}{Dt} = P_k - \varepsilon + D_k$$

Model ε-equation

$$\frac{D\tilde{\varepsilon}}{Dt} = \frac{\tilde{\varepsilon}}{k}\left(\boxed{f_1} C_{\varepsilon 1} P_k - \boxed{f_2} C_{\varepsilon 2}\tilde{\varepsilon}\right) + D_\varepsilon + \boxed{E}$$

Turbulent viscosity model

$$\mu_t = \boxed{f_\mu} C_\mu \rho \frac{k^2}{\tilde{\varepsilon}}$$

$$D, E, f_1, f_2 \text{ and } f_\mu ?$$

Not diffusion

$$\varepsilon = \tilde{\varepsilon} + \boxed{D}$$

$$\Downarrow$$

$\tilde{\varepsilon} = 0$ at boundary

k-production

$$P_k \equiv \frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j} = \frac{\mu_t}{\rho} S^2$$

k-diffusion

$$D_k \equiv \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

Dissipation

$$\varepsilon \equiv \frac{\mu}{\rho}\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle$$

ε-diffusion

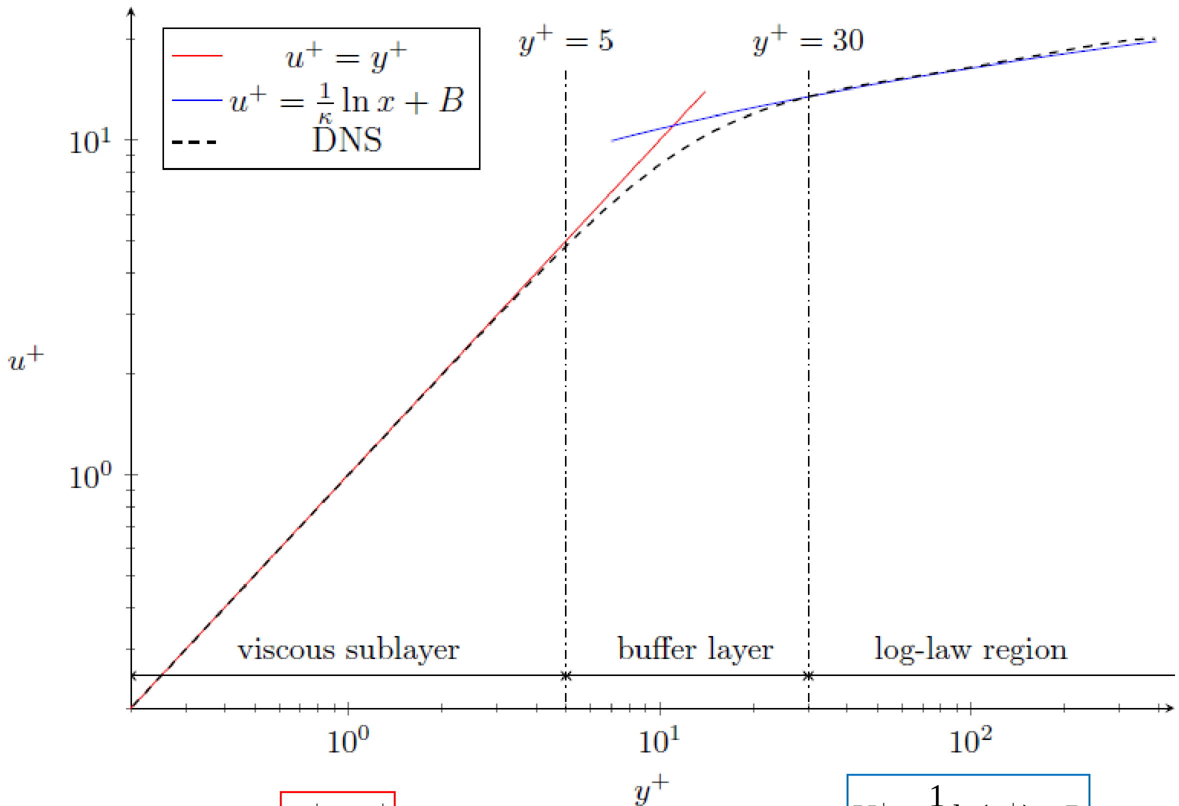
$$D_\varepsilon \equiv \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

$C_\mu = 0.09$

$C_{\varepsilon 1} = 1.44 \quad C_{\varepsilon 2} = 1.92$

$\sigma_k = 1.0 \quad \sigma_\varepsilon = 1.3$

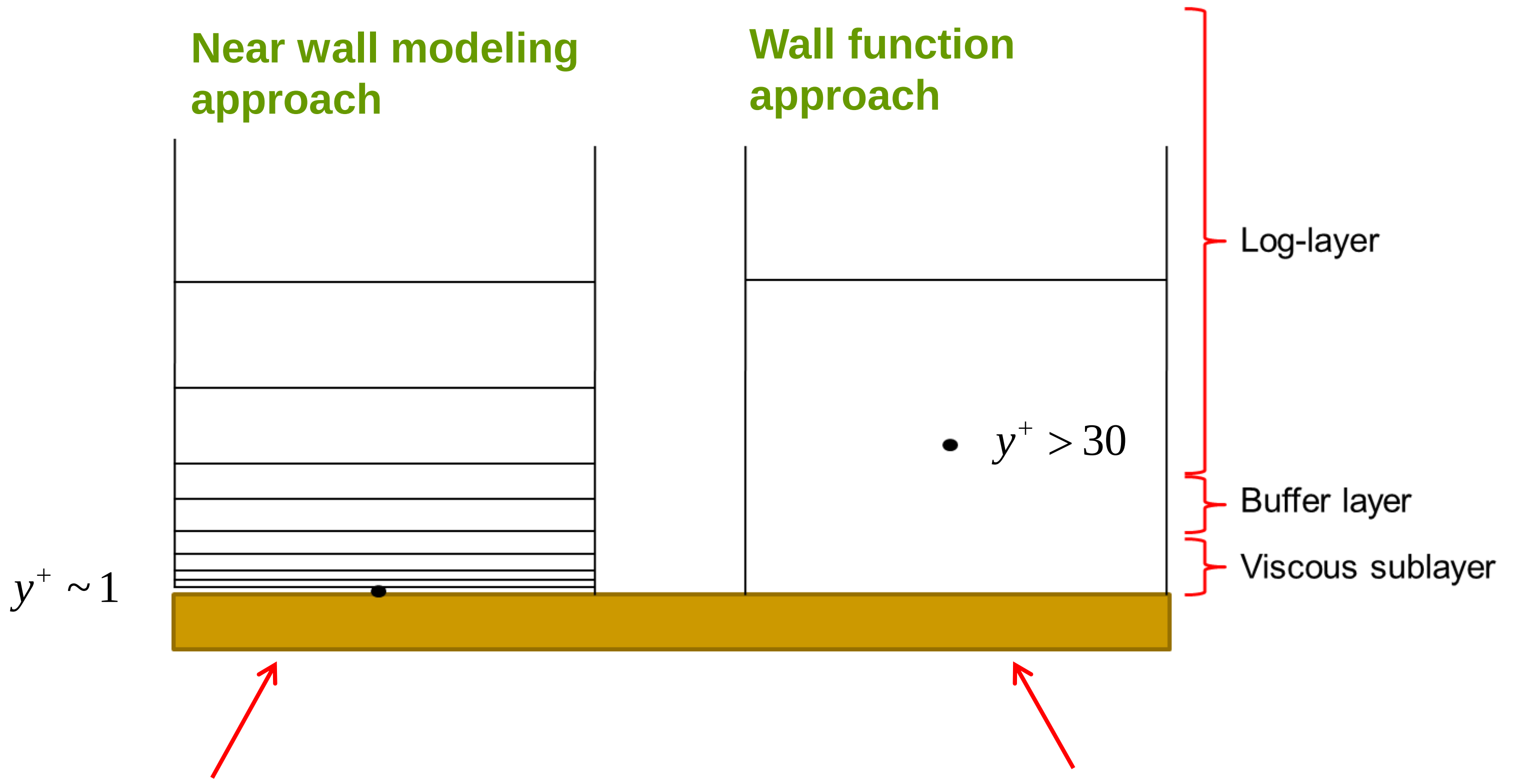
Recap: Turbulent boundary layer



$\boxed{U^+ = y^+}$

$$\boxed{U^+ = \frac{1}{\kappa}\ln(y^+) + B}$$

Recap: Near-wall modeling v.s. Wall functions



Low Re k-ε model

First grid point in viscous sublayer

$y^+ \sim 1$

Standard k-ε model

First grid point in log-layer

$y^+ > 30$

Recap: Near wall mesh size using skin friction along a flat plate

1) Calculate Reynolds number

2) Determine friction velocity

$$y^+ \equiv \frac{\rho u_\tau y}{\mu}$$

$$u_\tau \equiv \sqrt{\tau_w / \rho}$$

Wall shear stress on a flat plate

$$\tau_w = c_f \tfrac{1}{2} \rho U_0^2$$

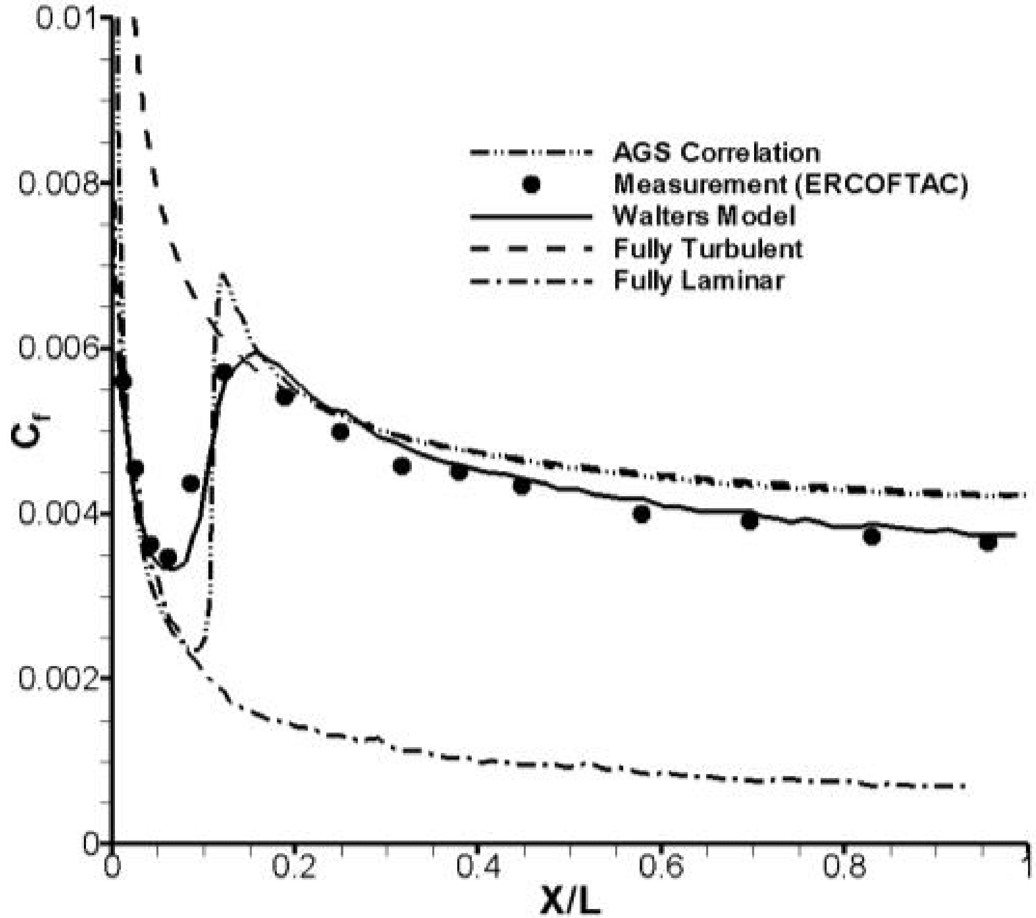
$$c_f = \frac{0.664}{\sqrt{\mathrm{Re}}} \quad \text{(Laminar, Blasius)}$$

$$c_f = 0.058\,\mathrm{Re}^{-0.2} \quad \text{(Turbulent)}$$

3) Determine height for the chosen model:

$$y^+ = 1 \;\rightarrow\; y_{1st} = \ldots$$

$$y^+ = 30 \rightarrow y_{1st} = \ldots$$



Near-wall modeling: Development of damping functions

Model k-equation	Model ε-equation	Turbulent viscosity
$\frac{Dk}{Dt} = P_k - \varepsilon + D_k$	$\frac{D\varepsilon}{Dt} = \frac{\varepsilon}{k}\left(f_1 C_{\varepsilon 1} P_k - f_2 C_{\varepsilon 2}\varepsilon \right) + D_\varepsilon$	$\mu_t = f_\mu C_\mu \rho \frac{k^2}{\varepsilon}$

By Near wall analysis it can be shown that: $f_\mu \sim y^{-1}$

How?
We will show it by comparing the exact and model versions of k-production

$$P_k \equiv \boxed{\frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j}} = \boxed{2\frac{\mu_t}{\rho} S_{ij} S_{ij}}$$

Exact　　　　Model

We thus need to know the near wall behavior of $P_k,\ R_{ij},\ \frac{\partial U_i}{\partial x_j},\ k,\ \varepsilon$

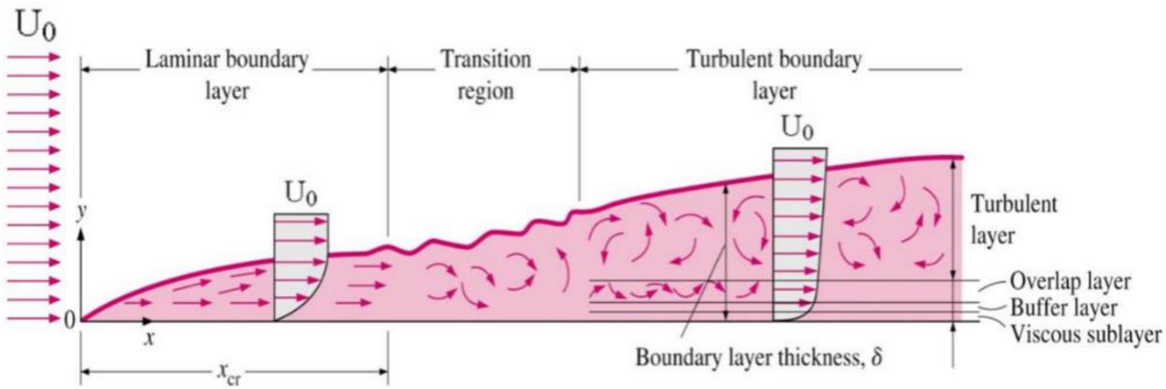
Last lecture

Near wall analysis of turbulence models in a boundary layer

Interesting quantities

$$k \equiv \frac{1}{2}\langle u_i' u_i' \rangle \qquad \varepsilon \equiv \frac{\mu}{\rho}\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j} \right\rangle$$

$$P_k \equiv R_{ij}\frac{\partial U_i}{\partial x_j} \qquad R_{ij} \equiv \rho \langle u_i' u_j' \rangle$$



Last lecture (Test case = 2D boundary layer)

Near wall (2D-assumption)

1.

$$u' = a_1 + b_1 y + c_1 y^2 + \dots$$
$$v' = a_2 + b_2 y + c_2 y^2 + \dots$$
$$w' = a_3 + b_3 y + c_3 y^2 + \dots$$

No-slip and no penetration at y=0

$$u' = a_1 = 0$$
$$v' = a_2 = 0$$
$$w' = a_3 = 0$$

2.

$$u_{i,i} = \underbrace{\frac{\partial u'}{dx}}_{=0} + \frac{\partial v'}{dy} + \underbrace{\frac{\partial w'}{dz}}_{=0} = 0$$

$$\left(\frac{\partial v'}{\partial y}\right)_{y=0} = b_2 = 0$$

$$u' = b_1 y + c_1 y^2 + \dots$$
$$v' = c_2 y^2 + \dots$$
$$w' = b_3 y + c_3 y^2 + \dots$$

Near wall analysis

$$U^+ \equiv \frac{U}{u_\tau} \qquad y^+ \equiv \frac{\rho u_\tau y}{\mu} \qquad u_\tau \equiv \sqrt{\tau_w / \rho}$$

$$u' = b_1 y + c_1 y^2 + \ldots$$
$$v' = c_2 y^2 + \ldots$$
$$w' = b_3 y + c_3 y^2 + \ldots$$

$$\boxed{k \equiv \frac{1}{2}\langle u_i' u_i' \rangle}$$

$$u_i' u_i' = u'^2 + v'^2 + w'^2 = b_1^2 y^2 + c_1^2 y^4 + b_3^2 y^2 + \ldots = \left(b_1^2 + b_3^2\right) y^2 + O(y^3)$$

$$\boxed{k \sim y^2}$$

$$\boxed{\varepsilon \equiv \frac{\mu}{\rho}\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j} \right\rangle}$$

2D-assumption

$$\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j} = \underbrace{\frac{\partial u'}{\partial x_j}\frac{\partial u'}{\partial x_j}}_{\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y} + \frac{\partial u'}{\partial z}\frac{\partial u'}{\partial z}} + \frac{\partial v'}{\partial x_j}\frac{\partial v'}{\partial x_j} + \frac{\partial w'}{\partial x_j}\frac{\partial w'}{\partial x_j} = \left(\frac{\partial u'}{\partial y}\right)^2 + \left(\frac{\partial v'}{\partial y}\right)^2 + \left(\frac{\partial w'}{\partial y}\right)^2 = b_1^2 + 4c_2^2 y^2 + b_3^2 + \ldots$$

$$\boxed{\varepsilon \sim y^0}$$

$$\boxed{P_k \equiv \frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j}}$$

Viscous sublayer

$$U^+ = y^+, \quad 0 \le y^+ \le 5 \quad \Rightarrow \quad U = \frac{\rho u_\tau^2}{\mu} y$$

$U = U(y), V = W = 0$

$$P_k = -\frac{\rho \langle u_i' u_j' \rangle}{\rho}\frac{\partial U_i}{\partial x_j} = -\underbrace{\langle u'v' \rangle}_{\sim y^3}\underbrace{\frac{\partial U}{\partial y}}_{\sim y^0}$$

$$\boxed{P_k \sim y^3}$$

Near wall analysis (cont.)

Exact vs Model eq.

a) $P_k \equiv \boxed{\frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j}} = \boxed{2\frac{\mu_t}{\rho}S_{ij}S_{ij}}$ ➡ $-\underbrace{\langle u'v'\rangle}_{\sim y^3}\underbrace{\frac{\partial U}{\partial y}}_{\sim y^0} = 2\frac{\mu_t}{\rho}\underbrace{\left(\frac{\partial U}{\partial y}\right)^2}_{\sim y^0}$ $\boxed{\mu_t \sim y^3}$

b) $\boxed{\mu_t = f_\mu C_\mu \rho \frac{k^2}{\varepsilon}}$ ➡ $\mu_t \sim f_\mu \frac{y^4}{y^0} = f_\mu y^4$ $\boxed{f_\mu \sim y^{-1}}$

Example: Asymptotic behavior of two terms in ε-eq.

$$\boxed{f_1 C_{\varepsilon 1}\frac{\varepsilon}{k}P_k} \sim f_1 \frac{y^0}{y^2} y^3 = f_1 y$$

$$\boxed{f_2 C_{\varepsilon 2}\frac{\varepsilon^2}{k}} \sim f_2 y^{-2}$$

f_1, f_2 can be used to tune these terms when $y \to 0$

How do we choose these damping functions, what do we compare with?

From lecture 8: Derivation of Exact ε-equation

Dissipation

$$\varepsilon \equiv \frac{\mu}{\rho}\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle$$

1) Operate on Navier-Stokes equation by $\frac{\mu}{\rho}\frac{\partial u_i'}{\partial x_j}\frac{\partial}{\partial x_j}$

2) Apply a time average on each term and drop terms with zero average…

$$\frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = (1) + (2) + (3) + (4)$$

Production $$(1) \equiv -2\frac{\mu}{\rho}\left[\left\langle u_{i,k}' u_{j,k}'\right\rangle + \left\langle u_{k,i}' u_{k,j}'\right\rangle\right] U_{i,j} - 2\frac{\mu}{\rho}\left\langle u_{i,k}' u_{i,m}' u_{k,m}'\right\rangle - 2\frac{\mu}{\rho}\left\langle u_k' u_{i,j}'\right\rangle U_{i,jk}$$

Destruction $$(2) \equiv -2\frac{\mu^2}{\rho^2}\left\langle u_{i,km}' u_{i,km}'\right\rangle$$

Diffusion $$(3) \equiv \frac{\mu}{\rho}\frac{\partial}{\partial x_j}\frac{\partial \varepsilon}{\partial x_j}$$

Turb.transport $$(4) \equiv -2\frac{\mu}{\rho}\frac{\partial}{\partial x_j}\left\langle p_{,m}' u_{j,m}'\right\rangle - \frac{\mu}{\rho}\frac{\partial}{\partial x_j}\left\langle u_j' u_{i,m}' u_{k,m}'\right\rangle$$

Near wall analysis of Exact equations

Details in Speziale etal (1990)
"A critical evaluation of two-equation models for near wall turbulence"

Cambro / Lecture Notes / References

As before, Taylor serie expansion of fluctuations inserted into the exact equations:

$$\boxed{\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j}=I+II+III+IV}$$

$$\boxed{\frac{\partial \varepsilon}{\partial t}+U_j\frac{\partial \varepsilon}{\partial x_j}=(1)+(2)+(3)+(4)}$$

Previous slides

$$\left.\begin{aligned}&\text{Production: } (I)\propto y^3\\ &\text{Dissipation: } (II)\propto y^0\end{aligned}\right.$$

$$\text{Diffusion: } (III)\propto y^0$$

$$\text{Turb. transp. } (IV)\propto y^1$$

$$\text{Production: } (1)\propto y^1$$

$$\text{Destruction: } (2)\propto y^0$$

$$\text{Diffusion: } (3)\propto y^0$$

$$\text{Turb. transp. } (4)\propto y^0$$

Any modeling of these terms should have the same y-dependency.
Such model is said to be *asymtotic consistent*.

Example

Terms from RHS of the Model ε-equation

$$\begin{aligned} k &\sim y^2 \\ \varepsilon &\sim y^0 \\ P_k &\sim y^3 \end{aligned} \Rightarrow \left\{ \begin{aligned} & f_1 C_{\varepsilon 1} \frac{\varepsilon}{k} P_k \sim f_1 \frac{y^0}{y^2} y^3 = f_1 y \\ & f_2 C_{\varepsilon 2} \frac{\varepsilon^2}{k} \sim f_2 y^{-2} \end{aligned} \right.$$

From Exact ε-equation

$$\Rightarrow \left\{ \begin{aligned} & f_1 y \sim y^1 \\ & f_2 y^{-2} \sim y^0 \end{aligned} \right. \Rightarrow \boxed{\left\{ \begin{aligned} & f_1 \sim 1 \\ & f_2 \sim y^2 \end{aligned} \right.}$$

Production

Destruction

Consistency conditions

Chien

$$f_\mu = 1 - \exp\left(-0.0115 y^+\right)$$

$$f_1 = 1$$

$$f_2 = 1 - 0.22 \exp\left(-\mathrm{Re}_T^2 / 36\right)$$

$$D = 2\nu \frac{k}{y^2}$$

$$E = -\frac{2\nu\varepsilon}{y^2} \exp\left(-0.5 y^+\right)$$

$$\mathrm{Re}_T \equiv \frac{\rho k^2}{\mu \varepsilon}$$

$$C_{\varepsilon 1} = 1.35$$

$$C_{\varepsilon 2} = 1.8$$

Previous slides

$$\boxed{f_\mu \sim y^{-1}}$$

Note!

www.cfd-online.com/Wiki/Low-Re_k-epsilon_models

From Lecture 10

$$\mu_t = f_\mu C_\mu \rho \frac{k^2}{\varepsilon}$$

AKN (1994)

$$f_\mu = \left(1 - e^{-l^*/14}\right)^2 \left[1 + 5\,\mathrm{Re}_T^{-3/4}\, e^{-(\mathrm{Re}_T/200)^2}\right]$$

$$f_1 = 1$$

$$f_2 = \left(1 - e^{-l^*/3.1}\right)^2 \left[1 - 0.3\, e^{-(\mathrm{Re}_T/6.5)^2}\right]$$

$$l^* = \frac{\rho u_\varepsilon l_w}{\mu} \qquad \mathrm{Re}_T \equiv \frac{\rho k^2}{\mu \varepsilon} \qquad u_\varepsilon = \left(\frac{\mu \varepsilon}{\rho}\right)^{1/4}$$

$$C_\mu = 0.09,\ \sigma_k = 1.4,\ \sigma_\varepsilon = 1.4$$

$$C_{\varepsilon 1} = 1.5,\ C_{\varepsilon 2} = 1.9$$

"A new turbulence model for predicting fluid flow and heat transfer in separating and reattaching flows", K. Abe, and T. Kondoh:

> The most important feature of the present k–ε model is the introduction of the Kolmogorov velocity scale, $u_\varepsilon = (\nu\varepsilon)^{1/4}$, instead of the friction velocity u_τ, to account for the near-wall and low-Reynolds-number effects in both attached and detached flows [9]. This model can reproduce the correct near-wall asymptotic relations of turbulence, i.e. $k \propto y^2$, $\varepsilon \propto y^0$, $\nu_t \propto y^3$ and $-\overline{uv} \propto y^3$ for $y \to 0$.

Why is it good idea to remove friction velocity from damping models?

$$u_\tau \equiv \sqrt{\frac{\tau_w}{\rho}}$$

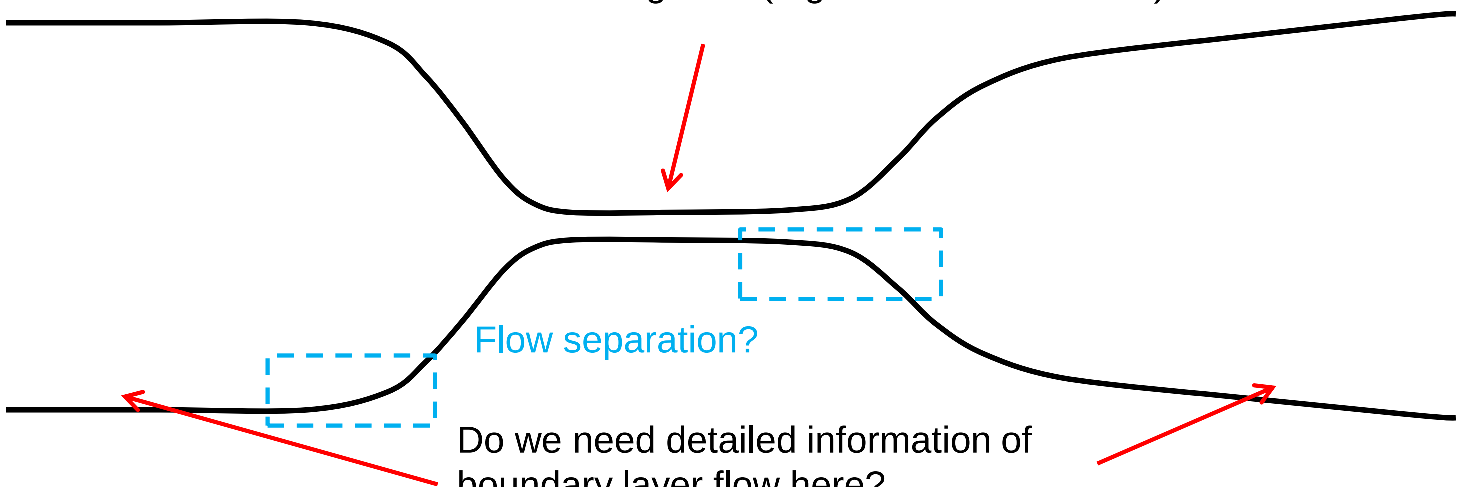
What do they mean by "correct near-wall asymptotic relations of turbulence"?

$$k \sim y^2 \qquad \varepsilon \sim y^0 \qquad \nu_T \sim y^3 \qquad -\langle u'v' \rangle \sim y^2$$

Example: Wall modeling vs Wall functions in large models

If the simulation geometry are large, the use of wall functions at strategic places reduces the computational time.

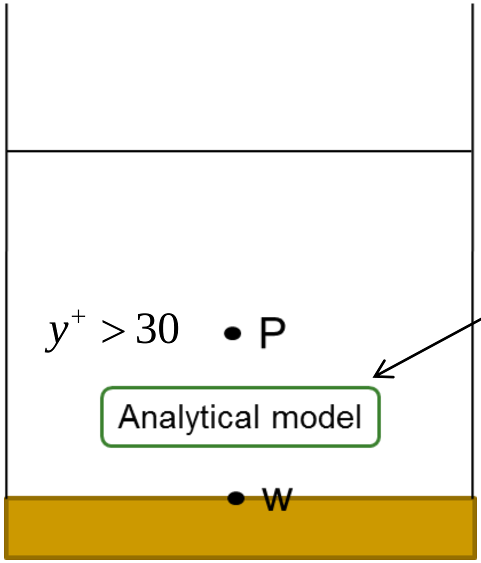
The channel is narrow here and details in the boundary layer flow might be important.
Use wall modeling here (e.g. Low Re-k-ε model)



Do we need detailed information of boundary layer flow here?
If No, apply wall functions in this region.

Many softwares (including Comsol) offers the choice of automatic wall treatment!

Wall functions



Assumptions

- Velocities parallell to walls obeys the ”Law of the wall”
- Total shear stress is constant and equal to wall shear stress throughout the grid cell nearest wall
- k is constant (se below) in the log-layer and has a quadratic y-dependency in the viscous sublayer
- The dissipation is modeled as below in log-layer and is constant in viscous sublayer

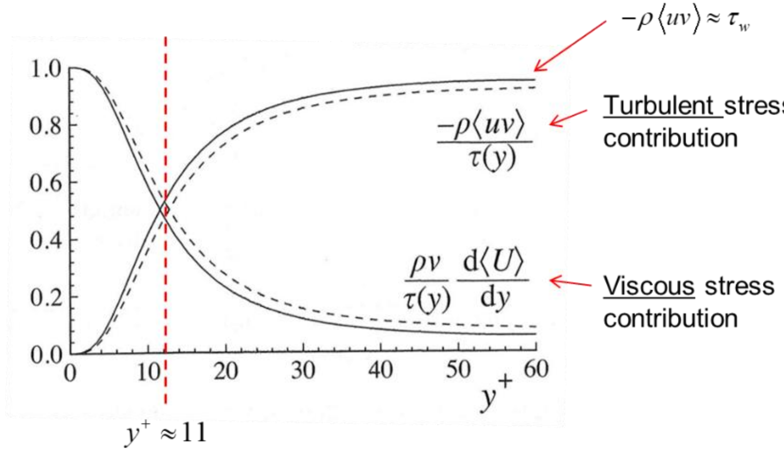
Analytical model at first grid cell

“Law of the wall”

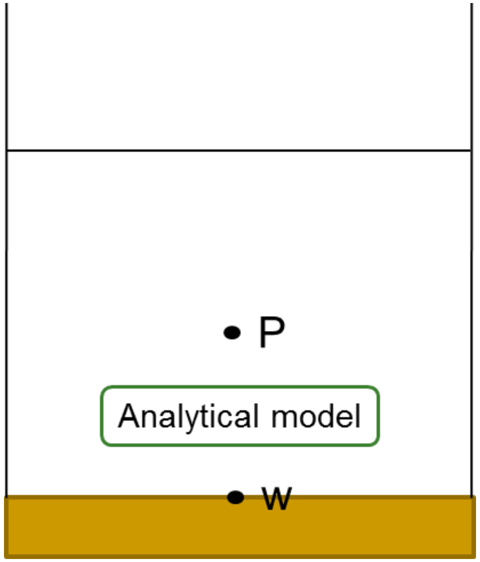
$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{u_\tau y}{\nu}\right) + B \qquad k = \frac{u_\tau^2}{\sqrt{C_\mu}} \qquad \varepsilon = \frac{u_\tau^3}{\kappa y}$$

$$\tau_w = \rho u_\tau^2$$

For a nice summary, see notes by H. Lacovides “*Current Practice and Recent Developments in Wall functions I*”



Simple implementation of wall functions



1) Assuming that y+>30 for the first cell and that Up and yp are known we find an estimate of the friction velocity by iteratively solving the log-layer equation

$$\frac{U_P}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{u_\tau y_P}{\nu}\right) + 5 \quad \Rightarrow \quad u_\tau$$

2) When we have an estimate of the friction velocity we determine the shear stress that it corresponds to, and use it as a source in the momentum equation:

$$\tau_w = \rho u_\tau^2$$

3) We then calculate k in the cell by $$k = \frac{u_\tau^2}{\sqrt{C_\mu}}$$

4) and calculate ε in the cell by $$\varepsilon = \frac{u_\tau^3}{\kappa y_P}$$

➡ No need to resolve the Viscous sublayer or Buffer layer

Derivation of wall functions from Log-layer properties

Exercise

$$a)\ U^+ = \frac{1}{\kappa}\ln y^+ + 5.5$$

$$b)\ P_k = \varepsilon$$

$$c)\ -\rho\langle u'v'\rangle \approx \tau_w$$

Recap

$$d)\ \color{red}{Exp} \Rightarrow \frac{u_\tau^2}{k} \approx 0.3$$

New

We want to derive these relations

Wall functions Boundary conditions (applied in Log-layer)

$$k = \frac{u_\tau^2}{\sqrt{C_\mu}} \qquad \varepsilon = \frac{u_\tau^3}{\kappa y}$$

Note: We consider a 2D-BL

Note: $C_\mu = \left(\frac{u_\tau^2}{k}\right)^2 \approx (0.3)^2 = 0.09$

Useful tools:

a) $$\frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y} \qquad (1)$$

b) $$\frac{\mu_t}{\rho}\left(\frac{\partial U}{\partial y}\right)^2 = \varepsilon \qquad (2)$$

c) $$P_k \approx u_\tau^2 \frac{\partial U}{\partial y} \qquad (3)$$

$$\tau_w = \rho u_\tau^2$$

$$\mu_t = C_\mu \rho \frac{k^2}{\varepsilon}$$

$$P_k \equiv \frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j} = 2\frac{\mu_t}{\rho}S_{ij}S_{ij}$$

Wall functions from Log-layer properties (cont.)

$b)\ P_k = \varepsilon$

$(1)\ \frac{\partial U}{\partial y} = \frac{u_\tau}{\kappa y}$

$(2)\ \frac{\mu_t}{\rho}\left(\frac{\partial U}{\partial y}\right)^2 = \varepsilon$

$(3)\ P_k \approx u_\tau^2 \frac{\partial U}{\partial y}$

b) & (3): ➡ $\varepsilon = u_\tau^2 \frac{\partial U}{\partial y} \overset{1)}{=} \frac{u_\tau^3}{\kappa y}$ ➡ $\boxed{\varepsilon = \frac{u_\tau^3}{\kappa y}}$

(2) ✚ $\boxed{\mu_t = C_\mu \rho \frac{k^2}{\varepsilon}}$ ➡ $C_\mu \frac{k^2}{\varepsilon}\left(\frac{\partial U}{\partial y}\right)^2 = \varepsilon$ ➡ $C_\mu k^2 \left(\frac{\partial U}{\partial y}\right)^2 = \varepsilon^2 \underset{b)\ \&\ (3)}{=} \left(u_\tau^2 \frac{\partial U}{\partial y}\right)^2$

➡ $k^2 = \frac{\left(u_\tau^2\right)^2}{C_\mu}$ ➡ $\boxed{k = \frac{u_\tau^2}{\sqrt{C_\mu}}}$

Summary: Wall functions

1) Assume that Up and yp is known in the first near wall cell

$$\frac{U_P}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{u_\tau y_P}{\nu}\right) + 5 \quad \Rightarrow \quad u_\tau$$

2) Use the corresponding shear stress as a source in the momentum equation:

$$\tau_w = \rho u_\tau^2$$

3) Calculate k in the cell and use it as a BC:

$$k = \frac{u_\tau^2}{\sqrt{C_\mu}}$$

4) Calculate ε in the cell and use it as a BC:

$$\varepsilon = \frac{u_\tau^3}{\kappa y_P}$$

RANS, k, ε PDE:s can be integrated one time step and the process can be repeated

Note: The momentum equation is affected by k and ε through $\mu_t = C_\mu \rho \frac{k^2}{\varepsilon}$

Turbulence models based on Boussinesq hypothesis

1. Mixing length model
2. k-equation model
3. Spalart-Allmaras one-equation model — A special model for airfoils
4. Standard k-ε model
5. k-ω model (Wilcox)

The two most basic two-eq. models

6. Low Re k-ε model — Correction for near wall behavior. Note: Wilcox k-ω model has not the same need for corrections near wall

7. Menter SST k-ω
   - Blending function
8. Realizable k-ε
   - Reynolds stress constraint

Guest lecturer from SpinChem use this

Many other modern models exists…

Study group question…

Wilcox *k-ω* model

Model k-equation

$$\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j}=2\frac{\mu_t}{\rho}S_{ij}S_{ij}-\beta^* k\omega+\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu+\sigma^*\mu_t\right)\frac{\partial k}{\partial x_j}\right]$$

Model ω-equation

$$\frac{\partial \omega}{\partial t}+U_j\frac{\partial \omega}{\partial x_j}=2\alpha\frac{\mu_t}{\rho}\frac{\omega}{k}S_{ij}S_{ij}-\beta\omega^2+\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu+\sigma\mu_t\right)\frac{\partial \omega}{\partial x_j}\right]$$

Production

Destruction
(or dissipation)

Re-distribution

Closure constants
(Wilcox)

$$\beta^*=9/100$$
$$a=5/9 \qquad \beta=3/40$$
$$\sigma=1/2 \qquad \sigma^*=1/2$$

$$\varepsilon=\beta^*\omega k$$

$$\boxed{\mu_t=\rho\frac{k}{\omega}}$$

$$[\omega]=s^{-1}$$

Low Re even without damping function
=> Possible to have first mesh node at y+ = 1

More turbulence models

Menter SST k-ω model

- Fully turbulent region: Use k-ε model
- Near wall region: Use k-ω model

Spalart-Allmaras

$$\frac{D\nu_T}{Dt} = \nabla \cdot \left(\frac{\nu + \nu_T}{\sigma} \nabla \nu_T \right) + S_\nu$$

$$S_\nu = S_\nu(\mu, \mu_T, \Omega, \nabla \nu_T, l_w)$$

Designed for aerodynamics, can handle boundary layers with separation

Realizable k-ε model

$$\boxed{R_{ij} \equiv -\rho \langle u_i' u_j' \rangle} \Rightarrow R_{11} = -\rho \langle u' u' \rangle < 0$$

$$\boxed{R_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij}} \Rightarrow R_{11} = 2\mu_t \frac{\partial U}{\partial x} - \frac{2}{3} \rho k$$

$$\Rightarrow \mu_t \frac{\partial U}{\partial x} - \frac{1}{3} \rho k < 0$$

$$\mu_t = C_\mu \rho \frac{k^2}{\varepsilon} \Rightarrow C_\mu \frac{k}{\varepsilon} \frac{\partial U}{\partial x} - \frac{1}{3} < 0$$

How do we avoid breaking this condition during a simulation?
Solution: Let C_μ be a function depending on the flow and adjust its value according to the condition.

Comparison of RANS turbulence models

Example: Lecture notes by Bakker

Model	Strengths	Weaknesses
Spalart-Allmaras	Economical (1-eq.); good track record for mildly complex B.L. type of flows.	Not very widely tested yet; lack of submodels (e.g. combustion, buoyancy).
STD k-ε	Robust, economical, reasonably accurate; long accumulated performance data.	Mediocre results for complex flows with severe pressure gradients, strong streamline curvature, swirl and rotation. Predicts that round jets spread 15% faster than planar jets whereas in actuality they spread 15% slower.
RNG k-ε	Good for moderately complex behavior like jet impingement, separating flows, swirling flows, and secondary flows.	Subjected to limitations due to isotropic eddy viscosity assumption. Same problem with round jets as standard k-ε.
Realizable k-ε	Offers largely the same benefits as RNG but also resolves the round-jet anomaly.	Subjected to limitations due to isotropic eddy viscosity assumption.
Reynolds Stress Model	Physically most complete model (history, transport, and anisotropy of turbulent stresses are all accounted for).	Requires more cpu effort (2-3x); tightly coupled momentum and turbulence equations.

End of lecture