Computational Fluid Dynamics
Fluid Mechanics III
Lecture 4

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SUMMARY OF LECTURE: Fluid mechanics III

- ❑ Be aware of Reynolds decomposition technique
- ❑ Be aware of the relation between Kinetic energy and Enstrophy
- ❑ Understand the relation between kinetic energy and dissipation
- ❑ Be aware of typical boundary conditions for Navier-Stokes
- ❑ Understand the physical flow conditions at a fluid-fluid interface

Note:
This lecture contains a little bit more theoretical details than you actually need at this stage, but do not worry, I just want to state some tools we need later, you do not have to understand all the derivations ☺

Short comment on turbulence

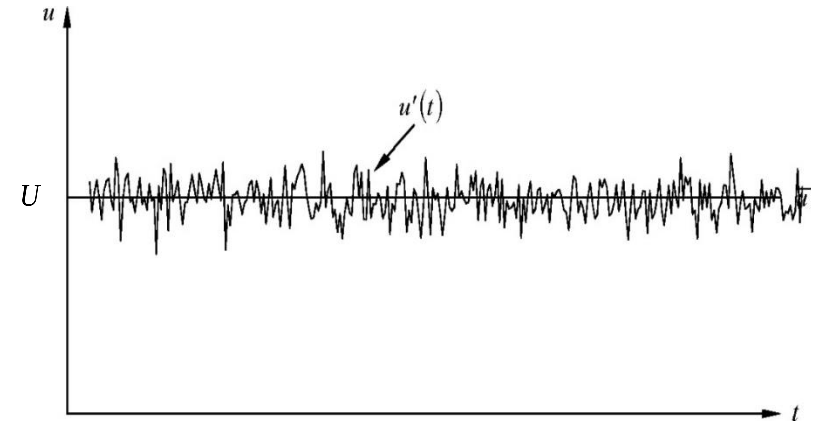
In turbulence one often divide the flow into a mean part and a random fluctuation part

Reynolds decomposition:

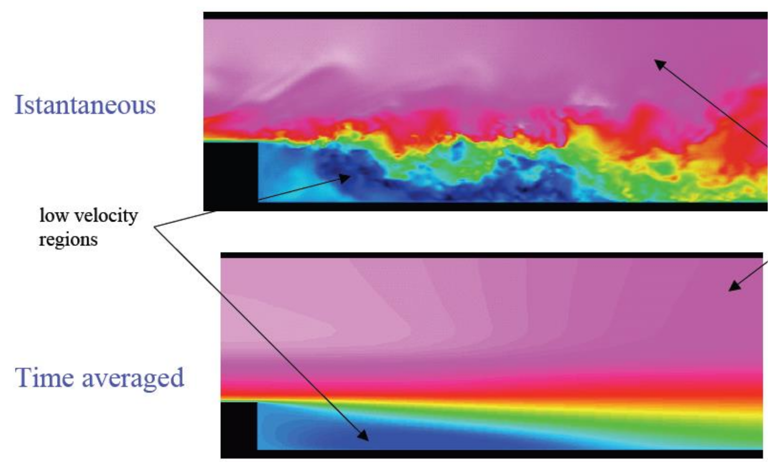
$$\begin{cases} u_i = U_i + u_i' \\ p = P + p' \\ \tau_{ij} = T_{ij} + \tau_{ij}' \end{cases}$$

Instantaneous values:	$u_i \, , p \, , \tau_{ij}$
Mean values:	$U_i \, , P \, , T_{ij}$
Fluctuation values:	$u_i' \, , p' \, , \tau_{ij}'$

In upcoming lectures we will see that the kinetic energy of turbulence *k*, (the kinetic energy of fluctuations), feeds on the kinetic energy of mean flow *K*.



Example of data measured at a fixed position in a fluid flow



Example: Preview of turbulence model equations

This model will be further investigated in Lab 2

Reynolds Averaged Navier-Stokes (RANS)

$$\rho\left(\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}\right)=-\frac{\partial P}{\partial x_i}+\frac{\partial T_{ij}}{\partial x_j}+\frac{\partial R_{ij}}{\partial x_j}+\left\langle f_i\right\rangle$$

Viscosity stress

$$\boxed{T_{ij}=\mu\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)}$$

Reynolds stress

$$\boxed{R_{ij}\equiv-\left\langle \rho u_i' u_j'\right\rangle}$$

Standard k-ε model

$$P_k\equiv 2\frac{\mu_t}{\rho}\bar{S}_{ij}\bar{S}_{ij}$$

$$\bar{S}_{ij}=\frac{1}{2}\left(U_{i,j}+U_{j,i}\right)$$

$$\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j}=\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu_k\frac{\partial k}{\partial x_j}\right)+P_k-\varepsilon$$

k = Kinetic energy of turbulence fluctuations

$$\frac{\partial \varepsilon}{\partial t}+U_j\frac{\partial \varepsilon}{\partial x_j}=\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu_\varepsilon\frac{\partial \varepsilon}{\partial x_j}\right)+\frac{\varepsilon}{k}\left(C_{\varepsilon 1}P_k-C_{\varepsilon 2}\varepsilon\right)$$

ε = Dissipation of turbulence fluctuations

Derivation of a kinetic energy equation

Navier-Stokes

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i \qquad \left(\frac{\partial u_i}{\partial x_i} = u_{i,i} = 0 \right)$$

$$\boxed{\begin{aligned} \tau_{ij} &= \mu\left(u_{i,j} + u_{j,i}\right) \\ \tau_{ij} &= 2\mu S_{ij} \end{aligned}}$$

1. Multiply NS with velocity and use $u_i \frac{Du_i}{Dt} = \frac{D}{Dt}\left(\frac{u_i^2}{2}\right)$

$$\boxed{\begin{aligned} &\frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij} \\ &\begin{cases} S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \\ \Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \end{cases} \end{aligned}}$$

2. Use $u_i \frac{\partial p}{\partial x_i} = \frac{\partial (u_i p)}{\partial x_i}$ and $u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial (u_i \tau_{ij})}{\partial x_j} - \tau_{ij} \frac{\partial u_i}{\partial x_j}$

$$\Rightarrow \quad \boxed{\rho \frac{D}{Dt}\left(\tfrac{1}{2} u_i^2\right) = \frac{\partial (u_i p)}{\partial x_i} + \frac{\partial (u_i \tau_{ij})}{\partial x_j} - \tau_{ij} \frac{\partial u_i}{\partial x_j} + u_i f_i}$$

Derivation of kinetic energy equation (cont)

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \frac{\partial (u_i p)}{\partial x_i} + \frac{\partial (u_i \tau_{ij})}{\partial x_j} - \tau_{ij}\frac{\partial u_i}{\partial x_j} + u_i f_i$$

See section 2.1.4 in Davidson

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \frac{\partial (u_i p)}{\partial x_i} + \frac{\partial (u_i \tau_{ij})}{\partial x_j} - 2\mu S_{ij}S_{ij} + u_i f_i$$

Exercise

$$\tau_{ij}\frac{\partial u_i}{\partial x_j} = \tau_{ij}S_{ij} + \tau_{ij}\Omega_{ij} = \tau_{ij}S_{ij} = 2\mu S_{ij}S_{ij}$$

Sym x Anti-sym =0

$$\tau_{ij}\Omega_{ij} = -\tau_{ji}\Omega_{ji} = -\tau_{km}\Omega_{km}$$

Rename again: $k = i, m = j$

$$\Rightarrow \tau_{ij}\Omega_{ij} = -\tau_{ij}\Omega_{ij}$$

$$\Rightarrow \tau_{ij}\Omega_{ij} = 0$$

1) After integration over a control volume (CV) the first two terms on RHS corresponds to rate of work on the CV.

2) The third term corresponds to rate of internal conversion of mechanical energy into heat. The term is negative which means that it decreases the kinetic energy.

Dissipation of energy per unit mass: $\varepsilon_{total} = 2\nu S_{ij}S_{ij}$

$\nu = \dfrac{\mu}{\rho}$ Kinematic viscosity

Vorticity, Kinetic energy and Enstrophy

Navier-Stokes equations

$$\boxed{\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{V} + \mathbf{f}}$$

The vorticity equation

$$\boxed{\rho \frac{D\boldsymbol{\omega}}{Dt} = \rho(\boldsymbol{\omega}\cdot\nabla)\mathbf{V} + \mu\nabla^2\boldsymbol{\omega}}$$

Equation for Kinetic energy $KE \equiv \int_{Vol} \tfrac{1}{2}V^2 dr$

1. Multiply NS with velocity

2. Use $\mathbf{V}\cdot\frac{D\mathbf{V}}{Dt} = \frac{D}{Dt}\left(\frac{\mathbf{V}^2}{2}\right)$

3. Use $\mathbf{V}\cdot\nabla p = \nabla\cdot(\mathbf{V}p)$ $(\nabla\cdot\mathbf{V}=0)$

$$\left\{\begin{array}{l} (\mathbf{V}\cdot\nabla)\mathbf{V} = \nabla\left(\tfrac{1}{2}V^2\right) - \mathbf{V}\times\nabla\times\mathbf{V} \\ \nabla^2\mathbf{V} = -\nabla\times\boldsymbol{\omega} \\ \mathbf{V}\cdot\nabla\times\boldsymbol{\omega} = \boldsymbol{\omega}^2 + \nabla\cdot[\mathbf{V}\times\boldsymbol{\omega}] \end{array}\right.$$

4. Integrate over a fixed domain with **no-slip**

➡ $\boxed{\frac{d}{dt}KE = -2\nu Z}$ $Z \equiv \int_{Vol} \tfrac{1}{2}\omega^2 dr$

Equation for Enstrophy $Z \equiv \int_{Vol} \tfrac{1}{2}\omega^2 dr$

1. Multiply Vorticity eq with vorticity

2. Use identities…

$$\frac{D}{Dt}\left(\tfrac{1}{2}\omega^2\right) = \boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\mathbf{V} - \nu(\nabla\times\boldsymbol{\omega})^2 + \nu\nabla\cdot[\boldsymbol{\omega}\times\nabla\times\boldsymbol{\omega}]$$

3. Integrate over a fixed domain with no-slip

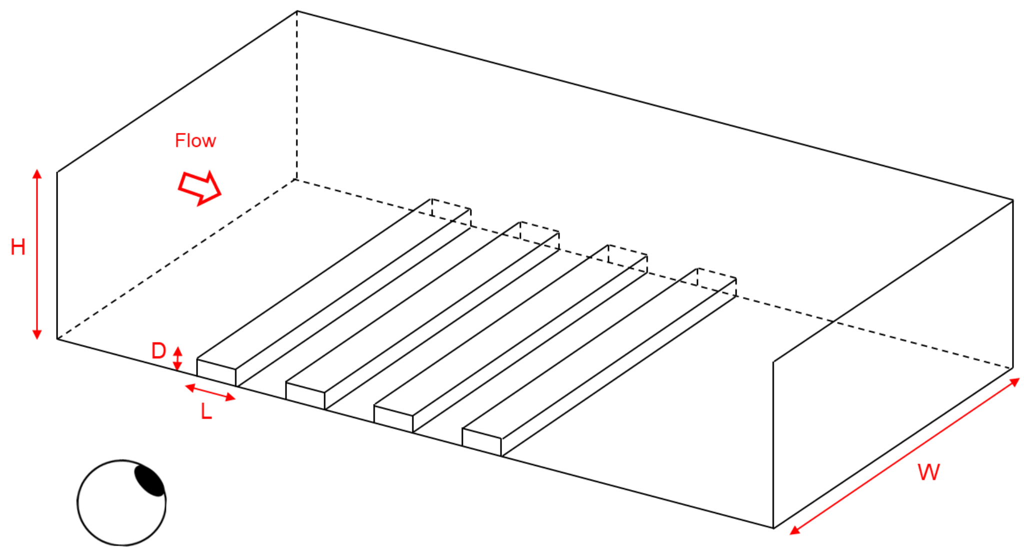
$$\boxed{\frac{dZ}{dt} = \int_{Vol}\left[\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\mathbf{V}\right]dr - \nu\int_{Vol}(\nabla\times\boldsymbol{\omega})^2\, dr}$$

Will be used in Lecture 6

Group discussion exercise

In the figure below a schematic of a reaction chamber is presented (details not in scale). The chamber is supposed to increase mixing rates near the corrugated surface (the small boxes) which should speed up chemical reactions in that region. We want to analyze how the flow behaves both near the corrugated surface, and a distance further out in the core flow.

Make a 2D sketch of the flow as seen by the indicated eye ball, and discuss the dynamics of velocity and vorticity.



Boundary conditions for fluids

Fluid-rigid body interface

Rigid body velocity

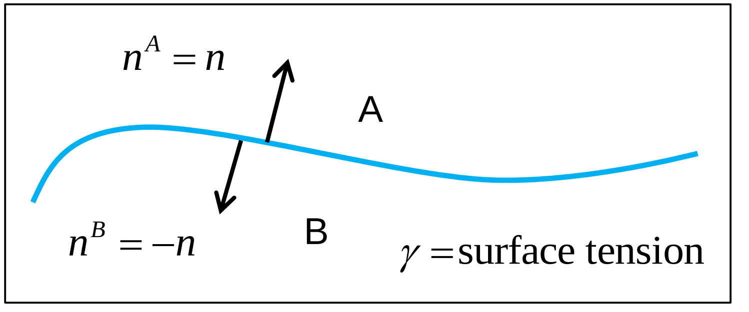
No mass transfer through surface: $\mathbf{u}\cdot\mathbf{n}=\mathbf{V}\cdot\mathbf{n}$

No slip at surface: $\mathbf{u}-(\mathbf{u}\cdot\mathbf{n})\mathbf{n}=\mathbf{V}-(\mathbf{V}\cdot\mathbf{n})\mathbf{n}$

$\Rightarrow$ $\mathbf{u}=\mathbf{V}$ at surface

Fluid-fluid interface

In a fluid-fluid surface case we have doubled the number of unknows compared to fluid-rigid surface case, and we need to double the number of BC at the interface.



Continuity of tangential velocity:

$$\mathbf{u}^A-(\mathbf{u}^A\cdot\mathbf{n})\mathbf{n}=\mathbf{u}^B-(\mathbf{u}^B\cdot\mathbf{n})\mathbf{n}$$

Continuity of normal velocity:

$$\mathbf{u}^A\cdot\mathbf{n}=\mathbf{u}^B\cdot\mathbf{n}$$

This is not enough, we need more BC:s…

Stress boundary conditions at fluid-fluid surfaces

Fluid stress is described by

$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$ and $\tau_{ij} = \mu(u_{i,j} + u_{j,i})$

The derivation comes later

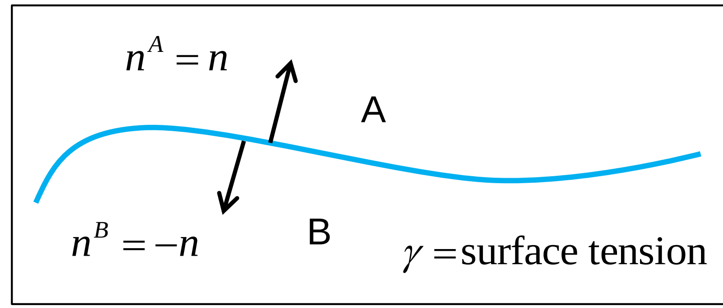
Stress balance between two fluids

$$\boxed{\sigma_{ij}{}^{A} n_j - \sigma_{ij}{}^{B} n_j = \gamma n_i \frac{\partial n_j}{\partial s_j} - \frac{\partial \gamma}{\partial s_i}}$$

Normal to surface

Measures the curvature

Tangential to surface



Normal stress balance:
(Multiplication by n_i)

$$n_i\sigma_{ij}{}^{A} n_j - n_i\sigma_{ij}{}^{B} n_j = \overset{=1}{\overbrace{n_i n_i}}\,\gamma \frac{\partial n_j}{\partial x_j} - n_i \frac{\partial \gamma}{\partial s_i}$$

Tangential stress balance:
(Multiplication by s_i)

$$s_i\sigma_{ij}{}^{A} n_j - s_i\sigma_{ij}{}^{B} n_j = \underset{=0}{\underbrace{s_i n_i}}\,\gamma \frac{\partial n_j}{\partial x_j} - s_i \frac{\partial \gamma}{\partial s_i}$$

A common special case

$$\frac{\partial \gamma}{\partial s_i} = 0$$

Ex.
Maragoni effect

$$\frac{\partial \gamma}{\partial s_i} \neq 0$$

Example: Liguid-gas interface

$$\left(\sigma_{ij}^A - \sigma_{ij}^B\right)n_j = \gamma n_i \left(\frac{\partial n_j}{\partial s_j}\right) - \frac{\partial \gamma}{\partial s_i} \qquad \sigma_{ij} = -p\delta_{ij} + \tau_{ij} \qquad \boxed{\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$$

Example 1: Consider a liquid-gas interface with constant surface tension (a common case): $\frac{\partial \gamma}{\partial s_i} = 0$

Multiplying the surface stress condition with the tangent vector s_i $\Rightarrow$ $s_i \tau_{ij}^A n_j = s_i \tau_{ij}^B n_j$ $\quad \left(s_i \delta_{ij} n_j = s_i n_i = 0\right)$

Ex. A flat interface with $s_i = \hat{\mathbf{x}}\ ,\ n_i = \hat{\mathbf{z}}:$ $\quad \tau_{xz}^A = \tau_{xz}^B$

Example 2: Assuming no viscous contribution (no shear stress) the normal component of stress (pressure) must satisfy:

$$n_i p^B \delta_{ij} n_j - n_i p^A \delta_{ij} n_j = \gamma \underbrace{\left(\frac{\partial n_j}{\partial s_j}\right)}_{\text{Curvature}} \Rightarrow p^B - p^A = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \underbrace{=}_{\text{For a sphere}} \frac{2\gamma}{R}$$

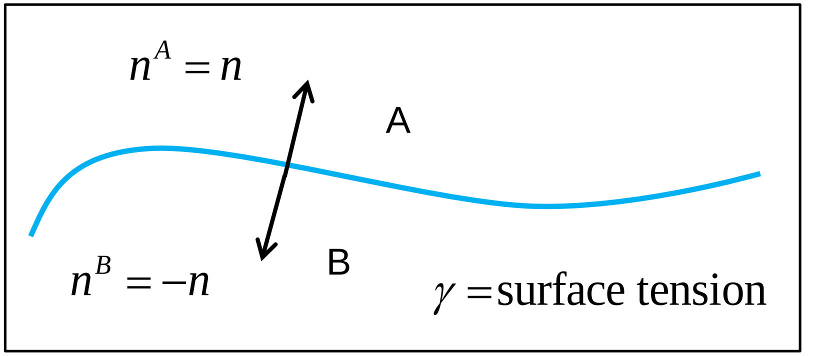
A small bubble has higher inner pressure than a large one.

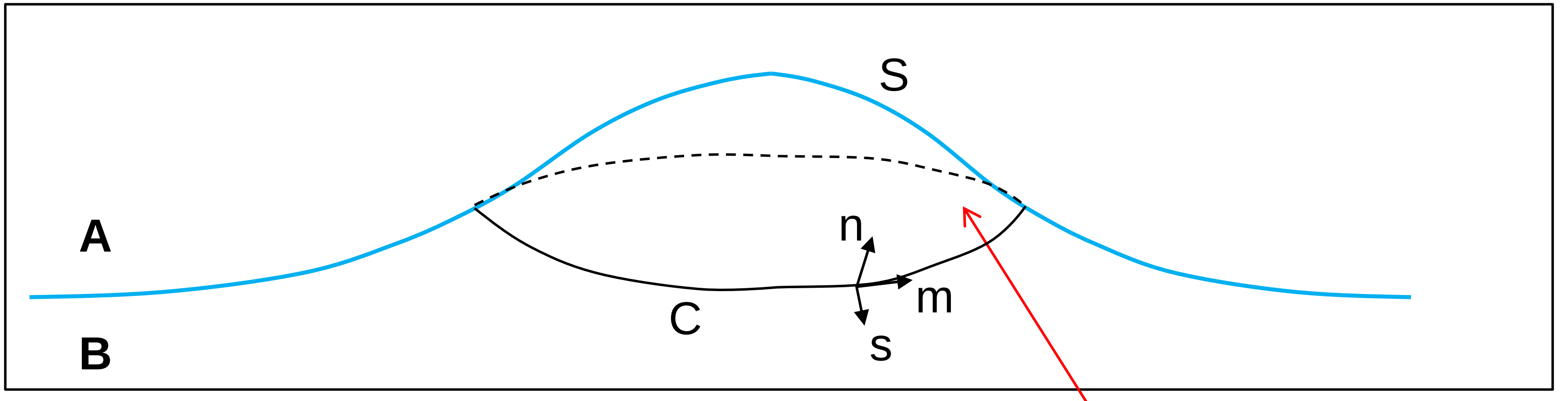
Derivation of stress conditions at surfaces

Extra material. Not mandatory

Show that the stress balance between the two fluids is:

$$\sigma_{ij}{}^{A} n_j - \sigma_{ij}{}^{B} n_j = \gamma n_i \frac{\partial n_j}{\partial x_j} - \frac{\partial \gamma}{\partial x_i}$$



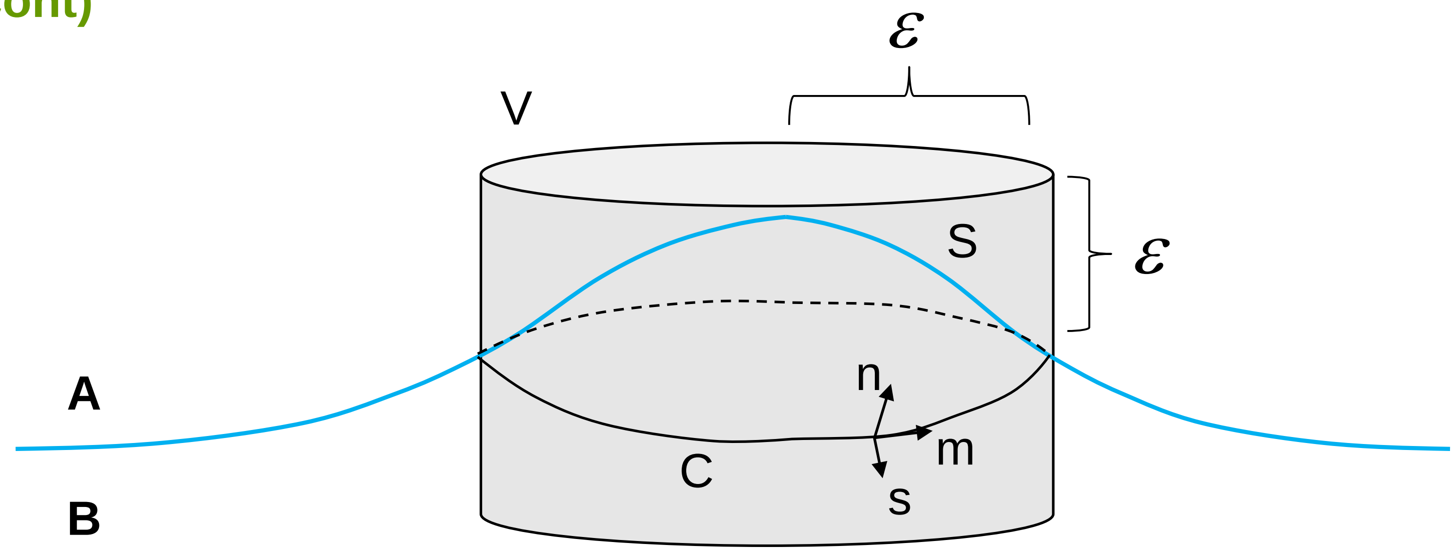


n = outward normal to surface S
m = tangent to curve C
s = normal to C and tangent to S

The surface tension acts in s-direction to make the surface S flat

Derivation of stress conditions at surfaces (cont)

Extra material. Not mandatory



Newtons 2:nd law

$$\underbrace{\int_V \rho \frac{Du_i}{Dt} dV}_{\sim \varepsilon^3} = \underbrace{\int_S \left[\sigma_{ij}^A n_j^A + \sigma_{ij}^B n_j^B \right] dS + \int_C \gamma s_i dl}_{\sim \varepsilon^2} + \underbrace{\int_V f_i dV}_{\sim \varepsilon^3}$$

Surface tension

$$[\gamma] = \frac{N}{m}$$

Derivation of stress conditions at surfaces (cont)

Extra material. Not mandatory

$$\underbrace{\int_V \rho \frac{Du_i}{Dt} dV}_{\sim \varepsilon^3} = \underbrace{\int_S \left[\sigma_{ij}^A n_j^A + \sigma_{ij}^B n_j^B \right] dS + \int_C \gamma s_i dl}_{\sim \varepsilon^2} + \underbrace{\int_V f_i dV}_{\sim \varepsilon^3}$$

Surface tension represents a force applied tangent to the interface

In the limit of ε→0 the acceleration term and body force term goes to zero faster than the surface terms. The surface terms must thus balance each other.

Convert line intergral to surface integral
Stokes Theorem + vector identities…

$$\int_C \gamma s_i dl = \ldots = \int_S \left[\frac{\partial \gamma}{\partial s_i} - \gamma n_i \left(\frac{\partial n_j}{\partial s_j} \right) \right] dS$$

$$\boxed{\frac{\partial}{\partial s_i} \equiv \frac{\partial}{\partial x_i} - n_i n_j \frac{\partial}{\partial x_j}}$$

$$0 = \int_S \left[\sigma_{ij}^A n_j^A + \sigma_{ij}^B n_j^B + \frac{\partial \gamma}{\partial s_i} - \gamma n_i \left(\frac{\partial n_j}{\partial s_j} \right) \right] dS$$

Twice the mean curvature of the surface

Surface element arbitrary

$$\boxed{\sigma_{ij}^A n_j^A + \sigma_{ij}^B n_j^B = \gamma n_i \left(\frac{\partial n_j}{\partial s_j} \right) - \frac{\partial \gamma}{\partial s_i}}$$

End of lecture