Computational Fluid Dynamics

Introduction to Turbulence II

Lecture 7

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SUMMARY OF LECTURE: INTRODUCTION TO TURBULENCE

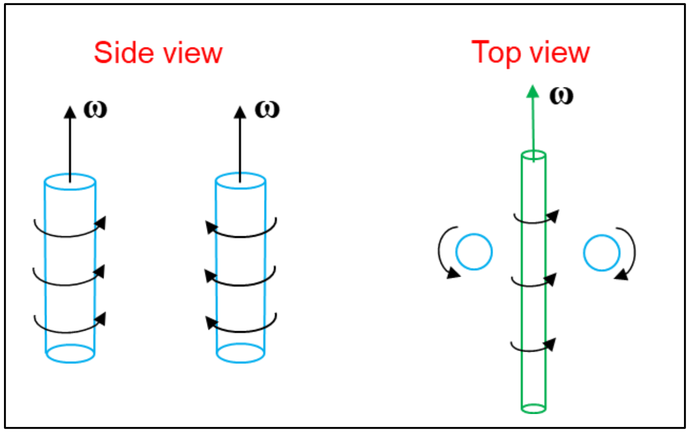
- Understand the origin of Kolmogorov length, time and velocity scales (the derivation)
- Understand the connection between Kolmogorov's second hypothesis and his famous "-5/3-law"
- Be aware of the three basic numerical approaches to turbulence, DNS, LES and RANS
- Be able to describe the process of deriving RANS including
  - Reynolds decomposition
  - Average of equations
- Be able to given a physical interpretation of each terms in RANS equation
- Understand the Boussinesq assumption and its importance to the Closure problem

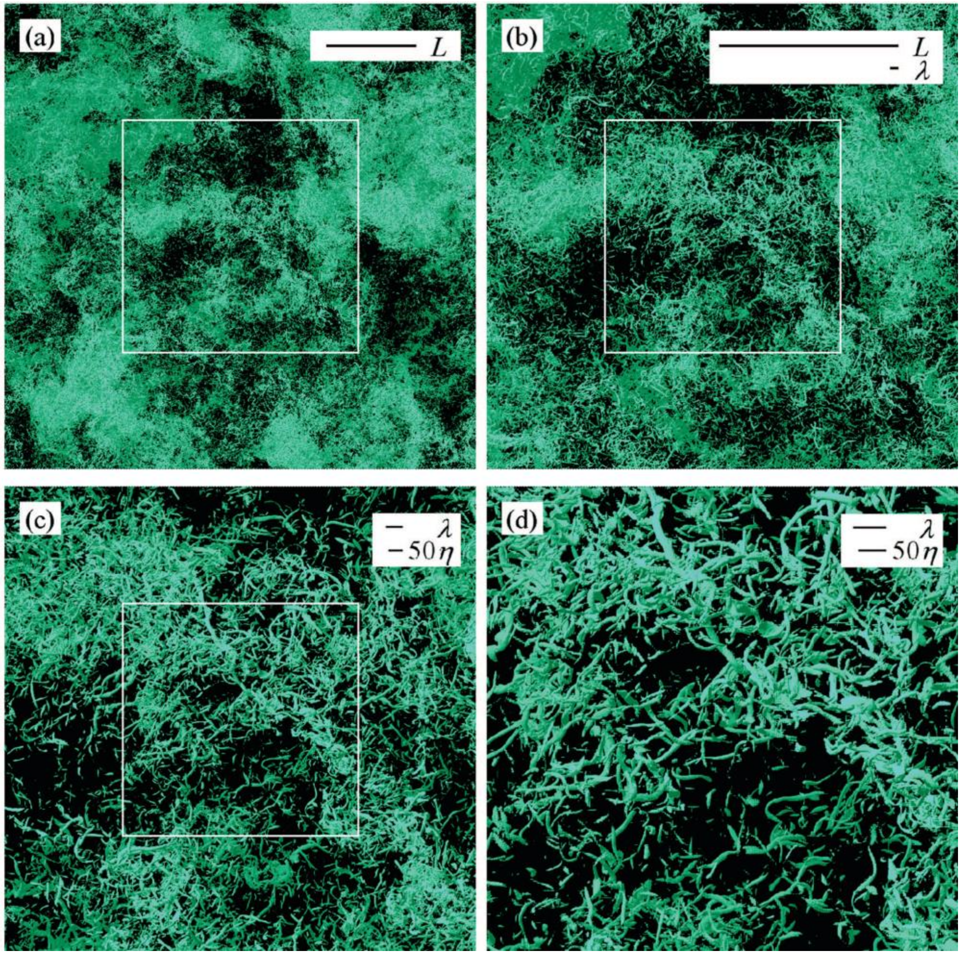
From: ”**Coherent vortices in high resolution direct numerical simulation of homogeneous isotropic turbulence: A wavelet viewpoint**”, N. Okamoto, K. Yoshimatsu

”We observe that the coherent vorticity, represented by 2.6%*N* wavelet coefficients, retains 99.8% of the energy and 79.8% of the enstrophy.”

Describing turbulence as interacting vortex tubes is a valid approximation

Example: Stretching large vortices in one direction generates a cascade of small scale stretching in random directions.





Different zooms of isosurfaces of vorticity

Kolmogorov theory 1941

K1. Kolmogorov's hypothesis of local isotropy
At sufficiently high Reynolds numbers, the **small-scale** turbulent motions are statistically isotropic.

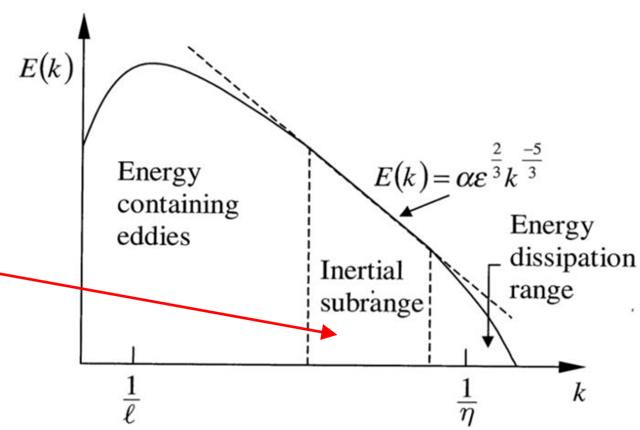
- Isotropic = tha same in all directions
- Large scale motion usually anisotropic

K2. Kolmogorov's first similarity hypothesis
In every turbulent flow at sufficiently high Reynolds number, the description of the **small scale** motions have a universal form that is uniquely determined by the *energy dissipation rate* and *viscosity*

- Universal form, see next slide
- U, L and T can be written in terms of ε and v

K3. Kolmogorov's second similarity hypothesis
In every turbulent flow at sufficiently high Reynolds number, the description of the motions of **intermediate scale** have a universal form that is uniquely determined by the *energy dissipation rate* only.

$$E(k) = \alpha \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$



Kolmogorov theory 1941 (cont)

K2: Kolmogorov micro scales

Time: $T_K = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$

Velocity: $U_K = (\varepsilon\nu)^{1/4}$

Length: $L_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

Normalization using Kolmogorov scales

$$E(k) \rightarrow E(k)/\left(U_K^2 L_K\right) = E(k)/\left(\varepsilon\nu^5\right)^{1/4}$$
$$k \rightarrow kL_K$$

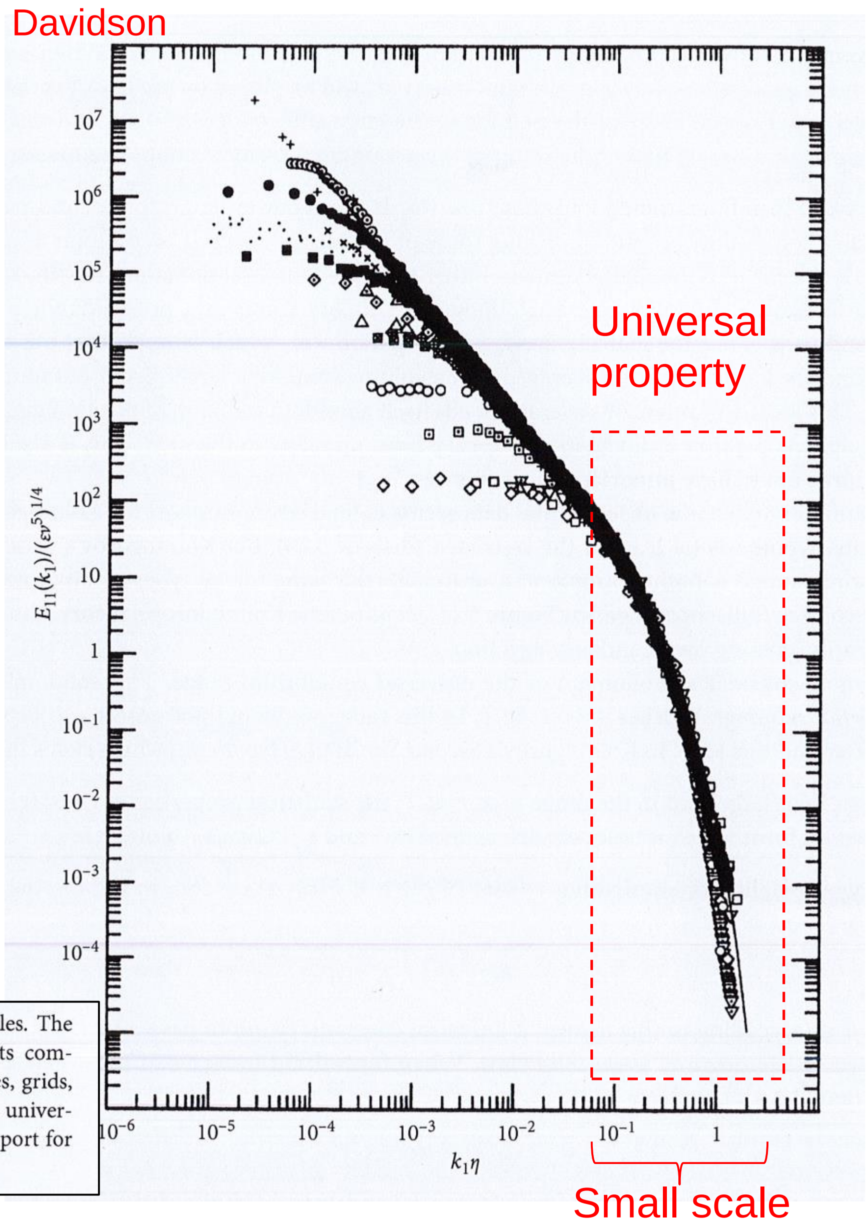


Figure 5.17 Energy spectrum versus wavenumber normalized by the Kolmogorov scales. The data is taken from Saddoughi and Veeravalli (1994) and incorporates measurements compiled from many experiments including measurements made in boundary layers, wakes, grids, ducts, pipes, jets, and the oceans. All of the data corresponding to $k\ell \gg 1$ fits on a universal curve when E and k are normalized by the Kolmogorov scales. This gives direct support for Equation (5.20) and Kolmogorov's universal equilibrium theorem.

Turbulence scales

Kolmogorov 2: Small scale eddies are controlled by energy dissipation rate and viscosity: ε, ν

1. Energy dissipation rate per unit mass: $\varepsilon \sim \frac{d}{dt}\left(u^2\right)$

$$\Rightarrow \quad \varepsilon \sim \frac{U_K{}^2}{T_K} = \frac{U_K{}^2}{L_K / U_K} = \frac{U_K{}^3}{L_K}$$

2. Viscosity important at small scales:

$$\mathrm{Re}_K = \frac{U_K L_K}{\nu} \sim 1 \quad \Rightarrow \quad U_K = \frac{\nu}{L_K}$$

$$U_K = \frac{\nu}{L_K}, \quad \varepsilon = \frac{U_K{}^3}{L_K} \quad \Rightarrow \quad \boxed{U_K = \left(\varepsilon\nu\right)^{1/4}}$$

$$\begin{cases} L_K = \dfrac{\nu}{U_K} \Rightarrow & L_K = \left(\dfrac{\nu^3}{\varepsilon}\right)^{1/4} \\ T_K = \dfrac{L_K}{U_K} \Rightarrow & T_K = \left(\dfrac{\nu}{\varepsilon}\right)^{1/2} \end{cases}$$

We want to show this:

Kolmogorov micro scales

Time: $T_K = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$

Velocity: $U_K = \left(\varepsilon\nu\right)^{1/4}$

Length: $L_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

We need to connect the dissipation rate to something measurable…

Energy cascade: Quasi-equilibrium

At large scales we have:

$$\boxed{\mathrm{Re} = \frac{UL}{\nu}} \qquad \boxed{\varepsilon_{\text{large}} \sim \frac{U^3}{L}}$$

$$\Rightarrow \quad \varepsilon_{\text{large}} \sim \frac{\nu^3 \mathrm{Re}^3}{L^4}$$

Kolmogorov micro scales

Time: $T_K = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$

Velocity: $U_K = (\varepsilon\nu)^{1/4}$

Length: $L_K = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

Assuming that the system is in a quasi-equilibrium during energy cascade we have:

$$\varepsilon_{small} = \frac{\nu^3}{L_K{}^4}$$

$$\varepsilon_{\text{large}} \sim \varepsilon_{\text{small}} \quad \Rightarrow \quad \frac{\nu^3 \mathrm{Re}^3}{L^4} \sim \frac{\nu^3}{L_K{}^4}$$

$$\boxed{\frac{L_K}{L} \sim \mathrm{Re}^{-3/4} \qquad \frac{T_K}{T} \sim \mathrm{Re}^{-1/2} \qquad \frac{U_K}{U} \sim \mathrm{Re}^{-1/4}}$$

Examples

What is the dissipation length scale?

$$\left.\begin{array}{l} \mathrm{Re} \sim 10^5 \\ L \sim 10^{-2}\, m \end{array}\right\} \Rightarrow L_K = L\,\mathrm{Re}^{-3/4} \sim 2\mu m$$

This length scale must be resolved in a simulation!

$$\frac{L}{L_K} = \mathrm{Re}^{3/4}$$

Number of grid points

$$\boxed{N \equiv \frac{L}{L_K} = 5000, \quad N^3 = 1.25 \cdot 10^{11}}$$

$$\frac{T_K}{T} = \mathrm{Re}^{-1/2} \Rightarrow T_K = \frac{T}{\sqrt{\mathrm{Re}}} \xrightarrow{\text{High Re}} T_K << T$$

The vortices at Kolmogorov scale has a higher frequency, they spin faster. They also have shorter life time, since when they spin kinetic energy is converted to heat through viscosity.

Lecture 4: Dissipation rate per unit mass: $\varepsilon = 2\nu S_{ij} S_{ij}$ $\quad S_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$\left.\begin{array}{l} S_{ij} \sim \dfrac{U}{L} = \dfrac{1}{T} \\ S_{ij}^K \sim \dfrac{U_K}{L_K} = \dfrac{1}{T_K} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \varepsilon \sim \nu T^{-2} \\ \varepsilon_K \sim \nu T_K^{-2} \end{array}\right\} \xrightarrow{\text{High Re}} \varepsilon << \varepsilon_K$$

The dissipation is dominated by the small scale vortices

Comment:
Here we compare the **viscous dissipation** at large and small scales, compared to previous slide where the dissipation at large scale also included vortex brakedown to smaller scale.

$$\varepsilon_{\text{large}} \sim \varepsilon_{\text{small}}$$

Turbulence modeling (and simulation)

Navier-Stokes equations

$$\rho \frac{\partial u_i}{\partial t} + \rho \left(u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + f_i$$

- N-S describes the total motion of a fluid, including turbulence
- By using an extremely fine mesh/grid the fluid motion at small scales can be resolved (Direct Numerical Simulation = DNS)
- Fine mesh => Huge computational cost => Fast computers and efficient algorithms are necessary

DNS can be viewed as a virtual experiment, sometimes even more exact than real world experiment.

Example:
Experimental probes might disturb the fluid of interest => large error in measurements

DNS produces a lot of data that can be post-processed, but the data contains all information we need.

An “engineering approach” to turbulence modeling

<u>Assumption</u>:
Exact representation of turbulence has to be weighted against computational effort

1. Use models (ODE, PDE or analytical expressions) for some parts of the fluid (e.g. for small scales)

2. Use experimental data (or DNS data) to find values of unknown parameters in models

3. Solve Navier-Stokes for the rest of fluid

Three simulation strategies to turbulence

- Direct Numerical Simulation (DNS)
- Large-Eddy Simulation (LES)
- Reynolds Averaged NS (RANS) ← This course

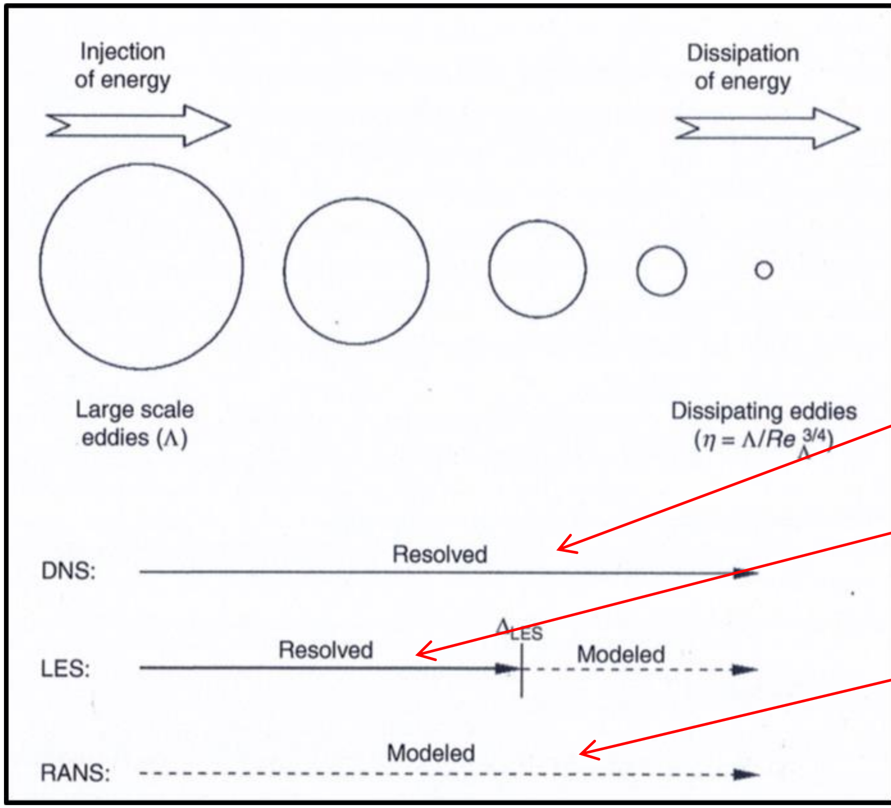


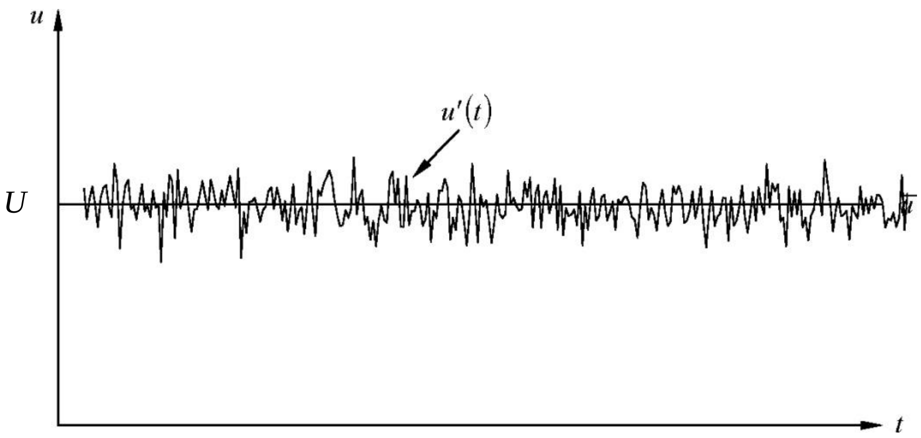
Fig 5.2 in "Basics of engineering turbulence" by D. Ting

Fine mesh for the whole domain and solve Navier-stokes

Solve Navier-Stokes on a part of the domain

Solve the **averaged** Navier-Stokes on the whole domain

Reynolds Averaged NS (RANS)



Example of data measured at a fixed position in a fluid flow

$$u = U + u'$$

Statistical averaging tools

- Time averages $\langle f \rangle(\mathbf{r}) = \lim_{T\to\infty}\left[\frac{1}{T}\int_t^{t+T} f(\vec{\mathbf{r}},t)dt\right]$
- Spatial averages $\langle f \rangle(t) = \lim_{V\to\infty}\left[\frac{1}{V}\iiint f(\vec{r},t)dV\right]$
- Ensemble averages $\langle f \rangle(\mathbf{r},t) = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=1}^{N} f_n(\mathbf{r},t)\right]$

Reynolds decomposition

$$\boxed{\begin{aligned} u_i &= U_i + u_i' \\ p &= P + p' \\ \tau_{ij} &= T_{ij} + \tau_{ij}' \end{aligned}}$$

Instantaneous values: $u_i\,, p\,, \tau_{ij}$

Mean values: $U_i\,, P\,, T_{ij}$

Fluctuation values: $u_i'\,, p'\,, \tau_{ij}'$

"Time average" of slow time-scale

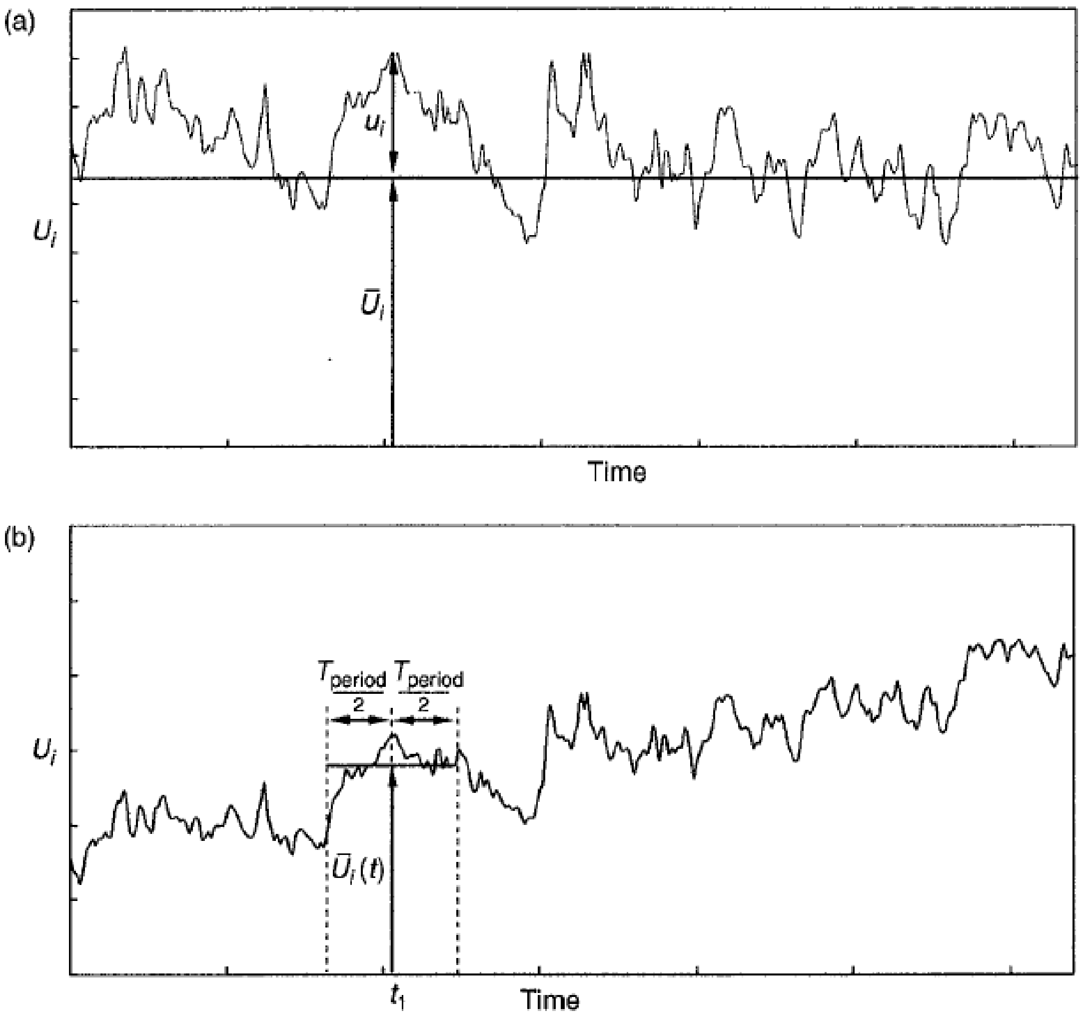


Fig 2.4 in "Basics of engineering turbulence" by D. Ting

$$u = U + u'$$

$$\langle u \rangle (\mathbf{r}, t_k) = \frac{1}{T_p} \int_{t_k - T_p/2}^{t_k + T_p/2} u(t) dt$$

Average properties

$$\langle u_i' \rangle = 0, \ \langle U_i \rangle = U_i$$
$$\langle U_i u_j' \rangle = 0, \ \langle u_i' u_j' \rangle \neq 0$$

See full list in Celik p.7-8

Summary of averaging rules

From "Statistical fluid mechanics: Mechanics of turbulence Vol I"
A. S. Monin and A. M. Yaglom 1971

The Reynolds conditions

$$\langle f+g\rangle = \langle f\rangle + \langle g\rangle$$

$$\langle af\rangle = a\langle f\rangle, \quad a = const$$

$$\langle a\rangle = a$$

$$\left\langle \frac{\partial f}{\partial s}\right\rangle = \frac{\partial\langle f\rangle}{\partial s}, \quad s = x, y, z \text{ or } t$$

$$\langle\langle f\rangle g\rangle = \langle f\rangle\langle g\rangle$$

$$\langle\langle f\rangle\rangle = \langle f\rangle$$

$$\langle f'\rangle = \langle f - \langle f\rangle\rangle = 0$$

$$\langle\langle f\rangle\langle g\rangle\rangle = \langle f\rangle\langle g\rangle$$

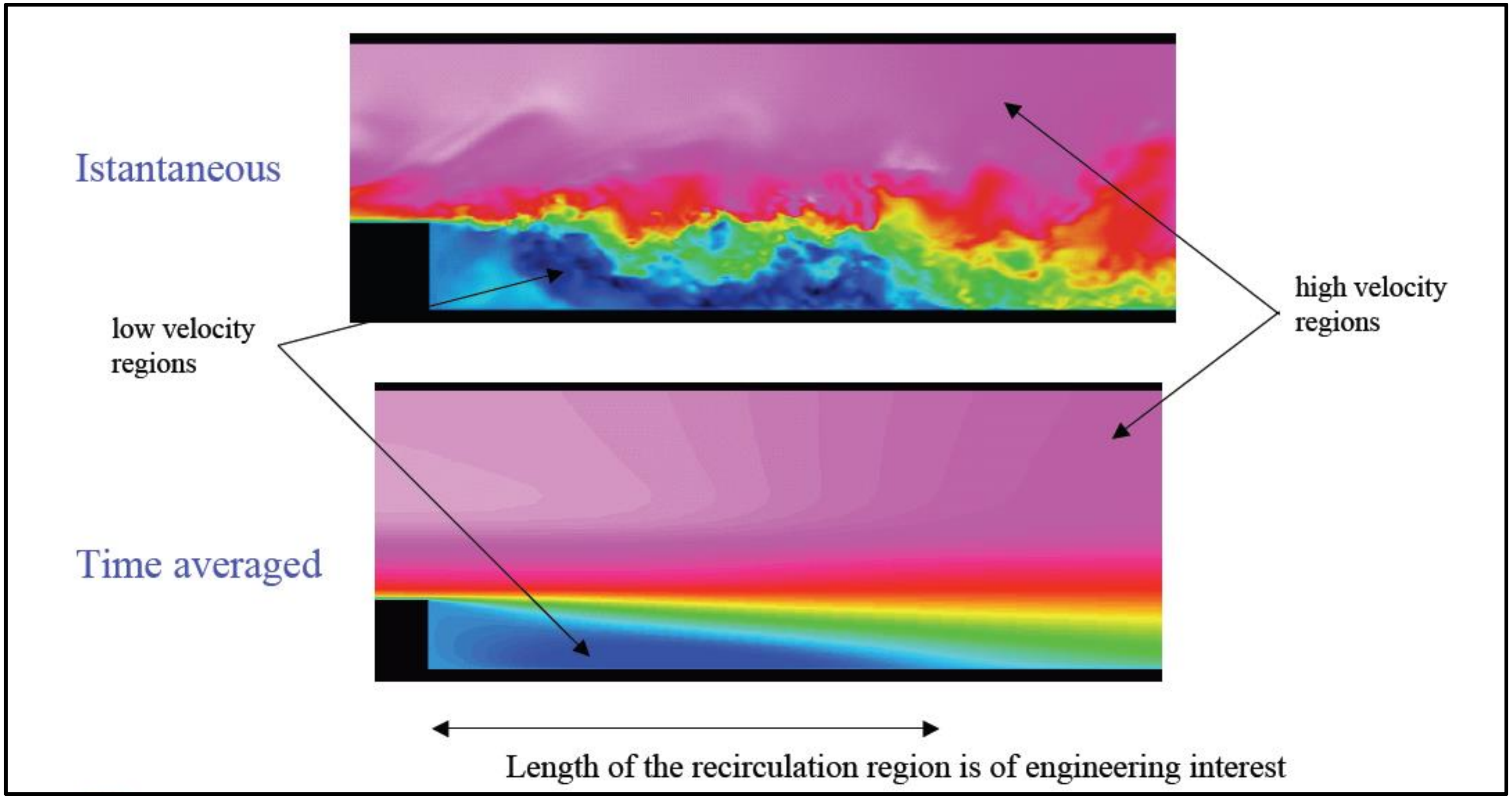
$$\langle\langle f\rangle g'\rangle = \langle f\rangle\langle g'\rangle = 0$$

Reynolds decomposition

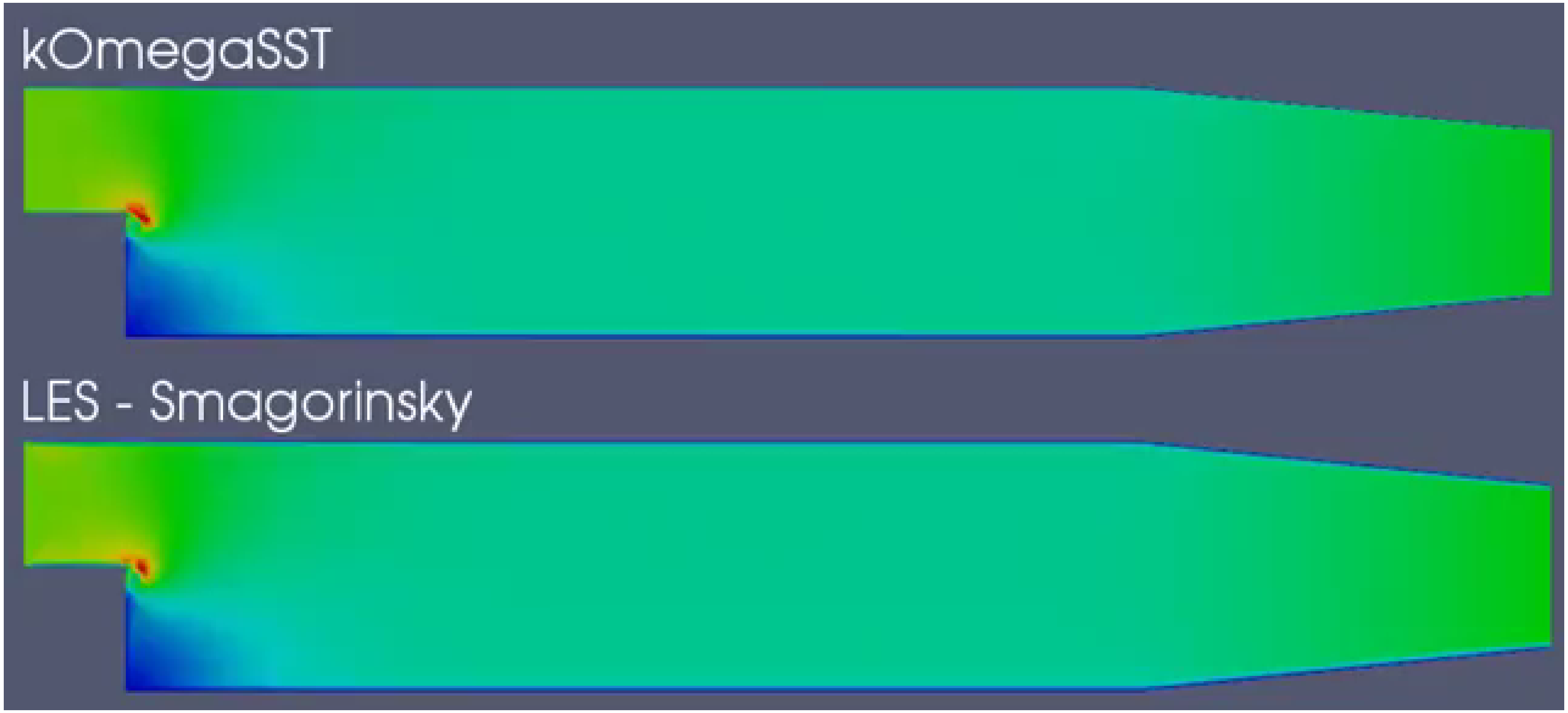
$$f = \langle f\rangle + f'$$

For time or spatial averages we must assume separated time or spatial scales of mean flow and fluctuations. Not always satisfied exact but often assumed in approximative sense.

Example: Time averaged fluid motion



Example: RANS vs LES



Averaging process of NS

Study group exercise

$$\rho \frac{\partial u_i}{\partial t} + \rho \left(u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j} + f_i$$

Reynolds decomposition

$$u_i = U_i + u_i'$$
$$p = P + p'$$
$$\tau_{ij} = T_{ij} + \tau_{ij}'$$

Trick:

$$\frac{\partial (u_j u_i)}{\partial x_j} = u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_j} = u_j \frac{\partial u_i}{\partial x_j}$$

$$LHS = \rho \frac{\partial (U_i + u_i')}{\partial t} + \rho \frac{\partial}{\partial x_j}(U_j + u_j')(U_i + u_i')$$

$$= \rho \frac{\partial (U_i + u_i')}{\partial t} + \rho \frac{\partial}{\partial x_j}(U_i U_j + U_i u_j' + U_j u_i' + u_i' u_j')$$

Next: Apply average on LHS and use properties of averaging integrals

For a detailed derivation see Appendix C in lecture notes “Introductory Turbulence Modeling” by Ismail B. Celik

Averaging process of NS (cont)

Study group exercise

$$LHS = \rho \frac{\partial \left(U_i + u_i' \right)}{\partial t} + \rho \frac{\partial}{\partial x_j} \left(U_i U_j + U_i u_j' + U_j u_i' + u_i' u_j' \right)$$

$$\langle u_i' \rangle = 0 , \langle U_i \rangle = U_i$$
$$\langle U_i u_j' \rangle = 0, \langle u_i' u_j' \rangle \neq 0$$

Average of LHS :

$$\langle LHS \rangle = \rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_j} \left(U_i U_j + \langle u_i' u_j' \rangle \right)$$

$$\left\langle \frac{\partial \left(U_i + u_i' \right)}{\partial t} \right\rangle = \frac{\partial}{\partial t} \langle U_i \rangle + \frac{\partial}{\partial t} \langle u_i' \rangle$$

$$\langle LHS \rangle = \rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} + \rho U_i \underbrace{\frac{\partial U_j}{\partial x_j}}_{=0} + \rho \frac{\partial}{\partial x_j} \langle u_i' u_j' \rangle$$

$$\langle LHS \rangle = \boxed{\rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} + \rho \frac{\partial}{\partial x_j} \langle u_i' u_j' \rangle}$$

Applying average on RHS and some operations give us the RANS equation…

Averaging process of NS (cont)

$$RHS = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j} + f_i \;\Rightarrow\; \langle RHS \rangle = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\frac{\partial U_i}{\partial x_j} + \langle f_i \rangle$$

$$\langle LHS \rangle = \langle RHS \rangle \;\Rightarrow\; \rho\frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} + \boxed{\rho \frac{\partial}{\partial x_j}\langle u_i' u_j' \rangle} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\frac{\partial U_i}{\partial x_j} + \langle f_i \rangle$$

$$\rho\frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\frac{\partial U_i}{\partial x_j} + \langle f_i \rangle - \rho \frac{\partial}{\partial x_j}\langle u_i' u_j' \rangle$$

Mean viscous stress

$$T_{ij} = \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$\boxed{\rho\frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}}{\partial x_j} + \frac{\partial R_{ij}}{\partial x_j} + \langle f_i \rangle}$$

Reynolds stress

$$R_{ij} \equiv -\langle \rho u_i' u_j' \rangle$$

RANS

The average of momentum equations becomes

$$\rho\left(\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}\right)=-\frac{\partial P}{\partial x_i}+\frac{\partial T_{ij}}{\partial x_j}+\frac{\partial R_{ij}}{\partial x_j}+\langle f_i\rangle$$

The average of continuity equations becomes

$$\frac{\partial U_k}{\partial x_k}=0 \qquad \frac{\partial u'_k}{\partial x_k}=0$$

Viscosity stress

$$\boxed{T_{ij}=\mu\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)}$$

Reynolds stress

$$\boxed{R_{ij}\equiv-\langle\rho u'_i u'_j\rangle}$$

$$\Downarrow$$

$$\boxed{RHS=\frac{\partial}{\partial x_j}\left(-p\delta_{ij}+T_{ij}+R_{ij}\right)}$$

Note!
Reynolds stress depends on the fluctuating velocities for which we do not have any governing equations…
The system is not closed

Closure using Boussinesq hypothesis

Assume a “simple” relationship between Reynolds stress and velocity gradients:

Turbulent kinetic energy

$$k \equiv \tfrac{1}{2}\langle u_i' u_i' \rangle$$

$$\boxed{R_{ij} = -\langle \rho u_i' u_j' \rangle = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij}}$$

Turbulent viscosity

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\frac{\mu + \mu_t}{\rho} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \right] + \langle f_i \rangle$$

Compare this turbulence approach with the modeling of ordinary molecular viscosity: $\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

Note:

$$P = P_{old} - 2\rho k$$

Note on Boussinesq assumption

Boussinesq assumption

$$\boxed{R_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij}}$$

?

The second term in Boussinesq makes it consistent with the definition of turbulent kinetic energy

Reynolds stress

$$R_{ij} \equiv -\langle \rho u_i' u_j' \rangle$$

Turbulent kinetic energy

$$k \equiv \tfrac{1}{2} \langle u_i' u_i' \rangle$$

$$\Downarrow$$

$$R_{ii} = -\rho \langle u_i' u_i' \rangle = -2\rho k$$

Test of Boussinesq assumption

$$R_{ii} = \mu_t \left(\frac{\partial U_i}{\partial x_i} + \frac{\partial U_i}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ii}$$

$$= 0 - \frac{2}{3} \rho k 3$$

$$= -2\rho k$$

Ok, since it is the same as what we get from pure definitions!

Summary of RANS

$$\frac{\partial R_{ij}}{\partial x_j} = ?$$

The RANS approach ends with a need of modeling the Reynolds stress term in the averaged Navier-Stoke equation.

1. **Boussinesq hypothesis models**
2. **Reynolds stress models (RSM)**

$$R_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij}$$

Further modelling

Zero-eq.

One-eq.

Two-eq.

This course

- Add six independent equations for the Reynolds stresses
- Derive their governing equations from NS
- Complicated and computational expensive

End of lecture