Computational Fluid Dynamics

Turbulence Models III

Lecture 14

Krister Wiklund
Department of Physics
Umeå University

SUMMARY OF LECTURE: TURBULENCE MODELS III

- Be able to describe how a boundary layer near a wall is divided into sublayers
- Understand how viscous stresses and turbulent stresses are related in viscous sublayer and log-layer
- Be able to describe how Kolmogorov theory is connected to near wall flows
- Be aware of different types of boundary conditions at solid surfaces

One aim for the course: Understand the equations in CFD-softwares

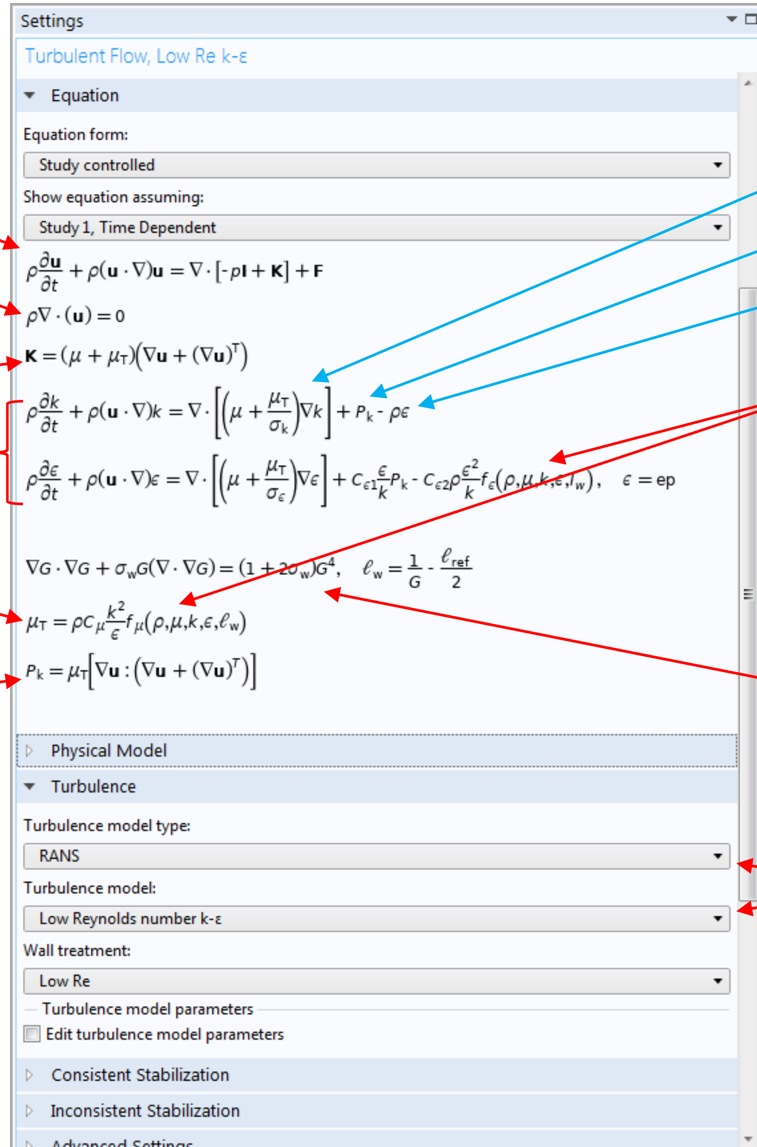
RANS

Incompressibility condition

Newtonian fluid + Boussinesq assumption

Turbulence model: *Low Re k-ε*

Turbulence production



Diffusion of k

Production of k

Dissipation of k

Damping function

Why is there only two damping functions?

Wall distance eq.

Turbulence model

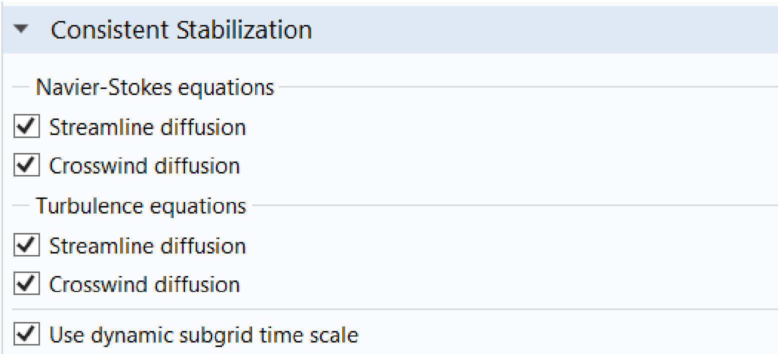
Wall treatment

"*Double dot product*"

$$(Matrix\,A):(Matrix\,B) \equiv A_{ij}B_{ij}$$

$$\nabla u : \nabla u = \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}$$

Comment on stabilization techniques

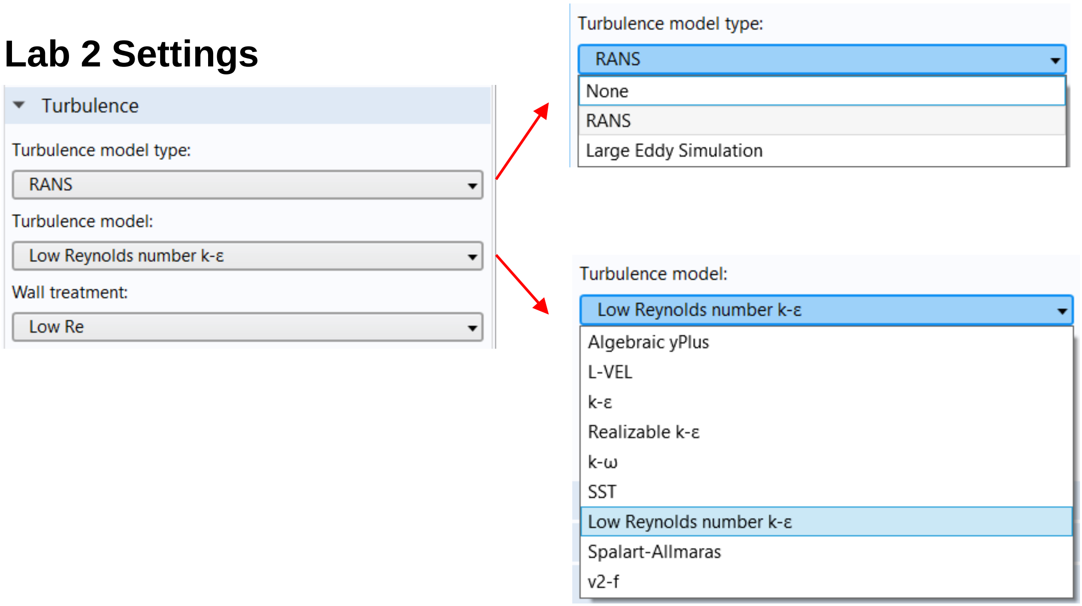


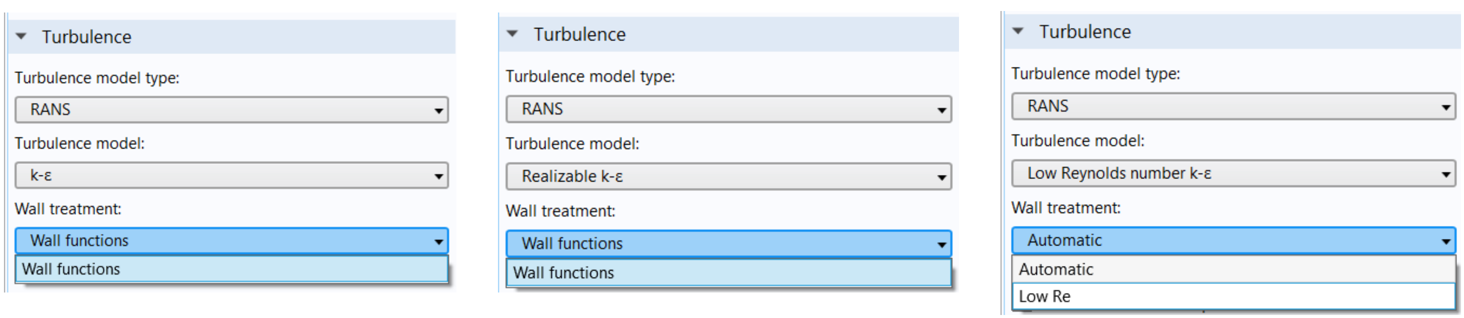
Streamline diffusion and Crosswind diffusion are essential in order to get stable simulations and correct results.

This was the reason for using “Heat transfer module” in Lab1, since the alternative was to add an PDE using the PDE-Interface module which does not have built-in stabilization for fluids.

COMSOL Blog
Understanding Stabilization Methods
https://www.comsol.com/blogs/understanding-stabilization-methods/

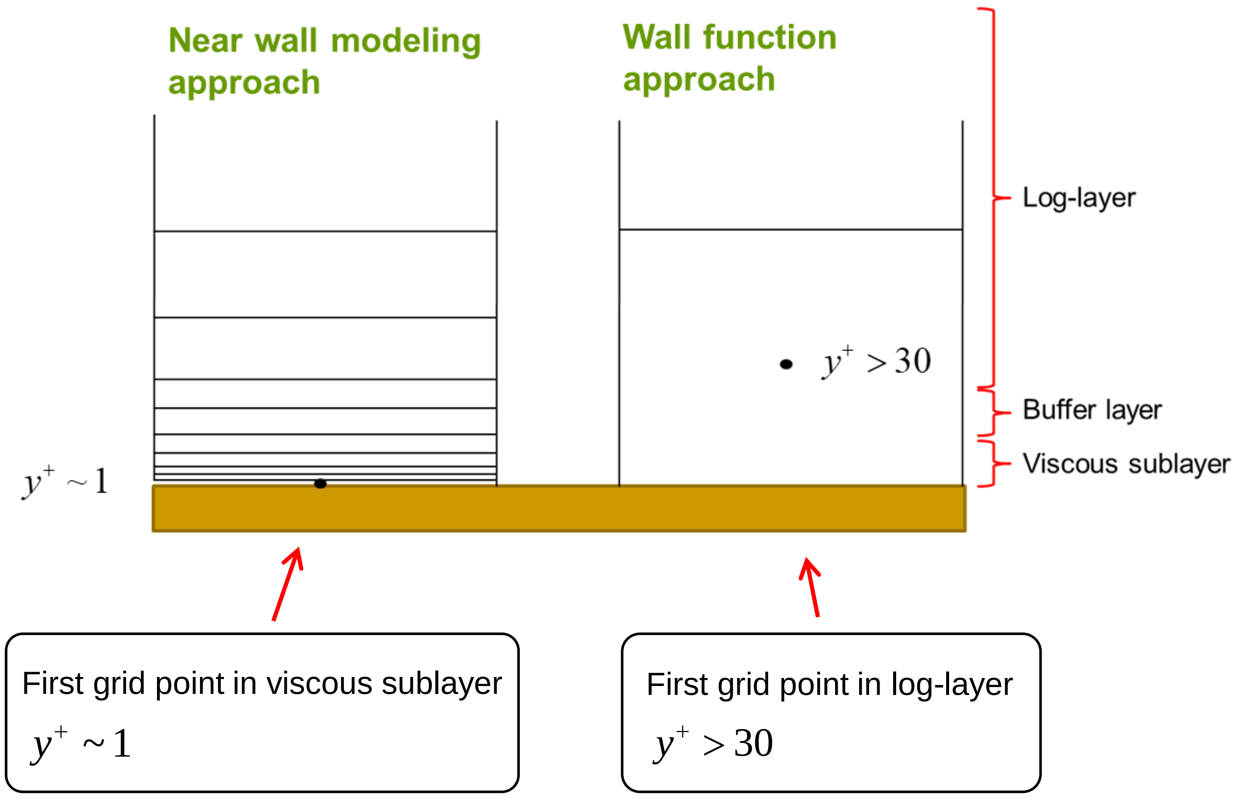
Lab 2 Settings





Wall treatment in Comsol:

1) Low Re ← Near wall modeling approach (or Damping function approach, AKN-model)
2) Automatic ← Uses Wall function approach for coarse mesh regions, and Near wall modeling approach for mesh that is fine enough. ⇨ Reduction in computational load



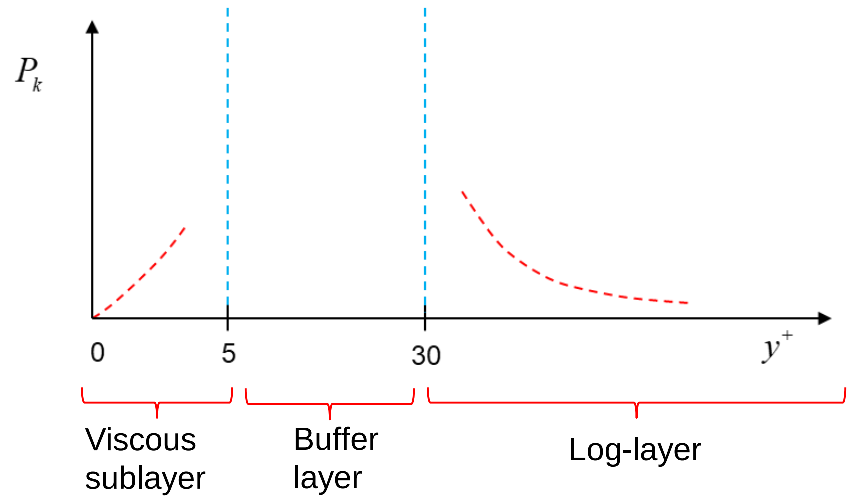
Recap: Log-layer properties in a 2D-BL

Wall function boundary conditions (log layer)

$$k = \frac{u_\tau^2}{\sqrt{C_\mu}} \qquad \varepsilon = \frac{u_\tau^3}{\kappa y} \qquad \boxed{P_k \approx \varepsilon \sim y^{-1}}$$

Near wall analysis in viscous sublayer

$$k \sim y^2 \qquad \varepsilon \sim y^0 \qquad \mu_t \sim y^3 \qquad \boxed{P_k \sim y^3}$$



- In the viscous sublayer the production goes to zero at the boundary
- In the log-layer the production decreases for large wall distances

➡ This means that the production has a maximum somewhere in the Buffer layer.

Recap: The Low-Re k-ε model equations

Model k-equation

$$\frac{Dk}{Dt} = P_k - \varepsilon + D_k$$

Model ε-equation

$$\frac{D\tilde{\varepsilon}}{Dt} = \frac{\tilde{\varepsilon}}{k}\left(f_1 C_{\varepsilon 1} P_k - f_2 C_{\varepsilon 2} \tilde{\varepsilon} \right) + D_\varepsilon + E$$

Turbulent viscosity

$$\mu_t = f_\mu C_\mu \rho \frac{k^2}{\tilde{\varepsilon}}$$

For some models: $\varepsilon = \tilde{\varepsilon} + D$ ➡ $\tilde{\varepsilon} = 0$ at boundary

Note: The AKN-model uses $E = 0, D = 0$

BC implemented in Comsol for dissipation: $\varepsilon = \dfrac{2\nu k}{y^2}$

k-production

$$P_k \equiv \frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j} = \frac{\mu_t}{\rho} S^2$$

k-diffusion

$$D_k \equiv \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

Dissipation

$$\varepsilon \equiv \frac{\mu}{\rho}\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle$$

ε-diffusion

$$D_\varepsilon \equiv \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

$C_\mu = 0.09$

$C_{\varepsilon 1} = 1.44$ $C_{\varepsilon 2} = 1.92$

$\sigma_k = 1.0$ $\sigma_\varepsilon = 1.3$

Additional comment on Boundary conditions

Details in Speziale etal 1990
"A critical evaluation of two-equation models for near wall turbulence"

Natural boundary conditions at a solid surface are: $\mathbf{u}=0,\ k=0,\ \varepsilon=?$

Model k-equation

$$\frac{Dk}{Dt}=P_k-\varepsilon+D_k$$

$$\boxed{D_k\equiv\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]}$$

At at a solid surface this equation becomes: $0=0-\varepsilon+D_k$

➡ A commonly used boundary condition: $\varepsilon=\dfrac{\mu}{\rho}\dfrac{\partial^2 k}{\partial y^2}$

The second order derivative can be numerically problematic and some use an alternative form: $\varepsilon=\dfrac{\mu}{\rho}\left(\dfrac{\partial\sqrt{k}}{\partial y}\right)^2$

An ad-hoc boundary condition have also been used: $\dfrac{\partial\varepsilon}{\partial y}=0$

DNS-Channel flow result at wall: $\dfrac{1}{\varepsilon^+}\dfrac{\partial\varepsilon^+}{\partial y^+}=-0.25$

Comsol referens manual:
This condition is numerically very unstable. Use instead the first term in a series expansion to obtain an analytical relation for the value at the first cell near walls:

$$\varepsilon=\frac{2\nu k}{y^2}$$

$$\left(k^+\equiv\frac{k}{u_\tau^2},\ \ \varepsilon^+\equiv\frac{\varepsilon\nu}{u_\tau^4}\right)$$

Group discussion: Research paper 3 reading exercise

"The prediction of laminarization with a two-equation model of turbulence"
W.P. Jones and B.E. Launder, 1972

1. Briefly read through section 1
2. Read section 2 carefully and note the that:
   - The molecular viscosity is not included in Eq. (5)-(6). (Why?)
   - The constant values in Table 1 are different from values presented at lectures
   - The last term in Eq. (9) seems to have a misprint
   - Paragraph 2 and 3 at page 306 includes a nice discussion of physical reason for the extra terms and the choice of damping functions.

Turbulence energy

$$\rho\frac{\mathrm{D}k}{\mathrm{D}t}=\frac{\partial}{\partial y}\left(\frac{\mu_T}{\sigma_k}\frac{\partial k}{\partial y}\right)+\mu_T\left(\frac{\partial u}{\partial y}\right)^2-\rho\varepsilon. \qquad (5)$$

Energy dissipation

$$\rho\frac{\mathrm{D}\varepsilon}{\mathrm{D}t}=\frac{\partial}{\partial y}\left(\frac{\mu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right)+c_1\frac{\varepsilon}{k}\mu_T\left(\frac{\partial \mu}{\partial y}\right)^2-c_2\frac{\rho\varepsilon^2}{k}. \qquad (6)$$

$$\mu_T=c_\mu\rho k^2/\varepsilon \qquad (7)$$

Table 1. The values of the empirical constants in the high-Reynolds-number form of the k ~ ε model of turbulence

c_μ	c_1	c_2	σ_k	σ_ε
0·09	1·55	2·0	1·0	1·3

Research paper 3 (cont)

Turbulence energy

$$\rho\frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + \underbrace{\mu_T\left(\frac{\partial u}{\partial y}\right)^2}_{P_k} - \rho\varepsilon - 2\mu\left(\frac{\partial k^{\frac{1}{2}}}{\partial y}\right)^2. \qquad (8)$$

Energy dissipation

$$\rho\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right] + c_1 f_1 \cdot \frac{\varepsilon}{k}\mu_T\left(\frac{\partial u}{\partial y}\right)^2 - c_2 f_2 \frac{\rho\varepsilon^2}{k}$$

2

$$+ 2{\cdot}0\, \mu\mu_T\left(\frac{\partial^2 u}{\partial y^2}\right). \qquad (9)$$

Turbulent viscosity formula

$$\mu_T = c_\mu f_\mu \rho k^2/\varepsilon. \qquad (10)$$

Turbulent thermal conductivity formula

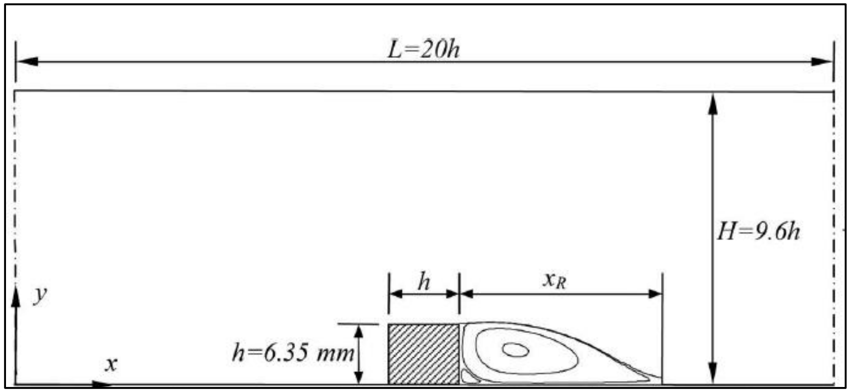
$$k_T = c_p \mu_T/0{\cdot}9. \qquad (11)$$

$$\left.\begin{aligned} f_1 &= 1{\cdot}0 \\ f_2 &= 1{\cdot}0 - 0{\cdot}3\exp(-R^2) \\ f_\mu &= \exp[-2{\cdot}5/(1 + R/50)] \end{aligned}\right\} \quad (12)$$

$$\boxed{R \equiv \rho k^2/\mu\varepsilon}$$

Comment on Research paper 4

“A comparative study of four low-Reynolds-number *k*-ε turbulence models for periodic fully developed duct flow and heat transfer”
A. A. Igci & M. E. Arici, 2016



1. INTRODUCTION
- Long intro with many references, just scan thorough this section
- However, read the last four paragraphs more carefully (starting from “For most engineering purposes…”)

2. MATHEMATICAL FORMULATION
- Understand the geometry (useful when you read the result section later)
- Skip everything concerning heat and heat flow, just focus on the RANS part and the turbulence models

3. PROBLEM DESCRIPTION
- The boundary conditions and flow parameters (Table 3) are always nice to know
- Be aware of how they define their Reynolds number

4. RESULT AND DISCUSSION
- Read through without getting stuck in details.
- Be aware of what type of parameters they choose to discuss
- The conclusion gives general comments about the turbulent models performance

Research paper 3 reading exercise (cont)

Table 1. Model constants and functions in the equations of the low-Re k-ε models.

Model	D	E	ε_w – B.C.	C_μ	C_1	C_2	σ_k	σ_ε
LS	$2\nu\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2$	$2\mu\nu_t\left(\frac{\partial^2 U}{\partial y^2}\right)$	0	0.09	1.44	1.92	1.0	1.3
LB	0	0	$\varepsilon_w = \nu\left(\frac{\partial^2 k}{\partial y^2}\right)$	0.09	1.44	1.92	1.0	1.3
AKN	0	0	$\varepsilon_w = \nu\left(\frac{\partial^2 k}{\partial y^2}\right)$	0.09	1.44	1.92	1.0	1.3
CH	$2\nu\frac{k}{y^2}$	$-2\mu\frac{\varepsilon}{y^2}\exp(-y^+/2)$	0	0.09	1.35	1.8	1.0	1.3

Table 2. Damping functions in the equations of the low-Re k-ε models.

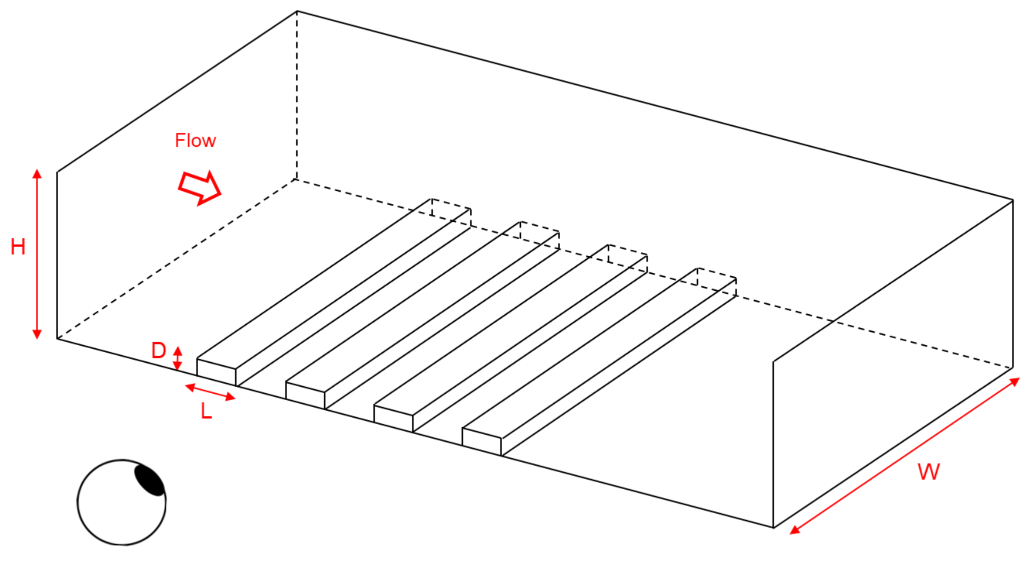
Model	f_μ	f_1	f_2
LS	$\exp[-3.4/(1+R_t/50)^2]$	1.0	$1-0.3\exp(-R_t^2)$
LB	$[1-\exp(-0.0165R_y)]^2[1+20.5/R_t]$	$1+(0.05/f_\mu)^3$	$1-\exp(-R_t^2)$
AKN	$[1-\exp(-R_\varepsilon/14)]^2$	1.0	$[1-\exp(-R_\varepsilon/3.1)]^2$
CH	$\left\{1+5/R_t^{3/4}\exp[-(R_t/200)^2]\right\}$ $1-\exp(-0.0115y^+)$	1.0	$\left\{1-0.3\exp[-(R_t/6.5)^2]\right\}$ $1-0.22\exp[-(R_t/6)^2]$

$$R_t = \frac{\rho k^2}{\mu\varepsilon} \quad R_y = \frac{\rho\sqrt{k}y}{\mu} \quad R_\varepsilon = \frac{\rho u_\varepsilon y}{\mu} \quad y^+ = \frac{\rho u_\tau y}{\mu}$$

From L4: Group discussion exercise

In the figure below a schematic of a reaction chamber is presented (details not in scale). The chamber is supposed to increase mixing rates near the corrugated surface (the small boxes) which should speed up chemical reactions in that region. We want to analyze how the flow behaves both near the corrugated surface, and a distance further out in the core flow.

Make a 2D sketch of the flow as seen by the indicated eye ball, and discuss the dynamics of velocity and vorticity.



Questions to be discussed next

- How do we physically interpret the Reynold stress term in RANS?
- How do we “explain” the flat velocity profile in a channel flow?
- Why does the viscous sublayer end at y+=5?

Reynolds stress in a **fully developed** Channel flow turbulence

RANS equations

$$\underbrace{\rho\left(\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}\right)}_{=0}=\underbrace{-\frac{\partial P}{\partial x_i}+\frac{\partial T_{ij}}{\partial x_j}+\frac{\partial R_{ij}}{\partial x_j}}_{\text{Balance between the driving pressure force and other forces}}\qquad \frac{\partial U_k}{\partial x_k}=0\qquad T_{ij}=\mu\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\qquad R_{ij}\equiv-\rho\left\langle u_i'u_j'\right\rangle$$

$$\boxed{x:\ \frac{\partial R_{1j}}{\partial x_j}=\frac{\partial R_{11}}{\partial x_1}+\frac{\partial R_{12}}{\partial x_2}+\frac{\partial R_{13}}{\partial x_3}=\frac{\partial R_{xx}}{\partial x}+\frac{\partial R_{xy}}{\partial y}+\frac{\partial R_{xz}}{\partial z}}$$

Consider a **simple shear profile** with a fully developed flow in x-direction (only pressure changes in x-direction).

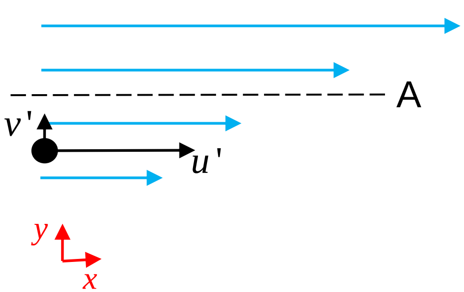
$$x:\ \ 0=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial y}\left(T_{xy}+R_{xy}\right)$$

$$T_{xy}=\mu\frac{\partial U}{\partial y}\qquad R_{xy}=-\rho\left\langle u'v'\right\rangle$$

Interpretation:
Friction between fluid layers, or momentum transfer between layers

Total shear stress in x-direction:

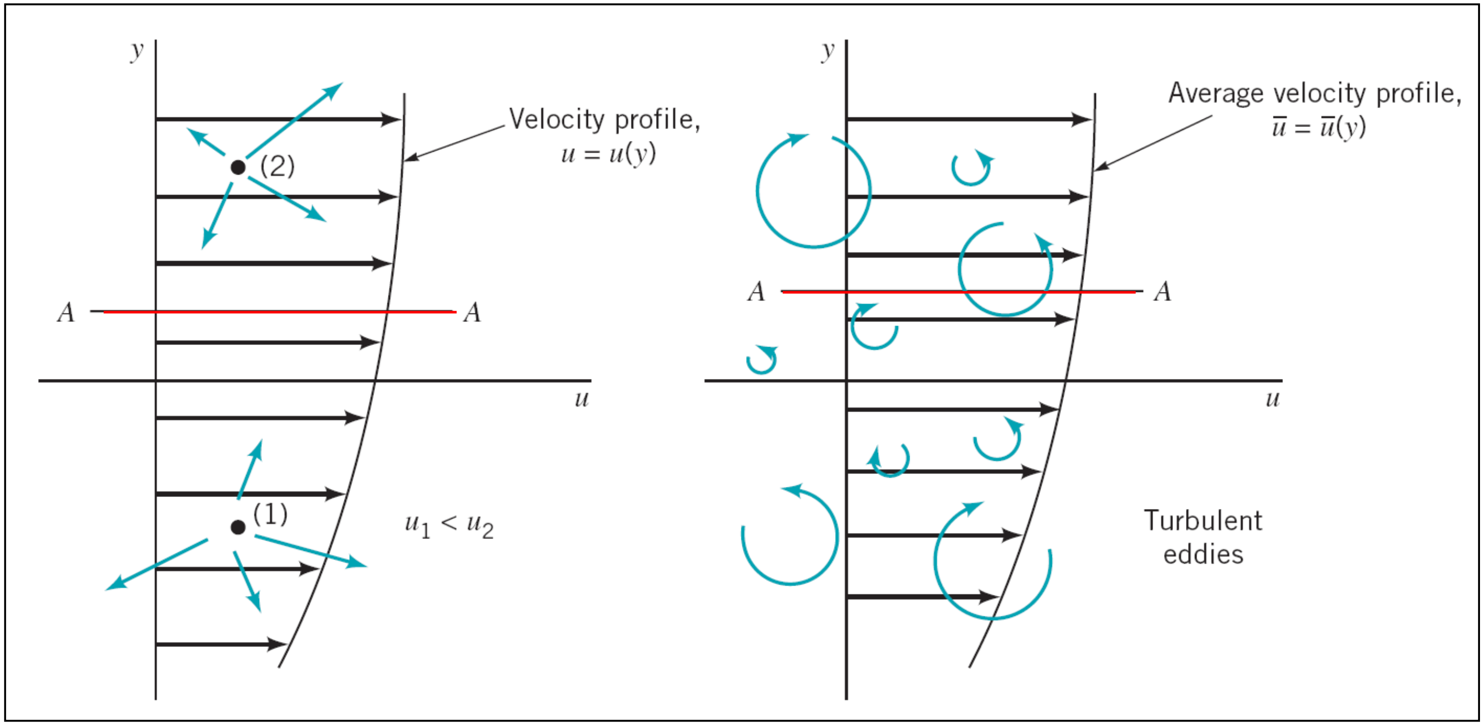
$$\boxed{T_{tot}\equiv\mu\frac{\partial U}{\partial y}-\rho\left\langle u'v'\right\rangle}$$



$(\rho u')Av'$ x-momentum transport rate through surface A $\Rightarrow$ $\rho\left\langle u'v'\right\rangle$ Average x-momentum transport rate per area

$$\left[(\rho u')Av'\right]=\frac{kg}{m^3}\frac{m}{s}m^2\frac{m}{s}=kg\frac{m}{s^2}=N$$

Viscous stress vs Turbulent stress



$$\tau_{lam} = \mu \frac{du}{dy}$$

Laminar flow:

Particles important

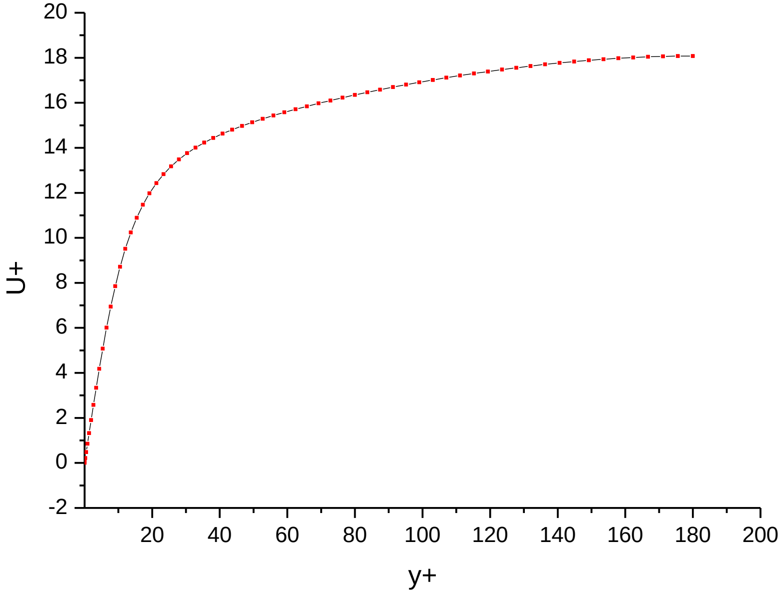
Turbulent flow:

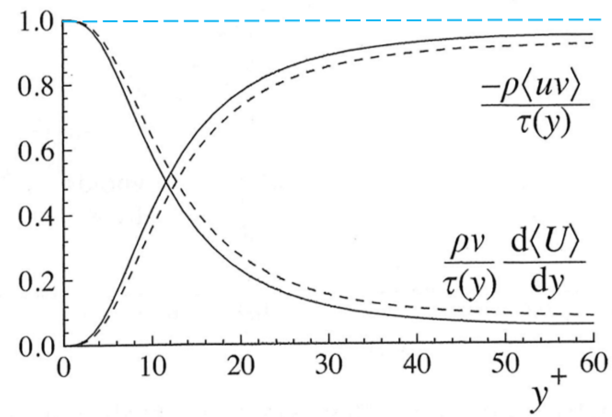
Eddies important

$$\tau = \tau_{lam} + \tau_{turb}$$

Laminar shear stress < Turbulent shear stress

How do we "explain" the flat velocity profile in a channel flow?





$$U^+ \equiv \frac{U}{u_\tau} \qquad y^+ \equiv \frac{\rho u_\tau y}{\mu} \qquad u_\tau \equiv \sqrt{\tau_w / \rho}$$

To discuss the flow profile we first need to revisit this figure.

Fully developed Channel flow

1) $T_{tot} = \mu \dfrac{\partial U}{\partial y} - \rho \langle u'v' \rangle$ Total shear stress in x-direction

2) No-slip condition ⇨ Reynolds stress is zero at boundary ⇨

$$\begin{cases} T_{tot}(0) = \mu \dfrac{\partial U(0)}{\partial y} = \tau_w \\ T_{tot}(2\delta) = \mu \dfrac{\partial U(2\delta)}{\partial y} = -\tau_w \end{cases}$$

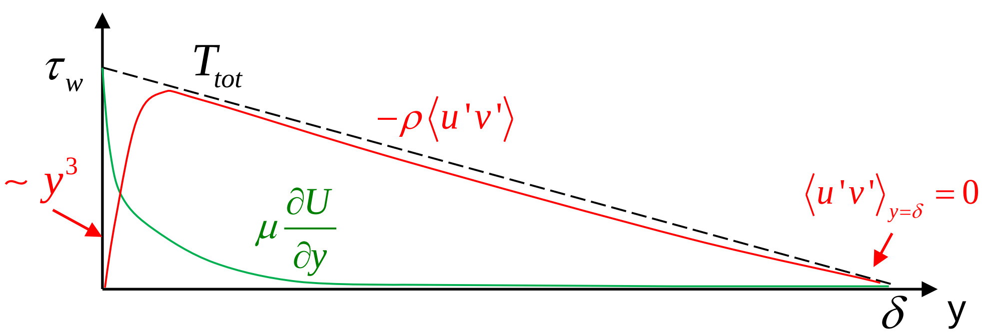
3) At center line:

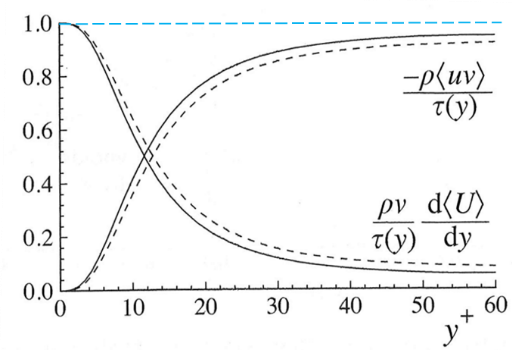
$$\left.\begin{aligned} T_{tot}(\delta) &= 0 \\ \frac{\partial U(\delta)}{\partial y} &= 0 \end{aligned}\right\} \Rightarrow \boxed{\langle u'v' \rangle_{y=\delta} = 0}$$

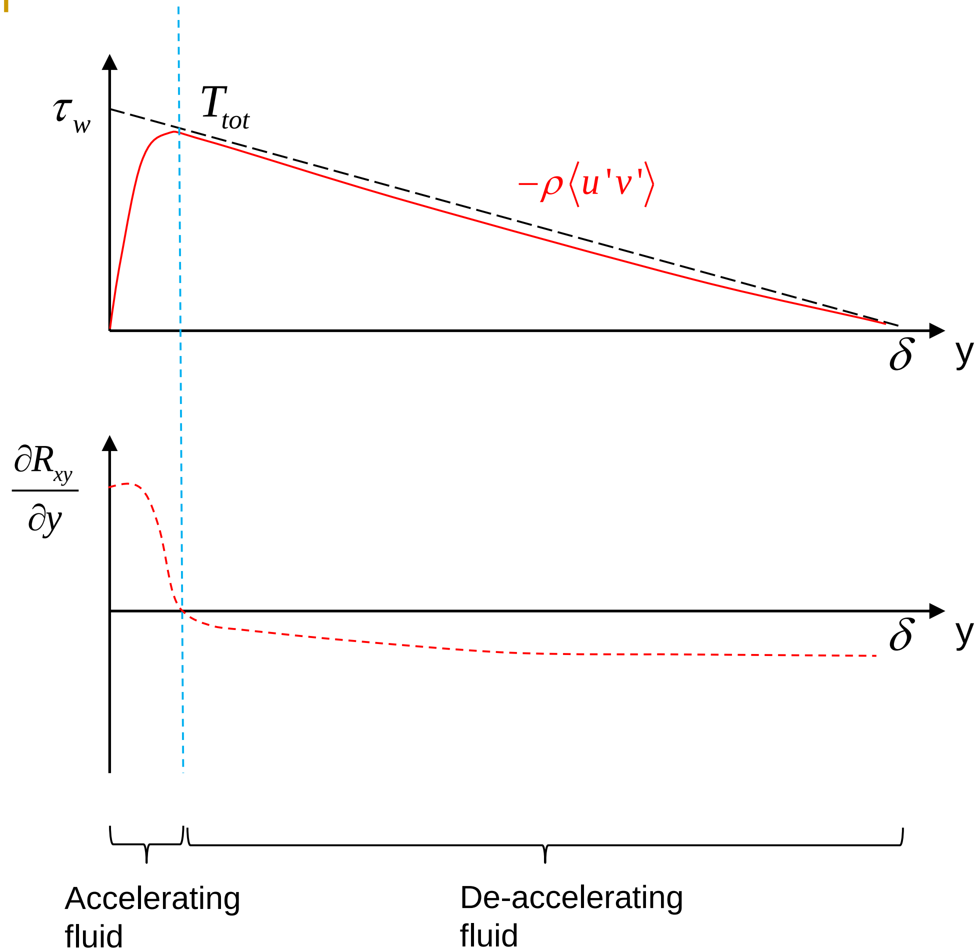
4) It can be shown using RANS that the total stress depends on y as:

$$\boxed{T_{tot}(y) = \tau_w \left(1 - \frac{y}{\delta}\right)}$$

Next lecture

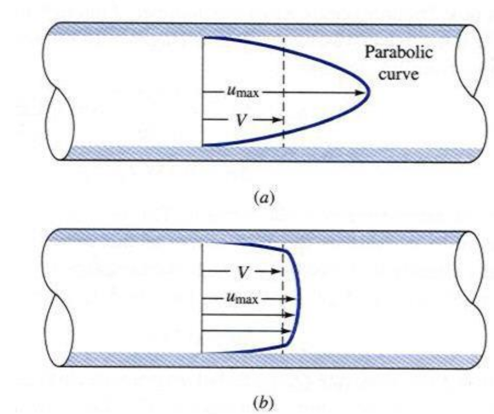




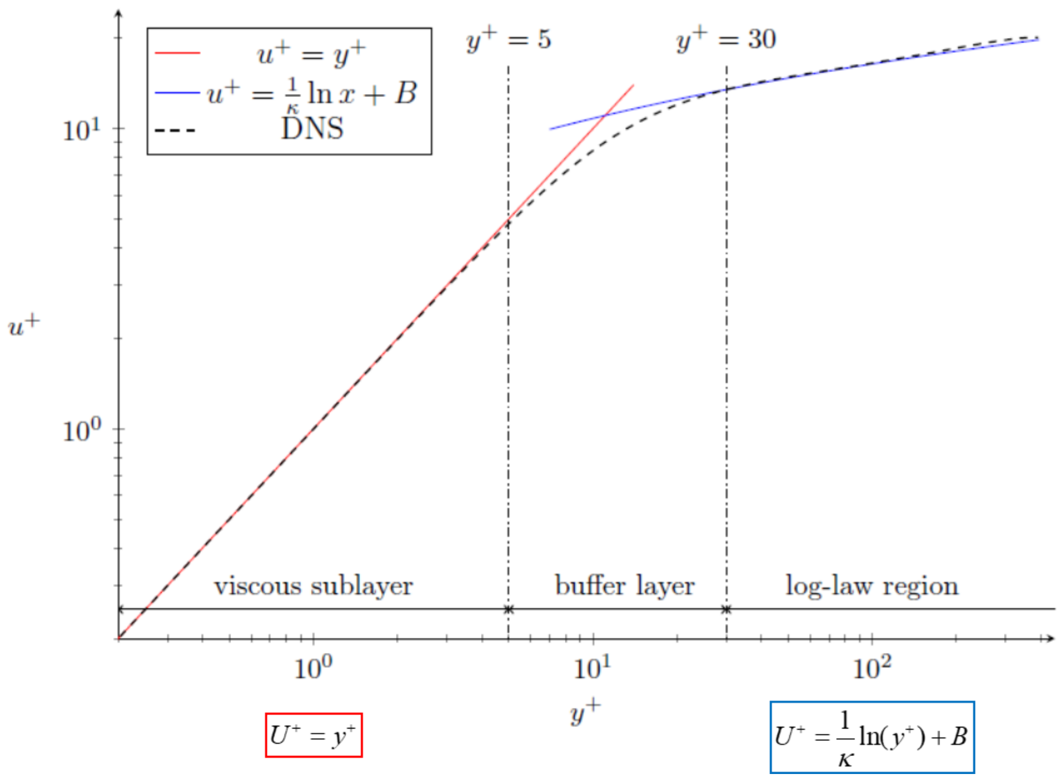


$$0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(T_{xy} + R_{xy}\right)$$

$$\frac{\partial R_{xy}}{\partial y} = \text{Force per unit volume}$$



Why does the viscous sublayer end at y+=5?



Layer models

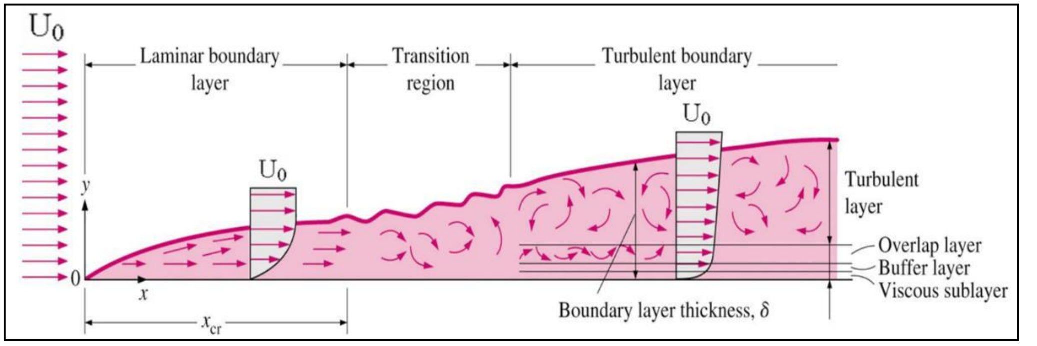
$U^+ = y^+$	$y^+ \leq 5$
$U^+ = \dfrac{1}{\kappa}\ln(y^+) + B$	$y^+ \geq 30$

”Kolmogorov behavior of near wall turbulence and its application in turbulence modeling”, T-H Shih, NASA.

$L_K = \left(\nu^3/\varepsilon\right)^{1/4}$ Kolmogorov scale

$$\left.\begin{aligned} L &= \kappa y \\ \varepsilon &= \frac{u_\tau^3}{\kappa y} \end{aligned}\right\}$$

Defined in log-layer. Here we investigate what they indicate closer to the wall.



L=Integral length = The size of energetic large eddies

Find the position where the integral length of turbulence equals Kolmogorv length scale.
This give us a rough estimate of the limit point of turbulence.

$$\boxed{L \sim L_K} \Rightarrow \kappa y_{\lim} \sim \left(\nu^3 \kappa y_{\lim} / u_\tau^3\right)^{1/4} \Rightarrow 1 \sim \left(\frac{\nu}{u_\tau \kappa y_{\lim}}\right)^3 = \frac{1}{\kappa^3}\left(\frac{1}{y_{\lim}^+}\right)^3$$

$$\boxed{y_{\lim}^+ \sim \frac{1}{\kappa} \approx 2.4}$$

Limit point of turbulence

Below this point the turbulence can not be self-sustained due to the large viscous action, and the flow is dominated by viscous stress.

End of lecture