Computational Fluid Dynamics

Fluid Mechanics I

Lecture 1

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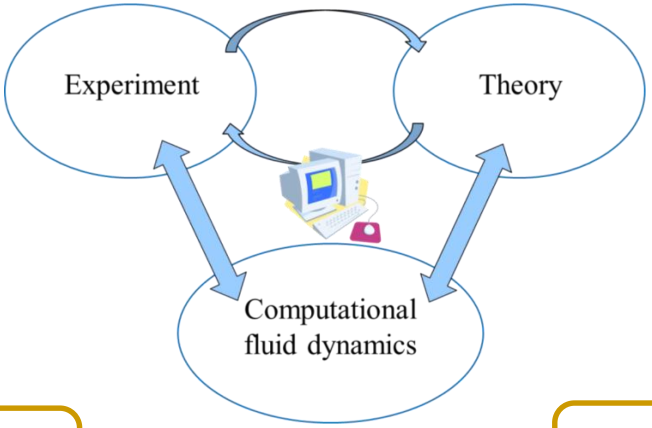
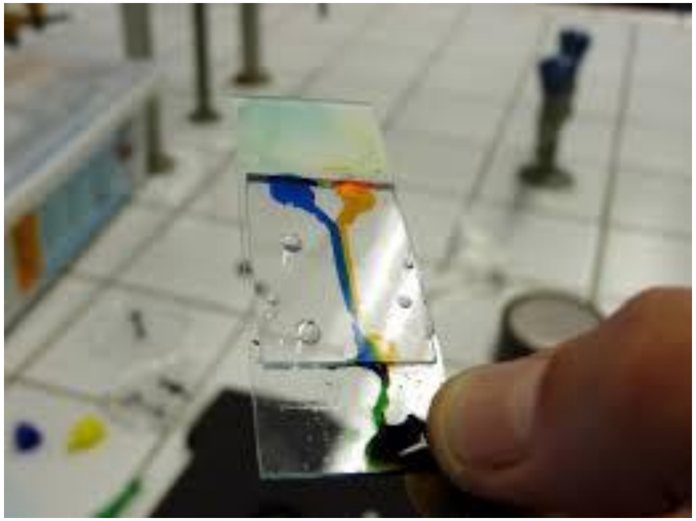
SUMMARY OF LECTURE: FLUID MECHANICS I

- ❑ Be able to give a physical interpretation of all terms in Navier-Stokes
- ❑ Be able to give a physical interpretation of all terms in Vorticity equation
- ❑ Understand vortex stretching
- ❑ Understand the relation between pressure and velocity
- ❑ Understand the concept of diffusion and convection of vorticity
- ❑ Be able to use Einsteins summation convention and Index formalism
- ❑ Understand the physical interpretation of the Reynolds number

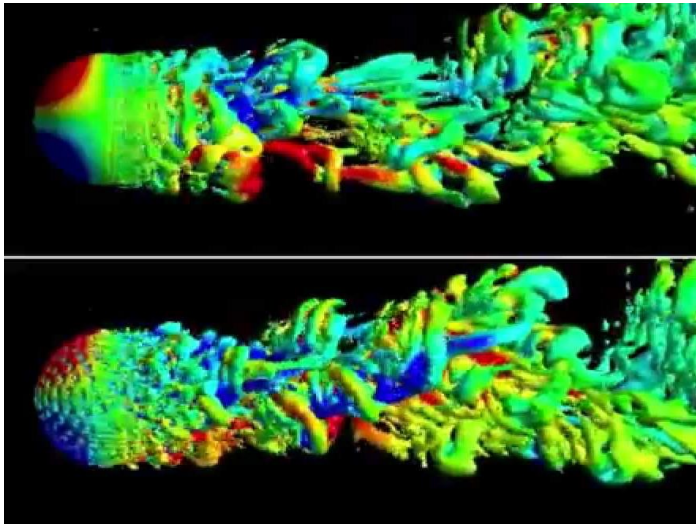
What is CFD?



Microfluidics



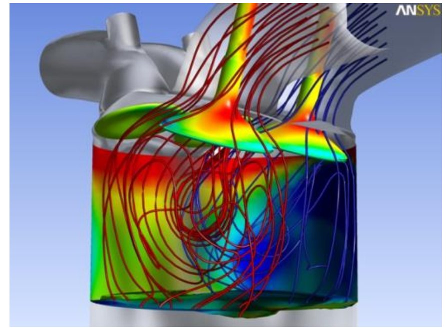
Turbulent flow

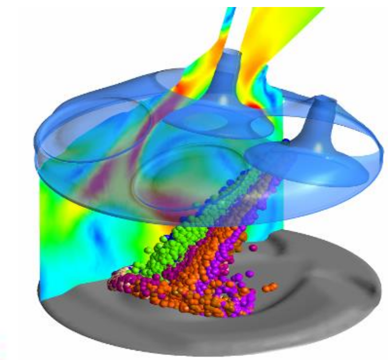


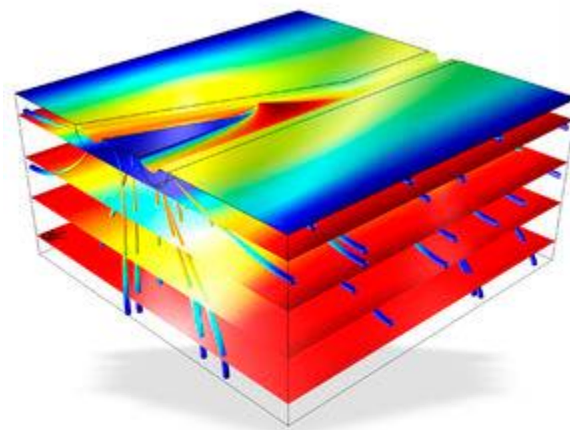
CFD-software

Different software:

- ANSYS CFX
- ANSYS FLUENT
- CD-adapco STAR-CD
- Open FOAM
- COMSOL
- Many others…







All of these numerically solves fluid equations, more about this later in the course…

Example: CFD-settings in Comsol

Settings

Turbulent Flow, Low Re k-ε

Equation

Equation form:

Study controlled

Show equation assuming:

Study 1, Time Dependent

Navier-Stokes

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot[-p\mathbf{I} + \mathbf{K}] + \mathbf{F}$$

Incompressibility condition

$$\rho\nabla\cdot(\mathbf{u}) = 0$$

”Newtonian fluid” assumption

$$\mathbf{K} = (\mu + \mu_T)\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

Turbulence model: Low Re k-ε

$$\rho\frac{\partial k}{\partial t} + \rho(\mathbf{u}\cdot\nabla)k = \nabla\cdot\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)\nabla k\right] + P_k - \rho\epsilon$$

$$\rho\frac{\partial \epsilon}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\epsilon = \nabla\cdot\left[\left(\mu + \frac{\mu_T}{\sigma_\epsilon}\right)\nabla \epsilon\right] + C_{\epsilon 1}\frac{\epsilon}{k}P_k - C_{\epsilon 2}\rho\frac{\epsilon^2}{k}f_\epsilon(\rho,\mu,k,\epsilon,l_w), \quad \epsilon = \mathrm{ep}$$

Wall distance eq.

$$\nabla G\cdot\nabla G + \sigma_w G(\nabla\cdot\nabla G) = (1+2\sigma_w)G^4, \quad \ell_w = \frac{1}{G} - \frac{\ell_{ref}}{2}$$

$$\mu_T = \rho C_\mu \frac{k^2}{\epsilon} f_\mu(\rho,\mu,k,\epsilon,\ell_w)$$

Turbulence production

$$P_k = \mu_T\left[\nabla\mathbf{u} : \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\right]$$

This complicated model is actually a simplified model of turbulence…

Physical Model

Turbulence

Turbulence model type:

RANS

Turbulence model:

Low Reynolds number k-ε

Wall treatment:

Low Re

Turbulence model parameters

- [ ] Edit turbulence model parameters

Consistent Stabilization

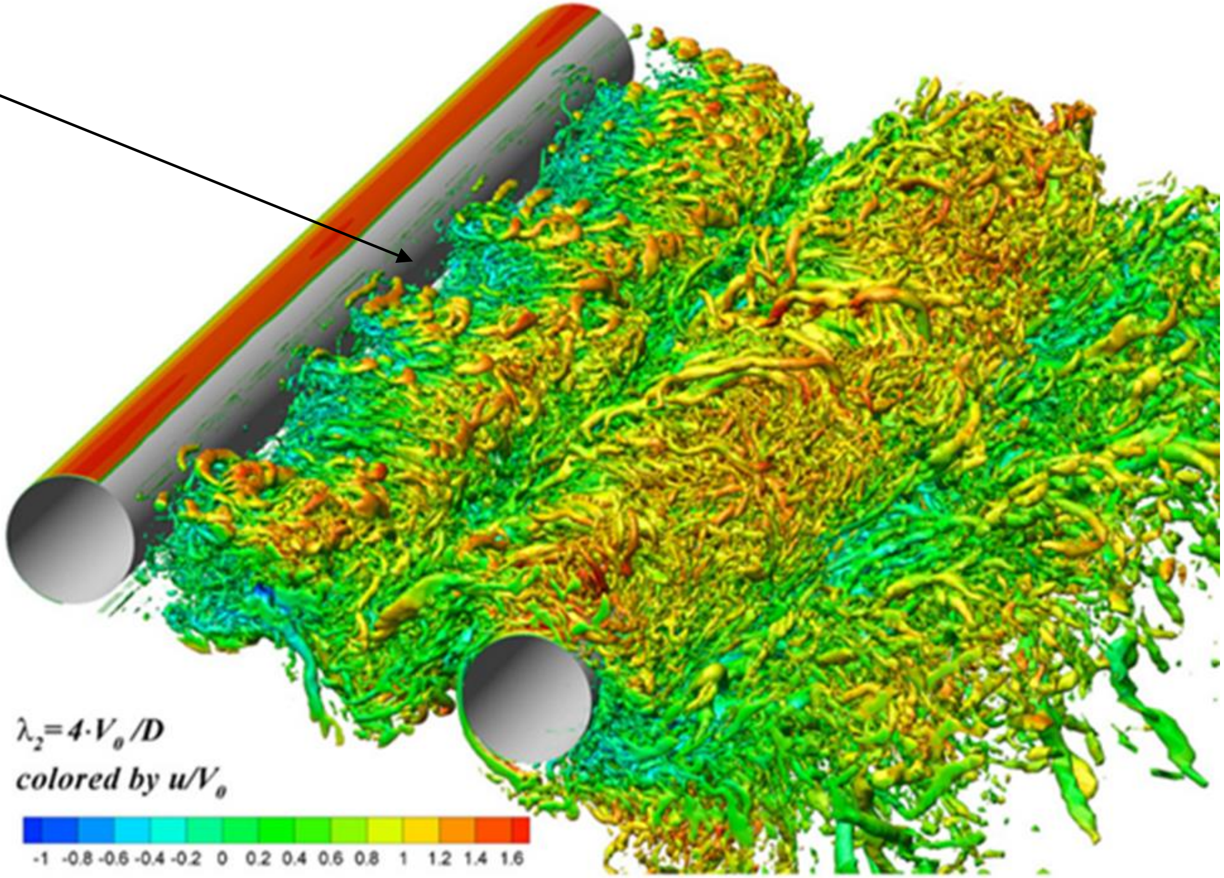
Inconsistent Stabilization

Advanced Settings

Picture of turbulence = Interacting vortex tubes

Small ”tornadoes”
or
Vortex tubes

Structures with
high vorticity



Fluid equations

Navier-Stokes equations $\xrightarrow{\nabla\times}$ The vorticity equation

$$\boxed{\begin{aligned}&\rho\frac{\partial \mathbf{V}}{\partial t}+\rho(\mathbf{V}\cdot\nabla)\mathbf{V}=-\nabla p+\mu\nabla^2\mathbf{V}+\mathbf{f}\\&\nabla\cdot\mathbf{V}=0\end{aligned}}$$

$$\boxed{\begin{aligned}&\rho\frac{\partial \boldsymbol{\omega}}{\partial t}+\rho(\mathbf{V}\cdot\nabla)\boldsymbol{\omega}=\rho(\boldsymbol{\omega}\cdot\nabla)\mathbf{V}+\mu\nabla^2\boldsymbol{\omega}\\&\nabla\cdot\boldsymbol{\omega}=0\end{aligned}}$$

(Assumed: $\nabla\times\mathbf{f}=0$) $\boldsymbol{\omega}\equiv\nabla\times\mathbf{V}$

$$\underbrace{\rho\left[\frac{\partial}{\partial t}+(\mathbf{V}\cdot\nabla)\right]}_{\rho\frac{D}{Dt}}\mathbf{V}=-\nabla p+\mu\nabla^2\mathbf{V}+\mathbf{f}$$

$$\rho\frac{D\mathbf{V}}{Dt}=-\nabla p+\mu\nabla^2\mathbf{V}+\mathbf{f}$$

$$\underbrace{\rho\left[\frac{\partial}{\partial t}+(\mathbf{V}\cdot\nabla)\right]}_{\rho\frac{D}{Dt}}\boldsymbol{\omega}=\rho(\boldsymbol{\omega}\cdot\nabla)\mathbf{V}+\mu\nabla^2\boldsymbol{\omega}$$

$$\rho\frac{D\boldsymbol{\omega}}{Dt}=\rho(\boldsymbol{\omega}\cdot\nabla)\mathbf{V}+\mu\nabla^2\boldsymbol{\omega}$$

Note:
No pressure term

Fluid equations (cont)

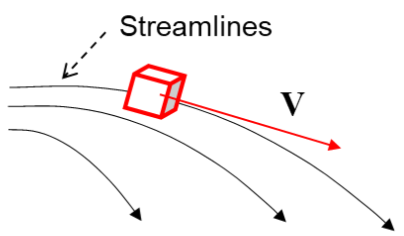
Study Group exercise:
Derivation of vorticity eq.

Navier-Stokes equations

$$\boxed{\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{V} + \mathbf{f}}$$

Ex:

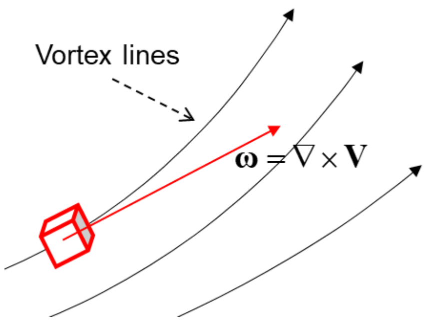
$f = \rho \nabla \Phi$

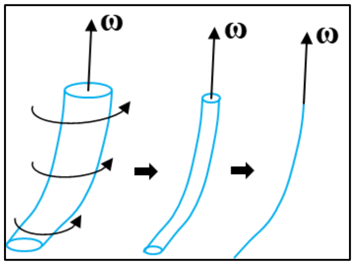


The vorticity equation

$$\boxed{\rho \frac{D\boldsymbol{\omega}}{Dt} = \rho (\boldsymbol{\omega} \cdot \nabla) \mathbf{V} + \mu \nabla^2 \boldsymbol{\omega}}$$

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{V}$$





Note: For the inviscid case (no viscosity term) we have

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{V} \Rightarrow$$

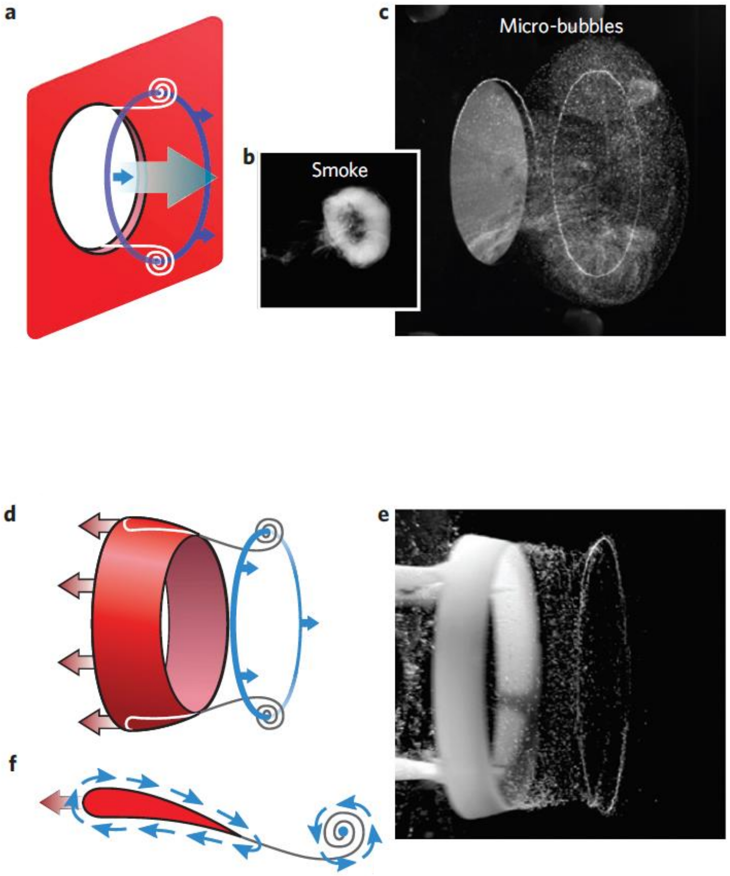
If we initially have no vorticity, then RHS is zero and we have:

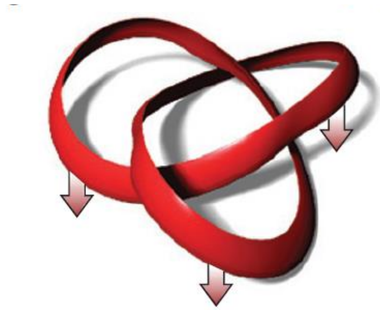
$\boldsymbol{\omega}(t) = 0$ or $\nabla \times \mathbf{V} = 0$ (Fundamental assumption in potential theory)

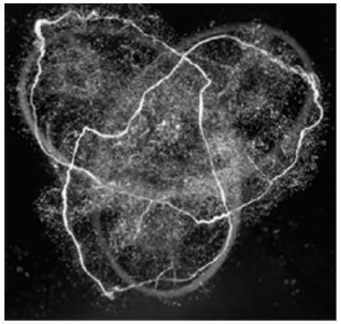
Other properites:

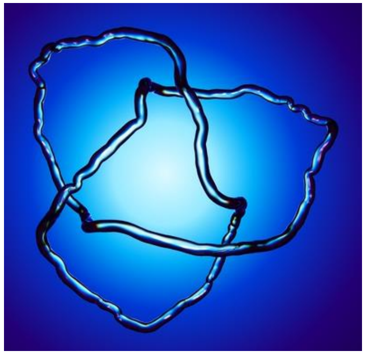
1. Vorticity measures the local rotation of the fluid
2. Vortex lines are ”frozen” into inviscid fluids, they move along with the fluid

Results produced at the Irvine Lab: http://irvinelab.uchicago.edu/



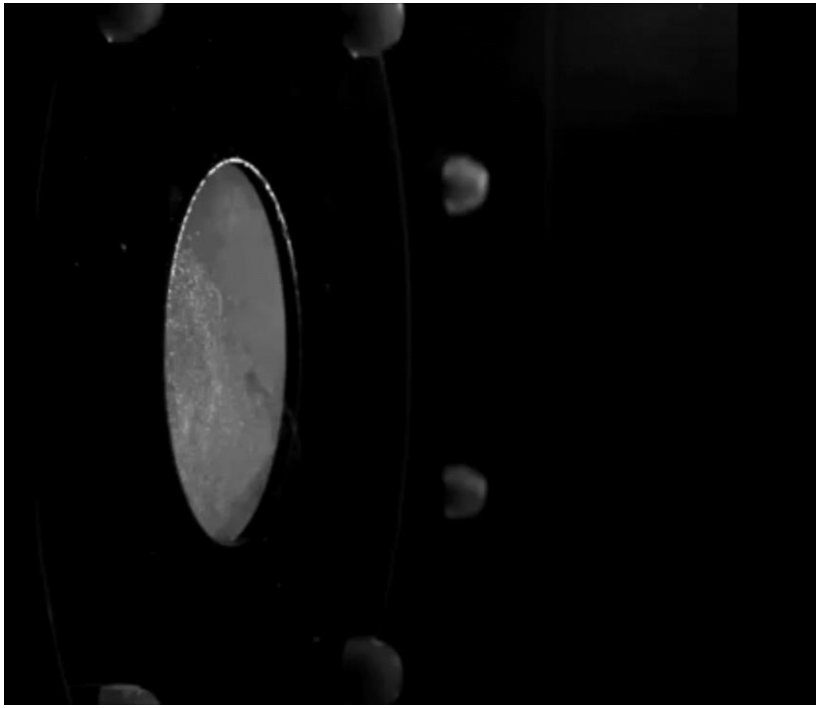


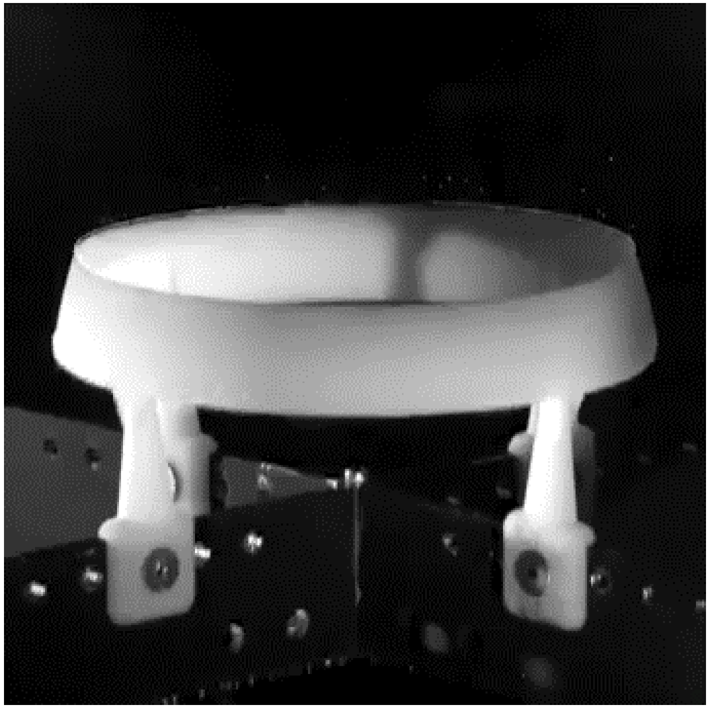


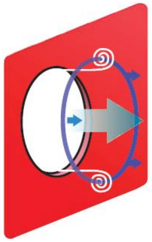


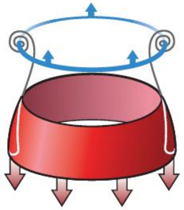
Kleckner, D., Irvine, W. Creation and dynamics of knotted vortices. *Nature Phys* **9,** 253–258 (2013).

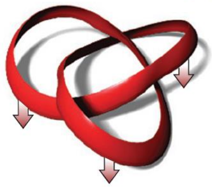
Irvine Lab: http://irvinelab.uchicago.edu/









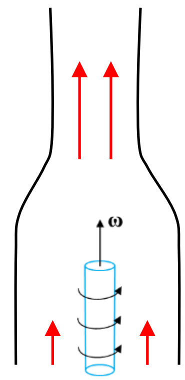


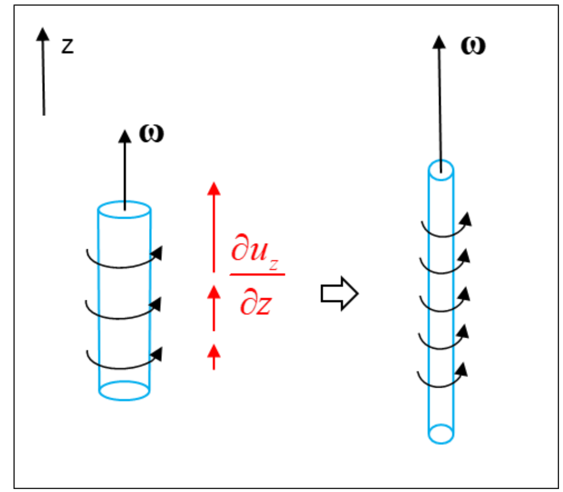
Vortex tube stretching

$$\rho \frac{D\boldsymbol{\omega}}{Dt} = \rho\left(\boldsymbol{\omega}\cdot\nabla\right)\mathbf{V} + \mu\nabla^2\boldsymbol{\omega}$$

Change of vorticity due to stretching or compression of a vortex

Change of vorticity due to viscous effects





Direction parallel to vortex

$$\rho\left(\boldsymbol{\omega}\cdot\nabla\right)u_z = \rho\left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)u_z = \rho\omega_z\frac{\partial u_z}{\partial z}$$

Stretching $\Leftrightarrow \dfrac{\partial u_z}{\partial z} > 0$ ➡ $\rho\left(\boldsymbol{\omega}\cdot\nabla\right)u_z > 0$ ➡ RHS of vorticity equation may become positive,which means that vorticity increases in time

The figure also indicate that the crossectional area of the vortex decreases during stretching. This is an combined effect of vortex tubes being material lines (move with the fluid) and mass conservation.

Pressure vs Velocity & Vorticity

In Navier-Stokes equations we have four unknown variabes, u,v,w and p, but we have no equation for p. Or do we…?

$$\rho\frac{\partial \mathbf{V}}{\partial t}+\rho\left(\mathbf{V}\cdot\nabla\right)\mathbf{V}=-\nabla p+\mu\nabla^2\mathbf{V}+\mathbf{f}$$

$$\nabla\cdot \quad \Downarrow \quad \nabla\cdot\mathbf{V}=0$$

By applying the divergence operator on NS, and using mass conservation, we get a Poisson equation for the pressure.

$$\nabla^2 p=-\nabla\cdot\left[\rho\left(\mathbf{V}\cdot\nabla\right)\mathbf{V}-\mathbf{f}\right]$$

The solution to this Poisson equation links the pressure to the velocity distribution for the whole domain. A change in the velocity at a point affects the whole pressure distribution.

Physical interpretation: Pressure waves generated by a change in velocity at a point sends information about the change throughout the whole fluid.

The vorticity equation do not have any pressure term.

$$\rho\frac{\partial \boldsymbol{\omega}}{\partial t}+\rho\left(\mathbf{V}\cdot\nabla\right)\boldsymbol{\omega}=\rho\left(\boldsymbol{\omega}\cdot\nabla\right)\mathbf{V}+\mu\nabla^2\boldsymbol{\omega}$$

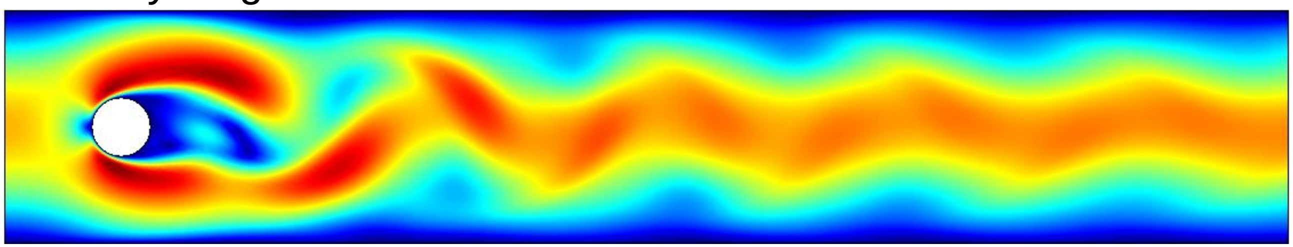
The velocity field cannot be considered localized in space, but the vorticity field can.

A localized velocity distribution will very fast affect the whole fluid thorugh pressure waves, but a localized vorticity distribution can only spread by advection or diffusion.

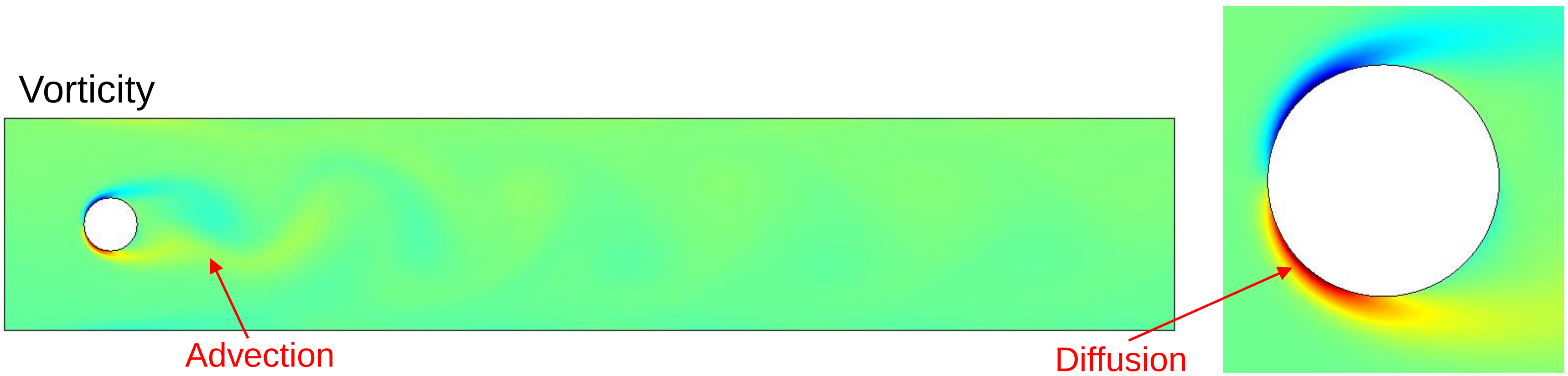
Vorticity generation in flow around a cylinder (Comsol Introduction lab)

Geometry

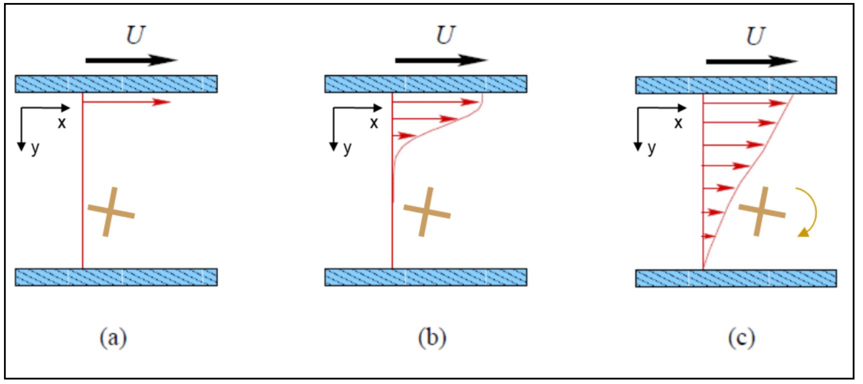
Velocity magnitude



Vorticity



Diffusion of vorticity



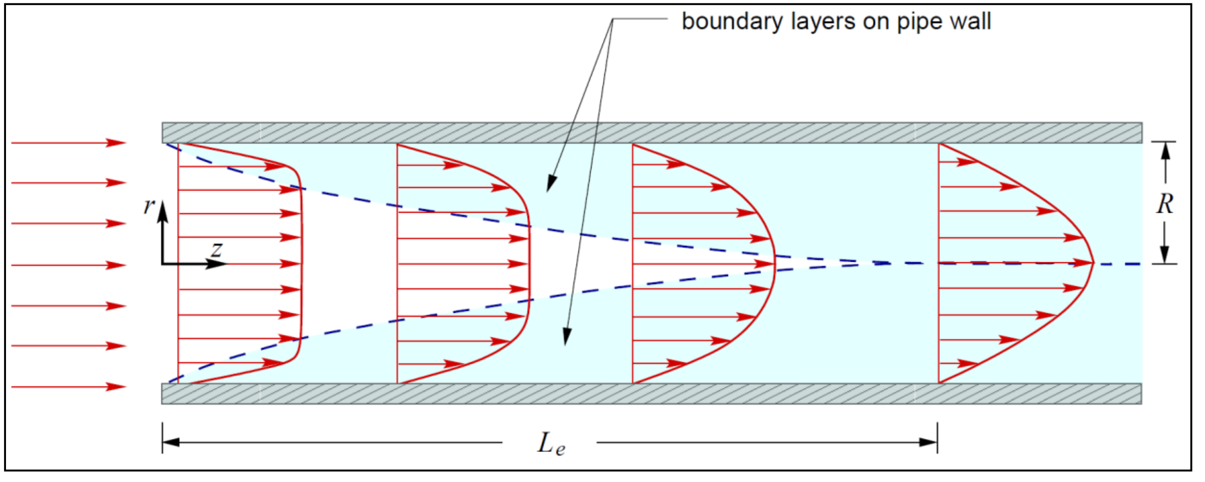
$$\rho \frac{\partial \boldsymbol{\omega}}{\partial t} + \rho(\mathbf{V} \cdot \nabla)\boldsymbol{\omega} = \rho(\boldsymbol{\omega} \cdot \nabla)\mathbf{V} + \mu \nabla^2 \boldsymbol{\omega}$$

$$\mathbf{V} = V(y)\hat{\mathbf{x}}$$
$$\boldsymbol{\omega} = \omega(y)\hat{\mathbf{z}}$$

$$\Rightarrow \quad (\mathbf{V} \cdot \nabla)\boldsymbol{\omega} = 0, \quad (\boldsymbol{\omega} \cdot \nabla)\mathbf{V} = 0$$

$$\Rightarrow \quad \frac{\partial \boldsymbol{\omega}}{\partial t} = \frac{\mu}{\rho} \nabla^2 \boldsymbol{\omega}$$ Diffusion of vorticity

$\frac{\mu}{\rho} \sim$ "diffusion coefficient"



No vorticity at the inlet, it is created at the walls, diffused inwards and advected.

At the outlet the vorticity is non-zero throughout the flow.

Diffusion of vorticity (cont)

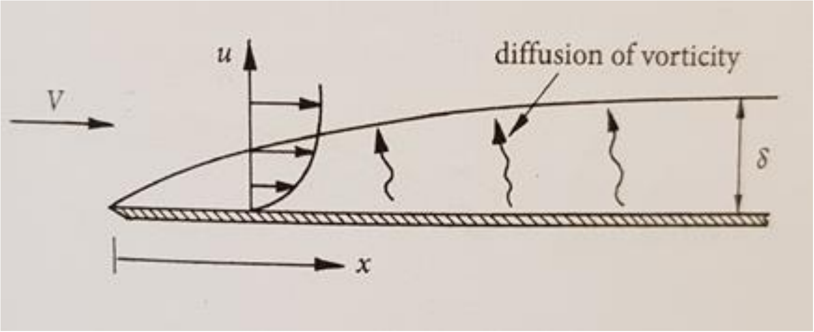


Figure 2.12 in Davidson

A boundary layer, laminar or turbulent, acts as a source of vorticity through diffusion.

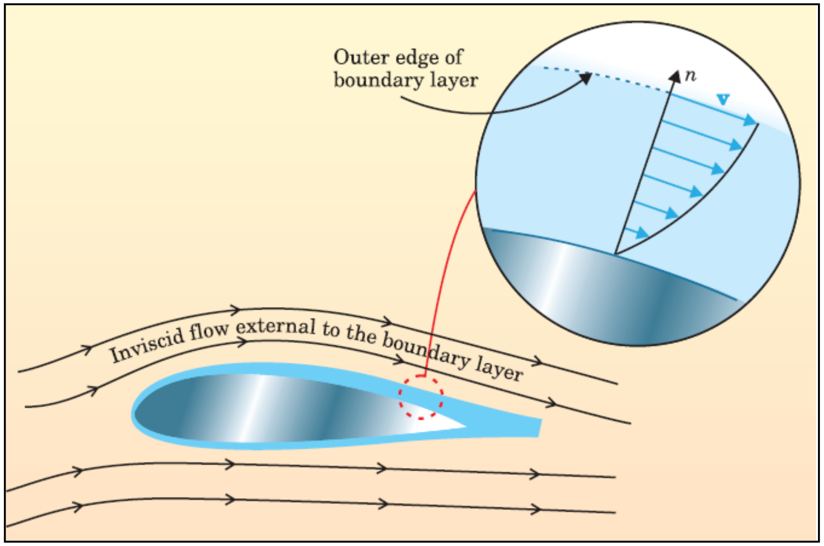
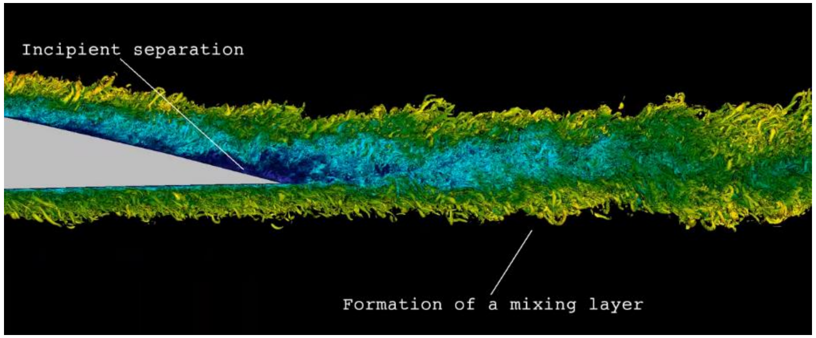
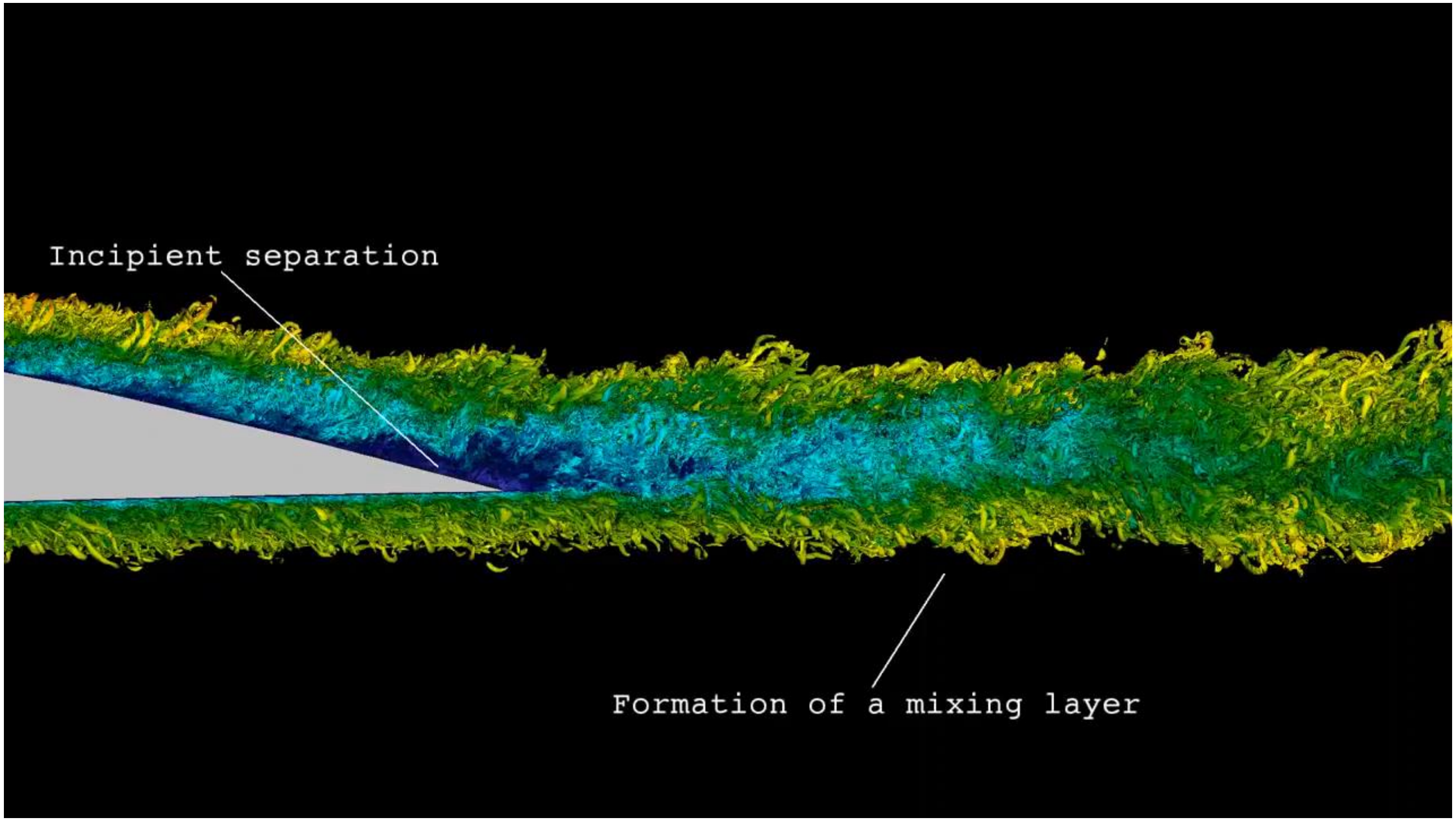


Figure 2 in ”Ludvig Prandtl’s Boundary Layer” by J.D. Andersson



DEMO: Visualization of vorticity



Navier-Stokes equations
Index notation

Navier-Stokes equations using index notation

$$\rho \frac{\partial u_i}{\partial t} + \rho \left(u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + f_i$$

$$u_{i,i} = 0$$

$$u_{i,k} \equiv \frac{\partial u_i}{\partial x_k}$$

$$i = 1,2,3 \qquad j = 1,2,3$$

$$\begin{pmatrix} i = x, y, z \\ j = x, y, z \end{pmatrix}$$

$$u_1 = u_x,\; u_2 = u_y,\; u_3 = u_z$$

$$x_1 = x, \quad x_2 = y, \quad x_3 = z$$

Example

$$u_{1,2} = u_{x,y} \equiv \frac{\partial u_x}{\partial y}$$

Example: Important physical quantities connected to velocity gradients

$$u_{i,j} = S_{ij} + \Omega_{ij}$$

$$S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Rate of strain (symmetric)

$$\Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

Rate of rotation (anti-symmetric)

Short tutorial: Index formalism

Einstein summation convention

$$A_i B_i \equiv \sum_i A_i B_i = A_1 B_1 + A_2 B_2 + \ldots$$

Not specified i, k

$A_i B_k$ are elements in a matrix: $\begin{pmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{pmatrix}$

$$A_k B_k = A_1 B_1 + A_2 B_2$$

Example:

$$u_{k,k} \equiv \frac{\partial u_k}{\partial x_k} = \sum_k \frac{\partial u_k}{\partial x_k} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla \cdot \mathbf{V}$$

$$\mathbf{V} = u_x \hat{\mathbf{x}} + u_y \hat{\mathbf{y}} + u_z \hat{\mathbf{z}}$$

Example:

$$u_k \frac{\partial u_i}{\partial x_k} = u_x \frac{\partial u_i}{\partial x} + u_y \frac{\partial u_i}{\partial y} + u_z \frac{\partial u_i}{\partial z} = \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) u_i = \left(\mathbf{V} \cdot \nabla \right) u_i$$

$$u_j \frac{\partial u_i}{\partial x_j} = ?$$

Example 1: Navier-Stokes equation- a physical interpretation of terms

$$\rho \frac{\partial u_i}{\partial t} + \rho \left(u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + f_i$$

Changes in time at fixed point

Convective acceleration

Pressure force

Viscous contribution

$$\mu \nabla^2 u_i$$

External forces

$$\rho \frac{Du_i}{Dt}$$

Changes in time following a fluid particle

Example 2: The vorticity equation- a physical interpretation of terms

$$\rho \frac{\partial \omega_i}{\partial t} + \rho \left(u_j \frac{\partial \omega_i}{\partial x_j} \right) = \rho \left(\omega_j \frac{\partial u_i}{\partial x_j} \right) + \mu \frac{\partial}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} + \xi_i$$

Changes in time at fixed point

Convective term

Vortex stretching

Viscous contribution

$$\mu \nabla^2 \omega_i$$

External source of vorticity

$$\rho \frac{D\omega_i}{Dt}$$

Changes in time following a fluid particle

Navier-Stokes equation (cont.)

$$\rho \frac{\partial u_i}{\partial t} + \rho \left(u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + f_i$$

Changes in time at fixed point

Convective acceleration

Pressure force

Viscous contribution

$\mu \nabla^2 u_i$

External forces

Reynolds number

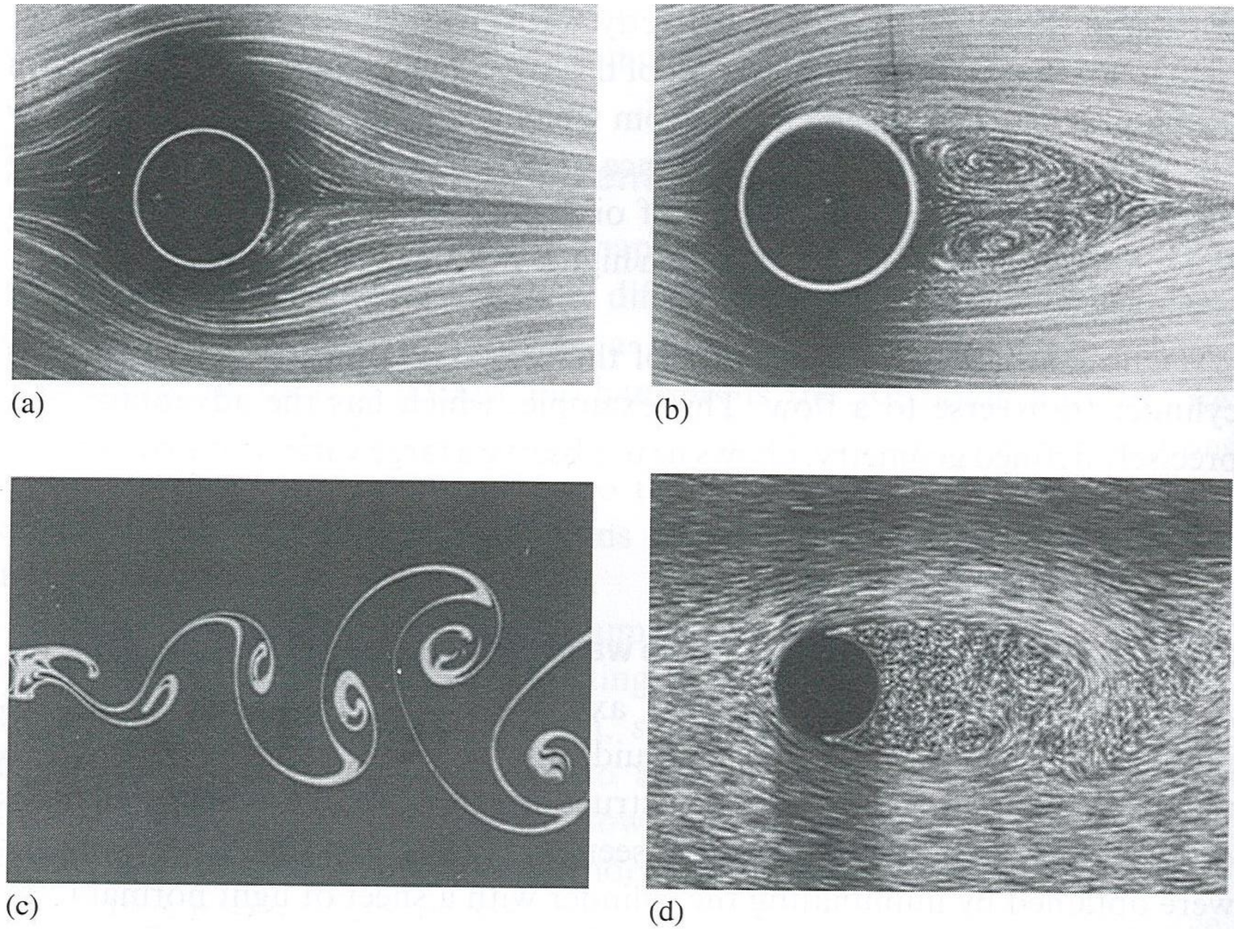
$$\boxed{\mathrm{Re} = \frac{\text{Convective term}}{\text{Viscous term}} \sim \rho \frac{U^2}{L} / \mu \frac{U}{L^2} = \frac{\rho L U}{\mu}}$$

Order of magnitude analysis

Reynolds number: Flow pattern around a cylinder

Re < 1

Re = 26

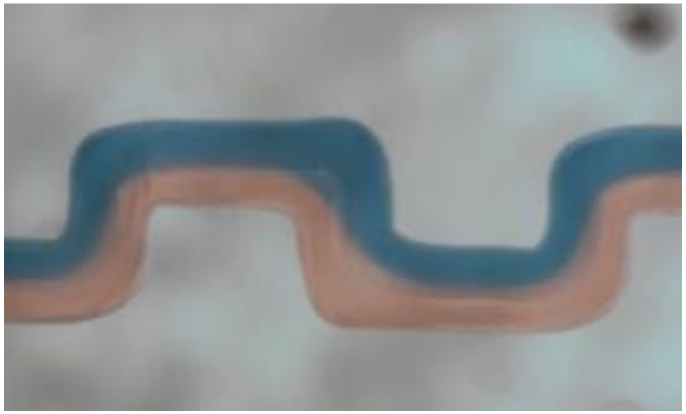


Re = 140

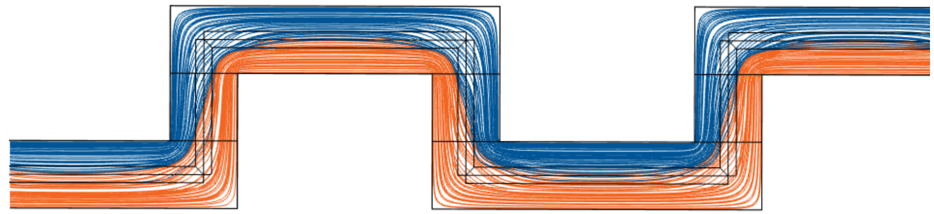
Re = 2000

Microfluidics in Biophysics: Experiment vs Simulation (Re << 1)

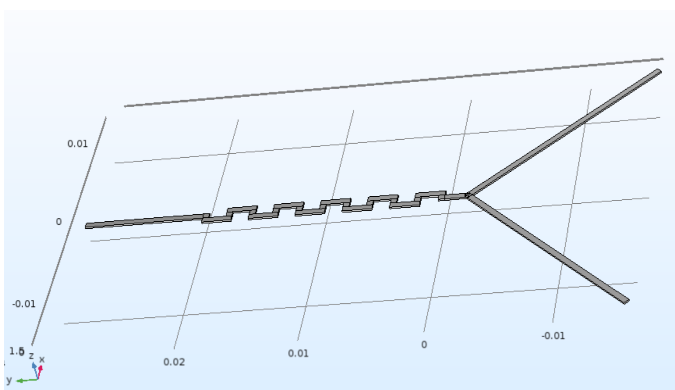
A CFD master thesis by Hampus Söderqvist, 2016

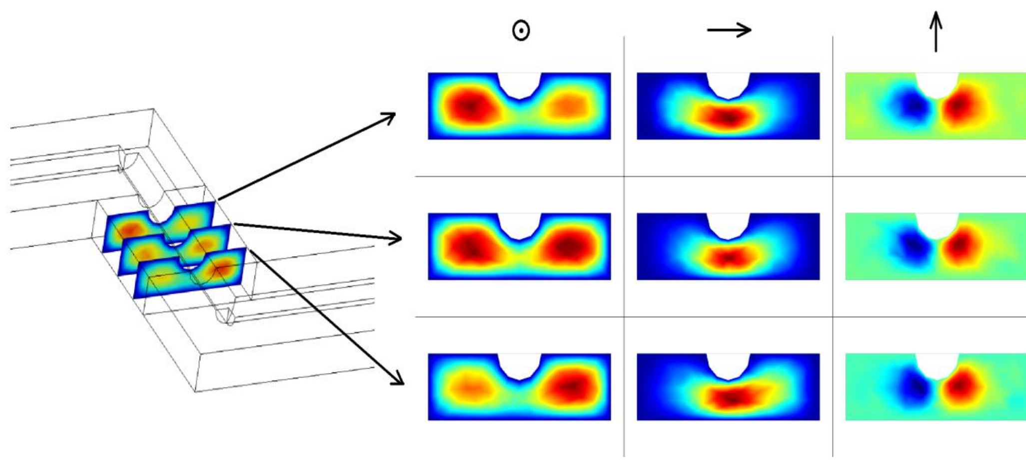


Experiment (3D-printed)

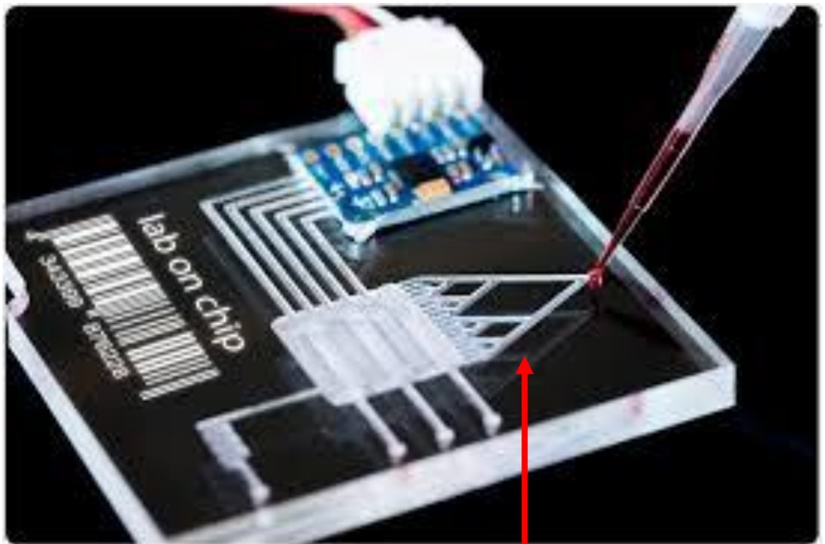


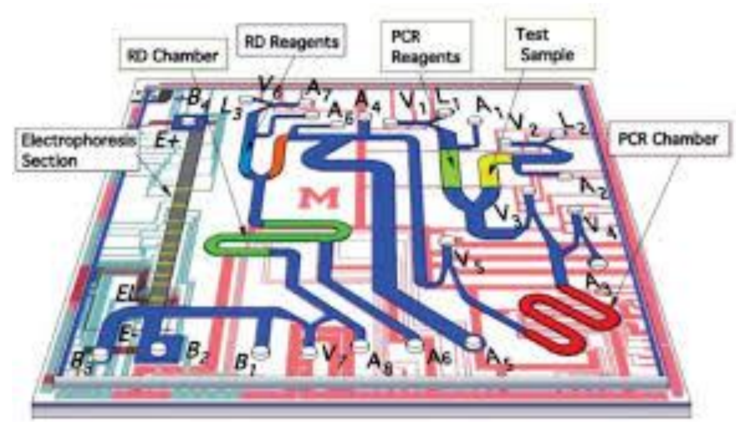
Simulation





Microfluidics in Biophysics: Bio-chip = ”mini factory”



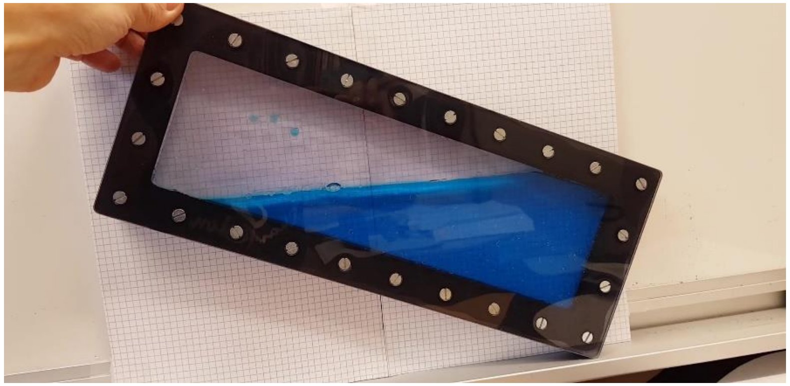


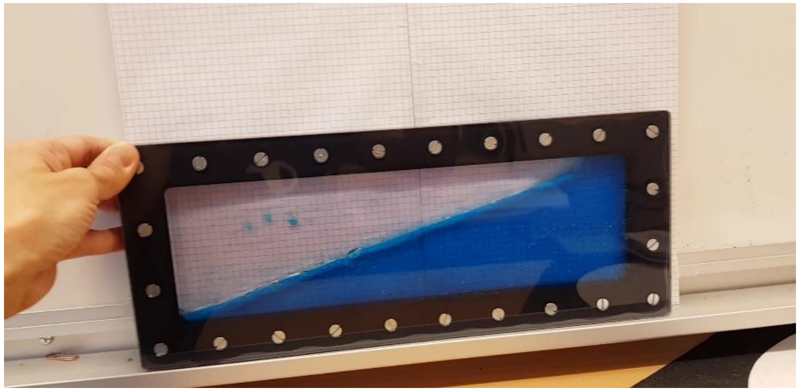
One drop of blood can be splitted into several tests

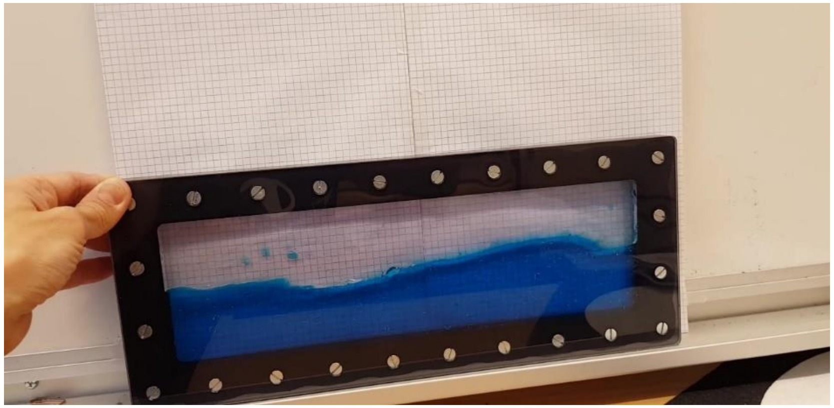
Many tests on a small chip:
Benefits = cheap and less volume of testing substrate needed

Comsol Lab 1: Two-fluid Interface Dynamics Experiment (Low Re)

Task: Find a way to compare experiment and simulation

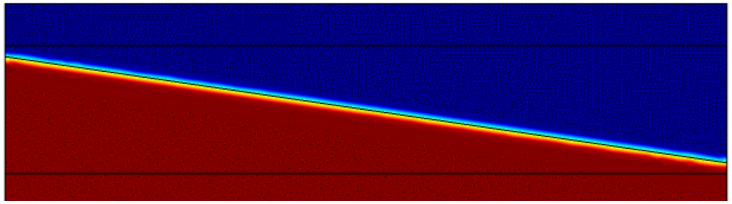




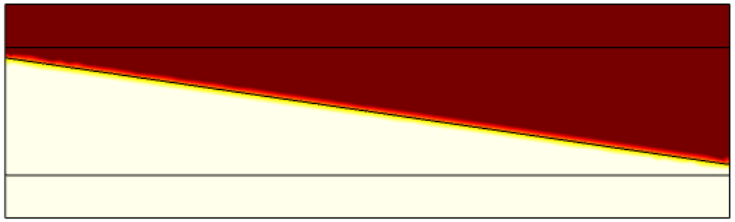


Two-fluid Interface Dynamics Simulation

MultiPhase-interface in Comsol

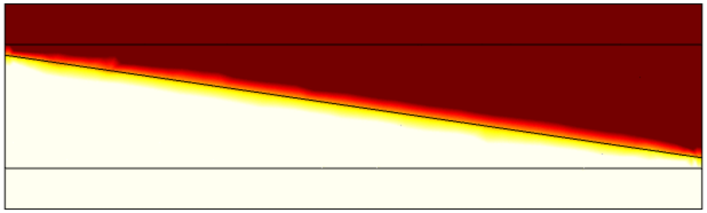


Single-phase-interface in Comsol



Fine mesh

Single-phase-interface in Comsol



Coarse mesh

End of lecture