Computational Fluid Dynamics

Fluid Mechanics II

Lecture 2

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SUMMARY OF LECTURE: FLUID MECHANICS II

- Understand the relation between Forces and Stresses in fluids
- Be able to give physical interpretation of Deformation and Rotation
- Be able to describe the Boundary layer concept
- Understand the origin of Laminar Boundary layer equations
- Understand the physical mechanism of Flow separation
- Be aware of the Bench-mark-system: Back step flow

Recap: Navier-Stokes equation and Vorticity equation

$$\rho \frac{\partial u_i}{\partial t} + \rho \left(u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + f_i$$

- $\rho \frac{\partial u_i}{\partial t}$: Changes in time at fixed point
- $\rho \left(u_j \frac{\partial u_i}{\partial x_j} \right)$: Convective acceleration
- $-\frac{\partial p}{\partial x_i}$: Pressure force
- $\mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j}$: Viscous contribution
- f_i: External forces

$$\rho \frac{\partial \omega_i}{\partial t} + \rho \left(u_j \frac{\partial \omega_i}{\partial x_j} \right) = \rho \left(\omega_j \frac{\partial u_i}{\partial x_j} \right) + \mu \frac{\partial}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} + \xi_i$$

- $\rho \frac{\partial \omega_i}{\partial t}$: Changes in time at fixed point
- $\rho \left(u_j \frac{\partial \omega_i}{\partial x_j} \right)$: Convective term
- $\rho \left(\omega_j \frac{\partial u_i}{\partial x_j} \right)$: Vortex stretching
- $\mu \frac{\partial}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}$: Viscous contribution
- ξ_i: External source of vorticity

Reynolds number

$$\boxed{\mathrm{Re} = \frac{\text{Convective term}}{\text{Viscous term}} \sim \rho \frac{U^2}{L} / \mu \frac{U}{L^2} = \frac{\rho L U}{\mu}}$$

Recap: Example of Index calculation

$$\frac{\partial}{\partial x_i}\boxed{\rho\frac{\partial u_i}{\partial t}+\rho\left(u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j}} \Rightarrow \boxed{-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2}=\left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right)}$$

$$LHS=\rho\frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial t}+\rho\frac{\partial}{\partial x_i}\left(u_j\frac{\partial u_i}{\partial x_j}\right)=\rho\frac{\partial}{\partial t}\underbrace{\frac{\partial u_i}{\partial x_i}}_{\nabla\cdot\mathbf{u}=0}+\rho\left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right)+\rho u_j\underbrace{\frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_j}}_{\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_i}}$$

$$LHS=\rho\left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right)$$

$$RHS=-\frac{\partial}{\partial x_i}\frac{\partial p}{\partial x_i}+\mu\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_i}=-\frac{\partial^2 p}{\partial x_i^2}$$

LHS=RHS: $$\boxed{-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2}=\left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right)}$$

This equation can be used to analyze the pressure distribution for colliding vortices and moving vortices…

Recap: Example of Index calculation (cont.)

$$\boxed{\frac{\partial u_i}{\partial x_k} = S_{ik} + \Omega_{ij}}$$

S_{ij} = Rate of strain

Ω_{ij} = Rate of rotation

$$\begin{cases} S_{ik} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_k} + \dfrac{\partial u_k}{\partial x_i}\right) \\ \Omega_{ik} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_k} - \dfrac{\partial u_k}{\partial x_i}\right) \end{cases}$$

Alternative notation

$$u_{i,j} = S_{ij} + \Omega_{ij} \qquad \begin{cases} S_{ij} = \dfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) \\ \Omega_{ij} = \dfrac{1}{2}\left(u_{i,j} - u_{j,i}\right) \end{cases}$$

$$u_{i,j} \equiv \frac{\partial u_i}{\partial x_j}$$

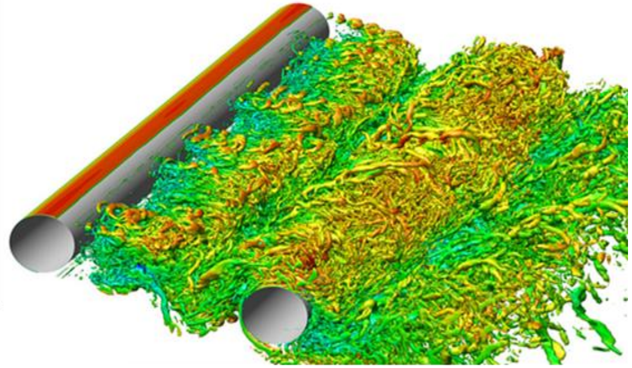
$$\Rightarrow \quad -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} = \left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right) = \left(\underbrace{S_{ji}}_{S_{ij}} + \underbrace{\Omega_{ji}}_{-\Omega_{ij}}\right)\left(S_{ij} + \Omega_{ij}\right) = \left(S_{ij} - \Omega_{ij}\right)\left(S_{ij} + \Omega_{ij}\right) = S_{ij}S_{ij} - \Omega_{ij}\Omega_{ij}$$

$$\Rightarrow \quad \boxed{-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} = S_{ij}S_{ij} - \Omega_{ij}\Omega_{ij}}$$

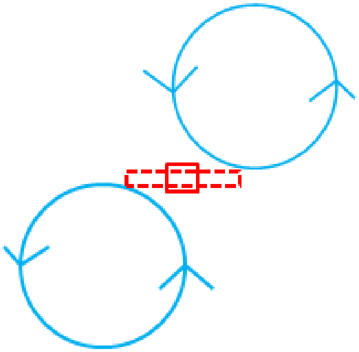
Clear physical interpretetion, the terms are connected to strain and rotation of fluid elements, respectively.

Recap: Example of Index calculation (cont.)

$$\boxed{-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} = S_{ij}S_{ij} - \Omega_{ij}\Omega_{ij}}$$



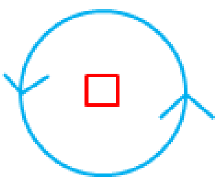
Colliding vortices



Fluid element
Small rotation
High strain

$\Rightarrow$ $$-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} > 0$$ $\Rightarrow$ Pressure maximum

Moving vortex



Fluid element
High rotation
Small strain

$\Rightarrow$ $$-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} < 0$$ $\Rightarrow$ Pressure minimum

More about the physical interpretation of strain and rotation later in this lecture…

Derivation of Navier-Stokes

Study group exercise

Navier-Stokes equation is derived from Newtons 2:nd law by summing all forces on a small fluid element and assuming relations between stress and velocity (see A and B).

$$\boxed{\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{\partial \sigma_{ij}}{\partial x_j}+f_i}$$

The momentum equations

A) Split the stress into two parts

$$\boxed{\sigma_{ij}=\underbrace{-p\delta_{ij}}_{\text{Stationary stress}}+\underbrace{\tau_{ij}}_{\text{Flow stress}}}$$

More comments on this model later

B) Assume a Newtonian fluid

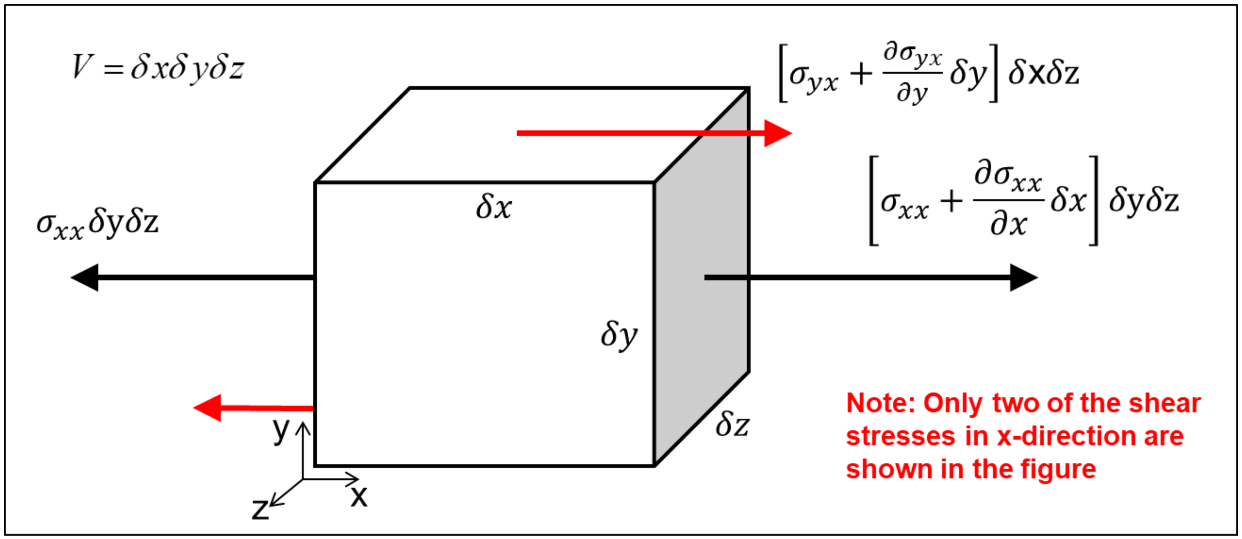
$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

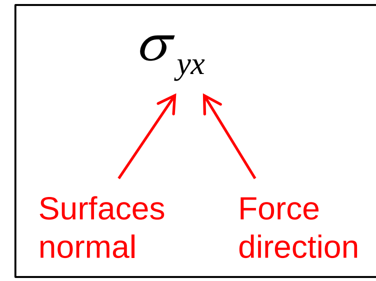
The momentum equations become closed, i.e. the unknown stresses are written in terms of known quantities

Comments on “A) Split the stress into two parts”

What do we mean by a stress in this context?

Acceleration of small fluid element in x-direction: $\rho V \dfrac{Du_x}{Dt} = F_x$





$$\left[\sigma_{ij}\right] = \frac{N}{m^2}$$

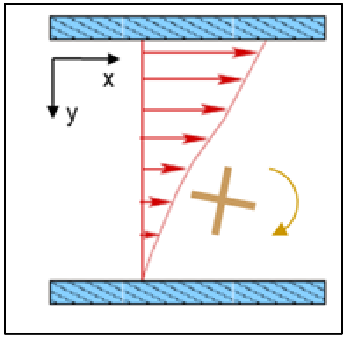
$$\boxed{\sigma_{ij} = \underbrace{-p\delta_{ij}}_{\text{Stationary stress}} + \underbrace{\tau_{ij}}_{\text{Flow stress}}}$$

Comments on ”B) Assume a Newtonian fluid”

Newtonian fluid 1D

$$\tau = \mu \frac{\partial u}{\partial y}$$

Dynamic viscosity [Pas]



Newtonian fluid

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = 2\mu S_{ij}$$

Non-Newtonian fluid: Paint, ketchup,…

Remember first step in the derivation of N-S:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i$$

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

Why is it important that the shear stress is modeled in terms of velocity gradients?

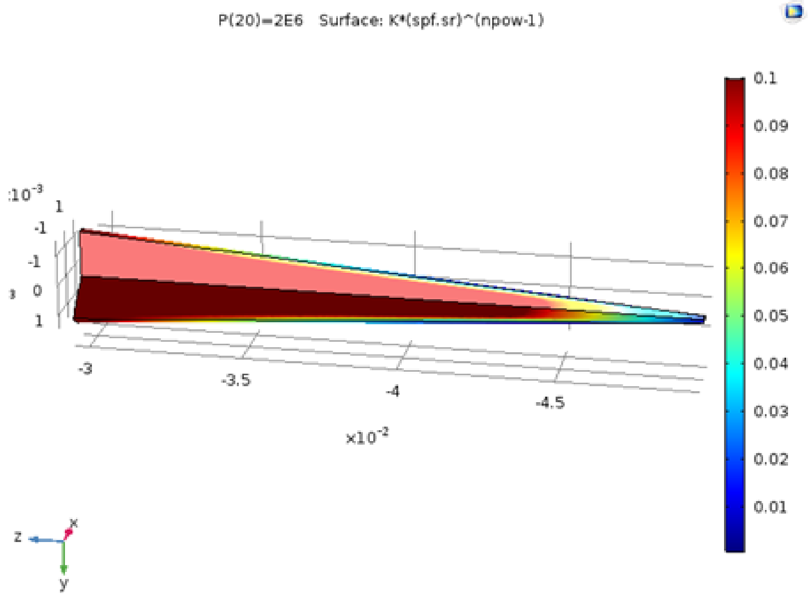
Exercise: Index calculations

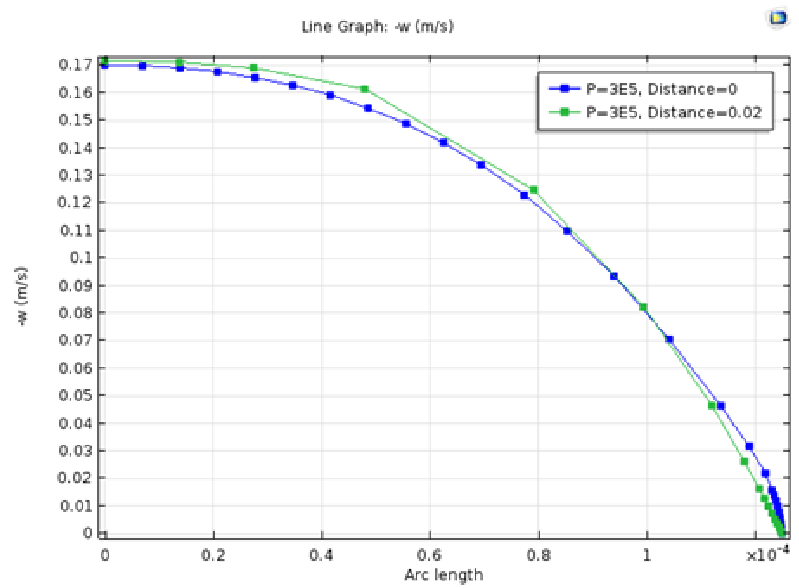
$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \mu \frac{\partial}{\partial x_j} \frac{\partial u_j}{\partial x_i} = \mu \nabla^2 u_i$$

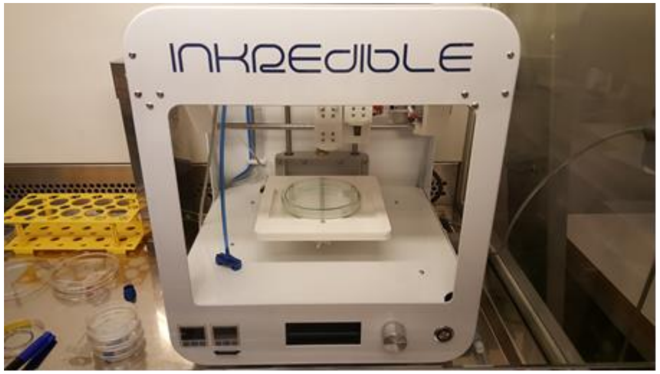
$$\mu \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} = 0$$

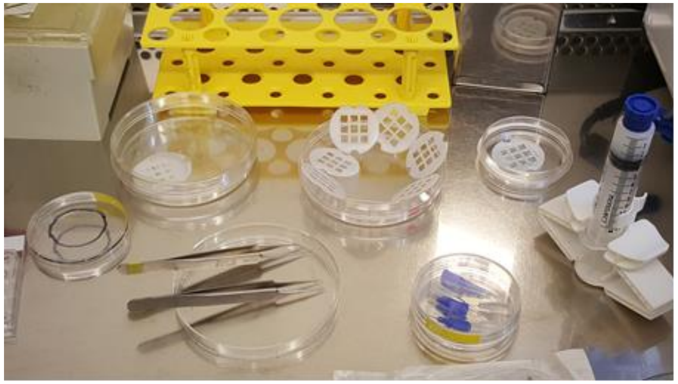
Example: Non-Newtonian flow in 3D-printers for biomaterials

A CFD master thesis by Anton Bahrd, 2016/17









Physical interpretation of *Rate of strain* and *Rate of rotation*

$$u_{i,j} = S_{ij} + \Omega_{ij} \qquad u_{i,j} \equiv \frac{\partial u_i}{\partial x_j}$$

$$S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$ **Rate of strain** (symmetric)

$$\Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$ **Rate of rotation** (anti-symmetric)

Claims

1 Sum of diagonal terms in the *Rate of strain* tensor measures <u>rate of volume expansion</u>

2 Off-diagonal terms in the *Rate of strain* tensor measures the <u>rate of shear deformation</u>

3 The *Rate of rotation* is a measure of the local angular velocity in a fluid

$$S_{ij} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix} \qquad \Omega_{ij} = \begin{pmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{pmatrix}$$

Summary

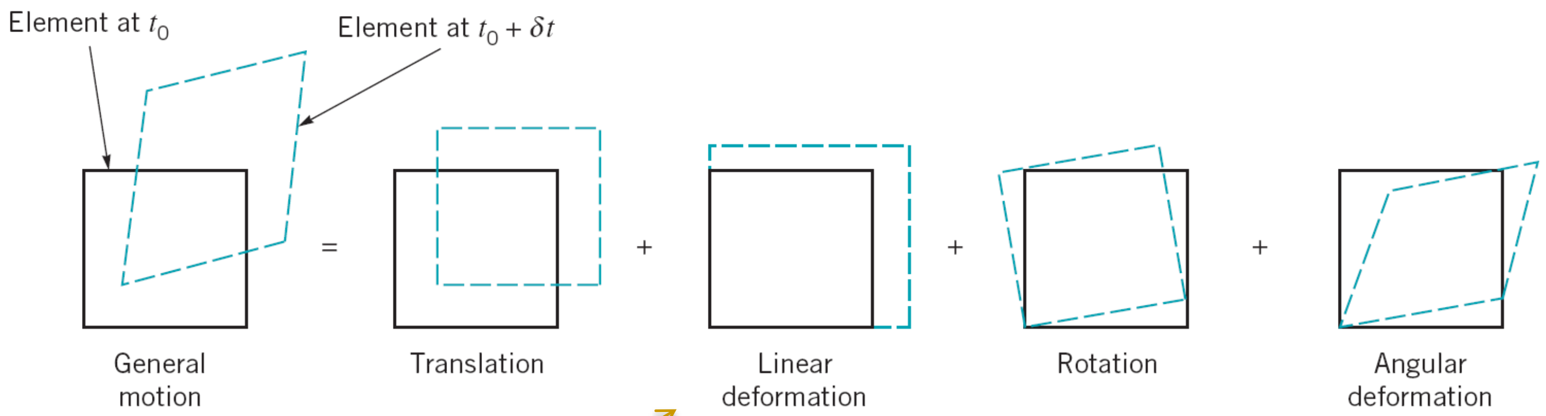
$$S_{ij} = S_{ji}$$

$$S_{ii} = \frac{1}{2}\left(u_{i,i} + u_{i,i}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{u}$$

$$\Omega_{ij} = -\Omega_{ji}$$

$$\Omega_{ii} = \frac{1}{2}\left(u_{i,i} - u_{i,i}\right) = 0$$

Summary: Deformation and rotation



Sum of diagonal terms

$$S_{ii} = \frac{1}{2}\left(u_{i,i} + u_{i,i}\right)$$

Rate of volume expansion

$$\Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

Rate of rotation

Off-diagonal terms

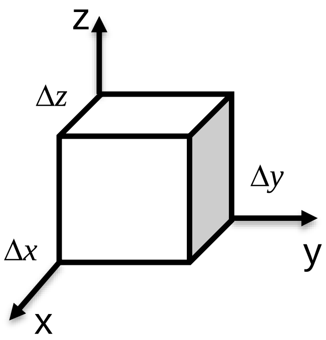
$$S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$i \neq j$$

Rate of shear strain

Claim 1: Rate of volume expansion

$$u_{i,j} = S_{ij} + \Omega_{ij} \quad S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad \Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$



Initial volume

$$Vol(0) = \Delta x \Delta y \Delta z$$

Final volume

$$Vol(\delta t) = \Delta x_{new} \Delta y_{new} \Delta z_{new} = ?$$

Expansion in x-direction

$$u(\Delta x) = u(0) + \frac{\partial u}{\partial x}\Delta x \quad \Rightarrow \quad u_{rel} = u(\Delta x) - u(0) = \frac{\partial u}{\partial x}\Delta x$$

New side length

$$\Delta x_{new} = \Delta x + u_{rel}\delta t = \Delta x + \frac{\partial u}{\partial x}\Delta x \delta t = \Delta x\left(1 + \frac{\partial u}{\partial x}\delta t\right)$$

Repeat for each side…

$$Vol(\delta t) = \Delta x\left(1 + \frac{\partial u}{\partial x}\delta t\right)\Delta y\left(1 + \frac{\partial v}{\partial y}\delta t\right)\Delta z\left(1 + \frac{\partial w}{\partial z}\delta t\right) = \underbrace{\Delta x \Delta y \Delta z}_{Vol(0)} + \Delta x \Delta y \Delta z \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{\nabla \cdot u}\delta t + O(\delta t^2)$$

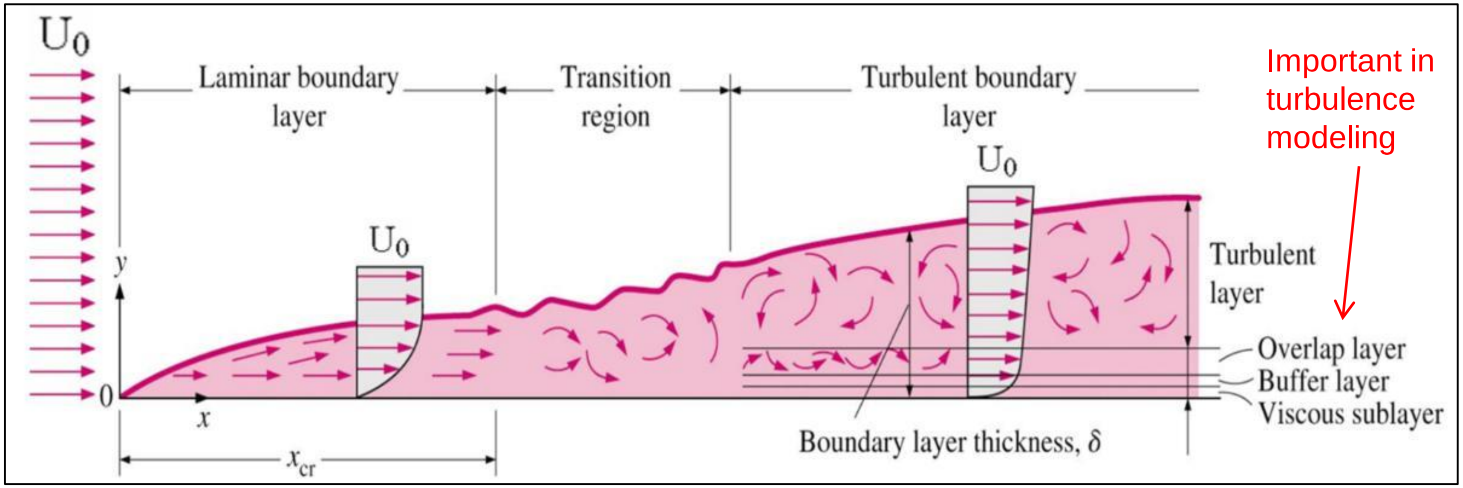
$$\frac{1}{Vol(0)}\frac{Vol(\delta t) - Vol(0)}{\delta t} = \nabla \cdot u + O(\delta t) \quad \Rightarrow \quad \boxed{\frac{1}{Vol}\frac{\partial Vol}{\partial t} = \nabla \cdot u = S_{ii}}$$

The sum of diagonal terms in the *Rate of strain* tensor measures rate of volume expansion. Incompressible flow means no expansion.

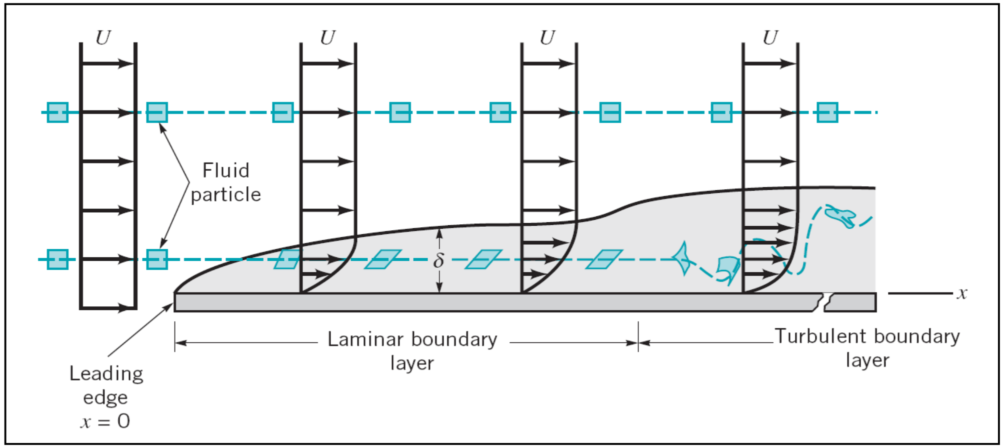
Boundary layers in fluid dynamics

The bounday layer is

- ❑ an example of a flow with large deformation and rotation of fluid particles
- ❑ a region where vorticity is created
- ❑ a region that can have huge impact on the rest of the flow



Laminar boundary layer



Assume

$\boxed{\delta << L}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Exercise

$$\boxed{V \sim \frac{\delta}{L} U}$$

U and V are typical velocity scales

$$\frac{\partial p}{\partial y} \approx 0$$

Pressure at upper part of BL is the same as pressure inside BL

See exercise below

Outside BL

$$\boxed{p + \tfrac{1}{2}\rho U^2 + \rho g y = const}$$

$$\left(\frac{dp}{dx} + \rho U \frac{dU}{dx} = 0 \right)$$

Inside BL

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

BL equations

BL assumption: $\delta << L$ $\Rightarrow$ $V \sim \frac{\delta}{L} U$

Momentum in x-direction

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

$$U\frac{U}{L} \qquad \underbrace{V\frac{U}{\delta}} \qquad \frac{\mu U}{\rho}\left(\cancel{\frac{1}{L^2}} + \frac{1}{\delta^2}\right)$$

$$\frac{\delta}{L}U\frac{U}{\delta} = \frac{U^2}{L}$$

$$\frac{\mu U}{\rho}\frac{1}{\delta^2} = \frac{U^2}{L}\left(\frac{\mu}{\rho U L}\frac{L^2}{\delta^2}\right) = \frac{U^2}{L}\left(\underbrace{\boxed{\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}}}_{\sim 1}\right)$$

In a BL the viscosity is assumed important by definition

$$\Rightarrow \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \approx -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

Momentum in y-direction

$$\boxed{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}$$

$$U\frac{V}{L} \qquad V\frac{V}{\delta} \qquad \frac{\mu V}{\rho}\left(\cancel{\frac{1}{L^2}} + \frac{1}{\delta^2}\right)$$

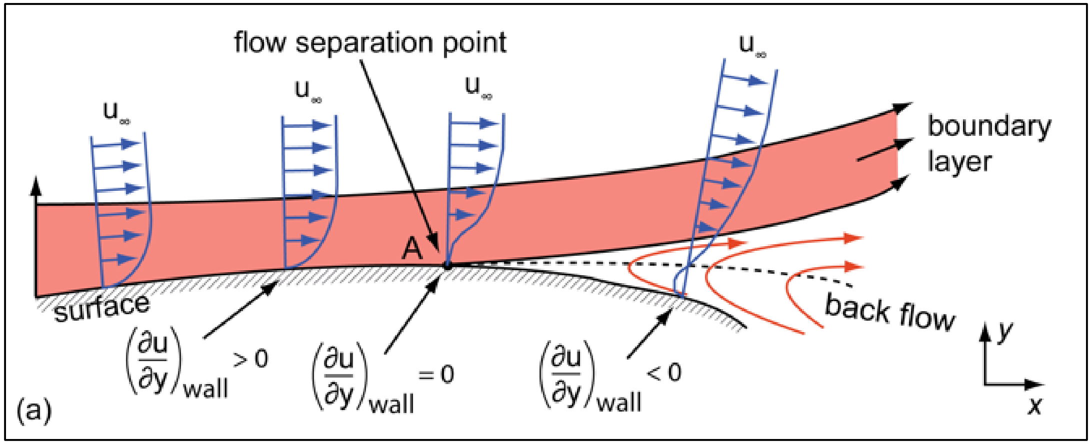
$$\frac{U^2}{L}\frac{\delta}{L} \qquad \boxed{\frac{1}{\mathrm{Re}}\frac{L^2}{\delta^2}}\frac{U^2}{L}\frac{\delta}{L}$$

Small

$$\frac{1}{\rho}\frac{\partial p}{\partial y} \approx 0$$

The pressure within the BL is the same as the pressure at the edge of BL.

Flow separation



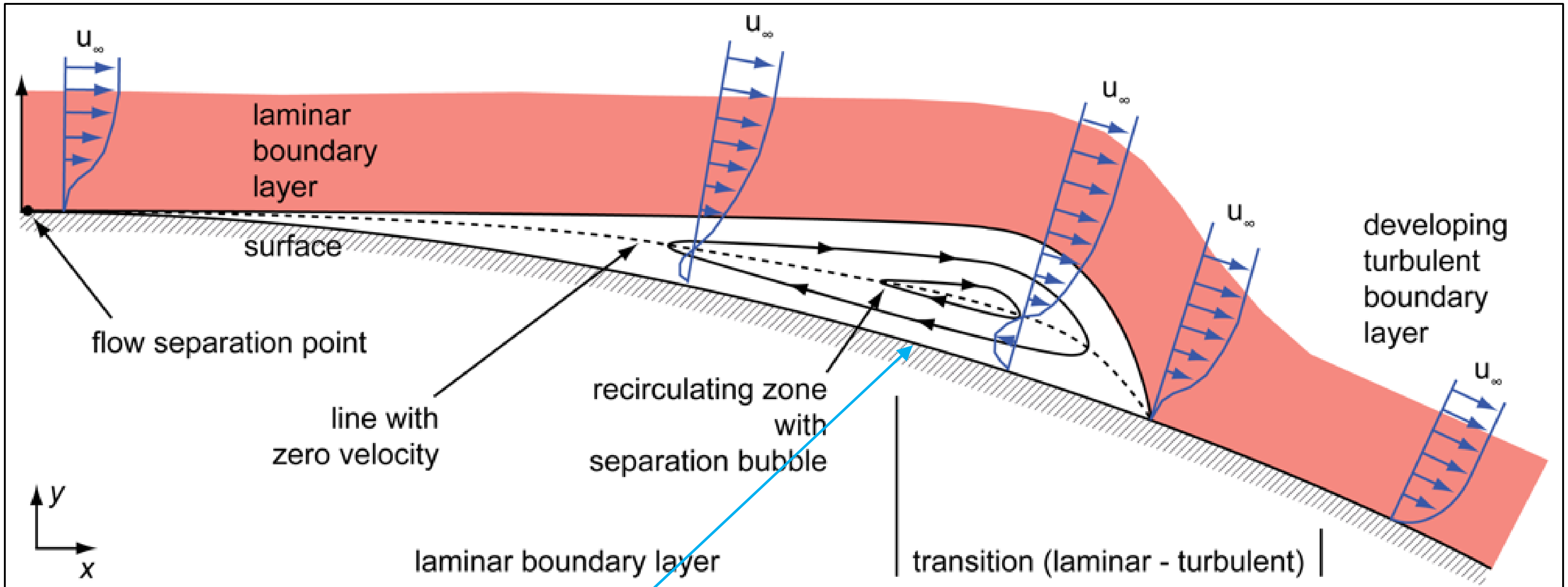
Outside of BL

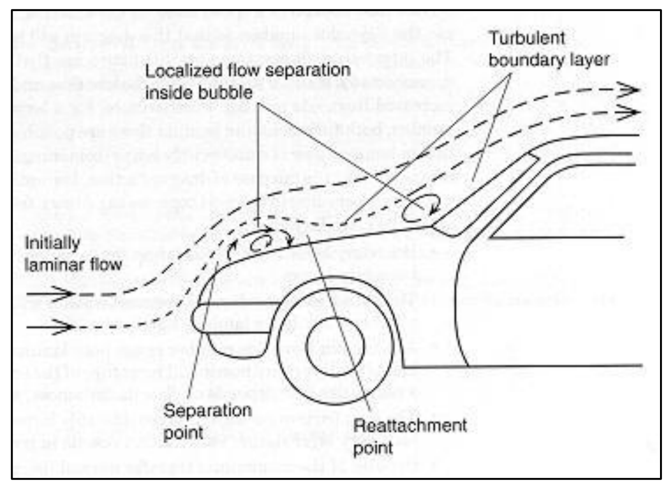
$$\boxed{\frac{dp}{dx} + \rho U \frac{dU}{dx} = 0}$$

Inside boundary layer

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}}$$

1. If the flow *U* outside BL is deaccelerated, a positive pressure gradient along the flow must be present.
2. The same pressure gradient is projected into the BL (no vertical pressure gradient within a BL).
3. Since the flow near the wall is slower compared to U, the flow near the wall can become zero even for minor changes in *U* (minor changes in pressure along the flow).
4. If the pressure gradient acts long enough, the velocity near the wall will reverse.
5. A reversed flow indicates that the BL has separated from the wall.



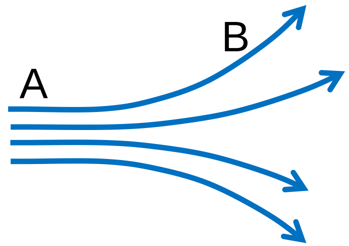


The circulating zone (separation bubble) acts as if it was a part of the body and has high impact on the outer flow.

Flow separation (cont.)

Bernoulli´s equation

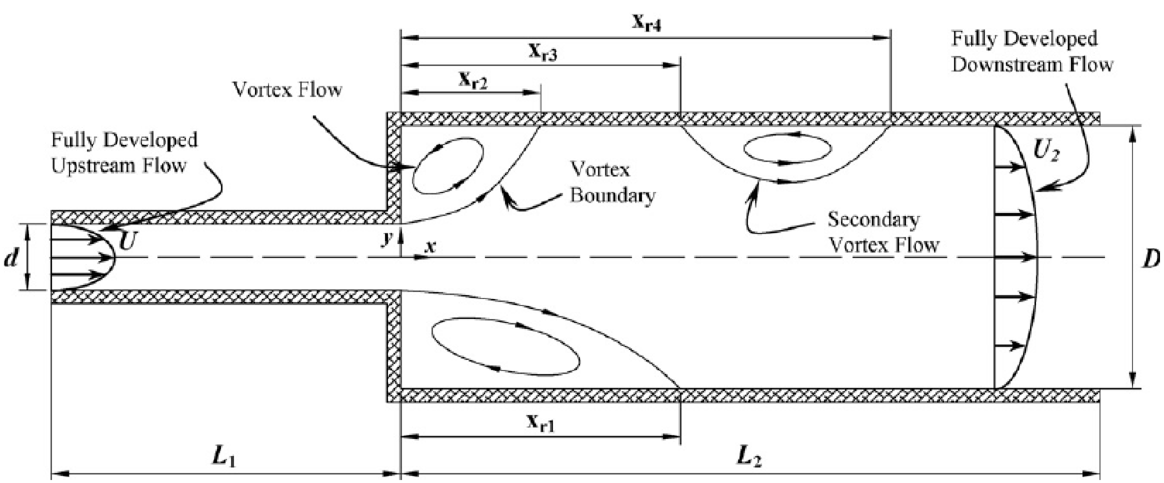
$$\boxed{\frac{U_A^2}{2g}+\frac{p_A}{\rho g}=\frac{U_B^2}{2g}+\frac{p_B}{\rho g}} \Rightarrow \rho\frac{U_A^2}{2}-\rho\frac{U_B^2}{2}=p_B-p_A$$

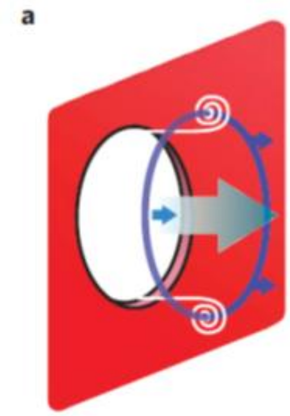


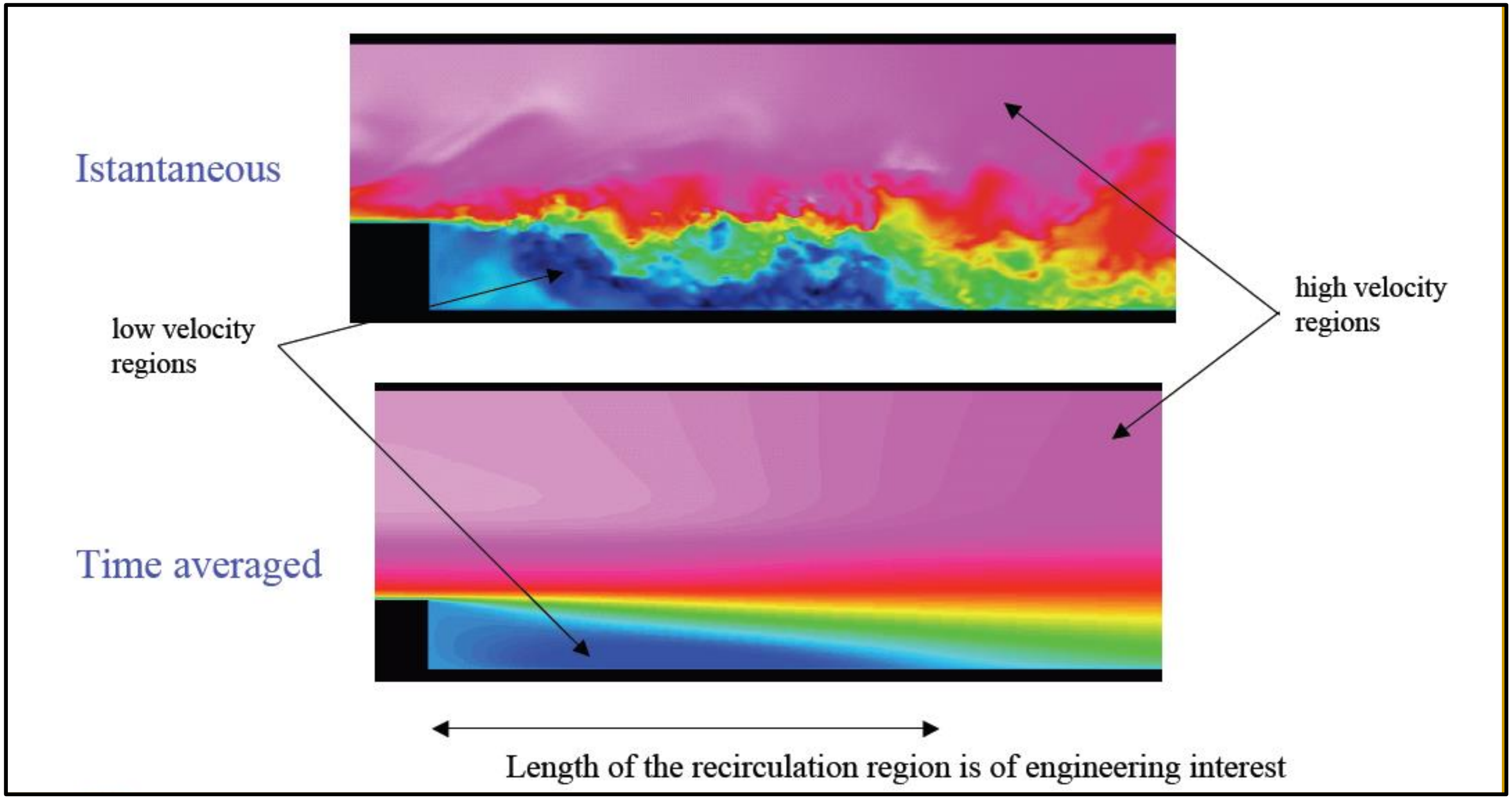
A deaccelerated flow implies an adverse pressure gradient

$$U_A > U_B \Rightarrow p_B > p_A$$

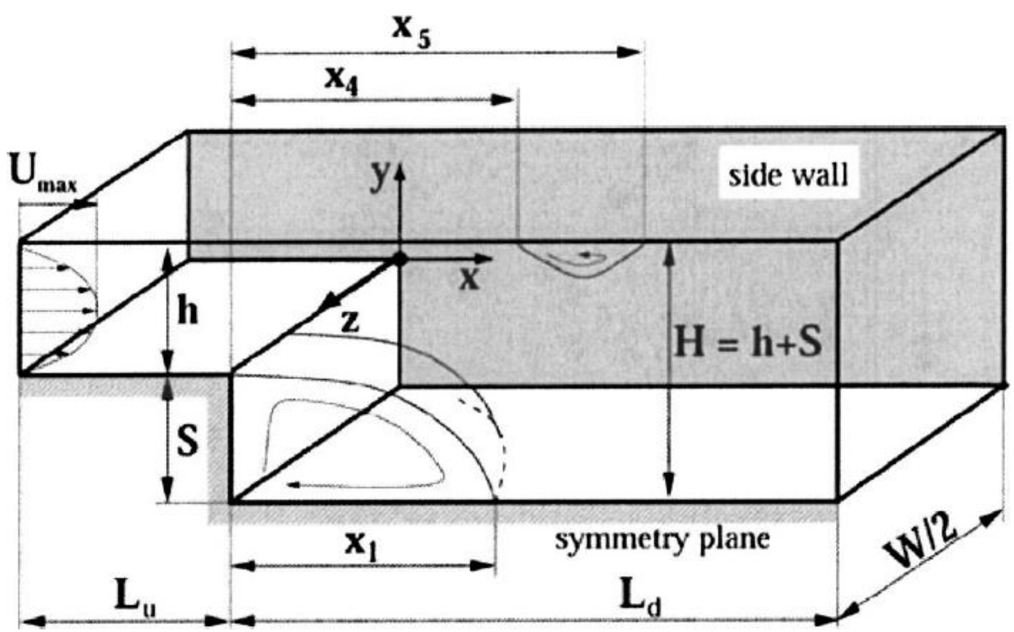
Ex. Diverging streamlines






Istantaneous
low velocity
regions
high velocity
regions
Time averaged
Length of the recirculation region is of engineering interest

Example: Flow over back step



$$\boxed{\mathrm{Re} = \frac{\rho U L}{\mu}}$$ Reynolds number

$$U_{\max} = \frac{3}{2} U_b$$

$$U_b = \text{mean velocity}$$

$$\mathrm{Re}_D = \frac{\rho U_b D}{\mu}, \quad D = 2h$$

$$\mathrm{Re}_h = \frac{\rho U_b h}{\mu}$$

$$\mathrm{Re}_S = \frac{\rho U_b S}{\mu}$$

Near wall simulations: $\mathrm{Re}_\tau = \frac{\rho U_\tau h}{\mu}$ $\quad U_\tau = \text{friction velocity}$

From Biswas, Breuer and Durst," Backward-Facing Step Flows for Various Expansion Ratios at Low and Moderate Reynolds Numbers", 2004

End of lecture