Computational Fluid Dynamics Turbulence Models I Free shear flows

Lecture 8

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SUMMARY OF LECTURE: TURBULENCE MODELS I

- ❑ Be aware of some different types of Free shear flows
- ❑ Understand the basic settings for the Prandtl Mixing length model
- ❑ Be aware of the strength and weakness of Prandtl mixing length model
- ❑ Understand why the k-equation model was introduced
- ❑ Be able to make a physical interpretation of the terms in the Modeled k-eq.
- ❑ Be aware of the strengths and weakness of the one-equation approach using the Modeled k-equation
- ❑ Understand why the equation for epsilon was introduced
- ❑ Be able to make a physical interpretation of the terms in the Standard k-epsilon model
- ❑ Be aware of the strengths and weakness of the Standard k-epsilon model
- ❑ Be aware of boundary and initial conditions for Standard k-epsilon model

Summary: Reynolds Averaged Navier-Stokes (RANS)

Navier-Stokes equations

$$\rho\frac{\partial u_i}{\partial t}+\rho\left(u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left(-p\delta_{ij}+\tau_{ij}\right)+f_i$$

$$u_{i,i}=0$$

$$\tau_{ij}=\mu(u_{i,j}+u_{j,i})$$

+

Reynolds decomposition

$$u_i=U_i+u_i'$$

$$p=P+p'$$

$$\tau_{ij}=T_{ij}+\tau_{ij}'$$

+

Time average

Averaging properties

RANS equations

$$\rho\left(\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}\right)=-\frac{\partial P}{\partial x_i}+\frac{\partial T_{ij}}{\partial x_j}+\frac{\partial R_{ij}}{\partial x_j}$$

$$\frac{\partial U_k}{\partial x_k}=0$$

Note: We need to model this term…

Reynolds stress

$$R_{ij}=-\left\langle \rho u_i' u_j' \right\rangle$$

$$\frac{\partial R_{ij}}{\partial x_j}$$

Summary of RANS (cont.)

$$\frac{\partial R_{ij}}{\partial x_j} = ?$$

The RANS approach ends with a need of modeling the Reynolds stress term in the averaged Navier-Stoke equation.

1. Boussinesq hypothesis models

2. Reynolds stress models (RSM)

$$R_{ij} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij}$$

Further modelling

Zero-eq.

One-eq.

Two-eq.

This course

- Add six independent equations for the Reynolds stresses
- Derive their governing equations from NS
- Complicated and computational expensive

Closure problem: Turbulence modeling

Many different approaches have been use to handle the closure problem.

Some of these are:

• Zero-equation / Algebraic models	(mixing length models,…)
• One-equation models	(k-model, μ_t-model,…)
• Two-equation models	(k-ε, k-ω,…)
• Reynolds stress models	(Not based on Boussinesq hypothesis)

Turbulence models based on Boussinesq hypothesis

1. Mixing length model
2. k-equation model
3. Spalart-Allmaras one-equation model
4. Standard k-ε model
5. k-ω model (Wilcox)
6. Low Re k-ε model
7. Menter SST k-ω
   - Blending function
8. Realizable k-ε
   - Reynolds stress constraint

Todays lecture

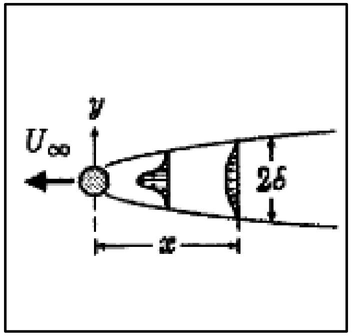
A special model for airfoils

The two most basic two-eq. models

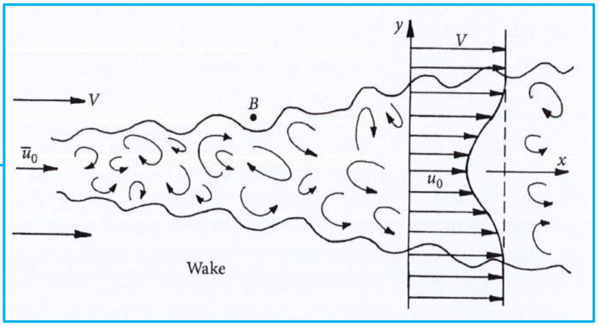
Correction for near wall behavior
(Wilcox k-ω model has not the same need for corrections near wall)

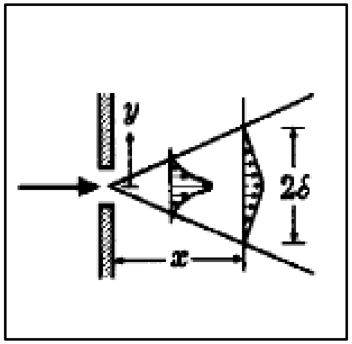
Many other modern models exists…

Turbulence models for Free shear flows

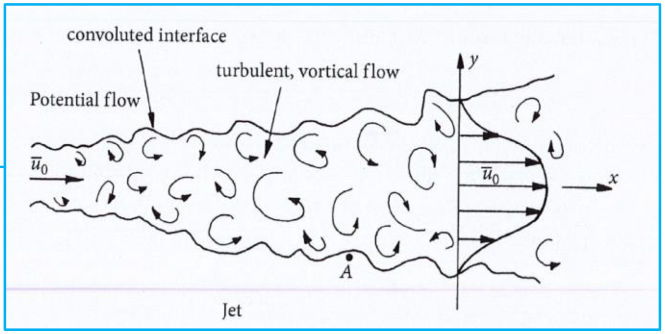


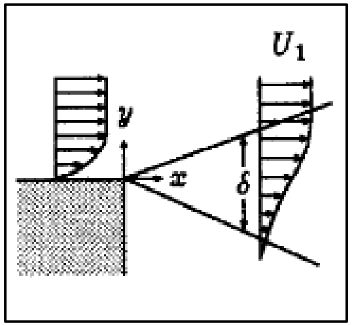
Far wake flow



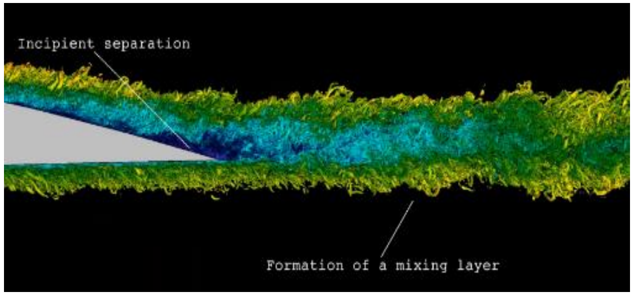


Jet flow





Mixing layer



Simulation
35 000 000 Core hours
(75 Tb of data)

Nr of cores at Kebnekaise:
~ 17 000 => 85 days

Zero-equation model

Consider shear stress from molecular momentum transport:

$$\tau = \mu \frac{\partial u}{\partial y} \qquad \mu = \frac{1}{2}\rho v_{th} l_{mfp}$$

v_{th} = thermal velocity

l_{mfp} = mean free path

Average distance between collisions

Prandtl mixing length model:

1. $\mu_t = \rho v_{mix} l_{mix}$
2. $\underbrace{v_{mix} \propto l_{mix} \frac{dU}{dy}}$

$$\mu_t = \rho l_{mix}{}^2 \frac{dU}{dy}$$

Now we instead have to find a suitable length scale…

This assumption will be upgraded in later models

Prandtl mixing length: $l_m = \alpha\delta$

Depends on situation, see figures in previous slide.

Closure coefficient to be determined.

Wilcox 1993

Far wake: 0.180
Plane Jet: 0.098
Radial Jet: 0.080
Mixing layer: 0.071

Summary of Zero-equation model
-The free shear turbulence case

- Easy to implement
- One closure coefficient needed
- The mixing length is determined by assumptions, not by the flow
- The mixing length model does not model transport of turbulence by the flow, all turbulence are created locally by the local gradient of the mean flow velocity
- Zero mean gradient implies that mixing velocity is zero. This is not correct in for example the middle of a pipe where the turbulent viscosity can be high

Improvement of mixing length model:
Use a velocity scale based on the turbulent kinetic energy instead of assuming it proportional to mean velocity gradient as in the mixing length model.

$$k \equiv \tfrac{1}{2}\langle u_i' u_i' \rangle \quad \Rightarrow \quad \sqrt{k} \sim \text{ velocity scale} \quad \Rightarrow \quad \mu_t = \rho\sqrt{k}\, l_{mix}$$

Derivation of k-equation

How do we obtain an evolution equation for the turbulent kinetic energy?

$$\boxed{k \equiv \tfrac{1}{2}\langle u_i' u_i' \rangle} \Rightarrow \frac{\partial k}{\partial t} = \left\langle u_i' \frac{\partial}{\partial t} u_i' \right\rangle$$

Remember

$$u_i = U_i + u_i'$$

Mean flow

Fluctuation

1) Multiply Navier-Stokes equation by u_i'

Ex. First term: $u_i' \rho \dfrac{\partial (U_i + u_i')}{\partial t}$

2) Apply a time average on each term

Ex. First term: $\rho \left\langle u_i' \dfrac{\partial (U_i + u_i')}{\partial t} \right\rangle = \rho \left\langle u_i' \dfrac{\partial U_i}{\partial t} \right\rangle + \rho \left\langle u_i' \dfrac{\partial u_i'}{\partial t} \right\rangle = \rho \langle u_i' \rangle \dfrac{\partial U_i}{\partial t} + \rho \dfrac{\partial k}{\partial t} = \rho \dfrac{\partial k}{\partial t}$

$$\Rightarrow \quad \rho \frac{\partial k}{\partial t} + \ldots = \langle u_i' \cdot \mathrm{RHS} \rangle$$

Derivation of k-equation (cont.)

3) Do the same for each term in Navier-Stokes equation…

$$\Rightarrow \quad \rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = \underset{\text{I}}{R_{ij} \frac{\partial U_i}{\partial x_j}} - \underset{\text{II}}{\mu \left\langle \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j} \right\rangle} + \frac{\partial}{\partial x_j} \left(\underset{\text{III}}{\mu \frac{\partial k}{\partial x_j}} \underbrace{- \frac{1}{2} \rho \left\langle u_i' u_i' u_j' \right\rangle - \left\langle p' u_j' \right\rangle}_{\text{IV}} \right)$$

Yet again we have a closure problem due to the primed variables…

Last time (in RANS) we solved the problem by using Boussinesq hypothesis.

This time the approach is to make different closure models for each term I-IV

Exact turbulent kinetic energy equation: Interpretation of terms

$$\underbrace{\rho\frac{\partial k}{\partial t}+\rho U_j\frac{\partial k}{\partial x_j}}_{\text{Rate of change of turbulent kinetic energy}} = \underbrace{R_{ij}\frac{\partial U_i}{\partial x_j}}_{\mathrm{I}} - \underbrace{\mu\left\langle\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle}_{\mathrm{II}} + \frac{\partial}{\partial x_j}\left(\underbrace{\mu\frac{\partial k}{\partial x_j}}_{\mathrm{III}} \underbrace{-\frac{1}{2}\rho\left\langle u_i'u_i'u_j'\right\rangle-\left\langle p'u_j'\right\rangle}_{\mathrm{IV}}\right)$$

Celik Lecture Notes
Section 7 + App. C

$$(I) = R_{ij}\frac{\partial U_i}{\partial x_j}$$

Production — "The rate at which the kinetic energy is transferred from the mean flow to the turbulence"

$$(II) = \mu\left\langle\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle \approx \rho\varepsilon > 0$$

Dissipation (ε) — "The rate at which the turbulent kinetic energy is converted into thermal energy"

$$\rho\varepsilon = 2\mu\left\langle S_{ij}^{'}S_{ij}^{'}\right\rangle = \mu\left\langle\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle + \mu\frac{\partial^2\left\langle u_i'u_j'\right\rangle}{\partial x_i\partial x_j}$$

$$(III) = \frac{\partial}{\partial x_j}\left(\mu\frac{\partial k}{\partial x_j}\right)$$

Diffusion — Diffusion by molecular motion

$$(IV) = \frac{\partial}{\partial x_j}\left(-\frac{1}{2}\rho\left\langle u_i'u_i'u_j'\right\rangle-\left\langle p'u_j'\right\rangle\right)$$

Turbulent transport
+ pressure fluctuations

Redistribution of kinetic energy

Example: Fluctuation Energy budget in a round jet

”Fingerprint” of turbulence

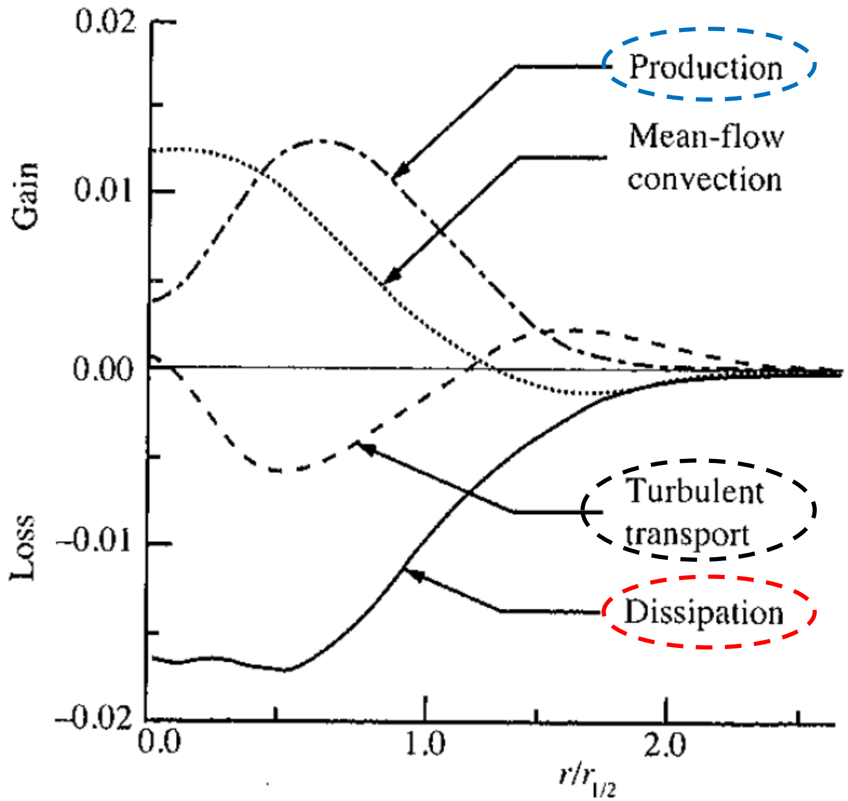
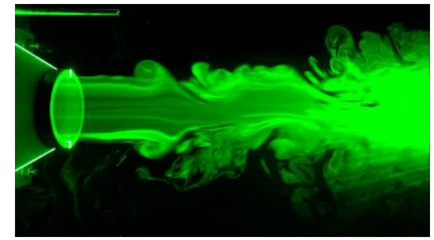
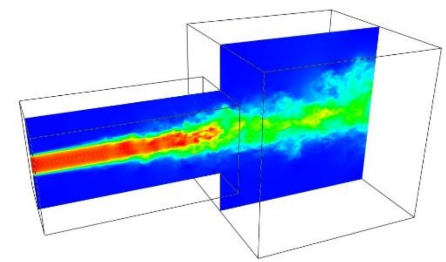


Fig. 5.16. The turbulent-kinetic-energy budget in the self-similar round jet. Quantities are normalized by U_0 and $r_{1/2}$. (From Panchapakesan and Lumley (1993a).)

From Pope, 2000





Note: For a large part production and dissipation is in balance.

What happens in the other regions?

Term-by-term modeling of k-equation

Production

$$(I) = R_{ij}\frac{\partial U_i}{\partial x_j}$$

Assumption

$$\boxed{\begin{aligned} &\mu_t = c\rho k^{1/2} l \\ &c = \text{constant} \\ &l = \text{turbulent length scale} \\ &k^{1/2} = \text{velocity scale} \end{aligned}}$$

Boussinesq hypothesis

$$\boxed{R_{ij} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho k \delta_{ij}}$$

$$\Downarrow$$

$$R_{ij}\frac{\partial U_i}{\partial x_j} = 2\mu_t S_{ij} S_{ij}$$

Dissipation

$$(II) = \mu\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle \approx \rho\varepsilon$$

Assumption

$$\boxed{\varepsilon \sim \frac{(\text{velocity scale})^3}{\text{length scale}} = C_D\frac{k^{3/2}}{l}}$$

Turbulent transport
+ pressure fluctuations

$$(IV) = \frac{\partial}{\partial x_j}\left(-\frac{1}{2}\rho\left\langle u_i' u_i' u_j'\right\rangle - \left\langle p' u_j'\right\rangle\right)$$

Assumption

$$\boxed{-\frac{1}{2}\rho\left\langle u_i' u_i' u_j'\right\rangle - \left\langle p' u_j'\right\rangle = \frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j}}$$

A tuning parameter

Again, similar approach as in Boussinesq hypothesis…

Modeled turbulent kinetic energy equation

$$\rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = R_{ij} \frac{\partial U_i}{\partial x_j} - \rho \varepsilon + \frac{\partial}{\partial x_j}\left((\mu + \frac{\mu_t}{\sigma_k}) \frac{\partial k}{\partial x_j} \right)$$

Lecture 7: $\varepsilon \sim \frac{U^3}{L}$

$l =$ turbulent length scale

$k^{1/2} =$ velocity scale

$\Rightarrow \varepsilon \sim \frac{k^{3/2}}{l}$

$$\boxed{\begin{aligned} \varepsilon &= C_D \frac{k^{3/2}}{l} \\ \mu_t &= c \rho l k^{1/2} \end{aligned}}$$

$\left.\begin{aligned} C_D &= const \\ c &= const \end{aligned}\right\}$ Closure constants

$\sigma_k = const$

Usually chosen to one

A common model

$$\boxed{\begin{aligned} C_D &= [0.07, 0.09] \\ c &= C_D^{1/3} \\ \sigma_k &= 1 \\ l &= l_{mix} \end{aligned}}$$

Note:

We still has to determine the turbulence length scale…

How well can this model predict experimental results?

The k-model compared with experiments

$$\rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = R_{ij} \frac{\partial U_i}{\partial x_j} - \rho\varepsilon + \frac{\partial}{\partial x_j}\left((\mu + \frac{\mu_t}{\sigma_k}) \frac{\partial k}{\partial x_j} \right)$$

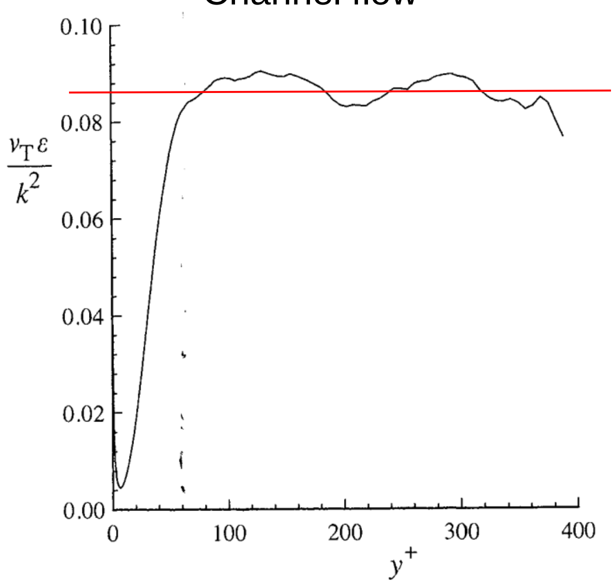
Difficult to measure

$$\varepsilon = C_D \frac{k^{3/2}}{l}, \qquad \mu_t = c\rho l k^{1/2}$$

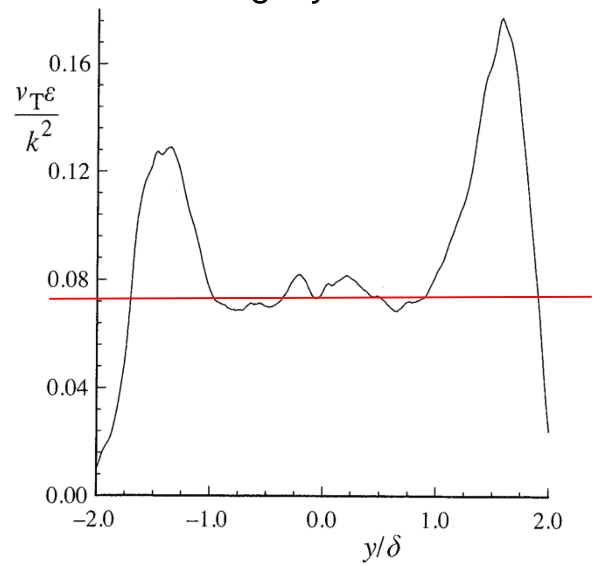
$$\mu_t = c\rho l k^{1/2} = c\rho C_D \frac{k^{3/2}}{\varepsilon} k^{1/2} = \underbrace{cC_D}_{\equiv C_\mu} \rho \frac{k^2}{\varepsilon} \Rightarrow C_\mu = \frac{\mu_t}{\rho}\frac{\varepsilon}{k^2} = \nu_t \frac{\varepsilon}{k^2}$$

Is the *RHS* constant everywhere according the experimental data?

Channel flow



Mixing layer



The experimental data is constant in some regions and here we have the value:

$$C_\mu \approx 0.08$$

From Pope Fig.10.3 & 10.4

A short note: Mean and fluctuation kinetic energy

Fluctuation kinetic energy: $k \equiv \frac{1}{2}\langle u_i' u_i' \rangle$

Mean flow kinetic energy: $K \equiv \frac{1}{2} U_i U_i$

$$K_{tot} = K + k$$

This term removes energy from mean flow

$$\rho \frac{\partial K}{\partial t} + \rho U_j \frac{\partial K}{\partial x_j} = -\mathrm{P} - \rho E + \frac{\partial}{\partial x_j} A$$

$$\rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = \mathrm{P} - \rho \varepsilon + \frac{\partial}{\partial x_j} B$$

Production

Dissipation

Turbulent transport

$$P \equiv R_{ij} \frac{\partial U_i}{\partial x_j}$$

$$E \equiv \frac{\mu}{\rho} \left\langle \frac{\partial U_i}{\partial x_j} \frac{\partial U_i}{\partial x_j} \right\rangle$$

$$\varepsilon \equiv \frac{\mu}{\rho} \left\langle \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j} \right\rangle$$

Summary of k-equation model

- Only a modest advantage in accuracy over mixing-length models
- Less adjustment is needed for different flows
- **Mixing length still has to be specified**
- It is for example not a good model for abrupt changes from wall-bounded to free shear flows (e.g. flow at trailing edge of an airfoil)

Next model improvement: Find a model where we account for transport effects of the turbulence length scale (important in separated flows).

This can for example be done by adding an extra equation for the dissipation

$$\varepsilon = C_D \frac{k^{3/2}}{l} \quad \Rightarrow \quad l = C_D \frac{k^{3/2}}{\varepsilon}$$

or by adding an equation for the **rate** of turbulent kinetic energy dissipation (Kolmogorov, 1942)

$$\omega = c \frac{k^{1/2}}{l} \quad \Rightarrow \quad l = c \frac{k^{1/2}}{\omega}$$

The Standard k-ε model

Important assumptions

1. Turbulent fluctuations are locally isotropic
   - A problem near a surface since both mean flow and fluctuations have to satisfy boundary condition that forces the flow to be anisotropic
2. Production and Dissipation are locally equal
   - As we shall see later this is not true near walls

Derivation of ε-equation

Dissipation

$$\varepsilon \equiv \frac{\mu}{\rho}\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle$$

1) Operate on Navier-Stokes equation by $\dfrac{\mu}{\rho}\dfrac{\partial u_i'}{\partial x_j}\dfrac{\partial}{\partial x_j}$

2) Apply a time average on each term and drop terms with zero average…

$$\frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = -2\mu\left[\left\langle u'_{i,k}u'_{j,k}\right\rangle + \left\langle u'_{k,i}u'_{k,j}\right\rangle\right]U_{i,j} - 2\mu\left\langle u'_k u'_{i,j}\right\rangle U_{i,kj}$$

$$-2\mu\left\langle u'_{i,k}u'_{i,m}u'_{k,m}\right\rangle - 2\frac{\mu^2}{\rho}\left\langle u'_{i,km}u'_{i,km}\right\rangle + \frac{\partial}{\partial x_j}\left[\mu\frac{\partial \varepsilon}{\partial x_j} - \mu\left\langle u'_j u'_{i,m}u'_{k,m}\right\rangle - 2\frac{\mu}{\rho}\left\langle p'_{,m}u'_{j,m}\right\rangle\right]$$

From here a term-by-term modeling approach can be used, but equally often one uses an empirical approach…

The standard *k-ε* model

Model k-equation

$$\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j}=\frac{\mu_t}{\rho}S^2-\varepsilon+\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

Model ε-equation

$$\frac{\partial \varepsilon}{\partial t}+U_j\frac{\partial \varepsilon}{\partial x_j}=C_{\varepsilon 1}\frac{\mu_t}{\rho}\frac{\varepsilon}{k}S^2-C_{\varepsilon 2}\frac{\varepsilon^2}{k}+\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

Production

Dissipation
(or destruction)

Redistribution
(diffusion)

Turbulent viscosity model

$$\mu_t=C_\mu\rho\frac{k^2}{\varepsilon}$$

Exercise
(see for example Celik)

$$P_k\equiv\frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j}=\ldots=\frac{\mu_t}{\rho}S^2$$

$$S^2\equiv 2S_{ij}S_{ij}$$

$$S_{ij}=\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$$

Closure constants
(Launder and Sharma)

$C_\mu=0.09$

$C_{\varepsilon 1}=1.44 \quad C_{\varepsilon 2}=1.92$

$\sigma_k=1.0 \quad \sigma_\varepsilon=1.3$

The standard *k*-*ε* model (cont.)

Model k-equation

$$\frac{Dk}{Dt} = P_k - \varepsilon + D_k$$

Model ε-equation

$$\frac{D\varepsilon}{Dt} = \frac{\varepsilon}{k}\left(C_{\varepsilon 1} P_k - C_{\varepsilon 2}\varepsilon\right) + D_\varepsilon$$

Turbulent viscosity model

$$\mu_t = C_\mu \rho \frac{k^2}{\varepsilon}$$

$$C_\mu = 0.09$$

$$C_{\varepsilon 1} = 1.44 \qquad C_{\varepsilon 2} = 1.92$$

$$\sigma_k = 1.0 \qquad \sigma_\varepsilon = 1.3$$

Closure constants
(Launder and Sharma)

k-production

$$P_k \equiv \frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j} = \frac{\mu_t}{\rho} S^2$$

k-diffusion

$$D_k \equiv \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

Dissipation

$$\varepsilon \equiv \frac{\mu}{\rho}\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle$$

ε-diffusion

$$D_\varepsilon \equiv \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

Some boundary and initial conditions

Standard k-ε model

Inlet:	Distributions of k and ε must be given
Outlet:	$\partial k / \partial n = 0,\ \partial \varepsilon / \partial n = 0$

$$\boxed{k = \frac{2}{3} U^2 I^2, \quad \varepsilon = C_\mu^{3/4} \frac{k^{3/2}}{l}, \quad l = 0.07L}$$

Turbulence intensity

$$\boxed{I \equiv \frac{u}{U}}$$

Usually a specified number at inlet (bc) or domain (initial value) depending on situation (channel flow, jet etc.)

Intensity and length scale dependency on conditions upstream

1) *Exhaust of a turbine*
Turbulence intensity = 20%. Length scale = 1 - 10 % of blade span.

2) *Downstream of perforated plate or screen*
Turbulence intensity = 10%. Length scale = screen/hole size.

3) *Fully-developed flow in a duct or pipe*
Turbulence intensity = 5%. Length scale = hydraulic diameter.

From notes by Bakker

Summary of *k*-ε model

- Simplest turbulence model for which only initial and/or boundary conditions need to be supplied
- Well established and tested
- More expensive to implement than mixing length model
- Five closure constants with standard values that represents a compromise, tuning may improve results but not recommended
- Overpredicts the spreading rate for a far wake and round jet (30%), mixing layer (15%) and plane jets (5%)
- Performs reasonable well for 2D thin layer shear flows in which the mean streamline curvature and mean pressure gradient are small
- For boundary layers with strong pressure gradients the model performs poorly

End of lecture